

**SOME RECENT DEVELOPMENTS  
IN THE THEORY OF  
ORE'S HARMONIC NUMBERS  
– A SURVEY**

**ABSTRACT**

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It is well-known that a natural number  $n$  is said to be a perfect number if the sum of all the positive divisors of  $n$  equals  $2n$ . Euclid of Alexandria (300-275 B.C.) proved that  $2^{n-1}(2^n - 1)$  is an even perfect number whenever  $2^n - 1$  is a prime. Two millennia after Euclid, Leonard Euler proved that this formula gives all the even perfect numbers. This result is often referred to as the “Euclid-Euler Theorem”. As of today only 44 even perfect numbers are known, the largest being  $2^{32582656} \times (2^{32582657} - 1)$  with 19616714 digits, discovered on September 4, 2006, by Curtis Cooper and Steven Boone. However, it is still open whether or not odd one does exist, though many necessary conditions for their existence have been found. For example, Brent, Cohen and de Riele [1] showed that odd perfect numbers must be greater than  $10^{300}$ . Perhaps because they were frustrated by their failure to disprove the existence of odd perfect numbers, numerous authors have defined a number of closely related concepts and produced a raft of problems, many of which seem no more tractable than the original. One such concept is what is called the theory of harmonic numbers introduced by Oystein Ore [18] in 1948. A positive integer is called harmonic if the harmonic mean of its positive divisors is an integer. Of course, it was Pomerance [19] who used the name “Harmonic numbers” for the first time in 1973. A. & E. Zachariou [22] called these “Ore numbers”. Ore himself proved that every perfect number is harmonic. He also showed that 6 is the only square-free harmonic number and a power of a prime cannot be harmonic. It was proposed in 1989 by Edgar [9] and solved in 1992 by Callan [2], that every harmonic number having two distinct prime factors is an even perfect number. However, it is not known whether there are infinitely many harmonic numbers or whether there is an

odd harmonic number apart from 1. Goto and Shibata [11] have computed all harmonic numbers having harmonic mean of their positive divisors not exceeding 300. They also have posed several interesting open questions. M. Garcia [10] has obtained all harmonic numbers less than  $10^7$  in 1954. In 1997, G. L. Cohen [5] has listed all harmonic numbers less than  $2 \times 10^9$ . Recently in 2007, Goto and Okeya [13] have listed all harmonic numbers less than  $10^{14}$ . The concept of harmonic numbers have been generalized in a number of ways by many authors like Hagsis and Lord [14], Hagsis and Cohen [15], Cohen and Moujje [6], Cohen and Sorli [7]. For example, harmonic seeds, unitary harmonic numbers, infinitary harmonic numbers (using infinitary divisors [4]) and k-harmonic numbers have already become quite popular. In [12], Goto has given certain upper bounds for unitary perfect numbers and unitary harmonic numbers.

In 1957, Kanold [16] has proved that there are only finitely many integers having a given fixed harmonic mean of their positive divisors. Recently Chishiki, Goto and Ohno [3] have proved that every odd harmonic number greater than 1 must be divisible by a prime greater than  $10^5$ .

In chapter 1, we have collected some basic concepts from the theory of numbers, which are used in the subsequent chapters. We also include some important inequalities in this chapter.

In chapter 2, we study some properties of harmonic numbers. Some of the important results are given below:

**Lemma 2.1.1.** *There is no harmonic numbers of the form  $p^a$ , where  $p$  is a prime and  $a > 0$ .*

**Theorem 2.1.2.** *Every perfect number is harmonic, but not conversely.*

**Lemma 2.1.4.**  *$H$  is monotonic in the sense that if  $e, f$  are positive integers and  $p, q$  are primes with  $e < f$  and  $p < q$  then*

$$H(p^e) < H(p^f) < H(q^f).$$

**Theorem 2.1.5.** *No harmonic number greater than 6 is squarefree.*

**Lemma 2.1.6.** *If  $H(n)$  is even then  $n$  is even.*

**Lemma 2.1.7.** *Suppose  $n \in \mathcal{H}$ . If  $p$  is an odd prime such that  $p \parallel n$  and  $p \mid H(n)$ , then  $\frac{n}{p}$  is harmonic. If  $2 \parallel n$  and  $4 \mid H(n)$ , then  $\frac{n}{2}$  is harmonic.*

**Theorem 2.1.9.**  *$H(p^e q^f) > H(p^f q^e)$  where  $p, q$  are primes and  $e, f$  are integers such that  $p < q, e < f$ .*

**Theorem 2.1.11.** *For any real number  $\alpha$ , there exist only finitely many positive integers  $n$  satisfying  $H(n)^\alpha > n$ .*

**Theorem 2.2.1.** *Let  $n$  be an odd harmonic number and  $p^e \parallel n$ . Then*

$$p^e \equiv 1 \pmod{4}.$$

**Lemma 2.2.7.** *If  $n$  is a harmonic number with  $w(n) = 2$  then  $n$  is an even perfect number.*

**Lemma 2.3.1.** *Let  $p$  be a prime. If  $H(n) = p$  then either  $p \mid n$  or  $n$  is a perfect number.*

**Theorem 2.3.2.** *Let  $p$  be a prime. If  $H(n) = 2p$  then  $2p \mid n$ . Also, if  $H(n) = 3p$ , then  $p \mid n$ .*

**Theorem 2.3.3.** *For any positive integer  $c$ , there exist only finitely many numbers satisfying  $H(n) = c$ .*

**Lemma 2.3.4.** *Let  $w(n)$  denote the number of distinct prime factors of  $n$ . For all  $n$ ,  $H(n) > \frac{2^{w(n)+1}}{w(n)+1}$ , with the following exceptions  $n = p, 2p, 6p$  ( $p \neq 3$ ),  $n = 30p$  ( $7 \leq p \leq 23$ ) and  $n = 1, 15, 21, 70$ .*

In chapter 3, we make a detailed study of  $k$ -harmonic numbers and some necessary conditions for existence of a  $k$ -harmonic number.

The main results in this connection are as follows:

**Lemma 3.1.2.** *A power harmonic number has atleast two distinct prime factors.*

**Lemma 3.1.3.** *If  $n$  is  $k$ -harmonic, then  $H_k(n) \leq \tau(n) - 1$ .*

**Lemma 3.1.6.** *For a fixed integer  $n$ ,  $H_k(n)$  is an increasing function of  $k$  and  $\lim_{k \rightarrow \infty} H_k(n) = \tau(n)$ .*

**Lemma 3.1.7.** *Let  $p$  be an odd prime and  $2^a p^b$  be a proper  $k$ -harmonic number (i.e.,  $k \geq 2$ ). If  $k$  is even then  $b \equiv 7 \pmod{8}$ . Also if  $k$  odd then  $b$  is odd and  $(p+1)(b+1) \equiv 0 \pmod{16}$ .*

**Lemma 3.1.8.** *If  $p, q$  are primes such that  $p < q$ , and  $a, b$  are integers such that  $a \geq b \geq 1$ , and  $k \geq 1$ , then  $H_k(p^a q^b) \leq H_k(p^b q^a)$  with equality if and only if  $a = b$ .*

**Lemma 3.1.10.** *Let the primes  $p, q$  and positive integers  $a, b, c, d$  satisfy  $p < q$ ,  $p^a q^b < p^c q^d$ ,  $8p^{a+1} > 9q^{d+1}$  and  $(a+1)(b+1) = (c+1)(d+1)$ , then for  $k \geq 2$  and  $c > a \geq b > d$ ,  $H_k(p^a q^b) < H_k(p^c q^d)$ .*

**Lemma 3.1.11.** *Let  $n_1 = \prod_{i=1}^u p_i^{a_i}$ ,  $n_2 = \prod_{i=1}^u q_i^{a_i}$  be prime factorisations with  $p_i \leq q_i$  for  $i = 1, 2, 3, \dots, u$ . Then  $H_k(n_1) \leq H_k(n_2)$ , for any  $k$ , with equality iff  $n_1 = n_2$ .*

**Theorem 3.2.1.** *If  $n$  is a proper  $k$ -harmonic number, then*

$$\tau(n) > 1 + \frac{k-1}{k+1} 2^k.$$

**Theorem 3.2.2.** *If  $n$  is a proper power-harmonic number, then  $\tau(n) \geq 60$ .*

**Theorem 3.2.3.** *If  $n$  is a proper power-harmonic number, then  $n > 10^{10}$ .*

In chapter 4, we discuss about the unitary and infinitary harmonic numbers. In section one, we study about the unitary,  $k$ -ary and infinitary divisors. Some important results about them are given below:

**Lemma 4.1.2.**  *$p^x \mid_k p^y$  if and only if  $p^{y-x} \mid_k p^y$ .*

**Theorem 4.1.3.** *For  $k \geq y - 1 \geq 0$ ,  $p^x \mid_k p^y$  if and only if  $p^x \mid_{y-1} p^y$ .*

**Theorem 4.1.5.**  *$p^x \mid_\infty p^y$  if and only if  $\gcd_\infty(p^x, p^{y-x}) = 1$ , where  $x \leq y$ .*

**Theorem 4.1.6.**  *$p \mid_\infty p^y$  if and only if  $y$  is odd.*

**Theorem 4.1.7.** *If  $y$  is even and  $p^x \mid_\infty p^y$ , then  $x$  is also even.*

**Theorem 4.1.9.**  *$p^x \mid_\infty p^y$  if and only if  $p^{2x} \mid_\infty p^{2y}$ .*

**Theorem 4.1.10.** *If  $p^x \mid_\infty p^y$  and  $y$  is divisible by  $2^j$ , for some  $j \geq 0$ , then  $x$  is divisible by  $2^j$ .*

**Theorem 4.1.13.**  *$p^{2^j} \mid_\infty p^y$  iff  $y \equiv 2^j$  or  $2^j + 1$  or  $2^j + 2$  or  $\dots$  or  $2^{j+1} - 1 \pmod{2^{j+1}}$*

**Theorem 4.1.19.**  *$p^x \mid_\infty p^y$  iff  $\binom{y}{x}$  is odd.*

In section two of this chapter, we discuss some properties of unitary and infinitary perfect as well as harmonic numbers (UPN, IPN, UHN and IHN), as in [14], [15], [20], and [21]. The main results in this context are as follows:

**Proposition 4.2.1.** *Every UPN is a UHN.*

**Proposition 4.2.2.** *If  $n$  is square-free and  $n \neq 6$  then  $n$  is not a unitary harmonic number.*

**Proposition 4.2.3.** *If  $n$  is an odd UHN then  $H^*(n)$  is also odd.*

**Proposition 4.2.4.** *If  $n$  is a UHN,  $\gcd(p, n) = 1$  and  $(p^a + 1) \mid 2H^*(n)$ . Then  $p^a n$  is also a UHN where  $p$  is a prime.*

**Theorem 4.2.7.** *If  $S_c$  is the set of natural numbers  $n$  such that  $H^*(n) = c$ , then  $S_c$  is finite ( may be empty ) for every real number  $c$ .*

**Proposition 4.2.8.**  *$T_1$  is empty, i.e., power of a prime is never a unitary harmonic number.*

**Proposition 4.2.9.**  *$T_2 = \{6, 45\}$ , i.e., 6 and 45 are the only unitary harmonic numbers having two distinct prime factor.*

**Proposition 4.2.10.** *The set of all unitary harmonic numbers with three distinct prime factors is given by*

$$T_3 = \{60, 90, 1512, 15925, 55925\}.$$

**Lemma 4.3.1.** *If  $p^a \parallel n$ , then  $p^a \geq \frac{H^*(n)}{2^k - H^*(n)}$  where  $w(n) = k$  with equality if and only if  $k = 1$ .*

**Lemma 4.3.2.** *If  $p^a \{r^c\}$  is the minimum (maximum) prime power divisor of  $n$  then*

$$p^a \leq \frac{kH^*(n)}{2^k - H^*(n)} \quad \{r^c \geq \frac{(k-1)2^k + H^*(n)}{2^k - H^*(n)}\}.$$

*with equality if and only if  $k = 1$  or  $n = p^a q^b r^c$  where  $q^b = p^a + 1$  ( so that  $2 \mid n$ ) and  $r^c = p^a + 2$  or  $c = 0$  { where  $q^b = r^c - 1$ ,  $p^a = r^c - 2$  or  $a = 0$  } .*

**Lemma 4.3.3.** *Let  $n$  be a UHN with  $k$  distinct prime factors. Then*

$$k \leq \frac{2^{k+1}}{k+2} \leq H^*(n) < 2^k.$$

*Furthermore, we have first equality only when  $k = 2$ , second equality only when  $n = 2$  or  $6$  and third equality only when  $n = 1$ .*

**Theorem 4.3.5.** *Suppose that  $n$  is a UHN with  $k$  distinct prime factors and  $H^*(n) = c$ . Then it follows that*

$$(a) n \leq c^{c^2},$$

$$(b) n \leq (2^{2^k})^k$$

*with each equality if and only if  $n = 1$ .*

**Theorem 4.3.6.** *There exists atmost finitely many unitary harmonic numbers with the specified number of distinct prime factors.*

**Corollary 4.3.7.** *There are atmost finitely many unitary perfect numbers with a specified number of prime factors.*

**Proposition 4.4.2.** *The set of infinitary perfect number (IPN) is a subset of IH.*

**Theorem 4.4.5.** *For all  $n$ ,  $H^*(n) \leq H_\infty(n) \leq H(n)$ . For  $n > 1$ , equality holds on the left iff  $p^y \parallel n$  implies  $y = 2^\alpha$ , and on the right iff  $p^y \parallel n$  implies  $y = 2^\beta - 1$ .*

**Proposition 4.4.8.** *If  $n \in IH$ ,  $\gcd(p, n) = 1$  and  $\sigma_\infty(p^y) \mid \tau_\infty(p^y)H_\infty(p^y)$ , then  $p^y n \in IH$ .*

**Theorem 4.4.9.** *If  $S_c$  is the set of natural numbers  $n$  such that  $H_\infty(n) = c$ , then  $S_c$  is finite (or empty) for every real number  $c$ .*

In the last chapter we discuss about the harmonic seeds [7]. Also we include some list of harmonic numbers and some result related to these lists. The main results in this connection are given below:

**Theorem 5.1.5.** *Suppose  $n$  and  $nq_1q_2q_3 \dots q_t$  are harmonic numbers, where  $q_1 < q_2 < \dots < q_t$  are primes not dividing  $n$ . Then  $nq_1$  is harmonic, except when  $t \geq 2$  and  $q_1q_2 = 6$ , in which case  $nq_1q_2$  is harmonic.*

**Theorem 5.2.1.** *The only harmonic numbers of the form  $2^a \cdot m$ , where  $m$  is odd and squarefree and  $1 \leq a \leq 11$ , are those listed in the Tables 1 and 2.*

**Theorem 5.2.4.** *Let  $n$  be harmonic and  $H(n) \leq 1200$ . Then  $n$  is one of the first 1376 numbers in the list which is available on the webpage [http://www.mnoda.tus.ac.jp/u/tg/files/list 3](http://www.mnoda.tus.ac.jp/u/tg/files/list%203).*

**Theorem 5.2.5.** *Let  $n$  be harmonic and  $H(n) \leq 13$ . Then  $n$  is one of the following 13 numbers:*

1      6      28      140   496   270   8128   672   1638   6200  
2970   105664   33550336.

*In particular, the numbers  $n$  with  $H(n) = 4$  or 12 do not exist.*

Finally, we would like to mention that we have posed certain interesting workable research problems in all the chapters.

Table 1:

$a$	All $2^a m \in \mathcal{H}$ with $m$ odd and squarefree
1	$2 \cdot 3 = 6$
2	$2^2 \cdot 7 = 28, 2^2 \cdot 5 \cdot 7 = 140$
3	none
4	$2^4 \cdot 31 = 496$
5	$2^5 \cdot 3 \cdot 7 = 672$
6	$2^6 \cdot 127 = 8128, 2^6 \cdot 13 \cdot 127 = 105664$
7	none
8	see Table 2
9–11	none

Table 2:

All $2^8 m \in \mathcal{H}$ with $m$ odd and squarefree	$H(n)$
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 37 \cdot 43 \cdot 73 = 15007087898880$	989
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 349002044160$	506
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 = 652482082560$	516
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 \cdot 257 = 167687895217920$	1028
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 \cdot 1031 = 672709027119360$	1031
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 713178090240$	517
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 1033 = 736712967217920$	1033

contd...

All $2^8 m \in \mathcal{H}$ with $m$ odd and squarefree	$H(n)$
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 = 15174001920$	264
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 131 = 1987794251520$	524
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 131 \cdot 523 = 1039616393544960$	1046
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 263 = 3990762504960$	526
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 263 \cdot 1051 = 4194291392712960$	1051
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 44345330883840$	1037
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 726972637440$	527
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 67 \cdot 73 = 1571198926080$	536
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 = 23450730240$	272
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 271 = 6355147895040$	542
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 271 \cdot 541 = 3438135011216640$	1082
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 31727458560$	276
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 = 4346661822720$	548
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 \cdot 547 = 2377624017027840$	1094
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 \cdot 547 \cdot 1093 = 2598743050611429120$	2186
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 2608548875520$	549
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 1097 = 2861578116445440$	1097
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 42763096320$	279
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 \cdot 557 = 23819044650240$	557
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 64834371840$	282
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 281 = 18218458487040$	562
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 281 \cdot 1123 = 20459328880945920$	1123

contd...

All $2^8 m \in \mathcal{H}$ with $m$ odd and squarefree	$H(n)$
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 563 = 36501751345920$	563
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 71 \cdot 73 = 97941285120$	284
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 71 \cdot 73 \cdot 283 = 27717383688960$	566
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 = 1379454720$	144
$2^8 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 9564679210240$	671
$2^8 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 156798019840$	341
$2^8 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 = 217494027520$	344
$2^8 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 = 5058000640$	176
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 10575819520$	184
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 367 = 3881325763840$	367
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 367 \cdot 733 = 2845011784894720$	733
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 869516291840$	366
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 14254365440$	186
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 21611457280$	188
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 = 459818240$	96
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 \cdot 191 = 87825283840$	191

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# CERTIFICATE

I certify that the dissertation entitled "SOME RECENT DEVELOPMENTS IN THE THEORY OF ORE'S HARMONIC NUMBERS - A SURVEY" submitted by Miss Kuwali Das in partial fulfilment of the requirement of the degree of Master of Philosophy in Mathematics is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.



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25<sup>th</sup> February, 2009.

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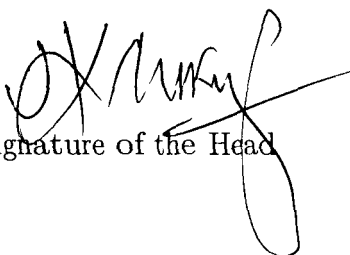
## DECLARATION

I, Kuwali Das, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.


This dissertation is being submitted to the North-Eastern Hill University for the degree of Master of Philosophy in Mathematics.

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# PREFACE

It is well-known that a natural number  $n$  is said to be a perfect number if the sum of all the positive divisors of  $n$  equals  $2n$ . Euclid of Alexandria (300-275 B.C.) proved that  $2^{n-1}(2^n - 1)$  is an even perfect number whenever  $2^n - 1$  is a prime. Two millennia after Euclid, Leonard Euler proved that this formula gives all the even perfect numbers. This result is often referred to as the “Euclid-Euler Theorem”. As of today only 44 even perfect numbers are known, the largest being  $2^{32582656} \times (2^{32582657} - 1)$  with 19616714 digits, discovered on September 4, 2006, by Curtis Cooper and Steven Boone. However, it is still open whether or not odd one does exist, though many necessary conditions for their existence have been found. For example, Brenti, Cohen and de Riele [1] showed that odd perfect numbers must be greater than  $10^{300}$ . Perhaps because they were frustrated by their failure to disprove the existence of odd perfect numbers, numerous authors have defined a number of closely related concepts and produced a raft of problems, many of which seem no more tractable than the original. One such concept is what is called the theory of harmonic numbers introduced by Oystein Ore [18] in 1948. A positive integer is called harmonic if the harmonic mean of its positive divisors is an integer. Of course, it was Pomerance [19] who used the name “Harmonic numbers” for the first time in 1973. A. & E. Zachariou [22] called these “Ore numbers”. Ore himself proved that every perfect number is harmonic. He also showed that 6 is the only square-free harmonic number and a power of a prime cannot be harmonic. It was proposed in 1989 by Edgar

[9] and solved in 1992 by Callan [2], that every harmonic number having two distinct prime factors is an even perfect number. However, it is not known whether there are infinitely many harmonic numbers or whether there is an odd harmonic number apart from 1. Goto and Shibata [11] have computed all harmonic numbers having harmonic mean of their positive divisors not exceeding 300. They also have posed several interesting open questions. M. Garcia [10] has obtained all harmonic numbers less than  $10^7$  in 1954. In 1997, G. L. Cohen [5] has listed all harmonic numbers less than  $2 \times 10^9$ . Recently in 2007, Goto and Okeya [13] have listed all harmonic numbers less than  $10^{14}$ . The concept of harmonic numbers have been generalized in a number of ways by many authors like Hagis and Lord [14], Hagis and Cohen [15], Cohen and Moujie [6], Cohen and Sorli [7]. For example, harmonic seeds, unitary harmonic numbers, infinitary harmonic numbers (using infinitary divisors [4]) and k-harmonic numbers have already become quite popular. In [12], Goto has given certain upper bounds for unitary perfect numbers and unitary harmonic numbers.

In 1957, Kanold [16] has proved that there are only finitely many integers having a given fixed harmonic mean of their positive divisors. Recently Chishiki, Goto and Ohno [3] have proved that every odd harmonic number greater than 1 must be divisible by a prime greater than  $10^5$ .

In chapter 1, we have collected some basic concepts from the theory of numbers, which are used in the subsequent chapters. We also include some important inequalities in this chapter.

In chapter 2, we study some properties of harmonic numbers. Some of the important results are given below:

**Lemma 2.1.1.** *There is no harmonic numbers of the form  $p^a$ , where  $p$  is a prime and  $a > 0$ .*

**Theorem 2.1.2.** *Every perfect number is harmonic, but not conversely.*

**Lemma 2.1.4.**  *$H$  is monotonic in the sense that if  $e, f$  are positive integers and  $p, q$  are primes with  $e < f$  and  $p < q$  then*

$$H(p^e) < H(p^f) < H(q^f).$$

**Theorem 2.1.5.** *No harmonic number greater than 6 is squarefree.*

**Lemma 2.1.6.** *If  $H(n)$  is even then  $n$  is even.*

**Lemma 2.1.7.** *Suppose  $n \in \mathcal{H}$ . If  $p$  is an odd prime such that  $p \parallel n$  and  $p \mid H(n)$ , then  $\frac{n}{p}$  is harmonic. If  $2 \parallel n$  and  $4 \mid H(n)$ , then  $\frac{n}{2}$  is harmonic.*

**Theorem 2.1.9.**  *$H(p^e q^f) > H(p^f q^e)$  where  $p, q$  are primes and  $e, f$  are integers such that  $p < q, e < f$ .*

**Theorem 2.1.11.** *For any real number  $\alpha$ , there exist only finitely many positive integers  $n$  satisfying  $H(n)^\alpha > n$ .*

**Theorem 2.2.1.** *Let  $n$  be an odd harmonic number and  $p^e \parallel n$ . Then*

$$p^e \equiv 1 \pmod{4}.$$

**Lemma 2.2.7.** *If  $n$  is a harmonic number with  $w(n) = 2$  then  $n$  is an even perfect number.*

**Lemma 2.3.1.** *Let  $p$  be a prime. If  $H(n) = p$  then either  $p \mid n$  or  $n$  is a perfect number.*

**Theorem 2.3.2.** *Let  $p$  be a prime. If  $H(n) = 2p$  then  $2p \mid n$ . Also, if  $H(n) = 3p$ , then  $p \mid n$ .*

**Theorem 2.3.3.** *For any positive integer  $c$ , there exist only finitely many numbers satisfying  $H(n) = c$ .*

**Lemma 2.3.4.** *Let  $w(n)$  denote the number of distinct prime factors of  $n$ . For all  $n$ ,  $H(n) > \frac{2^{w(n)+1}}{w(n)+1}$ , with the following exceptions  $n = p, 2p, 6p$  ( $p \neq 3$ ),  $n = 30p$  ( $7 \leq p \leq 23$ ) and  $n = 1, 15, 21, 70$ .*

In chapter 3, we make a detailed study of  $k$ -harmonic numbers and some necessary conditions for existence of a  $k$ -harmonic number.

The main results in this connection are as follows:

**Lemma 3.1.2.** *A power harmonic number has atleast two distinct prime factors.*

**Lemma 3.1.3.** *If  $n$  is  $k$ -harmonic, then  $H_k(n) \leq \tau(n) - 1$ .*

**Lemma 3.1.6.** *For a fixed integer  $n$ ,  $H_k(n)$  is an increasing function of  $k$  and  $\lim_{k \rightarrow \infty} H_k(n) = \tau(n)$*

**Lemma 3.1.7.** *Let  $p$  be an odd prime and  $2^a p^b$  be a proper  $k$ -harmonic number (i.e.,  $k \geq 2$ ). If  $k$  is even then  $b \equiv 7 \pmod{8}$ . Also if  $k$  odd then  $b$  is odd and  $(p+1)(b+1) \equiv 0 \pmod{16}$ .*

**Lemma 3.1.8.** *If  $p, q$  are primes such that  $p < q$ , and  $a, b$  are integers such that  $a \geq b \geq 1$ , and  $k \geq 1$ , then  $H_k(p^a q^b) \leq H_k(p^b q^a)$  with equality if and only if  $a = b$ .*

**Lemma 3.1.10.** *Let the primes  $p, q$  and positive integers  $a, b, c, d$  satisfy  $p < q$ ,  $p^a q^b < p^c q^d$ ,  $8p^{a+1} > 9q^{d+1}$  and  $(a+1)(b+1) = (c+1)(d+1)$ , then for  $k \geq 2$  and  $c > a \geq b > d$ ,  $H_k(p^a q^b) < H_k(p^c q^d)$ .*

**Lemma 3.1.11.** *Let  $n_1 = \prod_{i=1}^u p_i^{a_i}$ ,  $n_2 = \prod_{i=1}^u q_i^{a_i}$  be prime factorisations with  $p_i \leq q_i$  for  $i = 1, 2, 3, \dots, u$ . Then  $H_k(n_1) \leq H_k(n_2)$ , for any  $k$ , with equality iff  $n_1 = n_2$ .*

**Theorem 3.2.1.** *If  $n$  is a proper  $k$ -harmonic number, then*

$$\tau(n) > 1 + \frac{k-1}{k+1}2^k.$$

**Theorem 3.2.2.** *If  $n$  is a proper power-harmonic number, then  $\tau(n) \geq 60$ .*

**Theorem 3.2.3.** *If  $n$  is a proper power-harmonic number, then  $n > 10^{10}$ .*

In chapter 4, we discuss about the unitary and infinitary harmonic numbers. In section one, we study about the unitary,  $k$ -ary and infinitary divisors. Some important results about them are given below:

**Lemma 4.1.2.**  *$p^x \mid_k p^y$  if and only if  $p^{y-x} \mid_k p^y$ .*

**Theorem 4.1.3.** *For  $k \geq y - 1 \geq 0$ ,  $p^x \mid_k p^y$  if and only if  $p^x \mid_{y-1} p^y$ .*

**Theorem 4.1.5.**  *$p^x \mid_\infty p^y$  if and only if  $\gcd_\infty(p^x, p^{y-x}) = 1$ , where  $x \leq y$ .*

**Theorem 4.1.6.**  *$p \mid_\infty p^y$  if and only if  $y$  is odd.*

**Theorem 4.1.7.** *If  $y$  is even and  $p^x \mid_\infty p^y$ , then  $x$  is also even.*

**Theorem 4.1.9.**  *$p^x \mid_\infty p^y$  if and only if  $p^{2x} \mid_\infty p^{2y}$ .*

**Theorem 4.1.10.** *If  $p^x \mid_\infty p^y$  and  $y$  is divisible by  $2^j$ , for some  $j \geq 0$ , then  $x$  is also divisible by  $2^j$ .*

**Theorem 4.1.13.**  *$p^{2^j} \mid_\infty p^y$  iff  $y \equiv 2^j$  or  $2^j + 1$  or  $2^j + 2$  or  $\dots$  or  $2^{j+1} - 1 \pmod{2^{j+1}}$ .*

**Theorem 4.1.19.**  *$p^x \mid_\infty p^y$  iff  $\binom{y}{x}$  is odd.*

In section two of this chapter, we discuss some properties of unitary and infinitary perfect as well as harmonic numbers (UPN, IPN, UHN and IHN).

The main results in this context are as follows:

**Proposition 4.2.1.** *Every UPN is a UHN.*

**Proposition 4.2.2.** *If  $n$  is square-free and  $n \neq 6$  then  $n$  is not a unitary harmonic number.*

**Proposition 4.2.3.** *If  $n$  is an odd UHN then  $H^*(n)$  is also odd.*

**Proposition 4.2.4.** *If  $n$  is a UHN,  $\gcd(p, n) = 1$  and  $(p^a + 1) \mid 2H^*(n)$ .*

*Then  $p^a n$  is also a UHN where  $p$  is a prime.*

**Theorem 4.2.7.** *If  $S_c$  is the set of natural numbers  $n$  such that  $H^*(n) = c$ , then  $S_c$  is finite ( may be empty ) for every real number  $c$ .*

**Proposition 4.2.8.**  *$T_1$  is empty, i.e., power of a prime is never a unitary harmonic number.*

**Proposition 4.2.9.**  *$T_2 = \{6, 45\}$ , i.e., 6 and 45 are the only unitary harmonic numbers having two distinct prime factor.*

**Proposition 4.2.10.** *The set of all unitary harmonic numbers with three distinct prime factors is given by*

$$T_3 = \{60, 90, 1512, 15925, 55925\}.$$

**Lemma 4.3.1.** *If  $p^a \parallel n$ , then  $p^a \geq \frac{H^*(n)}{2^k - H^*(n)}$  where  $w(n) = k$  with equality if and only if  $k = 1$ .*

**Lemma 4.3.2.** *If  $p^a \{r^c\}$  is the minimum (maximum) prime power divisor of  $n$  then*

$$p^a \leq \frac{kH^*(n)}{2^k - H^*(n)} \quad \{r^c \geq \frac{(k-1)2^k + H^*(n)}{2^k - H^*(n)}\}$$

*with equality if and only if  $k = 1$  or  $n = p^a q^b r^c$  where  $q^b = p^a + 1$  ( so that  $2 \mid n$ ) and  $r^c = p^a + 2$  or  $c = 0$  { where  $q^b = r^c - 1$ ,  $p^a = r^c - 2$  or  $a = 0$  } .*

**Lemma 4.3.3.** *Let  $n$  be a UHN with  $k$  distinct prime factors. Then*

$$k \leq \frac{2^{k+1}}{k+2} \leq H^*(n) < 2^k.$$

Furthermore, we have first equality only when  $k = 2$ , second equality only when  $n = 2$  or  $6$  and third equality only when  $n = 1$ .

**Theorem 4.3.5.** *Suppose that  $n$  is a UHN with  $k$  distinct prime factors and  $H^*(n) = c$ . Then it follows that*

$$(a)n \leq c^{c^2},$$

$$(b)n \leq (2^{2^k})^k$$

with each equality if and only if  $n = 1$ .

**Theorem 4.3.6.** *There exists at most finitely many unitary harmonic numbers with the specified number of distinct prime factors.*

**Corollary 4.3.7.** *There are at most finitely many unitary perfect numbers with a specified number of prime factors.*

**Proposition 4.4.2.** *Every IPN is an IHN.*

**Theorem 4.4.5.** *For all  $n$ ,  $H^*(n) \leq H_\infty(n) \leq H(n)$ . For  $n > 1$ , equality holds on the left iff  $p^y \parallel n$  implies  $y = 2^\alpha$ , and on the right iff  $p^y \parallel n$  implies  $y = 2^\beta - 1$ .*

**Proposition 4.4.8.** *If  $n \in IH$ ,  $\gcd(p, n) = 1$  and  $\sigma_\infty(p^y) \mid \tau_\infty(p^y)H_\infty(p^y)$ , then  $p^y n \in IH$ .*

**Theorem 4.4.9.** *If  $S_c$  is the set of natural numbers  $n$  such that  $H_\infty(n) = c$ , then  $S_c$  is finite (or empty) for every real number  $c$ .*

In the last chapter we discuss about the harmonic seeds. Also we include some list of harmonic numbers and some result related to these lists. The main results in this connection are given below:

**Theorem 5.1.5.** *Suppose  $n$  and  $nq_1q_2q_3 \dots q_t$  are harmonic numbers, where  $q_1 < q_2 < \dots < q_t$  are primes not dividing  $n$ . Then  $nq_1$  is harmonic, except*

when  $t \geq 2$  and  $q_1q_2 = 6$ , in which case  $nq_1q_2$  is harmonic.

**Theorem 5.2.1.** *The only harmonic numbers of the form  $2^am$ , where  $m$  is odd and squarefree and  $1 \leq a \leq 11$ , are those listed in the Tables 5.1 and 5.2.*

**Theorem 5.2.4.** *Let  $n$  be harmonic and  $H(n) \leq 1200$ . Then  $n$  is one of the first 1376 numbers in the list which is available on the webpage [http://www.mnoda.tus.ac.jp/u/tg/files/list 3](http://www.mnoda.tus.ac.jp/u/tg/files/list3).*

**Theorem 5.2.5.** *Let  $n$  be harmonic and  $H(n) \leq 13$ . Then  $n$  is one of the following 13 numbers:*

1      6      28      140   496   270   8128   672   1638   6200  
2970   105664   33550336.

*In particular, the numbers  $n$  with  $H(n) = 4$  or 12 do not exist.*

Finally, we would like to mention that we have posed certain interesting workable research problems in all the chapters.

