

SOME PROBLEMS IN ALGEBRA - THE HOMOMORPHISM  
 $M \otimes N \rightarrow \text{HOM}(M, N)$ , SPLITTING OF RING EXTENSIONS,  
CRITERION FOR REGULARITY, HOMOLOGICAL  
DUALITY AND BNSI RINGS.

By *Bam*

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I certified that the thesis entitled "Some problems in Algebra - the homomorphism  $M^* \otimes N \rightarrow \text{Hom}(M, N)$ , splitting of ring extensions, criterion for regularity, homological duality and BNSI rings " submitted by Mr. Sib Nath Bose for the Degree of Philosophy of North Eastern Hill University, Shillong embodies the original work carried out by him under my supervision. He has been duly registered and the thesis presented is worthy of being considered for the Award of the Ph.D. Degree. This work has not been submitted for any degree of any other University.

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Place: Shillong

*P. Jothilingam*  
(DR. P. JOTHILINGAM)

Supervisor

DEDICATED  
TO  
MY PARENTS

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## Preface

The study of commutative algebra started with the works of famous authors like D. Hilbert, E. Noether, Macaulay and W. Krull. The latter progress in commutative algebra arises chiefly from quite different problems, issuing from Algebraic Geometry.

The concept of local rings was introduced by W. Krull in his paper "Dimensions Theorie in Stellen rigen". Krull conjectured a few problems in his paper. These were proved by C. Chevalley in "On the theory of Local rings" and by I.S. Cohen in his thesis "On the structure and ideal theory of complete local rings". Authors like Zariski, Nagata proved further important results with the study of local properties of Algebraic varieties. These developments are embodied in the famous work of Nagata "Local rings".

The second stage of development of the subject was ushered in by the lecture notes of J.P. Serre "Algebre - Localé Multiplicitiés", Classic papers of M. Auslander and D. Buchsbaum "Homological dimension in local rings" and "Maximal orders" by Auslander and Goldman. These authors introduced homological techniques into the subject. The proof of unique factorization in regular local rings by Auslander and Buchsbaum was an achievement signalling that these techniques have come to stay. Some more important results were proved by these

(ii)

authors by using homological techniques. Lichtenbaum's proof that  $\text{Tor}_i^R(M, N) = (0)$  implies  $\text{Tor}_j^R(M, N) = (0)$  for all  $j \geq i$  for finitely generated modules over a regular local ring is another achievement of this period. The subsequent appearance of the thesis of C. Peskin and L. Szpiro "Dimension Projective finite et Cohomologie Locale" and the work of M. Hochster on big Cohen Macaulay modules have all pointed out the effectiveness of homological techniques in Commutative Algebra. The "Zero-divisor Conjecture" by I. Kaplansky, the "Tor - Conjecture" by M. Auslander, the Conjecture of H. Bass on Cohen - Macaulay rings are still open problems to mention a few.

This dissertation consists of ~~six~~ chapters.

Chapter I deals with preliminary definitions and results used in the dissertation.

In Chapter II, we prove three theorems. In Theorem 2.1, we prove a result, a special case of which says that if  $M$  and  $N$  are two reflexive modules of finite projective dimensions over a Gorenstein local ring such that  $\text{Hom}(M, N)$  is a third module of syzygies, then the natural homomorphism  $M^* \otimes N \rightarrow \text{Hom}(M, N)$  is an isomorphism. This extends the result in [13]. In Theorem 2.2 we give a criterion for a module  $M$  over a regular local ring to have projective dimension less than or equal to an integer  $n$ . This extends the usual criterion for the projectivity of a module. In Theorem 2.3, we prove

that over a 1-dimensional Gorenstein local ring  $R$  if  $M, N$  are finitely generated  $R$ -modules such that  $\text{Hom}(M, N)$  is nonzero free then both  $M^*$  and  $N$  are free. This is a generalization of a result of W. Vasconceles [29, Theorem 3.1].

If  $R \subset S$  is an extension of rings (not necessarily commutative) making  $S$  into a projective  $R$ -module then  $R$  is a direct summand of  $S$  if and only if  $S$  is faithfully projective. This is a result of Cartzen and others [11, Theorem - 1]. The authors also give an example of a ring extension in which  $S$  is  $R$ -projective but not faithfully projective. The example involved noncommutative rings. In Chapter III, we show such a situation cannot happen if  $R$  is commutative i.e. if  $R \subset S$  is an extension of rings making  $S$  projective as  $R$ -module then  $R$  is a direct summand of  $S$  and  $S$  is faithfully projective. This is Theorem 3.1 of this Chapter. In Bourbaki [10, Ex 5.4, p-176] this is mentioned as an exercise when  $S$  is finitely generated as  $R$ -module.

In Chapter IV, we give a criterion for a noetherian local ring to be regular. This involves homological conditions on prime ideals of small height as compared to Hilbert-Serre Theorem which says that  $R$  is regular if and only if the maximal ideal of  $R$  has finite homological dimension. More precisely, suppose  $\text{pd}_R \mathcal{O}_{\mathfrak{p}} < \infty$  for every unmixed ideal  $\mathfrak{O}$  of height at most 2, then  $R$  is regular. W. Bruns [9] had earlier shown

that if  $\text{pd } \mathcal{O}_1 < \infty$  for every ideal  $\mathcal{O}$  all of whose associated prime ideals have depth at most 2, then  $R$  is regular. But our proposition is a direct improvement of M. Auslander's criterion given in [4, Theorem B, Corollary 5]. The proof also includes some results of independent interest giving criterion for a domain to be U.F.D.

In chapter - V, we give an application of homological duality to generalized M-regular sequences. The notion of M-sequence is generalized in [14] as follows: a sequence  $P_1, P_2, \dots, P_n$  of nonfree modules is said to be an M-sequence if  $\text{Tor}_1^R (M \otimes P_1 \otimes \dots \otimes P_{i-1}, P_i) = (0)$  for  $1 \leq i \leq n$ . The sum  $\sum \text{Pd } P_i$  is defined to be the length of the M-sequence  $P_1, P_2, \dots, P_n$ . We shall apply Strebel's homological duality ([28], §3, Theorem 13) to prove Theorem 5.1 on generalized M-sequences which states that if  $R$  is a regular local ring and  $M$  a nonzero  $R$ -module and  $\{P_1, P_2, \dots, P_n\}$  is an M-sequence of nonfree perfect modules of length  $\sum_{i=1}^n d_i$   $\mathcal{O}_i = \text{ann} P_i$ ,  $d_i = \text{Pd } P_i$ ; for  $1 \leq i \leq n$ . Then for every  $i$ ,  $1 \leq i \leq n$ ,  $\exists d_i$  elements  $x_1^{(i)}, x_2^{(i)}, \dots, x_{d_i}^{(i)}$  in  $\mathcal{O}_i$  such that the sequence of  $\sum d_i$  elements  $\{x_j^{(i)}\}$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq d_i$  form an M-sequence in the usual sense.

(v)

In Chapter VI, i.e. the last Chapter we take  $R$  to be a non-commutative local ring i.e.  $P$  is a ring with jacobson radical  $\mathcal{O}$  such that  $R/\mathcal{O} (= K)$  is a division ring. Mark Ramaras introduced the notion of BNSI rings in [24]. We have generalized this notion to non-commutative rings. A non-commutative local ring  $R$  is called a left BNSI ring if for every nonfree left  $P$ -module  $M$ , the sequence  $\{ \beta_i(M) \}_{i \geq 1}$  is strictly increasing where

$\beta_i(M) = \dim_K \text{Tor}_i^R(K, M)$  is called the  $i$ th Bettinumber of  $M$ . If  $R$  is a left and right BNSI-ring with jacobson radical  $\mathcal{O}$  nilpotent and  $M$  is an indecomposable left  $R$ -module such that  $\text{Ext}_R^1(M, R) = (0)$  then  $M$  is free. This is Theorem 6.6 of this Chapter which generalizes a theorem in [24].

## CONTENTS

CHAPTER - I	PRELIMINARIES	1-12
CHAPTER - II	THE HOMOMORPHISM $M^* \otimes N \rightarrow \text{Hom}(M, N)$	13-22
CHAPTER - III	SPLITTING OF RING EXTENSIONS	23-27
CHAPTER - IV	CRITERION FOR REGULARITY	28-32
CHAPTER - V	HOMOLOGICAL DUALITY	33-39
CHAPTER - VI	BNSI RINGS	40-46

## CHAPTER-I

### Preliminaries

In this chapter we give a brief outline of some preliminary definitions and results used in the dissertation. We also fix notations and other conventions here. Throughout this dissertation a ring  $R$  means a ring with identity and an  $R$ -module means an unitary module.

#### §1.1. Projective and injective modules, Tor. and Ext. functors:

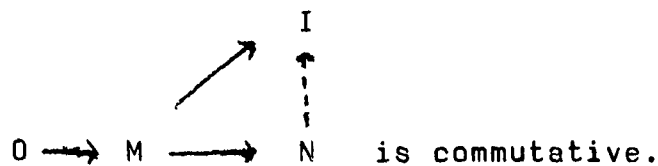
Definition 1.1.0 : An  $R$ -module  $P$  is called Projective if given an exact sequence of  $R$ -modules  $M \rightarrow N \rightarrow 0$  and an  $R$ -linear map  $P \rightarrow N$   $\exists$  an  $R$ -linear map  $P \rightarrow M$  such that the diagram

$$\begin{array}{ccccc} P & & & & \\ \downarrow & \searrow & & & \\ M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

is commutative.

Example: Any free module is projective.

Definition 1.1.1: An R-module  $I$  is injective if given an exact sequence  $0 \rightarrow M \rightarrow N$  and a homomorphism  $M \rightarrow I$  there exists a homomorphism  $N \rightarrow I$  such that the diagram



Example:  $\mathbb{Q}$  is  $\mathbb{Z}$ -injective.

Definition 1.1.2: An exact sequence of maps and modules

$\dots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$  where  $P_i$ 's are projective is called a projective resolution of the R-module  $M$ .

Example: Consider  $\mathbb{Z} / 2\mathbb{Z}$  as  $\mathbb{Z}$ -module. Then

$0 \rightarrow 2\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$  is a projective resolution of  $\mathbb{Z} / 2\mathbb{Z}$ .

It is easy to see that any module  $M$  has a projective resolution.

Definition 1.1.3: An exact sequence of maps and modules

$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$  where  $I_i$ 's are injective modules is called an injective resolution of  $M$ .

In this case also it can be shown that any module  $M$  has an injective resolution.

Example:  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$  is an injective resolution of  $\mathbb{Z}$  as  $\mathbb{Z}$ -module.

Definition 1.1.4: A sequence of maps and modules

$$M_* : \dots \rightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} M_{n-2} \rightarrow \dots$$

such that  $d_n \circ d_{n+1} = 0$  for every  $n$  is said to be a complex.

We denote this complex by  $M_*$ . Clearly  $\text{Im } d_{n+1} \subseteq \text{Ker } d_n$ .

The  $R$ -modules  $\frac{\text{Ker } d_n}{\text{Im } d_{n+1}}$  are said to be the  $n$ th homology of the

Complex  $M_*$  and is denoted by  $H_n(M_*)$ .

