

SOME PROBLEMS IN ALGEBRA - THE HOMOMORPHISM
 $M \otimes N \rightarrow \text{HOM}(M, N)$, SPLITTING OF RING EXTENSIONS,
CRITERION FOR REGULARITY, HOMOLOGICAL
DUALITY AND BNSI RINGS.

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I certified that the thesis entitled "Some problems in Algebra - the homomorphism $M^ \otimes N \rightarrow \text{Hom}(M, N)$, splitting of ring extensions, criterion for regularity, homological duality and BNSI rings " submitted by Mr. Sib Nath Bose for the Degree of Philosophy of North Eastern Hill University, Shillong embodies the original work carried out by him under my supervision. He has been duly registered and the thesis presented is worthy of being considered for the Award of the Ph.D. Degree. This work has not been submitted for any degree of any other University.*

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Preface

The study of commutative algebra started with the works of famous authors like D. Hilbert, E. Noether, Macaulay and W. Krull. The latter progress in commutative algebra arises chiefly from quite different problems, issuing from Algebraic Geometry.

The concept of local rings was introduced by W. Krull in his paper "Dimensions Theorie in Stellen ringen". Krull conjectured a few problems in his paper. These were proved by C. Chevalley in "On the theory of Local rings" and by I.S. Cohen in his thesis "On the structure and ideal theory of complete local rings". Authors like Zariski, Nagata proved further important results with the study of local properties of Algebraic varieties. These developments are embodied in the famous work of Nagata "Local rings".

The second stage of development of the subject was ushered in by the lecture notes of J.P. Serre "Algebre - Localé Multiplicités", Classic papers of M. Auslander and D. Buchsbaum "Homological dimension in local rings" and "Maximal orders" by Auslander and Goldman. These authors introduced homological techniques into the subject. The proof of unique factorization in regular local rings by Auslander and Buchsbaum was an achievement signalling that these techniques have come to stay. Some more important results were proved by these

authors by using homological techniques. Lichtenbaum's proof that $\text{Tor}_i^R(M, N) = (0)$ implies $\text{Tor}_j^R(M, N) = (0)$ for all $j \geq i$ for finitely generated modules over a regular local ring is another achievement of this period. The subsequent appearance of the thesis of C. Peskin and L. Szpiro "Dimension Projective finite et Cohomologie Locale" and the work of M. Hochster on big Cohen Macaulay modules have all pointed out the effectiveness of homological techniques in Commutative Algebra. The "Zero-divisor Conjecture" by I. Kaplansky, the "Tor - Conjecture" by M. Auslander, the Conjecture of H. Bass on Cohen - Macaulay rings are still open problems to mention a few.

This dissertation consists of ~~six~~ chapters.

Chapter I deals with preliminary definitions and results used in the dissertation.

In Chapter II, we prove three theorems. In Theorem 2.1, we prove a result, a special case of which says that if M and N are two reflexive modules of finite projective dimensions over a Gorenstein local ring such that $\text{Hom}(M, N)$ is a third module of syzygies, then the natural homomorphism $M^* \otimes N \rightarrow \text{Hom}(M, N)$ is an isomorphism. This extends the result in [13]. In Theorem 2.2 we give a criterion for a module M over a regular local ring to have projective dimension less than or equal to an integer n . This extends the usual criterion for the projectivity of a module. In Theorem 2.3, we prove

that over a 1-dimensional Gorenstein local ring R if M, N are finitely generated R -modules such that $\text{Hom}(M, N)$ is nonzero free then both M^* and N are free. This is a generalization of a result of W. Vasconceles [29, Theorem 3.1].

If $R \subset S$ is an extension of rings (not necessarily commutative) making S into a projective R -module then R is a direct summand of S if and only if S is faithfully projective. This is a result of Cartzen and others [11, Theorem - 1]. The authors also give an example of a ring extension in which S is R -projective but not faithfully projective. The example involved noncommutative rings. In Chapter III, we show such a situation cannot happen if R is commutative i.e. if $R \subset S$ is an extension of rings making S projective as R -module then R is a direct summand of S and S is faithfully projective. This is Theorem 3.1 of this Chapter. In Bourbaki [10, Ex 5.4, p-176] (this is mentioned as an exercise when S is finitely generated as R -module.