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# Chapter 1

## Preliminaries

In this chapter we recall some of the basic definitions, notations and conventions from the theory of numbers, which will be used in the forthcoming chapters.

### 1.1 Arithmetic functions and perfect numbers

A complex valued function, whose domain is the set of positive integers, is called an *arithmetic function*.

**Definition 1.1.1.** For positive integers  $n$  we make the following definitions:

$\tau(n)$  is the number of positive divisors of  $n$ .

$\sigma(n)$  is the sum of the positive divisors of  $n$ .

$\sigma_k(n)$  is the sum of the  $k$ -th powers of the positive divisors of  $n$ .

Thus we have

$$\tau(n) = \sum_{d|n} 1, \sigma(n) = \sum_{d|n} d, \sigma_k(n) = \sum_{d|n} d^k.$$

**Definition 1.1.2.** If  $f$  is an arithmetic function not identically zero such that  $f(mn) = f(m)f(n)$  for every pair of positive integers  $m, n$  satisfying  $\gcd(m, n) = 1$ , then  $f$  is said to be multiplicative. If  $f(mn) = f(m)f(n)$  whether  $m$  and  $n$  are relatively prime or not,  $f$  is said to be totally multiplicative or completely multiplicative.

$\tau$  and  $\sigma$  are multiplicative function. In fact it is well-known that if  $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$  is the prime factorization of  $n > 1$ , then

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_r + 1)$$

and

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \dots \frac{p_r^{\alpha_r+1} - 1}{p_r - 1}.$$

**Definition 1.1.3.** A positive integer  $n$  is said to be a *perfect number* if  $\sigma(n) = 2n$ .

Euclid and Euler proved the following theorem for a perfect number.

**Theorem 1.1.4.** *If  $2^k - 1$  is a prime ( $k > 1$ ) then  $n = 2^{k-1}(2^k - 1)$  is a perfect number and every even perfect number is of this form.*

*Proof.* Let  $p$  denotes the prime  $2^k - 1$ , then  $n = 2^{k-1}p$ . Now since  $\sigma$  is multiplicative,

$$\sigma(n) = \sigma(2^{k-1})\sigma(p) = (2^k - 1)2^k = 2n.$$

Hence  $n$  is a perfect number.

Conversely, let  $n$  be an even perfect number then  $n = 2^{k-1}m$  where  $m$  is an odd integer and  $k \geq 2$ . Since  $\sigma$  is multiplicative,  $\sigma(n) = (2^k - 1)\sigma(m)$ . Therefore,  $2^k m = (2^k - 1)\sigma(m)$ . This gives  $(2^k - 1) \mid 2^k m$ . But  $2^k - 1$  and  $2^k$  are relatively prime,  $2^k - 1 \mid m$ . Suppose,  $m = (2^k - 1)t$ , for some  $t \geq 1$ , then  $\sigma(m) = 2^k t$ . Now,

$$2^k t = \sigma(m) \geq m + t = 2^k t.$$

Thus,

$$\sigma(m) = m + t,$$

this implies that  $m$  has only two divisors  $m$  and  $t$ . Hence  $t = 1$ . This means that  $m = 2^k - 1$  is a prime. Thus  $n = 2^{k-1}(2^k - 1)$ , where  $2^k - 1$  is an odd prime. Hence complete the theorem.  $\square$

Note that the prime of the form  $2^k - 1$ ,  $k \geq 2$  are called mersenne primes. Till now there are 45 known mersenne primes. Also, as of october, 2006, 44 even perfect numbers are known, the largest one being  $2^{32582656} \times (2^{32582657} - 1)$  with 19616714 digits, discovered on September 4, 2006, by Curtis Cooper and Steven Boone. However it is still open whether or not odd perfect numbers exists, though many necessary conditions for their existence have been found. For example, Brent, Cohen and te Riele [1] showed that odd perfect numbers must be greater than  $10^{300}$ .

**Lemma 1.1.5.** *If  $n$  is a perfect number then  $\tau(n)$  can not be odd.*

*Proof.* Suppose,  $\tau(n)$  is odd for some perfect number  $n$ . Also let,  $n = \prod_{i=1}^r p_i^{\alpha_i}$  then  $\tau(n) = \prod_{i=1}^r (\alpha_i + 1)$  and  $\sigma(n) = \prod_{i=1}^r \left( \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \right)$ . Since  $\tau(n)$  is odd, all  $\alpha_i$ 's are even. Now since product of odd numbers of odd terms is odd, therefore  $\sigma(n) = \prod_{i=1}^r \left( \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \right)$  is odd, which is a contradiction to the fact that  $\sigma(n) = 2n$ . Hence, for all perfect number  $n$ ,  $\tau(n)$  must be even.  $\square$

## 1.2 $p$ -adic ordinal

Let  $p$  be a prime and let  $q$  be a rational number. Suppose that  $q = p^e m/n$  with  $p \nmid mn$ . Then  $e$  is called *the order* of  $p$  in  $q$  or, *the  $p$ -adic ordinal* of  $q$ , denoted by  $\text{ord}_p(q)$ .

$p$ -adic ordinal of  $q$  forms one of the basic ingredients for an entirely different theory known as  $p$ -adic analysis (see [17]).

**Theorem 1.2.1.** *Let  $p$  be an integer (not necessarily prime). If  $p \equiv 1 \pmod{4}$ , then*

$$\text{ord}_2(1 + p + p^2 + \cdots + p^e) = \text{ord}_2(e + 1).$$

*Again if  $p \equiv 1 \pmod{4}$  and  $\text{ord}_2(p + 1) = m$ , then*

$$\text{ord}_2(1 + p + p^2 + \cdots + p^e) = \begin{cases} \text{ord}_2(e + 1) + m - 1; & \text{if } e \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let  $e + 1 = 2^k l$  where  $l$  is an odd integer. If  $k = 0$  then  $e + 1$  is odd. So,  $\text{ord}_2(e + 1) = 0$ . Also,

$$1 + p + p^2 + \cdots + p^e = 1 + p + p^2 + \cdots + p^{l-1}.$$

Since sum of odd numbers of odd number is again an odd number,

$$\text{ord}_2(1 + p + p^2 + \cdots + p^e) = 0 = \text{ord}_2(e + 1).$$

Next, let  $k \geq 1$ . Now,

$$\begin{aligned} \frac{p^{(e+1)} - 1}{p - 1} &= \frac{(p^{2^k})^l - 1}{p^{2^k} - 1} \frac{p^{2^k} - 1}{p - 1} \\ \Rightarrow \text{ord}_2\left(\frac{p^{(e+1)} - 1}{p - 1}\right) &= \text{ord}_2\left(\frac{(p^{2^k})^l - 1}{p^{2^k} - 1}\right) + \text{ord}_2\left(\frac{p^{2^k} - 1}{p - 1}\right) = \text{ord}_2\left(\frac{p^{2^k} - 1}{p - 1}\right). \end{aligned}$$

Now,

$$\begin{aligned} \frac{p^{2^k} - 1}{p - 1} &= (p + 1)(p^2 + 1) \cdots (p^{2^{k-1}} + 1) \\ \Rightarrow \text{ord}_2\left(\frac{p^{2^k} - 1}{p - 1}\right) &= \text{ord}_2(p + 1) + \text{ord}_2(p^2 + 1) + \cdots + \text{ord}_2(p^{2^{k-1}} + 1) \\ &= k = \text{ord}_2(e + 1). \end{aligned}$$

Thus,

$$\text{ord}_2(1 + p + p^2 + \cdots + p^e) = \text{ord}_2(e + 1).$$

Since,  $p \equiv 3 \pmod{4}$  and  $\text{ord}_2(p + 1) = m$ , we have  $p^{2^i} + 1 = 2 \cdot \text{odd} \quad \forall i$ , therefore,

$$1 + p + \cdots + p^e = (p + 1)(p^2 + 1) + \cdots + (p^{2^{k-1}} + 1) = 2^m \cdot 2^{k-1} \cdot \text{odd}$$

Thus, if  $e$  is odd then  $\text{ord}_2(1 + p + \cdots + p^e) = m + \text{ord}_2(e + 1) - 1$ . Again if  $e$  is even then  $\text{ord}_2(1 + p + \cdots + p^e) = 0$ , as sum of odd numbers of odd term is odd. Hence the theorem.  $\square$

**Lemma 1.2.2.** *If  $p$  is an odd prime and  $k \equiv 1 \pmod{p}$  then for any positive integer  $n$ ,*

$$\text{ord}_p(n) = \text{ord}_p\left(\frac{k^n - 1}{k - 1}\right).$$

*Proof.* Let  $p$  and  $k$  be fixed. Also let,  $f(n) = \frac{k^n - 1}{k - 1}$  and  $x = \left[\frac{k}{p}\right]$ . Then  $px \leq k < px + p$ . But  $k \equiv 1 \pmod{p}$ ,  $k = px + 1$ . Now,

$$f(n) = \frac{k^n - 1}{k - 1} = \frac{(1 + px)^n - 1}{px} = n + \sum_{j=2}^n \binom{n}{j} (px)^{j-1}.$$

Thus,

$$\text{ord}_p\left(\frac{k^n - 1}{k - 1}\right) = \text{ord}_p\left(n + \sum_{j=2}^n \binom{n}{j} (px)^{j-1}\right).$$

It is sufficient to show that  $\text{ord}_p\left(\binom{n}{j}\right) + (j-1)(1 + \text{ord}_p(x)) > \text{ord}_p(n)$ ,  $\forall j \geq 2$ .

We have,

$$\binom{n}{j} = \frac{n(n-1)\dots(n-j+1)}{j!}.$$

So,

$$\begin{aligned} \text{ord}_p\left(\binom{n}{j}\right) &= \text{ord}_p(n) + \text{ord}_p(n-1) + \dots + \text{ord}_p(n-j+1) - \text{ord}_p(j!) \\ &\geq \text{ord}_p(n) - \text{ord}_p(j!). \end{aligned}$$

From elementary number theory we get,

$$\text{ord}_p(j!) = \sum_{i=1}^{\infty} \left[\frac{j}{p^i}\right] < \sum_{i=1}^{\infty} \frac{j}{p^i} = \frac{j}{p-1} \leq \frac{j}{2} \leq j-1.$$

Thus,

$$\text{ord}_p\left(\binom{n}{j}\right) \geq \text{ord}_p(n) - \text{ord}_p(j!) > \text{ord}_p(n) - j + 1.$$

This implies

$$\text{ord}_p\left(\binom{n}{j}\right) + j - 1 > \text{ord}_p(n), \quad \forall j \geq 2.$$

Since  $1 + \text{ord}_p(x) \geq 1$ ,  $\text{ord}_p\left(\binom{n}{j}\right) + (j - 1)(1 + \text{ord}_p(x)) \geq \text{ord}_p(n)$ ,  $\forall j \geq 2$ .

Hence the lemma. □

From above lemma, we can easily show that following result.

**Corollary 1.2.3.**  $\text{ord}_p(n!) = \text{ord}_p\left(\frac{k^{n!} - 1}{k - 1}\right)$  if  $k \equiv 1 \pmod{p}$ .

**Lemma 1.2.4.** *Suppose  $p$  is an odd prime,  $q$  is an integer not divisible by  $p$ , and  $n$  is a positive integer. Let  $l$  denote the multiplicative order of  $q \pmod{p}$ . Then  $p \mid (q^n - 1)$  if and only if  $l \mid n$ , in which case  $\text{ord}_p\left(\frac{q^n - 1}{q^l - 1}\right) = u$  if  $\text{ord}_p(n) = u$ .*

*Proof.* Since  $l$  is the multiplicative order of  $q \pmod{p}$ . Therefore  $q^l \equiv 1 \pmod{p}$ . First, let  $p \mid q^n - 1$  i.e.  $q^n \equiv 1 \pmod{p}$ . But  $l$  is the least positive integer such that  $q^l \equiv 1 \pmod{p}$ . Thus,  $l \mid n$ .

Conversely, let  $l \mid n$ , we have,  $n = lt$ , for some integer  $t$ . This gives  $q^n - 1 = (q^l)^t - 1 \equiv 0 \pmod{p}$ . So,  $p \mid q^n - 1$ .

Next, let  $k = q^l$ ,  $k \equiv 1 \pmod{p}$ . Also, let  $l = pr$  therefore  $q^l = q^{pr} \equiv q^r \equiv 1 \pmod{p}$ . Thus,  $l \mid r$ , which is a contradiction. So,  $l$  and  $p$  are relatively prime. Now  $p^u \parallel n$  gives,  $p^u \parallel t$ . Therefore by Lemma 1.2.2,  $p^u \parallel \frac{k^t - 1}{k - 1}$  i.e.,  $p^u \parallel \frac{q^n - 1}{q^l - 1}$ . Hence the lemma. □

**Lemma 1.2.5.** *Suppose  $q$  is an odd prime and  $n$  is a positive integer. Then  $2 \mid \frac{q^n - 1}{q - 1}$  iff  $2 \mid n$ , in which case  $\text{ord}_2\left(\frac{q^n - 1}{q - 1}\right) = \text{ord}_2(n) - 1$ .*

*Proof.* First let  $2 \mid \frac{q^n - 1}{q - 1}$ . This implies  $\frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} \equiv 0 \pmod{2}$ . Thus,  $n \equiv 0 \pmod{2}$  i.e.,  $2 \mid n$ .

Conversely, let  $2 \mid n$ . Now,

$$\frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} \equiv n \pmod{2} \equiv 0 \pmod{2}.$$

Thus,

$$2 \mid \frac{q^n - 1}{q - 1}.$$

Next, let  $q^2 = k$  and  $n = 2r$ . Now,  $2^u \parallel n$  gives  $2^{u-1} \parallel r$ . By first part,  $2^{u-1} \parallel \frac{k^r - 1}{k - 1}$ . So,  $2^{u-1} \parallel \frac{q^n - 1}{q^2 - 1}$ . Hence the lemma.  $\square$

**Lemma 1.2.6.** *If  $k \equiv 1 \pmod{4}$ , then for any positive integer  $n$ ,*

$$\text{ord}_2(n) = \text{ord}_2\left(\frac{k^n - 1}{k - 1}\right).$$

*Proof.* Suppose,  $l = \lfloor \frac{k}{4} \rfloor$ . This implies  $4l \leq k < 4l + 4$ . But  $k \equiv 1 \pmod{4}$ ,  $k = 4l + 1$ . Now,

$$\frac{k^n - 1}{k - 1} = \frac{(4l + 1)^n - 1}{4l} = n + \sum_{j=2}^n \binom{n}{j} (4l)^{j-1}.$$

It is sufficient to show that,

$$\text{ord}_2\left(\binom{n}{j}\right) + 2(j - 1) + \text{ord}_2(l)(j - 1) > \text{ord}_2(n).$$

Now,

$$\begin{aligned} \text{ord}_2\left(\binom{n}{j}\right) &= \text{ord}_2(n) + \text{ord}_2(n - 1) + \cdots + \text{ord}_2(n - j + 1) - \text{ord}_2(j!) \\ &\geq \text{ord}_2(n) - \text{ord}_2(j!) \end{aligned}$$

Now by elementary number theory,

$$\text{ord}_2(j!) = \sum_{i=1}^{\infty} \left\lfloor \frac{j}{2^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{j}{2^i} = j.$$

Therefore,

$$\text{ord}_2\left(\binom{n}{j}\right) \geq \text{ord}_2(n) - \text{ord}_2(j!) > \text{ord}_2(n) - j \geq \text{ord}_2(n) - (2j - 2).$$

Thus,

$$\text{ord}_2\left(\binom{n}{j}\right) + 2(j - 1) + \text{ord}_2(l)(j - 1) > \text{ord}_2(n).$$

Hence,

$$\text{ord}_2\left(\frac{k^n - 1}{k - 1}\right) = \text{ord}_2(n).$$

□

### 1.3 Harmonic mean

Most popular among all types of “mean” are arithmetic mean, geometric mean, harmonic mean.

Given a positive integer  $n$ , the arithmetic mean  $A(n)$ , the geometric mean  $G(n)$  and the harmonic mean  $H(n)$  of the positive divisors of  $n$  are defined as,

$$A(n) = \frac{\sigma(n)}{\tau(n)}, \quad G(n) = \left( \prod_{d|n} d \right)^{\frac{1}{\tau(n)}}, \quad \frac{1}{H(n)} = \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d}.$$

Note that  $A, G, H$  mentioned in the above definitions are arithmetic functions. In fact all of them are multiplicative functions.

Of the three means defined above, the harmonic mean  $H(n)$  plays the pivotal role in this dissertation.

**Lemma 1.3.1.** For all positive integer  $n$ ,  $H(n) = \frac{n\tau(n)}{\sigma(n)}$ .

*Proof.* The harmonic mean,  $H(n)$  of a positive integer  $n$  is defined by,

$$\frac{1}{H(n)} = \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d}.$$

Since

$$n \sum_{d|n} \frac{1}{d} = \sum_{d|n} \frac{n}{d} = \sum_{d|n} d = \sigma(n),$$

it follows that

$$\frac{1}{H(n)} = \frac{\sigma(n)}{n\tau(n)} \Rightarrow H(n) = \frac{n\tau(n)}{\sigma(n)}.$$

□

As  $A(n) = \frac{\sigma(n)}{\tau(n)}$  and  $G(n) = \sqrt{n}$ , we have, the following result.

**Corollary 1.3.2.** For all  $n$ ,

$$H(n)A(n) = G(n)^2.$$

*In other words, the geometric mean of the divisors is the geometric mean of the arithmetic and harmonic mean of the divisors.*

**Lemma 1.3.3.** If  $n$  is an even perfect number, then

$$H(n) = \text{ord}_2(n) + 1.$$

*Proof.* Since  $n$  is an even perfect, so by Euclid-Euler Theorem

$$n = 2^{a-1}(2^a - 1),$$

where  $(2^a - 1)$  is an odd prime. Now,

$$H(n) = \frac{n\tau(n)}{\sigma(n)} = \frac{n\tau(n)}{2n} = a = \text{ord}_2(n) + 1.$$

Hence the lemma. □

**Lemma 1.3.4.** If  $n = \prod_{i=1}^k p_i^{\alpha_i}$  then  $H(n) \geq \prod_{i=1}^k \frac{2p_i}{p_i + 1}$ .

*Proof.* We have,

$$\begin{aligned} H(n) &= \prod_{i=1}^k \frac{\alpha_i + 1}{1 + \frac{1}{p_i} + \cdots + \frac{1}{(p_i)^{\alpha_i}}} \geq \prod_{i=1}^k \frac{\alpha_i + 1}{1 + \frac{1}{p_i} + \cdots + \frac{1}{(p_i)^{\alpha_i}}} \\ &\geq \prod_{i=1}^k \frac{2}{1 + \frac{1}{p_i}} = \prod_{i=1}^k \frac{2p_i}{p_i + 1}. \end{aligned}$$

□

**Remark 1.3.5.** For  $n > 1$ ,  $H(n) \geq \frac{4}{3}$ .

Throughout this dissertation  $w(n)$  denotes the number of prime factors of any integer  $n$ .

**Lemma 1.3.6.** If  $w(n) = k > 2$ , then  $H(n) > k$ .

*Proof.* Let  $n = \prod_{i=1}^k p_i^{\alpha_i}$ , be the prime of decomposition of  $n$ . So,

$$H(n) \geq \prod_{i=1}^k \frac{2p_i}{p_i + 1} > \frac{4}{3} \cdot \frac{6}{4} \cdot \frac{10}{6} \cdots \frac{10}{6} = 2\left(\frac{10}{6}\right)^{k-2} > k.$$

□

**Lemma 1.3.7.**  $H(n) \leq \tau(n)$ ,  $\forall n$ .

*Proof.* If  $n = \prod_{i=1}^k p_i^{\alpha_i}$  is the prime decomposition of  $n$ . Then

$$H(n) = \prod_{i=1}^k \frac{\alpha_i + 1}{1 + \frac{1}{p_i} + \cdots + \frac{1}{(p_i)^{\alpha_i}}} \leq \prod_{i=1}^k (\alpha_i + 1) = \tau(n).$$

□

## 1.4 Some more results

**Lemma 1.4.1.** *Suppose  $p$  is a prime and  $t$  is any integer, then  $p \mid (1 + t + \dots + t^{p-1})$  iff  $t \equiv 1 \pmod{p}$ .*

*Proof.* First let,  $t \equiv 1 \pmod{p}$ . Then  $1 + t + \dots + t^{p-1} \equiv 0 \pmod{p}$ . Thus,  $p \mid (1 + t + t^2 + \dots + t^{p-1})$ .

Conversely, let  $p \mid (1 + t + \dots + t^{p-1})$ . Now, by Fermat's Theorem,  $t^p \equiv t \pmod{p}$ . This gives,  $(t-1)(1 + t + t^2 + \dots + t^{p-1}) \equiv (t-1) \pmod{p}$ . Therefore,  $1 + t + t^2 + \dots + t^{p-1} \equiv 1 \pmod{p}$ , which is a contradiction. Thus,  $t \equiv 1 \pmod{p}$ .  $\square$

**Corollary 1.4.2.** *If  $t \equiv 1 \pmod{p}$  then  $\text{ord}_p(1 + t + t^2 + \dots + t^{p-1}) = 1$ .*

*Proof.* If  $k \equiv 1$  and  $p$  is an odd prime, then by Lemma 1.2.2,

$$\text{ord}_p(n) = \text{ord}_p\left(\frac{k^n - 1}{k - 1}\right).$$

Now, putting  $k = t$  and  $n = p$ , we get the required result.  $\square$

**Lemma 1.4.3.** *Suppose  $k, l, p$  are primes with  $p \equiv 1 \pmod{l}$ . If  $l \mid (1 + p + \dots + p^{k-1})$ , then  $k = l$ .*

*Proof.* Since,  $p \equiv 1 \pmod{l}$ ,  $\sum_{i=0}^{k-1} p^i \equiv k \pmod{l}$ . If  $l \mid \sum_{i=0}^{k-1} p^i$ , then  $l \mid k$ , which is impossible unless  $k = l$ .  $\square$

**Proposition 1.4.4.** *Let  $r, a, b$  be positive integers. If integers  $x_1, x_2, \dots, x_r$  satisfy  $1 \leq x_1 < x_2 < \dots < x_r$  and  $1 \leq \prod_{i=1}^{r-1} (1 + \frac{1}{x_i}) < \frac{a}{b} \leq \prod_{i=1}^r (1 + \frac{1}{x_i})$  then it follows that*

$$\prod_{i=1}^r x_i \leq (b+1)^{2^r-1} - (b+1)^{2^{r-1}-1}$$

with equality if and only if  $x_i = m_i$ ,  $1 \leq i \leq r$ , where

$$m_i = \begin{cases} (b+1)^{2^{i-1}}; & \text{if } i = 1, 2, \dots, r-1 \\ (b+1)^{2^{i-1}-1} & i = r \end{cases}.$$

**Proposition 1.4.5.** *Let  $a, b, k$  be positive integers. Suppose that  $n$  is a positive integer with  $k$  distinct prime factors such that  $\sigma_{-1}^*(n) = \frac{a}{b}$ . Then it follows that  $n \leq (b+1)^{2^{k-1}-1}((b+1)^{2^{k-1}} - 1)$ . Furthermore, if the equality holds, then  $k \leq 2$ .*

## 1.5 Some inequalities

**Lemma 1.5.1.** *If  $a_1 > a_2 > \dots$  are real numbers then  $\forall m, n \geq 1$ ,*

$$\left( \frac{a_1 + a_2 + \dots + a_m}{m} \right) > \left( \frac{a_1 + a_2 + \dots + a_{m+n}}{m+n} \right).$$

*Proof.*  $\forall i \leq m$  and  $\forall j \leq n$ , we have  $a_i > a_{m+j}$ . Fixing  $i$  and summing over  $j$  we get

$$na_i > \sum_{j=1}^n a_{m+j} \Rightarrow a_i > \frac{1}{n} \sum_{j=1}^n a_{m+j}.$$

Now summing over  $i$ ,

$$\begin{aligned} \sum_{j=1}^n a_m > \frac{m}{n} \sum_{j=1}^n a_{m+j} &\Rightarrow (m+n) \sum_{i=1}^m a_i > m \sum_{k=1}^{m+n} a_k \\ &\Rightarrow \frac{1}{m} \sum_{i=1}^m a_i > \frac{1}{m+n} \sum_{k=1}^{m+n} a_k. \end{aligned}$$

□

**Lemma 1.5.2.** *Let  $a < b$  and  $c \geq 1$  are integers and  $p, q$  are primes such that  $p < q$ . Then,*

$$1 < S(q^a) < S(q^b) < S(\bar{q}) \leq S(p) \leq S(p^c).$$

*Proof.* We have,  $S(q^a) = \frac{\sigma(q^a)}{q^a} = 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^a} > 1.$  (I)

Now,

$$\begin{aligned} S(q^b) &= 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^b} = 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^a} + \cdots + \frac{1}{q^b} \\ &> 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^a} = S(q^a). \end{aligned} \quad (II)$$

Again,

$$S(q^b) = \frac{q^{b+1} - 1}{q^b(q - 1)} < \frac{q}{q - 1} = S(\bar{q}). \quad (III)$$

Aiso,

$$S(\bar{q}) = \frac{q}{q - 1} \leq 1 + \frac{1}{p} = S(p) \leq 1 + \frac{1}{p} + \cdots + \frac{1}{p^c} = S(p^c). \quad (IV)$$

Thus, (I), (II), (III) and (IV) gives the required result.  $\square$

**Lemma 1.5.3.** *Let  $p, q$  be primes and  $e, f$  positive integers. If  $p < q, e < f$ , then*

$$1 < S(q^e) < S(q^f) < S(p^e) < S(p^f) < 2.$$

*Proof.* We know that

$$S(q^e) = 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^e} > 1. \quad (I)$$

Also,

$$S(q^f) = 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^f} = 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^e} + \cdots + \frac{1}{q^f}$$

$$> 1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^e} = S(q^e). \quad (II)$$

Now,  $S(q^f) < \frac{q}{q-1} \leq S(p) < S(p^2) < \cdots < s(p^e) \leq S(p^f).$  (III).

Lastly,  $S(p^f) = 1 + \frac{1}{p} + \cdots + \frac{1}{p^f} \leq 1 + \left( \frac{1}{2} + \cdots + \frac{1}{2^f} \right) < 2.$  (IV)

Hence from (I), (II), (III) and (IV), we have

$$1 < S(q^e) < S(q^f) < S(p^e) < S(p^f) < 2.$$

□

**Lemma 1.5.4.**  $3^r \geq (r+1)^2, \forall r \geq 2$

*Proof.* For  $r = 2$ , we get the equality. Now,  $\forall r > 2$ ,

$$3^r = (1+2)^r \geq 1 + 2 \cdot r^2 > (r+1)^2.$$

□

**Lemma 1.5.5.** *If  $s \geq 1$ , then  $5^s \geq 1 + (s+1)^2$ .*

*Proof.* For  $s = 1$ , we get the equality. Suppose, the inequality is true for  $s$ .

Now,

$$5^{s+1} \geq 5(s+1)^2 + 5 > (s^2 + 4s + 5) \geq 1 + (s+1)^2.$$

□

**Lemma 1.5.6.** *If  $s \geq 2$  then  $5^s \geq (2s+1)^2$ .*



*Proof.* We have,

$$5^s = (1+4)^s \geq 1 + 4s^2 + 4s(s-1) \geq (2s+1)^2.$$

□

**Lemma 1.5.7.**  $5^{a-1} \geq a^2 + 1, \forall a \geq 2.$

*Proof.* We have,

$$5^{a-1} = (1+4)^{a-1} \geq 1 + 4(a-1) + \frac{(a-1)(a-2)}{2}4^2 = 8a^2 + 20a + 13 \geq a^2 + 1.$$

□

**Lemma 1.5.8.**  $3^a \geq 2a^2 + 1, \forall a.$

*Proof.* We have,

$$3^a = (1+2)^a \geq 1 + 2a + \frac{a(a-1)}{2}2^2 = 1 + 2a^2.$$

□

**Lemma 1.5.9.**  $5^a \geq 8a^2 + 1, \forall a \geq 3.$

*Proof.* We have,

$$5^a = (1+4)^a \geq 1 + 4a + \frac{a(a-1)}{2}4^2 + \frac{a(a-1)(a-2)}{6}4^3 \geq 1 + 8a^2.$$

□

**Lemma 1.5.10.**  $3^a - 1 > 4a, \forall a \geq 3.$

*Proof.* We have,

$$\begin{aligned} 3^a &= (1+2)^a \geq 1 + 2a + \frac{a(a-1)}{2}2^2 + \frac{a(a-1)(a-2)}{6}2^3 \\ &\geq 1 + \frac{4}{3}a^3 - 2a^2 + \frac{8}{3}a \\ &\geq 1 + 4a^2 - 2a^2 + \frac{8}{3}a \geq 1 + 2a^2 + \frac{8}{3}a > 4a. \end{aligned}$$

□

**Lemma 1.5.11.**  $2^{r+1} \geq (r+1)^2$ ,  $\forall r \geq 3$

*Proof.* For  $r = 3$  we get the equality. Let  $2^{r+1} \geq (r+1)^2$ . Now,

$$2^{r+2} = 2 \cdot 2^{r+1} \geq 2(r+1)^2 > r^2 + 4r + 4 = (r+2)^2.$$

Hence,

$$2^{r+1} \geq (r+1)^2, \quad \forall r \geq 3.$$

□

**Lemma 1.5.12.**  $3^s \geq (s+1)^2$ , for even  $s$ .

*Proof.* For  $s = 2$ , equality holds. Suppose,  $3^s \geq (s+1)^2$  for even  $s$  with  $s \geq 4$ . Now,

$$3^{s+2} = 9 \cdot 3^s \geq 9(s+1)^2 = 9s^2 + 18s + 9 > (s+3)^2.$$

Hence the inequality.

□

**Lemma 1.5.13.**  $2^{r+1} > \frac{8}{9}(r+1)^2$ , for  $r \geq 2$ .

*Proof.* For  $r = 1$  the result is true. Suppose it is true for  $r \geq 2$  i.e.,  $2^{r+1} > \frac{8}{9}(r+1)^2$ . Now,

$$2^{r+2} > 2 \frac{8}{9}(r+1)^2 > \frac{8}{9}(r^2 + 4r + 4) = \frac{8}{9}(r+2)^2.$$

□

**Lemma 1.5.14.**  $2^r \geq \frac{r(r+1)}{2} \quad \forall r \geq 1$

*Proof.* For  $r = 1$ , the result is true. Suppose the inequality is true for  $r \geq 2$ , i.e.,  $2^r \geq \frac{r(r+1)}{2}$ . Now,

$$2^{r+1} \geq 2 \frac{r(r+1)}{2} = \frac{(r+1)2r}{2} > \frac{(r+1)(r+2)}{2}.$$

Hence the inequality is true for all  $r$  with  $r \geq 1$ .

□

**Lemma 1.5.15.**  $3^s \geq \frac{27}{16}(s+1)^2, \quad \forall s \geq 3.$

*Proof.* It is sufficient to show that  $3^{s-3} \geq \frac{1}{16}(s+1)^2$ . For  $s = 3$ , we get the equality. Suppose the inequality is true for all  $s > 3$ . Now,

$$3^{s-2} = 3 \cdot 3^{s-3} \geq 3 \frac{1}{16}(s+1)^2 > \frac{1}{16}(s^2 + 4s + 4) = \frac{1}{16}(s+2)^2.$$

Hence,

$$3^{s-3} \geq \frac{1}{16}(s+1)^2 \text{ for } s \geq 3.$$

□

**Lemma 1.5.16.**  $2^{r+3} \geq \frac{32}{9}(r+1)^2, \quad \forall r \geq 1.$

*Proof.* It is sufficient to show that  $2^r \geq \frac{4}{9}(r+1)^2$ ,  $\forall r \geq 1$ . Now, the inequality is true for  $r = 1, 2$ . Suppose it is true for  $r$ , with  $r > 2$ , i.e.,  $2^r \geq \frac{4}{9}(r+1)^2$ . Now,

$$2^{r+1} \geq 2 \cdot \frac{4}{9}(r+1)^2 = \frac{4}{9}(2r^2 + 4r + 2) > \frac{4}{9}(r+2)^2.$$

Thus,

$$2^r \geq \frac{4}{9}(r+1)^2, \forall r \geq 1.$$

□

**Lemma 1.5.17.**  $3^s \geq \frac{3}{4}(s+3)^2$ ,  $\forall s \geq 3$ .

*Proof.* For  $s = 3$ , the equality holds. Suppose the inequality is true  $s$ ,  $s \geq 4$ .

Now,

$$3^{s+1} = 3 \cdot 3^s \geq 3 \cdot \frac{3}{4}(s+3)^2 = \frac{3}{4}(3s^2 + 18s + 27) > \frac{3}{4}(s+4)^2.$$

Hence,

$$3^s \geq \frac{3}{4}(s+3)^2, \forall s \geq 3.$$

□

**Lemma 1.5.18.**  $2^{r+3} \geq \frac{64}{9}(r+1)^2$ ,  $\forall r \geq 5$ .

*Proof.* It is sufficient to show that  $2^{r-3} \geq \frac{1}{9}(r+1)^2$ ,  $\forall r \geq 5$ . For  $r = 5$ , the equality holds. Suppose that the inequality is true  $\forall r \geq 6$ . Now,

$$2^{r-2} = 2 \cdot 2^{r-3} \geq 2 \cdot \frac{1}{9}(r+1)^2 > \frac{1}{9}(r^2 + 4r + 4) = \frac{1}{9}(r+2)^2.$$

Hence,

$$2^{r+3} \geq \frac{64}{9}(r+1)^2, \forall r \geq 5.$$

□

## Chapter 2

# Harmonic Numbers

Perhaps because a number of mathematicians were frustrated by their failure to disprove the existence of odd perfect numbers, many authors have defined a number of closely related concepts and produced a raft of problems, many of which seem no more tractable than the original. One such concept is the notion of harmonic numbers. Harmonic number is a kind of generalisation of perfect numbers. In this chapter we shall study harmonic numbers and their properties in detail.

