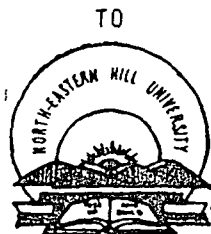


A STUDY OF EQUIVARIANCE PROBLEMS
AND INVARIANTS RELATED TO
GENERALIZATIONS OF
BORSUK-ULAM THEOREM
AND THE LEVEL OF SPACES

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I certify that the thesis entitled "A STUDY OF EQUIVARIANCE PROBLEMS AND INVARIANTS RELATED TO GENERALIZATIONS OF BORSUK ULAM THEOREM AND THE LEVEL OF SPACES", submitted by Mr. Basil S. Koikara for the Degree of Doctor of Philosophy of North-Eastern Hill University, Shillong, embodies the original work carried out by him under my supervision. He has been duly registered, and the thesis presented is worthy of being considered for the award of the Ph. D. degree. This work has not been submitted for any degree of any other university.

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Dedicated lovingly and respectfully

to my

Parents

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INTRODUCTION

Given two topological spaces together with certain groups acting on them, let us consider the question : Does there exist a continuous map between them which preserves these group actions, namely an equivariant map ? This general question in specific situations gives rise to several problems. For example, the existence of an equivariant map gets related to questions on the existence of embeddings of manifolds or existence of cross sections.

One may consider two situations : (1) One considers the extent to which a given continuous map between specific spaces with group actions can be made equivariant. (2) One looks for restrictions on specific spaces with group actions, so that he may have an affirmative answer to the above question.

The generalisations of the Borsuk-Ulam Theorem attempts to answer questions of type (1), as is made specific below :

The Classical Borsuk-Ulam Theorem states that a continuous map of an n -sphere into the Euclidean n -space maps some pair of antipodal points into a single point [2]. In a general setting, Borsuk-Ulam Theorem can be formulated in terms of the following data :

(i) A Domain which may be a space with group action or more generally, a fibre-bundle with a fibrewise action of a group.

(ii) A range which can be a space with group action or more generally, a fibre-bundle with fibrewise action of a group

(iii) A continuous map, which may or may not be equivariant, from the given domain to the range.

One then studies the "size" of the union of points of the domain space on which the map is equivariant or of the union of points of the orbits of the domain space which are mapped to a single point by the given map.

Determination of certain invariants of spaces, namely level or coindex answers questions of type (2) as is made specific below:

Another important aspect of problems related to equivariance of maps is the level of spaces. The level of a topological space X with a fixed point free involution T is defined to be the number $l(T, X) = \min \{ n : \text{there exists a } \mathbb{Z}_2\text{-equivariant map } f : X \longrightarrow S^{n-1} \}$, where the space X and the sphere S^{n-1} are considered to have \mathbb{Z}_2 -actions given by the involution T and the antipodal action respectively.

We now give below a brief historical survey of the progress of answers to the specific questions that have been answered in this thesis.

0.1 A BRIEF HISTORICAL SURVEY

While trying to study generalizations of the Borsuk-Ulam Theorem, one gives suitable meaning to the word "size" referred to above. By looking at the various developments centred around this problem, one notices that it is the (co)homological dimension that one attempts to estimate to a sufficiently satisfactory limit. To do this estimation, various algebraic invariants have been introduced. To name a few : homology index of C.T. Yang [41], cohomology co-index of P.E. Conner and E.E. Floyd [7], G-indices of Fadell-Husseini-Rabinowitz [16] and Jaworowski [22], characteristic polynomials [13], [33], certain ideals related to these polynomials, an ideal-valued index resulting from these [15], Nakaoka's invariant $\hat{I}(f)$ [31], the equivariant Lefschetz class $L(f)$ and the equivariant Euler class $e_G(f)$ of Necochea [34], and the self-intersection number $C(f)$ of Lin [25], [26] and Kahn [23], [24].

The last invariants attempt to measure the cohomology dimension of the Borsuk-Ulam set corresponding to a map of manifolds with group action. To be more specific, let $f : X \longrightarrow Y$ be a map of manifolds with \mathbb{Z}_p -action, p a prime, with the action of \mathbb{Z}_p free on X . Let T and T' denote respectively, the action of \mathbb{Z}_p on X and Y . The later generalisations of the Borsuk-Ulam Theorem attempt to study the cohomological dimension of the set $A(f) = \{x \in X / f(Tx) = T'f(x)\}$. In certain cases, for example, when X is a

product of spheres and Y , a manifold, none of the invariants that are at present available is able to give any information about the dimension of $A(f)$. In this work, we have used a deep analysis of the intrinsic nature of the problem, the natural properties and geometric interpretations of the Conner-Miller classes and the Bredon operation, to establish that the dimension of $A(f) \geq m+n-l$, where $X = S^m \times S^n$ and the dimension of Y is k . We state below the concerned result :

1.3.1 THEOREM

Let $f : S^m \times S^n \longrightarrow V^k$ ($k < m \leq n$) be a map of $S^m \times S^n$ into a manifold. Let $A(f) = \{(x,y) \in S^m \times S^n \mid f(-x,-y) = f(x,y)\}$. The cohom. dim $A(f) \geq m+n-k$.

We mentioned above the characteristic polynomials introduced by Dold and Nakaoka. They are used to generalize the Borsuk-Ulam Theorem to bundle maps of sphere bundles. In the present work, we have examined the possibility of generalising this to bundle maps of arbitrary manifold bundles. We have established that, as in the case when the manifold is a product of spheres, characteristic polynomials do not give satisfactory results. For example, when the manifold is $S^m \times S^n$, ($m \leq n$), the method of characteristic polynomials gives the dimension of the Borsuk-Ulam set associated to a fibre map to be greater than or equal to $l+m-k$, where l is the dimension of the base space and l , the fibre dimension of the range. We refer to the results :

2.3.1 THEOREM

Let $\pi : E \longrightarrow B$ be an $S^m \times S^n$ bundle ($1 < m \leq n$) with the fibrewise \mathbb{Z}_2 -action $A \times A$, and $\pi' : E' \longrightarrow B$ be a vector bundle of dimension k , with fibrewise antipodal action, over a paracompact space B , such that the quotient bundle $\bar{\pi} : \bar{E} \longrightarrow B$ has the cohomology extension property. Let $f : E \longrightarrow E'$ be a fibre-preserving equivariant map. Put $Z = f^{-1}(0)$ and let \bar{Z} denote the quotient by the action induced on Z . Now, if $q(t,s) \neq 0, \in H^*(B)[t,s]$ is a polynomial such that $q(t,s)|_{\bar{Z}} \neq 0$, then \exists polynomials $r_1(t,s), r_2(t,s) \in H^*(B)[t,s]$ such that $q(t,s) \cdot W'(t) = r_1(t,s) \cdot W_1(t,s) + r_2(t,s) \cdot W_2(t,s)$.

2.3.2 COROLLARY : If the fibre dimension of E' is l , then $\deg(W'(t)) = l$, $\deg(W_1(t,s)) = m+1$ and $\deg(W_2(t,s)) = 2n$. Then $q(t,s)|_{\bar{Z}} \neq 0$, for all polynomials whose degree in t and s is less than $m-k+1$.

This means that the $H^*(B)$ -homomorphism

$$\begin{array}{ccc} \bigoplus_{l+j = 0}^{m-k} H^*(B) \cdot t^l \cdot s^j & \longrightarrow & H^*(\bar{Z}) \\ t^l \cdot s^j & \longrightarrow & t^l \cdot s^j|_{\bar{Z}} \end{array}$$

is a monomorphism. Hence if $k \leq m$,

$$\text{Cohom. dim. } \bar{Z} \geq \text{Cohom. dim. } B + m - l.$$

In particular, if B is a point, we have an equivariant map $f : S^m \times S^n \longrightarrow \mathbb{R}^k$ and if $k \leq m \leq n$, then $\text{Cohom. dim. } \bar{Z} \geq m-k$.

We now proceed to use the Bredon operation and Conner-Miller classes, to extend the generalisations of Dold and Nakaoka, to bundle maps of arbitrary manifold bundles. We see, in fact, that this method in several cases, gives better estimates of the dimension of the Borsuk-Ulam set than the method of characteristic polynomials. For example, using this method, in the case mentioned above, we get the dimension of the Borsuk-Ulam set to be greater than or equal to $m+n+1-k$. The relevant results are

3.4.1 THEOREM

Let (X, π_1, M, K_1) and (Y, π_2, M, K_2) be G -bundles where M, K_1 and K_2 are closed differentiable manifolds of dimensions m, n_1, n_2 respectively, and such that there exist cohomology extensions of the fibres for both bundles. Let $f: X \longrightarrow Y$ be a fibre-preserving map and $A(f) = \{x \in X \mid f(T^l x) = T^l(f(x)), l = 0, 1, 2, \dots, p-1, T^l \in G\}$. Define $g: X \longrightarrow X \oplus Y$ by $g(x) = (x, f(x))$. Let \bar{g} denote the restriction of g to a fibre and let $\hat{Q}: H^k(K_1 \times K_2) \longrightarrow H^{pk}(K_1 \times_{\sigma} K_2)$ be the Bredon operation. Then if $\hat{Q}(\bar{g}_1(1))$ is non-zero, then $A(f)$ is non-empty and $\dim A(f) \geq m + n_1 - (p-1)n_2$.

3.5.1 COROLLARY

Let $f: X \longrightarrow Y$ be a fibre-preserving map from an n -sphere bundle with G action given by complex multiplication by p th roots of unity on S^n (n odd, when p odd), to a V^k -bundle with trivial G action, where V^k is a closed k -dimensional manifold, over the same closed m -dimensional manifold M .

Let $A(f) = \{x \in X \mid f(T^l x) = f(x), \quad l = 0, 1, 2, \dots, p-1 \}$.
 Then if $(p-1)k < n$, $\dim A(f) \geq m+n-(p-1)k$.

3.5.2 COROLLARY :

Let $f: X \longrightarrow Y$ be a fibre-preserving map from an $S^m \times S^n$ -bundle with G action induced by the diagonal action of complex multiplication by the p th roots of unity on $S^m \times S^n$ ($m \leq n$, m and n odd when p odd), to a V^k -bundle with trivial G action, where V^k is a closed k -dimensional manifold, over the same closed l -dimensional manifold M^l . Let $A(f) = \{(x,y) \in S^m \times S^n \mid f(T^l(x,y)) = f(x,y), \quad l = 0, 1, 2, \dots, p-1\}$. Now, if $(p-1)k < m \leq n$, then $\dim A(f) \geq l + m + n - (p-1)k$.

In earlier topological literature, the number $\ell-1$ was used under the name "co-index" (cfr. [7], [8]). In [8], Conner and Floyd computed the level of the real projective space $\mathbb{R}P^{2m-1}$ for certain lower dimensions ($\mathbb{R}P^{2m-1}$ here is considered to have a fixed point free involution induced by complex multiplication by i). A. Pfister and S. Stolz in [35] examine upper and lower bounds for the level of the Real Projective Space and the Complex Projective Space. Later Stolz in [39] computed fully, the level of the Real Projective Space $\mathbb{R}P^{2m-1}$. Other results on the level of spaces, in particular for Stieffel manifolds, are discussed by Dai and Lam in [12].

We have taken inspiration from the works of A. Pfister and S. Stolz and undertaken the computation of the level of the

Lens Space $L_k^{2m-1} \cong S^{2m-1}/\mathbb{Z}_k$. Using complex K-theory and some number theoretic arguments we establish a lower bound for the level of the Lens Space. We are also able to compute the level of L_k^{2m-1} , when k is odd. Using cohomotopy Euler classes and K-theoretic codegree of vector bundles, we are able to refine the lower bound obtained earlier, when k is even. To establish an upper bound, we use a "vanishing line" for the E_2 term of Adams' Spectral Sequence. We find that if $2|k$, but $2^2 \nmid k$, the computation of the upper and lower bounds and a little analysis, enables us to fix the level of the Lens space L_k^{2m-1} , k even. The results are as follows :

4.2.2 PROPOSITION.

- (i) The additive order of $\sigma_2 \in \mathbb{Z}[\sigma_2]/(1-(1+\sigma_2)^2, \sigma_2^l)$ is 2^{l-1} . (cfr. [35]).
- (ii) The additive order of $(1+\sigma_{2k})^{k-1} \in \mathbb{Z}[\sigma_{2k}]/(1-(1+\sigma_{2k})^{2k}, \sigma_{2k}^m)$ is $2^{\left[\frac{m-2}{2^r} \right] + 1}$, where $k = 2^r \cdot a$, $a \geq 1$, odd, $r \geq 0$.

In particular when k is odd, the additive order of $(1+\sigma_{2k})^{k-1}$ is 2^{m-1} .

4.2.4 THEOREM.

$\ell(m) \geq 2^{\left[\frac{m-2}{2^r} \right] + 3}$, where $k = 2^r \cdot a$, $a \geq 1$, odd, $r \geq 0$.

In particular $\ell(m) \geq 2m - 1$, if k is odd.

4.3.2 THEOREM

$\ell(m) = 2m$, where $\ell(m)$ denotes the level of the Lens space L_k^{2m-1} , k odd.

5.3.6 THEOREM

Let $\ell(m)$ denote the level of the Lens Space L_k^{2m-1} , $k = 2^r p$, p odd and $r > 1$. Also let $m-2 = 2^r a + b$, $a \geq 0$ and $0 \leq b \leq 2^r - 1$. Then,

$$\begin{aligned} \ell(m) &\geq 2a + 5 + c, & c < 2 \left(\frac{m-1}{r+1} - a \right), & c \text{ even} \\ & & & \text{when } 2m - (2a+2+c) \equiv 6 \pmod{8} \\ &\geq 2a + 4 + c, & c < 2 \left(\frac{m-1}{r+1} - a \right) - 1, & c \text{ odd,} \\ & & & \text{when } b = 2^r - 1 \\ &\geq 2a + 3 + c, & c < 2 \left(\frac{m-1}{r+1} - a \right), & c \text{ even} \\ & & & \text{in all other cases.} \end{aligned}$$

6.3.4 THEOREM

Let $\ell(m)$ be as above. Then,

$$\begin{aligned} \ell(m) &\leq m + 1 & \text{when } m \equiv 0, 2 \pmod{8} \\ &\leq m + 2 & \text{when } m \equiv 1, 3, 4, 5, 7 \pmod{8} \\ &\leq m + 3 & \text{when } m \equiv 6 \pmod{8}. \end{aligned}$$

6.4.1 THEOREM

Let $\ell(m)$ denote the level of the Lens space L_k^{2m-1} , $k = 2p$, p odd. Then,

$$\begin{aligned} \ell(m) &= m+1 & \text{if } m \equiv 0, 2 \pmod{8} \\ &= m+2 & \text{if } m \equiv 1, 3, 4, 5, 7 \pmod{8} \\ &= m+3 & \text{if } m \equiv 6 \pmod{8}. \end{aligned}$$

0.2 LAYOUT OF THE THESIS

We have tried to make each chapter self-sufficient, without repetition, of course, from chapter to chapter. The first three chapters deal with the Borsuk-Ulam Problem and the

last three address themselves to the computation of the level of the Lens Space. We also have an Appendix. We have numbered most of the definitions, results, Lemmas, Propositions and Theorems. In the numbers that we have used, the first number refers to the chapter, the second to the section and the third to the definition, result, Lemma, Proposition or Theorem as may be the case.

In the first chapter we start with the definitions of Steenrod's external cohomology operation P , the Bredon operation Q , generalized mod p Conner-Miller classes and mod p Bredon-Hattori classes leading to the definition of the Self-intersection number. We also state without proof certain results needed for the remaining work of the chapter. We then show that the existing invariants fail to give any information about the Borsuk-Ulam set for a map from a product of spheres to a closed manifold, with respect to the product antipodal action on the product of spheres and trivial action on the closed manifold. We finally prove the Borsuk-Ulam Theorem for a product of spheres. We conclude with a few remarks.

The second chapter attempts to parametrize the Borsuk-Ulam Theorem for a product of spheres using the method of characteristic polynomials. We first of all compute the cohomology structure of $S^m \times S^n / T$ and basing on this, define the characteristic polynomials. We then have the Theorem, and the Corollary which give the parametrized Borsuk-Ulam

Theorem for a product of spheres. We conclude by noting that the estimate for the dimension of the Borsuk-Ulam set obtained by using this theorem is weaker than that obtained in Chapter One.

Chapter Three proceeds logically from Chapters One and Two. Since the method of Chapter One gives a better estimate of the dimension of the Borsuk-Ulam set in the case under consideration, than the method of Chapter two, we seek to extend the use of the invariant of Chapter One, to fibre-preserving maps of arbitrary manifold bundles over the same closed manifold. To enable us to use the method of Chapter One in the context of bundles, we first of all define parametrized Reduced power operation and Bredon operation on a fibre bundle. We proceed then to state and prove certain lemmas on the (co)homology of fibre-preserving maps of fibre bundles, needed later in the chapter to prove the parametrized Borsuk-Ulam Theorem for fibre-preserving maps of arbitrary manifold bundles, with fibrewise \mathbb{Z}_p -action. As applications we have a parametrized \mathbb{Z}_p -Borsuk-Ulam Theorem for fibre-preserving maps from a sphere bundle to an arbitrary manifold bundle over the same closed manifold, and also a parametrized \mathbb{Z}_p -Borsuk-Ulam Theorem for fibre-preserving maps from an $S^m \times S^n$ -bundle to an arbitrary manifold bundle over the same closed manifold.

In Chapter Four, Five and Six, we address the problem of the level of spaces. In Chapter Four we use the complex

K-theory structure of the Lens Space and some Number Theoretic arguments to establish a lower bound for the level of Lens Spaces. We further use some results of Pfister and Stolz [35] adapted to the present situation and certain results of Conner and Floyd [7], to establish the level of the Lens Space L_k^{2m-1} when k is odd.

In Chapter five, we first formulate the problem of determining the level of a space, in terms of existence of a nowhere vanishing section for a certain vector bundle, an obstruction for which is the Cohomotopy Euler class. We proceed to establish preliminaries on Cohomotopy Euler Class and Codegree of vector bundles, and using these, we refine the lower bound for the level of L_k^{2m-1} , established in Chapter Four in certain cases, when k is even.

Chapter Six is devoted to establishing an upperbound for the level of L_k^{2m-1} , k even. We use Adams' spectral sequence, to do this. First of all, we present the necessary preliminaries on Adams' spectral sequence and certain results of Adams [1], and Stolz [39]. We proceed, using the \mathbb{Z}_p cohomology structure of the Lens Space, to establish an upper bound for the level of the Lens Space when k is even. A little elementary analysis using the results of Chapters Four and Five enables us to determine the level of L_k^{2m-1} when $2|k$ but $2^2 \nmid k$.

At the end, we present an Appendix which presents the

details of the computation of the integral and \mathbb{Z}_p -cohomology structure of the Lens Space. We have used the \mathbb{Z}_p -cohomology structure in the preceding work. We did the detailed computation because the necessary details were not available in a compact form anywhere.

ADDENDUM

After the preparation of the present thesis was completed, it was brought to our notice by S. Stolz that some of our results could be obtained by using obstruction theory as follows : Consider the natural projection map $\pi : L_k^{2m-1} \longrightarrow L_{kp}^{2m-1}$. If p is an odd integer, then π is a $\mathbb{Z}_{(2)}$ -equivalence, i.e., it induces an isomorphism on $\pi_1(\) \otimes \mathbb{Z}_{(2)}$, where $\mathbb{Z}_{(2)}$ stands for integers localised at 2. By obstruction theory, it follows that the level of L_{kp}^{2m-1} is the same as the level of L_k^{2m-1} . We are thankful to Stolz for bringing this to our notice.