In Chapter IV, we give a criterion for a noetherian local ring to be regular. This involves homological conditions on prime ideals of small height as compared to Hilbert-Serre Theorem which says that R is regular if and only if the maximal ideal of R has finite homological dimension. More precisely, suppose $\text{pd}_R \mathcal{O}_{\mathfrak{q}} \leq 2$ for every unmixed ideal \mathfrak{q} of height at most 2, then R is regular. W. Bruns [9] had earlier shown

that if $\text{pd } \mathcal{O}_i < \infty$ for every ideal \mathcal{O}_i all of whose associated prime ideals have depth at most 2, then R is regular. But our proposition is a direct improvement of M. Auslander's criterion given in [4, Theorem B, Corollary 5]. The proof also includes some results of independent interest giving criterion for a domain to be U.F.D.

In chapter - V, we give an application of homological duality to generalized M-regular sequences. The notion of M-sequence is generalized in [14] as follows: a sequence P_1, P_2, \dots, P_n of nonfree modules is said to be an M-sequence if $\text{Tor}_1^R (M \otimes P_1 \otimes \dots \otimes P_{i-1}, P_i) = (0)$ for $1 \leq i \leq n$. The sum $\sum \text{Pd } P_i$ is defined to be the length of the M-sequence P_1, P_2, \dots, P_n . We shall apply Strebel's homological duality ([28], §3, Theorem 13) to prove Theorem 5.1 on generalized M-sequences which states that if R is a regular local ring and M a nonzero R -module and $\{P_1, P_2, \dots, P_n\}$ is an M-sequence of nonfree perfect modules of length $\sum_{i=1}^n d_i$ $\mathcal{O}_i = \text{ann} P_i$, $d_i = \text{Pd } P_i$; for $1 \leq i \leq n$. Then for every i , $1 \leq i \leq n$, $\exists d_i$ elements $x_1^{(i)}, x_2^{(i)}, \dots, x_{d_i}^{(i)}$ in \mathcal{O}_i such that the sequence of $\sum d_i$ elements $\{x_j^{(i)}\}$, $1 \leq i \leq n$, $1 \leq j \leq d_i$ form an M-sequence in the usual sense.

(V)

In Chapter VI, i.e. the last Chapter we take R to be a non-commutative local ring i.e. R is a ring with jacobson radical \mathcal{O} such that $R/\mathcal{O} (= K)$ is a division ring. Mark Ramaras introduced the notion of BNSI rings in [24]. We have generalized this notion to non-commutative rings. A non-commutative local ring R is called a left BNSI ring if for every nonfree left R -module M , the sequence $\{\beta_i(M)\}_{i \geq 1}$ is strictly increasing where $\beta_i(M) = \dim_K \text{Tor}_i^R(K, M)$ is called the i th Bettinumber of M . If R is a left and right BNSI-ring with jacobson radical \mathcal{O} nilpotent and M is an indecomposable left R -module such that $\text{Ext}_R^1(M, R) = (0)$ then M is free. This is Theorem 6.6 of this Chapter which generalizes a theorem in [24].

CONTENTS

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CHAPTER-I

Preliminaries

In this chapter we give a brief outline of some preliminary definitions and results used in the dissertation. We also fix notations and other conventions here. Throughout this dissertation a ring R means a ring with identity and an R -module means an unitary module.