Definition 1.1.5: Let  $X_* \rightarrow N \rightarrow 0$  be a projective resolution of  $N$ . Consider the complex  $M \otimes X_*$  for any  $R$ -module  $M$ . Then the  $n$ th homology of this complex is defined to be  $\text{Tor}_n^R(M, N)$ . It can be shown easily that this definition is independent of the resolution  $X_* \rightarrow N \rightarrow 0$ . These are bifunctors on the category of  $R$ -modules and are called Tor functors. We mention here some properties of Tor functors:

Proposition 1.1.0: (i)  $\text{Tor}_0^R(M, N) = M \otimes_R N$ .

(ii) If  $0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0$  is an exact sequence of  $R$ -modules, then there exists a long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Tor}_n^R(M, N_1) \rightarrow \text{Tor}_n^R(M, N) \rightarrow \text{Tor}_n^R(M, N_2) \\ \rightarrow \text{Tor}_{n-1}^R(M, N_1) \rightarrow \dots \rightarrow M \otimes N_1 \rightarrow M \otimes N \rightarrow M \otimes N_2 \rightarrow 0. \end{aligned}$$

(iii)  $\text{Tor}_n^R(M, N) = \text{Tor}_n^R(N, M)$ , if  $R$  is a commutative ring.

(iv)  $\text{Tor}_n^R(M, N) = 0$  for  $n > 0$  if  $N$  is a flat module.

Definition 1.1.6: The projective dimension of a module  $M$  is defined to be infinite if it has no finite projective resolution. Otherwise the length of the shortest projective resolution of  $M$  is said to be the projective dimension of  $M$ . The notation for Projective dimension of  $M$  is  $\text{Pd } M$ .

Definition 1.1.7: If a module  $M$  has no finite injective resolution, we say that injective dimension of  $M$  is infinite. Otherwise the length of the shortest injective resolution of  $M$  is defined to be the injective dimension of  $M$ . The notation for injective dimension of  $M$  is  $\text{id } M$ .

Definition 1.1.8: Let  $0 \rightarrow N \rightarrow I_*$  be an injective resolution of  $N$ . Consider the complex  $\text{Hom}(M, I_*)$ . The  $n$ th homology of this complex is defined to be  $\text{Ext}_R^n(M, N)$ .  $\text{Ext}_R(M, \quad)$  is a functor in the category of  $R$ -modules and is called Ext functor.

We mention a few properties of this functor:

Proposition 1.1.1: (i)  $\text{Ext}_R^0(M, N) = \text{Hom}_R(M, N)$ .

(ii) If  $0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0$  is exact then there exists a long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(M, N_1) \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M, N_2) \rightarrow \text{Ext}_R^1(M, N_1) \\ \rightarrow \text{Ext}_R^1(M, N) \rightarrow \text{Ext}_R^1(M, N_2) \rightarrow \dots \end{aligned}$$

(iii) If  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  is exact then there exists a long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(M_2, N) \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M_1, N) \rightarrow \text{Ext}_R^1(M_2, N) \\ \rightarrow \text{Ext}_R^1(M, N) \rightarrow \text{Ext}_R^1(M_1, N) \rightarrow \dots \end{aligned}$$

§1.2: Primary decomposition of modules

Throughout this Section all rings are Commutative and modules are finitely generated. The set of prime ideals is called spectrum of  $R$  and is denoted as  $\text{Spec } R$ .

Definition 1.2.0. A prime ideal  $p$  of  $R$  is said to be an associated prime of a module  $M$  if one of the following equivalent conditions hold:

- (i) There exists an element  $x \in M$  with  $\text{Ann } x = p$ .
- (ii)  $M$  contains a submodule isomorphic to  $R/p$ . The set of associated prime ideals of  $M$  is denoted as  $\text{Ass } M$ .

Definition 1.2.1: The support of a module  $M$  denoted as  $\text{Supp } M$  is defined to be the set of all those prime ideals of  $R$  such that  $M_p \neq 0$ .

Definition 1.2.2: A module  $M$  is said to have finite length if it possesses Jordan-holfer series. We denote the length of a module  $M$  by  $l(M)$ .

Throughout the remaining part of this section  $R$  denotes a Noetherian ring.

Proposition 1.2.0: A module  $M$  over a ring  $R$  has finite length if  $\text{supp } M$  consists of maximal ideals only.

Definition 1.2.3: A nonzero module  $M$  is said to be co-primary if given any  $a \in R$ , the homothecy  $\lambda_a$  (multiplication by 'a' on  $M$ ) is either injective or nilpotent.

Example:  $\mathbb{Z}/2^3\mathbb{Z}$  is a co-primary  $\mathbb{Z}$ -module.

Definition 1.2.4: Suppose  $M$  is a co-primary module. Then the set  $\{a \in R / \lambda_a \text{ is nilpotent}\}$  is a prime ideal of  $R$ . This prime ideal  $p$  is said to be the associated prime ideal of  $M$  and  $M$  is called  $p$ -co-primary.

Definition 1.2.5: Suppose  $M$  is a nonzero module. A proper submodule  $N$  of  $M$  is said to be  $p$ -primary if  $M/N$  is  $p$ -co-primary.

Definition 1.2.6: A proper submodule  $N$  of  $M$  is said to be irreducible if there does not exist a representation

$$N = N_1 \cap N_2 \text{ with } N \subsetneq N_1, N \subsetneq N_2.$$

Example:  $(0)$  is irreducible in  $\mathbb{Z}/p^n\mathbb{Z}$  where  $p$  is a prime integer and  $n \geq 1$ .

Definition 1.2.7: Let  $N$  be a submodule of  $M$ . A primary decomposition of  $N$  is an equation  $N = Q_1 \cap Q_2 \cap \dots \cap Q_r$  where  $Q_i$ 's are primary in  $M$ . Such a decomposition is said to be irredundent or reduced if no  $Q_i$  can be omitted and if

the associated primes of  $M/Q_i$  ( $1 \leq i \leq r$ ) are all distinct. Any primary decomposition can be simplified to a reduced primary decomposition.

Proposition 1.2.1: Any proper submodule  $N$  of  $M$  has a primary decomposition.

### §1.3: Dimension of rings and modules

Definition 1.3.0: A finite sequence of  $(r+1)$  prime ideals  $P_0 \supset P_1 \supset P_2 \supset \dots \supset P_r$  is called a prime chain of length  $r$ . If  $p$  is a prime ideal of  $R$ , the supremum of the lengths of the prime chains descending from  $p$  is called <sup>the</sup> height of  $p$  and is denoted  $ht(p)$ . If  $ht(p) = 0$ ,  $p$  is called a minimal prime ideal. If  $I$  is a proper ideal of  $R$ , then <sup>the</sup> height of  $I$  is defined by  $ht(I) = \inf \left\{ ht(p) / p \supseteq I \right\}$ .

Definition 1.3.1: The Krull dimension of a ring  $R$  is defined to be the supremum of the heights of the prime ideals in  $R$ . Krull dimension will be abbreviated as  $\dim R$ . If  $R$  is a noetherian local ring then  $\dim R$  is finite. It follows from definition that  $ht(p) = \dim (R_p)$  for  $p \in \text{Spec } R$  and for any ideal  $I$  of  $R$ ,  $\dim R/I + ht(I) \leq \dim R$ .

Example: Krull dimension of a Principal ideal domain is one and K. dimension of a field is zero.

Definition 1.3.2: The Krull dimension of a module  $M$  is defined to be the supremum of the lengths of prime chains of primes in  $\text{Supp } M$ .

Proposition 1.3.0: Suppose  $R$  is a noetherian ring and  $M$  a finitely generated module. Then the following are equivalent:

- (i)  $M$  is a module of finite length;
- (ii) The ring  $R/\text{Ann } M$  is artinian.
- (iii)  $\dim M = 0$ .

Definition 1.3.3: Let  $R$  be a noetherian local ring. Let all  $R$ -modules be of finite projective dimensions. The supremum of the projective dimensions of all  $R$ -modules is called global dimension of the ring  $R$  and is denoted as  $\text{gl. dim } R$ .

Definition 1.3.4: Suppose  $R$  is a noetherian semilocal ring and  $J$  be the Jacobson radical of  $R$ . An ideal  $I$  of  $R$  is said to be an ideal of definition of  $R$  if  $\sqrt{I} = J$ .

Definition 1.3.5: Suppose  $R$  is a noetherian local ring of dimension  $d$ . If  $x_1, x_2, \dots, x_d$  generate an ideal of definition of  $R$ , then  $x_1, x_2, \dots, x_d$  are called system of parameters.

#### §1.4: Regular sequence

Definition 1.4.0: Let  $M$  be an  $R$ -module. A sequence  $a_1, a_2, \dots, a_r$  of elements of  $R$  is said to be  $M$ -regular if for

each  $1 \leq i \leq r$ ,  $a_i$  is not a zero divisor on  $M/(a_1 M + a_2 M + \dots + a_{i-1} M)$  and  $M \neq a_1 M + a_2 M + \dots + a_r M$ .

Definition 1.4.1: Let  $R$  be a noetherian local ring. The length of a maximal  $M$ -regular sequence in the maximal ideal  $\mathfrak{m}$  of  $R$  is called the depth of  $M$  and is denoted as  $\text{depth } M$ . Thus  $\text{depth } M = 0$  if and only if  $\mathfrak{m} \in \text{Ass}(M)$ .

If  $R$  is an arbitrary noetherian ring and  $\mathfrak{p} \in \text{Spec } R$ , we have  $\text{depth } M_{\mathfrak{p}} = 0$  as  $R_{\mathfrak{p}}$ -module if and only if  $\mathfrak{p} \in \text{Ass}(M)$ .

Proposition 1.4.0: Let  $R$  be a noetherian local ring and let  $M \neq 0$  be a finitely generated  $R$ -module. Then  $\text{depth } M \leq \dim(R/\mathfrak{p})$  for every  $\mathfrak{p} \in \text{Ass}(M)$ .

It follows from this proposition that for any finitely generated module,  $\text{depth } M \leq \dim M$ .

Definition 1.4.2: Let  $R$  be a noetherian local ring and  $M$  a finitely generated module.  $M$  is said to be a Cohen Macaulay module if  $M = 0$  or  $\text{depth } M = \dim M$ . If the local ring  $R$  is Cohen Macaulay as  $R$ -module we say that  $R$  is a Cohen Macaulay ring.

Example: Localization of the polynomial ring  $k[x_1, x_2, \dots, x_n]$  where  $k$  is a field at a prime ideal is a Cohen Macaulay ring.

Proposition 1.4.1: Let  $R$  be a Cohen Macaulay local ring. Then for any ideal  $I$ , we have  $\dim R/I + \text{ht } I = \dim R$ .

Definition 1.4.3: The grade of a module  $M$  over a noetherian local ring  $R$  is defined to be the length of a maximal  $R$ -sequence contained in the annihilator of  $M$ . If  $M$  has finite projective dimension then it is well known that  $\text{grade } M \leq \text{Proj. dim } M$ .