### 2.1 Definition and characterisation

A positive integer  $n$  is said to be *harmonic* if the harmonic mean  $H(n)$  of its divisors is an integer. The set of all harmonic number is denoted by  $\mathcal{H}$ . At the end of this dissertation we have listed all harmonic numbers up to  $10^{14}$ . From the table 5.4, we can also see that there are no powerful harmonic

numbers less than  $10^{12}$  where a number  $n$  is powerful if  $p \mid n$  implies  $p^2 \mid n$  ( $p$  is a prime). We now discuss some of the basic properties of harmonic numbers.

**Lemma 2.1.1.** *There is no harmonic number of the form  $p^a$ , where  $p$  is a prime and  $a > 0$ .*

*Proof.* Let  $n = p^a$  be a harmonic number. Then,

$$H(n) = \frac{p^a(a+1)}{1+p+p^2+\cdots+p^a}.$$

Since,  $1+p+p^2+\cdots+p^a$  and  $p^a$  are relatively prime,  $1+p+p^2+\cdots+p^a$  divides  $(a+1)$ . But,  $(1+p+p^2+\cdots+p^a) > (a+1)$ . Hence,  $n$  can not be a harmonic number.  $\square$

**Theorem 2.1.2.** *Every perfect number is harmonic, but not conversely.*

*Proof.* Let  $n$  be a perfect number. Then,  $\sigma(n) = 2n$ . Now,

$$H(n) = \frac{n\tau(n)}{\sigma(n)} = \frac{n\tau(n)}{2n} = \frac{\tau(n)}{2}.$$

First let  $n$  be an even perfect number,  $\tau(n)$  is even. Hence,  $H(n)$  is an integer. Next let  $n$  is odd then it is known that one of the exponents in the prime factor decomposition is odd so  $\tau(n)$  is even also in this case. Hence every perfect number is harmonic.

The converse of this theorem does not hold. For example, 140, 270 are not perfect numbers, but  $H(140) = 5, H(270) = 6$ .  $\square$

Form the above theorem we get the following result.

**Corollary 2.1.3.** *If  $n = 2^{a-1}(2^a - 1)$  is perfect (so that  $2^a - 1$  and  $a$  are primes), then  $H(n) = a$ .*

Till now, no odd harmonic number has been discovered. In fact, it is a conjecture that all harmonic numbers other than 1 are even.

**Lemma 2.1.4.**  *$\mathcal{H}$  is monotonic in the sense that if  $e, f$  are positive integers and  $p, q$  are primes with  $e < f$  and  $p < q$  then*

$$H(p^e) < H(p^f) < H(q^f).$$

*Proof.* Note that,  $\{\frac{1}{p^k}\}$  is a monotonically decreasing sequence. Therefore, if  $e < f$ , we have,

$$\begin{aligned} \frac{1}{e+1} \sum_{i=0}^e \frac{1}{p^i} &> \frac{1}{f+1} \sum_{i=0}^f \frac{1}{p^i} \\ \Rightarrow H(p^e) &> H(p^f) \end{aligned}$$

On the other hand, if  $p < q$ , we have,

$$\frac{1}{p^i} > \frac{1}{q^i}, \forall i \text{ with } 1 \leq i \leq f.$$

Therefore,

$$\begin{aligned} \frac{1}{f+1} \sum_{i=0}^f \frac{1}{q^i} &< \frac{1}{f+1} \sum_{i=0}^f \frac{1}{p^i} \\ \Rightarrow H(p^f) &< H(q^f). \end{aligned}$$

This proves the lemma. □

**Theorem 2.1.5.** *No harmonic number greater than 6 is squarefree.*

*Proof.* Let  $n > 6$  be a squarefree harmonic number. Then,  $n = p_1 p_2 \cdots p_r$  where  $p_i$ ,  $1 \leq i \leq r$ , are distinct primes arranged in ascending order. Now,

$$H(n) = 2^r \frac{p_1 p_2 \cdots p_r}{(p_1 + 1)(p_2 + 1) \cdots (p_r + 1)}.$$

Case 1:  $n$  is even. Without any loss we can take  $p_1 = 2$ . Now,

$$H(n) = \frac{4 p_2 \cdots p_r}{3 \frac{p_2 + 1}{2} \frac{p_3 + 1}{2} \cdots \frac{p_r + 1}{2}}.$$

Since  $H(n)$  is an integer, we must have  $p_2 = 3$ . So,

$$H(n) = \frac{2 p_3 \cdots p_r}{\frac{(p_3 + 1)}{2} \cdots \frac{(p_r + 1)}{2}}.$$

Therefore,  $\frac{p_3 + 1}{2} \mid (2 p_3 p_4 \cdots p_r)$ . This implies  $\frac{p_3 + 1}{2} \mid 2$ , which is a contradiction to the fact  $\frac{p_3 + 1}{2} > 2$ .

Case 2:  $n$  is odd. So,

$$H(n) = \frac{p_1 p_2 \cdots p_r}{\frac{p_1 + 1}{2} \frac{p_2 + 1}{2} \cdots \frac{p_r + 1}{2}}.$$

Here the denominator contains at least  $r$  prime factors less than  $p_r$ , but the numerator has  $(r-1)$  prime factors less than  $p_r$ . Hence  $H(n)$  is not an integer, which is a contradiction. This completes the theorem.  $\square$

**Lemma 2.1.6.** *If  $H(n)$  is even then  $n$  is even.*

*Proof.* Suppose  $n$  is odd. Also, let  $n = \prod_{i=1}^r p_i^{e_i}$  where  $p_i$ 's are distinct odd

primes. Now,

$$H(p_i^{e_i}) = \frac{p_i^{e_i}(e_i + 1)}{1 + p_i + p_i^2 + \cdots + p_i^{e_i}}$$

$$\Rightarrow \text{ord}_2(H(p_i^{e_i})) = \text{ord}_2(p_i^{e_i}) + \text{ord}_2(e_i + 1) - \text{ord}_2(1 + p_i + p_i^2 + \cdots + p_i^{e_i})$$

$$\leq 0 \quad \forall i$$

So,  $\text{ord}_2 H(n) = 0$ .

This implies that  $H(n)$  is odd, which is a contradiction to our hypothesis.

Hence,  $n$  is also even if  $H(n)$  is even.  $\square$

**Lemma 2.1.7.** *Suppose  $n \in \mathcal{H}$ . If  $p$  is an odd prime such that  $p \parallel n$  and  $p \mid H(n)$ , then  $\frac{n}{p}$  is harmonic. If  $2 \parallel n$  and  $4 \mid H(n)$ , then  $\frac{n}{2}$  is harmonic.*

*Proof.* Since,  $n \parallel p$ , so  $\frac{n}{p}$  is an integer. Now,  $\sigma\left(\frac{n}{p}\right) = \frac{\sigma(n)}{\sigma(p)}$  and  $\tau\left(\frac{n}{p}\right) = \frac{\tau(n)}{\tau(p)}$ .

Also,  $H\left(\frac{n}{p}\right) = \frac{\frac{n}{p} \tau\left(\frac{n}{p}\right)}{\sigma\left(\frac{n}{p}\right)} = H(n) \frac{p+1}{2p}$ , which is an integer as  $p \parallel H(n)$  and

$2 \mid (p+1)$ . Hence,  $\frac{n}{p} \in \mathcal{H}$ . Now, taking  $p = 2$  in the above proof and using the given hypothesis we can show easily that  $\frac{n}{2} \in \mathcal{H}$ .  $\square$

G. L. Cohen [5] mentioned the following result for a harmonic number  $n$  with  $w(n) = 3$  or 4.

**Lemma 2.1.8.** *Suppose  $w(n) = 3$  or 4. Then  $\sigma(n) \neq 2n$ . If  $\sigma(n) = 3n$  then  $n \in \{120, 672, 523776\}$ .*

**Theorem 2.1.9.**  *$H(p^e q^f) > H(p^f q^e)$  where  $p, q$  are primes and  $e, f$  are positive integers such that  $p < q, e < f$ .*

*Proof.* We are to show that

$$\begin{aligned} \frac{p^e q^f (e+1)(f+1)}{\frac{p^{e+1}-1}{p-1} \frac{q^{f+1}-1}{q-1}} &> \frac{p^f q^e (f+1)(e+1)}{\frac{p^{f+1}-1}{p-1} \frac{q^{e+1}-1}{q-1}} \\ \Rightarrow p^e q^f (p^{f+1}-1)(q^{e+1}-1) &> p^f q^e (p^{e+1}-1)(q^{f+1}-1) \end{aligned}$$

In general, if  $a > b$  and  $1 \leq x < y$ . Then,

$$\begin{aligned} x^b y^a (x^{a+1}-1)(y^{b+1}-1) &> x^a y^b (x^{b+1}-1)(y^{a+1}-1) \\ \text{or, } x^a y^b (y^{a+1}-x^{b+1}y^{a-b}) + x^b y^a &> x^b y^b (y^{a+1}-x^{a+1}) + x^a y^b \\ \text{or, } x^b y^b x^{a-b} (y^{a+1}-x^{b+1}y^{a-b}) + x^b y^a &> x^b y^b (y^{a+1}-x^{a+1}) + x^a y^b. \end{aligned} \quad (2.1.a)$$

Now,

$$1 - \left(\frac{x}{y}\right)^{b+1} \geq 1 - \left(\frac{x}{x+1}\right)^2 > 1 - \frac{x}{x+2} \geq \frac{1}{x} \geq \frac{1}{x^{a-b}}.$$

So,

$$x^{a-b} (y^{b+1} - x^{b+1}) > y^{b+1}.$$

Thus,

$$\begin{aligned} x^{a-b} (y^{a+1} - x^{b+1}y^{a-b}) &= x^{a-b} y^{a+1} (1 - \left(\frac{x}{y}\right)^{b+1}) > y^{a+1} > y^{a+1} - x^{a+1} \\ \text{or, } x^b y^b x^{a-b} (y^{a+1} - x^{b+1}y^{a-b}) &> x^b y^b (y^{a+1} - x^{a+1}) \end{aligned} \quad (2.1.b)$$

Again,  $x < y$  and  $a > b$  gives

$$x^b y^a > x^a y^b \quad (2.1.c)$$

(2.1.a), (2.1.b) and (2.1.c) gives

$$x^b y^a (x^{a+1}-1)(y^{b+1}-1) > x^a y^b (x^{b+1}-1)(y^{a+1}-1). \quad (2.1.d)$$

Now, taking  $x = p$ ,  $y = q$ ,  $e = b$  and  $f = a$  in (2.1.d), we get the required result.  $\square$

**Lemma 2.1.10.** *Let  $a$  be a positive integer such that for some  $p \equiv 2 \pmod{3}$ . We have,  $3p \mid \sigma(2^a)$  and  $\gcd(3p, a + 1) = 1$ . If  $m$  is an odd integer such that  $2^a m$  is harmonic then  $m$  is not squarefree.*

*Proof.* Note that  $p \neq 2$ , for if  $p = 2$  then  $2 \mid (2^{a+1} - 1)$ , which is absurd, therefore  $p$  is an odd prime. Let  $m$  be a squarefree odd integer such that  $n = 2^a m$  is harmonic. Now,  $\tau(n) = 2^{w(m)}(a + 1)$ . Since,  $H(n)\sigma(n) = n\tau(n)$ ,  $3p \mid \sigma(n)$  and  $\gcd(3p, a + 1) = 1$ , we have,  $3 \parallel n$  and  $p \parallel n$ . As  $3 \mid \sigma(2^a)$  and  $p \equiv 2 \pmod{3}$ ,  $3 \mid (p + 1)$ . Again  $p \parallel n$  gives  $(p + 1) \mid \sigma(m)$ ,  $3 \mid \sigma(n)$ . Thus,  $3^2 \mid H(n)\sigma(n)$ . But  $3^2 \nmid n\tau(n)$ , which is impossible as  $H(n)\sigma(n) = n\tau(n)$ . Thus,  $m$  cannot be squarefree.  $\square$

**Theorem 2.1.11.** *For any real number  $\alpha$ , there exist only finitely many positive integers  $n$  satisfying  $H(n)^\alpha > n$ .*

*Proof.* Since a harmonic mean is less than or equal to a geometric mean and the geometric mean of positive divisors of  $n$  is  $\sqrt{n}$ , there exist no harmonic numbers  $n$  satisfying  $H(n)^2 > n$ . Also if  $\alpha < 2$ , then  $H(n)^\alpha < H(n)^2 < n$ . So for  $\alpha \leq 2$ , the theorem is true. We may assume now that  $\alpha > 2$ . Now, fixing the real number  $\alpha$ , define a function  $f(\alpha, n) = \frac{H(n)^\alpha}{n}$ . Since  $H$  is multiplicative so also  $f$  is multiplicative in the second variable, that is  $f(\alpha, mn) = f(\alpha, n)f(\alpha, m)$  when  $\gcd(n, m) = 1$ . Thus for a prime  $p$  and a

positive integer  $e$ ,

$$f(\alpha, p^e) = \frac{H(p^e)^\alpha}{p^e} = \frac{p^{(\alpha-1)e}(e+1)^\alpha}{(p^e + p^{e-1} + \dots + 1)^\alpha}.$$

If  $p < q$  then

$$f(\alpha, p^e) = \frac{(e+1)^\alpha}{p^e(1 + \frac{1}{p} + \dots + \frac{1}{p^e})^\alpha} > \frac{(e+1)^\alpha}{q^e(1 + \frac{1}{q} + \dots + \frac{1}{q^e})^\alpha} = f(\alpha, q^e).$$

Also,

$$f(\alpha, p^e) = \frac{(1 + \frac{1}{e})^\alpha}{e^{-\alpha} p^e}.$$

Now by L.H. rule we can easily show that as  $e \rightarrow \infty$ ,  $e^{-\alpha} p^e \rightarrow \infty$ . Hence  $f(\alpha, p^e)$  is monotonically decreasing as a function of  $p$  and  $e$  for sufficiently large  $p$  and  $e$ . Furthermore, we have  $\lim_{p \rightarrow \infty} f(\alpha, p^e) = 0$  and  $\lim_{e \rightarrow \infty} f(\alpha, p^e) = 0$ . Hence there are only finitely many prime powers  $p^e$  satisfying  $f(\alpha, p^e) > 1$ . Let  $L$  be the set of integers whose all prime components satisfy this condition. Since  $L$  is finite, there exists the maximum value  $\max_{n \in L} f(\alpha, n)$ . Let  $A$  be this maximum value. There are also only finitely many prime powers  $q^f$  satisfying  $f(\alpha, q^f) > \frac{1}{A}$ . Let  $L^0$  be the set of integers whose all prime components satisfy this condition and  $M$  be the set of integers  $n$  satisfying the required condition  $f(\alpha, n) > 1$ . Clearly  $M \subset L^0$ . Since  $L^0$  is finite,  $M$  is also finite. Hence the theorem.  $\square$

## 2.2 Some results on odd harmonic number

**Theorem 2.2.1.** *Let  $n$  be an odd harmonic number and  $p^e \parallel n$ . Then,*

$$p^e \equiv 1 \pmod{4}.$$

*Proof.* If  $p^e \equiv 3 \pmod{4}$  then  $p \equiv 3 \pmod{4}$  and  $e$  is odd. Now,

$$\begin{aligned}
H(p^e) &= \frac{p^e(e+1)}{1+p+p^2+\cdots+p^e} \\
\Rightarrow \text{ord}_2(H(p^e)) &= \text{ord}_2(p^e) + \text{ord}_2(e+1) - \text{ord}_2(1+p+p^2+\cdots+p^e) \\
&= \text{ord}_2(e+1) - \text{ord}_2(1+p+p^2+\cdots+p^e) \\
&= -\text{ord}_2(p+1) + 1 \\
&\leq -1 < 0,
\end{aligned}$$

which is a contradiction. Hence  $p^e \equiv 1 \pmod{4}$ .  $\square$

**Lemma 2.2.2.** *Let  $n$  be an odd harmonic number of the form  $p^r q^s$  where  $p$  and  $q$  are distinct primes. Also let  $l$  be the multiplicative order of  $q \pmod{p}$  and  $k$  be the multiplicative order of  $p \pmod{q}$  then  $l \mid (s+1)$  and  $k \mid (r+1)$ .*

*Proof.* Let  $n = p^r q^s$  and  $H(n) = m$ . Then,

$$p^r q^s (r+1)(s+1) = m(1+p+p^2+\cdots+p^r)(1+q+q^2+\cdots+q^s) \quad (2.2.a)$$

Now, by Fermat's theorem we have,

$$p^{q-1} \equiv 1 \pmod{q} \quad \text{and} \quad q^{p-1} \equiv 1 \pmod{p}.$$

So,

$$l \mid (p-1) \quad \text{and} \quad k \mid (q-1).$$

Now, let  $l \nmid (s+1)$  then  $p \nmid (q^{s+1} - 1)$ , by Lemma (1.2.4). So, (2.2.a) gives  $p^r \mid m$ . Also,

$$(r+1)(s+1) = t(1+q+q^2+\cdots+q^s), \quad \text{where } t = \frac{m(1+p+p^2+\cdots+p^r)}{p^r q^s}.$$

Therefore,

$$q^s t = \frac{m(1 + p + p^2 + \cdots + p^r)}{p^r} > m \geq p^r.$$

Also,

$$q^s t < (r + 1)(s + 1).$$

Thus we get,

$$p^r q^s < p^r q^s t < (r + 1)^2 (s + 1)^2. \quad (2.2.b)$$

Now there are two cases arise:

case 1:  $p \geq 3$  and  $r \geq 2$  and  $q \geq 5$ . Now,

$$p^r \geq 3^r \geq (r + 1)^2.$$

Also,

$$q^s \geq 5^s > (s + 1)^2 + 1 > (s + 1)^2.$$

Hence,

$$p^r q^s > (r + 1)^2 (s + 1)^2,$$

which is a contradiction to (2.2.b).

Case 2 :  $p = 3, r = 1$ . Then

$$H(n) = \frac{(3q^s)2(s + 1)}{4 \frac{q^{s+1} - 1}{q - 1}} = m,$$

which gives

$$3(s + 1) = \frac{2m(1 + q + q^2 + \cdots + q^s)}{q^s} \geq 2m \geq 2q^s.$$

Now,

$$2s + 1 > \frac{3}{2}(s + 1) \geq q^s \geq 5^s \geq (s + 1)^2 + 1.$$

So, we get

$$s^2 + 1 < 0, \text{ absurd.}$$

Hence,

$$l \mid (s + 1).$$

Similarly we can prove that

$$k \mid (r + 1).$$

□

**Lemma 2.2.3.** *Suppose  $n = p^r q^s$  is an odd harmonic number,  $l$  is the multiplicative order of  $q \pmod p$ ,  $k$  is the multiplicative order of  $p \pmod q$ ,  $r + 1 = ak$  and  $s + 1 = bl$ . Then  $\max(a, b) \geq 3$  is impossible.*

*Proof.* Suppose  $\max(a, b) \geq 3$ . Without any loss we can assume  $b \geq 3$ . Now since  $n$  is harmonic,  $H(n) = m$ , say.

Thus,

$$p^r q^s (r + 1)(s + 1) = m \frac{p^{r+1} - 1}{p - 1} \frac{q^{s+1} - 1}{q - 1}$$

i.e.,

$$(r + 1)(s + 1) = u \frac{q^{s+1} - 1}{q - 1}. \quad (2.2.c)$$

where,  $u = \frac{m}{p^r q^s} \frac{p^{r+1} - 1}{p - 1} \frac{q^{s+1} - 1}{q - 1}$ . Clearly,  $u$  is an integer. Now,

$$p^r < 1 + p + p^2 + \cdots + p^r \leq uq^s \leq u \frac{q^{s+1} - 1}{q^l - 1} q^{l-1} = (r + 1)(s + 1)q^{l-1}. \quad (2.2.d)$$

Also,

$$q^{s+1-l} < \frac{q^{s+1} - 1}{q^l - 1} \leq u \frac{q^{s+1} - 1}{q^l - 1} = (r + 1)(s + 1). \quad (2.2.e)$$

From (2.2.d) and (2.2.e) we get,

$$p^r q^{s+1-l} < q^{l-1}(r+1)^2(s+1)^2.$$

i.e.,

$$p^r q^{(b-2)l+1} < (r+1)^2 b^2 l^2. \quad (2.2.f)$$

Next to establish that

$$p^r q^{(b-2)l+1} > (r+1)^2 b^2 l^2, \quad (2.2.g)$$

which is a contradiction to (2.2.x). Now, if in (2.2.y) a unit increase in  $b$  increases the left hand side of (2.2.y) by a factor  $q^l$  and increases the right hand side of (2.2.y) by a factor  $\frac{(b+1)^2}{b^2} \leq \frac{16}{9}$ . So it is enough to show that

$$q^l > \frac{(b+1)^2}{b^2}.$$

But,

$$\frac{(b+1)^2}{b^2} = \left(1 + \frac{1}{b}\right)^2 \leq \frac{16}{9} < 3 < q^l.$$

Hence it is enough to prove (2.2.y) for  $b = 3$ . i.e., it is enough to show that

$$p^r q^{l+1} > (r+1)^2 9l^2.$$

Now there are several cases arise.

Case 1 :  $q \geq 5, r \geq 2$ . Now,  $p^r \geq (r+1)^2$  and  $q^{l+1} > 20l^2 > 9l^2$ .

Therefore,

$$p^r q^{l+1} > 9(r+1)^2 l^2.$$

Case 2 :  $r = 1, q \geq 5$ . Then (2.2.x) becomes,

$$pq^{l+1} > 36l^2.$$

So,  $p^r q^{l+1} \geq 3.5^{l+1} > 36l^2 = 9.l^2(r+1)^2$ .

Case 3 :  $q = 3$ , then  $p^r q^{l+1} = p^r 3^{l+1} > 9(r+1)^2 l^2$ , i.e.,  $p^r 3^{l-1} > (r+1)^2 l^2$ .  
From above cases we can easily show (2.2.y), for all values of  $r, l, p, q$ , which is a contradiction to (2.2.x). Hence,  $\max(a, b) \geq 3$  is impossible.  $\square$

**Lemma 2.2.4.** *Suppose  $n = p^r q^s$  is an odd harmonic number,  $k$  is the multiplicative order of  $p \pmod q$ ,  $l$  the multiplicative order of  $q \pmod p$ ,  $r+1 = ak$  and  $s+1 = bl$ , then  $\max(a, b) \neq 2$ .*

*Proof.* Suppose,  $\max(a, b) = 2$ . Let  $b = 2$  and  $a \leq 2$ . Since  $n$  is harmonic,  $H(n) = m$ , say. i.e.,

$$p^{ak-1} q^{2l-1} (ak)(2l) = m \frac{p^{ak} - 1}{p - 1} \frac{(q^l + 1)(q^l - 1)}{q - 1}. \quad (2.2.h)$$

Since,  $\gcd(q, q^l + 1) = \gcd(p^{ak-1}, q^l + 1) = 1$  therefore (2.2.z) gives

$$(q^l + 1) \mid 2akl. \quad (2.2.i)$$

Now by Fermat's Theorem

$$p^{q-1} \equiv 1 \pmod q.$$

This gives

$$k \mid (q - 1). \quad (2.2.j)$$

Now,

$$q^l + 1 \leq 2akl \leq 4l(q - 1).$$

Claim:  $l \leq 2$ . If not then

$$\frac{q^l + 1}{q - 1} > \frac{q^l - 1}{q - 1} \geq \frac{3^l - 1}{3 - 1} > 4l.$$

Thus,

$$q^l + 1 > 4l(q - 1),$$

which is impossible as  $q^l + 1 \leq 4l(q - 1)$ . Hence the claim. Now there are four cases arise.

Case 1 :  $l = 1, a = 2$ . Now,  $k \mid q - 1 \Rightarrow q = ck + 1$ , where  $c$  is an positive integer. Also,  $q \geq 2p + 1$  and  $q + 1 \mid 2ak$ . Therefore, if  $a = 2$  then  $\frac{2ak}{q + 1}$  becomes  $\frac{4k}{ck + 2}$  which is an integer. Hence  $c \leq 3$ . Also,  $\frac{4k}{ck + 2} \leq 3$  because if  $\frac{4k}{ck + 2} \geq 4$  then we get an absurd. Now, if  $c = 1$  and  $\frac{4k}{ck + 2} = 2, 3$  then  $p \mid 2$ , which is not possible as  $p$  is odd. Again if  $c = 2$  and  $\frac{4k}{ck + 2} = 1$  then  $k = 1$  and  $p \mid 2$ , which is also not possible. Lastly if  $c = 3$  and  $\frac{4k}{ck + 2} = 1$  then we get  $13 \mid p^{ak-1}q^{2l-1}2akl$  which is impossible. Hence  $l = 1$  and  $a = 2$  is also not possible.

Case 2 :  $l = 2, a = 2$ . In this case (2.2.i) gives,

$$q^2 + 1 \leq 8k \leq 8(q - 1) \Rightarrow (q - 4)^2 \leq 7 \Rightarrow q = 3, 5.$$

Now, if  $q = 3$  then  $p \mid 8$ , impossible. Also if  $q = 5$  then  $13 \mid 4$ , which is absurd. Hence,  $l = 2$  and  $a = 2$  is impossible.

Case 3 :  $l = 1$  and  $a = 1$ . In this case (2.2.i) becomes

$$\frac{2k}{q + 1} = \frac{2k}{ck + 2},$$

which implies that  $c = 1, k = 2$  and  $q = 3$  which is impossible. Hence this case is also not possible.

Case 4 :  $l = 2, a = 1$ . In this case (2.2.i) becomes,

$$q^2 + 1 \mid 4k.$$

So,

$$q^2 + 1 \leq 4k \leq 4(q - 1),$$

which is not possible since

$$q^2 + 1 > 4(q - 1), \forall q.$$

Hence this case is impossible. Therefore our assumption  $\max(a, b) = 2$  is wrong. Hence the lemma. □

**Lemma 2.2.5.** *Suppose  $n = p^r q^s$  is an odd harmonic number with  $p < q$ ,  $l$  is the multiplicative order of  $q \pmod p$ ,  $k$  is the multiplicative order of  $p \pmod q$ ,  $r + 1 = ak$  and  $s + 1 = bl$ , then  $\max(a, b) \neq 1$ .*

*Proof.* Suppose  $a = b = 1$ . Then  $r + 1 = k$ ,  $s + 1 = l$ . Therefore,  $k > 1$  and  $l > 1$ . Now,

$$p^r q^s (r + 1)(s + 1) = m \frac{p^{r+1} - 1}{p - 1} \frac{q^{s+1} - 1}{q - 1},$$

where  $m = H(n)$ . So,

$$p^{k-1} q^{l-1} kl = m \frac{p^k - 1}{p - 1} \frac{q^l - 1}{q - 1}. \quad (2.2.k)$$

Claim:  $l$  and  $k$  are primes. Suppose  $l$  is a composite number  $> 4$ . Also let  $l_1$  be the largest proper divisor of  $l$ . Then  $2 < l_1 < l$ . Clearly,  $p \nmid q^{l_1} - 1$  since  $l_1 < l$ . From (2.2.k) we get

$$\frac{q^{l_1} - 1}{q - 1} \mid kl. \quad (2.2.1)$$

Thus,

$$q^{l_1-1} < \frac{q^{l_1} - 1}{q - 1} \leq kl < qp < q^2.$$

So,  $l_1 < 3$ , impossible as  $2 < l_1$ . Thus  $l$  can not be a composite number  $> 4$ .

Now let  $l = 4$  then  $l_1 = 2$  and  $\frac{q^{l_1} - 1}{q - 1} = q + 1$ . Also, (2.2.k) gives,

$$(q + 1) \leq 4k \Rightarrow dk + 2 \leq 4k \Rightarrow d \leq 3,$$

where  $q = dk + 1$  for some positive integer  $d$ . So,

$$q = k + 1, 2k + 1, 3k + 1.$$

Now if  $k$  is odd and  $q = 2k + 1$  then  $k + 1 \mid 2$  and so  $k = 1$  and  $q = 3$  which is not possible as  $p < q$  is an odd prime. Hence  $k$  must be even. Now, since  $k$  is even,

$$k + 2 \mid 4k \tag{2.2.m}$$

or,

$$2k + 2 \mid 4k \tag{2.2.n}$$

or,

$$3k + 2 \mid 4k. \tag{2.2.o}$$

Now if  $\frac{4k}{k+2} \in N$  then  $k \leq 6$  i.e.,  $k = 2, 4, 6$ . So,  $q = 3, 5, 7$  which are not possible as  $p < q$ ,  $5 \nmid 2$  and  $7 \nmid 4$ . So, (2.2.m) is not possible if  $k$  is even. Next (2.2.n) gives  $\frac{2k}{k+1} \in N$  which implies  $k = 1$ , absurd. So, (2.2.n) can not be hold. Lastly, (2.2.o) gives  $k = 2$ , which is also not possible as  $5 \nmid 2$ . Hence  $l$  must be a prime number. Next let  $k = l^u v$ , where  $u \geq 0$ ,  $l \nmid v$  and  $k$  is not prime. Also let  $d$  be the largest proper divisor of  $k$ . Then  $q \nmid (p^d - 1)$  and so  $\frac{p^d - 1}{p - 1} \mid kl = l^{u+1}v$ . Now first let  $l$  be an odd prime. Since  $l^u \parallel k$  therefore  $l^u \parallel \frac{p^d - 1}{p - 1}$ . Also since  $\frac{p^d - 1}{p - 1} \mid l^{u+1}v$ ,  $\frac{p^d - 1}{p - 1} \mid l^u v$ . Hence,

$$p^{d-1} < \frac{p^d - 1}{p - 1} \leq l^u v = k. \tag{2.2.p}$$

Now since  $l$  is odd so  $p \neq 3$ . Therefore  $p \geq 5$ . Now,

$$p^{d-1} \geq 5^{d-1} \geq d^2 + 1 \geq k + 1 > k,$$

which is a contradiction to (2.2.p). Thus if  $l$  is an odd prime then  $k$  is also a prime. On the other hand if  $l = 2$  then

$$\frac{p^d - 1}{p - 1} \leq 2k. \quad (2.2.q)$$

Now  $p \geq 3$  gives  $\frac{p^d - 1}{p - 1} \geq k$  which contradicts (2.2.q). Now several cases arise.

Case I: Suppose  $k = 4$  then  $d = 2$  and  $s = 1, r = 3$ . Now,

$$\frac{p^2 - 1}{p - 1} = p + 1 \mid 8 \Rightarrow p = 3, 7.$$

Now if  $p = 3$  then  $q = 5$  and then  $n = 3^3 \cdot 5$  which is not harmonic. Again if  $p = 7$  then  $q = 5$ , which is also not possible as  $p < q$ .

Case II:  $k > 4$  then  $d \geq 3$ . In this case  $p \neq 3$  because if  $p = 3$  then  $3^d - 1 \leq 4d$ .

So  $p \geq 5$  and then  $\frac{p^d - 1}{p - 1} \geq \frac{5^d - 1}{4} > 2d^2 > 2k$ , which contradicts (2.2.q).

Hence  $k$  can not be 4. Thus,  $k$  is also a prime number. Now we can easily show that

$$\max(k, l) < \frac{8kl}{15} < m < kl < \min(kp, ql) < \max(kp, ql) < pq < q^2. \quad (2.2.r)$$

From (2.2.1), we get

$$m \mid p^{k-1}q^{l-1}kl.$$

So, the only possible values of  $m$  are

(I)  $q$ , (II)  $p^c$  with  $1 < p < kl$  and (III)  $p^c l$  with  $1 < p^c < k$ .