§1.1. Projective and injective modules, Tor. and Ext. functors:

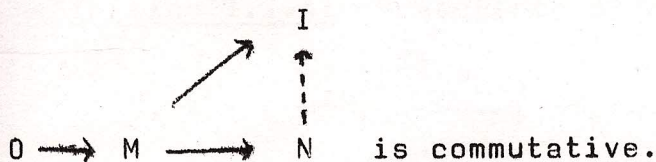
Definition 1.1.0 : An R -module P is called Projective if given an exact sequence of R -modules $M \rightarrow N \rightarrow O$ and an R -linear map $P \rightarrow N$ \exists an R -linear map $P \rightarrow M$ such that the diagram

$$\begin{array}{ccccc} P & & & & \\ \downarrow & \searrow & & & \\ M & \longrightarrow & N & \longrightarrow & O \end{array}$$

is commutative.

Example: Any free module is projective.

Definition 1.1.1 : An R-module I is injective if given an exact sequence $0 \rightarrow M \rightarrow N$ and a homomorphism $M \rightarrow I$ there exists a homomorphism $N \rightarrow I$ such that the diagram



Example: \mathbb{Q} is \mathbb{Z} -injective.

Definition 1.1.2: An exact sequence of maps and modules

$$\dots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where

P_i 's are projective is called a projective resolution of the R-module M.

Example: Consider $\mathbb{Z} / 2\mathbb{Z}$ as \mathbb{Z} -module. Then

$$0 \rightarrow 2\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

is a projective resolution of $\mathbb{Z} / 2\mathbb{Z}$.

It is easy to see that any module M has a projective resolution.

Definition 1.1.3: An exact sequence of maps and modules

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

where I_i 's are injective modules is called an injective resolution of M.

In this case also it can be shown that any module M has an injective resolution.

Example: $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ is an injective resolution of \mathbb{Z} as \mathbb{Z} -module.

Definition 1.1.4: A sequence of maps and modules

$$M_* : \dots \rightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} M_{n-2} \rightarrow \dots$$

such that $d_n \circ d_{n+1} = 0$ for every n is said to be a complex.

We denote this complex by M_* . Clearly $\text{Im } d_{n+1} \subseteq \text{Ker } d_n$.

The R -modules $\frac{\text{Ker } d_n}{\text{Im } d_{n+1}}$ are said to be the n th homology of the

Complex M_* and is denoted by $H_n(M_*)$.

Definition 1.1.5: Let $X_* \rightarrow N \rightarrow 0$ be a projective resolution of

N . Consider the complex $M \otimes X_*$ for any R -module M . Then the

n th homology of this complex is defined to be $\text{Tor}_n^R(M, N)$. It

can be shown easily that this definition is independent of the

resolution $X_* \rightarrow N \rightarrow 0$. These are bifunctors on the

category of R -modules and are called Tor functors. We

mention here some properties of Tor functors:

Proposition 1.1.0: (i) $\text{Tor}_0^R(M, N) = M \otimes_R N$.

(ii) If $0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0$ is an exact sequence of

R -modules, then there exists a long exact sequence

$$\dots \rightarrow \text{Tor}_n^R(M, N_1) \rightarrow \text{Tor}_n^R(M, N) \rightarrow \text{Tor}_n^R(M, N_2)$$

$$\rightarrow \text{Tor}_{n-1}^R(M, N_1) \rightarrow \dots \rightarrow M \otimes N_1 \rightarrow M \otimes N \rightarrow M \otimes N_2 \rightarrow 0.$$

(iii) $\text{Tor}_n^R(M, N) = \text{Tor}_n^R(N, M)$, if R is a commutative ring.

(iv) $\text{Tor}_n^R(M, N) = 0$ for $n > 0$ if N is a flat module.

Definition 1.1.6: The projective dimension of a module M is defined to be infinite if it has no finite projective resolution. Otherwise the length of the shortest projective resolution of M is said to be the projective dimension of M . The notation for Projective dimension of M is $\text{Pd } M$.

Definition 1.1.7: If a module M has no finite injective resolution, we say that injective dimension of M is infinite. Otherwise the length of the shortest injective resolution of M is defined to be the injective dimension of M . The notation for injective dimension of M is $\text{id } M$.