We have also come to learn that some of the results of Chapter Four is included in a result of Vicks (Bull AMS 75(1969) p 1017, Cor 3.3). Our proof is, however, more elementary. As it is also of independent interest and does not use any obstruction theory, we decided to retain it in the thesis.

CHAPTER ONE

A BORSUK ULAM THEOREM FOR A PRODUCT OF SPHERES

1.1 INTRODUCTION

The classical Borsuk-Ulam Theorem has been generalized in several ways. One of the more recent results in this direction is due to Kahn and Lin (cfr. [23] and [26]). They have generalized the Borsuk-Ulam Theorem to maps of manifolds. The result for \mathbb{Z}_2 -action is as follows :

1.1.1 Let $f : (T, M) \longrightarrow (T_1, N)$ be a map of closed manifolds where T is a fixed point free involution on M and T_1 is an involution, not necessarily free, on N . Let $A(f) = \{x \in M^n \mid f(Tx) = T_1 f(x)\}$. Now, if $m > n$, then $\dim A(f) \geq m - n$, provided the "self-intersection number" $C(f) \stackrel{\text{def}}{=} \langle c^{m-n} Q(q_1(1)), [\frac{M \times N}{T}] \rangle \neq 0$. (The notations are to be explained in 1.2.

The result has also been generalised by Kahn and Lin to \mathbb{Z}_p -actions (cfr. [24], [25]).

In the course of this chapter, it will become clear that this result does not yield any information about the dimension of $A(f)$ when M is replaced by a product of two spheres $S^m \times S^n$ and $T = A \times A$, where A is the antipodal action on the sphere. In this chapter, therefore, we prove generalisations of the Borsuk-Ulam Theorem for maps from

$S^m \times S^n$ into a manifold V^k using Conner-Miller classes and self-intersection numbers.

1.2 THE GENERALIZED CONNER-MILLER CLASSES
AND THE SELF-INTERSECTION NUMBER

In [9] Conner and Miller defined certain characteristic classes which he later called the Conner-Miller classes. Lin in [26] introduced a generalization of these classes for involutions and later in [25], he introduced the Generalized Conner-Miller classes for \mathbb{Z}_p -actions. In this section we recall the necessary details.

Let $G = \mathbb{Z}_p$ for a prime p .

It is known that $H^*(BG) = \mathbb{Z}_p[a, b]/a^2$, if p is odd
 and $H^*(BG) = \mathbb{Z}_2[c]$, if $p = 2$,



where $\deg a = \deg c = 1$, $\deg b = 2$ and $b = \beta a$, where β is the Bockstein homomorphism corresponding to the exact sequence $0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \xrightarrow{2} \mathbb{Z}_p \longrightarrow 0$.

We shall write $w_{2h} = b^h$, $w_{2h+1} = ab^h$ for p odd, and $w_l = c^l$ when $p = 2$. As long as there is no confusion, we shall write $w_l \in H^l(M/G)$ for the images of $w_l \in H^l(BG)$ under the map induced by $M \longrightarrow \{pt\}$.

Let X be a free G -space and $\nu : X \longrightarrow X/G$, the natural projection. We have the "umkehr homomorphism":

$$\text{tr} : H^*(X) \longrightarrow H^*(X/G).$$

We shall call it transfer. Since EG is contractible, the

projection $\nu : X^p \times EG \longrightarrow X^p \times_G EG$ gives rise to

$$\nu^* : H^*(X^p \times_G EG) = H_G^*(X^p) \longrightarrow H^*(X^p)$$

and the transfer

$$\text{tr} : H^*(X^p) \longrightarrow H_G^*(X^p) = H^*(X^p \times_G EG).$$

We have the external cohomology operation of Steenrod

$$P : H^q(X) \longrightarrow H_G^{pq}(X^p)$$

satisfying the following properties:

- a) For any map $f : Y \longrightarrow X$, $(f^p \times_G 1)^* P = P f^*$
- b) $\nu^* P(\alpha) = \alpha \times \alpha \times \dots \times \alpha$ (p times)
- c) $P(\alpha + \beta) = P(\alpha) + P(\beta) + z$, where z is in the image of tr .
- d) $P(\alpha \cdot \beta) = P(\alpha) \cdot P(\beta)$.

(cfr. [38]).

1.2.1 The Bredon Operation

The diagonal action $(G, X \times EG)$, has an equivariant embedding

$$e : X \times EG \longrightarrow X^p$$

given by $e(x, s) = (x, Ts, \dots, T^{p-1}x, s)$, where T is the generator of G . This induces an embedding,

$$E : X \times_G EG \longrightarrow X^p \times_G EG.$$

Further, the equivariant projection $X \times EG \longrightarrow X$ will induce a fibration $p : X \times_G EG \longrightarrow X/G$ with fibre EG . p is a homotopy equivalence. Following Bredon [3], we define the Bredon operation

$$Q : H^q(X) \longrightarrow H^{pq}(X/G)$$

by postulating that for $\alpha \in H^q(X)$, $Q(\alpha) \in H^{pq}(X/G)$ is the unique cohomology class for which

$$p^*(Q(\alpha)) = E^*P(\alpha).$$

We list below some of the properties of Q that we shall be using :

- a) Q commutes with homomorphism induced by equivariant maps
- b) $\nu^* Q(\alpha) = \alpha \cdot T^*(\alpha) \dots T^{*p-1}(\alpha)$.
- c) $Q(\alpha+\beta) = Q(\alpha) + Q(\beta) + z$, where z is in the image of the transfer homomorphism.
- d) $Q(\alpha \cdot \beta) = Q(\alpha) \cdot Q(\beta)$

1.2.2 Generalized mod p Conner-Miller Classes

Let M^n be an n -dimensional manifold and let G act freely on M . If $\frac{1}{p} \{(p-1)n + s\} < k \leq n$, then for a fixed $z \in H^s(M^n/G)$, the *mod p Conner-Miller Class* $\theta_k(z) \in H^k(M)$ is the unique cohomology class for which

$$\langle \theta_k(z) \cdot \alpha, [M^n] \rangle = \langle w_{pk-(p-1)n-s} \cdot z \cdot Q(\alpha), [M^n/G] \rangle,$$

for all $\alpha \in H^{n-k}(M^n)$. When $z = 1 \in H^0(M/G)$, or $w_1 \in H^1(M/G)$, the above defines the usual Conner-Miller class and we denote it by θ_k .

In the above definition $[M^n]$ and $[M^n/G]$ denote respectively, the fundamental homology classes of M and M/G .

We state without proof the following proposition. We shall use it later in our work.

1.2.3 Proposition

Let $f : (T_1, M_1^n) \longrightarrow (T, M^m)$ be an equivariant map between two fixed point free involutions on closed manifolds. Let $\bar{f} : M_1^n/T_1 \longrightarrow M^m/T$ be the induced map on orbit spaces. If

$z \in H^a(M_1^n/T_1)$, then for all $k > \frac{1}{p}\{(p-1)n+s\}$,

$$\theta_{k+(m-n)}(\bar{f}_1(z)) = f_1(\theta_k(z)).$$

Note that for any map $f: M_1^n \rightarrow M^m$ between closed manifolds, the "umkehr" homomorphism $f_!: H^k(M_1^n) \rightarrow H^{m+k-n}(M^m)$ is defined by the commutativity of the diagram

$$\begin{array}{ccc} H^k(M_1^n) & \xrightarrow{f_!} & H^{m+k-n}(M^m) \\ \downarrow [M_1^n] \frown & & \downarrow [M^m] \frown \\ H_{n-k}(M_1^n) & \xrightarrow{f_*} & H_{n-k}(M^m) \end{array}$$

where the vertical maps are Poincaré duality isomorphisms. i.e., for $x \in H^k(M_1^n)$, $f_!(x) \in H^{m+k-n}(M^m)$ is the unique cohomology class for which $\langle x \cdot f^*(y), [M_1^n] \rangle = \langle f_!(x) \cdot y, [M^m] \rangle$ for all $y \in H^{n-k}(M^m)$. (Using the well known identities relating cup and cap products).

(Proof of Proposition cfr. [26]). □

1.2.4 Mod p Bredon-Hattori Classes.

Let V^n be a closed manifold with G -action (not necessarily free). With $m > n$, let S^m (m odd when p odd), have G -action (given by complex multiplication by p th roots of unity if p is odd and the antipodal action when $p = 2$). The *mod p Bredon-Hattori class* $B_q(V) \in H^q(V)$ is defined to be the unique cohomology class for which is in ind. of m

$$\langle B_q(V) \cdot \alpha, [V^n] \rangle = \langle w_{pq+m-(p-1)n} \cdot Q(\alpha \otimes 1), [V^n \times_\tau S^m] \rangle$$

for all $\alpha \in H^{n-q}(V^n)$, where τ denotes the free diagonal action of G on $V^n \times S^m$.

We also have the following product formula verified by Kahn (cfr. [23] and [24]) which we state without proof.

1.2.5 PROPOSITION

Let M^m and N^n be closed manifolds with G -action, the former free and the latter, not necessarily free. On $M \times N$, we have the fixed point free diagonal action. Then

$$\theta_k(M \times N) = \sum_{j=0}^{((p-1)n/p)} \theta_{k-j}(M) \otimes \underline{B_j(N)} \quad ?$$

provided the classes are defined.

1.2.6 Self Intersection Number

Let M^m and V^k be closed manifolds with G -action, the former free and the latter, not necessarily free. On $M^m \times V^k$ we have the free diagonal action. Let $f : M^m \rightarrow V^k$, $m > (p-1)k$ be any map and $g : M^m \rightarrow M^m \times V^k$, a map defined $g(x) = (x, f(x))$ for all $x \in M^m$. Put $A(f) = \{x \mid f(T^i(x)) = T^i f(x), i = 1, 2, \dots, p-1\}$, where T is a generator of G . For $1 \in H^0(M^m)$, $g_!(1) \in H^k(M \times V)$ is the cohomology class dual to the image \mathcal{G} of g . The "self-intersection number" of $\bigcap_{i=0}^{p-1} T^i(\mathcal{G})$

$\bigcap_{i=0}^{p-1} T^i(\mathcal{G})$ can easily be seen to be homeomorphic to $A(f)$ is defined to be

$$C(f) = \langle w_{m-(p-1)k} Q(g_!(1)), [M \times_G V] \rangle$$

where w and Q relate to the diagonal G action on $M \times V$. (Note that in the above we have used T^i to denote the G -action on M , V and $M \times V$).

Using the Product formula 1.2.5, we get that

$$C(f) = \sum_{j=0}^{((p-1)k/p)} \langle \theta_{m-j}^{m-j} (M) \cdot f^*(B_j(V)), [M] \rangle.$$

We now have the preliminaries required to prove the main result of this chapter. In the remaining work, we shall let $G = \mathbb{Z}_2$. We shall also write c^i for w_i .

1.3 A BORSUK-ULAM THEOREM FOR A PRODUCT OF SPHERES

Consider a map $f: S^m \times S^n \longrightarrow V^k$, $k < m \leq n$. We have the diagonal antipodal fixed point free action on $S^m \times S^n$. On V^k , we consider the trivial action.

$$\text{Now, } C(f) = \sum_{j=0}^{(k/2)} \langle \theta_{m+n-j}^{m+n-j} (S^m \times S^n) \cdot f^*(B_j(V^k)), [S^m \times S^n] \rangle$$

Since the action on V^k is trivial by supposition, $B_0(V^k)$ $Wu_0(V^k) = 1$ (cfr. [26]). Further, since $k < m$,

$$\begin{aligned} C(f) &= \langle \theta_{m+n}^{m+n} (S^m \times S^n) \cdot f^*(B_0(V^k)), [S^m \times S^n] \rangle \\ &= \langle c^{2(m+n)-(m+n)}, [S^m \times_A S^n] \rangle \\ &= \langle c^{m+n}, [S^m \times_A S^n] \rangle \\ &= 0, \text{ since as we shall see in 2.2, } c^{m+1} = 0. \end{aligned}$$

Thus, Theorem 1.1.1 is not able to give us any conclusion about the dimension of $A(f)$. However, a deeper examination of the given data and the machinery developed does yield the following:

1.3.1 THEOREM

Let $f : S^m \times S^n \longrightarrow V^k$ ($k < m \leq n$) be a map of $S^m \times S^n$ into a manifold. Let $A(f) = \{(x,y) \in S^m \times S^n \mid f(-x,-y) = f(x,y)\}$. The cohom. dim. $A(f) \geq m+n-k$.

Before we go to the proof of the theorem itself, we state and prove two lemmas which will be used in the proof of the theorem.

Consider the equivariant map

$$\boxed{1 \times i \times 1} : S^m \times V^k \longrightarrow S^m \times S^n \times V^k, \quad \text{bad notation.}$$

where $i : S^m \times S^n$ is the natural inclusion, and the induced map

$$\overline{1 \times i \times 1} : \frac{S^m \times V^k}{T} \longrightarrow \frac{S^m \times S^n \times V^k}{T}$$

of quotient spaces. (We shall, for facility of notation, denote the diagonal antipodal action on $S^m \times S^n$, the diagonal action on $S^m \times V^k$ and the diagonal action on $S^m \times S^n \times V^k$, by T).

1.3.2 LEMMA

$(\overline{1 \times i \times 1})_! (1) \in H^n \left(\frac{S^m \times S^n \times V^k}{T} \right)$ is non zero

Proof : Note that the composite

$$S^m \times V^k \xrightarrow{1 \times i \times 1} S^m \times S^n \times V^k \xrightarrow{\pi} S^m \times V^k$$

where π is the projection $S^m \times S^n \longrightarrow S^m$, is equivariant and equal to the identity. Passing onto quotients, we get a map

$$\frac{S^m \times V^k}{T} \xrightarrow{\overline{1 \times i \times 1}} \frac{S^m \times S^n \times V^k}{T} \xrightarrow{\bar{\pi}} \frac{S^m \times V^k}{T}$$

and $\bar{\pi} \circ \overline{1 \times i \times 1} = \text{Id}_{\frac{S^m \times V^k}{T}}$

In homology therefore, $\bar{\pi}_* \circ \overline{1 \times i \times 1}_* \left(\left[\frac{S^m \times V^k}{T} \right] \right) = \left[\frac{S^m \times V^k}{T} \right]$

so that $\overline{1 \times i \times 1}_* \left(\left[\frac{S^m \times V^k}{T} \right] \right)$ is non zero.

The commutative diagram

$$\begin{array}{ccc} H^0 \left(\frac{S^m \times V^k}{T} \right) & \xrightarrow{(\overline{1 \times i \times 1})!} & H^n \left(\frac{S^m \times S^n \times V^k}{T} \right) \\ \cong \downarrow & & \cong \downarrow \\ H_{m+k} \left(\frac{S^m \times V^k}{T} \right) & \xrightarrow{(\overline{1 \times i \times 1})_*} & H_{m+k} \left(\frac{S^m \times S^n \times V^k}{T} \right) \end{array}$$

which defines the map $\overline{1 \times i \times 1}_!$ gives us that $\overline{1 \times i \times 1}_!(1)$ is non zero. □

Now, let $f: S^m \times S^n \longrightarrow V^k$, $k < m \leq n$ be any map.

Define $g: S^m \times S^n \longrightarrow S^m \times S^n \times V^k$

by $g(x, y) = ((x, y), f(x, y))$.

1.3.3 LEMMA

$(1 \times i \times 1)(g_!^*(1)) = 1 \otimes [V^k]^*$, where $[V^k]^* \in H^k(V^k)$ is the cohomology class (vector space) dual to the fundamental homology class $[V^k]$.

Proof: Note that $g_*([S^m \times S^n]) = [S^m \times S^n] \otimes f_*(1) + 1 \otimes f_*([S^m \times S^n]) +$

$$\begin{array}{l} \sum x_i \otimes y_i \quad , \quad x_i \in H_i(S^m \times S^n) \\ \dim x_i + \dim y_i = m+n \quad \text{and } y_i \in H_i(V^k) \end{array}$$

Since by assumption, $k < m \leq n$, the last two terms are zero.

Thus $g_*([S^m \times S^n]) = [S^m \times S^n] \otimes 1$. And hence,

$$\begin{aligned} (1 \times i \times 1)^*(g_!(1)) &= (1 \times i \times 1)^*([S^m \times S^n \times V^k] \cap -)^{-1} g_*([S^m \times S^n]) \\ &= (1 \times i \times 1)^*(1 \otimes [V^k]^*) \\ &= 1 \otimes [V^k]^* \end{aligned}$$

where $[V^k]^*$ is the cohomology class dual to the fundamental homology class $[V^k]$. □

Proof of Theorem 1.3.1

By lemma 1.3.2 we have that $(\overline{1 \times i \times 1})_!(1) \in H^n\left(\frac{S^m \times S^n \times V^k}{T}\right)$ is non zero.

$$\begin{aligned}
 & \text{Hence, } \langle \theta_{m+n}(S^m \times S^n \times V^k) ((\overline{1 \times i \times 1})_!(1)) \cdot g_!(1), [S^m \times S^n \times V^k] \rangle \\
 &= \langle (1 \times i \times 1)_!(\theta_m(S^m \times V^k)(1)) \cdot g_!(1), [S^m \times S^n \times V^k] \rangle \\
 & \quad \text{by proposition 1.2.3.} \\
 &= \langle \theta_m(S^m \times V^k) \cdot (1 \times i \times 1)^*(g_!(1)), [S^m \times V^k] \rangle \\
 & \quad \text{by the definition of } (1 \times i \times 1)_! \\
 &= \langle \theta(S^m \times V^k) \cdot (1 \otimes [V^k]^*), [S^m] \otimes [V^k] \rangle, \text{ by lemma 1.3.3.} \\
 &= \langle \left(\sum_{j=0}^{[k/2]} \theta_{m-j} \otimes B_j(V^k) \right) \cdot (1 \otimes [V^k]^*), [S^m] \otimes [V^k]^* \rangle \\
 & \quad \text{by Proposition 1.2.5} \\
 &= \langle \theta_m(S^m) \otimes B_0(V^k) \cdot (1 \otimes [V^k]^*), [S^m] \otimes [V^k] \rangle \\
 & \quad \text{as } k < m \text{ and hence } [k/2] < m. \\
 &= \langle \theta_m(S^m) \cdot (1 \otimes [V^k]^*), [S^m] \otimes [V^k] \rangle \\
 & \quad \text{as the action on } V^k \text{ is trivial and so } B_0(V^k) = \\
 & \quad Wu_0(V^k) = 1. \text{ (cfr. 1.3).} \\
 &= \langle \theta_m(S^m), [S^m] \rangle \cdot \langle [V^k]^*, [V^k] \rangle = 1.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 1 &= \langle \theta_{m+n}(S^m \times S^n \times V^k) (\overline{1 \times i \times 1})_!(1) \cdot g_!(1), [S^m \times S^n \times V^k] \rangle \\
 &= \langle c^{m-k} \cdot (\overline{1 \times i \times 1})_!(1) \cdot Q(g_!(1)), \left[\frac{S^m \times S^n \times V^k}{T} \right] \rangle \\
 & \quad \text{by definition.} \\
 &= \langle c^{m-k} \cdot (\overline{1 \times i \times 1})_!(1), Q(g_!(1)) \cap \left[\frac{S^m \times S^n \times V^k}{T} \right] \rangle
 \end{aligned}$$

The geometric meaning of the Bredon operation as equivariant self-intersection, (cfr. [25] and [26]), implies that

$$Q(g_1(1)) \cap \left[\frac{S^m \times S^n \times V^k}{T} \right] = g_{1*} \left(\left[\frac{A(f)}{T} \right] \right), \text{ where } g_1 = g|_{A(f)}.$$

Thus the above expression can be written as

$$\begin{aligned} & \langle c^{m-k} \cdot (\overline{1 \times 1 \times 1})_!(1), g_{1*} \left(\left[\frac{A(f)}{T} \right] \right) \rangle \\ &= \langle g_1^* (c^{m-k} \cdot (\overline{1 \times 1 \times 1})_!(1)), \left[\frac{A(f)}{T} \right] \rangle \end{aligned}$$

Hence, $g_1^* (c^{m-k} \cdot (\overline{1 \times 1 \times 1})_!(1)) \in H^{m+n-k}(A(f)/T)$ is non zero,

so that the cohom dim. $(A(f)) \geq m + n - k$. □

1.4 REMARKS

1.4.1 There are various invariants which have been introduced to measure the dimension of $A(f)$ for maps of manifolds with involution. We have mentioned those of Kahn and Lin which use the "Self-intersection Number." We may also mention the Equivariant Lefschetz class $L(f)$ of Necochea (cfr. [34]) which is motivated by Nakaoka's invariant $\hat{I}(f)$ (cfr. [31]) and the Equivariant Euler class $e_G(f)$, also of Necochea [34]. Theorem 1.3.1 is necessitated because the results of Kahn, Lin, Nakaoka and Necochea do not yield any information about $\dim A(f)$ for a map $f: S^m \times S^n \longrightarrow V^k$. In fact, the cohomological structure of $\frac{S^m \times S^n}{A \times A}$ (as we shall see in 2.2), makes $C(f) = 0$, so that the Kahn-Lin result fails to give any information about $A(f)$. Moreover for the same reason, in the case we are considering, $\hat{I}(f) = L(f) = e_{\mathbb{Z}_2}(f) = 0$. Hence, neither the results of Nakaoka nor those of Necochea are able to give any information about $A(f)$ for maps $f: S^m \times S^n \longrightarrow V^k$.