Now if  $m = q$  then (2.2.1) gives

$$\frac{p^k - 1}{p - 1} \mid P^{k-1} q^{l-2} kl.$$

But

$$\frac{p^k - 1}{p - 1} \nmid p^{k-1}.$$

Also from (2.2.1) we get

$$q^{l-2} \mid \frac{p^k - 1}{p - 1}.$$

Therefore, the possible values of  $\frac{p^k - 1}{p - 1}$  are

$$q^{l-2}, kq^{l-2}, lq^{l-2} \text{ and } klq^{l-2}.$$

If  $\frac{p^k - 1}{p - 1} = q^{l-2}$  then  $q^{l-2} \equiv 1 \pmod{p}$ , which is a contradiction to the fact that  $l$  is the multiplicative order of  $q \pmod{p}$ . Again if  $\frac{p^k - 1}{p - 1} = kq^{l-2}$  then  $l = p$ , which is impossible as  $l < p < q$ . Similarly if  $\frac{p^k - 1}{p - 1} = lq^{l-2}$  then also  $k = p = l$ , a contradiction. So,  $m \neq q$ . Now if  $m = p^c$  where  $1 < p^c < kl$  then (2.2.1) gives  $\frac{p^k - 1}{p - 1} \mid q^{l-1} kl$  and so  $\frac{p^k - 1}{p - 1} = q^{l-1}, q^{l-1}k, q^{l-1}l, q^{l-1}kl$ . Now if  $\frac{p^k - 1}{p - 1} = q^{l-1}kl$  then  $\frac{q^l - 1}{q - 1} = p^{k-c-1}$  and so  $p^{k-c-1} \equiv 1 \pmod{q}$ , impossible. Again if  $\frac{p^k - 1}{p - 1} = q^{l-1}l$  then  $l = q$  and so  $q \mid p - 1$ , impossible. Next if  $\frac{p^k - 1}{p - 1} = q^{l-1}k$  then  $k \mid \frac{p^k - 1}{p - 1}$  and then by Lemma 1.4.1 we get  $p - 1$  is divisible by  $k$ . Also we have  $l \mid p - 1$ . But,

$$p^{l-1} < q^{l-1} = \frac{p^k - 1}{(p - 1)k} < p^{k-1}.$$

Thus  $l < k$ . Hence we get  $l$  and  $k$  are two distinct prime factors of  $p - 1$  and so  $p > kl$ , which contradicts the fact that  $p^c < kl$ . Next if  $\frac{p^k - 1}{p - 1} = q^{l-1}$  then

$q^{l-1} \equiv 1 \pmod{p}$  which is also not possible. Hence  $m = p^c$  where  $1 < p^c < kl$  is not possible. Finally, if  $m = p^c l$  where  $1 < p^c < k$  then (2.2.1) becomes

$$p^{k-c-1} q^{l-1} k = \frac{p^k - 1}{p - 1} \frac{q^l - 1}{q - 1},$$

where  $c + 1 \leq 3^c \leq p^c < k$ . Thus  $k - c - 1 > 0$ . Now

$$q^l - 1 \mid \frac{p^k - 1}{p - 1}.$$

Also

$$\frac{p^k - 1}{p - 1} \mid p^{k-c-1} q^{l-1} k$$

and so

$$\frac{p^k - 1}{p - 1} \mid q^{l-1} k.$$

Thus

$$\frac{p^k - 1}{p - 1} = q^{l-1} \text{ or } q^{l-1} k.$$

Now if  $\frac{p^k - 1}{p - 1} = q^{l-1}$  then  $q^{l-1} \equiv 1 \pmod{p}$  a contradiction. Again, if  $\frac{p^k - 1}{p - 1} = q^{l-1} k$  then  $k \mid \frac{p^k - 1}{p - 1} = q^{l-1}$  and so  $p \equiv 1 \pmod{k}$ . Also  $l \mid p - 1$  and so  $kl \leq p - 1 < p$  which is impossible as  $p^c < kl$ . Hence all the possibilities of  $m$  are denied. Therefore  $\max(a, b) = 1$  is impossible.  $\square$

In view of Lemmas 2.1.1, 2.2.2, 2.2.3, 2.2.4 and 2.2.5, we have the following theorem.

**Theorem 2.2.6.** *If  $n$  is an odd harmonic number, then  $w(n) > 2$ .*

From above results we can conclude that if  $n$  is a harmonic number with  $w(n) = 2$ , then  $n$  must be even.

**Lemma 2.2.7.** *If  $n$  is a harmonic number with  $w(n) = 2$  then  $n$  is an even perfect number.*

*Proof.* In view of the above result, it is enough to show that  $n$  is perfect. Let  $n = 2^r q^s$ , where  $q$  is odd prime and  $H(n) = m$ , then

$$2^r q^s (r+1)(s+1) = m(2^{r+1} - 1) \frac{q^{s+1} - 1}{q - 1}. \quad (2.2.s)$$

Claim :  $s = 1$ . Suppose  $s > 1$  is an odd integer. So,  $s + 1$  is even. Let  $s + 1 = 2^u v$  where  $u > 0$  and  $v$  is an odd integer. Clearly  $2^u \parallel (s + 1)$ ,  $2^{u-1} \parallel \frac{q^{s+1} - 1}{q^2 - 1}$  by Lemma 1.2.4, also  $2^{u-1} \parallel \frac{s+1}{2}$ . So (2.2.s) gives

$$\begin{aligned} \frac{q^{s+1} - 1}{q^2 - 1} & \mid \frac{(r+1)(s+1)}{2} \\ \Rightarrow \frac{(r+1)(s+1)}{2} & = t \frac{q^{s+1} - 1}{q^2 - 1}. \end{aligned} \quad (2.2.t)$$

Now,

$$q^{s-1} \leq tq^{s-1} < t(1 + t + t^2 + \dots + q^{s-1}) = \frac{(r+1)(s+1)}{2} \dots \quad (2.2.u)$$

From (2.2.s), (2.2.t) and (2.2.u) we get

$$2^{r+1} \leq 1 + q^s t < q + q^s t \leq q \frac{(r+1)(s+1)}{2}.$$

Thus

$$2^{r+3} q^{s-2} < (r+1)^2 (s+1)^2. \quad (2.2.v)$$

But

$$2^{r+3} \geq \frac{32}{9} (r+1)^2 \forall r \geq 1.$$

Also

$$q^{s-2} \geq \frac{3}{4}(s+1)^2 \quad \forall s \geq 5.$$

So,

$$2^{r+3}q^{s-2} \geq 4(r+1)^2(s+1)^2,$$

which contradicts (2.2.v). Hence  $s < 5$  i.e.,  $s = 3$ . Now if  $r \geq 5$  then

$$2^{r+3}q^{s-1} \geq \frac{64}{3}(r+1)^2.$$

So,

$$(r+1)^2(s+1)^2 = 16(r+1)^2 < \frac{64}{3}(r+1)^2 \leq 2^{r+3}q,$$

which is a contradiction to (2.2.v). So,  $r \leq 4$ . Now (2.2.s) gives

$$2^{r+2}q^3(r+1) = m(2^{r+1} - 1)\frac{q^4 - 1}{q - 1}. \quad (2.2.w)$$

Now if  $r = 4$  then  $q = 31$  and then we get an absurd result. Again if  $r = 3$  then  $q = 5$  and then also we get an absurd result. Similarly  $r = 2$  and  $r = 1$  are also not possible. Hence  $s$  can not be an odd integer greater than 1. Next let  $s$  is even and then  $\gcd(q^s, 1 + q + \dots + q^s) = 1$  gives

$$\frac{q^{s+1} - 1}{q - 1} \mid (r+1)(s+1)$$

and so

$$(r+1)(s+1) = t\frac{q^{s+1} - 1}{q - 1}.$$

Now (2.2.s) gives

$$(2^{r+1} - 1) \mid tq^s.$$

Now,

$$q^s \leq tq^s < t(1 + q + \dots + q^s) \leq (r+1)(s+1).$$

Thus,

$$2^{r+1}q^s < (r+1)^2(s+1)^2, \quad (2.2.x)$$

which is a contradiction to the fact that  $2^{r+1}q^s \geq (r+1)^2(s+1)^2$ , for  $r \geq 3$ . So if  $r \geq 3$  then  $s$  can not be even. Therefore  $r \leq 2$  and  $s$  is even. Now  $2^{r+1} \geq \frac{8}{9}(r+1)^2$  for  $r \geq 1$  and  $q^s \geq 3^s \geq \frac{27}{16}(s+1)^2$  for  $s \geq 2$ . So for even  $s$ ,  $s \neq 2$ ,

$$2^{r+1}q^s \geq \frac{3}{2}(r+1)^2(s+1)^2,$$

which contradicts (2.2.x). Hence  $s = 2$  and  $r = 1, 2$ . Now if  $s = 2$  and  $r = 1$  then (2.2.s) gives  $1 + q + q^2 = 1$ , which has no solution. Again if  $r = 2$  then  $q = 7$  and we get  $1 + q + q^2 = 57$ , which is not possible. Hence  $s$  must be 1. Now putting  $s = 1$  in (2.2.s) we get

$$\begin{aligned} 2^{r+1}q(r+1) &= m(2^{r+1} - 1)(q+1) & (2.2.y) \\ \Rightarrow q &| m(2^{r+1} - 1) \\ \Rightarrow q &| m \text{ or } q | (2^{r+1} - 1) \end{aligned}$$

Now if  $q | m$  then  $m = qx$  and then (2.2.y) becomes,

$$2^{r+1}(r+1) = x(2^{r+1} - 1)(q+1) \rightarrow (2^{r+1} - 1) | (r+1),$$

which is not possible since  $2^{r+1} > (r+2)$ ,  $\forall r \geq 2$ . Hence,  $q | (2^{r+1} - 1)$  and so  $(2^{r+1} - 1) = qu$  where  $u$  is an odd integer. Putting this in (2.2.y), we get  $u | (r+1)$  and

$$2^{r+1}(r+1) = m(2^{r+1} + u - 1). \quad (2.2.z)$$

Now take  $v = \lceil \frac{\log r}{\log 2} \rceil$  so that  $r = 2^{v+\theta}$ , for some  $\theta \in [0, 1)$ . Now since  $u \leq r+1$ ,  $u - 1 \leq 2^{v+\theta}$ . If  $u \neq 0$  then taking  $\log$  we get  $\log_2(u - 1) \leq v + \theta$ . Now let

$u-1 = 2^k$ . Clearly  $k \leq r+1$ , and so highest power of 2 dividing  $2^{r+1}+u-1$  is atmost  $2^v$ . Hence  $2^{r+1-v} \mid m$  i.e.,  $m = 2^{r+1-v}m_0$ ,  $m_0 \in N$ . Now,

$$\begin{aligned} 2^{r+1}(r+1) &= 2^{r+1-v}m_0(2^{r+1}+u-1) \\ \Rightarrow 2^v 2^\theta(r+1) &= 2^\theta m_0(2^{r+1}+u-1) \\ \Rightarrow r(r+1) &= 2^\theta m_0(2^{r+1}+u-1) \\ \Rightarrow r(r+1) &\geq 2^{r+1} \end{aligned}$$

which is absurd. Thus our assumption  $u \neq 1$  is untenable and so  $u = 1$ . Therefore  $q = 2^{r+1} - 1$ . Also  $(r+1)$  is an odd prime since  $q$  is an odd prime. Hence  $n = 2^r(2^{r+1} - 1)$ , which is an even perfect number by Euclid- Euler Theorem. Hence the theorem.  $\square$

**PROBLEM:** Does a nontrivial odd harmonic number exist?

**PROBLEM:** Given any two primes  $p$  and  $q$ , how many harmonic numbers are there with  $p$  and  $q$  as the only prime factors?

## 2.3 Harmonic numbers with a given harmonic mean

**Lemma 2.3.1.** *Let  $p$  be a prime. If  $H(n) = p$  then either  $p \mid n$  or  $n$  is a perfect number.*

*Proof.* Since  $p = H(n) = \frac{n\sigma(n)}{\tau(n)}$  gives  $p \mid n\tau(n) \Rightarrow p \mid n$  or  $p \mid \tau(n)$ . Now, if  $p \mid n$  then done. If not then  $p \mid \tau(n)$ . If  $n = \prod_{i=1}^r p^{e_i}$  then  $\tau(n) = \prod_{i=1}^r (e_i + 1)$

and so  $p \mid (e_j + 1)$  for some  $j$ ,  $e_j = pk - 1$  for some positive integer  $k$ . Without any loss take  $e_1 = pk - 1$  then

$$\tau(n) = kp \prod_{i=2}^r (e_i + 1) \geq kp2^{w(n)-1} \geq p2^{w(n)-1}. \quad (2.3.a)$$

Now,

$$\begin{aligned} \tau(n) = H(n)S(n) &= H(n) \prod_{i=1}^r S(p^{e_i}) < p \prod_{i=1}^{w(n)} \frac{p_i}{p_i - 1} \\ &< p \prod_{i=1}^{w(n)} \frac{i+1}{i} = p(w(n) + 1). \end{aligned} \quad (2.3.b)$$

From (2.3.a) and (2.3.b) we get  $2^{w(n)-1} < w(n) + 1$ , which is not true if  $w(n) \geq 3$ . Hence  $w(n) = 2$ . But the only harmonic numbers of the form  $p^a q^b$  are perfect numbers by Lemma 2.2.7. Hence if  $p \nmid n$  then  $n$  must be a perfect number.  $\square$

**PROBLEM:** Is it true that given any prime  $p \exists$  a harmonic number  $n$  such that  $p \mid n$ ?

**Theorem 2.3.2.** *Let  $p$  be a prime. If  $H(n) = 2p$  then  $2p \mid n$ . Also, if  $H(n) = 3p$  then  $p \mid n$ .*

*Proof.*  $H(n)$  is even so by Lemma (2.1.6),  $n$  is also even. Therefore,  $2 \mid n$ . It is sufficient to show that  $p \mid n$ . Suppose,  $p \nmid n$ , then  $p \mid \tau(n)$ . Now, if  $n = \prod_{i=1}^r p_i^{e_i}$  then  $\tau(n) = \prod_{i=1}^r (e_i + 1)$  and so  $p \mid (e_i + 1)$ , for some  $i$ ,  $1 \leq i \leq r$  i, e.,  $e_i = kp - 1$ , for some  $k$ . Without any lose, we can take  $e_1 = kp - 1$ ,  $p_1 = q$  and  $m = \prod_{i=2}^r p_i^{e_i}$ . Then  $n = q^{kp-1}m$  with  $q \nmid m$ . If  $w(n) \leq 2$ ,  $n$  is an

even perfect number and then  $H(n)$  is a prime number which is not the case.

Therefore,  $w(n) \geq 3$  i.e.,  $w(m) \geq 2$ . Now,

$$H(n) = H(q^{kp-1}m) \geq H(q^{kp-1})H(6) \geq H(2^{kp-1})H(6) > kp.$$

Thus,  $2p > kp$  implies  $k = 1$ . Also,  $H(q^{p-1}) > \frac{q-1}{q}p$  gives

$$\frac{q-1}{q}p < H(q^{p-1}) < p.$$

Now,

$$\begin{aligned} H(m) &= \frac{2p}{H(q^{p-1})} \\ \text{i.e., } 2 < H(m) &< \frac{2q}{q-1} \\ \text{i.e., } \frac{20}{9} = H(10) &\leq H(m) < \frac{2q}{q-1} \end{aligned}$$

Since  $m \geq 10$ ,  $q < 10$ . Therefore,  $q = 2, 3, 5, 7$ .

Case1:  $q = 7$ . In this case,  $n = 7^{p-1}m$  with  $7 \nmid m$ . So,  $\frac{6}{7} < H(7^{p-1}) < p$ .

Also,  $2 < H(m) < \frac{7}{3}$ . Suppose  $m > 1$ . If  $w(m) = 2$  then  $H(m) > 2.5$ . Also, if  $w(m) \geq 3$  then  $H(m) > \frac{10}{3}$ . Therefore,  $m = 10$  and  $n = 7^{p-1} \cdot 10$ . But  $7^{p-1} = 3$  has no solution. Hence,  $q \neq 7$ .

Case2:  $q = 5$ . In this case,  $2 < H(m) < \frac{5}{2}$ . If  $p = 2, 3, 5$ , the statement holds by Theorem 5.2.5. So, we assume  $p \geq 7$ . Therefore,

$$H(5^{p-1}) \leq \frac{4.5^6 p}{5^7 - 1}; \text{ whence, } H(m) > 2.49.$$

Claim 1: There does not exist an integer  $m$  satisfying  $2.49 < H(m) < 2.5$ .

If  $w(m) \geq 3$  then  $H(m) \geq H(2)H(3)H(5) = \frac{10}{3} > 2.5$ . So, we have,  $w(m) \leq$

2. Let  $w(m) = 2$  then  $m = p^e q^f$ , where  $e, f \geq 1$ . Now, if  $\max(e, f) \geq 2$  then  $H(p^e q^f) \geq H(2^2)H(3) = \frac{18}{7} > 2.5$ . So,  $\max(e, f) = 1$ . If  $m$  is odd then  $H(m) \geq H(3)H(5) = 2.5$ . Again if  $m$  even then  $H(26) = 2.47$  and  $H(30) = 2.51$ . Similarly,  $w(m) = 1$  is also not possible. Hence the claim.

Case 3: Let  $q = 3$  then  $n = 3^{p-1}m$  with  $3 \nmid m$ . Then,

$$\frac{2}{3}p < H(3^{p-1}) < p$$

$$\text{i.e. } 2 < H(m) < 3$$

Now,  $H(3^{p-1}) = \frac{23^{p-1}p}{3^p - 1} \leq \frac{2p3^6}{3^7 - 1}$ . Hence,

$$H(m) = \frac{2p}{H(3^{p-1})} \geq 2.9984 > 2.998.$$

Therefore,

$$2.998 < H(m) < 3.$$

Claim 2: There does not exist an integer  $m$  satisfying

$$2.998 < H(m) < 3.$$

If  $w(m) \geq 3$  then  $H(m) \geq H(2)H(3)H(5) = \frac{10}{3} = 3.13 > 3$ . So,  $w(m) \leq 2$ .

Now, let  $w(m) = 2$  then  $m = 2^{e_1} p_2^{e_2}$ . Therefore,

$$H(m) = H(2^{e_1} H(p_2^{e_2})) \geq H(2^{e_1})H(5^{e_2}) > 4,$$

which is not possible as  $H(m) < 3$ . So,  $w(m) = 1$ . If  $m = 2^{e_1}$  then  $H(m) = \frac{2^{e_1}(e_1 + 1)}{2^{e_1+1} - 1}$ .  $e_1 \geq 5$  gives  $e_1 > 5 - \frac{3}{2^{e_1}}$  i.e.,  $e_1 + 1 > 6 - \frac{3}{2^{e_1}}$  and so  $H(m) > 3$ , which is impossible. Also if  $e_1 = 1, 2, 3, 4$  then  $H(m) < 2.998$ .

Hence there exist no integer  $m$  satisfying  $2.998 < H(m) < 3$ .

Case 4: Let  $n = 2^{p-1}m$  with  $2 \nmid m$ . Since

$$\frac{1}{2^p} < H(2^{p-1}) = \frac{2^{p-1}p}{2^p - 1} \leq \frac{2^6 p}{2^7 - 1},$$

we have

$$3.96875 < H(m) < 4.$$

Such integers  $m$  satisfying  $w(m) \geq 2$  and  $2 \nmid m$  are only  $3^2 \cdot 23$  and products of two distinct odd primes. First, put  $m = 3^2 \cdot 23$  then we get  $2^{p-1} = 104$  which has no solution. So,  $m \neq 3^2 \cdot 23$ . Next put  $m = p_1 p_2$  then

$$2^{p-2} p_1 p_2 = (2^p - 1) \frac{p_1 + 1}{2} \frac{p_2 + 1}{2}.$$

Hence the odd integer  $2^p - 1$  is equal to either  $p_1$ ,  $p_2$  or  $p_1 p_2$ . If  $2^p - 1 = p_1$  or  $p_2$  then  $2^{p-1} = 0$ , impossible. Also, if  $2^p - 1 = p_1 p_2$  then

$$p_1 p_2 + 1 = (p_1 + 1)(p_2 + 1), \text{ which has no solution.}$$

So, all possibilities of  $p \nmid n$  are denied. Hence if  $H(n) = 2p$  then  $p \mid n$ . By the similar argument we can show that if  $H(n) = 3p$  then  $p \mid n$ .  $\square$

**PROBLEM:** To determine whether or not any integer  $m$  such that  $m$  is relatively prime to  $p$  ( $p$  is a prime) and  $H(n) = mp$  implies  $p \mid n$ .

**PROBLEM:** To determine whether  $p \mid H(n)$  implies  $p \mid n$ .

**PROBLEM:** Assume that  $H(n)$  is a triple of a prime. Is  $n$  arithmetic?

Note that a positive integer  $n$  is said to be *arithmetic* if the arithmetic mean of its positive  $A(n) = \frac{\sigma(n)}{\tau(n)}$  is an integer.

Kanold [16] proved the following interesting result.

**Theorem 2.3.3.** *For any positive integer  $c$ , there exist only finitely many numbers satisfying  $H(n) = c$ .*

*Proof.* Clearly,  $c > 0$ . Let  $S_c$  be the set of all natural number  $n$  such that  $H(n) = c$ . Also let  $S_c$  is an infinite set. We have  $w(n) < H(n) \forall n$  with  $w(n) > 2$ . So with out any loss assume  $\forall n \in S_c, w(n) \leq k$  where  $k > 2$  is a fixed positive integer. Now let  $n_i = \prod_{j=1}^k p_{ij}^{\alpha_{ij}}$  with  $n_i \in S_c$  then

$$c = \prod_{j=1}^k H(p_{ij}^{\alpha_{ij}}) \geq H(p_{ij}^{\alpha_{ij}}).$$

Now  $H(p_{ij}^{\alpha_{ij}}) > \frac{\alpha_{ij} + 1}{2}, \forall j = 1, 2 \dots k$  as  $\{\frac{x}{x+1}\}$  is a monotonically increasing sequence. Thus,  $\alpha_{ij} < 2c - 1, \forall j = 1, 2 \dots k$ . Since  $2c - 1$  is a fixed quantity, the components  $\alpha_{ij}$  of  $p_{ij}$  can attain only finitely many values. Therefore we can construct a subsequence  $n_{i_t}$  of  $n_i$  for which  $\alpha_{i_t j} = \alpha_j, \forall j = 1, 2, \dots k$ . In other words,  $n_{i_t} = p_{i_t 1}^{\alpha_1} \cdot p_{i_t 2}^{\alpha_2} \cdot \dots \cdot p_{i_t k}^{\alpha_k}$ . With out any loss we can replace  $i_t$  by  $i$  then we have  $n_i = p_{i 1}^{\alpha_1} \cdot p_{i 2}^{\alpha_2} \cdot \dots \cdot p_{i k}^{\alpha_k}$ . Now, the sequence  $\{n_i = p_{i 1}^{\alpha_1} p_{i 2}^{\alpha_2} \dots p_{i k}^{\alpha_k}\}$  is equivalent to sequence  $\{(p_{i 1}^{\alpha_1}, p_{i 2}^{\alpha_2}, \dots, p_{i k}^{\alpha_k})\}$ . Now if range  $\{p_{ij}^{\alpha_j}\}$  is finite then  $\exists$  subsequence which is infinitely constant. Also if range of  $\{p_{ij}^{\alpha_j}\}$  is infinite then  $\exists$  subsequence which is strictly increasing. So we can construct a strictly increasing subsequence of sequence  $\{n_i\}$ . With out any loss we can take the sequence  $\{n_i\}$  is strictly increasing. Now let  $n_1, n_2 \in \{n_i\}$  with  $n_1 \neq n_2$ . Also let  $n_1 = \prod_{j=1}^k p_{1j}^{\alpha_j}$  and  $n_2 = \prod_{j=1}^k p_{2j}^{\alpha_j}$ . Now suppose  $n_2 > n_1$  then  $p_{2j}^{\alpha_j} \geq p_{1j}^{\alpha_j}$ , for all  $j$  with strict inequality for atleast one  $j$ . Now put  $b_j = H(p_{1j}^{\alpha_j}) > 0 \forall j$  and  $a_j = H(p_{2j}^{\alpha_j}) > 0 \forall j$ . Thus

$$a_j \geq b_j, \forall j \text{ and strict for atleast one } j. \quad (2.3.c)$$

Therefore  $c = a_1 a_2 \cdots a_k = b_1 b_2 \cdots b_k$ . Now,

$$\begin{aligned} 0 &= a_1 a_2 \cdots a_k - b_1 b_2 \cdots b_k \\ &= (a_1 - b_1) a_2 \cdots a_k + (a_2 - b_2) b_1 a_3 \cdots a_k + \cdots + (a_k - b_k) b_1 b_2 \cdots b_{k-1} \end{aligned}$$

Therefore,  $a_i = b_i \forall i = 1, 2, \dots, k$ , which is a contradiction to (2.3.c). Hence  $S_c$  can not be infinite.  $\square$

**Lemma 2.3.4.** *Let  $w(n)$  denote the number of distinct prime factors of  $n$ . For all  $n$ ,  $H(n) > \frac{2^{w(n)+1}}{w(n)+1}$  with the following exceptions  $n = p, 2p, 6p$  ( $p \neq 3$ ),  $n = 30p$  ( $7 \leq p \leq 23$ ) and  $n = 1, 15, 21, 70$ .*

*Proof.* Suppose  $n > 1$  and  $n = \prod_{i=1}^t p_i^{a_i}$  where  $p_i$ 's are distinct primes. Now,

$$\frac{\sigma(n)}{n} = S(n) = \prod_{i=1}^t S(p_i^{a_i}) \leq \prod_{i=1}^t \frac{p_i}{p_i - 1} \leq \prod_{i=1}^t \frac{i}{i + 1} = t + 1 = w(n) + 1.$$

Now,  $p_i \geq i + 1$  gives  $\frac{p_i}{p_i - 1} \leq \frac{i + 1}{i}$  therefore,  $H(n) \geq \frac{\tau(n)}{1 + w(n)} \forall n$ . Thus, the theorem is proved for those  $n$  for which  $\tau(n) \geq 2^{w(n)+1}$ . In fact, this is always the case except if  $n$  is squarefree or of the form  $p^2 m$  where  $m$  is squarefree and  $p \nmid m$ . Now, suppose  $n = p_1 p_2 \cdots p_r$ , then  $\tau(n) = 2^{w(n)} < 2^{w(n)+1}$ . Again, if  $n = p^2 q_1 q_2 \cdots q_r$ , then  $\tau(n) = 3 \cdot 2^{w(n)-1} < 2^{w(n)+1}$ . Also if  $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$  where either some  $a_i \geq 3$  or  $r \geq 2$  and there exist  $i$  and  $j$  such that  $a_i = a_j = 2$ . If  $a_i \geq 3$  for some  $i$  then with out any loss take  $a_1 \geq 3$ . So,  $\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1) \geq 2^{w(n)+1}$ . Again if  $a_i = a_j = 2$ , for some  $i \neq j$  we can take  $a_1 = a_2 = 2$  then  $\tau(n) \geq 9 \cdot 2^{w(n)-2} \geq 2^{w(n)+1}$ . Hence the theorem.

By inspection, we can show easily that the theorem is not true for the given values of  $n$ .  $\square$

# Chapter 3

## k-Harmonic Numbers

In this chapter we study a generalization of harmonic numbers called  $k$ -harmonic numbers. This study is all about finding a necessary condition for the existence of a  $k$ -harmonic number as no  $k$ -harmonic number could be determined so far for  $k \geq 2$ .

### 3.1 Definition and basic properties

For any positive integer  $n > 1$ , the sum of the  $k$ -th powers of the positive divisors of  $n$  is denoted by  $\sigma_k(n)$  i.e.,  $\sigma_k(n) = \sum_{d|n} d^k$ . Clearly  $\sigma_0(n) = \tau(n)$  and  $\sigma_1(n) = \sigma(n)$ .

Cohen and Moujje [6] have defined a number  $n > 1$  to be a  $k$ -harmonic number or power harmonic number if the harmonic mean of the  $k$ -th powers of the positive divisors of  $n$  is an integer and it is denoted as

$$H_k(n) = \frac{n^k \tau(n)}{\sigma_k(n)}.$$

Till now, no  $k$ -harmonic number with  $k \geq 2$  could be found. Only we can able to know a number of necessary conditions that must be satisfied by such numbers.

**Definition 3.1.1.** A power- harmonic number is said to be proper if it is  $k$ -harmonic for some integer  $k \geq 2$ .

We can easily verify that  $\sigma_k(n)$  for  $k \geq 0$  is multiplicative. Since  $\sigma_k(n)$  is multiplicative so is  $H_k(n)$ .

Now we discuss some basic properties of  $k$ -harmonic number.

**Lemma 3.1.2.** *A power harmonic number has atleast two distinct prime factors.*

*Proof.* Let  $n$  be a number such that  $w(n) = 1$ . Also let  $n = p^a$ . Then

$$H_k(n) = \frac{n^k \tau(n)}{\sigma_k(n)} = \frac{p^{ak}(a+1)}{1 + p^k + p^{2k} + p^{ak}}.$$

Now,  $p^{ak} \nmid 1 + p^k + p^{2k} + p^{ak}$ . So,  $1 + p^k + p^{2k} + p^{ak} \mid (a+1)$ , which is not possible. Hence, a power harmonic number has atleast two distinct prime factors.  $\square$

**Lemma 3.1.3.** *If  $n$  is  $k$ -harmonic, then  $H_k(n) \leq \tau(n) - 1$ .*

*Proof.* Suppose  $n$  is a  $k$ -harmonic number, then  $w(n) \geq 2$ . Now,  $H_k(n) = \frac{n^k \tau(n)}{\sigma_k(n)} < \tau(n)$  as  $\frac{n^k}{\sigma_k(n)} < 1$ . But  $H_k(n)$  is an integer. Hence,  $H_k(n) \leq \tau(n) - 1$ .  $\square$

**Lemma 3.1.4.** *For  $n > 1$ ,  $H_1(n) \leq \tau(n) - 8$  for all even harmonic number except for  $n = 6, 28, 140, 496, 8128$ .*

*Proof.* Since all harmonic numbers  $n$  with  $H_1(n) \leq 13$  satisfies the above inequality except  $n = 6, 28, 140, 496, 8128$ . So we can take  $H_1(n) \geq 14$ . Now  $n > 2$ , we have,

$$\frac{3}{2} = 1 + \frac{1}{2} < \sum_{d|n} \frac{1}{d} < \frac{\tau(n)}{H_1(n)} \leq \frac{\tau(n)}{14}.$$

Thus,

$$\tau(n) \geq 22.$$

Now,

$$H_1(n) = \frac{n\tau(n)}{\sigma(n)} < \frac{2}{3}\tau(n) \leq \tau(n) - \frac{22}{3}.$$

Hence,

$$H_1(n) \leq \tau(n) - 8.$$

□

**Remark 3.1.5.**  $H_1(n) \leq \tau(n) - 10$  for all even harmonic numbers with the above exceptions.

**Lemma 3.1.6.** For a fixed integer  $n$ ,  $H_k(n)$  is an increasing function of  $k$  and  $\lim_{k \rightarrow \infty} H_k(n) = \tau(n)$ .

*Proof.* Let  $d$  be a divisor of  $n$ . Now,

$$H_k(n) = \frac{n^k \tau(n)}{\sum_{d|n} \frac{1}{d^k}} > \frac{n^k \tau(n)}{\sum_{d|n} \frac{1}{d^{k-1}}} > \frac{n^{k-1} \tau(n)}{\sum_{d|n} \frac{1}{d^{k-1}}} = H_{k-1}(n).$$

Hence,  $H_k(n)$  is an increasing function of  $k$ . Now,

$$\frac{\sigma_k(n)}{n^k} = 1 + \sum_{d|n, d>1} \frac{1}{d^k} \rightarrow 1, \text{ as } k \rightarrow \infty.$$

So,

$$H_k(n) = \frac{\tau(n)}{\frac{\sigma_k(n)}{n^k}} \rightarrow \tau(n) \text{ as } k \rightarrow \infty.$$

Hence,

$$\lim_{k \rightarrow \infty} H_k(n) = \tau(n).$$

□

**Lemma 3.1.7.** *Let  $p$  be an odd prime and  $2^a p^b$  be a proper  $k$ -harmonic number ( i.e.,  $k \geq 2$  ). If  $k$  is even then  $b \equiv 7 \pmod{8}$ . Also if  $k$  odd then  $b$  is odd and  $(p+1)(b+1) \equiv 0 \pmod{16}$ .*

*Proof.* Let  $n = 2^a p^b$ . Also let  $b$  is even. Now,

$$H_k(n) = \frac{2^{ak} p^{bk} (a+1)(b+1)}{\sigma_k(2^a) \sigma_k(p^b)},$$

which is an integer. Since  $\gcd(\sigma_k(p^b), p^{bk}) = \gcd(\sigma_k(p^b), 2^{ak}) = 1$ ,

$$\sigma_k(p^b) \mid (a+1)(b+1). \quad (3.1.a)$$

Also,

$$\gcd(\sigma_k(2^a), 2^{ak}) = \gcd(\sigma_k(p^b), 2^{ak}) = 1,$$

$$\sigma_k(p^b) \sigma_k(2^a) \mid p^{bk} (a+1)(b+1).$$

$$\text{i.e., } H_k(n) \geq 2^{ak} \quad (3.1.b)$$

Now, if  $a \leq b$  then  $\sigma_k(p^b) > p^{bk} \geq p^{2b} \geq 3^{2b} > (2b+1)^2 > (a+1)(b+1)$ , which contradicts ( 3.1.a ). Also if  $a > b$  then  $H_k(n) \geq 2^{ak} \geq 2^{2a} > (a+1)^2$ , which contradicts the fact that if  $n$  is  $k$ -harmonic then  $H_k(n) \leq \tau(n) - 1$ . Hence  $b$  must be odd. Next let  $k$  be even. Also if  $b \equiv 7 \pmod{8}$  is not

true, then  $\sigma_k(p^b) = 1 + p^k + \dots + p^{bk} \equiv (b+1) \pmod{8}$ . i.e.,  $8 \nmid \sigma_k(p^b)$ . So,  $\sigma_k(p^b) = 2^r \text{ odd}$ ,  $r \leq 2$ . Hence 3.1.a gives

$$\frac{\sigma_k(p^b)}{2^r} \mid (a+1)(b+1). \quad (3.1.c)$$

Also,

$$H_k(n) = \frac{\frac{2^{ak}}{2^r} p^{bk} (a+1)(b+1)}{\sigma_k(2^a) \frac{\sigma_k(p^b)}{2^r}} = \frac{2^{ak} p^{bk} (a+1) \frac{(b+1)}{2^r}}{\sigma_k(2^a) \frac{\sigma_k(p^b)}{2^r}}.$$

Now,  $\sigma_k(2^a) \frac{\sigma_k(p^b)}{2^r}$  is odd,  $\sigma_k(2^a) \frac{\sigma_k(p^b)}{2^r} \mid p^{bk} (a+1) \frac{(b+1)}{2^r}$ .

$$\text{i.e., } H_k(n) \geq 2^{ak}. \quad (3.1.d)$$

If  $a \leq b$  then (3.1.c) gives a contradiction. Again if  $a > b$  then also we get a contradiction. Thus  $b \equiv 7 \pmod{8}$ . Next if  $k$  is odd then  $\sigma_k(p^b) = 1 + p + \dots + p^{bk} \equiv \frac{b+1}{2} (p+1) \pmod{8}$ . If  $(b+1)(p+1) \equiv 0 \pmod{16}$  is not true then  $8 \nmid \sigma_k(p^b)$  and so  $\sigma_k(p^b) = 2^r \text{ odd}$ ,  $r \leq 2$ . Now,

$$\frac{\sigma_k(p^b)}{2^r} \mid (a+1) \frac{(b+1)}{2} 2$$

and so

$$\frac{\sigma_k(p^b)}{2} \mid (a+1)(b+1), \text{ (if } r = 2\text{)}. \quad (3.1.e)$$

Claim:  $H_k(n) \geq 2^{ak-1}$ . Now,

$$H_k(n) = \frac{2^{ak} p^{bk} (a+1)(b+1)}{\sigma_k(2^a) \sigma_k(p^b)} = \frac{\frac{2^{ak}}{2^r} p^{bk} (a+1)(b+1)}{\frac{\sigma_k(2^a) \sigma_k(p^b)}{2^r}}. \quad (3.1.f)$$

Thus,  $H_k(n) \geq 2^{ak-r}$ . If  $r = 1$  then done. If  $r = 2$  then (3.1.f) gives  $H_k(n) \geq 2^{ak-1}$ . Hence the claim.