Definition 1.1.8: Let $0 \rightarrow N \rightarrow I_*$ be an injective resolution of N . Consider the complex $\text{Hom}(M, I_*)$. The n th homology of this complex is defined to be $\text{Ext}_R^n(M, N)$. $\text{Ext}_R(M, \quad)$ is a functor in the category of R -modules and is called Ext functor.

We mention a few properties of this functor:

Proposition 1.1.1: (i) $\text{Ext}_R^0(M, N) = \text{Hom}_R(M, N)$.

(ii) If $0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0$ is exact then there exists a long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(M, N_1) \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M, N_2) \rightarrow \text{Ext}_R^1(M, N_1) \\ \rightarrow \text{Ext}_R^1(M, N) \rightarrow \text{Ext}_R^1(M, N_2) \rightarrow \dots \end{aligned}$$

(iii) If $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is exact then there exists a long exact sequence

$$0 \rightarrow \text{Hom}(M_2, N) \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M_1, N) \rightarrow \text{Ext}_R^1(M_2, N) \\ \rightarrow \text{Ext}_R^1(M, N) \rightarrow \text{Ext}_R^1(M_1, N) \rightarrow \dots$$

§1.2: Primary decomposition of modules

Throughout this Section all rings are Commutative and modules are finitely generated. The set of prime ideals is called spectrum of R and is denoted as $\text{Spec } R$.

Definition 1.2.0. A prime ideal p of R is said to be an associated prime of a module M if one of the following equivalent conditions hold:

- (i) There exists an element $x \in M$ with $\text{Ann } x = p$.
- (ii) M contains a submodule isomorphic to R/p . The set of associated prime ideals of M is denoted as $\text{Ass } M$.

Definition 1.2.1: The support of a module M denoted as $\text{Supp } M$ is defined to be the set of all those prime ideals of R such that $M_p \neq 0$.

Definition 1.2.2: A module M is said to have finite length if it possesses Jordan-holfer series. We denote the length of a module M by $l(M)$.

Throughout the remaining part of this section R denotes a Noetherian ring.

Proposition 1.2.0: A module M over a ring R has finite length if $\text{supp } M$ consists of maximal ideals only.

Definition 1.2.3: A nonzero module M is said to be co-primary if given any $a \in R$, the homothety λ_a (multiplication by 'a' on M) is either injective or nilpotent.

Example: $\mathbb{Z}/2^3\mathbb{Z}$ is a co-primary \mathbb{Z} -module.

Definition 1.2.4: Suppose M is a co-primary module. Then the set $\{a \in R / \lambda_a \text{ is nilpotent}\}$ is a prime ideal of R . This prime ideal p is said to be the associated prime ideal of M and M is called p -co-primary.

Definition 1.2.5: Suppose M is a nonzero module. A proper submodule N of M is said to be p -primary if M/N is p -co-primary.

Definition 1.2.6: A proper submodule N of M is said to be irreducible if there does not exist a representation $N = N_1 \cap N_2$ with $N \subsetneq N_1$, $N \subsetneq N_2$.

Example: (0) is irreducible in $\mathbb{Z}/p^n\mathbb{Z}$ where p is a prime integer and $n \geq 1$.

Definition 1.2.7: Let N be a submodule of M . A primary decomposition of N is an equation $N = Q_1 \cap Q_2 \cap \dots \cap Q_r$ where Q_i 's are primary in M . Such a decomposition is said to be irredundant or reduced if no Q_i can be omitted and if

the associated primes of M/Q_i ($1 \leq i \leq r$) are all distinct. Any primary decomposition can be simplified to a reduced primary decomposition.

Proposition 1.2.1: Any proper submodule N of M has a primary decomposition.

§1.3: Dimension of rings and modules

Definition 1.3.0: A finite sequence of $(r+1)$ prime ideals $P_0 \supset P_1 \supset P_2 \supset \dots \supset P_r$ is called a prime chain of length r .

If p is a prime ideal of R , the supremum of the lengths of the prime chains descending from p is called ^{the} height of p and is denoted $ht(p)$. If $ht(p) = 0$, p is called a minimal prime ideal.