Definition 1.4.4: Let  $R$  be noetherian local. An  $R$ -module  $M$  is said to be perfect if  $\text{Pd } M$  is finite and is equal to the grade of the module. For a perfect module we have  $\text{Ext}_R^i(M, R) = 0$  for  $i \neq \text{Pd } M$ .

Example: If  $\underline{x} = (x_1, x_2, \dots, x_r)$  is an  $R$ -sequence, then  $R/\underline{x}$  is a perfect  $R$ -module.

Definition 1.4.5: A module  $M$  is said to be torsionless if the natural mapping  $M \rightarrow M^{**}$  where  $*$  denotes duals, is injective.

Example: Any torsion-free module over a domain is torsionless.

Definition 1.4.6: A module  $M$  is said to be reflexive if the natural map  $M \rightarrow M^{**}$  is an isomorphism.

Example: Any free module is reflexive.

Proposition 1.4.2: (Rees formula) Let  $M$  and  $N$  be  $R$ -modules. Then  $\text{Ann } N$  contains an  $M$ -regular sequence of length  $t$  if and only if  $\text{Ext}_R^i(M, N) = 0$  for  $i = 0, 1, 2, \dots, t-1$ .

Proposition 1.4.3 (Auslander's depth formula):

Let  $M$  be a module over a noetherian local ring  $R$  having finite projective dimension. Then

$$\text{Pd } M + \text{depth } M = \text{depth } R.$$

§1.5: Regular local rings and Gorenstein rings

Definition 1.5.0. Let  $R$  be a noetherian local ring. If there exists a system of parameters generating the maximal ideal of the ring  $R$ , we say that  $R$  is regular local and such a system of parameters is called a regular system of parameters.

Example: If  $K$  is a field, then the power series ring

$K[[x_1, x_2, \dots, x_n]]$  is regular local.

Proposition 1.5.0 (Hilbert - Serre's Theorem):

A noetherian local ring  $R$  is regular local if and only if global dimension of  $R$  is finite.

Definition 1.5.1: A noetherian local ring  $R$  is said to be a Gorenstein ring if injective dimension of the ring is finite as  $R$ -module.

Example: Let  $R$  be regular local and  $\underline{x} = (x_1, x_2, \dots, x_r)$  is an  $R$ -sequence. Then  $R/\underline{x}R$  is Gorenstein.

Proposition 1.5.1: Let  $R$  be a local Gorenstein ring and  $M$  a nonzero  $R$ -module. Then  $\text{depth } M = \dim R - t$ , where  $t$  is the largest integer such that  $\text{Ext}_R^t(M, R) \neq (0)$ .

Definition 1.5.2: The integer  $t$  such that  $\text{Ext}_R^t(M, R) \neq (0)$  mentioned above is called the G-dimension of the module  $M$ .

Definition 1.5.3: A module  $M$  is said to be an  $n$ th syzygy for a positive integer  $n$  if there exists an exact sequence

$$0 \rightarrow M \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0,$$

where  $F_i$ 's are free modules.

If  $\cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$  is a projective resolution of  $M$ , the Kernel of  $F_i \rightarrow F_{i-1}$  will be denoted as  $\Omega^{i+1} M$  with the understanding that  $F_{-1} = M$  and  $\Omega^0 M = M$ ; the Kernel of  $F_i^* \rightarrow F_{i+1}^*$  will be denoted as  $D \Omega^i M$ .

Definition 1.5.4: A module  $M$  is said to be rigid if whenever  $\text{Tor}_1^R(M, N) = 0$  for a module  $M$ , then  $\text{Tor}_i^R(M, N) = 0$  for all  $i \geq 1$ . Over a regular local ring any module is rigid. This is a theorem of Lichtenbaum (Ref.?)

We shall make use of the following exact sequences of M. Auslander and M. Bridger [1] in Chapter II.

Proposition 1.5.2: For every pair of modules  $M$  and  $N$  and for every integer  $n \geq 0$ , there exist exact sequences:

$$\begin{aligned} \text{Tor}_2^R(D \Omega^n M, N) &\rightarrow \text{Ext}_R^n(M, R) \otimes N \rightarrow \text{Ext}_R^n(M, N) \\ &\rightarrow \text{Tor}_1^R(D \Omega^n M, N) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow \text{Ext}_R^1(D \Omega^n M, N) &\rightarrow \text{Tor}_n^R(M, N) \rightarrow \text{Hom}(\text{Ext}_R^n(M, R), N) \\ &\rightarrow \text{Ext}_R^2(D \Omega^n M, N). \end{aligned}$$

## CHAPTER - II

### THE HOMOMORPHISM $M^* \otimes N \rightarrow \text{Hom}(M, N)$

Throughout this chapter  $R$  denotes a Commutative ring and modules are finitely generated.

In the study of orders in simple algebras over the quotient field of a regular local ring  $R$ , M. Auslander and O. Goldman were led to show that if  $M$  is a reflexive  $R$ -module such that  $\text{Hom}_R(M, M)$  is free then  $M$  itself is free. This is generalized in [13] to assert that if  $M$  and  $N$  are reflexive modules over a regular local ring  $R$  such that  $\text{Hom}_R(M, N)$  is nonzero free, then  $M$  and  $N$  are both free. This was proved by showing that under the given conditions, the natural homomorphism  $M^* \otimes N \rightarrow \text{Hom}(M, N)$  is an isomorphism. This can be improved further. This is our Theorem 2.1 of this chapter. The projectivity of a module  $M$  is equivalent to the surjectivity of the natural homomorphism

$$M^* \otimes M \rightarrow \text{Hom}(M, M).$$

Theorem 2.2 of this chapter is an extension of this result giving a criterion for a module over a regular local ring to have projective dimension less than or equal to a given non-negative integer.

If  $R$  is a 1-dimensional Gorenstein local ring and  $M$  is a finitely generated  $R$ -module such that  $\text{Hom}(M, M)$  is free then  $M$  is free. This is a result of W. Vasconcelos [29, Th-3.1]. Theorem-2.3 is a generalization of this result.

Theorem 2.1: Suppose  $M$  and  $N$  are reflexive modules of finite projective dimensions over a Cohen - Macaulay local Ring  $R$  and  $N$  is rigid. Then if either  $\text{Ext}_R^1(\text{Hom}(M,N)^*, R) = (0)$ , or the ring  $R$  is Gorenstein and  $\text{Hom}(M,N)$  is a third module of syzygies, then the natural homomorphism  $M^* \otimes N \rightarrow \text{Hom}(M,N)$  is an isomorphism.

Proof: We first claim that the given conditions imply  $\text{Ext}_R^1(M,N) = (0)$ . This is proved by induction on Krull dimension of  $R$ . Firstly, if  $q$  is a prime ideal of height atmost two, then  $M_q$  being a reflexive  $R_q$ -module,  $\text{depth } M_q = \text{depth } R_q$ . Then Auslander's depth formula,  $\text{Pd } M_q + \text{depth } M_q = \text{depth } R_q$  implies that  $M_q$  is  $R_q$ - free. Hence  $\text{Ext}_{R_q}^1(M_q, N_q) = (0)$ . We can therefore assume  $\dim R \geq 3$ , and that  $\text{Ext}_{R_p}^1(M_p, N_p) = (0)$ , for all non-maximal prime ideals  $p$  of  $R$ . Then either  $\text{Ext}_R^1(M,N) = (0)$ , or it is a nonzero module of finite length.

$$\text{Let } 0 \rightarrow L \rightarrow F \rightarrow M \rightarrow 0 \quad \dots \quad (1)$$

be an exact sequence with  $F$  free. We apply the functor  $\text{Hom}(\quad, N)$  to (1) and get the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(M,N) \rightarrow \text{Hom}(F,N) \rightarrow \text{Hom}(L,N) \rightarrow \text{Ext}_R^1(M,N) \\ \rightarrow 0 \quad \dots \quad (2) \end{aligned}$$

The exact sequence (2) can be broken into two short exact sequences as follows:

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(F, N) \rightarrow T \rightarrow 0 \dots \quad (3)$$

and

$$0 \rightarrow T \rightarrow \text{Hom}(L, N) \rightarrow \text{Ext}_R^1(M, N) \rightarrow 0 \dots \quad (4)$$

By the assumption on  $\text{Ext}_R^1(M, N)$ , some powers of the maximal ideal of  $R$  annihilates  $\text{Ext}_R^1(M, N)$ . So,  $\dim \text{Ext}_R^1(M, N) = 0$ . Since  $R$  is Cohen-Macaulay,  $\text{grade } \text{Ext}_R^1(M, N) = \dim R \geq 3$ . This shows that  $\text{Ext}_R^j(\text{Ext}_R^1(M, N), R) = (0)$  for  $j = 0, 1, 2$ . Therefore, by dualizing (3) and (4), we get the following exact sequences

$$0 \rightarrow T^* \rightarrow \text{Hom}(F, N)^* \rightarrow \text{Hom}(M, N)^* \rightarrow \text{Ext}_R^1(T, R) \rightarrow \dots \quad (5)$$

$$0 \rightarrow \text{Hom}(L, N)^* \rightarrow T^* \rightarrow 0 \rightarrow \dots \quad (6)$$

Putting (5) and (6) together we get the exact sequences:

$$0 \rightarrow \text{Hom}(L, N)^* \rightarrow \text{Hom}(F, N)^* \rightarrow \text{Hom}(M, N)^* \rightarrow \text{Ext}_R^1(T, R) \rightarrow \dots \quad (7)$$

Now if  $\mathfrak{p}$  is a prime ideal of height at most two, then  $M_{\mathfrak{p}}$  and  $N_{\mathfrak{p}}$  are free  $R_{\mathfrak{p}}$ -modules, as in the beginning of the proof. Hence (1) and (4) imply that  $T_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}$ -free, i.e.  $(\text{Ext}_R^1(T, R))_{\mathfrak{p}} = (0)$ . Denoting by  $P$ , the image of the homomorphism  $\text{Hom}(M, N)^* \rightarrow \text{Ext}_R^1(T, R)$ , then by what it



precedes,  $P_p = (0)$ . Thus if  $\mathcal{O}_P$  = annihilator of  $P$ , then height  $\mathcal{O}_P \geq 3$ . Cohen - Macaulay property of  $R$  implies that grade  $P = \text{height } \mathcal{O}_P \geq 3$ . Hence, we have  $\text{Ext}_R^i(P, R) = (0)$  for  $i = 0, 1, 2$ . Breaking (7) into two short exact sequences, as in the case of (2), we obtain

$$0 \rightarrow \text{Hom}(L, N)^* \rightarrow \text{Hom}(F, N)^* \rightarrow U \rightarrow 0 \quad \dots \quad (8)$$

$$0 \rightarrow U \rightarrow \text{Hom}(M, N)^* \rightarrow P \rightarrow 0 \quad \dots \quad (9)$$

Dualizing (8) and (9) and making use of the fact that  $\text{Ext}_R^i(P, R) = 0$  for  $i = 0, 1$  in (9) and putting together the resulting exact sequences we get,

$$\begin{aligned} 0 \rightarrow \text{Hom}(M, N)^{**} \rightarrow \text{Hom}(F, N)^{**} \rightarrow \text{Hom}(L, N)^{**} \\ \rightarrow \text{Ext}^1(U, R) \quad \dots \quad (10) \end{aligned}$$