Now if  $k \geq 3$  and  $a \leq b$  then  $\frac{1}{2}\sigma_k(p^b) > \frac{1}{2}p^{bk} > p^{2b} \geq 3^{2b} \geq (2b+1)^2 > (a+1)(b+1)$ , which contradicts (3.1.e). Again if  $a > b$  then  $H_k(n) \geq 2^{ak-1} \geq 2^{2a} > (a+1)^2 > (a+1)(b+1) = \tau(n)$ , which contradicts the Lemma 3.1.3. Hence  $(b+1)(p+1) \equiv 0 \pmod{16}$ .  $\square$

**Lemma 3.1.8.** *If  $p, q$  are primes such that  $p < q$ , and  $a, b$  are integers such that  $a \geq b \geq 1$  and  $k \geq 1$ , then  $H_k(p^a q^b) \leq H_k(p^b q^a)$ , with equality if and only if  $a = b$ .*

*Proof.* If  $a = b$  then clearly  $H_k(p^a q^b) = H_k(p^b q^a)$ . Now suppose  $a > b$  and  $x = p^k, y = q^b$  then the condition  $H_k(p^a q^b) \leq H_k(p^b q^a)$  is equivalent to,

$$\begin{aligned} p^{ka} q^{kb} (p^{k(b+1)} - 1)(q^{k(a+1)} - 1) &< p^{kb} q^{ka} (p^{k(a+1)} - 1)(q^{k(b+1)} - 1) \\ \text{i.e., } x^a y^b (x^{b+1} - 1)(y^{a+1} - 1) &< x^b y^a (x^{a+1} - 1)(y^{b+1} - 1) \\ \text{i.e., } x^a y^b (y^{a+1} - x^{a+1}) + x^a y^b &< x^b y^b x^{a-b} (y^{a+1} - x^{b+1} y^{a-b}) + x^b y^a. \end{aligned} \quad (3.1.g)$$

Now,

$$\begin{aligned} 1 - \left(\frac{x}{y}\right)^{b+1} &\geq 1 - \left(\frac{x}{y}\right)^2 \geq 1 - \left(\frac{x}{x+1}\right)^2 > \frac{2}{x+2} > \frac{1}{x} > \frac{1}{x^{a-b}}, \\ x^{a-b} (y^{a+1} - x^{b+1} y^{a-b}) &> x^{a-b} y^{a+1} \frac{1}{x^{a-b}} = y^{a+1} > y^{a+1} - x^{a+1} \end{aligned} \quad (3.1.h)$$

Also,  $x^{a-b} < y^{a-b}$  gives

$$x^a y^b < x^b y^a. \quad (3.1.i)$$

Using (3.1.h) and (3.1.i), we can show that (3.1.g) is true with strict inequality.  $\square$

From the above result we get the following immediate corollary.

**Corollary 3.1.9.** *If  $p_1, p_2, \dots, p_u$  are primes such that  $p_1 < p_2 < \dots < p_u$ , and  $a_1, a_2, \dots, a_u$  are integers such that  $a_1 \geq a_2 \geq \dots \geq a_u \geq 1$ , and  $k \geq 1$ , then  $H_k(p_1^{a_1} p_2^{a_2} \dots p_u^{a_u}) \leq H_k(p_1^{a_{i_1}} p_2^{a_{i_2}} \dots p_u^{a_{i_u}})$ , where  $i_1, i_2, \dots, i_u$  is any permutation of  $1, 2, \dots, u$ . There is equality if and only if this is the identity permutation.*

**Lemma 3.1.10.** *Let the primes  $p, q$  and positive integers  $a, b, c, d$  satisfy  $p < q$ ,  $p^a q^b < p^c q^d$ ,  $8p^{a+1} > 9q^{d+1}$  and  $(a+1)(b+1) = (c+1)(d+1)$ , then for  $k \geq 2$  and  $c > a \geq b > d$ ,  $H_k(p^a q^b) < H_k(p^c q^d)$ .*

*Proof.* Let  $x = p^k, y = q^b$ . Then the condition  $H_k(p^a q^b) < H_k(p^c q^d)$  becomes

$$\begin{aligned} x^a y^b (x^{c+1} - 1)(y^{d+1} - 1) &< x^c y^d (x^{a+1} - 1)(y^{b+1} - 1) \\ \text{i.e., } x^{b+d+1} (x^c - y^a) + x^a y^b &< x^{a+c+1} (y^b - y^d) + x^c y^d. \end{aligned}$$

Now,  $8p^{a+1} > 9q^{d+1}$  gives,

$$8^k p^{k(a+1)} > 9^k q^{k(d+1)} \Rightarrow \frac{x^{a+1}}{y^{d+1}} > \left(\frac{9}{8}\right)^k > \frac{9}{8} \Rightarrow 8x^{a+1} > 9y^{d+1}.$$

Again,  $q \geq 3$ , so  $y \geq 3^2$ . Also,

$$x^{a+1} \left(1 - \frac{1}{y^{b-d}}\right) > \frac{9}{8} y^{d+1} \left(1 - \frac{1}{y^{b-d}}\right) > y^{d+1} > y^{d+1} \left(1 - \frac{1}{x^{c-a}}\right).$$

Multiplying  $x^c y^b$ , we have,

$$x^{a+c+1} (y^b - y^d) > y^{b+d+1} (x^c - x^a). \quad (3.1.j)$$

Now,

$$x^a y^b = p^{ak} q^{bk} < p^{ck} q^{dk} = x^c y^d. \quad (3.1.k)$$

(3.1.j) and (3.1.k) gives

$$x^{b+d+1}(x^c - y^a) + x^a y^b < x^{a+c+1}(y^b - y^d) + x^c y^d.$$

Hence for  $k \geq 2$ ,

$$H_k(p^a q^b) < H_k(p^c q^d).$$

□

**Lemma 3.1.11.** Let  $n_1 = \prod_{i=1}^u p_i^{a_i}$ ,  $n_2 = \prod_{i=1}^u q_i^{a_i}$  be prime factorisations with  $p_i \leq q_i$  for  $i = 1, 2, 3, \dots, u$ . Then  $H_k(n_1) \leq H_k(n_2)$ , for any  $k$ , with equality iff  $n_1 = n_2$ .

*Proof.* An average is larger if the numbers it averages are larger. Since  $p_i \leq q_i \forall i$ , therefore  $H_k(p_i^{a_i}) \leq H_k(q_i^{a_i})$ ,  $\forall i$ . Since  $H_k$  is multiplicative,  $H_k(n_1) \leq H_k(n_2)$ , for any  $k$  and the equality holds whenever  $n_1 = n_2$ . □

## 3.2 Some necessary conditions for existence.

In this section we mention certain necessary conditions for the existence of  $k$ -harmonic numbers with  $k \geq 2$ .

**Theorem 3.2.1.** If  $n$  is a proper  $k$ -harmonic number, then

$$\tau(n) > 1 + \frac{k-1}{k+1} 2^k.$$

*Proof.* Since  $n$  is  $k$ -harmonic so by Lemma 3.1.3,

$$H_k(n) \leq \tau(n) - 1.$$

Therefore,

$$\begin{aligned}\tau(n) &\geq \frac{\sigma_k(n)}{\sigma_k(n) - n^k} = 1 + \frac{1}{\frac{\sigma_k(n)}{n^k} - 1} = 1 + \frac{1}{\sum_{d|n} \left(\frac{d}{n}\right)^k - 1} \\ &= 1 + \left( \sum_{d|n, d>1} \frac{1}{d^k} \right)^{-1}.\end{aligned}\tag{3.2.a}$$

Also,

$$\sum_{d|n, d>1} \frac{1}{d^k} < \sum_{m=2}^{\infty} \frac{1}{m^k} = \xi(k) - 1$$

where  $\xi(k)$  is Reimann's zeta function. Now,

$$\begin{aligned}\xi(k) &= \sum_{m=1}^{\infty} \frac{1}{m^k} = 1 + \frac{1}{2^k} + \sum_{m=3}^{\infty} \frac{1}{m^k} \\ &< 1 + \frac{1}{2^k} + \lim_{c \rightarrow \infty} \int_2^c \frac{1}{x^k} dx \\ &= 1 + \frac{1}{2^k} + \frac{2}{(k-1)2^k} = 1 + \frac{k+1}{(k-1)2^k} \\ \text{or, } \xi(k) - 1 &< \frac{k+1}{(k-1)2^k} \\ \text{or, } \sum_{d|n, d>1} \frac{1}{d^k} &< \frac{k+1}{(k-1)2^k}.\end{aligned}\tag{3.2.b}$$

Hence (3.2.a) and (3.2.b) gives  $\tau(n) > 1 + \frac{k-1}{k+1} 2^k$ .

□

If for example,  $n$  is even but not divisible by 3, then

$$\sum_{d|n, d>1} \frac{1}{d^k} < \frac{1}{2^k} + \frac{1}{4^k} + \frac{1}{5^k} + \frac{1}{7^k} + \frac{1}{8^k} + \frac{1}{10^k} + \dots$$

and we may use (3.2.a) to obtain a better lower bound of  $\tau(n)$ , for a given value of  $k$ . When  $n$  is  $k$ -harmonic with  $k \geq k_0$  and  $2 \leq k_0 \leq 8$ , we get the following values for a lower bound of  $\tau(n)$ .

		$k_0$						
		2	3	4	5	6	7	8
$2 n,$	$3 n$	3	6	14	29	59	121	247
$2 n,$	$3 \nmid n$	4	8	16	32	64	128	256
$2 \nmid n,$	$3 n$	6	21	70	223	693	2122	6446
$2 \nmid n,$	$3 \nmid n$	12	79	466	2566	13635	70996	365021

For example, if  $n$  is an odd  $k$ -harmonic number with  $k \geq 6$ , then  $\tau(n) \geq 693$ .

**Theorem 3.2.2.** *If  $n$  is a proper power-harmonic number, then  $\tau(n) \geq 60$ .*

*Proof.* Essentially, we consider all  $n$  with  $2 \leq \tau(n) \leq 59$ . Let us assume that  $n$  is a proper power-harmonic number and we get contradictions. If  $\tau(n)$  is prime, then  $n$  has only one prime factor and can not be a power-harmonic by Lemma 3.1.2. Among the other 41 values, the most complicated is that for which  $\tau(n) = 48$  (corresponding to the fact  $\tau(\tau(n))$ , for  $2 \leq \tau(n) \leq 59$ , is greatest when  $\tau(n) = 48$ ) and we give certain details only in this case, except for one situation to be described more fully later. There are eleven subcases to be considered (apart from the possibility  $n = p^{47}$ , which cannot be power-harmonic by Lemma 3.1.2). These correspond to prime factorisations of  $n$  of the following forms: (a)  $n = p_1 p_2^{23}$ , (b)  $n = p_1^2 p_2^{15}$ , (c)  $n = p_1^3 p_2^{11}$ , (d)  $p_1^5 p_2^7$ , (e)  $n = p_1 p_2 p_3^{11}$ , (f)  $n = p_1 p_2^2 p_3^7$ , (g)  $n = p_1 p_2^3 p_3^5$ , (h)  $n = p_1^2 p_2^3 p_3^3$ , (i)  $n = p_1 p_2 p_3 p_4^5$ ,

(j)  $n = p_1 p_2 p_3^2 p_4^3$ , (k)  $n = p_1 p_2 p_3 p_4 p_5^2$ . We treat them in turn, although not in the order indicated.

By Lemma 3.1.6, for any integer  $m > 1$  we may define  $k^*(m) = \min\{k : H_k(m) > \tau(m) - 1\}$ . Again by Lemma 3.1.3, if  $k^*(m) = 2$  then  $m$  is not proper  $k$ -harmonic number for any  $k$  and, from the above table,  $k^*(m) \leq 5$  if  $m$  is a proper  $k$ -harmonic number and  $\tau(m) \leq 58$  (accounting for our choice of “60” in the statement of the theorem, since 59 is prime). For integers  $m_i > 1, i = 1, 2, \dots, v$ , we shall use the notation  $K(m_1, m_2, \dots, m_v) = (k_1, k_2, \dots, k_v)$  to mean that  $k^*(m_i) = k_i$  for  $i = 1, 2, \dots, v$ .

(d) Suppose  $n = p_1^5 p_2^7$ , where, first,  $p_1 > p_2$ . Calculation shows that

$$K(2^7 3^5, 3^7 5^5, 5^7 7^5, 7^7 11^5, 11^7 13^5) = (6, 4, 3, 3, 2). \quad (3.2.c)$$

If  $n = 2^7 p^5$  for  $p > 2$ , then by Lemma 3.1.11, this confirms that  $n$  cannot be  $k$ -harmonic for  $k \geq 6$ . If  $k = 2$  or  $k = 4$ , Lemma 3.1.7 implies that  $n$  cannot be  $k$ -harmonic; if  $k = 3$  or  $k = 5$ , it may be verified by computer that  $\sigma_k(2^7)$  has at least three distinct odd prime factors so that

$$H_k(n) = \frac{2^{7k+4} p^{5k} 3}{\sigma_k(2^7 p^5)}$$

cannot be an integer. A more general approach is available here: since

$$2^{2k} + 1 = (2^k + 1)^2 - 2^{k+1}$$

and

$$2^{4k} + 1 = (2^{2k} + 1)^2 - 2^{k+1} = (2^k + 1)^4 - 2^{k+2}(2^k + 1)^2 + 2^{2k+1},$$

then

$$\sigma_k(2^7) = \frac{2^{8k} - 1}{2^k - 1} = (2^k + 1)(2^{2k} + 1)(2^{4k} + 1).$$

has at least three distinct odd prime factors for any  $k \geq 1$ . If  $n = 3^7 p^5$ , with  $p \geq 5$ , then, as above, we need only check that  $\sigma_k(3^5)$  has at least three distinct odd prime factors for  $k = 2$  and  $k = 3$ . We use a similar approach if  $n = 5^7 p^5$ , with  $p \geq 7$ , or  $n = 7^7 p^5$ , with  $p \geq 11$ . If  $n = p^7 q^5$  for  $p, q$  with  $11 \leq p < q$ , then using Lemma 3.1.11 and (3.2.c), we have  $k^*(n) = 2$ , and  $n$  is not proper  $k$ -harmonic. We suppose next that  $p_1 < p_2$ , and write  $K(2^5 3^7, 3^5 5^7, 5^5 7^7, 7^5 11^7, 11^5 13^7) = (k_1, k_2, k_3, k_4, k_5)$ . It follows from (3.2.c) and Lemma 3.1.8 that  $k_1 \leq 6, k_2 \leq 4, k_3 \leq 3, k_4 \leq 3$  and  $k_5 = 2$ . We can then proceed numerically as before for the remaining cases. (c) Suppose  $n = p_1^3 p_2^{11}$ . We assume first that  $p_1 > p_2$ , and write  $K(2^{11} 3^3, 3^{11} 5^3, 5^{11} 7^3, 7^{11} 11^3, 11^{11} 13^3) = (k_1, k_2, k_3, k_4, k_5)$ . Since  $2^7 3^5 < 2^{11} 3^3$ ,  $8 \cdot 2^8 > 9 \cdot 3^4$  and  $k^*(2^7 3^5) = 6$ , we have  $k_1 \leq 6$ , by Lemma 3.1.10. We may similarly estimate  $k_2, \dots, k_5$ , and complete the proof as above. When  $p_1 < p_2$ , we proceed in the manner of the corresponding part of (d). (a, b) In case (a), where  $n = p_1 p_2^{23}$ , and case (b), where  $n = p_1^2 p_2^{15}$ , we again proceed as before, except that in (b) we need not consider the subcase  $n = 2^{15} p^2$ , by Lemma (3.1.7). (h) Suppose  $n = p_1^2 p_2^3 p_3^3$ . Assume first that  $\max(p_1, p_2, p_3) = p_1$ . We calculate that

$$K(2^3 3^3 5^2, 3^3 5^3 7^2, 5^3 7^3 11^2, 7^3 11^3 13^2, 11^3 13^3 17^2) = (6, 4, 3, 3, 2). \quad (3.2.d)$$

If  $n = 2^3 p^3 q^2$  with  $3 \leq p < q$ , then this implies that we need only show that  $H_k(n)$  is not an integer for  $2 \leq k \leq 5$ . Note here that

$$H_k(n) = \frac{2^{3k+4} 3 p^{3k} q^{2k}}{(1+2^k)(1+2^{2k})\sigma_k(p^3 q^2)}. \quad (3.2.e)$$

If  $k = 2$ , then  $(1+2^k)(1+2^{2k}) = 5 \cdot 17$ , so that  $p = 5$  and  $q = 17$  if  $H_2(n)$  is to be an integer; but then  $(1+17^2+17^4 \nmid 5^6)$ , a contradiction. The situation

is similar if  $k = 4$ . If  $k = 3$ , then  $(1 + 2^k)(1 + 2^{2k}) = 3^2 \cdot 5 \cdot 13$ ; cancelling 3, the denominator of the right-side of (3.2.e) has at least three distinct odd prime factors so that  $H_3(n)$  cannot be an integer. The situation is similar if  $k = 5$ . We use similar arguments when  $n = 3^3 p^3 q^2$ , for  $5 \leq p < q$ , or  $n = 5^3 q^3 p^2$ , for  $7 \leq p < q$ , or  $n = 7^3 p^3 q^2$ , for  $11 \leq p < q$ , or  $n = r^3 p^3 q^2$ , for  $11 \leq r < p < q$ . Assume next that  $p_1$  is between  $p_2$  and  $p_3$ , or smaller than both  $p_2$  and  $p_3$ , and suppose

$$K(2^3 3^2 5^3, 3^3 5^2 7^3, 5^3 7^2 11^3, 7^3 11^2 13^3, 11^3 13^2 17^3) = (k_1, k_2, k_3, k_4, k_5)$$

and

$$K(2^2 3^3 5^3, 3^2 5^3 7^3, 5^2 7^3 11^3, 7^2 11^3 13^3, 11^2 13^3 17^3) = (l_1, l_2, l_3, l_4, l_5).$$

By the Corollary of Lemma 3.1.8,  $H_k(2^3 3^3 5^2) < H_k(2^3 3^2 5^3)$  and  $H_k(2^3 3^3 5^2) < H_k(2^2 3^3 5^3)$ , and then, from (3.2.d),  $k_1 \leq 6$  and  $l_1 \leq 6$ . Similarly,  $k_2, l_2 \leq 4$ ,  $k_3, l_3 \leq 3$ ,  $k_4, l_4 \leq 3$  and  $k_5, l_5 = 2$ , we then proceed as above. (g) Here,  $n = p_1 p_2^3 p_3^5$ , and we suppose first that  $p_1 > p_2 > p_3$ . We calculate

$$K(2^5 3^3 5, 3^5 5^3 7, 5^5 7^3 11, 7^5 11^3 13, 11^5 13^3 17) = (6, 4, 3, 3, 2). \quad (3.2.f)$$

and eliminate remaining cases as before. Also as before, the corollary to Lemma 3.1.8 allows us to treat other orderings of  $p_1, p_2, p_3$  similarly, with no further calculations of specific values of  $k^*(n)$ . (f) We could proceed as in (g) in this case,  $n = p_1 p_2^2 p_3^7$ , or we can make use of Lemma 3.1.8 and (3.2.f), as follows. Suppose  $p_1 > p_2 > p_3$  and write

$$K(2^7 3^2 5, 3^7 5^2 7, 5^7 7^2 11, 7^7 11^2 13, 11^7 13^2 17) = (k_1, k_2, k_3, k_4, k_5).$$

Since  $k^*(2^5 3^3 5) = 6$ ,  $2^5 3^3 < 2^7 3^2$  and  $8 \cdot 2^6 > 9 \cdot 3^3$ , we have  $k_1 \leq 6$ , and similarly  $k_2 \leq 4$ ,  $k_3 \leq 3$ ,  $k_4 \leq 3$ ,  $k_5 = 2$ . The rest proceeds in the usual fashion. ( e, i, j ) The case (e), as well as the two cases (i) and (j), in which  $n$  has four distinct prime factors, may also be handled in this manner. We concentrate finally on the case in which  $n$  has five distinct prime factors. (k) We have  $n = p_1 p_2 p_3 p_4 p_5^2$ . Suppose first that  $\min(p_1, p_2, p_3, p_4, p_5) = p_5$ . Calculations show that

$$K(2^2 3 \cdot 5 \cdot 7 \cdot 11, 3^2 5 \cdot 7 \cdot 11 \cdot 13, 5^2 7 \cdot 11 \cdot 13 \cdot 17, 7^2 11 \cdot 13 \cdot 17 \cdot 19, 11^2 13 \cdot 17 \cdot 19 \cdot 23, 13^2 17 \cdot 19 \cdot 23 \cdot 29) = (6, 4, 3, 3, 3, 2). \quad (3.2.g)$$

If  $n = 2^2 p q r s$ , with  $3 \leq p < q < r < s$ , then this means that we need consider  $H_k(n)$  only for  $2 \leq k \leq 5$ . We have

$$H_k(n) = \frac{2^{2k+4} 3 \cdot p^k q^k r^k s^k}{(1 + 2^k + 2^{2k})(1 + p^k)(1 + q^k)(1 + r^k)(1 + s^k)}.$$

Say  $k = 2$ . Since  $1 + 2^2 + 2^4 = 3 \cdot 7$ , then  $7 \mid p q r s$ , if  $H_2(n)$  is to be an integer, and then, since  $1 + 7^2 = 2 \cdot 5^2$  ( so  $5 \mid n$  ),  $1 + 5^2 = 2 \cdot 13$  ( so  $13 \mid n$  ) and  $1 + 13^2 = 2 \cdot 5 \cdot 17$ , we are lead to the contradiction that  $5^3 \mid n^2$ . If  $k = 3$  and  $H_3(n)$  is an integer, then, since  $1 + 2^3 + 2^6 = 73$ ,  $1 + 73^3 = 2 \cdot 7 \cdot 37 \cdot 751$  and  $1 + 7^3 = 2^3 \cdot 43$ , we have  $n$  divisible by  $7 \cdot 37 \cdot 43 \cdot 73 \cdot 751$ ; however,  $n$  has only four distinct odd prime factors, so this is a contradiction. It is similarly impossible to have  $k = 4$  or  $5$ . We similarly use (3.2.g) to show that  $n$  is not proper  $k$ -harmonic if  $n = 3^2 p q r s$  where  $7 \leq p < q < r < s$ , or  $n = 7^2 p q r s$  where  $11 \leq p < q < r < s$ , and it follows immediately from (3.2.g) that this is the case if  $n = 13^2 p q r s$  where  $17 \leq p < q < r < s$ . The Corollary to Lemma (3.1.8) allows all other orderings of  $p_1, \dots, p_5$  to be handled in a

manner corresponding to that described above in cases (h) and (g), but with special numerical arguments in each of the numerous subcases. We omit all remaining details concerning the case  $\tau(n) = 48$ . Elimination of all the above subcases follows since numerical factors appear in the denominators of the expressions for  $H_k(n)$  ( or by use of Lemma (3.1.7)). there are occasions in the complete proof when this is not possible. In these cases,  $\tau(n)$  has a divisor of the form  $1 + p^k$ . ( For larger values of  $\tau(n)$ , there would be a corresponding problem when  $\tau(n)$  has a divisor of the form  $\sigma_k(p^a)$ , for  $a \geq 1$ . ) When  $n$  has four distinct prime factors, for example, this happens three times: when  $p_1, p_2, p_3$  are distinct odd primes and ( i )  $n = 2p_1p_2p_3^4, k = 2$ ; ( ii )  $n = 3p_1p_2p_3^4$  ( and  $3 \nmid p_1p_2p_3$  ),  $k = 2$ ; and ( iii )  $n = 2p_1p_2^2p_3^2, k = 3$ . Of these, ( ii ) is most easily eliminated, but the elimination is indicative of general approach. In this case,

$$H_2(n) = \frac{3^2 p_1^2 p_2^2 p_3^8 \cdot 2^3 5}{(1 + 3^2) \sigma_2(p_1 p_2 p_3^4)} = \frac{2^2 3^3 p_1^2 p_2^2 p_3^8}{\sigma_2(p_1 p_2 p_3^4)}. \quad (3.2.h)$$

If  $H_2(n)$  is an integer, then this shows that  $9 \mid H_2(n)$ , since  $3 \nmid \sigma_2(p_1 p_2 p_3^4)$ . But, using Lemma (3.1.11), the corollary to Lemma (3.1.8), and ( 3.2.h), we have

$$33 < H_2(3 \cdot 5^4 \cdot 7 \cdot 11) \leq H_2(n) < 36,$$

so  $H_2(n) = 34$  or  $35$ , this is a contradiction.

□

We remark that the smallest number with at least 60 divisors is  $5040 = 2^4 3^2 5 \cdot 7$ , and the smallest such odd number is  $135135 = 3^3 5 \cdot 7 \cdot 11 \cdot 13$ .

**PROBLEM:** To determine whether or not 60 is the greatest lower bound for  $\tau(n)$  in Theorem 3.2.2.

**Theorem 3.2.3.** *If  $n$  is a proper power-harmonic number, then  $n > 10^{10}$ .*

*Proof.* A program was written with an outer loop incrementing  $n$  by 1 from 2, but by virtue of Theorem (3.2.2), considering only those  $n$  with  $\tau(n) \geq 60$ . ( These are the numbers 5040, 7560, 7920, 8400,  $\dots$  ) Within an inner loop incrementing  $k$  by 1 from 2,  $H_k(n)$  was calculated until it occurred that  $H_k(n)$  exceeding  $\tau(n) - 1$ . This must happen, according to Lemma (3.1.6). No integer values were found for  $H_k(n)$ . By Lemma (3.1.3) and (3.1.6), the inner loop could then be terminated. A number of computers were used with different ranges of values for  $n$ , and the process was stopped when  $n$  exceeded  $10^{10}$ . The program was written in UBASIC, making use of its facility for rational arithmetic to check that the obtained values of  $H_k(n)$  were indeed not integral. □

We conclude this chapter with the following conjecture.

**Conjecture:** There does exist any proper power harmonic number.

## Chapter 4

# Unitary and Infinitary Harmonic Numbers

In this chapter we discuss two generalizations of harmonic numbers, namely, unitary and infinitary harmonic numbers. We also list these numbers up to  $10^6$ .

### 4.1 Unitary and infinitary divisors

A positive divisor  $d$  of  $n$  is said to be a *unitary or 1-ary divisor* if  $\gcd(d, \frac{n}{d}) =$

1. Using this, one defines the *unitary divisor function*  $\sigma_j^*$  as

$$\sigma_j^*(n) = \sum_{d|n, \gcd(d, \frac{n}{d})=1} d^j.$$

**Definition 4.1.1.** A positive integer  $n$  is said to be a *unitary perfect number* (in short, UPN) if  $\sigma_1^*(n) = 2n$ .

Subbarao and Warren [20] listed four UPNs : 6, 60, 90 and 87360. They showed that every UPN is even and conjectured that there exist only four UPNs; however Wall [21] discovered the fifth one:  $2^{18}.3.5^4.7.11.13.19.37.79.109.157.313$ , an integer with 24 digits. He showed that this is the exact fifth one, that is, there exist no unknown UPNs less than the number above. It is still open whether or not there exist other UPNs.

A positive divisor  $d$  of  $n$  is said to be a *2-ary divisor* of  $n$  if  $\gcd_1(d, \frac{n}{d}) = 1$ , i.e., the greatest common 1-ary divisor of  $d$  and  $\frac{n}{d}$  is 1; and in that case we write  $d \mid_2 n$ .

Inductively, we can also define the *k-ary divisor* of a number  $n$ . More precisely, we write  $d \mid_k n$  if  $\gcd_{k-1}(d, \frac{n}{d}) = 1$ . Note that *k-ary* divisors occur in pair, i.e.,  $d$  is a *k-ary* divisor of  $n$  if and only if  $\frac{n}{d}$  is also so.

We mention the following two results for *k-ary* divisors.

**Lemma 4.1.2.**  $p^x \mid_k p^y$  if and only if  $p^{y-x} \mid_k p^y$ .

*Proof.* We have,

$$\begin{aligned} p^x \mid_k p^y &\Leftrightarrow p^x \mid p^y \text{ and } \gcd_{k-1}(p^x, p^{y-x}) = 1 \\ &\Leftrightarrow p^{y-x} \mid p^y \text{ and } \gcd_{k-1}(p^{y-(y-x)}, p^{y-x}) = 1 \\ &\Leftrightarrow p^{y-x} \mid_k p^y. \end{aligned}$$

□

**Theorem 4.1.3.** For  $k \geq y - 1 \geq 0$ ,  $p^x \mid_k p^y$  if and only if  $p^x \mid_{y-1} p^y$ .

*Proof.* The proof is by induction on  $y$  and  $k$ . For  $k = 1$ , the result is true. Now suppose the result is true for  $y \leq Y - 1$ . For  $k = Y - 1$ , there is nothing

to prove. Let the result be true for  $Y - 1 \leq k \leq K - 1$ . Now consider  $y = Y$  and  $k = K$ . Suppose,  $p^x \mid_K p^Y$ . Also let  $p^x \nmid_{Y-1} p^Y$ . Then  $1 \leq x \leq Y - 1$  and  $\gcd_{Y-2}(p^x, p^{Y-x}) \neq 1$ . Let  $\gcd_{Y-2}(p^x, p^{Y-x}) = p^a$  where  $a \geq 1$ . Since  $p^a \mid_{Y-2} p^x$ , the induction hypothesis shows that  $p^a \mid_{x-1} p^x$ , then again by induction we get  $p^a \mid_{K-1} p^x$ . Similarly,  $p^a \mid_{Y-2} p^{Y-x}$  and  $y - x \leq Y - 1$ ,  $p^a \mid_{Y-x-1} p^Y$  and then  $p^a \mid_{K-1} p^{Y-x}$ . Hence,  $\gcd_{K-1}(p^x, p^{Y-x}) \geq p^a > 1$ , which contradicts the fact  $p^x \mid_K p^Y$ . Thus  $p^x \mid_{Y-1} p^Y$ . Suppose next that  $p^x \nmid_{Y-1} p^Y$ . Also let  $p^x \mid_K p^Y$ , then  $x \leq Y - 1$  and  $\gcd_{K-1}(p^x, p^{Y-x}) = p^b$ , where  $b \geq 1$ , then  $p^b \mid_{K-1} p^x$ , and the induction hypotheses give  $p^b \mid_{x-1} p^x$  and  $p^b \mid_{Y-2} p^x$ . Similarly, we can show that  $p^b \mid_{Y-2} p^{Y-x}$ , which is a contradiction to  $p^x \mid_{Y-1} p^Y$ . This completes the proof of the theorem.

□

In view of the above results, we have the following definition due to G. L. Cohen [4]

**Definition 4.1.4.** Given any two integers  $x \geq 0$  and  $y > 0$ ,  $p^x$  is said to be an *infinitary divisor* of  $p^y$  if  $p^x \mid_{y-1} p^y$ , and in that case we write  $p^x \mid_\infty p^y$ .

The greatest common infinitary divisor of  $p^i$  and  $p^j$  is denoted by  $\gcd_\infty(p^i, p^j)$ . By convention,  $1 \mid_\infty 1$ . We then have the following results for infinitary divisors.

**Theorem 4.1.5.**  $p^x \mid_\infty p^y$  if and only if  $\gcd_\infty(p^x, p^{y-x}) = 1$ , where  $x \leq y$ .

*Proof.* If  $y = 0, 1$  or if  $x = 0, y$  then the result is trivial. Now, let  $y \geq 2$  and  $1 \leq x \leq y - 1$ . If  $p^x \nmid_\infty p^y$  then  $p^x \nmid_{y-1} p^y$  and so  $\gcd_{y-2}(p^x, p^{y-x}) = p^a > 1$ .