If I is a proper ideal of R , then ^{the} height of I is defined by $ht(I) = \inf \left\{ ht(p) / p \supseteq I \right\}$.

Definition 1.3.1: The Krull dimension of a ring R is defined to be the supremum of the heights of the prime ideals in R . Krull dimension will be abbreviated as $\dim R$. If R is a noetherian local ring then $\dim R$ is finite. It follows from definition that $ht(p) = \dim (R_p)$ for $p \in \text{Spec } R$ and for any ideal I of R , $\dim R/I + ht(I) \leq \dim R$.

Example: Krull dimension of a Principal ideal domain is one and K.dimension of a field is zero.

Definition 1.3.2: The Krull dimension of a module M is defined to be the supremum of the lengths of prime chains of primes in $\text{Supp } M$.

Proposition 1.3.0: Suppose R is a noetherian ring and M a finitely generated module. Then the following are equivalent:

- (i) M is a module of finite length;
- (ii) The ring $R/\text{Ann } M$ is artinian.
- (iii) $\dim M = 0$.

Definition 1.3.3: Let R be a noetherian local ring. Let all R -modules be of finite projective dimensions. The supremum of the projective dimensions of all R -modules is called global dimension of the ring R and is denoted as $\text{gl. dim } R$.

Definition 1.3.4: Suppose R is a noetherian semilocal ring and J be the Jacobson radical of R . An ideal I of R is said to be an ideal of definition of R if $\sqrt{I} = J$.

Definition 1.3.5: Suppose R is a noetherian local ring of dimension d . If x_1, x_2, \dots, x_d generate an ideal of definition of R , then x_1, x_2, \dots, x_d are called system of parameters.

§1.4: Regular sequence

Definition 1.4.0: Let M be an R -module. A sequence a_1, a_2, \dots, a_r of elements of R is said to be M -regular if for

each $1 \leq i \leq r$, a_i is not a zero divisor on $M/(a_1 M + a_2 M + \dots + a_{i-1} M)$ and $M \neq a_1 M + a_2 M + \dots + a_r M$.

Definition 1.4.1: Let R be a noetherian local ring. The length of a maximal M -regular sequence in the maximal ideal \mathfrak{m} of R is called the depth of M and is denoted as $\text{depth } M$. Thus $\text{depth } M = 0$ if and only if $\mathfrak{m} \in \text{Ass}(M)$.

If R is an arbitrary noetherian ring and $\mathfrak{p} \in \text{Spec } R$, we have $\text{depth } M_{\mathfrak{p}} = 0$ as $R_{\mathfrak{p}}$ -module if and only if $\mathfrak{p} \in \text{Ass}(M)$.

Proposition 1.4.0: Let R be a noetherian local ring and let $M \neq 0$ be a finitely generated R -module. Then $\text{depth } M \leq \dim(R/\mathfrak{p})$ for every $\mathfrak{p} \in \text{Ass}(M)$.

It follows from this proposition that for any finitely generated module, $\text{depth } M \leq \dim M$.

Definition 1.4.2: Let R be a noetherian local ring and M a finitely generated module. M is said to be a Cohen Macaulay module if $M = 0$ or $\text{depth } M = \dim M$. If the local ring R is Cohen Macaulay as R -module we say that R is a Cohen Macaulay ring.

Example: Localization of the polynomial ring $k[x_1, x_2, \dots, x_n]$ where k is a field at a prime ideal is a Cohen Macaulay ring.

Proposition 1.4.1: Let R be a Cohen Macaulay local ring. Then for any ideal I , we have $\dim R/I + \text{ht } I = \dim R$.

Definition 1.4.3: The grade of a module M over a noetherian local ring R is defined to be the length of a maximal R -sequence contained in the annihilator of M . If M has finite projective dimension then it is well known that $\text{grade } M \leq \text{Proj. dim } M$.