From (9) and the facts that  $\text{Ext}^1(\text{Hom}(M, N)^*, R) = (0)$ , we get  $\text{Ext}^1(U, R) = (0)$ , i.e. the exact sequence (10) becomes

$$\begin{aligned} 0 \rightarrow \text{Hom}(M, N)^{**} \rightarrow \text{Hom}(F, N)^{**} \rightarrow \text{Hom}(L, N)^{**} \rightarrow 0 \\ \dots \quad (11) \end{aligned}$$

From (1), applying the functor  $\text{Ext}(\quad, N)$  we get the exact sequence,

$$\text{Hom}(F, N) \rightarrow \text{Hom}(L, N) \rightarrow \text{Ext}^1(M, N) \rightarrow 0$$

Then with obvious maps, the diagram

*Page 17?*

Proof: Suppose  $\text{Pd. } M = j$ . If  $j < n$ , clearly

$\text{Ext}_R^n(M, R) = 0 = \text{Ext}_R^n(M, M)$ . If  $j = n$ , then it is well known that the natural homomorphism

$$\text{Ext}_R^n(M, R) \otimes M \rightarrow \text{Ext}_R^n(M, M) \text{ is an isomorphism [1].}$$

Conversely, suppose that the latter homomorphism is surjective, we make use of fundamental exact sequences of M. Auslander and M. Bridger [1] which we have stated in Chapter-I.

$$\begin{aligned} \text{Tor}_2^R(D \Omega^n M, N) &\rightarrow \text{Ext}_R^n(M, R) \otimes N \rightarrow \text{Ext}_R^n(M, N) \\ &\rightarrow \text{Tor}_1^R(D \Omega^n M, N) \rightarrow 0 \quad \dots \quad (12) \end{aligned}$$

$$\begin{aligned} 0 \rightarrow \text{Ext}^1(D \Omega^n M, N) &\rightarrow \text{Tor}_n^R(M, N) \rightarrow \text{Hom}(\text{Ext}_R^n(M, R), N) \\ &\rightarrow \text{Ext}_R^2(D \Omega^n M, N) \quad \dots \quad (13) \end{aligned}$$

Taking  $M = N$  in (12) and using the fact that

$\text{Ext}_R^n(M, R) \otimes N \rightarrow \text{Ext}_R^n(M, N)$  is surjective, we get,

$\text{Tor}_1^R(D \Omega^n M, M) = (0)$ . Since  $R$  is regular local,

$\text{Tor}_j^R(D \Omega^n M, M) = (0)$  for  $j \geq 1$ . In particular

$\text{Tor}_2^R(D \Omega^n M, M) = (0)$ . Then (13) yields  $\text{Ext}_R^1(D \Omega^n M, D \Omega^n M) = (0)$ .

Applying [13] we find that  $D \Omega^n M$  is projective, i.e.  $\Omega^n M$  is projective. Hence  $\text{P.d } M \leq n$ .

Is  $M_2$  reflexive?

Now we give a short proof of a result of Jineah [12, Theorem-2.2] by applying the above Theorems. Our result is more general than that of the author.

Proposition 2.3: Let  $R$  be regular local.  $M$  be a finitely generated reflexive  $R$ -module. Suppose  $\text{Ext}_R^1(\text{Hom}(M, \mathcal{O}_1)^*, R) = 0$  for every unmixed ideal  $\mathcal{O}_1$  of height 2. Then  $M$  is free.

Proof: As in the proof of Theorem 2.1 above, the given conditions mean  $\text{Ext}^1(M, \mathcal{O}_1) = 0$ . By [4, Theorem B] there exists an exact sequence  $0 \rightarrow F \rightarrow M \rightarrow \mathcal{O}_1 \rightarrow 0$  where  $F$  is finitely generated and free. Apply  $\otimes M^*$  and  $\text{Hom}(\_, M)$  to the above exact sequence we get,

$$\begin{array}{ccccccc} F \otimes M^* & \rightarrow & M \otimes M^* & \rightarrow & \mathcal{O}_1 \otimes M^* & \rightarrow & 0 \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ 0 & \rightarrow & \text{Hom}(M, F) & \rightarrow & \text{Hom}(M, M) & \rightarrow & \text{Hom}(M, \mathcal{O}_1) \end{array}$$

~~Is this an assumption or bad notation~~

Since  $F$  is free,  $\alpha$  is an isomorphism. Now  $M$  is free if  $\beta$  is surjective by Theorem 2.2. Since  $\text{Ext}^1(M, \mathcal{O}_1) = 0$ , applying Lemma [13] we get  $M^* \otimes \mathcal{O}_1 = \text{Hom}(M, \mathcal{O}_1)$ .

So,  $\gamma$  is an isomorphism. This means that  $\beta$  is surjective.

The next theorem is a generalization of a result of W. Vasconcelos which says that over a 1-dimensional Gorenstein local ring  $R$ , a finitely generated  $R$ -module  $M$  is free if and only if  $\text{Hom}(M, M_\lambda)$  is free. Our proof runs parallel to the argument of Vasconcelos [29, Theorem 3.1].

Theorem 2.4. Suppose  $R$  is a 1-dimensional Gorenstein local ring and  $M, N$  are finitely generated  $R$ -modules such that  $\text{Hom}(M, N)$  is free. Then both  $M^*$  and  $N$  are free.

Proof: Let  $\mathfrak{m}$  be the maximal ideal of  $R$ . Since  $\text{Hom}(M, N)$  is free, by Auslander's depth formula we see that  $\text{depth of Hom}(M, N) > 0$ . So  $\mathfrak{m}$  is not associated to  $\text{Hom}(M, N)$ . Since  $\text{Ass}(\text{Hom}(M, N)) = \text{Ass } N \cap \text{Supp } M$ ,  $\mathfrak{m}$  is not associated to  $N$ . Also  $\mathfrak{m}$  is not associated to  $R$ . So  $\mathfrak{m}$  is not in the union of prime ideals associated to either  $R$  or  $N$  i.e.  $\exists a \in \mathfrak{m}$ , a nonzero divisor on  $R$  as well as on  $N$ . The exact sequence

$$0 \rightarrow R \xrightarrow{a} R \rightarrow \bar{R} \rightarrow 0 \quad ; \dots \quad (14)$$

implies that the sequence

$$0 \rightarrow \text{Hom}(N, R) \xrightarrow{a} \text{Hom}(N, R) \rightarrow \text{Hom}_{\bar{R}}(\bar{N}, \bar{R}) \rightarrow \text{Ext}_R^1(N, R) \dots \quad (15)$$

is exact, where  $\bar{R} = R/aR$ ,  $\bar{N} = N/aN$ . Since  $\text{depth } N = 1$  and  $G$ -dimension of  $N$  is 0, we get  $\text{Ext}_R^1(N, R) = (0)$ .

Then (15) gives the isomorphism

$$\text{Hom}_R(\bar{N}, \bar{R}) \cong N^* \otimes_R R \dots \quad (16)$$

If  $\mathfrak{p}$  is finitely generated projective then the homomorphism  $P^* \otimes P \rightarrow R$  given by  $f \otimes a \rightarrow f(a)$  is called the trace map and its image is an ideal of  $R$ , called the trace ideal of  $P$ .

*why free?*

The trace map above is denoted  $T_P$ .  $P$  has a direct summand isomorphic to  $R$  if and only if  $T_P$  is surjective. This is easy to see by using the fact that  $R$  is local. Then we have the following commutative diagram.

$$\begin{array}{ccc}
 N^* \otimes N & \xrightarrow{T_N} & R \\
 \downarrow \text{natural} & & \downarrow \\
 N^* \otimes N \otimes_R R & & \\
 \downarrow \wr & & \\
 N^* \otimes_R N & \xrightarrow{T_{\bar{N}}} & \bar{R}
 \end{array}$$

To show  $N$  is free, sufficient to show that  $N$  has a free direct summand isomorphic to  $R$ . This is equivalent to show  $T_N$  is surjective. Now  $T_N$  is surjective if and only if  $T_{\bar{N}}$  is surjective, via Nakayama lemma. The exact sequence,

$$0 \rightarrow N \xrightarrow{a} N \rightarrow \bar{N} \rightarrow 0 \quad \text{gives rise to the exact}$$

sequence,

$$0 \rightarrow \text{Hom}(M, N) \xrightarrow{a} \text{Hom}(M, N) \rightarrow \text{Hom}_{\bar{R}}(\bar{M}, \bar{N}) \quad \dots \quad (17)$$

(17) implies that  $\text{Hom}_{\bar{R}}(\bar{M}, \bar{N})$  contains  $\text{Hom}(M, N) \otimes_R \bar{R}$

as a submodule. Now  $\text{Hom}(M, N)$  is nonzero free implies  $\text{Hom}(M, N) \otimes_R \bar{R}$  is nonzero  $\bar{R}$ -free so that  $\text{Hom}(\bar{M}, \bar{N})$

contains a faithful  $\bar{R}$ -module. So,  $\text{Hom}(\bar{M}, \bar{N})$  is faithful. This shows that  $\bar{M}$  and  $\bar{N}$  are both faithful. Since  $\bar{R}$  is a 0-dimensional Gorenstein ring,  $(0)$  is an irreducible ideal. Let  $x_1, x_2, \dots, x_r$  be a set of generators of  $\bar{N}$ . Then  $\text{Ann } \bar{N} = \bigcap \text{Ann } x_i = (0)$ . As  $(0)$  is irreducible,  $\text{Ann } x_i = (0)$  for some  $x_i$ . So the map  $\bar{R} \xrightarrow{x_i} \bar{N}, \lambda \rightarrow \lambda x_i$  is injective. This shows that  $\bar{N}$  contains  $\bar{R}$  as a direct summand. So,  $T_{\bar{N}}$  is surjective. This shows that  $T_N$  is surjective. So,  $N$  is free. As  $N$  is free and finitely generated,  $N$  can be written as a direct sum of copies of  $R$ . Also  $\text{Hom}(M, N)$  is free i.e.  $\text{Hom}(M, \bigoplus R)$  is free. So,  $M^*$  is free.

## CHAPTER - III

### SPLITTING OF RING EXTENSIONS

If  $R \subset S$  is an extension of rings making  $S$  into a projective  $R$ -module then  $R$  is a direct summand of  $S$  if and only if  $S$  is faithfully projective. This is shown by Cartzen and others [11]. The authors also give an example of a ring in which  $S$  is  $R$ -projective but not faithfully projective. The example involved non-commutative rings. In this Chapter we show that such a situation cannot happen if  $R$  is commutative. This is shown in Theorem 3.1 of this chapter. In Bourbaki [10, P-176] this is mentioned as an exercise when  $S$  is finitely generated as  $R$ -module.

First we state a Lemma the proof of which is given in Bourbaki[10].

Lemma: Suppose  $R \subset S$  is a ring extension. Then  $S$  is faithfully flat over  $R$  if and only if  $S/R$  is  $R$ -flat.

Theorem 3.1: Suppose  $R$  is a commutative ring and  $R \subset S$ , a ring extension making  $S$  into a projective  $R$ -module. Then  $R$  is a direct summand of  $S$  as  $R$ -module.