Now,

$$\begin{aligned}
\gcd_{y-2}(p^x, p^{y-x}) = p^a &\Leftrightarrow p^a \mid_{y-2} p^x \text{ and } p^a \mid_{y-2} p^{y-x} \\
&\Leftrightarrow p^a \mid_{x-1} p^x \text{ and } p^a \mid_{(y-x)-1} p^{y-x}, \text{ by Theorem(4.1.3)} \\
&\Leftrightarrow p^a \mid_{\infty} p^x \text{ and } p^a \mid_{\infty} p^{y-x} \\
&\Leftrightarrow \gcd_{\infty}(p^x, p^{y-x}) \geq p^a > 1.
\end{aligned}$$

Hence the theorem. □

**Theorem 4.1.6.**  $p \mid_{\infty} p^y$  if and only if  $y$  is odd.

*Proof.* If  $y = 1$  then  $p \mid_{\infty} p$ , so we can assume  $y \geq 3$ . Now we have,

$$\begin{aligned}
p \mid_{\infty} p^y &\Leftrightarrow \gcd_{\infty}(p, p^{y-1}) = 1 \Leftrightarrow p \nmid_{\infty} p^{y-1} \\
&\Leftrightarrow \gcd_{\infty}(p, p^{y-2}) > 1, \text{ by Theorem (4.1.5)} \\
&\Leftrightarrow \gcd_{\infty}(p, p^{y-2}) = p \Leftrightarrow p \mid_{\infty} p^{y-2}.
\end{aligned}$$

Continuing this process we get,

$$p \mid_{\infty} p^y \Leftrightarrow p \mid_{\infty} p, \text{ which is true.}$$

Hence the theorem. □

**Theorem 4.1.7.** If  $y$  is even and  $p^x \mid_{\infty} p^y$ , then  $x$  is also even.

*Proof.* Suppose  $x$  is odd then  $y - x$  is also odd. By theorem (4.1.6), we have,

$$\begin{aligned}
p \mid_{\infty} p^y \text{ and } p \mid_{\infty} p^{y-x} &\Rightarrow \gcd_{\infty}(p^x, p^{y-x}) \geq p > 1 \\
&\Rightarrow p^x \nmid_{\infty} p^y,
\end{aligned}$$

which contradicts the fact  $p^x \mid_{\infty} p^y$ . Thus  $x$  must be even. □

**Remark 4.1.8.** If  $x$  is even and  $p^x \mid_{\infty} p^y$  then  $y$  may not be even. For example  $p^2 \mid_{\infty} p^3$  since  $\gcd_{\infty}(p^2, p) = 1$ . That is the converse of the above theorem is not true.

**Theorem 4.1.9.**  $p^x \mid_{\infty} p^y$  if and only if  $p^{2x} \mid_{\infty} p^{2y}$ .

*Proof.* We use induction on  $y$ . The result is trivial for  $y = 0$ . Now suppose the theorem is true for  $y \leq Y - 1$ . Consider  $y = Y$ . Clearly, we may assume  $1 \leq x \leq Y - 1$ . Suppose,  $p^x \mid_{\infty} p^y$  but  $p^{2x} \nmid_{\infty} p^{2Y}$ . Then  $\gcd_{\infty}(p^{2x}, p^{2Y-2x}) = p^a$  where  $a \geq 1$ . This implies  $p^a \mid_{\infty} p^{2x}$  and  $p^a \mid_{\infty} p^{2Y-2x}$ . Now by Theorem 4.1.7,  $a$  is even. Let  $a = 2b$ . Since  $p^{2b} \mid_{\infty} p^{2x}$  and  $p^{2b} \mid_{\infty} p^{2Y-2x}$ , so by induction hypothesis we have,  $p^b \mid_{\infty} p^x$  and  $p^b \mid_{\infty} p^{Y-x}$ . Then  $\gcd_{\infty}(p^x, p^{Y-x}) \geq p^b > 1$ , which is a contradiction to the fact  $p^x \mid_{\infty} p^Y$ . Next suppose that  $p^x \nmid_{\infty} p^Y$  then  $\gcd_{\infty}(p^x, p^{Y-x}) = p^c$  for some  $c \geq 1$ . Therefore,  $p^c \mid_{\infty} p^x$  and  $p^c \mid_{\infty} p^{Y-x}$ . So by induction hypothesis,  $p^{2c} \mid_{\infty} p^{2x}$  and  $p^{2c} \mid_{\infty} p^{2Y-2x}$ . So,  $\gcd_{\infty}(p^{2x}, p^{2Y-2x}) \geq p^{2c}$ , this implies  $p^{2x} \nmid_{\infty} p^{2Y}$ . Thus if  $p^x \mid_{\infty} p^Y$  then  $p^{2x} \mid_{\infty} p^{2Y}$ . Hence the theorem.  $\square$

**Theorem 4.1.10.** If  $p^x \mid_{\infty} p^y$  and  $y$  is divisible by  $2^j$ , for some  $j \geq 0$ , then  $x$  is divisible by  $2^j$ .

*Proof.* The result is trivial for  $j = 0$ . Suppose it is true for  $j = k$ . Now consider  $j = k + 1$  then let  $y = 2^{k+1}a$ . Since  $y$  is even and  $p^x \mid_{\infty} p^y$  so by Theorem 4.1.7,  $x$  is even say  $x = 2w$ . Then by last theorem,  $p^w \mid_{\infty} p^{2^k a}$ . Therefore by induction hypothesis  $w$  is divisible by  $2^k$ . Hence  $x = 2^k w = 2^k \cdot 2t = 2^{k+1}t$ . i.e.,  $2^{k+1} \mid x$ . Hence the theorem.  $\square$

As a consequence, we have the following two corollaries.

**Corollary 4.1.11.** *The infinitary divisors of  $p^{2^a}$  are 1 and  $p^{2^a}$ .*

*Proof.* Since  $y = 2^a$  i.e.,  $y$  is divisible by  $2^a$ , therefore by the last theorem,  $x$  is also divisible by  $2^a$ . Thus,  $x = 2^a$ . Also  $1 \mid_{\infty} p^{2^a}$ . Hence the corollary.  $\square$

**Corollary 4.1.12.** *For  $0 \leq k < 2^j$ ,  $p^{2^j} \mid_{\infty} p^{2^j+k}$ ; for  $2^j \leq k < 2^{j+1}$ ,  $p^{2^j} \nmid_{\infty} p^{2^j+k}$ .*

*Proof.* Since  $p^{2^j}$  has only two infinitary divisors 1 and  $p^{2^j}$  itself. Also since  $0 \leq k < 2^j$  and  $\gcd_{\infty}(p^{2^j}, p^k) = 1$ , this implies  $p^{2^j} \mid_{\infty} p^{2^j+k}$ . For the next part it is enough to show that  $p^{2^j} \nmid_{\infty} p^k$ . Since  $2^j \leq k < 2^{j+1}$ ,  $k = 2^j + t$  where  $0 \leq t < 2^j$ . Therefore it is enough to show that  $p^{2^j} \nmid_{\infty} p^{2^j+t}$  where  $0 \leq t < 2^j$ . Using the first part we get  $\gcd_{\infty}(p^{2^j}, p^{2^j+t}) \neq 1$ . This implies that  $p^{2^j} \nmid_{\infty} p^{2^j+t}$ . Hence the corollary.  $\square$

**Theorem 4.1.13.**  $p^{2^j} \mid_{\infty} p^y$  iff  $y \equiv 2^j$  or  $2^j + 1$  or  $2^j + 2$  or  $\dots$  or  $2^{j+1} - 1 \pmod{2^{j+1}}$ .

*Proof.* We have,

$$\begin{aligned} p^{2^j} \mid_{\infty} p^y &\Leftrightarrow \gcd_{\infty}(p^{2^j}, p^{y-2^j}) = 1 \Leftrightarrow p^{2^j} \nmid_{\infty} p^{y-2^j} \\ &\Leftrightarrow \gcd_{\infty}(p^{2^j}, p^{y-2^j+1}) = p^{2^j} \\ &\Leftrightarrow p^{2^j} \mid_{\infty} p^{y-2^j+1} \end{aligned}$$

Replacing  $y$  by  $y - 2^{j+1}$ , we have

$$p^{2^j} \mid_{\infty} p^{y-2^j+1} \Leftrightarrow p^{2^j} \mid_{\infty} p^{y-2^j+2}.$$

In this way we shall have

$$p^{2^j} \mid_{\infty} p^y \Leftrightarrow p^{2^j} \mid_{\infty} p^{y-2^{j+1}l}, \quad (4.1.a)$$

where  $l \geq 0$  is the largest integer such that  $y - 2^{j+1}l \geq 2^j$ . Then  $2^j \leq y - 2^{j+1}l < 2^{j+1}$ . Now using the first part of last corollary we get the required result. □

**Theorem 4.1.14.** *Let  $y = \sum y_j 2^j$ . Then  $p^{2^j} \mid_{\infty} p^y$  if and only if  $y_j = 1$ .*

*Proof.* In the previous theorem we have shown that  $p^{2^j} \mid_{\infty} p^y$  if and only if  $2^j + 2^{j+1}l \leq y < 2^{j+1}(l+1)$  where  $l \geq 0$  is the largest integer such that  $y - 2^{j+1}l \geq 2^j$ . Now,  $2^j + 2^{j+1}l \leq y < 2^{j+1}(l+1)$  means  $y = \sum_{k=0}^{\infty} y_k 2^k$  where  $y_j = 1$ . Now,

$$\begin{aligned} y &= 2^j + (y_0 2^0 + y_1 2^1 + \cdots + y_{j-1} 2^{j-1}) + y_{j+1} 2^{j+1} + y_{j+2} 2^{j+2} + \cdots \\ &= 2^j + \alpha + 2^{j+1} \beta. \end{aligned}$$

where  $\alpha = y_0 2^0 + y_1 2^1 + \cdots + y_{j-1} 2^{j-1}$  and  $\beta = y_{j+1} + 2y_{j+2} + \cdots$ . Now,

$$\begin{aligned} \alpha &= y_0 2^0 + y_1 2^1 + \cdots + y_{j-1} 2^{j-1} \\ &\leq 1 \cdot 2^0 + 1 \cdot 2^1 + \cdots + 1 \cdot 2^{j-1} \\ &= 2^j - 1 < 2^j. \end{aligned}$$

Putting  $\beta = l$  we have,

$$\begin{aligned} 2^j + 0 + 2^{j+1}l &\geq 2^j + \alpha + 2^{j+1}\beta < 2^{j+1} + 2^{j+1}l \\ \text{or, } 2^j + 2^{j+1}l &\leq y < 2^{j+1}(l+1). \end{aligned}$$

Hence  $p^{2^j} \mid_{\infty} p^y$ . □

**Theorem 4.1.15.** Let  $x = \sum x_j 2^j$  and  $y - x = \sum z_j 2^j$ . Then  $p^x \mid_{\infty} p^y$  iff  $\sum x_j z_j = 0$ .

*Proof.* Suppose first that  $\sum x_j z_j \neq 0$ . Then  $\exists j$  such that  $x_j = z_j = 1$ . So,  $p^{2^j} \mid_{\infty} p^x$  and  $p^{2^j} \mid_{\infty} p^{y-x}$ , by last theorem. Therefore,  $\gcd_{\infty}(p^x, p^{y-x}) \geq p^{2^j} > 1$ . Hence  $p^x \nmid_{\infty} p^y$ .

Conversely, suppose  $p^x \nmid_{\infty} p^y$ . Let  $\gcd_{\infty}(p^x, p^{y-x}) = p^a$  with  $a \geq 1$ . Also let  $a = \sum a_j 2^j$ ,  $x - a = \sum b_j 2^j$  and  $y - x - a = \sum c_j 2^j$ . Now since  $a = \sum a_j 2^j \geq 1$  so  $a_j = 1$ , for some  $j$ . Also since  $p^a \mid_{\infty} p^x$ , by first part we get  $\sum a_j b_j = 0$ . Thus  $b_j = 0 \forall j$ . Now,  $x_j = a_j + b_j$  for each  $j$ . Hence  $x_j = 1$ . Similarly,  $p^a \mid_{\infty} p^{y-x}$  gives  $\sum a_j c_j = 0$ , i.e.,  $c_j = 0 \forall j$ . Hence,  $z_j = a_j + b_j = 1$ . Thus  $\sum x_j z_j \neq 0$ . Hence the theorem. □

**Corollary 4.1.16.** The infinitary divisors of  $p^{2^a-1}$  are  $p^x$ ,  $0 \leq x \leq 2^a - 1$ .

*Proof.* Here,  $y = 2^a - 1$ . Let  $x = \sum_{j=0}^{a-1} x_j 2^j$  and  $2^a - 1 - x = \sum_{j=0}^{a-1} z_j 2^j$ . So  $2^a - 1 = \sum_{j=0}^{a-1} (x_j + z_j) 2^j$ . This implies  $x_j + z_j = 1 \forall j$ ,  $0 \leq j \leq a - 1$ . Now if  $x_j = 0$  then  $z_j = 1$  and if  $x_j = 1$  then  $z_j = 0$ . Hence in both case,  $\sum x_j z_j = 0$ . □

**Theorem 4.1.17.** If  $p^x \mid_{\infty} p^y$ ,  $p^y \mid_{\infty} p^z$  then  $p^x \mid_{\infty} p^z$ .

*Proof.* Let  $x = \sum x_j 2^j$ ,  $y = \sum y_j 2^j$ ,  $y - x = \sum r_j 2^j$ ,  $z - y = \sum s_j 2^j$  and  $z - x = \sum t_j 2^j$ . Since  $p^x \mid_{\infty} p^y$ ,

$$\sum x_j r_j = 0. \tag{4.1.b}$$

Also,  $p^y \mid_{\infty} p^z$  gives

$$\sum y_j s_j = 0. \quad (4.1.c)$$

We are to show that  $\sum x_j t_j = 0$ . Now  $y = \sum (r_j + x_j) 2^j$ . Since  $\forall j, x_j = 1$ , therefore  $r_j = 0$ , by (4.1.b). We have,  $y_j = r_j + x_j, \forall j$ . Now suppose that  $x_k = 1$ , for some  $k$  therefore  $r_k = 0$  and then  $y_k = 1$  and so  $s_k = 0$ . Now,

$$z - x = (z - y) + (y - x) = \sum (r_j + s_j) 2^j.$$

Therefore, if  $k = 0, r_0 + s_0 = 0$  which implies  $t_0 = 0$ . Again if  $k > 0$  then  $t_k = r_k + s_k = 0$  if we do not have  $r_i = s_i = 1$  for any  $i < k$ . If  $r_i = s_i = 1$  for some  $i < k$  then by (4.1.b),  $s_i = 0$  and by (4.1.c),  $y_i = 0$  and so  $y_i = r_i + x_i$  is not possible. Hence it follows that  $\sum x_j t_j = 0$ . Thus  $p^x \mid_{\infty} p^z$ . □

From the above theorem we see that the infinitary divisors satisfy transitivity relation. However, in general,  $k$ -ary divisors do not satisfy transitivity. For,  $p \mid_5 p^3$ , and  $p^3 \mid_5 p^7$ , but  $p \nmid_5 p^7$ .

**Theorem 4.1.18.** *Suppose  $2^a \leq y < 2^{a+1}$ . If  $p^x \mid_{\infty} p^{y-2^a}$ , then  $p^x \mid_{\infty} p^y$  and  $p^{2^a+x} \mid_{\infty} p^y$ ; If  $x \leq y - 2^a$  and  $p^x \mid_{\infty} p^y$ , then  $p^x \mid_{\infty} p^{y-2^a}$ .*

*Proof.* Since  $2^a \leq y < 2^{a+1}$ ,  $y = 2^a + t$ , where  $0 \leq t < 2^a$ . Now by corollary 4.1.16 we get  $p^{2^a} \mid_{\infty} p^{2^a+t}$ , i.e.,  $p^{2^a} \mid_{\infty} p^y$ . So,  $\gcd_{\infty}(p^{2^a}, p^{y-2^a}) = 1$ . This gives  $p^{y-2^a} \mid_{\infty} p^y$ . Now, since  $p^x \mid_{\infty} p^{y-2^a}$  and  $p^{y-2^a} \mid_{\infty} p^y$ , so by previous theorem  $p^x \mid_{\infty} p^y$ . Now, let  $x = \sum x_j 2^j$  and  $y - 2^a - x = \sum z_j 2^j$ . Since  $p^x \mid_{\infty} p^{y-2^a}$  so  $\sum x_j z_j = 0$ . □

**Theorem 4.1.19.**  $p^x \mid_{\infty} p^y$  iff  $\binom{y}{x}$  is odd.

*Proof.* We have,

$$\begin{aligned} \binom{y}{x} \text{ is odd} &\Leftrightarrow \binom{y-x+x}{x} = \text{odd} \\ &\Leftrightarrow y-x \text{ and } x \text{ has no powers of 2 common} \\ &\Leftrightarrow \sum x_j z_j = 0 \\ &\Leftrightarrow p^x \mid_{\infty} p^y. \end{aligned}$$

□

## 4.2 Unitary harmonic number and its properties

Let  $\tau^*(n)$  and  $\sigma^*(n)$  be the number of unitary divisors of  $n$  and sum of the unitary divisors of  $n$  respectively. Also let  $H^*(n)$  be the harmonic mean of the unitary divisor of  $n$ . Then we can easily verify that

$$H^*(n) = \frac{n\tau^*(n)}{\sigma^*(n)}.$$

Following Hagis and Lord [14], if  $H^*(n)$  is integer then  $n$  is said to be a *unitary harmonic number* (in short, UHN). Note that  $\tau^*$ ,  $\sigma^*$  and  $H^*$  are multiplicative functions. If  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$  then we can easily show that

$$\tau^*(n) = 2^k, \sigma^*(n) = (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2}) \cdots (1 + p_k^{\alpha_k})$$

and

$$H^*(n) = \frac{n\tau^*(n)}{\sigma^*(n)} = \prod_{i=1}^k \frac{2p_i^{\alpha_i}}{1 + p_i^{\alpha_i}}.$$

**Proposition 4.2.1.** *Every UPN is a UHN.*

*Proof.* Let  $n$  be a unitary perfect number. Now,

$$H^*(n) = \frac{n\tau^*(n)}{\sigma^*(n)} = \frac{n2^{w(n)}}{2n} = 2^{w(n)-1} \quad (4.2.a)$$

where  $w(n)$  is the number of distinct primes of  $n$ . From (4.2.a), we see that  $H^*(n)$  is an integer, i.e.,  $n$  is a UHN.  $\square$

Since  $H(n) = H^*(n)$  if and only if  $n$  is square-free, we can reformulate Theorem 2.1.5 as follows:

**Proposition 4.2.2.** *If  $n$  is square-free and  $n \neq 6$  then  $n$  is not a UHN.*

**Proposition 4.2.3.** *If  $n$  is an odd UHN then  $H^*(n)$  is also odd.*

*Proof.* Let  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ , where  $p_i$ 's are odd primes. Then

$$H^*(n) = \frac{2^r p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}}{(1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2}) \cdots (1 + p_r^{\alpha_r})}. \quad (4.2.b)$$

Since  $p_i$ 's are all odd so the denominator of (4.2.b) is a multiple of  $2^r$  and hence it follows that  $H^*(n)$  is odd.  $\square$

**Proposition 4.2.4.** *If  $n$  is an odd UHN,  $p^a \parallel n$ , and  $p = 4j + 3$  then  $a$  is even.*

*Proof.* Let  $n = p^a p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ . Then

$$H^*(n) = \frac{p^a p_2^{\alpha_2} \cdots p_r^{\alpha_r}}{\frac{p^a + 1}{2} \frac{p_2^{\alpha_2} + 1}{2} \cdots \frac{p_r^{\alpha_r} + 1}{2}}. \quad (4.2.c)$$

If  $a$  is odd then  $p^a \equiv 3 \pmod{4}$  and then  $\frac{p^a + 1}{2}$  is even. But numerator of (4.2.c) is odd, if  $a$  is odd  $H^*(n)$  can not be integer. Thus  $a$  must be even if  $n$  is a UHN.  $\square$

**Remark 4.2.5.** If  $n$  is an even UHN and  $p^a \parallel n$  where  $p$  is the largest prime dividing  $n$ , then largest prime dividing  $H^*(n)$  is less than or equal to  $p$  with exponent not exceeding  $a$ .

**Proposition 4.2.6.** If  $n$  is a UHN,  $\gcd(p, n) = 1$  and  $(p^a + 1) \mid 2H^*(n)$ . Then  $p^a n$  is also a UHN where  $p$  is a prime.

*Proof.* Since  $H^*(n)$  is multiplicative,  $H^*(p^a n) = H^*(p^a)H^*(n) = p^a \frac{2H^*(n)}{1 + p^a}$ . Since  $(p^a + 1) \mid 2H^*(n)$  so  $H^*(p^a n)$  is an integer. Hence  $p^a n$  is a UHN.  $\square$

Analogous to Theorem 2.3.3, Hagis and Lord [14] proved the following result.

**Theorem 4.2.7.** If  $S_c$  is the set of natural numbers  $n$  such that  $H^*(n) = c$ , then  $S_c$  is finite (may be empty) for every real number  $c$ .

*Proof.* Suppose  $S_c$  is an infinite set then there exists an infinite subset  $S_{cm}$  of  $S_c$  whose elements have exactly  $m$  distinct prime factors with  $1 \leq m \leq [c]$ . Therefore it follows that there exists an infinite sequence  $\{n_1, n_2, \dots\}$  of distinct integers with the following properties

(I)  $n_i \in S_{cm}$ .

(II)  $n_i = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{s-1}^{\alpha_{s-1}} p_{is}^{\alpha_{is}} p_{is+1}^{\alpha_{is+1}} \dots p_{im}^{\alpha_{im}} = P \prod_{j=s}^m p_{ij}^{\alpha_{ij}}$

where  $p_1^{\alpha_1} < p_2^{\alpha_2} < \dots < p_{s-1}^{\alpha_{s-1}} < \dots < p_{im}^{\alpha_{im}}$  and  $P$  may be empty i.e.,  $s = 1$

but  $s - 1 \neq m$

and (III)  $p_{ij}^{\alpha_{ij}} \rightarrow \infty$  as  $i \rightarrow \infty$  for  $j = s, \dots, m$ .

Now,

$$\frac{c}{H^*(P)} = \frac{H^*(n)}{H^*(P)} = \prod_{j=s}^m H^*(p_{ij}^{\alpha_{ij}}) < 2^{m+1-s}. \quad (4.2.d)$$

Therefore there exists a fixed positive integer  $v$  such that

$$\begin{aligned} \prod_{j=s}^m H^*(p_{ij}^{\alpha_{ij}}) &= 2^{m+1-s} - v, \text{ for } i = 1, 2, \dots \\ \Rightarrow \prod_{j=s}^m \frac{2p_{ij}^{\alpha_{ij}}}{1 + p_{ij}^{\alpha_{ij}}} &= 2^{m+1-s} - v, \text{ for } i = 1, 2, \dots \\ \Rightarrow v &= 0 \text{ (by (III))} \end{aligned}$$

So,  $\prod_{j=s}^m H^*(p_{ij}^{\alpha_{ij}}) = 2^{m+1-s}$ , which is a contradiction to (4.2.d). This contradiction implies that  $S_c$  must be a finite set.  $\square$

Now, let  $T_k$  be the set of all unitary harmonic numbers which have exactly  $k$  prime factors. Hagis and Lord [14] gave the results for the sets  $T_1$ ,  $T_2$  and  $T_3$ .

**Proposition 4.2.8.**  $T_1$  is empty, i.e., power of a prime is never a unitary harmonic number.

*Proof.* Let  $n = p^a$ . So,  $H^*(p^a) = \frac{2p^a}{1 + p^a}$ . Since  $1 + p^a \nmid 2p^a$ , so  $H^*(p^a)$  can not be an integer. Hence,  $T_1$  is empty.  $\square$

**Proposition 4.2.9.**  $T_2 = \{6, 45\}$ , i.e., 6 and 45 are the only unitary harmonic numbers having two distinct prime factor.

*Proof.* We have,  $\frac{2^{k+1}}{k+2} \leq H^*(n) < 2^k$ . Here,  $k = 2$  and  $H^*(n)$  is integer,  $H^*(n) = 2$  or  $3$ . Now let  $n = p^a q^b$ . Without any loss assume,  $p^a < q^b$ .

Case 1 :  $H^*(n) = 2$ . Then  $p^a \leq 2$ , by Lemma 4.3.2. So,  $p^a = 2$ . Then  $q^b \geq 3$ .

Now

$$H^*(2q^b) = 2 \Rightarrow \frac{8q^b}{3(1+q^b)} = 2 \Rightarrow q^b = 3.$$

Hence,  $n = 6$ .

Case 2:  $H^*(n) = 3$ , then  $p^a \leq 6$ . So,  $p^a = 2, 3, 4, 5$ .  $p^a = 2$  gives  $q^b = -9$ , impossible. Also  $p^a = 3$  and  $p^a = 4$  are not possible. Now, if  $p^a = 5$  then  $q^b = 9$ . Therefore  $n = 5 \times 9 = 3^2 \times 5$ .

Hence,  $T_2 = \{6, 45\}$ . □

**Proposition 4.2.10.** *The set of all unitary harmonic numbers with three distinct prime factors is given by*

$$T_3 = \{60, 90, 1512, 15925, 55925\}.$$

*Proof.* By Lemma 4.3.3,  $H^*(n) = 4, 5, 6$ , or  $7$  if  $n \in T_3$ . Let  $n = p^a q^b r^c$ . With out any loss assume  $p^a$  is the minimum prime power of  $p$  dividing  $n$ . We have

$$\frac{H^*(n)}{2^k - H^*(n)} \leq p^a \leq \frac{kH^*(n)}{2^k - H^*(n)}. \quad (4.2.e)$$

Case1:  $H^*(n) = 4$  and then  $p^a = 2, 3$ .  $p^a = 2$  gives  $H^*(n) = H^*(2q^b r^c)$  and then  $3 \mid n$ . With out any loss assume  $q = 3$  then  $n = 2.3^b r^c$ . Since  $n$  can not be squarefree so  $b > 1$  or  $c > 1$ . If  $b > 1$  then

$$4 = \frac{2.3^b.r^c.8}{3(1+3^b)(1+r^c)} \geq \frac{2}{3} \frac{3^2.5.8}{(1+3^2)(1+5)}.$$

This gives  $b = 2$  and  $r^c = 5$ . Thus  $n = 2.3^2.5 = 90$ . If  $c > 1$  then as seen above  $b = 1$  and so  $n = 2.3.r^c$ . Thus,

$$4 = \frac{2.3.r^c.8}{3.4.(1+r^c)}.$$

which is not possible. So,  $c > 1$  is not possible. If  $p^a = 3$  then Lemma (4.3.2) gives  $q^b = 1 + 3 = 4$  and  $r^c = 3 + 2 = 5$  and so  $n = 3.2^2.5 = 60$ .

Case2:  $H^*(n) = 5$ . By Lemmas (4.3.1) and (4.3.2) gives,  $p^a = 2, 3, 4, 5$ . If  $p^a = 5$  then Lemma (4.3.2) gives  $q^b = 6$ , impossible. So  $p^a \neq 5$ . Again  $H^*(n) = 5$  gives  $5 \mid n$ . Assume  $q = 5$ . Now  $p^a = 2$  gives  $r = 3$  and  $n = 2 \cdot 5^b \cdot 3^c$ . Also,  $H^*(2 \cdot 5^b \cdot 3^c) = 5$  gives  $(5^{b-1} - 3)(3^{c-1} - 5) = 16$ , which is not possible. Similarly  $p^a = 3, 4$  are not possible. Hence if  $H^*(n) = 5$  then there does not exist any  $n \in T_3$ .

Case 3:  $H^*(n) = 6$ . Then  $n$  is even and  $p^a = 4, 5, 7, 8$ . As above we can show that  $p^a = 4, 5, 8$  are not possible. Now let  $p^a = 7$ . Also let  $q = 2$ . Then we get  $r^c = 3^c$ ,  $c \geq 1$ . Now,

$$6 = \frac{7 \cdot 2^b \cdot 3^c}{(1 + 2^b)(1 + 3^c)} \Rightarrow (3^{c-1} - 2)(2^{b-1} - 3) = 7 \Rightarrow b = c = 3.$$

Hence,  $n = 2^3 \cdot 3^3 \cdot 7 = 1512$ .

Case 4:  $H^*(n) = 7$  then  $7 \mid n$ . With out any loss we can take  $q = 7$  and then  $7 \leq p^a \leq 21$  i.e.,  $p^a = 8, 9, 11, 13, 16, 17, 19$ . Now let  $n = p^a 7^b r^c$  with  $b \geq 2$ . If  $p^a = 8$  then  $r^c = 3^c$  with  $c \geq 2$ . So,  $H^*(8 \cdot 7^b 3^c) = 7$  gives  $(3^{c-2} - 7)(7^{b-1} - 9) = 2^6$ , which has no solution. Next if  $p^a = 9$  then  $H^*(9 \cdot 7^b \cdot r^c) = 7$  gives  $5 \mid r^c$ . Let  $r = 5$  then we get  $(7^{b-1} - 5)(5^{c-1} - 7) = 36$  which gives  $b = 2$  and  $c = 3$ . Therefore,  $n = 9 \cdot 7^2 \cdot 5^3 = 55125$ . Now, for  $p^a = 11, 16, 17, 19$ , there is no solution for  $H^*(n) = 7$ . Again if  $p^a = 13$ , then  $n = 13 \cdot 7^b \cdot r^c$ . So,  $(1 + r^c)(1 + 7^b) = 13 \cdot 7^{b-2} \cdot r^c \cdot 4$  which gives  $7^{b-2} \mid (1 + r^c)$ . If  $b \geq 4$  then  $49 \mid (1 + r^c)$  gives  $r^c \leq 97$ . But  $H^*(13 \cdot 7^4 \cdot 97) > 7$ . Therefore  $b = 2$ , then we get  $r^c = 25$ . So,  $n = 13 \cdot 7^2 \cdot 5^2 = 15925$ . Hence the proposition.  $\square$

**PROBLEM:** To determine  $T_k$  where  $k \geq 4$ .

**PROBLEM:** To find a relation between suffix  $k$  in  $T_k$  and number of elements in the set  $T_k$ .

**PROBLEM:** To verify whether the number of harmonic numbers having a fixed harmonic mean  $c$  depends on  $k$  where  $k = w(n)$ .

**PROBLEM:** How many harmonic numbers are there with  $w(n) = 2$ ?

### 4.3 Upper bound for UHN:

In this section, we prove first the following lemmas.

**Lemma 4.3.1.** *If  $p^a \parallel n$ , then  $p^a \geq \frac{H^*(n)}{2^k - H^*(n)}$  where  $w(n) = k$  with equality if and only if  $k = 1$ .*

*Proof.* Let  $k > 1$ . Since  $p^a \parallel n$ ,  $n = p^a x$  with  $p \nmid x$ . Now,

$$H^*(n) = H^*(p^a)H^*(x) = \frac{2p^a}{1+p^a}H^*(x) \leq \frac{2p^a}{1+p^a}2^{k-1} = \frac{2^k p^a}{1+p^a}.$$

This implies that  $p^a = \frac{H^*(n)}{2 - H^*(n)}$ . Now if  $k = 1$  then  $n = p^a$  and then  $H^*(n) = \frac{2p^a}{1+p^a}$  and hence  $p^a = \frac{H^*(n)}{2 - H^*(n)}$ .  $\square$

**Lemma 4.3.2.** *If  $p^a \{r^c\}$  is the minimum (maximum) prime power divisor of  $n$  then*

$$p^a \leq \frac{kH^*(n)}{2^k - H^*(n)} \left\{ r^c \geq \frac{(k-1)2^k + H^*(n)}{2^k - H^*(n)} \right\},$$

*with equality if and only if  $k = 1$  or  $n = p^a q^b r^c$  where  $q^b = p^a + 1$  ( so that  $2 \mid n$  ) and  $r^c = p^a + 2$  or  $c = 0$  {where  $q^b = r^c - 1$ ,  $p^a = r^c - 2$  or  $a = 0$ }*

*Proof.* Let  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ . Take  $p_1^{\alpha_1} = p^a$ . Then

$$\begin{aligned} H^*(n) &= H^*(p^a) H^*(p_2^{\alpha_2}) \cdots H^*(p_k^{\alpha_k}) = \frac{2p^a}{1+p^a} \frac{2p_2^{\alpha_2}}{1+p_2^{\alpha_2}} \cdots \frac{2p_k^{\alpha_k}}{1+p_k^{\alpha_k}} \\ &\geq 2^k \frac{p^a}{(p^a+1)} \frac{p^a+1}{p^a+2} \cdots \frac{p^a+k-1}{p^a+k} \\ &= \frac{2^k p^a}{p^a+k} \end{aligned}$$

From this we get,

$$p^a \leq \frac{kH^*(n)}{2^k - H^*(n)}. \quad (4.3.a)$$

Again taking  $p_k^{\alpha_k} = r^c$ , we get

$$\begin{aligned} H^*(n) &= H^*(p^a) H^*(p_2^{\alpha_2}) \cdots H^*(r^c) \\ &= \frac{2p_1^{\alpha_1}}{1+p_1^{\alpha_1}} \frac{2p_2^{\alpha_2}}{1+p_2^{\alpha_2}} \cdots \frac{2p_{k-1}^{\alpha_{k-1}}}{1+p_{k-1}^{\alpha_{k-1}}} \frac{r^c}{1+r^c} \\ &\leq 2^k \frac{r^c - k + 1}{r^c + 1} \end{aligned}$$

From above we get

$$r^c \geq \frac{2^k(k-1) + H^*(n)}{2^k - H^*(n)}. \quad (4.3.b)$$

Hence (4.3.a) and (4.3.b) are the required inequality. The equality holds only in the specified "exceptional" cases.  $\square$

**Lemma 4.3.3.** *Let  $n$  be a UHN with  $k$  distinct prime factors. Then*

$$k \leq \frac{2^{k+1}}{k+2} \leq H^*(n) < 2^k.$$

*Furthermore, we have first equality only when  $k = 2$ , second equality only when  $n = 2$  or  $6$ , third equality only when  $n = 1$ .*

*Proof.* We have  $\forall k, 2^{k+1} \geq k^2 + 2k$ . Now, let  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ . Since  $\{\frac{x}{x+1}\}$  is a monotonically increasing sequence of real numbers and bounded by 1. Thus,

$$H^*(n) = \prod_{i=1}^k \frac{2p_i^{\alpha_i}}{1+p_i^{\alpha_i}} < \prod_{i=1}^k 2 = 2^k. \quad (4.3.c)$$

Again,

$$\begin{aligned} H^*(n) &= \frac{2^k}{\prod_{i=1}^k \frac{\sigma^*(p_i^{\alpha_i})}{p_i^{\alpha_i}}} = \frac{2^k}{\prod_{i=1}^k \frac{1+p_i^{\alpha_i}}{p_i^{\alpha_i}}} \geq \frac{2^k}{\prod_{i=1}^k 1 + \frac{1}{p_i}} \\ &\geq \frac{2^k}{(1 + \frac{1}{2})(1 + \frac{1}{3}) \cdots (1 + \frac{1}{k+1})} \\ &= \frac{2^k}{k+2}. \end{aligned} \quad (4.3.d)$$

From (4.3.c) and (4.3.d), we get the required result. Now, if  $n = 2$  then  $H^*(n) = \frac{4}{3} = \frac{2^k}{k+2}$ . Again if  $n = 6$  then  $H^*(n) = 2 = \frac{2^k}{k+2}$ . This completes the lemma.  $\square$

**Lemma 4.3.4.** *Let  $n$  be a UHN with  $k$  distinct prime factors, and  $H^*(n) = c$ . Then the following facts hold,*

- (a) *If  $k \leq 3$ , then  $n \in \{1, 6, 45, 60, 90, 1512, 15925, 55125\}$ .*
- (b) *If  $c \leq 5$  then  $n \in \{1, 6, 45, 60, 90\}$ .*

*Proof.* (a) If  $k = 0$  then  $n$  must be equal to 1. Now by Proposition (4.2.8), we have there is no UHN with  $k = 1$ . Again if  $k = 2$  then by Proposition (4.2.9),  $n = 6, 45$ . Finally if  $k = 3$  then by Proposition (4.2.10), we have  $n = 60, 90, 1512, 15925, 55125$ . This complete the proves of part(a).