Definition 1.4.4: Let R be noetherian local. An R -module M is said to be perfect if $\text{Pd } M$ is finite and is equal to the grade of the module. For a perfect module we have $\text{Ext}_R^i(M, R) = 0$ for $i \neq \text{Pd } M$.

Example: If $\underline{x} = (x_1, x_2, \dots, x_r)$ is an R -sequence, then R/\underline{x} is a perfect R -module.

Definition 1.4.5: A module M is said to be torsionless if the natural mapping $M \rightarrow M^{**}$ where $*$ denotes duals, is injective.

Example: Any torsion-free module over a domain is torsionless.

Definition 1.4.6: A module M is said to be reflexive if the natural map $M \rightarrow M^{**}$ is an isomorphism.

Example: Any free module is reflexive.

Proposition 1.4.2: (Rees formula) Let M and N be R -modules. Then $\text{Ann } N$ contains an M -regular sequence of length t if and only if $\text{Ext}_R^i(M, N) = 0$ for $i = 0, 1, 2, \dots, t-1$.

Proposition 1.4.3 (Auslander's depth formula):

Let M be a module over a noetherian local ring R having finite projective dimension. Then

$$\text{Pd } M + \text{depth } M = \text{depth } R.$$

§1.5: Regular local rings and Gorenstein rings

Definition 1.5.0. Let R be a noetherian local ring. If there exists a system of parameters generating the maximal ideal of the ring R , we say that R is regular local and such a system of parameters is called a regular system of parameters.

Example: If K is a field, then the power series ring

$K[[x_1, x_2, \dots, x_n]]$ is regular local.

Proposition 1.5.0 (Hilbert - Serre's Theorem):

A noetherian local ring R is regular local if and only if global dimension of R is finite.

Definition 1.5.1: A noetherian local ring R is said to be a Gorenstein ring if injective dimension of the ring is finite as R -module.

Example: Let R be regular local and $\underline{x} = (x_1, x_2, \dots, x_r)$ is an R -sequence. Then $R/\underline{x}R$ is Gorenstein.

Proposition 1.5.1: Let R be a local Gorenstein ring and M a nonzero R -module. Then $\text{depth } M = \dim R - t$, where t is the largest integer such that $\text{Ext}_R^t(M, R) \neq (0)$.

Definition 1.5.2: The integer t such that $\text{Ext}_R^t(M, R) \neq (0)$ mentioned above is called the G-dimension of the module M .

Definition 1.5.3: A module M is said to be an n th syzygy for a positive integer n if there exists an exact sequence

$$0 \rightarrow M \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0,$$

where F_i 's are free modules.

If $\dots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$ is a projective resolution of M , the Kernel of $F_i \rightarrow F_{i-1}$ will be denoted as $\Omega^{i+1} M$ with the understanding that $F_{-1} = M$ and $\Omega^0 M = M$; the Kernel of $F_i^* \rightarrow F_{i+1}^*$ will be denoted as $D \Omega^i M$.

Definition 1.5.4: A module M is said to be rigid if whenever $\text{Tor}_1^R(M, N) = 0$ for a module M , then $\text{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Over a regular local ring any module is rigid. This is a theorem of Lichtenbaum (Ref.?)

We shall make use of the following exact sequences of M . Auslander and M. Bridger [1] in Chapter II.

Proposition 1.5.2: For every pair of modules M and N and for every integer $n \geq 0$, there exist exact sequences:

$$\begin{aligned} \text{Tor}_2^R(D \Omega^n M, N) &\rightarrow \text{Ext}_R^n(M, R) \otimes N \rightarrow \text{Ext}_R^n(M, N) \\ &\rightarrow \text{Tor}_1^R(D \Omega^n M, N) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow \text{Ext}_R^1(D \Omega^n M, N) &\rightarrow \text{Tor}_n^R(M, N) \rightarrow \text{Hom}(\text{Ext}_R^n(M, R), N) \\ &\rightarrow \text{Ext}_R^2(D \Omega^n M, N). \end{aligned}$$