Proof: Let  $p \in \text{Spec } R$ . Localizing the exact sequence

$0 \rightarrow R \rightarrow S \rightarrow S/R \rightarrow 0$ , we get the exact sequence

$0 \rightarrow R_p \rightarrow S_p \rightarrow (S/R)_p \rightarrow 0$  of  $R_p$ -modules.

By a Theorem of Kaplansky [17], the  $R_p$ -projective module  $S_p$  must be free and hence faithfully flat over  $R_p$ . By above lemma,  $(S/R)_p$  is  $R_p$ -flat. This being true for every  $p \in \text{Spec } R$ ,  $S/R$  must be  $R$ -flat. Hence by lemma above  $S$  must be faithfully flat as  $R$ -module. To conclude that  $R$  as a direct summand of  $S$  as  $R$ -module, one could make use of Theorem-I of Cartzen and others or one could use the criterion for projectivity given in [16] to conclude that  $S/R$  is  $R$ -projective and so the exact sequence

$0 \rightarrow R \rightarrow S \rightarrow S/R \rightarrow 0$  splits and hence  $R$  is a direct summand of  $S$ .

Corollary 3.2: Let  $R_0$  be a commutative ring and  $R_0 \subset R$  be an extension of rings. If there exists a ring  $S$  such that  $R_0 \subset R \subset S$  with  $S$  projective as  $R_0$ -module, then  $R_0$  is a direct summand of  $R$  as  $R_0$ -module.

Proof: By Theorem 3.1 there exists a splitting  $S \rightarrow R_0$  as  $R_0$ -module. Restricting this to  $R$  we get a splitting  $R \rightarrow R_0$ .

Corollary 3.3: Let  $R_0 \subset R \subset S$  be as above with  $S$  projective as  $R_0$ -module, then  $R$  is a direct summand of  $S \oplus R/R_0$  as  $R_0$ -module.

Proof: Consider a decomposition of  $S$ :  $S = R_0 \oplus S_1$  as  $R_0$ -modules. Then  $R = R_0 \oplus R \cap S_1$ . We obtain a monomorphism  $f$  of  $R$  to  $S \oplus R/R_0 = R_0 \oplus R/R_0 \oplus S_1$  by setting  $f(x+y) = x + (y + R_0) + y$ , where  $x \in R_0, y \in R \cap S_1$ . It is clear that the mapping  $g: R/R_0 \rightarrow R \cap S_1$  defined by  $g(y+R_0) = y$  is an isomorphism. On the other hand,  $S_1 \oplus R/R_0 = S_1 \oplus R/R_0(g)$ , where  $R/R_0(g) = \{y+g(y) / y \in R/R_0\}$ . Hence  $R \cong R_0 \oplus R/R_0(g)$  and  $R_0 \oplus R/R_0(g) \oplus S_1 = R_0 \oplus R/R_0 \oplus S_1 = S \oplus R/R_0$ . Hence  $R$  is a direct summand of  $S \oplus R/R_0$  as  $R_0$ -module.

The proof of the Corollaries 3.4 and 3.5 are immediate if we use [S, Cor<sup>2</sup> 13] and [23, P-88].

Corollary 3.4:  $R \subset S$  extension of noetherian local rings.  $S$  is a finitely generated  $R$ -module of finite projective dimension. If  $S$  is regular local, Then  $R$  is a direct summand of  $S$ .

Corollary 3.5:  $R \subset S$  extension of rings.  $R$  is regular local, If  $S$  is Cohen Macaulay local ring, then  $R$  is a direct summand of  $S$  if  $S$  is finite as  $R$ -module.

Remark: The next proposition is proved in [20]. The authors assume that  $R$  is a direct summand of the extension  $S$ . Our theorem above shows that the condition is not necessary. For the sake of completeness we give below the full proof of this statement.

If  $P$  is  $R$ -projective an element  $u \in P$  is said to be unimodular if  $\exists f \in \text{Hom}(P, R)$  with  $f(u) = 1$ .

Proposition 3.6: Let  $R \subset S$  be a projective extension of commutative rings. Consider the following statements:

(i). If  $P$  is a projective  $S$ -module, then an  $R$ -unimodular element of  $P$  is also  $S$ -unimodular.

(ii) If  $\mathfrak{m}$  is a maximal ideal of  $R$ , then  $\mathfrak{m}S$  is a maximal ideal of  $S$ .

Then (i) implies (ii).

Proof: Let  $\mathfrak{m}$  be a maximal ideal of  $R$ . Clearly  $\mathfrak{m}S \neq S$ , for suppose  $\mathfrak{m}S = S$ . Since  $S$  is  $R$ -projective, the extension  $F/\subset S$  splits. Let  $\Theta: S \rightarrow R$  be a splitting as  $R$ -module.

Then  $\Theta/R = 1_R$ . Here  $\mathfrak{m}S = S$  implies  $\Theta(\mathfrak{m}S) = \Theta(S)$

$$\Rightarrow \mathfrak{m}\Theta(S) = R \Rightarrow \mathfrak{m}R = R$$

$$\Rightarrow \mathfrak{m} = R, \text{ contradiction.}$$

So,  $\mathfrak{m}S \neq S$ . Let  $x \in S \setminus \mathfrak{m}S$ . We show that  $\mathfrak{m}S + xS = S$ .

Notice

that  $\text{Hom}_R(S, R) \rightarrow \text{Hom}_{R/\mathfrak{m}}(S/\mathfrak{m}S, R/\mathfrak{m})$  is surjective,

Since  $S$  is  $R$ -projective,  $S/\mathfrak{m}S$  is a  $R/\mathfrak{m}$ -vector space and

$\bar{x}$  is nonzero in  $S/\mathfrak{m}S$  where  $\bar{x} = x + \mathfrak{m}S$ ,  $\exists R/\mathfrak{m}$ -linear map on  $S/\mathfrak{m}S$  which does not vanish at  $\bar{x}$ ; lifting it to

$\text{Hom}_R(S, R)$  there exists  $f: S \rightarrow R$ ,  $R$ -linear map such that

$f(x) \notin \mathfrak{m}$ . Hence if  $O_R(x) = \left\{ \lambda(x) / \lambda \in \text{Hom}_R(S, R) \right\}$ ,

Then  $O_R(x) \not\subset \mathfrak{m}$ . Hence  $\mathfrak{m} + O_R(x) = R$ .

Let  $u = (a, x) \in P = S \oplus S$ . If  $\pi_1$ , and  $\pi_2$  are the projections of  $S \oplus S$ ,  $\theta\pi_1 + \lambda\pi_2 \in \text{Hom}(P, R)$ .

$$\begin{aligned}\text{Also, } (\theta\pi_1 + \lambda\pi_2)(u) &= \theta(a) + \lambda(x) \\ &= a + \lambda(x) = 1.\end{aligned}$$

Hence  $u$  is  $R$ -unimodular element. Then by assumption it is also  $S$ -unimodular i.e. there exists  $g \in \text{Hom}_S(P, S)$  such that  $g(u) = 1$ ; But  $g(u) = g(a, x) = g(a(1, 0) + x(0, 1))$

$$= a g(1, 0) + xg(0, 1)$$

$$\in aS + xS \subset \mathfrak{m}_S + xS.$$

i.e.  $1 \in S + xS$ .

Hence  $S + xS = S$ .

Thus  $\mathfrak{m}_S$  is maximal in  $S$ .

## CHAPTER - IV

### CRITERION FOR REGULARITY

In this chapter, by a ring we mean a commutative ring and modules are finitely generated. Here we give a criterion for a local ring  $R$  to be regular. M. Auslander [4, Theorem B, Corollary-5] proved that a Cohen Macaulay unique factorization domain  $R$  is regular if projective dimension of  $\mathcal{O}_\mathfrak{p}$  is finite for every unmixed ideal  $\mathfrak{O}$  of height less than 2. Besides Bourbaki's theorem on torsion free modules over integrally closed domains, which is an essential point in the proof of M. Auslander, we make use of Macrae's result [21, Prop-4.3] on projective dimension of grade one grade unmixed ideals in a noetherian domain to prove the theorem which gives the criterion for a local ring to be regular. W. Bruns [9] had shown that if  $\text{Pd } \mathfrak{O} < \infty$  for every ideal  $\mathfrak{O}$  all of whose associated prime ideals have height at most 2, then  $R$  is regular local. Here we have deleted the additional assumption that  $R$  is a Cohen - Macaulay unique factorization domain. We state the theorem.

Theorem: A local ring  $R$  is regular if projective dimension of  $\mathcal{O}_\mathfrak{p}$  is finite for every unmixed ideal  $\mathfrak{O}$  of height less than  $\dim R = 2$ .

Remark:= Let  $k$  be a nonnegative integer. A local ring  $R$

is said to satisfy the  $S_k$ -condition if whenever  $\mathfrak{p}$  is a prime with height  $\geq k$ , then  $\text{depth } R_{\mathfrak{p}} \geq k$ . If  $R_{\mathfrak{p}}$  is regular local for all prime ideals  $\mathfrak{p}$  with height  $\leq k$ , then  $R$  is said to satisfy the  $R_k$ -condition. Serre's criterion on normality says that a local ring is an integrally closed domain if and only if it satisfies  $S_2$  and  $R_1$  [22, Theorem - 39]. Notice that our  $S_k$ -condition is somewhat different from that of [22].

Lemma 4.1: A local ring  $R$  satisfies the  $S_{k+1}$ -condition if projective dimension of  $\mathfrak{p}$  is finite for all prime ideals  $\mathfrak{p}$  of height  $k$ .

Proof: Suppose  $\mathfrak{q}$  is a prime ideal of height  $\geq k+1$ . Let  $\mathfrak{p}$  be any prime ideal of height  $k$  contained in  $\mathfrak{q}$ . Clearly  $\text{Pd}_{R_{\mathfrak{p}}}(pR_{\mathfrak{p}})$  and  $\text{Pd}_{R_{\mathfrak{q}}}(pR_{\mathfrak{q}})$  are both finite and

$$\text{pd}_{R_{\mathfrak{q}}}(pR_{\mathfrak{q}}) \geq \text{pd}_{R_{\mathfrak{p}}}(pR_{\mathfrak{p}}) = k-1, \text{ the latter holds since } R_{\mathfrak{p}} \text{ is}$$

regular local. Hence  $\text{pd}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/pR_{\mathfrak{q}}) \geq k$ . Also  $R_{\mathfrak{q}}/pR_{\mathfrak{q}}$  being

an integral domain and not a field, we see that

$\text{depth}(R_{\mathfrak{q}}/pR_{\mathfrak{q}}) \geq 1$ . Then the well known depth formula

$$\text{depth}(R_{\mathfrak{q}}/pR_{\mathfrak{q}}) + \text{pd}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/pR_{\mathfrak{q}}) = \text{depth } R_{\mathfrak{q}}$$

implies that

depth  $R_q \geq k+1$ , i.e.  $R$  satisfies  $S_{k+1}$ -condition.

Lemma 4.2: A local ring  $R$  is a unique factorization domain if  $\text{pd}_R p$  is finite for every prime ideal  $p$  of height  $\leq 1$ .