1.4.2 In Theorem 1.3.1, if $k \geq m$, and $m \leq n$, then as will be evident from the proof of the theorem, we cannot make any conclusion about the dimension of $A(f)$. In fact one can directly see that if $k \geq m$, then the map $f : S^m \times S^n \longrightarrow S^k$, defined by $f(x,y) = ix$, where $i: S^m \longrightarrow S^k$, is the natural inclusion, has $A(f) = \phi$.

$$\begin{aligned} \text{For, } A(f) &= \{(x,y) \in S^m \times S^n \mid f(-x,-y) = f(x,y)\} \\ &= \{(x,y) \in S^m \times S^n \mid i(-x) = i(x)\} \\ &= \phi, \text{ as } k \geq m. \end{aligned}$$

Thus the result of the theorem is valid only for $k < m \leq n$.

CHARACTERISTIC POLYNOMIALS
CHAPTER TWO AND A PARAMETRIZED BORSUK-ULAM THEOREM
FOR $S^m \times S^n$ BUNDLES

2.1 INTRODUCTION

Parametrization of the Borsuk-Ulam Theorem is another direction in which the Borsuk-Ulam Theorem has been generalized. The works of Jan Jaworowski, Dold and Nakaoka (cfr. [19], [20], [13], [31] and [32]) are in this direction. Dold [13] parametrizes the Borsuk-Ulam result considering fibre-preserving equivariant maps from sphere bundles to plane bundles, equivariance being considered with respect to fibre-preserving involutions. Nakaoka [33] generalizes the result to \mathbb{Z}_p -actions.

The relevant result of Dold runs as follows :

2.1.1 Let $\pi : E \longrightarrow B \longleftarrow E' : \pi'$ be two vector bundles over the same paracompact space B and $f : SE \longrightarrow E'$, a fibre-preserving equivariant map w.r.t the antipodal action, where $SE \subseteq E$ is the sphere bundle of E . If m and n are the fibre dimensions of E and E' respectively, and if $m > n$, then the cohomological dimension of $f^{-1}(0)$ in terms of a continuous cohomology theory with coefficients in \mathbb{Z}_2 satisfies

$$\text{Cohom dim } (f^{-1}(0)) \geq \text{Cohom. dim } B + m - n - 1.$$

If B is taken to be a point, we obtain a generalisation of the classical Borsuk-Ulam result.

We know that $H^*(X) = \mathbb{Z}_2$, $*$ = 0, m , n , $m+n$
 = 0, otherwise.

Hence the section of the sequence for $0 \leq k \leq m$ gives us that $H^0(\bar{X}) \cong H^1(\bar{X}) \cong \dots \cong H^{m-1}(\bar{X}) \cong \mathbb{Z}_2$.

Further, since $m < n$, let $\iota : S^m \longrightarrow S^n$ be the natural inclusion and let $d : S^m \longrightarrow S^m \times S^n$ be defined by

$$d(x) = (x, \iota x).$$

Since d is clearly equivariant, we have a commutative cohomology diagram

$$\begin{array}{ccc} H^m(S^m \times S^n) & \xrightarrow{d^*} & H^m(S^m) \\ p^* \uparrow & & \uparrow \bar{p}^* \\ H^m\left(\frac{S^m \times S^n}{T}\right) & \xrightarrow{\bar{d}^*} & H^m\left(\frac{S^m}{T}\right) \end{array}$$

This, together with the Gysin sequence for the 0-sphere bundle, $S^m \longrightarrow S^m/T$ gives us that \bar{p}^* and hence p^* is the zero map (note that d^* is an isomorphism).

Now, consider the following section of the above Gysin sequence :

$$\begin{array}{ccccccc} \dots \longrightarrow & H^{m-1}(X) & \longrightarrow & H^{m-1}(\bar{X}) & \xrightarrow{s^*} & H^m(\bar{X}) & \xrightarrow{p^*} & H^m(X) & \xrightarrow{r^*} & \dots \\ & & & & & H^m(\bar{X}) & \xrightarrow{s_1^*} & H^{m+1}(\bar{X}) & \longrightarrow & H^{m+1}(X) & \longrightarrow & \dots \end{array}$$

Since $H^{m-1}(X) = 0$ and p^* is the zero map, $\mathbb{Z}_2 = H^{m-1}(\bar{X}) = H^m(\bar{X})$. Since r^* is injective and hence an isomorphism, and since $H^{m+1}(X) = 0$, s_1^* is zero; but it is also onto. Hence $H^{m+1}(\bar{X}) = 0$.

It is easy to see that $H^{m+1}(\bar{X}) = H^{m+2}(\bar{X}) = \dots = H^{n-1}(\bar{X}) = 0$.

Further, the equivariant map $d : S^m \times S^n \longrightarrow S^n \times S^n$ given by

$$d(x, y) = (x, y)$$

gives rise to a commutative cohomology diagram

$$\begin{array}{ccc} H^n(S^n \times S^n) & \xrightarrow{d^*} & H^n(S^m \times S^n) \\ \uparrow p^* & & \uparrow \bar{p}^* \\ H^n\left(\frac{S^n \times S^n}{T}\right) & \xrightarrow{\bar{d}^*} & H^n\left(\frac{S^m \times S^n}{T}\right) \end{array}$$

The fact that $\text{im } \bar{p}^* = \{0, a_1 + a_2\}$, where $\{a_1, a_2\}$ generate $H^n(S^n \times S^n) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (vide case 2) gives us that p^* is onto (as d^* maps $a_1 + a_2$ onto the generator of $H^n(S^m \times S^n)$).

A look at the Gysin sequence,

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k(\bar{X}) & \xrightarrow{s^*} & H^{k+1}(\bar{X}) & \xrightarrow{p^*} & H^{k+1}(X) & \xrightarrow{r^*} & H^{k+1}(\bar{X}) \\ & & & & \xrightarrow{s^*} & H^{k+2}(\bar{X}) & \xrightarrow{p^*} & H^{k+2}(X) & \longrightarrow \dots \end{array}$$

shows us that p^* is an isomorphism for $k = n - 1$ and that s^* is an isomorphism for $n-1 \leq k \leq n+m-2$

Hence $\mathbb{Z}_2 = H^n(\bar{X}) = H^{n+1}(\bar{X}) = \dots = H^{n+m}(\bar{X})$.

Thus, $H^k(\bar{X}) = \mathbb{Z}_2$, $0 \leq k \leq m$; $n \leq k \leq m+n$.
 $= 0$, otherwise.

Case 2 : $n = m$

As in case 1, we shall obtain that $H^k(\bar{X}) = \mathbb{Z}_2$, $0 \leq k \leq m-1$.

Now, consider the diagram :

$$\begin{array}{ccc} H^m(S^m \times S^m) & \xrightarrow{d^*} & H^m(S^m) \\ \uparrow p^* & & \uparrow \bar{p}^* \\ H^m\left(\frac{S^m \times S^n}{T}\right) & \xrightarrow{\bar{d}^*} & H^m\left(\frac{S^m}{T}\right) \end{array}$$

We have that \bar{p}^* is the zero map. (cfr. case 1). Hence $\text{Im } p^* \subseteq \{0, a_1 + a_2\}$ where $\{a_1, a_2\}$ generate $H^m(S^m \times S^m) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. But $p^* \neq 0$ (as may be easily seen from the Gysin sequence section given below). Hence,

$$\text{Im } p^* = \{0, a_1 + a_2\} \cong \mathbb{Z}_2.$$

The following section of the Gysin sequence,

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^{m-1}(X) & \longrightarrow & H^{m-1}(\bar{X}) & \xrightarrow{s^*} & H^m(\bar{X}) & \xrightarrow{p^*} & H^m(X) & \xrightarrow{r^*} \\ & & & & & & & & & \\ & & & & H^m(X) & \xrightarrow{s_1^*} & H^{m+1}(X) & \xrightarrow{p_1^*} & H^{m+1}(\bar{X}) & \longrightarrow \dots \end{array}$$

now gives that since $\text{Im } p^* \cong \mathbb{Z}_2$ and s_1^* , a monomorphism, rank considerations make $H^m(\bar{X}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Thus, $H^k(X) = \mathbb{Z}_2$, $0 \leq k \leq m-1$; $m+1 \leq k \leq 2m$

$$= \mathbb{Z}_2 \oplus \mathbb{Z}_2, k = m$$

$$\cong 0, \text{ otherwise.}$$

Case 3 : $n = m+1$

A consideration of the two commutative diagrams of case 1 and the Gysin sequence sections that follow shows us that in this case

$$H^k(\bar{X}) = \mathbb{Z}_2, 0 \leq k \leq 2m+1$$

$$= 0, \text{ otherwise.}$$

Finally, we find the generators. Let $c \in H^1(\bar{X})$ be the fundamental class of the involution (T, X) (i.e., the characteristic class of the zero-sphere bundle $X \longrightarrow \bar{X}$) and let $d \in H^n(\bar{X})$ be the n -dimensional generator. The fact that $s^*(x) = x.c$ for $x \in H^l(\bar{X})$, $l \geq 1$ and $s^*(x) = 0$, $x \in H^m(\bar{X})$, $n \neq m$ gives us that $H^*(\bar{X})$ is generated as a free-module by $\{c^l, c^{n-l}.d\}$, subject to the condition $c^{m+1} = 0$. Evidently also, $d^2 = 0$.

The case when $n = m$ can be seen as follows :

Consider the section of the Gysin sequence we considered in case 2. If b_1, b_2 generate $H^m(\bar{X}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $p^*(b_\nu) = a_1 + a_2$ (notations as in case 2), for $\nu = 1$ or 2 , since p^* is not the zero map. Call that b_ν, d . Then it can be easily

seen that $H^m(\bar{X})$ is generated by $\{c^m, d\}$. That $c^{m+1} = 0$ follows from the naturality of the characteristic classes and a consideration of the cohomology diagram induced from the commutative diagram

$$\begin{array}{ccc} S^m \times S^n & \xrightarrow{\pi} & S^m \\ \downarrow p & & \downarrow \bar{p} \\ \frac{S^m \times S^m}{\Gamma} & \xrightarrow{\bar{\pi}} & \frac{S^m}{\Gamma} \end{array}$$

where π is the projection. □

Now, we proceed to form the characteristic polynomials. We shall assume that we can apply the Lerray-Hirsch Theorem to the fibre bundle $(\bar{E}, \bar{\pi}, \bar{B}, \bar{X})$. For this, it suffices to postulate that B is 1-connected. Otherwise, we assume that a cohomology extension of the fibre exists (cfr. [36]). We consider a continuous cohomology theory with coefficients in \mathbb{Z}_2 .

2.3 CHARACTERISTIC POLYNOMIALS

By the Lerray-Hirsch Theorem, there exist elements $\bar{c} \in H^1(\bar{E})$ and $\bar{d} \in H^n(\bar{E})$ such that the natural homomorphism $\bar{j}^* : H^*(\bar{E}) \longrightarrow H^*(\bar{X})$ maps \bar{c} to c and \bar{d} to d , and that $H^*(\bar{E})$ is an $H^*(B)$ -module (via $\bar{\pi}^*$), generated by $\{\bar{c}^i, \bar{c}^{n-1} \bar{d}\}$, $0 \leq i \leq m+n$. To simplify notation, we shall use c and d to denote \bar{c} and \bar{d} .

We first of all let $1 < m < n$.

We may write $c^{m+1} \in H^{m+1}(\bar{E})$ and $d^2 \in H^{2m}(\bar{E})$ respectively as $c^{m+1} = w_{m+1} + w_m c + \dots + w_1 c^m + \alpha d$, where $\alpha \in \mathbb{Z}_2$, $\alpha = 0$ if $n > m+1$, and

$$d^2 = v_{2n} + v_{2n-1}c + \dots + v_{2n-m}c^m + v_n d + v_{n-1}c.d + \dots + v_{n-m}c^m d$$

for unique $w_i, v_i \in H^i(B)$

We define the following polynomials, which we shall call characteristic polynomials, in the indeterminates t , of degree 1 and s , (of degree n) (Compare Nakaoka, [33]).

$$W_1(t,s) = w_{m+1} + w_m t + \dots + w_1 t^m + t^{m+1} + \alpha.s$$

$$W_2(t,s) = v_{2n} + v_{2n-1}t + \dots + v_{2n-m}t^m + v_n s + \dots + v_{n-m}t^m.s + s^2.$$

[For $1 < m = n$, the relations will be of the form

$$c^{m+1} = w_{m+1} + w_m c + \dots + w_1 c^m + \bar{w}_1 d + \alpha.c.d$$

and

$$d^2 = v_{2m} + v_{2m-1}c + \dots + v_m c^m + \bar{v}_m d + v_{m-1}cd + \dots + \beta.c^m d.$$

for $c^{m+1} \in H^{m+1}(\bar{E})$ and $d^2 \in H^{2m}(\bar{E})$ and unique w_i, \bar{w}_i, v_i and $\bar{v}_i \in H^i(B)$, $\alpha, \beta \in \mathbb{Z}_2$.

The Characteristic Polynomials will now have the form

$$W_1(t,s) = w_{m+1} + w_m t + \dots + w_1 t^m + t^{m+1} + \bar{w}_1 s + \alpha.t.s$$

and

$$W_2(t,s) = v_{2m} + v_{2m-1}t + \dots + v_m t^m + \bar{v}_m s + v_{m-1}ts + \dots + \beta.t^m s + s^2.]$$

It is easy to see that $(t,s) \longrightarrow (c,d)$ gives an isomorphism of $H^*(B)$ -algebras:

$$\frac{H^*(B)[t,s]}{(W_1(t,s), W_2(t,s))} \cong H^*(\bar{E})$$

Now let $E' \xrightarrow{\pi'} B$ be a vector bundle (of dimension k) over B . Let $f: E \longrightarrow E'$ be a fibre preserving map, equivariant with respect to the antipodal action on E' . Let $Z = f^{-1}(0)$

$\subseteq E$, and put \bar{Z} , the quotient of the fixed point free action induced from E . We denote by SE' the sphere bundle of E' . Then $\overline{SE'}$ is a bundle of $(k-1)$ -dimensional projective spaces. We know that $H^*(\mathbb{R}P^{k-1}) = \mathbb{Z}_2[c']/c'^k$ where $c' \in H^1(\mathbb{R}P^{k-1})$ is the fundamental class of the antipodal involution on S^{k-1} . Since we can define the fundamental class of the fibrewise involution on SE' , mapping c' to this element gives us a cohomology extension of the fibre. Thus we can apply the Lerray-Hirsch Theorem to the fibre bundle $SE' \rightarrow B$ and get that $H^*(\overline{SE'})$ is freely generated as an $H^*(B)$ -module by $1, c'^1, c'^2, \dots, c'^{k-1}$, where we have denoted the fundamental class of the involution on SE' also by $c' \in H^1(\overline{SE'})$.

We may write $c'^k \in H^k(\overline{SE'})$ as

$$c'^k = w'_k + w'_{k-1}c' + \dots + w'_1c'^{k-1}.$$

for unique $w'_i \in H^i(B)$. We define the characteristic polynomial

$$W'(t) = w'_k + w'_{k-1}t + \dots + w'_1t^{k-1} + t^k,$$

and have that

$$\frac{H^*(B)[t]}{(W'(t))} \cong H^*(\overline{SE'})$$

$$t \longmapsto c' \quad , \quad \text{as } H^*(B)\text{-algebras.}$$

To simplify expression, for a polynomial $q(t,s) \in H^*(B)[t,s]$, we shall denote by $q(t,s)|\bar{E}$ or $q(t,s)|\bar{Z}$, the images of $q(t,s)$ by the first map or the composite respectively, in the following :

$$\begin{array}{ccccc}
H^*(B)[t,s] & \longrightarrow & H^*(\bar{E}) & \xrightarrow{i^*} & H^*(\bar{Z}) \\
(t,s) & \longrightarrow & (c,d) & \longrightarrow & (i^*c, i^*d)
\end{array}$$

Now that the preliminary definitions have been made and notations established, we are in a position to proceed to the main Theorem of this Chapter.

2.3.1 THEOREM

Let $\pi : E \longrightarrow B$ be an $S^m \times S^n$ bundle ($1 < m \leq n$) with the fibrewise \mathbb{Z}_2 -action $A \times A$, and $\pi' : E' \longrightarrow B$ be a vector bundle of dimension k , with fibrewise antipodal action, over a paracompact space B , such that the quotient bundle $\bar{\pi} : \bar{E} \longrightarrow B$ has the cohomology extension property. Let $f : E \longrightarrow E'$ be a fibre-preserving equivariant map. Put $Z = f^{-1}(0)$ and let \bar{Z} denote the quotient by the action induced on Z . Now, if $q(t,s) \neq 0, \in H^*(B)[t,s]$ is a polynomial such that $q(t,s)|_{\bar{Z}} \neq 0$, then \exists polynomials $r_1(t,s), r_2(t,s) \in H^*(B)[t,s]$ such that $q(t,s).W(t) = r_1(t,s).W_1(t,s) + r_2(t,s).W_2(t,s)$.

Proof:

Let $q(t,s) \in H^*(B)[t,s]$ be such that $q(t,s)|_{\bar{Z}} = 0$. By the continuity of the cohomology theory we are considering, $\exists V$ open in \bar{E} , $V \supset \bar{Z}$, such that $q(t,s)|_V = 0$.

The exact sequence

$$\dots \longrightarrow H^*(\bar{E}, V) \xrightarrow{j_1^*} H^*(\bar{E}) \longrightarrow H^*(V) \longrightarrow \dots$$

now gives $\alpha \in H^*(\bar{E}, V)$, such that $j_1^*(\alpha) = q(t,s)|_{\bar{E}}$.