(b) If  $c = 1$ , then by Lemma (4.3.3),  $k \leq c = 1$ , which gives  $k = 0, 1$ . But for  $k = 1$ , there is no UHN by Propositon (4.2.8). Thus  $n = 1$  when  $c = 1$ .

Now if  $c = 2$  then by case 1 of Proposition (4.2.9), we get  $n = 6$ . Again if  $c = 3$ , then from case 2 of Proposition (4.2.9), we have  $n = 45$ . Next if  $c = 4$  then  $n = 60, 90$ , by case 1 of Proposition (4.2.9). Finally, if  $c = 5$  there is no UHN, by case 2 of Proposition (4.2.10). Hence completes the part(b).  $\square$

Now, we have prove the following two theorems for the upper bounds of a UHN.

**Theorem 4.3.5.** *Suppose that  $n$  is a UHN with  $k$  distinct prime factors and  $H^*(n) = c$ . Then it follows that*

(a)  $n \leq c^{c^2}$ ,

(b)  $n \leq (2^{2^k})^k$  with each equality if and only if  $n = 1$ .

*Proof.* Since  $\sigma_{-1}^*(n) = \frac{\sigma_0^*(n)}{H^*(n)} = \frac{2^k}{c}$ , then by Proposition (1.4.5),

$$n \leq (c+1)^{2^{k-1}-1}((c+1)^{2^{k-1}} - 1) < (c+1)^{2^k}. \quad (4.3.e)$$

If  $k \leq 3$  or  $c \leq 5$  then by Lemma (4.3.4), we get the required inequalities. So, we may assume that  $k \geq 4$  and  $c \geq 6$ . now since by Lemma (4.3.3),  $c < 2^k$  i.e.,  $c+1 \leq 2^k$  then (4.3.e) implies that  $n < (2^k)^{2^k} = (2^{2^k})^k$ . Now, since  $k \leq 4$ ,

$$(k+2)^4 < 2^{2k+3} \Rightarrow k+2 < 2^{\frac{2k+3}{4}}.$$

Hence by Lemma (4.3.3) implies that

$$c \geq \frac{2^{k+1}}{k+2} > \frac{2^{k+1}}{2^{\frac{2k+3}{4}}} = 2^{\frac{2k+1}{4}}.$$

Therefore,

$$2^k < \frac{c^2}{\sqrt{2}}.$$

Now,  $c + 1 < c^{\sqrt{2}}$  if  $c \geq 6$ . Hence it follows that

$$n < (c + 1)^{2^k} < (c + 1)^{\frac{c^2}{\sqrt{2}}} < c^{c^2}.$$

□

**Theorem 4.3.6.** *There exists at most finitely many unitary harmonic numbers with the specified number of distinct prime factors.*

*Proof.* Let  $k$  be a fixed positive integer. Since  $\frac{2^{k+1}}{k+2} \leq H^*(n) < 2^k \forall n$  with  $w(n) = k$ . It follows that  $c = \{H^*(n) \mid n \text{ is a UHN with } w(n) = k\}$  is a finite set. Therefore by Theorem 4.2.7 the result follows. □

**Corollary 4.3.7.** *There are at most finitely many unitary perfect numbers with a specified number of prime factors.*

*Proof.* It follows immediately since the set of unitary perfect numbers is a subset of the set of unitary harmonic numbers. □

**PROBLEM:** To find a result analogous to Theorem 4.3.6 for harmonic numbers.

We conclude this section with the following table of UHN up to  $10^6$ .

Table 4.1: List of unitary harmonic numbers upto  $10^6$ .

$n$	$H^*(n)$	$n$	$H^*(n)$	$n$	$H^*(n)$	$n$	$H^*(n)$
1	1	9100	10	87360	16	598500	19
6	2	15925	7	95550	14	646425	13
45	3	16632	11	143640	19	661500	12
60	4	27300	15	163800	20	716625	13
90	4	315000	10	172900	19	790398	17
420	7	40950	15	185976	12	791700	29
630	7	46494	9	232470	16	859950	18
1512	6	51408	12	257040	28	900900	33
3780	9	55125	7	330750	10	929880	20
5460	13	64260	17	332640	20		
7560	10	66528	12	464940	18		
8190	13	81900	18	565488	22		

#### 4.4 Infinitary harmonic number and its properties

Let  $P$  be the set of all primes and let  $I = \{p^{2^\alpha} : p \in P, \alpha \in N_0\}$ . From the fundamental theorem of arithmetic and the fact that the binary representation of a natural number is unique, it follows that if  $n > 1$  then we can write it in exactly one way (except for the order of the factors) as the product of distinct elements from  $I$ . Each element of  $I$  in this product is called an

I-component of  $n$ .

Let the number of I-components of  $n$  be denoted by  $J(n)$ . Then clearly,  $J(1) = 0$  and if  $y = \sum_{i=0}^{\infty} y_i 2^i$  where  $y_i = 0$  or  $1$ , then we can easily show that  $J(p^y) = \sum y_i$ .  $J$  is an additive function so that if  $n = \prod_{p^y || n} p^y$  then  $J(n) = \sum_{p^y || n} J(p^y)$ .

**Definition 4.4.1.** A positive integer  $d$  is called an I-divisor of  $n$  if every I-component of  $d$  is also an I-component of  $n$ .

Now if  $\sigma_I(n)$  is the sum of the I-divisors of  $n$ , then  $\sigma_I(1) = 1$  and  $\sigma_I(p^y) = \prod_{y_i=1} (1 + p^{2^i})$  if  $y = \sum y_i 2^i$ . We can easily show that  $\sigma_I$  is a multiplicative function so that if  $n = \prod_{p^y || n} p^y$  then  $\sigma_I(n) = \prod_{p^y || n} \prod_{y_i=1} (1 + p^{2^i})$ . Also if  $\tau_I(n)$  is the number of I-divisors of  $n$  then  $\tau_I(n) = \prod_{p^y || n} 2^{J(p^y)} = 2^{J(n)}$ . The set of infinitary divisors of  $n$  is equal to the set of I-divisors of  $n$ . Therefore, if  $\tau_{\infty}(n)$  and  $\sigma_{\infty}(n)$  denote the number and sum, respectively, of the infinitary divisors of  $n$ , we have  $\tau_{\infty}(p^y) = 2^{\sum y_j}$ ,  $\sigma_{\infty}(p^y) = \sum_{y_j=1} (1 + p^{2^j})$  where  $y = \sum y_j 2^j$ .

Now, if  $n = \prod_{p^y || n} p^y$  and  $y = \sum y_j 2^j$ , then

$$\tau_{\infty}(n) = \prod_{p^y || n} 2^{J(p^y)} = 2^{J(n)},$$

where  $J(n) = \sum_{p^y || n} J(p^y) = \sum_{p^y || n} \sum y_i$

and

$$\sigma_{\infty}(n) = \prod_{p^y || n} \prod_{y_i=1} (1 + p^{2^i}).$$

The infinitary harmonic mean of  $n$  or the harmonic mean of the infinitary

divisors of  $n$  is given by

$$H_{\infty}(n) = \frac{n\tau_{\infty}(n)}{\sigma_{\infty}(n)} = \frac{n2^{J(n)}}{\prod_{p^{\nu}||ny_i=1} \prod (1+p^{2^i})} = 2^{J(n)} \prod_{p^{\nu}||ny_i=1} \prod \frac{p^{2^i}}{1+p^{2^i}}.$$

If  $H_{\infty}(n)$  is an integer then  $n$  is said to be an *infinitary harmonic number* (IHN) and the set of all infinitary harmonic number is denoted by  $\mathcal{I}$ . It may also be mentioned here that a positive integer  $n$  is said to be an *infinitary perfect number* (IPN) if  $\sigma_{\infty}(n) = 2n$ .

The following two results are similar to Proposition 4.2.1 and Lemma 4.3.3.

**Proposition 4.4.2.** *Every IPN is an IHN.*

*Proof.* Let  $n \in IPN$ , then  $\sigma_{\infty}(n) = 2n$ . Now,

$$H_{\infty}(n) = \frac{n\tau_{\infty}(n)}{\sigma_{\infty}(n)} = \frac{n2^{J(n)}}{2n} = 2^{J(n)-1}$$

which is an integer as  $J(n) \geq 1$ . Thus  $n \in \mathcal{I}$ . □

**Lemma 4.4.3.** *Let  $J(n) = J$ . Then, if  $n > 1$ ,*

$$\frac{2^{J+1}}{J+2} \leq H_{\infty}(n) < 2^J.$$

*Proof.* Since  $\left\{\frac{x}{x+1}\right\}$  is monotonically increasing and bounded above by 1 for positive values of  $x$ , we have,

$$H_{\infty}(n) = 2^{J(n)} \prod_{p^{\nu}||ny_i=1} \prod \frac{p^{2^i}}{1+p^{2^i}} < 2^J. \quad (4.4.a)$$

Also,

$$H_{\infty}(n) = 2^J \prod_{p^{\nu}||ny_i=1} \prod \frac{p^{2^i}}{1+p^{2^i}} \geq 2^J \frac{2}{3} \frac{3}{4} \cdots \frac{J+1}{J+2} = \frac{2^{J+1}}{J+2}. \quad (4.4.b)$$

Hence the lemma follows from (4.4.a) and (4.4.b). □

Note that we have equality on the left in the above lemma if and only if  $n = 2$  or  $n = 2.3$  or  $n = 2^3.3$  or  $n = 2^3.3.5$ . Also  $H_\infty(n) = 1$  if and only if  $n = 1$ .

**Lemma 4.4.4.** *Suppose that there are  $s$  zeroes in the binary representation of  $y$ . Then*

$$\frac{\tau(p^y)}{\tau_\infty(p^y)} \geq \frac{2^s + 1}{2}.$$

*Proof.* Let  $y = \sum_{i=0}^t y_i 2^i$  where  $y_t = 1$ . Since There are  $s$  zeroes in the binary representation of  $y$ ,

$$y \geq 1 + 2 + 2^2 + \cdots + 2^{t-s-1} + 2^t = 2^t + 2^{t-s} - 1.$$

Therefore,

$$\frac{\tau(p^y)}{\tau_\infty(p^y)} = \frac{y+1}{2^{\sum y_i}} \geq \frac{2^t + 2^{t-s}}{2^{t+1-s}} = \frac{2^s + 1}{2}.$$

□

**Theorem 4.4.5.** *For all  $n$ ,  $H^*(n) \leq H_\infty(n) \leq H(n)$ . For  $n > 1$ , equality holds on the left iff  $p^y \parallel n$  implies  $y = 2^\alpha$ , and on the right iff  $p^y \parallel n$  implies  $y = 2^\beta - 1$ .*

*Proof.* Since  $H^*(1) = H_\infty(1) = H(1)$ , so let  $n > 1$ . If  $p^y \parallel n$  implies that  $y = 2^\alpha$ , then

$$H^*(p^{2^\alpha}) = \frac{p^{2^\alpha} 2}{1 + p^{2^\alpha}} = H_\infty(p^{2^\alpha}).$$

Since  $H^*$  and  $H_\infty$  are each multiplicative, it follows that  $H^*(n) = H_\infty(n)$ .

Now suppose  $p^y \parallel n$  and  $y \neq 2^\alpha$  then  $y = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_u}$  where  $\alpha_1 > \alpha_2 > \cdots > \alpha_u \geq 0$  and  $u \geq 2$ . It follows that

$$\begin{aligned}
\frac{H^*(p^y)}{H_\infty(p^y)} &= \frac{2p^y}{1+p^y} \frac{(1+p^{2\alpha_1})(1+p^{2\alpha_2})\cdots(1+p^{2\alpha_u})}{2^u p^y} \\
&< \frac{1}{2^{u-1}} \frac{p^y + p^{y-1} + \cdots + p + 1}{p^y} \\
&< \frac{1}{2^{u-1}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \\
&\leq \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = 1
\end{aligned}$$

Thus,

$$H^*(p^y) < H_\infty(p^y).$$

Since,  $H^*$  and  $H_\infty$  are multiplicative,

$$H^*(n) < H_\infty(n).$$

If  $p^y \parallel n$  implies that  $y = 2^\beta - 1$  then

$$H(p^{2^\beta-1}) = p^{2^\beta-1} \frac{2^\beta(p-1)}{p^{2^\beta}-1} = H_\infty(p^{2^\beta-1}).$$

Since,  $H^*$  and  $H_\infty$  are multiplicative, it follows that

$$H^*(n) < H_\infty(n).$$

Now, suppose that  $p^y \parallel n$  and  $y \neq 2^\beta - 1$ . There are several cases arise .

Suppose first that there are  $s$  zeroes in the binary representation of  $y$  where

$s \geq 2$ . Then since  $p \geq 2$ ,

$$\begin{aligned}
\frac{H(p^y)}{H_\infty(p^y)} &= \frac{\tau(p^y)}{\tau_\infty(p^y)} \frac{\sigma_\infty(p^y)}{\sigma(p^y)} \geq \frac{2^2+1}{2} \frac{(1+p^y)(p-1)}{p^{y+1}-1} \\
&= 1 + \frac{3(p^{y+1}-5p^y+5p-3)}{2(p^{y+1}-1)} > 1.
\end{aligned}$$

Now suppose that  $y$  is odd then by Theorem 4.1.6, we have  $p \parallel_{\infty} p^y$  and so  $p^{y-1} \parallel_{\infty} p^y$ . Therefore, using Lemma 4.4.4 with  $s \geq 1$  we get

$$\begin{aligned} \frac{H(p^y)}{H_{\infty}(p^y)} &= \frac{\tau(p^y)}{\tau_{\infty}(p^y)} \frac{\sigma_{\infty}(p^y)}{\sigma(p^y)} \geq \frac{2^1 + 1}{2} \frac{(1 + p^y)(p - 1)}{p^{y+1} - 1} \\ &= 1 + \frac{p^{y+1} - 3p^{y-1} + 3p - 1}{2(p^{y+1} - 1)} > 1. \end{aligned}$$

Lastly,  $s = 1$  and  $y$  is even. Then the binary representation of  $y$  has the form  $11 \cdots 110$ , so that  $2^{\gamma} - 2$  where  $\gamma \geq 2$ . If  $\gamma = 2$ , then

$$\frac{H(p^y)}{H_{\infty}(p^y)} = \frac{H(p^2)}{H_{\infty}(p^2)} = 1 + \frac{(p - 1)^2}{2(p^2 + p + 1)} > 1.$$

Next if  $\gamma \geq 3$ , then

$$\begin{aligned} \frac{H(p^y)}{H_{\infty}(p^y)} &= \frac{H(p^{2^{\gamma}-2})}{H_{\infty}(p^{2^{\gamma}-2})} = \frac{2^{\gamma} - 1}{2^{\gamma-1}} \frac{(p^{2^{\gamma}} - 1)(p - 1)}{(p^2 - 1)(p^{2^{\gamma}-1})} \\ &> \frac{2^{\gamma} - 1}{2^{\gamma-1}} \frac{p}{p + 1} \geq \frac{4}{3} \left(1 - \frac{1}{2^{\gamma}}\right) \geq \frac{47}{38} > 1. \end{aligned}$$

Therefore in all cases,

$$H(p^y) > H_{\infty}(p^y).$$

Since  $H$  and  $H_{\infty}$  are multiplicative, so it follows that

$$H_{\infty}(n) < H(n).$$

This completes the proof of the theorem.  $\square$

From the Theorem 4.4.5 and Lemma 2.1.5, we can easily obtain the following result .

**Lemma 4.4.6.** *The only square-free infinitary harmonic number is 6.*

**Proposition 4.4.7.** *If  $n$  is odd,  $n \in \mathcal{I}$ ,  $p^y \parallel n$  and  $p = 4m + 3$ , then  $y$  is even.*

The following Proposition gives a criterion for obtaining one IHN from a given IHN.

**Proposition 4.4.8.** *If  $n \in \mathcal{I}$ ,  $\gcd(p, n) = 1$  and  $\sigma_\infty(p^y) \mid \tau_\infty(p^y)H_\infty(p^y)$ , then  $p^y n \in \mathcal{I}$ .*

*Proof.* Since  $H_\infty$  is a multiplicative function,

$$H_\infty(p^y n) = H_\infty(p^y)H_\infty(n) = \frac{p^y \tau_\infty(p^y)}{\sigma_\infty(p^y)} H_\infty(n) = \text{integer}.$$

Hence the lemma. □

Analogous to Theorem 2.3.3, Hagis and Cohen [15] proved the following result.

**Theorem 4.4.9.** *If  $S_c$  is the set of natural numbers  $n$  such that  $H_\infty(n) = c$ , then  $S_c$  is finite ( or empty ) for every real number  $c$ .*

*Proof.* By Lemma (4.4.3) and the fact that  $c \leq \frac{2^{c+1}}{c+2} \forall c > 0$ , we can say that the number of I-components of  $n$  is bounded above by  $c$ . Assume that  $S_c$  is infinite. Then  $S_c$  must contain an infinite subset  $S_{cm}$ , each of whose elements has exactly  $m$  I-components. So there exist an infinite sequence  $n_1, n_2, n_3 \dots$  of distinct integers with the following properties.

(i)  $n_i \in S_{cm}$

(ii)  $n_i = p_1^{2^{\alpha_1}} \cdots p_{s-1}^{2^{\alpha_{s-1}}} p_{i_s}^{2^{\alpha_{i_s}}} \cdots p_{i_m}^{2^{\alpha_{i_m}}} = P \prod_{j=s}^m p_{ij}^{2^{\alpha_{ij}}}$ , where  $p_1^{2^{\alpha_1}} < \cdots < p_{s-1}^{2^{\alpha_{s-1}}} < p_{i_s}^{2^{\alpha_{i_s}}} < p_{i_m}^{2^{\alpha_{i_m}}}$  for  $i = 1, 2, \dots$ . The  $p'_i$ 's are primes which are not necessarily

distinct;  $P$  may be an empty product, but  $s - 1 \neq m$ .

(iii)  $p_{ij}^{2^{\alpha_{ij}}} \rightarrow \infty$  as  $i \rightarrow \infty$  for  $j = s, \dots, m$ .

We know that

$$H_{\infty}(n) = 2^{J(n)} \prod_{p^{\nu} \parallel n} \prod_{y_i=1} \frac{p^{2^i}}{1 + p^{2^i}}. \quad (4.4.c)$$

From (i), (ii), (4.4.c) and the fact that  $H_{\infty}$  is multiplicative, we have

$$\frac{c}{H_{\infty}(P)} = \prod_{j=s}^m H_{\infty}(p_{ij}^{2^{\alpha_{ij}}}) = 2^{m-s+1} \prod_{j=s}^m \frac{p_{ij}^{2^{\alpha_{ij}}}}{1 + p_{ij}^{2^{\alpha_{ij}}}} < 2^{m-s+1}.$$

Therefore, there exists a fixed positive integer  $v$  such that

$$\prod_{j=s}^m H_{\infty}(p_{ij}^{2^{\alpha_{ij}}}) = 2^{m-s+1} - v. \quad (4.4.d)$$

But from (iii), it follows that

$$\lim_{i \rightarrow \infty} H_{\infty}(p_{ij}^{2^{\alpha_{ij}}}) = 2,$$

for  $j = s, \dots, m$ . Therefore, for large  $i$ ,

$$\prod_{j=s}^m H_{\infty}(p_{ij}^{2^{\alpha_{ij}}}) > 2^{m-s+1} - v.$$

This is a contradiction to (4.4.d). Hence,  $S_c$  must be a finite set.  $\square$

Analogous to Theorem 4.3.6 we mention the following result.

**Theorem 4.4.10.** *There exist at most finitely many infinitary harmonic numbers with a specified number of  $I$ -components.*

*Proof.* Consider the elements of  $\mathcal{I}$  with precisely  $k$  components. There are only finitely many integers between  $\frac{2^{k+1}}{k+2}$  and  $2^k$ . If  $l$  is one of these integers then by previous theorem,  $S_l$  must be finite ( or, empty ). Hence the theorem.  $\square$

**Corollary 4.4.11.** *There is atmost a finite number of infinitary perfect numbers with a specified number of I-components.*

*Proof.* Since every IPN is an IHN, the corollary follows from the previous theorem. □

We conclude this chapter with the following list of IHN up to  $10^6$ .

Table 4.2: List of infinitary harmonic numbers up to  $10^6$ .

$n$	$H_\infty(n)$	$n$	$H_\infty(n)$	$n$	$H_\infty(n)$	$n$	$H_\infty(n)$
1	1	9100	10	163800	24	716625	21
6	2	15925	7	172900	19	790398	17
45	3	27300	15	204750	25	791700	29
60	4	36720	16	232470	16	819000	40
90	4	40950	15	245700	27	900900	33
270	6	46494	9	257040	28	929880	24
420	7	54600	20	409500	30	955500	28
630	7	81900	18	464940	18		
2970	11	95550	14	491400	36		
5460	13	136500	25	646425	13		

# Chapter 5

## Harmonic Seeds and List of Harmonic Numbers

In this chapter, we study a special type of harmonic numbers called harmonic seeds. As the name suggests this type of numbers seems to form the nucleus of all harmonic numbers. We also tabulate all harmonic numbers up to  $10^{14}$ .

### 5.1 Harmonic seed

Although, it is not easy to extend the direct search for harmonic numbers, we shall show through the introduction of harmonic seeds, which was introduced by Cohen and Sorli [7].

**Definition 5.1.1.** Let  $n$  be an positive integer then a divisor of  $n$  is said to be a *unitary divisor* of  $n$  and  $n$  is a *unitary multiple* of  $d$  if  $d \mid n$  and  $\gcd(d, \frac{n}{d}) = 1$ . Also  $d$  is called proper unitary divisor if  $d > 1$ .

**Definition 5.1.2.** A harmonic number is called a *harmonic seed* if it does not have a smaller proper unitary divisor which is harmonic.

Every harmonic number is unitary multiple of a certain seed. It is conjectured that such a harmonic seed is unique. In Table 5.4, we marked all the harmonic seeds with an asterisk.

**Remark 5.1.3.** Any harmonic number is either itself a harmonic seed or a unitary multiple of a harmonic seed. For example,  $n = 2^3 \cdot 3^3 \cdot 5^2 \cdot 31$  is harmonic with  $H(n) = 27$ ; the proper unitary divisors of  $n$  are the various product of  $2^3, 3^3, 5^2$  and  $31$ . Since  $2^3 \cdot 5^2 \cdot 31$  is harmonic and does not itself have a proper unitary harmonic divisor. So it is a harmonic seed of  $n$  and in all of these cases the seed is unique.

**Remark 5.1.4.** Since there are no harmonic numbers of the form  $p^a$  and the only harmonic numbers of the form  $p^a q^b$  where  $p \neq q$  are primes, are even perfect numbers. So even perfect numbers are harmonic seeds and besides 1, all other seeds have atleast three distinct prime factors.

Now we are going to discuss an algorithm for determining all harmonic seeds less than  $10^{12}$ . From the second remark we are able to know that even perfect numbers are harmonic seeds and all other harmonic seeds, besides 1, have atleast three prime factors. Now let  $2^a \parallel n$ , where  $a \leq 35$  since  $2^{36} \cdot 3 \cdot 5 > 10^{12}$ . We are now going to construct an even harmonic seed  $n$ , based on specific components  $2^a$ ,  $1 \leq a \leq 35$ , by calculating  $H(n)$  simultaneously with  $n$  until  $H(n)$  is an integer. Take  $a = 13$  then  $H(2^{13}) = \frac{2^{13} \tau(2^{13})}{\sigma(2^{13})} = \frac{2^{14} 7}{3 \cdot 43 \cdot 127}$ . Choosing the largest prime in the denominator, either  $127^b \parallel n$  for

$1 \leq b \leq 3$  since  $2^{13} \cdot 3 \cdot 127^4 > 10^{12}$  or  $p^{126} \mid n$  for some  $p$  so that  $127 \mid \tau(n)$ . Now if  $b = 1$  then  $H(2^{13} \cdot 127) = \frac{2^{28} \cdot 7}{3 \cdot 43}$  so that  $43^c \parallel n$  for  $1 \leq c \leq 3$  since  $2^{13} \cdot 127 \cdot 43 \cdot 11 > 10^{12}$  or  $p^{42} \mid n$ . In the similar way, we then take, in particular  $H(2^{13} \cdot 127 \cdot 43) = \frac{2^7 \cdot 7}{3 \cdot 11}$ ,  $H(2^{13} \cdot 127 \cdot 43 \cdot 11) = \frac{2^6 \cdot 7}{3^2}$ . In this stage we must have  $3^d \parallel n$  for  $1 \leq d \leq 6$ , or  $p^2 \mid n$  for two primes  $p$ , or  $p^8 \mid n$  for some prime  $p$ . Considering all possibilities we get  $H(2^{13} \cdot 127 \cdot 43 \cdot 11 \cdot 3^3) = \frac{2^5 \cdot 3 \cdot 7}{5}$ ,  $H(2^{13} \cdot 127 \cdot 43 \cdot 11 \cdot 3^3 \cdot 5) = 2^5 \cdot 7$  and so  $2^{13} \cdot 127 \cdot 43 \cdot 11 \cdot 3^3 \cdot 5$  is a harmonic seed. Odd harmonic seeds upto  $10^{12}$  were sought in the same way.

**Theorem 5.1.5.** *Suppose  $n$  and  $nq_1q_2q_3 \cdots q_t$  are harmonic numbers, where  $q_1 < q_2 < \cdots < q_t$  are primes not dividing  $n$ . Then  $nq_1$  is harmonic, except when  $t \geq 2$  and  $q_1q_2 = 6$ , in which case  $nq_1q_2$  is harmonic.*

*Proof.* Assume that  $t \geq 2$ . First suppose that  $q_1 \geq 3$ . Since  $nq_1q_2q_3 \cdots q_t$  is harmonic number and  $\mathcal{H}$  is multiplicative,

$$\begin{aligned} H(nq_1q_2q_3 \cdots q_t) &= H(n)H(q_1) \cdots H(q_t) \\ &= H(n) \frac{2q_1}{q_1+1} \frac{2q_2}{q_2+1} \cdots \frac{2q_t}{q_t+1} = h, \text{ say.} \end{aligned}$$

Then

$$H(n)q_1q_2q_3 \cdots q_t = h \frac{q_1+1}{2} \frac{q_2+1}{2} \cdots \frac{q_t+1}{2}.$$

Since

$$\frac{q_1+1}{2} < \frac{q_2+1}{2} < \cdots < \frac{q_t+1}{2} < q_t,$$

so  $q_t \mid h$  and then

$$H(nq_1q_2q_3 \cdots q_{t-1}) = \frac{h}{q_t} \frac{q_t+1}{2}, \text{ an integer.}$$

Applying the same argument to the harmonic number  $nq_1q_2q_3 \cdots q_{t-1}$  and repeating as necessary, we get  $H(nq_t) = \text{integer}$ . i.e.,  $nq_1 \in \mathcal{H}$ . For the next part we take  $q_1 = 2$  then  $q_2 = 3$  and  $n$  is odd. Now,

$$H(nq_1q_2) = H(n)H(q_1)H(q_2) = 2H(n),$$

which is an integer since  $n$  is harmonic. Hence  $nq_1q_2$  is harmonic.  $\square$

From the above theorem we get the following remark.

**Remark 5.1.6.** If  $n$  and  $nq_1$  are harmonic numbers with  $q_1 > 2$ ,  $q_1 \nmid n$  then  $\frac{q_1 + 1}{2} \mid H(n)$  i.e.,  $q_1 \leq 2H(n) - 1$ .

**PROBLEM:** Are there infinitely many harmonic seeds  $n$  with  $w(n) = 3$ ?  
If not, find all such  $n$ . Does an odd one exist?

**PROBLEM:** Does every harmonic number have a unique harmonic seed?

## 5.2 List of harmonic numbers

We first make a list of all harmonic numbers of the form  $2^a m$  with  $m$  square-free and  $1 \leq a \leq 11$ .

**Theorem 5.2.1.** *The only harmonic numbers of the form  $2^a m$ , where  $m$  is odd and squarefree and  $1 \leq a \leq 11$ , are those listed in the following tables:*

Table 5.1:

$a$	All $2^a m \in \mathcal{H}$ with $m$ odd and squarefree
1	$2 \cdot 3 = 6$
2	$2^2 \cdot 7 = 28, 2^2 \cdot 5 \cdot 7 = 140$
3	none
4	$2^4 \cdot 31 = 496$
5	$2^5 \cdot 3 \cdot 7 = 672$
6	$2^6 \cdot 127 = 8128, 2^6 \cdot 13 \cdot 127 = 105664$
7	none
8	see Table 5.2
9-11	none

Table 5.2:

All $2^8 m \in \mathcal{H}$ with $m$ odd and squarefree	$H(n)$
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 37 \cdot 43 \cdot 73 = 15007087898880$	989
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 349002044160$	506
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 = 652482082560$	516
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 \cdot 257 = 167687895217920$	1028
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 \cdot 1031 = 672709027119360$	1031
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 713178090240$	517
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 1033 = 736712967217920$	1033

contd...

All $2^8 m \in \mathcal{H}$ with $m$ odd and squarefree	$H(n)$
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 = 15174001920$	264
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 131 = 1987794251520$	524
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 131 \cdot 523 = 1039616393544960$	1046
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 263 = 3990762504960$	526
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 \cdot 263 \cdot 1051 = 4194291392712960$	1051
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 44345330883840$	1037
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 726972637440$	527
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 67 \cdot 73 = 1571198926080$	536
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 = 23450730240$	272
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 271 = 6355147895040$	542
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 271 \cdot 541 = 3438135011216640$	1082
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 31727458560$	276
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 = 4346661822720$	548
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 \cdot 547 = 2377624017027840$	1094
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 137 \cdot 547 \cdot 1093 = 2598743050611429120$	2186
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 2608548875520$	549
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 1097 = 2861578116445440$	1097
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 42763096320$	279
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 \cdot 557 = 23819044650240$	557
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 64834371840$	282
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 281 = 18218458487040$	562
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 281 \cdot 1123 = 20459328880945920$	1123

contd...