Proof: Lemma 4.1 implies that  $R$  satisfies  $S_2$ -condition. Also  $\text{pd}_R p$  is finite implies  $R_p$  is regular local. Hence  $R$  also satisfies the condition  $R_1$ . Then by Serre's normality criterion quoted above,  $R$  is a (normal) domain. Now if  $p$  is a prime ideal of height 1, it is clearly grade one grade unmixed. Such a prime ideal is principal by Macrae's theorem [21, Prop-4.3]. Then it is well known that  $R$  must be a unique factorization domain.

Proof of the Theorem: The rest of the proof relies upon the material developed in M. Auslander's article quoted in the beginning. Lemmas 4.1 and 4.2 show that  $R$  is a unique factorization domain satisfying the condition  $S_3$ . We shall conclude  $R$  is regular by showing that  $\text{pd}_R M$  is finite for every nonzero module  $M$ . Let  $A$  be a second syzygy of  $M$  so that there exists an exact sequence  $0 \rightarrow A \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$   
..... (\*)

with  $F_0$  and  $F_1$  free modules. If  $A$  were free, we would have  $\text{pd}_R M \leq 2$  and we would be through. Suppose  $A$  is nonfree.  $R$  being integrally closed,  $A$  is a reflexive module.

Bourbaki's Theorem [21, Theorem-6, §4] then gives an exact

sequence

$$0 \rightarrow F \rightarrow A \rightarrow \mathcal{O}_1 \rightarrow 0 \dots \quad (**)$$

where  $F$  is a free module and  $\mathcal{O}_1$  is an ideal of  $R$ . Now  $\mathcal{O}_1$  cannot be a principal ideal, for otherwise the sequence (\*\*) will split and  $A$  will be free. This means on the one hand  $\mathcal{O}_1$  is neither  $0$  nor  $R$  and on the other hand  $F \neq (0)$ , the latter holds since  $F = (0)$  forces  $\mathcal{O}_1$  to be reflexive, i.e. principal,  $R$  being a unique factorization domain. Let  $p \in \text{Ass}(R/\mathcal{O}_1)$ . We claim  $\text{height } p \leq 2$ . For suppose  $\text{height } p \geq 3$ . Since  $R$  satisfies  $S_3$ -condition,  $\text{depth } R_p \geq 3$ . Clearly also  $\text{depth } R_p/\mathcal{O}_1 R_p = 0$ , so that  $\text{depth } \mathcal{O}_1 R_p = 1$ . Localizing (\*\*) at  $p$ , we get the exact sequence  $0 \rightarrow F_p \rightarrow A_p \rightarrow \mathcal{O}_1 R_p \rightarrow 0 \dots (***)$  Since  $A$  is a nonzero reflexive module and  $\text{depth } R_p > 2$ , it follows that  $\text{depth } A_p \geq 2$  [4, Lemma 2]. Then the exact sequence (\*\*\*) implies  $\text{depth } F_p = 2$ . But  $F$  being a nonzero free module  $\text{depth } F_p = \text{depth } R_p \geq 3$ , a contradiction. Hence the claim is proved.

Next  $R$  being a unique factorization domain, one can apply procedure [4, Proposition 4] to find an unmixed ideal  $\mathcal{D}$  of height 2 isomorphic to  $\mathcal{O}_1$ . Applying the long exact sequence for  $\text{Ext}$ . to the short exact sequences (\*) and (\*\*) we get for  $j > 2$ , the isomorphisms  $\text{Ext}_R^j(\mathcal{D}, \quad) \cong \text{Ext}_R^j(\mathcal{O}_1, \quad) \cong \text{Ext}_R^j(A, \quad) \cong \text{Ext}_R^{j+2}(M, \quad)$ . Since  $\mathcal{D}$  is an unmixed

ideal of height 2,  $\text{pd}_R \mathfrak{b}$  is finite by assumption so that

$\text{Ext}_R^j(\mathfrak{b}, ) = (0)$  for  $j$  sufficiently large, i.e.

$\text{Ext}_R^{j+2}(M, ) = (0)$  for  $j$  sufficiently large. Hence projective dimension of  $M$  is finite i.e.  $R$  is regular local.

□

## CHAPTER - V

### AN APPLICATION OF HOMOLOGICAL DUALITY TO GENERALIZED M-REGULAR SEQUENCES

Throughout this chapter modules are assumed to be finitely generated. Suppose  $R$  is a regular local ring and  $M$  a nonzero  $R$ -module. The familiar notion of  $M$ -sequence is generalized in ([14], §2, Definition ) as follows: a sequence  $P_1, P_2, \dots, P_n$  of non-free modules is said to be an  $M$ -sequence if  $\text{Tor}_1^R ( M \otimes P_1 \otimes \dots \otimes P_{i-1}, P_i ) = (0)$  for  $1 \leq i \leq n$ . The sum  $\sum P_i$  is defined to be the length of the  $M$ -sequence  $P_1, P_2, \dots, P_n$ . It is shown in [14] that the length of an  $M$ -sequence  $P_1, P_2, \dots, P_n$  is at most depth  $M$ , that any  $M$ -sequence can be extended to a maximal  $M$ -sequence, and that any two maximal  $M$ -sequences have the same length namely depth  $M$ . Suppose  $\mathcal{O}_i$  denotes the annihilator of  $P_i$  for  $1 \leq i \leq n$ . We assume that the ideals  $\mathcal{O}_i$  are nonzero for all  $1 \leq i \leq n$ . It is shown in ([15], Theorem ?) that  $\exists x_i \in \mathcal{O}_i$  such that  $x_1, x_2, \dots, x_n$  is an  $M$ -sequence. In this chapter, we shall apply Strebel's homological duality ([28], §3, Theorem B) to establish the following Theorem:

Theorem 5.1: Suppose  $R$  is a regular local ring and  $M$  a nonzero  $R$ -module. Let  $\{P_1, P_2, \dots, P_n\}$  be a sequence of nonfree perfect modules over  $R$ , forming an  $M$ -sequence of length  $d$  say. Let  $\mathcal{O}_i = \text{annihilator of } P_i$  (which are clearly nonzero) and let  $d_i = \text{Pd } P_i$  for  $1 \leq i \leq n$ . Then for every  $i$  in  $1 \leq i \leq n$  there exist  $d_i$  elements  $x_1^{(i)}, x_2^{(i)}, \dots, x_{d_i}^{(i)}$  in  $\mathcal{O}_i$  such that the sequence of  $\sum_{i=1}^n d_i$  elements  $\{x_j^{(i)}\}, 1 \leq i \leq n, 1 \leq j \leq d_i$  form an  $M$ -sequence in the usual sense.

Let  $R$  be any noetherian local ring and  $A$  be a perfect  $R$ -module of projective dimension  $d$ . Denote by  $A^*$ , the  $R$ -module  $\text{Ext}_R^d(A, R)$ . Then Strebel's homological duality ([28], 3, Theorem B) asserts that there exists natural isomorphism.

$$\text{Tor}_{d-j}^R(A, \ ) \cong \text{Ext}_R^j(A^*, \ ) \dots\dots\dots (*)$$

of connected sequences of functors for  $j \geq 0$ . Moreover  $\text{Ext}_R^d(A^*, R)$  is naturally isomorphic to  $A$ .

Lemma 5.2. Suppose  $P$  is a nonzero perfect module of projective dimension  $d$  over the regular local ring  $R$ . Let  $M$  be any  $R$ -module such that  $\text{Tor}_1^R(P, M) = (0)$ . Then  $\text{Ext}_R^j(P, M) \neq (0)$  for  $0 \leq j \leq d-1$ .

Proof: By 'rigidity',  $\text{Tor}_j^R(P, M) = (0)$  for  $j \geq 1$

Then the duality isomorphism (\*) above means in this case

$$\text{Tor}_{d-j}^R(P, M) \cong \text{Ext}_R^j(P^*, M)$$

Hence  $\text{Ext}_R^j(P^*, M) = (0)$  for  $0 \leq j \leq d-1$ .

By D.Rees [26] or [22, Theorem 26], this is equivalent to the existence of an M-sequence of length  $d$  in the annihilator of  $P^*$ . Since  $\text{Ext}_R^d(P^*, R) \cong P$ , it is easily seen that the annihilator of  $P$  and  $P^*$  are the same. Hence by applying D.Rees's theorem again we get,  $\text{Ext}_R^j(P, M) = (0)$  for  $0 \leq j \leq d-1$ .

Lemma 5.3. Let  $R$  be any noetherian local ring. Let  $P, M_1, M_2, \dots, M_n$  be a sequence of nonzero finitely generated  $R$ -modules such that  $\text{Ext}_R^j(P, M_i) = (0)$  for  $j = 0, 1, 2, \dots, t-1$ , where  $t \geq 1$ . Then  $\exists x_1, x_2, \dots, x_t$  in the annihilator of  $P$  forming an  $M_i$ -sequence simultaneously for all  $1 \leq i \leq n$ .

Proof: Let  $\mathcal{O}$  denote the annihilator of  $P$ . Suppose  $t = 1$  so that  $\text{Hom}(P, M_i) = (0)$  for  $1 \leq i \leq n$ . Then  $\mathcal{O}$  is not contained in any prime ideal associated to  $M_i$ , this being true for all  $1 \leq i \leq n$ . Hence there exists  $x \in \mathcal{O}$  which is not in any prime ideal associated to any of the modules  $M_1, M_2, \dots, M_n$ . Such an element  $x$  is a nonzero divisor of every one of the modules  $M_1, M_2, \dots, M_n$ . This proves the lemma when  $t = 1$ .

Suppose  $t > 1$ . We use induction on  $t$ . By what precedes, there exists  $x_1 \in \mathcal{O}$  which is not a zero divisor of every one of the modules  $M_1, M_2, \dots, M_n$ . Then it is easily seen that  $\text{Ext}_R^j(P, M_i/x_1 M_i) = (0)$  for  $1 \leq i \leq n$  and  $j = 0, 1, 2, \dots, t-2$ . Inductively, we can assume the existence of elements  $x_2, x_3, \dots, x_t$  in  $\mathcal{O}$  forming an  $M_i/x_1 M_i$ -sequence simultaneously for  $1 \leq i \leq n$ . Then  $x_1 \cdot x_2 \cdot \dots \cdot x_t$  is an  $M_i$ -sequence simultaneously for all  $1 \leq i \leq n$ .