The map $f : E \longrightarrow E'$ induces $\bar{f} : \bar{E} - \bar{Z} \longrightarrow \bar{E}' - (0)$ of the quotient spaces. Since the induced cohomology homomorphism is a $H^*(B)$ -homomorphism, we shall have that

$$W'(t)|_{\bar{E} - \bar{Z}} = W'(c) = W'(\bar{f}^*(c')) = \bar{f}^*(W'(c')) = 0 \text{ as } W'(c) = 0.$$

The exact sequence

$$\dots \longrightarrow H^*(\bar{E}, \bar{E} - \bar{Z}) \xrightarrow{j_2^*} H^*(\bar{E}) \longrightarrow H^*(\bar{E} - \bar{Z}) \longrightarrow \dots$$

gives $\beta \in H^*(\bar{E}, \bar{E} - \bar{Z})$, such that $j_2^*(\beta) = W'(t)|_{\bar{E}}$.

$$\begin{aligned} \text{Hence, } q(t,s).W'(t)|_{\bar{E}} &= j_1^*(\alpha).j_2^*(\beta) \\ &= \bar{j}^*(\alpha \cup \beta), \text{ by naturality of cup products.} \end{aligned}$$

But, $\alpha \cup \beta \in H^*(\bar{E}, \bigvee \cup (\bar{E} - \bar{Z})) = H^*(\bar{E}, \bar{E})$, so that $\alpha \cup \beta = 0$.

$$\text{Thus, } q(t,s).W'(t)|_{\bar{E}} = 0. \text{ But } H^*(\bar{E}) = \frac{H^*(B)[t,s]}{(W_1(t,s), W_2(t,s))}$$

Hence \exists polynomials $r_1(t,s)$ and $r_2(t,s) \in H^*(B)[t,s]$ such that $q(t,s).W'(t) = r_1(t,s).W_1(t,s) + r_2(t,s).W_2(t,s)$. \square

We have the following Corollary which gives us the parametrized version of the Borsuk-Ulam Theorem.

2.3.2 COROLLARY : If the fibre dimension of E' is k , then $\deg(W'(t)) = k$, $\deg(W_1(t,s)) = m+1$ and $\deg(W_2(t,s)) = 2n$. Then $q(t,s)|_{\bar{Z}} \neq 0$, for all polynomials whose degree in t and s is less than $m-k+1$.

This means that the $H^*(B)$ -homomorphism

$$\begin{aligned} \oplus_{l+j = 0}^{m-k} H^*(B).t^l.s^j &\longrightarrow H^*(\bar{Z}) \\ t^l.s^j &\longmapsto t^l.s^j|_{\bar{Z}} \end{aligned}$$

is a monomorphism. Hence if $k \leq m$,

$$\text{Cohom. dim. } \bar{Z} \geq \text{Cohom. dim. } B + m - k.$$

In particular, ^{then} if B is a point, ^{if} we have an equivariant map
 $f: S^m \times S^n \longrightarrow \mathbb{R}^k$ and if $k \leq m \leq n$, then $\text{Cohom. dim. } \bar{Z} \geq m - k$.

2.4 REMARKS

2.4.1 Corollary 2.3.2 can be used to find a lower bound for $\dim A(f)$ for any map $f: S^m \times S^n \longrightarrow \mathbb{R}^k$, $k \leq m \leq n$ as follows: Denote by T , the $A \times A$ action on $S^m \times S^n$. Consider the map $g: S^m \times S^n \longrightarrow \mathbb{R}^k$ defined by $g(x) = f(x) - f(Tx)$. g is clearly equivariant with respect to the action T on $S^m \times S^n$ and the antipodal action on \mathbb{R}^k . Also, $A(f)$ is precisely $g^{-1}(0)$ and hence $\dim A(f) = \dim (g^{-1}(0)) \geq m - k$.

Thus the corollary gives a weaker estimate for the lower bound of $A(f)$ than theorem 1.3.1. However theorem 1.3.1 gives information only when $k < m$ (cfr. 1.4.2) while Theorem 2.3.1 is valid for all k and hence gives information, whatever the value of k . *for $k=m$.*

A BORSUK-ULAM THEOREM
CHAPTER THREE FOR MAPS OF FIBRE-BUNDLES
WITH MANIFOLDS AS FIBRES

3.1 INTRODUCTION

As we have seen in the previous chapters, Dold [13] and Nakaoka [33], parametrized the Borsuk-Ulam Theorem using characteristic polynomials. An extension of this method to maps of bundles where the fibres are arbitrary closed manifolds could entail great difficulty, and may often give less than satisfactory results. In Chapter Two, we have seen a particular example which illustrates this fact. In this chapter we generalize the Borsuk-Ulam Theorem to maps of fibre bundles over a closed manifold, where the fibres are arbitrary closed manifolds.

We let $G = \mathbb{Z}_p$, p a prime, and we consider (co)homology with coefficients in \mathbb{Z}_p . We start with a few necessary prerequisites.

3.2 PRELIMINARIES

3.2.1 A FIBRE-BUNDLE WITH G ACTION

Let X and K be paracompact G -spaces. Let $\pi : X \longrightarrow B$ be a continuous map of X to a paracompact space B . Then $\xi = (X, \pi, B, K)$ is called a *fibre-bundle with G action*, if the following condition is satisfied. For each point $b \in B$, $\pi^{-1}(b)$ is an invariant subspace of X , and there exists a neighbourhood U of b and an equivariant homeomorphism

$\varphi : U \times K \longrightarrow \pi^{-1}(U)$ such that $\pi\varphi^{-1}(u, k) = u$, where $U \times K$ is considered as a G -space by $g(u, k) = (u, gk)$.

Let $\xi = (X, \pi, B, K)$ be a fibre bundle with G action. Consider its p -fold Whitney sum with itself $\xi_p = (\oplus_p X, \bar{\pi}, B, K^p)$. An element of $\oplus_p X$ can be written as $x = (x_1, x_2, \dots, x_p)$ such that $\bar{\pi}(x) = \pi_i(x_i), \forall i$. The fibre of ξ_p is K^p , the p -fold product of K with itself.

If we identify K with a typical fibre, then we have the inclusion $i: K \longrightarrow X$ and the corresponding inclusion $\oplus_p i: K^p \longrightarrow \oplus_p X$. G acts on K^p by $T(x_1, x_2, \dots, x_p) = (x_2, x_3, \dots, x_p, x_1)$. Consider the diagonal action of G on $K^p \times EG$, which is free. The map $\nu : K^p \times EG \longrightarrow K^p \times_G EG$ induces $\nu^* : H^*_G(K^p) \longrightarrow H^*(K^p)$, (as EG is contractible) and the transfer $\text{tr} : H^*(K^p) \longrightarrow H^*_G(K^p)$. We can define a similar action of G on $\oplus_p X$ and the diagonal action on $\oplus_p X \times EG$. Since the inclusion

$$\oplus_p i \times 1 : K^p \times EG \longrightarrow \oplus_p X \times EG$$

is equivariant, we get $\oplus_p i \times_G 1 : K^p \times_G EG \longrightarrow \oplus_p X \times_G EG$. The projection $\bar{\nu} : \oplus_p X \times EG \longrightarrow \oplus_p X \times_G EG$ gives rise to the homomorphism $\bar{\nu}^* : H^*_G(\oplus_p X) \longrightarrow H^*(\oplus_p X)$ and the transfer $\bar{\text{tr}} : H^*(\oplus_p X) \longrightarrow H^*_G(\oplus_p X)$. We also have the commutative diagram

$$\begin{array}{ccc} H^*_G(\oplus_p X) & \xrightarrow{\bar{\nu}^*} & H^*(\oplus_p X) \\ (\oplus_p i \times_G 1)^* \downarrow & & \downarrow \oplus_p i^* \\ H^*_G(K^p) & \xrightarrow{\nu^*} & H^*(K^p) \end{array}$$

3.2.2 Parametrized Reduced Power Operation

The inclusion $j : \bigoplus_p X \longrightarrow X^p$ gives rise to an equivariant inclusion $j \times 1 : (\bigoplus_p X) \times EG \longrightarrow X^p \times EG$. Passing to quotient spaces, we obtain $j \times_G 1 : (\bigoplus_p X) \times_G EG \longrightarrow X^p \times_G EG$.

Define the *Parametrised Reduced Power Operation* \bar{P} on X as the composition :

$$\bar{P} : H^k(X) \xrightarrow{P} H_G^{pk}(X^p) \xrightarrow{(j \times_G 1)^*} H_G^{pk}(\bigoplus_p X)$$

where $P : H^k(X) \longrightarrow H_G^{pk}(X^p)$ is Steenrod's Reduced Power operation. (cfr. [10]). \bar{P} has properties analogous to those of P . We list below those which we shall use.

3.2.3 Properties of \bar{P} .

- For any bundle map $f : Y \longrightarrow X$, $(\bigoplus_p f \times_G 1)^* \bar{P} = \bar{P} f^*$.
- $\bar{P}^*(u) = j^*(u \times_p u \times_{p-1} \dots \times u)$.
- $\bar{P}(u + v) = \bar{P}(u) + \bar{P}(v) + z$, where z is in the image of \bar{tr} .
- $\bar{P}(u \cdot v) = \bar{P}(u) \cdot \bar{P}(v)$.

3.2.4 Bredon Operation on a Fibre Bundle.

Define an equivariant embedding $\bar{e} : X \times EG \longrightarrow X^p \times EG$ by

$$\bar{e}(x, g) = (x, Tx, \dots, T^{p-1}x, g).$$

Note that since the action of G is fibre preserving, the embedding factors through $\bigoplus_p X \times EG$. It will also induce an equivariant embedding of $K \times EG$ in $K^p \times EG$. Passing onto the quotient spaces, we get the following maps :

$$\begin{aligned} \hat{E} : K \times_G EG &\longrightarrow K^p \times_G EG \\ E : X \times_G EG &\longrightarrow (\bigoplus_p X) \times_G EG \\ \bar{E} : X \times_G EG &\longrightarrow X^p \times_G EG. \end{aligned}$$

We also have the commutative diagram :

$$\begin{array}{ccc}
 K \times_G EG & \xrightarrow{\hat{E}} & K^p \times_G EG \\
 \downarrow i \times_G 1 & & \downarrow \oplus_i \times_G 1 \\
 X \times_G EG & \xrightarrow{E} & (\oplus_p X) \times_G EG
 \end{array}$$

Further, consider the G-bundles

$$K \longrightarrow K/G$$

and $X \longrightarrow X/G$

and the associated EG-bundles

$$\hat{p} : K \times_G EG \longrightarrow K/G$$

and $p : X \times_G EG \longrightarrow X/G$.

We obtain the commutative diagram :

$$\begin{array}{ccc}
 K \times_G EG & \xrightarrow{i \times_G 1} & X \times_G EG \\
 \downarrow \hat{p} & & \downarrow p \\
 K/G & \xrightarrow{\bar{i}} & X/G
 \end{array}$$

Define the *Bredon Operation* \bar{Q} on the fibre bundle X by the commutativity of the diagram :

$$\begin{array}{ccc}
 H^k(X) & \xrightarrow{\bar{P}} & H_G^{pk}(\oplus_p X) \\
 \downarrow \bar{Q} & & \downarrow E^* \\
 H^{pk}(X/G) & \xrightarrow{P^*} & H_G^{pk}(X)
 \end{array}$$

Note that the commutativity of

$$\begin{array}{ccc}
 H^k(X) & \xrightarrow{\bar{P}} & H_G^{pk}(\oplus_p X) \\
 \downarrow P & & \downarrow E^* \\
 H_G^{pk}(X^p) & \xrightarrow{\bar{E}^*} & H_G^{pk}(X)
 \end{array}$$

ensures that \bar{Q} is essentially the same as Q , the usual Bredon Operation generalised to G-action (cfr. 1.2).

3.3 SOME TECHNICAL RESULTS ON THE (CO)HOMOLOGY OF FIBRE-PRESERVING MAPS OF FIBRE-BUNDLES

Let $\xi_1 = (X, \pi_1, K_1, M)$, $\xi_2 = (Y, \pi_2, K_2, M)$ be G -fibre bundles with fibres, closed differentiable manifolds K_1 and K_2 of dimensions n_1 and n_2 respectively, over a closed differentiable manifold M , of dimension m . On the Whitney sum $\xi_1 \oplus \xi_2 = (X \oplus Y, \pi, K_1 \times K_2, M)$ we have the fibrewise diagonal action.

Let $f : X \longrightarrow Y$ be a fibre preserving map, and let $A(f) = \{x \in X \mid f(T^i x) = T^i f(x), i = 0, 1, 2, \dots, p-1\}$. Define $g : X \longrightarrow X \oplus Y$, also a fibre-preserving map by $g(x) = (x, f(x))$. The image \mathcal{G} of g is evidently the graph of f .

We assume that there exist $\{e_i\}_{i \in I} \in H^*(X)$ and $\{f_j\}_{j \in J} \in H^*(Y)$ such that $\{i_1^*(e_i)\} \in H^*(K_1)$ and $\{i_2^*(f_j)\} \in H^*(K_2)$ form a free basis for $H^*(K_1)$ and $H^*(K_2)$ respectively as modules over $H^*(pt)$, where $i_1 : K_1 \longrightarrow X$ and $i_2 : K_2 \longrightarrow Y$ are fibre inclusions. We know that if M is 1-connected, this is always true; otherwise this is equivalent to the assumption of the existence of cohomology extensions of the fibres. (cfr. [36] and [37]).

We have the following diagram of induced bundles :

$$\begin{array}{ccc} \Delta^*(X \times Y) = X \oplus Y & \xrightarrow{j} & X \times Y \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

where Δ is the diagonal map, $\Delta(m) = (m,m)$, and j is induced on the top spaces making the diagram commutative. Now $\{e_i \otimes f_j\}_{i \in I, j \in J}$ are elements in $H^*(X \times Y)$ such that $\{i_1^*(e_i) \otimes i_2^*(f_j)\} \in H^*(K_1 \times K_2)$ are free generators over $H^*(pt)$. Then $\{j^*(e_i \otimes f_j)\} \in H^*(X \oplus Y)$ are elements such that $\{i_1^*(e_i) \otimes i_2^*(f_j)\}$ generate $H^*(K_1 \times K_2)$ as an $H^*(pt)$ -module. (cfr. [37]).

Let Φ_{1*}, Φ_1^* and Φ_{2*}, Φ_2^* denote respectively the homology and cohomology Lerray-Hirsch isomorphisms relating to the bundles ζ_1 and $\zeta_1 \oplus \zeta_2$.

3.3.1 PROPOSITION

The diagram

$$\begin{array}{ccc} H_*(X) & \xrightarrow{g_*} & H_*(X \oplus Y) \\ \downarrow \Phi_{1*} & & \downarrow \Phi_{2*} \\ H_*(M) \otimes H_*(K_1) & \xrightarrow{1 \otimes \bar{g}_*} & H_*(M) \otimes H_*(K_1 \times K_2) \end{array}$$

is commutative, where $\bar{g}: K_1 \longrightarrow K_1 \times K_2$ is the restriction of $g: X \longrightarrow X \oplus Y$ to a typical fibre.

Proof: We note first of all, that the Lerray-Hirsch isomorphisms for homology and cohomology are induced by the same chain level map (cfr. [37]). Further, since the coefficients are in the field \mathbb{Z}_p , the (co)homology groups are in fact vector spaces and duals of each other. Hence it suffices to prove the result in cohomology.

i.e., we are to prove the commutativity of

$$\begin{array}{ccc} H^*(X) & \xleftarrow{g^*} & H^*(X \oplus Y) \\ \downarrow \Phi_1^* & & \downarrow \Phi_2^* \\ H^*(M) \otimes H^*(K_1) & \xleftarrow{1 \otimes \bar{g}^*} & H^*(M) \otimes H^*(K_1 \times K_2) \end{array}$$

We define $\hat{g} : X \longrightarrow X \times Y$ by the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\hat{g}} & X \times Y \\ & \searrow g & \nearrow j \\ & X \oplus Y & \end{array}$$

\hat{g} is, of course, essentially g , as f is fibre-preserving.

The induced cohomology diagram gives that for $a \otimes b \in H^*(X \times Y)$, $a \cdot f^*(b) = \hat{g}^*(a \otimes b) = g^* j^*(a \otimes b)$, by the definition of g .

Now, take $\sum m_k \otimes (i_1^*(e_l) \otimes i_2^*(f_j)) \in H^*(M) \otimes H^*(K_1 \times K_2)$.

$$\begin{aligned} g^* \Phi_2^* \left(\sum m_k \otimes (i_1^*(e_l) \otimes i_2^*(f_j)) \right) &= g^* \left(\sum \pi_1^* m_k \cdot j^*(e_l \otimes f_j) \right) \\ &= \sum g^* \pi_1^* m_k \cdot g^* j^*(e_l \otimes f_j) \\ &= \sum \pi_1^* m_k \cdot \hat{g}^*(e_l \otimes f_j) \\ &= \sum \pi_1^* m_k \cdot e_l \cdot f^*(f_j) \end{aligned}$$

$$\begin{aligned} \text{and } \Phi_1^* (1 \otimes \bar{g}^*) \left(\sum m_k \otimes (i_1^*(e_l) \otimes i_2^*(f_j)) \right) &= \Phi_1^* \left(\sum m_k \otimes \bar{g}^*(i_1^*(e_l) \otimes i_2^*(f_j)) \right) \\ &= \Phi_1^* \left(\sum m_k \otimes i_1^*(e_l) \cdot \bar{f}^*(i_2^*(f_j)) \right) \\ &= \Phi_1^* \left(\sum m_k \otimes i_1^*(e_l \cdot f^*(f_j)) \right) \\ &= \sum \pi_1^* m_k \cdot e_l \cdot f^*(f_j) \end{aligned}$$

This proves the commutativity of the above diagram and hence the proposition. □

The *Umkehr* homomorphism $g_! : H^k(X) \longrightarrow H^{n_2+k}(X \oplus Y)$

is defined by the commutativity of the diagram

$$\begin{array}{ccc} H^k(X) & \xrightarrow{g_!} & H^{n_2+k}(X \oplus Y) \\ \downarrow [X] \cap - & & \downarrow [X \oplus Y] \cap - \\ H_{m+n_2-k}^{m+n_2-k}(X) & \xrightarrow{g_*} & H_{m+n_2-k}^{m+n_2-k}(X \oplus Y) \end{array}$$

where the Poincaré duality isomorphisms $[X] \cap -$ and $[X \oplus Y] \cap -$

are defined by the commutative rectangles

$$\begin{array}{ccc}
 H^*(M) \otimes H^*(K_1) & \xrightarrow{\Phi_1^*} & H^*(X) \\
 \downarrow [M] \otimes [K_1] \cap - & & \downarrow [X] \cap - \\
 H_*(M) \otimes H_*(K_1) & \xleftarrow{\Phi_{1*}} & H_{m+n_1}^{*-*}(X)
 \end{array}$$

and

$$\begin{array}{ccc}
 H^*(M) \otimes H^*(K_1 \times K_2) & \xrightarrow{\Phi_2^*} & H^*(X \oplus Y) \\
 \downarrow [M] \otimes [K_1 \times K_2] \cap - & & \downarrow [X \oplus Y] \cap - \\
 H_*(M) \otimes H_*(K_1 \times K_2) & \xleftarrow{\Phi_{2*}} & H_{m+n_1+n_2}^{*-*}(X \oplus Y)
 \end{array}$$

3.3.2 LEMMA

For $1 \in H^0(X)$, $g_1(1) = \Phi_2^*(1 \otimes \bar{g}_1(1))$.