All $2^8 m \in \mathcal{H}$ with $m$ odd and squarefree	$H(n)$
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 \cdot 563 = 36501751345920$	563
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 71 \cdot 73 = 97941285120$	284
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 71 \cdot 73 \cdot 283 = 27717383688960$	566
$2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 = 1379454720$	144
$2^8 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 9564679210240$	671
$2^8 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 156798019840$	341
$2^8 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 43 \cdot 73 = 217494027520$	344
$2^8 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 37 \cdot 73 = 5058000640$	176
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 = 10575819520$	184
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 367 = 3881325763840$	367
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 367 \cdot 733 = 2845011784894720$	733
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 = 869516291840$	366
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73 = 14254365440$	186
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 47 \cdot 73 = 21611457280$	188
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 = 459818240$	96
$2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 \cdot 191 = 87825283840$	191

*Proof.* For each value of  $a$  in turn, we put  $n = 2^a m$ , where  $m$  is odd and squarefree. Also let  $n$  is harmonic. Now if  $a = 1$  then  $n = 6$ . Next if  $a = 2$  then  $n = 2^2 m$ . Now let  $q' = \frac{\sigma(q)}{\tau(q)}$  where  $q$  is a prime and  $Q_{j,s} = \prod_{i=j}^s \frac{q_i}{q'_i}$  where  $q'_i$ 's are all distinct primes and  $1 \leq j \leq s$ . Then clearly  $Q_{j,s} = H(q_j q_{j+1} \cdots q_s)$ . Now  $\sigma(2^2) = 7$  and so  $7 \mid n\tau(n)$  but  $7 \nmid \tau(n)$  and so  $7 \parallel m$ . Therefore  $7 \parallel n$ . We can write  $n = 2^2 \cdot 7k$  where  $k$  is an integer such that either  $k = 1$  or  $k$  is squarefree and  $\gcd(k, 14) = 1$ . Now if  $k = 1$  then  $n = 2^2 \cdot 7 = 28$  which is a harmonic number. If  $k \neq 1$  then let  $k = q_1 q_2 \cdots q_s$  where  $q_i$ 's are distinct primes with  $q_1 < q_2 < \cdots < q_s$ . Then  $H(n) = 3H(k) = 3 \frac{q_1 q_2 \cdots q_s}{q'_1 q'_2 \cdots q'_s}$ . Since  $H(n)$  is an integer therefore the denominator can be fully factored into the numerator to produce an integer. It follows that  $q'_1 = 3$  i.e.,  $q_1 = 5$ . Now if  $s = 1$  then  $n = 2^2 \cdot 5 \cdot 7 = 140$  which is a harmonic number. If  $s > 1$  then  $H(n) = 5 \frac{q_2 \cdots q_s}{q'_2 \cdots q'_s}$ . Since  $q'_1 < q_1 < q_s, \forall j \geq 1$ . If we take  $q'_2 = 5$  then  $q_2 = 9$  which is not possible. So,  $s > 1$  is not possible. Hence if  $a = 2$  then the only harmonic numbers of the form  $2^a \cdot m$  are 28 and 140. Now if  $a = 3, 7, 9$  then  $\sigma(2^3) = 3 \cdot 5, \sigma(2^7) = 3 \cdot 5 \cdot 17, \sigma(2^9) = 3 \cdot 11 \cdot 31$ . We know that if  $a$  is a positive integer such that for some prime  $p, p \equiv 2 \pmod{3}, 3p \mid \sigma(2^a)$  and  $\gcd(3p, a+1) = 1$ . Also if  $m$  is an odd integer such that  $2^a m$  is harmonic then  $m$  can not be squarefree, by Lemma 2.1.10. Using this result we can easily show that if  $m$  is squarefree then there is no harmonic number of the form  $2^a m$  where  $a = 3, 7, 9$ . Next let  $a = 4$  then  $n = 2^4 m$ . Then  $\sigma(2^4) = 31$  and  $31 \parallel n$ . We can write  $n = 2^4 \cdot 31 \cdot k$  where  $k$  is an odd integer such that  $\gcd(k, 31) = 1$  and either  $k = 1$  or  $k$  is squarefree. If  $k = 1$  then  $n = 2^4 \cdot 31 = 496$ , which is a harmonic number. If  $k$  is squarefree  $> 1$ , then let



$k = q_1 q_2 \cdots q_s$  where  $q_i$ 's are distinct primes such that  $q_1 < q_2 < \cdots < q_s$ . In this case  $H(n) = 5Q_{1,s} = 5 \frac{q_1 q_2 \cdots q_s}{q'_1 q'_2 \cdots q'_s}$ . To get  $H(n)$  integer let  $q'_1 = 5$  then  $q_1 = 9$ , impossible. So  $s \geq 1$  is not possible. Similarly we can show the theorem for  $a = 5, 6, 10$  and  $11$ . Now consider the case  $a = 8$  and  $\sigma(2^8) = 7.73$  then we can show that  $7 \parallel n$  and  $73 \parallel n$ . Let  $n = 2^8 \cdot 7.73 \cdot k$  where  $k = q_1 q_2 \cdots q_s$  where  $q_i$ 's are distinct primes not  $2, 7, 73$ . Now  $H(2^8 \cdot 7.73) = \frac{2^6 \cdot 3^2}{37}$ , so  $k \neq 1$ . Hence  $k$  is squarefree. Now  $H(n) = \frac{2^6 \cdot 3^2}{37} \frac{q_1 q_2 \cdots q_s}{q'_1 q'_2 \cdots q'_s}$ . Let  $q_1 = 37$  then  $q'_1 = 19$  and  $H(n) = \frac{2^6 \cdot 3^2}{19} \frac{q_2 \cdots q_s}{q'_2 \cdots q'_s}$ . If  $q_2 = 19$  then  $q'_2 = 10$  then  $H(n) = \frac{2^5 \cdot 3^2}{5} \frac{q_3 \cdots q_s}{q'_3 \cdots q'_s}$ . Put  $q_3 = 5$  then  $q'_3 = 3$  and  $H(n) = 2^5 \cdot 3^2 \frac{q_4 \cdots q_s}{q'_4 \cdots q'_s}$ . If  $s = 3$  then  $n = 2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$ , which is a harmonic seed and all other harmonic numbers arise from this "seed" and depend on finding a value for  $q'_i$  which divides the numerical part of the current numerator and for which  $q_i$  is a prime different from those already encountered. There are the following possibilities for  $q'_4$ :  $2, 2^4, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3$  and  $2^5 \cdot 3$  ( these are the only acceptable divisors of  $2^5 \cdot 3$ )

If  $q'_4 = 2$  then  $q_4 = 3$  and  $H(n) = 2^4 \cdot 3^2 \cdot Q_{5,s}$ . If  $s = 4$  then we found the solution  $n = 2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$  otherwise , there are the following possibilities for  $q'_5$ :  $2^4, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3, 2^2 \cdot 3^2$  and  $3^2$  ( these are the acceptable divisors of  $2^4 \cdot 3^2$ . ) Now if  $q'_5 = 2^4$  then  $q_5 = 31$  and then  $H(n) = 3^2 \cdot 31 \cdot Q_{6,s}$ . If  $s = 5$  then  $n = 2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 37 \cdot 73$ , otherwise we must consider possible values of  $q'_6$  and so on. Then we must consider the other possible values of  $q_4$  and then other possible values of  $q'_5$ . The proof continues until all possibilities have been considered. □

G. L. Cohen [5] mentioned the following result for a harmonic number  $n > 2 \cdot 10^9$ .

**Lemma 5.2.2.** *Suppose  $n$  is a harmonic number, satisfying  $n > 2 \cdot 10^9$  and  $H(n) \leq 13$ . Then  $n$  has a prime factor exceeding 20.*

Goto and Okeya [13] have listed all harmonic numbers  $n$  with the property  $H(n)^{4.55} > n$  as follows

**Theorem 5.2.3.** *Let  $n$  be harmonic and  $H(n)^{4.55} > n$ . Then  $n$  is one of the first 1643 numbers in the list which is available on the webpage [http://www.mn.noda.tus.ac.jp/u/tg/files/list 5](http://www.mn.noda.tus.ac.jp/u/tg/files/list%205).*

They also listed all harmonic numbers with the property  $H(n) < 1200$ .

**Theorem 5.2.4.** *Let  $n$  be harmonic and  $H(n) \leq 1200$ . Then  $n$  is one of the first 1376 numbers in the list which is available on the webpage [http://www.mn.noda.tus.ac.jp/u/tg/files/list 3](http://www.mn.noda.tus.ac.jp/u/tg/files/list%203).*

For the proof of next theorem, we need a tabulated improvement of last theorem, for harmonic numbers  $n$  with small values of  $w(n)$ . Let  $P_i$  denote the  $i$ th prime, so that  $P_1 = 2, P_2 = 3, \dots$ . Then  $p_i \geq P_i$  for each  $i$ , and our improvement is based on exact calculations with  $\frac{P_i}{P_i - 1}$ , rather than  $\frac{i + 1}{i}$  as in the above proof. For example, if  $2 \parallel n$  and  $w(n) = 3$ , then we get  $S(n) = S(2p_2^{a_2} p_3^{a_3}) < S(2 \cdot 3 \cdot 5)$ , so that

$$H(n) = \frac{\tau(n)}{S(n)} > \frac{2^2 \cdot 3}{\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{5}{4}} = \frac{64}{15}.$$

In this way, we have constructed the column of lower bounds for  $H(n)$  headed  $2 \parallel n$  in Table 5.3. For each value of  $w(n)$ ,  $H(n)$  is not less than the corresponding entry in this column. The other columns treat the special cases in which  $2^a \parallel n$  for  $2 \leq a \leq 11$ ,  $2^{12} \mid n$  and  $2 \nmid n$  respectively.

Suppose  $2^2 \parallel n$ ,  $w(n) \geq 3$  and  $n \neq 140$ . By Theorem 5.2.1,  $\tau(n) \geq 2^{w(n)-2}3^2$ . Also,  $S(n) < S(2^2) \prod_{i=2}^{w(n)} S(\bar{P}_i)$ . This is used in the calculations for the third column of the above table, and in this fashion this table may be completed. We notice from the table that  $w(n) \leq 4$  if  $H(n) \leq 13$ . Not having to consider  $w(n) \geq 5$  was the main reason for seeking only those  $n \in \mathcal{H}$  with  $H(n) \leq 13$  in next theorem. The number of columns in the above table was determined by continuing until it could be asserted that if  $n$  is harmonic with  $H(n) \leq 13$ , and  $n$  is even, then  $2^6 \nmid n$  ( except if  $n = 1056664$  ).

**Theorem 5.2.5.** *Let  $n$  be harmonic and  $H(n) \leq 13$ . Then  $n$  is one of the following 13 numbers:*

1      6            28      140 496 270 8128 672 1638 6200  
2970 105664 33550336.

*In particular, the numbers  $n$  with  $H(n) = 4$  or 12 do not exist.*

*Proof.* By Theorem 2.3.4, we have, for all  $n$ ,

$$H(n) > \frac{2^{w(n)+1}}{w(n)+1} \tag{5.2.a}$$

with the following exceptions  $n = p, 2p, 6p$  ( $p \neq 3$ ),  $n = 30p$  ( $7 \leq p \leq 23$ ) and  $n = 1, 15, 21, 70$ . As a function of  $w(n)$ , the right-hand side of (5.2.a) is increasing function, and equals to 4 when  $w(n) = 3$ . Also,  $H(n) \geq 5$  when

Table 5.3: lower bounds for  $H(n)$  if  $n \in \mathcal{H}$

$\omega(n)$	$2 \parallel n$	$2^2 \parallel n$	$2^3 \parallel n$	$2^4 \parallel n$	$2^5 \parallel n$	$2^6 \parallel n$
3	5	$a = 6$	7	10	$b = 11$	$c = 14$
4	8	10	12	16	17	23
5	14	18	22	28	31	39
6	25	32	40	50	57	70
7	47	60	74	92	106	129
$\geq 8$	88	113	141	174	201	243
$n \neq 140$ for $a$ , $n \neq 672$ for $b$ , and $n \neq 105664$ for $c$						

$\omega(n)$	$2^7 \parallel n$	$2^8 \parallel n$	$2^9 \parallel n$	$2^{10} \parallel n$	$2^{11} \parallel n$	$2^{12} \mid n$	$2 \nmid n$
3	16	23	27	32	23	14	6
4	25	31	36	42	36	24	10
5	42	49	57	67	61	44	19
6	76	89	97	115	111	80	35
7	140	$d = 164$	179	208	209	151	66
$\geq 8$	264	264	336	384	395	285	126
$n \neq 1379454720$ for $d$							

$w(n) \geq 3$ . Now by Lemma 2.1.1, if  $H(n) \leq 4$ , we must have  $n = 1$ , or  $w(n) = 2$ . Then by Lemma 2.2.7 and corollary of Lemma 2.1.3,  $n = 6$  or  $28$ . In particular, there is no solution of the equation  $H(n) = 4$ . Suppose  $H(n) = 5$ , so  $5\sigma(n) = n\tau(n)$ . Two solutions  $n = 140$  and  $n = 496$  are shown from table 5.4. For any other solution, from table 5.4, we have  $w(n) = 3$  and  $2 \parallel n$  and from Lemma 2.3.1, we have  $5 \mid n$ . By Lemma 5.2.2, the remaining prime factor exceeds 20. Then  $\tau(n) = 5S(n) < 5S(2.\bar{5}2\bar{3}) < 9.9$ , which is a contradiction to Lemma (2.1.5). The proof continues in this manner, considering in turn each possible value of  $H(n)$ .

Finally, suppose  $H(n) = 12$ , so that  $12\sigma(n) = n\tau(n)$ . Now, using Lemma 2.1.7, we deduce that we can not have  $2 \parallel n$  or  $3 \parallel n$ . and from Lemma 2.1.8,  $\tau(n) \neq 24$  or  $36$ . Of course when  $\tau(n) \neq 12$ , we note that there are no solution in table 5.4. Now suppose  $n$  is odd. Then we can not have  $w(n) = 4$ , since this implies that  $32 \leq \tau(n) = 12S(n) < 12S(\bar{3}.\bar{5}.\bar{7}.\bar{2}\bar{3}) < 27.5$ . So, if  $n$  is odd, then  $w(n) = 3$  and  $\tau(n) = 12S(n) < S(\bar{3}.\bar{5}.\bar{2}\bar{3}) < 23.6$ . Note that,  $4 \parallel \tau(n)$ , since otherwise  $2 \mid n$  then  $3 \nmid \tau(n)$ . Thus  $3^2 \mid n$ , but we cannot have  $3^2 \parallel n$ , else  $3 \mid \tau(n)$ , or  $3^3 \parallel n$ , by Theorem 2.2.1. The only possibility is then  $n = 3^4.pq$ , where  $p, q$  are distinct odd primes exceeding 3; then we have  $\sigma(3^4) = 11^2 \mid n$ , a contradiction. Suppose  $n$  is even and  $w(n) = 4$ . From Table 5.3,  $2^4 \nmid n$ , so  $\tau(n) < 12S(2^3.\bar{3}.\bar{5}.\bar{2}\bar{3}) < 44.2$ . The only possible exponents for the four prime factors of  $n$  are arrangements of  $3, 1, 1, 1$  or  $4, 1, 1, 1$ . Since  $2^2 \mid n$ , these arrangements are impossible, by Theorem 5.2.1. Hence  $w(n) = 3$ . Again from Table 5.3,  $2^6 \nmid n$ , so that  $\tau(n) < 12S(2^5.\bar{3}.\bar{2}\bar{3}) < 37.1$ . Suppose first that  $3 \mid n$ . Then, since  $2^2.3^2 \mid n$ ,  $n$  must equal to one of the following  $2^3.3^3.p, 2^4.3^2.p, 2^2.3^2.p^2, 2^2.3^4.p$

or  $2^2.3^2.p$ . But all of which are easily eliminated, whatever the prime  $p$ . Then we may now suppose that  $2^2 \mid n$ ,  $3 \nmid n$  and  $w(n) = 3$ . Then  $3 \mid \tau(n)$  and  $\tau(n) < 12S(2^5.5.2\bar{3}) < 30.9$ . Now by Theorem 5.2.1, it follows that we cannot have  $2^5 \parallel n$  or  $2^3 \parallel n$ . If  $2^4 \parallel n$ , then  $\sigma(2^4) = 31 \mid n$ , and either  $n = 2^4.31.p^2$  or  $n = 2^4.31^2.p$ , but they are not harmonic for any  $p$ . If  $2^2 \parallel n$ , then  $\sigma(2^2) = 7 \mid n$ , since  $\sigma(7^4) = 2801 \parallel n$ , but  $2^2.7^4.2801$  is not harmonic. Again we cannot have  $7^2 \parallel n$ , since then  $\frac{1}{3}\sigma(7^2) = 19 \mid n$ , a contradiction to Lemma 5.2.2, also we can not have  $7 \parallel n$  since then  $2^2.7.p^a$  for  $a = 2$  or  $4$  and  $\gcd(14, p) = 1$ , but substituting this into  $12\sigma(n) = n\tau(n)$  we get  $4 \mid (a + 1)$ , a contradiction. Hence the theorem.  $\square$

**Theorem 5.2.6.** *Let  $n$  be harmonic number less than  $10^{14}$ . Then  $n$  is one of the 937 numbers given in Table 5.4.*

*Proof.* Let,

$$H = \{n \in N : H(n) \in N\}.$$

$$H_1 = \{n \in H : n < 10^{14}\}.$$

$$H_2 = \{n \in H : H(n)^{4.55} > n\}.$$

$$H_3 = \{n \in H : H(n) \leq 1200\}.$$

Clearly,  $H_1 \subset H_2 \cup H_3$ . Indeed, suppose that  $n \in H_1$ . If  $n \notin H_3$ , then

$$H(n)^{4.55} > 1200^{4.55} > 10^{14} > n,$$

and hence  $n \in H_2$ . Now by Theorem 5.2.3 and Theorem 5.2.4, the sets  $H_2$  and  $H_3$  are known. Therefore we can give the set  $H_1$ . Hence the theorem.  $\square$

**PROBLEM:** Are there infinitely many harmonic numbers? How about harmonic seeds?

We conclude this chapter and also the dissertation with the following table which lists all the harmonic numbers up to  $10^{14}$ . In this table, harmonic seeds are marked with asterisks.

Table 5.4: List of harmonic numbers up to  $10^{14}$ .

$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
1	1	167400	27	*4713984	48	44660070	82
*6	2	*173600	25	4754880	45	*45532800	96
*28	3	237510	29	5772200	49	46683000	114
140	5	242060	26	*6051500	50	50401728	53
*270	6	332640	44	*8506400	49	*52141320	108
*496	5	360360	44	8872200	53	56511000	115
*672	8	539400	29	11981970	77	69266400	105
*1638	9	695520	46	14303520	86	71253000	116
2970	11	726180	39	15495480	86	75038600	91
*6200	10	753480	46	16166592	51	80832960	85
*8128	7	950976	27	*17428320	96	*81695250	105
8190	15	*1089270	42	18154500	75	90409410	83
18600	15	1421280	47	*23088800	70	108421632	92
*18620	14	1539720	47	23569920	80	110583200	91
27846	17	2178540	54	23963940	99	*115048440	78
*30240	24	*2229500	35	27027000	110	115462620	106
*32760	24	2290260	41	*29410290	81	137891520	87
55860	21	*2457000	60	32997888	84	*142990848	120
105664	13	2845800	51	*33550336	13	144963000	118
117800	19	4358600	37	37035180	102	163390500	135

contd...

$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
164989440	140	540277920	186	1558745370	159
191711520	176	559903400	97	*1630964808	99
221557248	94	623397600	189	1632825792	101
233103780	107	*644271264	117	1727271000	222
*255428096	88	675347400	189	1862023680	158
287425800	101	714954240	200	*1867650048	128
300154400	130	758951424	161	2008725600	203
301953024	27	766284288	132	2140041600	188
318177800	73	819131040	188	2144862720	260
318729600	168	825120800	97	2369162250	203
*326781000	168	886402440	204	2481357060	201
400851360	184	900463200	195	2701389600	270
407386980	187	995248800	189	2705020500	149
423184320	89	1047254400	184	2716826112	228
428972544	156	1162161000	215	2738824704	166
447828480	152	1199250360	207	2763489960	212
*459818240	96	1265532840	143	2777638500	255
*481572000	168	*1307124000	240	2839922400	205
499974930	153	1352913408	164	*2876211000	150
500860800	176	1379454720	144	2945943000	218
513513000	209	*1381161600	240	3134799360	266
526480500	145	1509765120	45	3209343200	139

contd...

$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
3221356320	195	7330780800	322	12941019000	229
3288789504	230	7515963000	322	13067913600	328
3328809120	191	8104168800	351	13073550336	224
3349505250	205	8154824040	165	13398021000	328
3506025600	308	8243595360	344	13581986600	181
3594591000	308	*8410907232	171	13584130560	380
3702033720	213	8436460032	236	13660770240	169
3740553180	202	*8589869056	17	*14182439040	384
3831421440	220	*8628633000	195	14254365440	186
4143484800	312	8659696500	265	14378364000	440
4146734592	232	8696764800	191	14541754500	267
4720896180	197	*8698459616	121	14980291200	329
4738324500	261	9866368512	299	15174001920	264
5058000640	176	*10200236032	96	15192777600	440
5133201408	51	10575819520	184	15358707000	329
5275179000	226	10597041000	227	16003510272	53
5297292000	308	10597759200	357	16569653760	296
5510647296	167	10952611488	221	16919229600	357
5579121240	214	10983408128	172	17624538624	253
5943057120	341	11076156000	322	18999981000	407
6720569856	235	11296276992	237	*19017782784	336
7279591410	163	11480905800	357	*19209881600	256

contd...

$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
19744452000	328	31638321000	275	45578332800	572
20015559200	181	31727458560	276	45923623200	510
20387256120	391	31766716800	460	50497467930	303
21537014400	344	32950224384	258	51001180160	160
21611457280	188	32956953120	366	52748186400	371
21943595520	392	33040072800	371	53227843200	334
22633884000	329	34174812672	239	53621568000	500
22933532160	278	34482792960	396	54572427000	334
23450730240	272	*35032757760	392	54648009000	285
23855232960	173	35793412200	371	56481384960	395
24362612820	211	37906596000	464	*57575890944	192
25559301600	369	39970476000	332	57629644800	384
25666007040	85	40053686400	464	*57648181500	273
26113432800	377	40520844000	465	57897151488	248
26242070400	456	40752391680	494	59388963480	402
26454556800	332	40805200800	369	61434828000	470
27122823000	332	42054536160	285	62487000576	437
27689243400	369	42763096320	279	64834371840	282
27726401736	187	43783188480	87	64914595200	470
29715285600	495	*43861478400	264	*66433720320	224
30063852000	460	43952044500	269	67622100480	302
30600708096	144	44184172032	309	*71271827200	270

contd...

$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
*73924348400	125	*109585986048	324	159248314400	193
77120316000	472	*110886522600	155	159381986400	531
*77924700000	375	112202596352	176	164297299320	411
78340298400	522	115987576320	518	164751121920	430
80422524000	334	*123014892000	484	169696449000	295
80533908000	375	*124406100000	375	169956154368	416
80551516500	493	126090783000	438	*183694492800	672
*81417705600	484	133410461184	311	194743785600	611
81488534400	472	134369095680	89	201532767744	263
83410119000	290	*137438691328	19	*206166804480	384
*84418425000	375	137770869600	663	213815481600	405
87825283840	191	142275893760	398	217494027520	344
89526646440	404	142985422944	323	220524885504	326
93419333280	377	143173648800	530	220920860160	515
95088913920	560	147112449120	367	*221908282624	171
95300150400	598	150115204512	233	227783556000	602
97941285120	284	150759100800	602	234605428736	184
100383241728	262	151955343540	373	236489897160	319
100522566144	444	153003540480	240	237191556096	254
103262796000	474	154567413000	602	240423674400	534
108061356200	193	*156473635500	390	250230357000	377
109111766400	474	156798019840	341	*271309925250	405

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$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
280541488500	505	417624936960	436	583096381560	422
285266741760	728	426778934400	618	586207480320	748
287879454720	320	*428440390560	546	603567619200	874
288662774400	836	428555439000	298	616719527424	454
289048687200	535	429520946400	689	618269652000	860
292337717760	314	434508127200	697	626112396000	479
307001350656	452	437409004032	644	633926092800	704
307030348800	462	439655610240	744	652482082560	516
311203567584	333	*443622427776	352	653289436800	860
312402636000	478	465036042240	392	661576406400	479
321300067176	197	*469420906500	507	666574634880	752
326196097920	736	470717137800	697	*677701763200	340
330097622400	478	479411093504	188	693688413600	697
336607789056	264	482476262400	484	703816286208	276
341519256000	325	483548738400	537	704575228896	405
349002044160	506	494122282290	317	713178090240	517
350280184800	389	502612830720	740	726673802400	538
362526484320	671	505159855200	935	726972637440	527
384342364800	367	*513480135168	648	753132796416	458
403031236608	336	518453342208	101	765181053000	443
405280060416	434	520212037632	272	779729094144	656
410240742912	453	547929930240	540	783990099200	495

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$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
793104238080	759	1135890756000	869	1447428787200	600
819730138500	519	*1144136294400	350	*1480003190400	529
*830350521000	756	1159571485800	707	1482760097280	774
861743282400	957	1161528261600	409	1507838492160	962
863638364160	416	1175104476000	899	*1517389419000	529
869516291840	366	*1179832600464	217	*1542738616320	352
888875820360	327	1200229430400	869	*1553357978368	252
888988066400	277	1209584724480	584	1556017837920	555
893835790848	658	1211621062400	510	1567241676000	872
906550977024	331	1219581548640	551	1571198926080	536
*945884459520	756	1233377308800	893	1578475971072	664
950432517216	339	1253107608480	389	1584792261000	551
970956604800	888	1288623772800	622	*1599300612000	648
995024181060	401	1324245491712	368	1626268644000	614
*997978703400	279	1325481830400	736	1656012758400	872
1018809792000	950	*1330464844800	660	1681994012160	439
*1058501001600	648	1331785072800	986	1683038945280	440
1070373679200	707	1369947647250	409	1708842189600	707
1076349859200	614	1377031864320	432	1721209990500	715
*1085239701000	648	1386998613000	803	1773515487744	471
1103539437000	614	1413817996500	509	1784852619264	372
1109541413120	285	1438233280512	282	1801169758080	762

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$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
1862961762816	612	2183877423000	652	2709493768800	1003
1886043571200	516	*2198278051200	1080	*2827553208480	686
1888271330400	699	*2236152828000	529	2915401724928	446
1919938116096	463	2259816300000	725	2965353955200	904
1924339334400	729	2267834849280	704	3076882754400	1005
1948245082112	191	2312019021312	851	3105356994432	616
1959868310400	1118	2335483332000	725	3175969724928	668
1987794251520	524	2363575441500	533	3218345676000	652
2015156183040	560	2439654963200	508	3238966130400	981
2020639420800	1232	2448134325000	725	*3321402084000	1080
2021976333000	555	2448278300160	781	3356538237000	389
2033105289600	510	2468667064500	521	3377333836800	847
2051203714560	755	2471771484000	915	3398177502720	776
2059445329920	434	2520477679104	621	3448576989000	545
2061489484800	517	2567400675840	1080	3500961340800	946
2066882988800	522	2608548875520	549	3519081431040	460
2070303429600	729	2627456832000	980	3522876144480	675
2096328767456	241	2644660418400	979	3531726240768	488
*2112394079250	585	2677752441000	735	3607776900000	725
2128528765440	776	2706066874368	376	*3622293071600	245
2130069916800	652	2708593305600	752	3634863187200	765
2172650274816	284	2708845856640	764	3772440804608	323

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$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
3777406841600	530	4713692054400	643	6045468549120	728
3881325763840	367	4741836503040	736	*6073712944992	693
*3946161492000	735	4752162586080	565	6175225017000	565
3962552630400	906	4824711643136	344	6200648966400	783
3990762504960	526	4832764209000	643	6312101796000	878
3991394534400	858	4903097162600	361	6343192620800	534
4029093232640	316	4959751305600	1296	6352588408320	554
4205037804800	531	5085231579136	37	6355147895040	542
4224973334400	1288	*5111051997870	366	6669629366400	878
4240965560832	669	5148385482240	758	6734495875072	49
*4314435969536	385	5268640785408	806	6764077878600	305
4346661822720	548	5289640356000	946	6793110213120	788
*4409499089268	147	5290460648928	629	*6844445080704	684
4437102673920	464	5431874152320	766	6884622108000	916
4517245877760	786	5469709639680	608	7121968308000	643
4537735429500	754	5681022328800	701	7131668544000	1400
4603679570880	337	5745853670400	524	7191166402560	470
4612268729250	765	5808057260544	636	7274578147200	916
4638285943200	1010	5853911263200	985	7318964889600	762
4660073935104	378	5914045683000	457	*7322605472000	672
4694568278400	1133	*5914410203520	936	7338147328512	876
4712844296160	1001	5956949980800	908	7512024199680	1106

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$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
7531474204800	1312	9564679210240	671	11610780300000	745
7574491607040	173	9689839810560	752	11643511017600	1188
7626085510400	535	*9831938337200	350	11725700507136	642
7741979148288	506	10112079035520	1426	11810043108864	1242
7761092320800	1014	10132001510400	1050	11937636711000	1188
7766789891840	420	10256659997220	421	11977778891232	765
7780605009408	639	*10297226649600	630	11999552292000	745
7867987832250	783	10341947847528	373	12087279697920	474
*8449576317000	936	10410668674560	1026	12412499299200	1634
8467093071360	1022	10434320851500	543	*12452007204000	936
8468207666688	514	*10461217539500	305	12493968334848	651
8633641161600	1316	10670692032000	995	12578345325000	745
8729162297856	1224	10680522652800	1628	12588244300800	902
8756458300800	662	*10711009764000	1050	12602388395520	1035
8867577438720	785	10799170314240	616	12757657068800	537
8924263096320	620	*10881843388416	648	12876333500800	646
8977654413000	662	10996995170304	382	13202304998400	903
9027208888320	472	11007262156800	764	13217359034880	1112
9068974548480	789	11332220524800	795	13230227556000	662
9231944494500	767	*11484718245000	1125	13327831686400	935
9269718441984	644	11535568819200	526	13552871623200	1038
9314808814080	1020	*11567890545120	1053	*13661860101120	1056

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$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
13914857829600	1313	17505483899904	824	*21733758429600	434
14115958857000	1428	17550753948000	926	21738589593600	665
14379426038250	795	17566056012960	1066	21755342568960	1449
14474134929408	516	17592306732000	1188	21967816416000	1008
14635113292800	1210	18218458487040	562	22047495446340	245
*14747907505800	434	18297947606400	918	22051566231552	383
14814719631360	446	18449074917000	1224	22072153958400	766
14873771827200	650	18536508900000	745	22332001910400	1702
15007087898880	989	18544856803200	926	*22385029489560	198
15147350507520	792	18942468120576	658	22717860433632	657
15246642902400	1328	19029577862400	801	22735712876800	957
15337823806032	403	19075764394368	688	22742476922880	632
*15462510336000	960	19098061983000	1449	23300369675520	630
15820566085632	517	19172121516800	538	23375124208800	1018
15889967976960	1044	19621667049600	964	23409541693440	816
16080035811840	1428	20193653718784	468	23814974355480	802
16212258972000	942	20432681637984	783	23819044650240	557
16524280700928	656	*20662005324800	506	*23885971200000	960
*16924847940000	1125	20663813681280	1457	23906526134400	911
16965637957800	527	20746479283200	946	23929031075040	569
17086937762048	462	21204827804160	1115	24133566352896	1269
17130547324800	942	*21590959104000	800	24146583347250	801

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$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
24345523036800	1242	*28103080287744	496	35137010809600	986
*24613169545216	285	29040286302720	1060	35195158303200	1377
24722083685376	854	29193612739200	919	35560552416480	1079
24960513123000	1242	29382474401280	1442	35727233502464	483
25075635936512	339	29495815011600	525	36457089596928	1278
25206921653760	1462	29646588972000	964	36501751345920	563
25278832051200	1254	30209639896800	1055	36567846174720	478
25483518950400	913	*30233275380000	1155	36690736642560	1460
26025228028800	1142	30368564724960	1155	36783914076000	1242
26183184168960	762	30676980297600	1336	37342487131488	795
26407085632256	476	31094717121000	569	37643864076000	929
26757162432000	998	31671732879360	828	37695962304000	1850
26772789288960	806	32133029292000	1365	38287967477760	1064
26818992224640	1464	32176700980480	551	38583480499200	1276
27184083544800	1041	32327865884160	1062	38629000502400	969
27188110404000	1224	32713768684800	1377	38781262840320	1476
27214447163904	1272	*32752714995000	1155	38903025047040	1065
27258821990400	1376	33451592638464	664	39275901181440	1474
27261634143744	617	34044371361000	1476	39377859655680	764
27501146956800	1140	3422225403520	1140	40220975692800	917
27628679988000	919	34854206521344	536	40369640927616	1144
27717383688960	566	35085648124800	1337	40815295466400	1042

contd...

$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
40963871894400	1338	47089809930800	455	56463835428000	2040
41051610243072	713	47824897105920	682	56483529822720	794
41975434828800	1160	*47911115564928	1008	57209988326400	1290
42848120544768	1173	48765763791360	1068	57516364550400	807
*43180427911400	403	49749547075200	1269	57517704153000	1272
43588078934400	1661	49921679808000	1862	58628502535680	1070
43645811489280	2040	50131876354560	1484	58637313657000	1073
43865704602720	1331	50560395177600	2070	58954886991000	1491
43905339878400	1573	51006265947000	1269	58960053398400	986
*43947421401888	216	51260813286144	693	*58991630023200	620
44008577613000	1484	51633280258560	1448	59050215544320	2070
44095620366336	524	52500435423744	523	59888894456160	1275
44345330883840	1037	*53092467020880	651	60580961156352	702
44531496801792	886	54409216942080	1080	60868244209500	778
44621315481600	900	54557264361600	2376	60876700907400	549
45108496097280	776	*54934276752360	252	61015386432000	1540
45124517299680	1073	54942374462976	1284	61259428298250	1131
45406018134400	670	55062424216800	1853	61986015974400	759
45634960425600	1739	56087667603968	715	62469841674240	1085
*46013471418096	558	56100553084800	1272	62707195371744	801
*46353444300800	760	56221571976570	671	63687677113088	492
46796172040800	1019	56261841199200	1059	63750063484800	2088

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$n$	$H(n)$	$n$	$H(n)$	$n$	$H(n)$
64720497623040	846	77052018771000	1278	86100192346112	367
65058512238720	1716	77212128389760	1368	86887495094400	2408
67157796625920	1491	77547710691360	1086	87548446375936	91
67306216513536	668	77654820789000	1421	88527848521728	526
68029152998400	1175	78429196876800	1331	88847505747000	1498
68919093243000	1494	78497425843200	1298	88954751508480	824
69775118828800	979	78508410140160	1494	89163516169728	887
70879832150400	1752	78730921315200	967	89229599877600	1943
70902132973056	647	78958268284896	1287	89875965763584	669
71485642681600	1003	78961886115840	766	*90134334505600	952
71681373036000	1769	79221256896768	714	90606219580800	1202
71969788628304	427	*79708161843200	660	91501705658880	626
72370674647040	860	80508613785600	1397	92895015213000	1202
72874680721920	1512	80548660192000	1232	92945339487000	1716
72982369892250	807	81178611180800	1005	94258317081600	921
72982688808960	848	*84761657875440	651	96320660436000	2088
73288695889920	479	84762932436000	1272	96422831210496	548
74454619599360	2088	85372015528800	1859	97516898519040	1376
75153571113600	1278	85398298444800	1180	97769262366720	852
75167128764000	1269	85434688810240	770	97789867812000	947
76343936628960	1323	85454920812800	981	98079457512960	1074
76392247932000	2070	85626151750656	722		

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  - (ii) 96<sup>th</sup> Indian Science Congress held in NEHU, Shillong, from 3<sup>rd</sup> to 7<sup>th</sup> January, 2009.