Proof of Theorem 5.1: Since any permutation of

$\{P_1, P_2, \dots, P_n\}$  is still an  $M$ -sequence [14, §2, Theorem], we get  $\text{Tor}_1^R(M \otimes P_1 \otimes P_2 \otimes \dots \otimes P_i, P_n) = (0)$  for

$0 \leq i \leq n-1$ , where the interpretation when  $i = 0$  is

$\text{Tor}_1^R(M, P_n) = (0)$ . By Lemma 5.2 this means that

$\text{Ext}_R^j(P_n, M \otimes P_1 \otimes \dots \otimes P_i) = (0)$  for  $0 \leq j \leq d_n - 1$  and

$0 \leq i \leq n-1$ . Applying Lemma 5.3, we can find elements

$x_1^{(n)}, x_2^{(n)}, \dots, x_{d_n}^{(n)}$  in  $\mathcal{O}_n$  forming an

$M \otimes P_1 \otimes P_2 \otimes \dots \otimes P_i$ -sequence simultaneously for

$0 \leq i \leq n-1$ . For the sake of simplicity let  $I_k$  be the ideal

generated by  $x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)}$ . Using induction on

$k$  and the short exact sequences

$$\begin{array}{c}
 0 \rightarrow (M/I_k M) \otimes P_1 \otimes \dots \otimes P_{i-1} \xrightarrow{x_{k+1}^{(n)}} (M/I_k M) \otimes \\
 P_1 \otimes \dots \otimes P_{i-1} \rightarrow (M/I_{k+1} M) \otimes P_1 \otimes \dots \otimes P_{i-1} \rightarrow 0
 \end{array}$$

one can easily see that

$$\text{Tor}_1^R (M/I_k M \otimes P_1 \otimes \dots \otimes P_{i-1}, P_i) = (0) \text{ for}$$

$k = 1, 2, \dots, d_n$ . Taking  $k = d_n$ , this means

$P_1, P_2, \dots, P_{n-1}$  is a  $M/(x_1^{(n)}, x_2^{(n)}, \dots, x_{d_n}^{(n)})M$  - sequence

Inductively on  $n$  one can assume the existence of elements

$x_1^{(i)}, x_2^{(i)}, \dots, x_{d_i}^{(i)}$  in  $\mathcal{O}_i$  for  $1 \leq i \leq n-1$ , such that

$\{x_j^{(i)}\}_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq d_i}}$  is a  $M/(x_1^{(n)}, x_2^{(n)}, \dots, x_{d_n}^{(n)})M$  - sequence.

Then clearly the  $\sum d_i$  elements  $\left\{ x_j^{(i)} \right\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq d_i}}$

form an M-sequence.

**Corollary 5.4:** Suppose  $P_1, P_2, \dots, P_n$  is a sequence of nonzero perfect modules over a regular local ring  $R$  such that  $\text{Tor}_1^R (P_1 \otimes P_2 \otimes \dots \otimes P_{i-1}, P_i) = (0)$  for  $1 \leq i \leq n$ . Then,  $P_1 \otimes P_2 \otimes \dots \otimes P_n$  is a perfect module of grade equal to  $\sum_{i=1}^n \text{grade } P_i$ .

**Proof:** It is easy to see that one can assume without loss of generality that none of the  $P_i$  is a free module. Then  $P_1, P_2, \dots, P_n$  is an R-sequence of modules. Let  $\text{Pd } P_i = d_i$

for  $1 \leq i \leq n$ . Let  $\sigma_i =$  annihilator of  $P_i$  for  $1 \leq i \leq n$ . Clearly  $\sigma_i \neq (0)$  for every  $i$ . By above Theorem, there exist  $x_1^{(i)}, x_2^{(i)}, \dots, x_{d_i}^{(i)}$  in  $\sigma_i$  such that the elements

$\{x_j^{(i)}\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq d_i}}$  form an  $R$ -sequence in the usual sense.

Clearly  $\sigma_i$  annihilates  $P_1 \otimes P_2 \otimes P_3 \otimes \dots \otimes P_n$  for every  $i$  in the range  $1 \leq i \leq n$ . Hence  $\sigma_1 + \sigma_2 + \dots + \sigma_n$  annihilates  $P_1 \otimes P_2 \otimes \dots \otimes P_n$ . Since  $\sigma_1 + \sigma_2 + \dots + \sigma_n$  contains the elements  $x_j^{(i)}$  for  $1 \leq i \leq n$  and  $1 \leq j \leq d_i$ , we find that  $\text{grade}(P_1 \otimes P_2 \otimes \dots \otimes P_n) \geq d_1 + d_2 + \dots + d_n$ . Also the conditions

$\text{Tor}_j^R(P_1 \otimes P_2 \otimes \dots \otimes P_{i-1}, P_i) = (0)$  for  $j \geq 1$  and  $1 \leq i \leq n$  imply that

$$\text{Pd}(P_1 \otimes P_2 \otimes \dots \otimes P_i) = \text{Pd}(P_1 \otimes P_2 \otimes \dots \otimes P_{i-1}) + \text{Pd} P_i.$$

Adding we get,

$$\text{Pd}(P_1 \otimes P_2 \otimes \dots \otimes P_n) = \text{Pd} P_1 + \text{Pd} P_2 + \dots + \text{Pd} P_n = d_1 + d_2 + \dots + d_n.$$

Hence,  $\text{grade}(P_1 \otimes P_2 \otimes \dots \otimes P_n)$

$\geq \text{Pd}(P_1 \otimes P_2 \otimes \dots \otimes P_n)$ . Since the inequality in the other direction is clear, we find that  $P_1 \otimes P_2 \otimes \dots \otimes P_n$  is a perfect module of grade equal to  $\text{Pd}(P_1 \otimes P_2 \otimes \dots \otimes P_n) = \sum \text{grade} P_i$ .

Corollary 5.5: Let  $R$  be a regular local ring and  $M$  a nonzero  $R$ -module. Suppose  $P_1, P_2, \dots, P_n$  is a sequence of nonfree perfect modules over  $R$  forming an  $M$ -sequence.

Let  $d = \sum_{i=1}^n \text{Pd } P_i$ . Then the annihilator  $P_1 \otimes P_2 \otimes \dots \otimes P_n$  contains an  $M$ -sequence, in the usual sense of length  $d$ .

Proof: By [14, §2, Theorem 2], one sees that  $P_1, P_2, \dots, P_n$  is an  $R$ -sequence and

$$\text{Tor}_j^R (P_1 \otimes P_2 \otimes \dots \otimes P_n, M) = (0) \text{ for } j > 1.$$

Corollary 5.4 implies that  $P_1 \otimes P_2 \otimes \dots \otimes P_n$  is a perfect module of grade equal to  $d$ . Then by lemma 5.2, we get

$$\text{Ext}_R^j (P_1 \otimes P_2 \otimes \dots \otimes P_n, M) = (0) \text{ for } 0 \leq j \leq d-1.$$

Hence the annihilator of  $P_1 \otimes P_2 \otimes \dots \otimes P_n$  contains an  $M$ -sequence of length  $d$ .

CHAPTER - VI

BNSI Rings.

Throughout this chapter  $R$  is assumed to be a non-commutative local ring i.e.  $R$  is a ring with Jacobson radical  $\mathcal{O}_r$  such that  $R/\mathcal{O}_r$  is a division ring. All  $R$ -modules are assumed to be finitely generated.

Let  $M$  be a left  $R$ -module. Since  $M$  is finitely generated,  $M/\mathcal{O}_r M \cong R/\mathcal{O}_r \otimes_R M$  is a finite dimensional (left) vector space over  $R/\mathcal{O}_r$ . Let  $[M/\mathcal{O}_r M : R/\mathcal{O}_r] = r$ . We take a free module  $F_0$  of rank  $r$  with basis  $\{e_i\}$ . Suppose  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$  be a set of generators of  $M/\mathcal{O}_r M$ .

Let  $x_1, x_2, \dots, x_r$  be lifts of  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_r$  to  $M$ . Then the mapping  $F_0 \xrightarrow{f_0} M, e_i \rightarrow x_i$  is an epimorphism with  $\text{Ker } f_0 \subseteq \mathcal{O}_r F_0$ . Such an epimorphism is called a minimal epimorphism. Thus for any left  $P$ -module  $M$  there exists a minimal epimorphism  $P \rightarrow M \rightarrow 0$ . Let  $F_0 \xrightarrow{f_0} M$  be a minimal epimorphism and  $K_1$  be its kernel. Also, let  $F_1 \rightarrow K_1 \rightarrow 0$  be a minimal epimorphism. So we get a minimal free resolution by (left) free  $P$ -modules,

$$F_* \quad \cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

Let  $R/\mathfrak{A} = K$ . Consider the complex

$$M \otimes F_* \rightarrow \dots \rightarrow K \otimes F_n \rightarrow K \otimes F_{n-1} \rightarrow \dots \rightarrow K \otimes F_1 \rightarrow K \otimes F_0 \rightarrow K \otimes M \rightarrow 0$$

where each of the maps  $K \otimes F_n \rightarrow K \otimes F_{n-1}$  are zero.

$$\text{Then } \text{Tor}_i^R(K, M) \cong K \otimes F_i.$$

So,

$$\begin{aligned} \dim_K \text{Tor}_i^R(K, M) &= \dim (K \otimes F_i) \\ &= \text{rank } F_i. \end{aligned}$$

Two left  $R$ -modules  $A$  and  $B$  are projectively equivalent if there exist projective left  $R$ -modules  $P$  and  $Q$  such that  $A \oplus P \cong B \oplus Q$ . If  $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$  and  $Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0$  are any two presentations of  $M$ , then  $\text{Co-Ker} (P_0^* \rightarrow P_1^*)$  and  $\text{Co-Ker} (Q_0^* \rightarrow Q_1^*)$  are projectively equivalent as right  $R$ -modules. If both of these presentations are minimal, the co-kernels are isomorphic.

Let  $DM = \text{Coker} (F_0^* \rightarrow F_1^*)$  where  $F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$  is a minimal presentation of  $M$ . Thus  $F_0^* \rightarrow F_1^* \rightarrow DM \rightarrow 0$  is a presentation of  $DM$ . Since free modules are reflexive,  $\text{coker} (F_1^{**} \rightarrow F_0^{**}) = \text{Coker}(F_1 \rightarrow F_0) = M$ .

So,  $M$  and  $D(DM)$  are projectively equivalent. It follows that  $M$  is free if and only if  $DM = 0$ . The importance of  $DM$  for our purpose, comes from the following exact sequence

established in more general form by Auslander [2]

$$0 \rightarrow \text{Ext}_R^1(DM, R) \rightarrow M \rightarrow M^{**} \rightarrow \text{Ext}_R^2(M, R) \rightarrow 0$$

Thus  $M$  is reflexive if and only if

$$\text{Ext}_R^i(DM, R) = 0 \text{ for } i = 1, 2.$$

The proof of the following proposition is identical with proposition 1.1 [24].

Proposition 6.1: The following are equivalent:

- (i) Every reflexive left  $R$ -module is free.
- (ii) For every left  $R$ -module  $M$ , if

$$\text{Ext}_R^i(M, R) = 0 \text{ for } i = 1, 2, \text{ then } M \text{ is free.}$$

Proposition 6.2: Let  $R$  be a ring. Suppose  $M$  is a left  $R$ -module contained in a left free  $R$ -module  $P$ . If  $M \not\subseteq \mathcal{O}_1 P$ , then  $M$  is free.