Proof: We simply consider the following diagram where all the quadrilaterals are now commutative. We get the result by chasing $1 \in H^0(M) \otimes H^0(K_1)$ corresponding to $1 \in H^0(X)$, around the outer rectangle, and keeping in mind certain dimensional considerations.

$$\begin{array}{ccccccc}
 H^0(M) \otimes H^0(K_1) & \xrightarrow{\Phi_1^*} & H^0(X) & \xrightarrow{g_1} & H^{n_2}(X \oplus Y) & \xleftarrow{\Phi_2^*} & \sum_{i+j=n_2} H^i(M) \otimes H^j(K_1 \times K_2) \\
 \downarrow [M] \otimes [K_1] \cap - & & \downarrow [X] \cap - & & \downarrow [X \oplus Y] \cap - & & \downarrow [M] \otimes [K_1 \times K_2] \cap - \\
 H_m(M) \otimes H_{n_1}(K_1) & \xleftarrow{\Phi_{1*}} & H_{m+n_1}(X) & \xrightarrow{g_*} & H_{m+n_1}(X \oplus Y) & \xleftarrow{\Phi_{2*}} & \sum_{i+j=m+n_1} H_i(M) \otimes H_j(K_1 \times K_2) \\
 & & \downarrow 1 \otimes \bar{g}_* & & & & \downarrow
 \end{array}$$

For example, $(1 \otimes \bar{g}_*) ([M] \otimes [K_1] \cap 1) = [M] \otimes \bar{g}_*([K_1])$
 $= [M] \otimes [K_1 \times K_2] \cap (1 \otimes \bar{g}_1(1))$
 $\in H_m(M) \otimes H_{n_1}(K_1 \times K_2)$,
 by the definition of $\bar{g}_1(1)$.

And hence, $g_1(1) = \Phi_2^*(1 \otimes \bar{g}_1(1))$ □

Write $\bar{g}_1(1) = \sum_{i,j} \alpha_{ij} (i_1^* e_i \otimes i_2^* f_j)$, where $\alpha_{ij} \in \mathbb{Z}_p$,
 $i \in \hat{I} \subset I$ and $j \in \hat{J} \subset J$.

$$\begin{aligned} \text{Then, } g_1(1) &= \Phi_2^*(1 \otimes \bar{g}_1(1)) \\ &= \pi^*(1) \cup \sum \alpha_{ij} (j^*(e_i \otimes f_j)). \end{aligned}$$

3.3.3 PROPOSITION

If $\hat{Q}(\bar{g}_1(1))$ is non-zero, then $Q(g_1(1))$ is non-zero, where
 $\hat{Q} : H^k(K_1 \times K_2) \longrightarrow H^{pk}(K_1 \times_{\sigma} K_2)$ and $Q : H^k(X \otimes Y) \longrightarrow H^{pk}(X \otimes_{\sigma} Y)$
denote the Bredon operations on the fibres and top spaces respectively.

Proof : Consider the diagram :

$$\begin{array}{ccc} H^{n_2}(K_1 \times K_2) & \xleftarrow{\bar{i}^*} & H^{n_2}(X \otimes Y) \\ \downarrow \hat{Q} & & \downarrow Q \\ H^{pn_2}(K_1 \times_{\sigma} K_2) & \xleftarrow{\bar{i}_{\sigma}^*} & H^{pn_2}(X \otimes_{\sigma} Y) \end{array}$$

where \bar{i}^* and \bar{i}_{σ}^* are homomorphisms induced by inclusions.

The diagram is commutative since $\bar{i} : K_1 \times K_2 \longrightarrow X \otimes Y$ is equivariant.

$$\begin{aligned} \text{Hence, } \bar{i}_{\sigma}^* Q(g_1(1)) &= \hat{Q} \bar{i}^*(g_1(1)) \\ &= \hat{Q} \bar{i}^* \left[\pi^*(1) \cup \sum \alpha_{ij} (j^*(e_i \otimes f_j)) \right] \\ &= \hat{Q} \left[\bar{i}^* (\sum \alpha_{ij} j^*(e_i \otimes f_j)) \right] \\ &= \hat{Q} (\sum \alpha_{ij} \bar{i}^* j^*(e_i \otimes f_j)) \\ &= \hat{Q} (\sum \alpha_{ij} (i_1^* e_i \otimes i_2^* f_j)) \\ &= \hat{Q} (\bar{g}_1(1)). \end{aligned}$$

The proposition follows. □

3.4 THE PARAMETRIZED BORSUK-ULAM THEOREM

3.4.1 THEOREM

Let (X, π_1, M, K_1) and (Y, π_2, M, K_2) be G -bundles where M, K_1 and K_2 are closed differentiable manifolds of dimensions m, n_1, n_2 respectively, and such that there exist cohomology extensions of the fibres for both bundles. Let $f: X \longrightarrow Y$ be a fibre-preserving map and $A(f) = \{x \in X \mid f(T^i x) = T^i(f(x)), i = 0, 1, 2, \dots, p-1, T^i \in G\}$. Define $g : X \longrightarrow X \oplus Y$ by $g(x) = (x, f(x))$. Let \bar{g}_i denote the restriction of g to a fibre and let $\hat{Q} : H^k(K_1 \times K_2) \longrightarrow H^{pk}(K_1 \times_G K_2)$ be the Bredon operation. Then if $\hat{Q}(\bar{g}_i(1))$ is non-zero, then $A(f)$ is non-empty and $\dim A(f) \geq m + n_1 - (p-1)n_2$.

Proof:

Let $\hat{Q}(\bar{g}_i(1))$ be non-zero. Then by proposition 3.3.3, $Q(\bar{g}_i(1))$ is non-zero. Hence it suffices to prove that if $Q(\bar{g}_i(1)) \neq 0$, then $\dim A(f) \geq m + n_1 - (p-1)n_2$.

This follows from the following series of lemmas:

3.4.2 LEMMA. \mathcal{S} supports $g_i(1)$.

Proof: By \mathcal{S} supports $g_i(1)$, we mean that $g_i(1)$ is in the kernel of the map $H^{n_2}(X \oplus Y) \longrightarrow H^{n_2}(X \oplus Y \setminus U)$ for all neighbourhoods U of \mathcal{S} in $X \oplus Y$.

This is easy to see, as $g_i(1)$ is the cohomology class dual the fundamental class of \mathcal{S} . ■

3.4.3 LEMMA. $\bigcap_{i=0}^{p-1} T^i(\mathcal{S})/T$ supports $Q(g_i(1))$.

Proof: This has been proved by S. Kahn ([24]) using the geometrical interpretation of the Bredon operation. In fact, he proves that if (T, M^m) be a fixed point free

diffeomorphism of prime period p , and if a closed subset $S \subset M$ supports a class $\alpha \in H^k(M)$, then $\bigcap_{l=0}^{p-1} T^l(S)/T$ supports $Q(\alpha) \in H^{pk}(M/T)$. ■

3.4.4 LEMMA. $\bigcap_{l=0}^{p-1} T^l(\mathcal{G})$ is equivariantly homeomorphic to $A(f)$.
Proof: It is easy to see that $g|A(f)$ is an equivariant homeomorphism of $A(f)$ onto $\bigcap_{l=0}^{p-1} T^l(\mathcal{G})$. ■

Now, to complete the proof of the theorem, we note that since $\bigcap_{l=0}^{p-1} T^l(\mathcal{G})/T$ supports $Q(g_1(1))$ and $Q(g_1(1)) \neq 0$, $\dim \left[\bigcap_{l=0}^{p-1} T^l(\mathcal{G})/T \right]$ and hence $\dim \left[\bigcap_{l=0}^{p-1} T^l(\mathcal{G}) \right] \geq m + n_1 + n_2 - pn_2 = m + n_1 - (p-1)n_2$. (This can be seen using some duality arguments). The theorem now follows from lemma 3.4.4. ■

3.5 APPLICATIONS

Using mod p Conner-Miller classes and Bredon-Hattori classes (cfr. 1.2), the main theorem enables us to prove certain parametrized generalisations of the Borsuk-Ulam theorem. As the first application, we have the following Corollary which gives a parametrized generalization of the Borsuk-Ulam Theorem for fibre-preserving maps from sphere bundles to arbitrary manifold bundles over the same closed manifold.

3.5.1 COROLLARY

Let $f: X \longrightarrow Y$ be a fibre-preserving map from an n -sphere bundle with G action given by complex multiplication by p th

roots of unity on S^n (n odd, when p odd), to a V^k -bundle with trivial G action, where V^k is a closed k -dimensional manifold, over the same closed m -dimensional manifold M . Let $A(f) = \{x \in X \mid f(T^v x) = f(x), v = 0, 1, 2, \dots, p-1\}$. Then if $(p-1)k < n$, $\dim A(f) \geq m+n-(p-1)k$.

Proof :

Let g be defined as earlier. We have the map \bar{g} , the restriction of g to fibres. Lin considers the "self-intersection number"

$$\langle w_{n-(p-1)k} \cdot \hat{Q}(\bar{g}_1(1)), [S^n \times_{\tau} V^k] \rangle,$$

where τ is the free diagonal action. This number, by definition is

$$\langle \theta_n(S^n \times V^k) \cdot \bar{g}_1(1), [S^n \times V^k] \rangle.$$

Using the product formula of 1.2.5, this is shown to be non-zero (cfr. [251]).

Hence $\hat{Q}(\bar{g}_1(1))$ is non-zero.

The corollary follows on the application of the Main Theorem to this situation. ■

We also get a parametrized version of the result of Theorem 1.3.1 as another application.

3.5.2 COROLLARY :

Let $f: X \longrightarrow Y$ be a fibre-preserving map from an $S^m \times S^n$ -bundle with G action induced by the diagonal action of complex multiplication by the p th roots of unity on $S^m \times S^n$ ($m \leq n$, m and n odd when p odd), to a V^k -bundle with trivial G action, where V^k is a closed k -dimensional manifold, over the same closed l -dimensional manifold M^l . Let $A(f) = \{(x, y)$

$\in S^m \times S^n \mid f(T^l(x,y)) = f(x,y), l = 0, 1, 2, \dots, p-1\}$. Now, if $(p-1)k < m \leq n$, then $\dim A(f) \geq l + m + n - (p-1)k$.

Proof:

For $p = 2$, let $k < m \leq n$. We consider the number

$$\langle \theta_{m+n} (S^m \times S^n \times V^k) ((\overline{1 \times 1 \times 1})_1(1)) \cdot \bar{g}_1(1), [S^m \times S^n \times V^k] \rangle$$

which is equal to

$$\langle \omega_{m-k} \cdot (\overline{1 \times 1 \times 1})_1(1) \cdot \hat{Q}(\bar{g}_1(1)), \left[\frac{S^m \times S^n \times V^k}{T} \right] \rangle$$

In 1.3.1, we have proved that this number is non-zero.

Hence $\hat{Q}(\bar{g}_1(1))$ is non-zero.

For p odd, the same proof as in 1.3.1. with slight changes, proves that in this case too $\hat{Q}(\bar{g}_1(1))$ is non-zero when $(p-1)k < m \leq n$.

The corollary now follows from the Theorem 3.4.1. ■

3.5.3 Remark

We note that in the situation of corollary 3.5.2 ($p=2$), the existing invariants are not able to give any information about the dimension of $A(f)$ (cfr. 1.4.1 for details about this). Also, comparing corollary 3.5.2 with theorem 2.3.1, and Corollary 2.3.2 ($p = 2$), which was proved using the method of characteristic polynomials of [13], it is clear that in many situations, Theorem 3.4.1 may help to get better estimates of the size of the Borsuk-Ulam set, than that obtained by characteristic polynomials.

CHAPTER FOUR

LEVEL OF THE LENS SPACE

$$\underline{L_{-k}^{2m-1}}, \text{ FOR ODD } k$$

4.1 INTRODUCTION

In the first three chapters we considered one aspect of equivariance of maps - the Borsuk-Ulam Problem. Now we turn our attention to another aspect - the Level of Spaces.

The level of a Topological space X with a fixed point free involution T is defined to be the number $\ell(T, X) = \text{Min } \{n : \text{there exists a } \mathbb{Z}_2\text{-equivariant map } f: X \rightarrow S^{n-1}\}$, where the space X and the space S^{n-1} are considered to have \mathbb{Z}_2 -actions given by the involution T and the antipodal action respectively.

In earlier topological literature, the number $\ell-1$ was used under the name "co-index" (cfr. [7], [8]). In [8], Conner and Floyd proved that $\ell(\mathbb{R}P^5) = 5$. (Here $\mathbb{R}P^{2m-1}$ is considered to have a fixed point free involution induced by complex multiplication by i). For greater dimensions, the level of the Real Projective Space and the Complex Projective Space was explored by A. Pfister and S. Stolz in [35], and later determined for the Real Projective Space by S. Stolz in [39]. Other results on the level, in particular for Stieffel Manifolds, are to be found in [12]. Here, we examine the level of the Lens Space.

The Lens space L_k^{2m-1} is the orbit space of the \mathbb{Z}_k -action on $S^{2m-1} \subseteq \mathbb{C}^m$, given by multiplication by the k^{th} root of unity $e^{2\pi i/k} \in \mathbb{C}$. On L_k^{2m-1} we can have a fixed point free \mathbb{Z}_2 -action induced by multiplication by $e^{\pi i/k}$, the $2k^{\text{th}}$ root of unity. We are interested in calculating the level of the Lens space L_k^{2m-1} , with respect to this \mathbb{Z}_2 -action and the antipodal action on the sphere. We shall denote by $\ell(m)$, the level of L_k^{2m-1} . In this Chapter we shall establish a lower bound for the level of the Lens Space using Complex K-theory and some number theoretic arguments. We shall also be able to determine the level when k is odd.

4.2 A LOWER ESTIMATE FOR $\ell(m)$

Let H be the Hopf bundle over $\mathbb{C}P^{m-1}$ and H_k denote the pullback p^*H , where $p: L_k^{2m-1} = S^{2m-1}/\mathbb{Z}_k \longrightarrow S^{2m-1}/S^1 = \mathbb{C}P^{m-1}$ is the natural projection induced by the inclusion $\mathbb{Z}_k \subseteq S^1$. We denote by η , the class of H in $K(\mathbb{C}P^{m-1})$ and by η_k , the class of H_k in $K(L_k^{2m-1})$. Here $K(-)$ denotes the complex K-theory. As a ring, $K(L_k^{2m-1}) = \mathbb{Z}(\sigma_k)/(1 - (1 + \sigma_k)^k, \sigma_k^m)$, where $\sigma_k = \eta_k - 1$.

Note that H is the vector bundle over $\mathbb{C}P^{m-1}$, associated with the representation $h: S^1 \times \mathbb{C} \longrightarrow \mathbb{C}$, given by multiplication by $z \in S^1$. Also H_k is the vector bundle over L_k^{2m-1} associated with the representation $h_k: \mathbb{Z}_k \times \mathbb{C} \longrightarrow \mathbb{C}$, which is the restriction of h to \mathbb{Z}_k . Since this construction is compatible with tensor products, we have that $\bigoplus_k H_{2k} = (H_{2k})^k$

is the vector bundle over L_{2k}^{2m-1} associated with the representation $\otimes_k h_{2k} : \mathbb{Z}_{2k} \times \otimes_k \mathbb{C} \longrightarrow \otimes_k \mathbb{C} \cong \mathbb{C}$.

Now, suppose $f : L_k^{2m-1} \longrightarrow S^{n-1}$ is an equivariant map with respect to the \mathbb{Z}_2 action on L_k^{2m-1} described earlier and the antipodal action on S^{n-1} . Let $\bar{f} : S^{2m-1} \xrightarrow{\nu} L_k^{2m-1} \xrightarrow{f} S^{n-1}$ be the composition of f with the projection $\nu : S^{2m-1} \longrightarrow L_k^{2m-1}$. Then \bar{f} is equivariant with respect to the \mathbb{Z}_{2k} -action on S^{2m-1} given by multiplication by $e^{\pi i/k}$ and by -1 on S^{n-1} . Passing to orbits we have a map $\tilde{f} : L_{2k}^{2m-1} \longrightarrow \mathbb{R}P^{n-1}$.

We let n be even, $n = 2t$, say. Then the map

$$\begin{array}{ccc} F : \otimes_k H_{2k} & \longrightarrow & H_2 \\ \parallel & & \parallel \\ S^{2m-1} \times_{\mathbb{Z}_{2k}} \otimes_k \mathbb{C} & \longrightarrow & S^{2t-1} \times_{\mathbb{Z}_2} \mathbb{C} \end{array}$$

given by $F([x, z_1 \otimes \dots \otimes z_k]) = [\bar{f}(x), z_1, \dots, z_k]$ may be seen to be a well-defined bundle map covering

$$\tilde{f} : L_{2k}^{2m-1} \longrightarrow \mathbb{R}P^{2t-1} = L_2^{2t-1}.$$

Hence $\tilde{f}^*(\eta_2) = (\eta_{2k})^k$, where \tilde{f}^* is the induced map in K-theory. Thus we have proved :

4.2.1 PROPOSITION . Let $f : L_k^{2m-1} \longrightarrow S^{2t-1}$ be a \mathbb{Z}_2 -equivariant map and let $\tilde{f} : L_{2k}^{2m-1} \longrightarrow L_2^{2t-1}$ be the induced map on orbit spaces. Then in complex K-theory $\tilde{f}^*(\eta_2) = (\eta_{2k})^k$. (cfr. [35]). \square

Let the situation now be as in 4.2.1, with t as small as possible. This means that $l(m) = 2t$ or $l(m) = 2t-1$

according as $l(m)$ is even or odd respectively. Using Proposition 4.2.1, we get that

$$\begin{aligned} \tilde{f}^*(\sigma_2) &= \tilde{f}^*(\eta_2 - 1) = (\eta_{2k})^k - 1 \\ &= (1 + \sigma_{2k})^k - 1. \end{aligned}$$

Hence the additive order of $(1 + \sigma_{2k})^k - 1 \in \mathbb{Z}[\sigma_{2k}]/(1 - (1 + \sigma_{2k})^{2k}, \sigma_{2k}^m)$ is a divisor of the additive order of $\sigma_2 \in \mathbb{Z}[\sigma_2]/(1 - (1 + \sigma_2)^2, \sigma_2^l)$.

4.2.2 PROPOSITION.

- (i) The additive order of $\sigma_2 \in \mathbb{Z}[\sigma_2]/(1 - (1 + \sigma_2)^2, \sigma_2^l)$ is 2^{l-1} . (cfr. [35]).
- (ii) The additive order of $(1 + \sigma_{2k})^k - 1 \in \mathbb{Z}[\sigma_{2k}]/(1 - (1 + \sigma_{2k})^{2k}, \sigma_{2k}^m)$ is $2^{\left[\frac{m-2}{2^r} \right] + 1}$, where $k = 2^r \cdot a$, $a \geq 1$, odd, $r \geq 0$.

and $[x]$ denotes the integral part of x .

In particular when k is odd, the additive order of $(1 + \sigma_{2k})^k - 1$ is 2^{m-1} .

Proof :

To evaluate the order of an element of $\mathbb{Z}[a]/(1 - (1+a)^n, a^m)$, represented by a polynomial $p \in \mathbb{Z}[a]$, with vanishing constant term, we expand the quotient $p/(1 - (1+a)^n)$ in a formal power series $\sum_{i=0}^{\infty} b_i a^i$, $b_i \in \mathbb{Q}$. Now, p will be contained in the ideal generated by $1 - (1+a)^n$, i.e., p will be zero, if and only if b_0, b_1, \dots, b_{m-2} are integers. Hence the order of p will be

$$\text{l.c.m. } \{ \text{denominators of } b_0, b_1, \dots, b_{m-2} \}.$$

Proof of (i)

Note that
$$\begin{aligned} \frac{\sigma_2}{1-(1+\sigma_2)^2} &= \frac{-1}{2+\sigma_2} = -\frac{1}{2} \left[1 + \sigma_2/2\right]^{-1} \\ &= -\frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \cdot \sigma_2^j / 2^j . \\ &= \sum_{j=0}^{\infty} (-1)^{j+1} \cdot \sigma_2^j / 2^{j+1} . \end{aligned}$$

Hence the order of σ_2 is l.c.m. $\{2^{j+1}, 0 \leq j \leq t-2\}$, i.e., 2^{t-1} .

Proof of (ii)

We have that
$$\begin{aligned} \frac{(1 + \sigma_{2k})^k - 1}{1 - (1 + \sigma_{2k})^{2k}} &= \frac{-1}{1 + (1 + \sigma_{2k})^k} \\ &= \frac{-1}{\left[2 + \binom{k}{1} \sigma_{2k} + \dots + \sigma_{2k}^k\right]} \\ &= -\frac{1}{2} \left[1 + \frac{1}{2} \left(\binom{k}{1} \sigma_{2k} + \dots + \sigma_{2k}^k\right)\right]^{-1} \\ &= \sum_{n=0}^{\infty} \left[\sum_{\substack{s_1 + \dots + s_k = n \\ s_k \leq s_{k-1} \leq \dots \leq s_1}} (-1)^{s_1+1} \frac{\sigma_{2k}^{s_1 + \dots + s_k}}{2^{s_1+1}} \binom{s_1}{s_2} \dots \binom{s_{k-1}}{s_k} \cdot \binom{k}{1}^{s_1-s_2} \dots \binom{k}{k-1}^{s_{k-1}-s_k} \right] \\ &= \sum_{n=0}^{\infty} b_n \sigma_{2k}^n . \end{aligned}$$

To avoid cumbersome repetition, we shall refer to the above expression as expression [A]. We will need to consider two other expressions which are parts of expression [A].

Expression [B] :
$$\frac{1}{2^{s_1}} \binom{s_1}{s_2} \dots \binom{s_{k-1}}{s_k} \cdot \binom{k}{1}^{s_1-s_2} \dots \binom{k}{k-1}^{s_{k-1}-s_k} .$$

and expression [C] :

$$\sum_{\substack{s_1 + \dots + s_k = n \\ s_k \leq s_{k-1} \leq \dots \leq s_1}} \frac{1}{2^{s_1}} \binom{s_1}{s_2} \dots \binom{s_{k-1}}{s_k} \cdot \binom{k}{1}^{s_1-s_2} \dots \binom{k}{k-1}^{s_{k-1}-s_k} .$$

We shall also call a partition $s_1 + s_2 + \dots + s_k$ of n , admissible, if and only if $s_k \leq s_{k-1} \leq \dots \leq s_1$.

Case 1: $k \equiv 1 \pmod{2}$.

Consider expression [B]. We note that $\binom{k}{1} = k \equiv 1 \pmod{2}$. Hence the power of 2 in the denominator of $\frac{1}{2^n} \binom{k}{1}^n$, which is the value of [B] for the admissible partition $0+0+\dots+0+n$ of n , is n . It is evident that for a fixed n and for other admissible partitions $s_1+s_2+\dots+s_k$ of n , the power of 2 in the denominator of [B] is less than n . Hence expression [C] = $\frac{1}{2^n}$ where $1 \equiv 1 \pmod{2}$. Thus, in expression [A], l.c.m. {denominators of b_0, \dots, b_{m-2} } is $2^{m-2+1} = 2^{m-1}$.

Case 2 : $k \equiv 0 \pmod{2}$.

Let $k = 2^r a$, $a \geq 1$, odd and $r \geq 1$.

For the proof, we need the following results on binomial coefficients.

4.2.3 Lemma

$\binom{2^j a}{b} \equiv 0 \pmod{2}$ if $a \geq 1$, odd, and $b < 2^j$; and $\binom{2^j a}{2^j} \equiv 1 \pmod{2}$.

In particular, putting $a = 1$, we get that the binomial coefficients $\binom{2^j}{b}$, $b < 2^j$, are even.

Proof of Lemma

We know that if $m = \sum_{i \geq 0} 2^i m_i$ and $n = \sum_{i \geq 0} 2^i n_i$ be the diadic expansions of m and n , then

$$\binom{m}{n} \equiv \prod_{i \geq 0} \binom{m_i}{n_i} \pmod{2}. \quad (\text{cfr. [40]}).$$

Hence $\binom{m}{n}$ is odd if and only if $m_i \geq n_i$ for each i . Note that for $m = 2^j a$, $m_i = 0$, $0 \leq i \leq j-1$, $m_j = 1$. If $n = b < 2^j$,

then for some $i < j$, $n_i = 1$, so that $\binom{2^j a}{b} \equiv 0 \pmod{2}$.

Further, if $n = 2^j$, then $n_i = 0$ for $i > j$. Thus $m_i \geq n_i$ for each i and hence $\binom{2^j a}{b} \equiv 1 \pmod{2}$. \square

We proceed with the proof of the proposition.

For a fixed n and for an admissible partition of n , the power of 2 in the denominator of expression [B] is

$$\leq s_1 - r(s_1 - s_2) - r_1(s_2 - s_3) - \dots - r_{2^r-2} (s_{2^r-1} - s_{2^r}).$$

where each $r_i \geq 1$, by Lemma 4.2.3.

$$\leq s_1 - r(s_1 - s_2) - (s_2 - s_3) - \dots - (s_{2^r-1} - s_{2^r}).$$

$$= -(r-1)(s_1 - s_2) + s_{2^r}$$

$$\leq s_{2^r}.$$

Note that $s_{2^r} \leq \left\lfloor \frac{n}{2^r} \right\rfloor$, as the partition of n we are considering is admissible.

If $n \equiv 0 \pmod{2^r}$, for the admissible partition

$s_1 + s_2 + \dots + s_{2^r} = n$, where $s_1 = \dots = s_{2^r} = \frac{n}{2^r}$, expression [B]

reduces to $\frac{1}{2} / 2^r \cdot \binom{2^r a}{2^r} s_{2^r}^{2^r}$. Hence by Lemma 4.2.3,

the power of 2 in the denominator is $\frac{n}{2^r}$. Further, in this case, for any other admissible partition, $s_{2^r} < \frac{n}{2^r}$, so that

the power of 2 in the denominator of expression [B] is less than $\frac{n}{2^r}$. Therefore expression [C] is of the form $\frac{1}{2} / 2^r$

where 1 is odd. Hence in expression [A],

l.c.m. {denominators of b_0, \dots, b_{m-2} } is $\left[\frac{m-2}{2^r} \right] + 1$.

This completes the proof of Proposition 4.2.2 ■

We now proceed to the Theorem which establishes a lower bound for the level of the Lens Space.

4.2.4 THEOREM.

$l(m) \geq 2 \left[\frac{m-2}{2^r} \right] + 3$, where $k = 2^r \cdot a$, $a \geq 1$, odd, $r \geq 0$.

In particular $l(m) \geq 2m - 1$, if k is odd.

Proof: By Propositions 4.2.1 and 4.2.2 we have that

$$t - 1 \geq \left[\frac{m-2}{2^r} \right] + 1$$

$$\text{i.e., } 2t - 1 \geq 2 \left[\frac{m-2}{2^r} \right] + 3. \quad \square$$

4.2.5 Remark

Let $f : L_k^{2m-1} \longrightarrow S^{p-1}$ and $g : L_k^{2n-1} \longrightarrow S^{q-1}$ be \mathbb{Z}_2 -equivariant maps with respect to multiplication by $e^{\pi i/k}$ on L_k and -1 on S . Denote by $\bar{f} : S^{2m-1} \longrightarrow S^{p-1}$ and $\bar{g} : S^{2n-1} \longrightarrow S^{q-1}$, the composition of f and g with the projection maps from the spheres to the Lens spaces respectively. Then \bar{f} and \bar{g} are \mathbb{Z}_{2k} -equivariant with respect to multiplication by $e^{\pi i/k}$ on the domains and by -1 on the ranges. Taking their joins, we get a \mathbb{Z}_{2k} -equivariant map

$$\bar{f}^* * \bar{g}^* : S^{2m-1} * S^{2n-1} \longrightarrow S^{p-1} * S^{q-1}.$$

$$\text{i.e., } \bar{f}^* * \bar{g}^* : S^{2(m+n)-1} \longrightarrow S^{(p+q)-1}.$$

Passing to quotients we get a \mathbb{Z}_2 -equivariant map

$$\overline{f * g} : L_k^{2(m+n)-1} \longrightarrow S^{(p+q)-1}$$

Using this construction we can improve the lower bound for $l(m)$ when $m-2$ is of the form $2^r(1+1)-1$ as follows : According to our present estimate, if $m-2 = 2^r(1+1)-1$, then $l(m) \geq 2l+3$. Suppose, if possible, $l(m) = 2l + 3$. Then there exists a \mathbb{Z}_2 -equivariant map $f : L_k^{2(2^r(l+1)+1)-1} \longrightarrow S^{(2l+3)-1}$. Applying the above construction to f , we get a \mathbb{Z}_2 -equivariant map

$$\overline{f*f} : L_k^{2(2^{r+1}(l+1)+2)-1} \longrightarrow S^{(4l+6)-1}.$$

This contradicts the estimate of 4.2.4, that

$$l(2^{r+1}(1+1)+2) \geq 4(1+1)+3 = 4l+7.$$

Hence $l(m) \geq 2l+4$ if $m-2$ is of the form $2^r(1+1)-1$.

4.3 LEVEL OF L_k^{2m-1} , k ODD

We first of all use the construction of 4.2.5 to prove a lemma which will be used here to determine the level of L_k^{2m-1} for odd k . The lemma will also be used in later work.

4.3.1 Lemma

$l(m+n) \leq l(m) + l(n)$ where $l(m)$ denotes the level of L_k^{2m-1} , for a fixed k . (cfr. [39]).

Proof: Let $f : L_k^{2m-1} \longrightarrow S^{l(m)-1}$ and $g : L_k^{2n-1} \longrightarrow S^{l(n)-1}$

be \mathbb{Z}_2 -equivariant maps with respect to multiplication by $e^{\pi i/k}$ on L_k and -1 on S . Then by the join construction of

4.2.5, we shall get a \mathbb{Z}_2 -equivariant map

$$\overline{f*g} : L_k^{2(m+n)-1} \longrightarrow S^{l(m)+l(n)-1}.$$

Hence, $l(m+n) \leq l(m) + l(n)$. □

Now we are in a position to proceed to the Theorem determining the level of L_k^{2m-1} for odd k .

4.3.2 THEOREM

$\ell(m) = 2m$, where $\ell(m)$ denotes the level of the Lens space L_k^{2m-1} , k odd.

Proof : By a result of P.E. Conner and E.E. Floyd, we know that $\ell(m) \leq \dim(L_k^{2m-1}) + 1$. (cfr. [7], 3.7).

This, in conjunction with Theorem 4.2.4 tells us that $2m-1 \leq \ell(m) \leq 2m$.

Suppose $\ell(m) = 2m-1$.

$$\begin{aligned} \text{Then, } 2m+1 = 2(m+1)-1 &= \ell(m+1) \leq \ell(m) + \ell(1) \\ &= (2m-1) + (2-1) = 2m, \end{aligned}$$

obviously a contradiction.

Hence $\ell(m) = 2m$. □

A REFINEMENT OF THE LOWER BOUND
 CHAPTER FIVE FOR THE LEVEL OF
THE LENS SPACE L_k^{2m-1} , k EVEN

5.1 INTRODUCTION

In the preceding chapter we established a lower bound for the level of the Lens Space L_k^{2m-1} . In several situations, we can better this lower bound by using some obstruction theory and K-theoretic codegree of vector bundles. This is done by first of all reducing the problem of determining the level of the Lens Space, to that of the existence of a nowhere vanishing cross section of a certain vector bundle. In this chapter, we do precisely this, and refine the lower bound for the level of the Lens Space L_k^{2m-1} , k even.

5.1.2 Formulation of the Problem.

Consider the line bundle $L = L_k^{2m-1} \times_{\mathbb{Z}/2} \mathbb{R} \longrightarrow L_{2k}^{2m-1}$. If $f : L_k^{2m-1} \longrightarrow S^{n-1}$ is an equivariant map, then $\text{id} \times f : L_k^{2m-1} \longrightarrow L_k^{2m-1} \times S^{n-1}$ is an equivariant map, and passing to quotients, we get a map $L_{2k}^{2m-1} \longrightarrow L_k^{2m-1} \times_{\mathbb{Z}/2} S^{n-1}$, and hence a nowhere vanishing section of $nL = L_k^{2m-1} \times_{\mathbb{Z}/2} \mathbb{R}^n \longrightarrow L_{2k}^{2m-1}$. Conversely, a nowhere vanishing section of nL gives rise to an equivariant map. Hence we can characterise the level of L_k^{2m-1} as the smallest n such that nL has a nowhere vanishing section. We shall denote the level of L_k^{2m-1} by $\ell(m)$.

5.2 COHOMOTOPY EULER CLASS AND

K-THEORETIC CODEGREE OF VECTOR BUNDLES

In the following we briefly recall some preliminaries on the Cohomotopy Euler Class and K-theoretic codegree of Vector Bundles that we shall need in this chapter. For further details we refer to [10], [11] and [39] etc..

5.2.1 Cohomotopy Euler Class

Let X be a finite CW complex and let E be an n -dimensional vector bundle over X . Let $T(E)$ denote its Thom Space. The zero section of E induces a map $i : X \longrightarrow T(E)$, or, more generally, a map $i : T(F) \longrightarrow T(E \oplus F)$ for a vector bundle F over X . If E' is an n' -dimensional inverse bundle of E , then a trivialization of $E \oplus E'$ induces a map $t : T(E \oplus E') \longrightarrow S^{n+n'}$. For n' large, the bundle E' is unique and we define the cohomotopy Euler class of $e(E)$ to be the composition

$$T(E') \xrightarrow{i} T(E \oplus E') \xrightarrow{t} S^{n+n'}$$

The Thom spectrum $M(E)$ is by definition the n th desuspension of the suspension spectrum of $T(E)$. The definition of the Thom spectrum may be extended to virtual bundles, by writing $M(-E) = M(E')$ where E' is an inverse to E . For large n' , the set of homotopy classes of maps from $T(E')$ to $S^{n+n'}$, $[T(E'), S^{n+n'}]$, is isomorphic to $\{T(E'), S^{n+n'}\}$, the group of homotopy classes of maps from the suspension spectrum of $T(E')$ to the suspension spectrum of $S^{n+n'}$. By suspension isomorphism, this is isomorphic to $\{M(-E), S^n\} = \pi^n(M(-E))$. Using these identifications, we consider the cohomotopy Euler class $e(E)$ to be an element of $\pi^n(M(-E))$. As we

consider $\pi^n(M(-E))$ to be a "twisted" cohomotopy group of X , we denote it as $\pi^n(X; -E)$.

5.2.2 The ordinary Euler class and the cohomotopy Euler class are related as follows : The Hurewicz homomorphism

$$h : \pi^n(X; -E) = \pi^n(M(-E)) \longrightarrow H^n(M(E); \mathbb{Z}) = H^n(X; -E)$$

maps $e(E)$ to a (twisted) cohomology class $e_{\mathbb{Z}}(E)$ which is called the cohomology Euler class of E . If E is orientable, the Thom isomorphism $H^n(X; -E) \cong H^n(X; \mathbb{Z})$ maps $e_{\mathbb{Z}}(E)$ to the usual Euler class. If we replace integral cohomology with \mathbb{Z}_2 -cohomology, the corresponding Hurewicz homomorphism maps $e(E)$ to the n th Stiefel-Whitney class of E (here, of course, we do not need E to be oriented).

5.2.3 Properties of the Cohomotopy Euler Class

The following are the important properties of the cohomotopy Euler class.

(i) Multiplicative Property

$e(E \oplus F) = e(E) \cdot e(F)$, where the product on the right is the cup product for (twisted) cohomotopy, and E and F are vector bundles over X .

(ii) Vanishing of the Cohomotopy Euler Class

If E is an n -dimensional vector bundle over a finite CW-complex X and $\dim(X) < 2(n-1)$, then E has a nowhere vanishing section if and only if its cohomotopy Euler Class vanishes (cfr. [10] 2.4).

It is this property that enables us to use the cohomotopy Euler Class to establish bounds for the level of a space.

5.2.4 Gysin Sequences

If $E \xrightarrow{p} X$ is a vector bundle of dimension n over X and F is another vector bundle over X , then there is a cofibration

$$T(p^*F) \xrightarrow{p} T(F) \xrightarrow{i} T(E \oplus F),$$

inducing long exact sequences :

$$\begin{aligned} \longrightarrow \pi^{l-n}(X; E \oplus F) \xrightarrow{i^*} \pi^l(X; F) \xrightarrow{p^*} \pi^l(S(E); p^*(F)) \xrightarrow{\partial} \\ \pi^{l-n+1}(X; E \oplus F) \longrightarrow \dots \end{aligned}$$

and

$$\begin{aligned} \longrightarrow H^{l-n}(X; E \oplus F) \xrightarrow{i^*} H^l(X; F) \xrightarrow{p^*} H^l(S(E); p^*(F)) \xrightarrow{\partial} \\ H^{l-n+1}(X; E \oplus F) \longrightarrow \dots \end{aligned}$$

which are respectively called the cohomotopy and cohomology Gysin sequences for $S(E)$ with coefficients in F . The maps i^* in these sequences are multiplications respectively by the cohomotopy and the cohomology Euler classes of E . These sequences will be needed later in the chapter.

5.2.5 Codegree of an Orientable Vector Bundle

If E is an orientable (virtual) vector bundle over a space X , the *codegree of E* denoted by $cd(E)$, is defined as the index of the Hurewicz homomorphism $h : \pi^0(M(E)) \longrightarrow H^0((M(E); \mathbb{Z}))$.

5.2.6 K-Theoretic Codegree of a Vector Bundle

Note that the composition $\pi^0(M(E)) \xrightarrow{h} H^0(M(E); \mathbb{Z}) \xrightarrow{i} H^*(M(E); \mathbb{Q})$ is the same as the composition $\pi^0(M(E)) \xrightarrow{h_K} K^0(M(E)) \xrightarrow{ch} H^*(M(E); \mathbb{Q})$ where h_K is the K-theory Hurewicz homomorphism and ch is the Chern character. The index of $im(i) \cap im(ch)$ in $im(i)$ is defined to be the *K-theory codegree of E* and is denoted by $cd^K(E)$.

In the above definition, if we replace the Chern character $ch:K^{\circ}(M(E)) \longrightarrow H^*(M(E);\mathbb{Q})$ by the Pontrjagin character $Ph:KO^{\circ}(M(E)) \longrightarrow H^*(M(E);\mathbb{Q})$ which is the composition of the complexification map $KO^{\circ}(M(E)) \longrightarrow K^{\circ}(M(E))$ and the Chern character, we obtain the *KO-theoretic codegree* of E which is denoted by $cd^{KO}(E)$.

5.2.7 Lemma : $cd(E)$ is a multiple of $cd^{KO}(E)$, which in turn is a multiple of $cd^K(E)$. ■

$cd^K(E)$ is characterised by

5.2.8 Lemma : Let E be a complex vector bundle over a finite CW space X with torsion free homology. Then $cd^K(E) = \min\{m \in \mathbb{N} \mid m \cdot ch^{-1}(\text{Todd}(-E)) \in K^{\circ}(X) \otimes \mathbb{Q} \text{ is integral}\}$ where $\text{Todd}(E) \in H^*(X;\mathbb{Q})$ denotes the Todd genus of E .