Proof: We see that  $M/M \cap \mathcal{O}_1 P$  is a nonzero  $R/\mathcal{O}_1$ -submodule of  $P/\mathcal{O}_1 P$ . Since  $R/\mathcal{O}_1$  is a division ring,  $M/M \cap \mathcal{O}_1 P$  is a nonzero left vector space over  $R/\mathcal{O}_1$ . Hence there exists a finitely generated nonzero free left  $R$ -module  $F$  and an epimorphism  $f: F \rightarrow M$  such that the induced homomorphism

$f: F/\mathcal{O}_1 F \xrightarrow{f} M/M \cap \mathcal{O}_1 P$  is an isomorphism. Let the composite  $f \circ \text{incl.}$   
 $F \rightarrow M \xrightarrow{\text{incl.}} P$  map be  $\theta$ ; then the induced homomorphism

$\bar{\theta} : F/\mathcal{O}_1 F \rightarrow P/\mathcal{O}_1 P$  is the composite of the isomorphism  $F/\mathcal{O}_1 F \cong M/M \cap \mathcal{O}_1 P$  and the monomorphism  $M/M \cap \mathcal{O}_1 P \rightarrow P/\mathcal{O}_1 P$  and so  $\bar{\theta}$  is a monomorphism. It is a split monomorphism since  $R/\mathcal{O}_1$  is a division ring. Let  $\bar{\phi} : P/\mathcal{O}_1 P \rightarrow F/\mathcal{O}_1 F$  be an  $R$ -linear splitting of  $\bar{\theta}$  so that

$\bar{\phi} \circ \bar{\theta} : F/\mathcal{O}_1 F \rightarrow F/\mathcal{O}_1 F$  is the identity map. This

$\bar{\phi}$  is induced from an  $R$ -linear map

$\phi : P \rightarrow F$ . Then  $\phi \circ \theta : F \rightarrow F$  is such that

$\overline{\phi \circ \theta} = \bar{\phi} \circ \bar{\theta} = \text{identity}$ . Hence  $\phi \circ \theta$  is an automorphism of  $F$ . Let  $\phi \circ \theta = h$ . Define  $\tilde{\phi} : P \rightarrow F$  as  $\tilde{\phi} = h^{-1} \circ \phi$ .

Then  $\tilde{\phi} \circ \theta = h^{-1} \circ \phi \circ \theta = \text{identity map on } F$ . So,  $\theta : F \rightarrow P$  is a monomorphism. This shows that the map  $F \xrightarrow{f} M$  is a monomorphism. Since it is already surjective, so an isomorphism.

Definition 6.3: For any left  $R$ -module  $M$  and for any  $i \geq 0$ , let  $\beta_i(M) = \dim_K \text{Tor}_i^R(K, M)$  as a left vector space over  $K$ .  $\beta_i(M)$  is called the  $i$ th Betti number of  $M$ .

Definition 6.4:  $R$  is called a (left) BNSI ring (Betti numbers strictly increasing) ring if for every nonfree left  $R$ -module  $M$ , the sequence  $\{\beta_i(M)\}_{i \geq 1}$  is strictly increasing.

Proposition 6.5: Suppose  $R$  is a left and right BNSI ring with Jacobson radical  $\mathcal{O}$  nilpotent. Let  $M$  be a left  $R$ -module such that  $M \subseteq \mathcal{O}F_0$  for some left <sup>free</sup>  $R$ -module  $F_0$ . Then for any right free  $R$ -module  $F$ ,  $DM \not\subseteq F\mathcal{O}$ .

Proof: Let  $R^m \rightarrow R^n \rightarrow M \rightarrow 0$  ----- (\*)

be a minimal presentation of  $M$ . Since  $M \subseteq \mathcal{O}F_0$  and

$0 \rightarrow M \rightarrow F_0 \rightarrow F_0/M \rightarrow 0$  exact,

$R^m \rightarrow R^n \rightarrow F_0 \rightarrow F_0/M \rightarrow 0$  is a part of a minimal resolution  $F_0/M$ . Then  $\beta_1(F_0/M) = n$  and  $\beta_2(F_0/M) = m$ .

Since  $R$  is a BNSI ring,  $m > n$ . Dualizing (\*) we get

$0 \rightarrow M^* \xrightarrow{\theta} R^n \rightarrow R^m \rightarrow DM \rightarrow 0$  exact

Since  $M \subseteq \mathcal{O}F_0$ ,  $\text{ann } M \supseteq \text{ann } \mathcal{O}$ . By hypothesis  $\mathcal{O}$  is nilpotent. Let  $t$  be the index of nilpotency of  $\mathcal{O}$ .

So,  $\mathcal{O}^t = (0)$  and  $\mathcal{O}^{t-1} \neq (0)$ . Then  $(0) = \mathcal{O}^t = \mathcal{O}^{t-1}\mathcal{O}$ .

Thus there exists  $x (\neq 0) \in \mathcal{O}^{t-1}$  such that  $x\mathcal{O} = 0$ .

So,  $\text{ann } \mathcal{O} \neq 0$ . As  $\text{ann } M \supseteq \text{ann } \mathcal{O}$ ,  $\text{ann } M \neq 0$ .

Claim: For any  $f \in M^*$ ,  $f(M) \subseteq \mathcal{O}$ . For if  $f(M) \not\subseteq \mathcal{O}$ , there exists  $x \in M$  such that  $f(x) \notin \mathcal{O}$ . Then  $\overline{f(M)} = R/\mathcal{O}$ . So,  $\mathcal{O} + f(M) = R$ . Thus  $1 = \lambda + y$  where  $\lambda \in f(M)$ ,  $y \in \mathcal{O}$ . <sup>This</sup> implies  $\lambda = 1 - y$ , a unit. This shows that  $f(M) = R$  contradicts the

fact that  $\text{ann} M \neq 0$ .

Since  $\text{ann} M \neq 0$ ,  $\text{ann} M^* \neq 0$  as a right  $R$ -module.

So, by above argument  $f(M^*) \subseteq \mathcal{O}_1$ . This shows that

$$\text{Hom}(M^*, \mathcal{O}_1) = \text{Hom}(M^*, R).$$

$$\Rightarrow \text{Hom}(M^*, \mathcal{O}_1 R^n) = \text{Hom}(M^*, R^n).$$

Thus the image of  $M^*$  under  $\theta$  is contained in  $\mathcal{O}_1 R^n$ . Using this it can be seen that the presentation  $R^n \rightarrow R^m \rightarrow DM \rightarrow 0$  is minimal. Now if  $DM \subseteq F\mathcal{O}_1$ , for a right free module  $F$ , the argument as above can be used to show that  $n > m$ , which ~~contradicts~~ the fact that  $m > n$ . Thus  $DM \not\subseteq F\mathcal{O}_1$  for any right free module  $F$ .

Theorem 6.6 : Let  $R$  be a left and right BNSI ring with  $\mathcal{O}_1$  nilpotent and let  $M$  be an indecomposable<sup>?</sup> left  $R$ -module contained in a left free  $R$ -module such that  $\text{Ext}^1(M, R) = (0)$ , then  $M$  is free.

Proof :  $M$  is contained in a left free  $R$ -module  $F_0$ . If  $M \not\subseteq \mathcal{O}_1 F_0$ , Proposition 6.2 shows that  $M$  is free. Suppose  $M \subseteq \mathcal{O}_1 F_0$ . Now for any module  $T$ ,  $T$  is a submodule of a free module if and only if  $\text{Ext}^1(DT, R) = 0$ . By hypothesis,  $\text{Ext}^1(M, R) = 0$ . Since  $M$  is projectively equivalent to  $D(DM)$  we get  $\text{Ext}^1(D(DM), R) = 0$ . So  $DM$  is a submodule of a free right  $R$ -module  $F$ . Since by assumption  $M \subseteq \mathcal{O}_1 F_0$

and  $M \neq 0$ , by Proposition 6.5,  $DM \not\subseteq F07$  for any right free module  $F$ . So, by Proposition 6.2,  $DM$  is free. This shows that  $M$  is free.

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## BIBLIOGRAPHY

1. Auslander, M. and Bridger, M. : Stable module Theory, Amer. Math. Soc. Providence, Rhode Island, 1969.
2. Auslander, M. Coherent functors, "Proc. of the Conference on Categorical Algebra", La Jolla 1965 ( Springer-Verlag), 189-231.
3. Auslander, M. and Goldman, O. : Maximal Orders, Trans. Amer. Math. Soc. Vol-97(1960)
4. Auslander, M.: Remarks on a Theorem of Bourbaki, Nagoya J. Math. 27, (1966)
5. Auslander, M. and Buchsbaum, D. : Co-dimension and multiplicity, Vol-68, No. 3, Nov-1958.
6. Bass, H. : On the ubiquity of Gorenstein rings, Math. Zeit. 82 (1963).
7. Bose, S. : Splitting of ring extensions, Ind. J. Pure appl. Math 16(4), April 1985.
8. Bose, S. : The homomorphism  $M^* \otimes N \rightarrow \text{Hom}(M, N)$ , Proc. Indian Aca. Sci (Math. Sci), Vol-94, No.1, Sept. 1985.
9. Bruns, W. : Die verallgemeinerung eines satzes von Bourbaki und einige Anwendungen, Manuscripta Mathematica, No. 4, 17(1975).

10. Bourbaki, N: Algebra Comm., Herman Paris (1961)
11. Cartzen, B., Small, L.W. and Stafford, J.T. :  
Proc. Am. Math. Soc. 82 (1981) 28.
12. Jinnu, M.I. : Reflexive modules over regular local rings, Arch. Math. Vol-XXVI, 1973.
13. Jothilingam, P. : A note on grade, Nagoya Math. J. Vol-59(1975).
14. Jothilingam, P. : A note on M-sequences, Journal of the Indian Math. Soc. 34(1970).
15. Jothilingam, P.: A note on M-sequences II, Journal of the Indian Math. Soc. 37 (1973).
16. Jothilingam, P. : When is a flat module projective?  
Indian Journal of Pure and Applied Math. 16(1984)
17. Kaplansky, I: Projective modules,  
Ann. Math. 68 (1958).
18. Lam T.Y.: Serre's Conjecture, Lecture notes in Math. Vol-635, Springer-Verlag(1978)
19. Lichtenbaum, S: On the vanishing of Tor. in regular local rings, Illinois J. of Math. Vol-10(1966)
20. Lissener, D and Lønsted, K: J. Algebra 50,  
454-462 (1978)
21. Macrae, R.E. : The homological dimension of certain ideals,  
Proc. Amer. Math. Soc. Vol-14, 1963.

22. Matsumura, H. : Commutative Algebra, W.A. Benjamin, Inc. New York, 1970.
23. Nagata, M: Local rings, Inter sc. Pub., New York, 1962.
24. Ramaras, M: Bettinnumbers and reflexive modules, Proc. Ring Theory edited by Robert Gordon March 2-6, 1971.
25. Ramaras, M: Maximal orders over regular local rings of dimension 2, Trans. Amer. Math. Soc., 141-142(1969)
26. Rees, D.: The grade of an ideal or module, Proc. Camb. Phil.Soc. 84(1957).
27. Serre, J.P.: Algèbre Locale - Multiplicités, Lecture notes in Mathematics (Springer-Verlag) No. 11(1965).
28. Sirebel R: On homological duality, Journal of Pure and Applied algebra 8(1976).
29. Vas conceles, W.V.: Reflexive modules over Gorenstein rings, Proc. Amer. Math. Soc. 19(1349-1355), 1968.