Todd genus is multiplicative

i.e., $\text{Todd}(E \oplus F) = \text{Todd}(E) \cdot \text{Todd}(F)$

and if L is a line bundle, then

$$\text{Todd}(L) = (\exp(c_1(L)) - 1) / c_1(L). \quad \blacksquare$$

5.2.9 Lemma : Let L be a complex line bundle. Then $ch^{-1}(\text{Todd}(-L)) = \log(\lambda+1)/\lambda \in K^{\circ}(X) \otimes \mathbb{Q}$, where $\lambda = L^{-1} \in K^{\circ}(X)$ and $\log(\lambda+1)$ is the usual power series of the natural logarithm. ■

5.2.10 Lemma : If E is some (virtual) vector bundle over $\mathbb{C}P^{m-1}$, then the index of the Hurewicz homomorphism $h : \pi^{2s}(\mathbb{C}P^{m-1}; E) \longrightarrow H^{2s}(\mathbb{C}P^{m-1}; \mathbb{Z})$ is the codegree of $E + sH$ over $\mathbb{C}P^{m-s-1}$. ■

5.3 A REFINEMENT OF THE LOWER BOUND FOR $\ell(m)$

We have now introduced the preliminaries. We proceed to the refinement of the lower bound.

Let $k = 2^r \cdot p$, p , p odd, $r \geq 1$. We may identify L_{2k}^{2m-1} with the sphere bundle of H^{2k} , the $2k^{\text{th}}$ tensor product of the Hopf bundle $H \longrightarrow \mathbb{C}P^{m-1}$ over the Complex Projective space. We know that the Hopf bundle H is the vector bundle associated to the standard one-dimensional complex representation of S^1 given by multiplication by $e^{it} \in S^1$. Then H^k , k^{th} tensor power of H is the vector bundle associated to the multiplication by e^{ikt} , for $e^{it} \in S^1$. Also $p^*(H^k)$, the vector bundle over L_{2k}^{2m-1} induced by the projection $p : L_{2k}^{2m-1} = S(H^{2k}) \longrightarrow \mathbb{C}P^{m-1}$ is the vector bundle associated to multiplication by e^{ikt} for $e^{it} \in \mathbb{Z}_{2k}$, the group of $2k^{\text{th}}$ roots of unity $\subseteq S^1$.

Now, for a generator $e^{\pi i/k}$ of \mathbb{Z}_{2k} , and $z \in \mathbb{C}$, $z \cdot e^{ik \cdot \pi/k} = -z$ lies on the straight line passing through z and the origin in the complex plane. Hence the above representation of \mathbb{Z}_{2k} is the sum of two copies of the one-dimensional real representation of \mathbb{Z}_2 whose associated line bundle is L (cfr. 5.1.2), i.e., $p^*(H^k) = 2L$.

For arbitrary q , we write down the cohomotopy and the cohomology Gysin sequences of the sphere bundle $S(H^{2k})$ with coefficients in $-qH^k$ (cfr. 5.2.4).

$$\begin{array}{ccccccc}
\longrightarrow & \pi^{2q-2}(\mathbb{C}P^{m-1}; H^{2k} - qH^k) & \xrightarrow{i_1^*} & \pi^{2q}(\mathbb{C}P^{m-1}; -qH^k) & \xrightarrow{p_1^*} & \pi^{2q}(L_{2k}^{2m-1}; -2qL) & \longrightarrow \\
& \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & \\
\longrightarrow & H^{2q-2}(\mathbb{C}P^{m-1}; \mathbb{Z}) & \xrightarrow{i_2^*} & H^{2q}(\mathbb{C}P^{m-1}; \mathbb{Z}) & \xrightarrow{p_2^*} & H^{2q}(L_{2k}^{2m-1}; \mathbb{Z}) & \longrightarrow
\end{array}$$

The naturality of the cohomotopy Euler class gives us that

$$\begin{aligned}
\text{for } e(qH^k) \in \pi^{2q}(\mathbb{C}P^{m-1}; -qH^k), \quad p_1^*(e(qH^k)) &= e(p^*(qH^k)) \\
&= e(2qL) \in \pi^{2q}(L_{2k}^{2m-1}; -2qL).
\end{aligned}$$

Further, we know that $H^*(\mathbb{C}P^{m-1}; \mathbb{Z}) = \mathbb{Z}[x]/x^m$, where $x \in H^2(\mathbb{C}P^{m-1})$ is the first chern class of the Hopf bundle. Then the cohomology Euler class $e_{\mathbb{Z}}(H^k) = c_1(H^k) = kx = 2^r p \cdot x$, and $e_{\mathbb{Z}}(qH^k) = (2^r p \cdot x)^q = 2^{rq} \cdot p^q \cdot x^q$.

We also have that i_2^* is multiplication by $e_{\mathbb{Z}}(H^{2k}) = 2kx = 2^{r+1} p \cdot x$.

Now, suppose that $e(2qL) = p_1^*(e(qH^k)) = 0$. Then by exactness, we get a $y \in \pi^{2q-2}(\mathbb{C}P^{m-1}; H^{2k} - qH^k)$ such that $i_1^*(y) = e(qH^k)$. Commutativity of the diagram gives us that

$$\begin{aligned}
h_2^*(e(qH^k)) &= i_2^*(h_1(y)) \\
\text{i.e., } e_{\mathbb{Z}}(qH^k) &= h_1(y) \cdot 2^{r+1} p \cdot x \\
\text{i.e., } 2^{rq} \cdot p^q \cdot x^q &= h_1(y) \cdot 2^{r+1} p \cdot x, \quad \text{so that,}
\end{aligned}$$

$$5.3.1 \quad h_1(y) = 2^{rq-r-1} p^{q-1} x^{q-1}.$$

In Chapter Four we have established lower bounds for $\ell(m)$ for two different cases (cfr. 4.2.4 and 4.2.5). We shall denote this lower bound by l . Take $q = \left\lceil \frac{1+c}{2} \right\rceil$, $c \geq 0$ where $[x]$ denotes the integral part of x . Then

$$2q = \begin{cases} 1+c & \text{if } 1+c \text{ is even} \\ 1+c-1 & \text{if } 1+c \text{ is odd} \end{cases}$$

We want to find the greatest value of c which will give a contradiction to 5.3.1. It will then follow that for this value of c , $e(2qL) \neq 0$ and hence 5.2.3 (ii) that

$$5.3.2 \quad l(m) \geq \begin{cases} 1+c+1, & \text{if } 1+c \text{ is even} \\ 1+c, & \text{if } 1+c \text{ is odd.} \end{cases}$$

Using 5.2.10, we get that the index of the Hurewicz homomorphism $h_1 : \pi^{2q-2}(\mathbb{C}P^{m-1}; H^{2k} - qH^k) \rightarrow H^{2q-2}(\mathbb{C}P^{m-1}; \mathbb{Z})$ is the codegree of $H^{2k} - qH^k + (q-1)H$ over $\mathbb{C}P^{m-(q-1)-1}$, i.e. $\mathbb{C}P^{m-q}$.

5.3.3 Lemma : $Ch^{-1} \text{Todd} (-(H^{2k} - qH^k + (q-1)H)) = (1-y)^{q-1} \pmod{2 \mathbb{Z}_{(2)}[y]}$ where $y = \eta/2$, $\eta = H - 1$, and $\mathbb{Z}_{(2)}$ the integers localized at 2.

Proof : Note that

$$\frac{\log(\eta-1)}{\eta} = 1 - \frac{\eta}{2} + \frac{\eta^2}{3} - \frac{\eta^3}{4} + \dots$$

If we write $\eta/2 = y$, we see that the above power series is an element of $\mathbb{Z}_{(2)}[y]$. Further, if we compute modulo $2\mathbb{Z}_{(2)}$, we find that $Ch^{-1} \text{Todd} (-H) = \frac{\log(\eta+1)}{\eta}$, by 5.2.9.

$$= (1-y) \pmod{2\mathbb{Z}_{(2)}[y]}.$$

In general, we can see that if $\lambda \in \mathbb{Z}[\eta]$, with vanishing constant term, then

$$\frac{\log(1+\lambda)}{\lambda} = \left(1 - \frac{\lambda}{2}\right) \pmod{2 \mathbb{Z}_{(2)}[y]}.$$

$$\begin{aligned} \text{Hence, } Ch^{-1} \text{Todd} (-H^{2k}) &= \frac{\log(\eta+1)^{2k}}{(\eta+1)^{2k} - 1} \\ &= \left[1 - \frac{(\eta+1)^{2k} - 1}{2} \right] \pmod{2 \mathbb{Z}_{(2)}[y]} \end{aligned}$$

$$= \left[1 - \frac{\eta^{2k} + 2k \cdot \eta^{2k-1} + \dots + 2k \cdot \eta}{2} \right] \text{ mod } 2\mathbb{Z}_{(2)}[y]$$

$$= 1 \text{ mod } 2\mathbb{Z}_{(2)}[y].$$

Also, $\text{Ch}^{-1}\text{Todd}(-H^k) = \frac{\log(\eta+1)^k}{(\eta+1)^k - 1}$

$$= \left[1 - \frac{(\eta+1)^k - 1}{2} \right] \text{ mod } 2\mathbb{Z}_{(2)}[y]$$

$$= \left[1 - \frac{\eta^k + 1\eta^{k-1} + \dots + 1\eta}{2} \right] \text{ mod } 2\mathbb{Z}_{(2)}[y]$$

$$= 1 \text{ mod } 2\mathbb{Z}_{(2)}[y] \quad \text{as } k \equiv 0 \text{ mod } 2.$$

Since the Chern character is a ring homomorphism, using the multiplicativity of the Todd Genus, we get that

$$\text{Ch}^{-1}\text{Todd}(-(H^{2k} - qH^k + (q-1)H)) = (1-y)^{q-1} \text{ mod } 2\mathbb{Z}_{(2)}[y].$$

□

5.3.4 PROPOSITION

The codegree of $H^{2k} - qH^k + (q-1)H$ over $\mathbb{C}P^{m-q}$ (and hence the index of the Hurewicz homomorphism $h_1 : \pi^{2q-2}(\mathbb{C}P^{m-1}; H^{2k} - qH^k) \longrightarrow H^{2q-2}(\mathbb{C}P^{m-1}; \mathbb{Z})$ is a multiple of 2^{m-q-1} or 2^{m-q} according as $m \geq 2q$ or $m < 2q$.

Proof : We know that $K^0(\mathbb{C}P^{m-q})$ is the truncated polynomial ring $\mathbb{Z}[\eta]/\eta^{m-q+1}$ where $\eta = H - 1$. When we express $(1-y)^{q-1}$, $y = \eta/2$, as a power series in η , we see that $q-1$ is the smallest power of 2 such that $2^{q-1}(1-y)^{q-1} \in \mathbb{Z}_{(2)}[\eta]$. Also 2^{m-q-1} is the smallest power of 2 such that $2^{m-q-1}(2\mathbb{Z}_{(2)}[\eta]) \subseteq \mathbb{Z}_{(2)}[\eta]/\eta^{m-q+1}$. It follows that 2^{m-q-1} and 2^{m-q} respectively are the smallest powers of 2 such that $(1-y)^{q-1} \text{ mod } 2\mathbb{Z}_{(2)}[y]$ is contained in $\mathbb{Z}_{(2)}[\eta]/\eta^{m-q+1}$,

according as $m-q-1 \geq q-1$ or $m-q-1 < q-1$ respectively.

The proposition follows from 5.2.8 and 5.3.3. ■

5.3.5 Lemma : Let m, q, l, c be as defined earlier in the chapter. Then

a) When $b = 2^r - 1$,

(i) $m \geq 2q$ according as $c \leq m-2a-4$ if c is even

or as $c \leq m-2a-3$ if c is odd.

(ii) $m-q > rq-r-1$ according as $c < 2\left(\frac{m}{r+1} - a\right) - 2$,

if c is even

or as $c < 2\left(\frac{m}{r+1} - a\right) - 1$ if c is odd.

(iii) $m-q-1 > rq-r-1$ according as $c < 2\left(\frac{m-1}{r+1} - a\right) - 2$,

if c is even

or as $c < 2\left(\frac{m-1}{r+1} - a\right) - 1$, if c is odd.

b) In the other cases,

i) $m \geq 2q$ according as $c \leq m-2a-3$, if c is odd

or as $c \leq m-2a-2$, if c is even

ii) $m-q > rq-r-1$ according as $c < 2\left(\frac{m}{r+1} - a\right) - 1$, if c is odd

or as $c < 2\left(\frac{m}{r+1} - a\right)$, if c is even.

iii) $m-q-1 > rq-r-1$ according as $c < 2\left(\frac{m-1}{r+1} - a\right) - 1$, if c is odd

or as $c < 2\left(\frac{m-1}{r+1} - a\right)$, if c is even.

Proof : In (a), $l = 2a+4$ and in (b), $l = 2a+3$, by 4.2.4 and 4.2.5. Thus, in (a), $l+c$ is even or odd according as c is even or odd and in (b) $l+c$ is even or odd according as c is odd or even. Elementary calculations now prove (i) in both (a) and (b).

For (ii) in both cases, we have that $m-q > rq-r-1 \Leftrightarrow q < \frac{m}{r+1} + 1$.
 We note further that if $1+c$ is even $q = \frac{1+c}{2}$, and when $1+c$ is odd, $q = \frac{1+c}{2} - \frac{1}{2}$. (cfr. 5.3.2).

Thus, for example, when $l = 2a+4$ and if c is odd, we have that $m-q > rq-r-1 \Leftrightarrow q < \frac{m}{r+1} + 1$

$$\Leftrightarrow \frac{2a+4+c}{2} - \frac{1}{2} < \frac{m}{r+1} + 1$$

$$\Leftrightarrow c < 2 \left(\frac{m}{r+1} - a \right) - 1.$$

Similar calculations prove (ii) in all other cases.

For (iii) in both cases, we have that

$$m-q-1 < rq-r-1 \Leftrightarrow q < \frac{m+r}{r+1} = \frac{m-1}{r+1} + 1.$$

Considerations similar to those above now prove (iii) in both cases. □

Finally we come to the theorem which establishes the refined lower bound.

5.3.6 THEOREM

Let $k = 2^r p$, p odd and $r \geq 1$. Write $m-2 = 2^r a+b$, where $a \geq 0$, $0 \leq b \leq 2^r - 1$. Then $\ell(m)$, the level of the Lens space L_k^{2m-1} satisfies the following :

when $r = 1$, $\ell(m) \geq m+1$, $m = 0, 2, 4 \pmod{8}$

$\geq m+2$, $m = 1, 3, 5, 7 \pmod{8}$

$\geq m+3$, $m = 6 \pmod{8}$.

When $r > 1$, $\ell(m) \geq 2a+5+c$, $c < 2 \left(\frac{m-1}{r+1} - a \right)$, c even,

$2m-(2a+2+c) = 6 \pmod{8}$.

$$\begin{aligned} &\geq 2a+4+c, \quad c < 2\left(\frac{m-1}{r+1} - a\right) - 1, \quad c \text{ odd,} \\ &\quad \text{when } b = 2^r - 1, \\ &\geq 2a+3+c, \quad c < 2\left(\frac{m-1}{r+1} - a\right), \quad c \text{ even,} \\ &\quad \text{in all other cases.} \end{aligned}$$

Proof :

Case 1 : We first of all consider the case when $b = 2^r - 1$.

By 4.2.5, $l = 2a+4$.

When $r = 1$, by lemma 5.3.5 (a), (i) and (ii), $m < 2q$ and $m-q > rq-r-1$ hold simultaneously when $-1 < c < 1$, i.e., when $c = 0$ if c is even, and when $0 < c < 2$, i.e., when $c = 1$ if c is odd. Hence proposition 5.3.4 contradicts 5.3.1 if $2q = 2a+4+c$ where $c = 0$ if c is even and if $2q = 2a+3+c$ where $c = 1$ if c is odd. In either case $l(m) \geq 2a+5$ (cfr. 5.3.2.) i.e., $l(m) \geq m+2$, as in this case, since $r = 1$, $m = 2a+3$.

On the other hand by lemma 5.3.5 (a), (i) and (iii), $m \geq 2q$ and $m-q-1 > rq-r-1$ hold when $c \leq -1$ and $c < 0$ if c is even and when $c \leq 0$ and $c < 1$ if c is odd. As we assume $c \geq 0$, we cannot draw any conclusions from these results.

When $r \geq 1$, $m < 2q$ and $m-q > rq-r-1$ hold simultaneously when $m-2a-4 < c < 2\left(\frac{m}{r+1} - a\right) - 2$ if c is even and when $m-2a-3 < c < 2\left(\frac{m}{r+1} - a\right) - 1$ if c is odd. Elementary calculations show that these restrictions do not yield any integral value of c .

The conditions $m \geq 2q$ and $m-q-1 > rq-r-1$ yield $c < \left(\frac{m-1}{r+1} - a\right) - 2$ if c is even and $c < 2\left(\frac{m-1}{r+1} - a\right) - 1$ if c is odd. Hence a contradiction of 5.3.1 is obtained when $2q = 2a+4+c$ or $2a+3+c$

according as c is even or odd, c satisfying the above restrictions. 5.3.2 now yields the required result.

Case 2 : $l = 2a+3$ (Cfr. 4.2.4).

When $r = 1$, by lemma 5.3.5 (b), (i) and (ii), $m < 2q$ and $m-q > rq-r-1$ hold simultaneously when $0 < c < 2$ if c is even and when $-1 < c < 1$ if c is odd. These do not give any values for c . On the other hand, $m \leq 2q$ and $m-q-1 > rq-r-1$ hold simultaneously when $c \leq 0$ if c is even and $c \leq -1$ if c is odd. This gives us that $c = 0$. Hence proposition 5.3.4 contradicts 5.3.1 if $2q = 2a+3+c$ where $c = 0$. Hence $l(m) \geq 2a+3$ (cfr. 5.3.2). i.e., $l(m) \geq m+1$, as in this case, since $r = 1$, $m = 2a+2$.

When $r > 1$, considerations similar to those in case 1 give us that no integral values of c are obtained when $m < 2q$ and $m-q > rq-r-1$. When $m \leq 2q$ and $m-q-1 > rq-r-1$, simple calculations give the required modifications in the lower bound.

Case 3: Put $2q = 2a+4+c$, $c \geq 0$ (evidently c is even).

We want to obtain conclusions about the codegree of $H^{2k-q}H^k + (q-1)H$ over $\mathbb{C}P^{m-q} = \mathbb{C}P^{m-a-2-c/2}$. Write s for $m-a-1-c/2$. Consider the cofibration

$$\mathbb{C}P^{s-1} \longrightarrow \mathbb{C}P^s \longrightarrow \mathbb{C}P^s/\mathbb{C}P^{s-1} \cong S^{2s},$$

and the induced long exact sequence in KO-theory

$$\dots \longrightarrow KO^{-1}(S^{2s}) \longrightarrow KO^0(\mathbb{C}P^s) \longrightarrow KO^0(\mathbb{C}P^{s-1}) \longrightarrow KO^0(S^{2s}) \longrightarrow$$

This gives us an isomorphism

$$KO^0(\mathbb{C}P^s) \xrightarrow{\cong} KO^0(\mathbb{C}P^{s-1})$$

by Bott periodicity in KO theory, provided that $2s = 6 \pmod{8}$.

Hence the KO-theoretic co-degree of $H^{2k} - qH^k + (q-1)H$ over $\mathbb{C}P^{s-1}$ (i.e., $\mathbb{C}P^{m-\alpha-2-c/2} = \mathbb{C}P^{m-q}$ when $2q = 2\alpha+4+c$) is the same as its KO-theoretic codegree over $\mathbb{C}P^s$ (i.e., $\mathbb{C}P^{m-\alpha-1-c/2} = \mathbb{C}P^{m-q}$ when $2q = 2\alpha+2+c$).

The proof of Theorem 5.3.6, case 2, in conjunction with lemma 5.2.7 gives us the required results both when $r = 1$ and when $r > 1$. ■

AN UPPERBOUND FOR THE
 CHAPTER SIX LEVEL OF THE LENS SPACE
 L_k^{2m-1} , k EVEN

6.1 INTRODUCTION

In this chapter, we use Adams' Spectral Sequence, especially a "vanishing line" for its E_2 term to establish an upper bound for the level of the Lens Space L_k^{2m-1} when k is even. The determination of the upper bound will enable us to establish the level when $2 \mid k$ but $2^2 \nmid k$. We first of all establish a few preliminaries on Adams' Spectral Sequence. For details we refer to [39]

6.2 PRELIMINARIES ON ADAMS' SPECTRAL SEQUENCE

Let X and Y be finite spectra and let p be a fixed prime. We say that a map $X \longrightarrow Y$ has \mathbb{Z}_p -Adams' filtration $\geq s$, if it can be written as a composition

$$X \longrightarrow Z_1 \longrightarrow \dots \longrightarrow Z_{s-1} \longrightarrow Y$$

of s maps which are all trivial in \mathbb{Z}_p -cohomology. This defines a filtration on $[\Sigma^n X; Y] \stackrel{\text{def}}{=} [X; Y]^n$. We shall denote by $F_s [X; Y]_n$, the subgroup of elements of filtration $\geq s$, in $[X; Y]_n$. When X or Y is the sphere spectrum S^0 , this defines a filtration on the homotopy or the cohomotopy groups of spectra respectively.

The following lemma relates to the compatibility of the Adams' filtration with the cup product.

6.2.1 Lemma

If α and α' are vector bundles over a space X and f and f' are elements of $\pi^n(X; \alpha)$ and $\pi^{n'}(X; \alpha')$ of Adams' filtration $\geq s$ and $\geq s'$ respectively, then their cup product has filtration $\geq s+s'$. (cfr. [39], 4.3) \square

Associated to the Adams' filtration on $[X; Y]_n$, there is a spectral sequence $E_r^{s,t}(X, Y)$, called the Adams' spectral sequence such that

$$E_\infty^{s,t}(X, Y) \cong F_s[X; Y]_{t-s} / F_{s+1}[X; Y]_{t-s},$$

where $F_s[X; Y]_{t-s}$ denotes the elements of filtration s in $[X; Y]_{t-s}$. Also the intersection of all $F_s[X; Y]_{t-s}$ consists of the torsion elements of $[X; Y]_{t-s}$, whose order is prime to p . The E_2 term of the spectral sequence is

$$E_2^{s,t}(X, Y) = \text{Ext}_A^{s,t}(H^*Y, H^*X)$$

where H^*Y and H^*X denote the cohomology of Y and X respectively with coefficients in \mathbb{Z}_p , which is a module over the mod p Steenrod Algebra A . The differentials have the form

$$d_r : E_r^{s,t}(X, Y) \longrightarrow E_r^{s+r, t+r-1}(X, Y).$$

We need the following results for subsequent work.

For $p = 2$, let A_0 denote the subalgebra of A generated by Sq^1 . Note that since Sq^1Sq^1 is zero, this is an exterior algebra.

6.2.2 Proposition : (cfr. [1], Thm 3, p. 62.)

Let M be a graded A -module which is free over A_0 and $(1-1)$ -connected, i.e., trivial in dimensions < 1 . Then $\text{Ext}_A^{s,t}(M, \mathbb{Z}_2)$ is zero if $t-s < 1+F(s)$, where $F(s)$ is the

numerical function defined by $F(4r) = 8r$, $F(4r+1) = 8r+1$,
 $F(4r+2) = 8r+2$ and $F(4r+3) = 8r+4$. □

6.2.3 Corollary : (cfr. [39]).

Let X be a finite spectrum whose \mathbb{Z}_p -cohomology vanishes for p odd and whose \mathbb{Z}_2 -cohomology is free as an A_0 -module and trivial above dimension d . Let $\alpha \in \pi^n(S)$ be an element of Adam's filtration s . Then $\alpha = 0$, provided that $d-n < F(s)$. □

6.3 AN UPPERBOUND FOR $\ell(m)$

Earlier (cfr. 5.3), we have denoted the lower limit for the level of the Lens space we obtained in Chapter Four (vide 4.2.4 and 4.2.5). What we do is use corollary 6.2.3 above, to find the smallest f for which the cohomotopy Euler Class $e((1+f)L) \in \pi^n(M(-(1+f)L))$ is trivial. Hence using 5.2.3 (ii), we shall be able to establish an upper bound for the level of the Lens Space.

We first of all recall the cohomology structure of L_k^{2m-1} . We have discussed the details of the computation of the Cohomology structure of L_k^{2m-1} in the Appendix. We present the structure here for ready reference.

6.3.1 The \mathbb{Z}_p -cohomology structure of L_k^{2m-1} .

$$H^*(L_k^{2m-1}; \mathbb{Z}_p) = \begin{cases} H^*(S^{2m-1}; \mathbb{Z}_p), & \text{if } p \nmid k \\ H^*(\mathbb{R}P^{2m-1}; \mathbb{Z}_2), & \text{if } p = 2, 2 \mid k \\ & \text{but } 2^2 \nmid k \end{cases}$$

$$\frac{\mathbb{Z}_p[a,b]}{a^2, b^m}, \quad \beta_p^*(a) = b, \text{ if } p \text{ is odd, } p|k,$$

but $p^2 \nmid k$, where $a \in H^1(L_k^{2m-1}; \mathbb{Z}_p)$,

$b \in H^2(L_k^{2m-1}; \mathbb{Z}_p)$ and β_p^* is the mod p

Bockstein.

$$\frac{\mathbb{Z}_p[a,b]}{a^2, b^m}, \text{ if } p^2|k, \text{ where } a \in H^1(L_k^{2m-1}; \mathbb{Z}_p)$$

and $b \in H^2(L_k^{2m-1}; \mathbb{Z}_p)$. Moreover when

$p = 2$, a is the first S-W class of

the line bundle $L_{k/2}^{2m-1} \times_{\mathbb{Z}_2} \mathbb{R} \longrightarrow L_k^{2m-1}$.

6.3.2 Proposition : $M(-(2q+1)L)$ satisfies :

- (i) $H^*(M(-(2q+1)L); \mathbb{Z}_2)$ is free as an A_0 -module
- (ii) $H^*(M(-(2q+1)L); \mathbb{Z}_p)$ is trivial for an odd prime p .

Proof :

(i) We note that $H^*(M(-(2q+1)L); \mathbb{Z}_2) \cong H^*(L_{2k}^{2m-1}; \mathbb{Z}_2)$ by the Thom isomorphism given by cup with the Thom class $U \in H^0(M(-(2q+1)L); \mathbb{Z}_2)$. Now, $Sq^1(\phi(1)) = Sq^1(U) = a.U$, from 6.3.1, by the definition of the S-W classes. (ϕ denotes the Thom isomorphism). For $s < m$, $Sq^1(\phi(b^s)) = Sq^1(b^s.U) = Sq^1(b^s).Sq^0(U) + Sq^0(b^s).Sq^1(U) = b^s.a.U$, as $Sq^1(b) = 0$.

(Note that even though $H^*(M(-(2q+1)L); \mathbb{Z}_2)$ is free over A_0 , $H^*(L_{2k}^{2m-1}; \mathbb{Z}_2)$ is not, as $Sq^1(b) = 0$).

This proves (i).

(ii) To prove the second part, we use the Gysin sequence for $S(L)$ with coefficients in $-(2q+2)L$ (cohomology with \mathbb{Z}_p -coefficients. Cfr 5.2.4).

$$\dots \longrightarrow H^{l-1}(L_{2k}^{2m-1}; -(2q+1)L) \longrightarrow H^l(L_{2k}^{2m-1}; -(2q+2)L) \xrightarrow{p^*} H^l(S(L); p^*(-(2q+2)L)) \longrightarrow \dots$$

We note that (a) $S(L) = L_k^{2m-1}$

(b) $-(2q+2)L$ is orientable.

Since p is an odd prime, $p|k$ (or $p \nmid k$) $\Leftrightarrow p|2k$ (or $p \nmid 2k$). Hence by (a), the untwisted \mathbb{Z}_p -cohomology of $S(L) = L_k^{2m-1}$ and L_{2k}^{2m-1} are isomorphic. Because of (b), p^* may be considered to be induced from the corresponding map in untwisted cohomology, and hence is an isomorphism. (cfr. [3], III 2.4).

The exactness of the sequence now implies that

$$H^*(L_{2k}^{2m-1}; -(2q+1)L), \text{ i.e., } H^*(M(-(2q+1)L); \mathbb{Z}_p)$$

is trivial. ■

6.3.3 Proposition : The cohomotopy Euler class $e((2q+1)L)$ has Adams' filtration at least q .

Proof: $w_2(2L)$, the second S-W class, is the image of $e(2L)$ under the Hurewicz homomorphism and $w_2(2L) = (w_1(L))^2 = a^2 = 0$. Hence, $e(2L)$ has Adams' filtration at least 1. Therefore, $e(2qL) = (e(2L))^q$ has Adams' filtration at least q . Note that $e(L)$ is trivial in \mathbb{Z}_p cohomology (cfr. 6.3.2) and hence has Adams' filtration at least zero. By 6.2.1, it follows that $e((2q+1)L)$ has filtration at least q . ■

6.3.4 THEOREM

Let $l(m)$ denote the level of the Lens Space L_k^{2m-1} , $k = 2^r p$, p odd, $r \geq 1$. The following hold:

$$\begin{aligned}
l(m) &\leq m+1 && \text{when } m = 0, 2 \pmod 8 \\
&\leq m+2 && \text{when } m = 1, 3, 4, 5, 7 \pmod 8 \\
&\leq m+3 && \text{when } m = 6 \pmod 8.
\end{aligned}$$

Proof : Put $m-2 = 2^r a + b$ where $0 \leq b \leq 2^r - 1$, $a \geq 0$. In general, we put $2q+1 = 1+f$. We want to find the smallest value of f for which $e((1+f)L) \in \pi^{l+f}(M(-(1+f)L))$ vanishes.

Using the notations of 6.2.3 we have $d = 2m-1$, $n = 2q+1 = 1+f$. By 6.3.3, $s = q = \frac{1+f-1}{2}$. In particular, when $b = 2^r - 1$, $l = 2a+4$ (cfr. 4.2.5). Hence $n = 2a+4+f$ and $s = \frac{2a+3+f}{2}$.

In other cases, $l = 2a+3$ (cfr. 4.2.4). Then $n = 2a+3+f$ and $s = \frac{2a+2+f}{2}$. Elementary calculations using 6.2.2 give us that $d-n < F(s)$ holds, and hence that the cohomotopy Euler class $e((1+f)L)$ vanishes according to the following conditions :

$$l(m) \leq 2a+4+f \quad \text{when } b = 2^r - 1$$

$$\begin{aligned}
&\text{where } f > m-2a-4, && \text{when } 2a+3+f = 0 \pmod 8 \\
&> \frac{2m-4a-7}{2}, && \text{when } 2a+3+f = 2 \pmod 8 \\
&> m-2a-3, && \text{when } 2a+3+f = 4, 6 \pmod 8.
\end{aligned}$$

and $l(m) \leq 2a+3+f$, in all other cases

$$\begin{aligned}
&\text{where } f > m-2a-3, && \text{when } 2a+2+f = 0 \pmod 8 \\
&> \frac{2m-4a-5}{2}, && \text{when } 2a+2+f = 2 \pmod 8 \\
&> m-2a-2, && \text{when } 2a+2+f = 4, 6 \pmod 8.
\end{aligned}$$

Let $b = 2^r - 1$. Then $m = 3$ or $7 \pmod 8$. Put $f = m-2a-4+1$, $1 > 0$. We get from the first condition that $l(m) \leq m+1$ if $m+1-1 = 0 \pmod 8$. We get the least values for 1 to be 6 or 2 according as $m = 3$ or $7 \pmod 8$. Thus, $l(m) \leq m+6$ or $m+2$ according as $m = 3$ or $7 \pmod 8$. The examination of other conditions using similar consideration, give us the following results :

$$\ell(m) \geq m+8 \text{ or } m+4$$

$$\geq m+2 \text{ or } m+4$$

$$\geq m+6 \text{ or } m+8$$

according as $m = 3$ or $7 \pmod 8$. Choosing the least upper bound, we get $\ell(m) \leq m+2$ when $m = 3$ or $7 \pmod 8$.

Similar calculations in the general case give us the following conclusions :

$$\ell(m) \leq m+1, m+8, m+7, m+6, m+5, m+4, m+3, m+2$$

$$\ell(m) \leq m+3, m+2, m+9, m+8, m+7, m+6, m+5, m+4$$

$$\ell(m) \leq m+5, m+4, m+3, m+2, m+9, m+8, m+7, m+6 \text{ and}$$

$$\ell(m) \leq m+7, m+6, m+5, m+4, m+3, m+2, m+9, m+8$$

according as $m = 0, 1, 2, \dots, 7 \pmod 8$. A comparison of these results give the theorem except when $m = 2$ and $4 \pmod 8$. When $m = 2 \pmod 8$, we put $2q+1 = m+1$. Then we find that the inequality $d-n \cdot F(s)$ holds. When $m = 4 \pmod 8$,

$$\ell(m) \leq \ell(m-2) + \ell(2) \text{ (by 4.3.1)}$$

$$\leq m-1+3 = m+2$$

This completes the proof of the theorem. ■

6.4 THE LEVEL OF $L_k^{2^{m-1}}$, WHEN $k = 2p$, p ODD

We see that Theorem 6.3.4 combined with Theorem 5.3.6 fixes the level of the Lens space $L_k^{2^{m-1}}$, $k = 2p$, p odd, except when $m = 4 \pmod 8$. For fixing this, we note that

$$(m+2)+3 = \ell(m+2) \leq \ell(m) + \ell(2) \text{ by 3.3.1}$$

$$= \ell(m) + 3$$

Thus, $m + 2 \leq \ell(m)$.

Hence we have the following :

6.4.1 THEOREM :

Let $l(m)$ denote the level of the Lens Space L_k^{2m-1} , $k = 2p$, p odd. Then, $l(m) = m+1$ if $m = 0, 2 \pmod 8$
 $= m+2$ if $m = 1, 3, 4, 5, 7 \pmod 8$
 $= m+3$ if $m = 6 \pmod 8$. □

6.4.2 REMARKS

When $k = 2^r p$, $r > 1$, p odd, we note that the lower and upper bounds for the level of L_k^{2m-1} move farther apart as r increases. We have at present established the bounds by using obstructions to the existence of cross sections. It does not seem that the obstructions to the existence of cross sections which can be obtained using existing techniques like Obstruction theory, Singularity methods, and Postnikov decompositions, can be effectively computed in order to bridge this gap.

APPENDIX COHOMOLOGY STRUCTURE
OF THE LENS SPACE

In this appendix, we present a detailed computation of the Cohomology structure of the Lens Space with both integral coefficients and \mathbb{Z}_p coefficients for a prime p . When we needed it in our work, we could not find it in a compact form in any available literature. Hence we present it here so that other students who may require it can find it easily.

Let $L_k^{2m-1} = S^{2m-1}/\mathbb{Z}_k$, k any positive integer denote the Lens Space. Consider the fibrations

$$S^1 \longrightarrow L_k^{2m-1} \longrightarrow L_k^{2m-1}/S^1 \cong \mathbb{C}P^{m-1}$$

and $S^1 \longrightarrow S^{2m-1} \longrightarrow \mathbb{C}P^{m-1}$

and the diagram

$$\begin{array}{ccccc} \mathbb{Z}_k & \longrightarrow & S^1 & \xrightarrow{\pi} & S^1/\mathbb{Z}_k \cong S^1 \\ & & \downarrow & & \downarrow \\ \mathbb{Z}_k & \longrightarrow & S^{2m-1} & \xrightarrow{\pi'} & L_k^{2m-1} \\ & & \downarrow & & \downarrow \\ & & \mathbb{C}P^{m-1} & & \mathbb{C}P^{m-1} \end{array}$$

(diagram 1)

Note that π is a k -fold covering.

Since $\mathbb{C}P^{m-1}$ is simply connected, we have the Lerray-Serre spectral sequences for the two fibrations above : The spectral sequence $\{E_r^{p,q}, d_r^L\}$ converging to $H^*(L_k^{2m-1})$ where $E_2^{p,q} \cong H^p(\mathbb{C}P^{m-1}; H^q(S^1))$ and the spectral sequence $\{E_r^{p,q}, d_r^S\}$ converging to $H^*(S^{2m-1})$ where $E_2^{p,q} \cong H^p(\mathbb{C}P^{m-1};$

Note further that $\pi^* : H^0(S^1) \longrightarrow H^0(S^1)$ is the identity.

Hence, $d_2^L(a) = kb$. Also, the product formula gives $d_2^L(ab) = kb^2$. Thus we get that the differentials d_2^L are one-to-one (in fact a multiplication by k). _____ (A)

INTEGRAL COHOMOLOGY

The E_2 terms of the second spectral sequence (integral coefficients) are given by :

$$\begin{array}{c}
 \mathbb{Z} \\
 | \\
 \begin{array}{cccccc}
 & 0 & \mathbb{Z} & 0 & & \mathbb{Z} & 0 \\
 & + & + & + & & + & + \\
 \hline
 0 & 1 & 2 & 3 & & 2m-2 & 2m-1 \\
 \mathbb{Z} & 0 & \mathbb{Z} & 0 & & \mathbb{Z} & 0
 \end{array}
 \end{array}$$

zeroes elsewhere.

We have examined d_2^L above. Because of its nature, we see that the E_∞ terms are as follows :

$$\begin{array}{c}
 0 \\
 | \\
 \begin{array}{cccccc}
 & 0 & 0 & 0 & & \mathbb{Z} & 0 \\
 & + & + & + & & + & + \\
 \hline
 0 & 1 & 2 & 3 & & 2m-2 & 2m-1 \\
 \mathbb{Z} & 0 & \mathbb{Z}_k & 0 & & \mathbb{Z}_k & 0
 \end{array}
 \end{array}$$

zeroes elsewhere.

We have the following filtration :

$$H^p(L_k^{2m-1}) = F^{0,p} \supseteq F^{1,p-1} \supseteq \dots \supseteq F^{p,0} \supseteq 0 \quad \text{_____ (B)}$$

where $E_\infty^{i,j} = \frac{F^{i,j}}{F^{i+1,j-1}} = 0$ when $j \neq 0$, and i not even

except when $i = 2m-2$.

$$E_\infty^{2m-2,1} = \mathbb{Z}.$$

Also, when i is even, $F^{i,j} = F^{i+1,j-1}$ for each i when $j \neq 0$ and $F^{i,0} \cong E_\infty^{i,0} = \mathbb{Z}_k$.

An analysis of the filtration (B) gives us the following integral cohomology structure of L_k^{2m-1} .

$$H^i(L_k^{2m-1}) = \begin{cases} \mathbb{Z}_k, & i \neq 0, i \text{ even}, i \leq 2m-1 \\ \mathbb{Z}, & i = 0, 2m-1 \\ 0, & \text{otherwise} \end{cases}$$

Ring Structure of $H^*(L_k^{2m-1}; \mathbb{Z})$

It is easy to see that $H^*(L_k^{2m-1}; \mathbb{Z}) = \frac{\mathbb{Z}_k[x]}{x^m} \otimes \Lambda[y]$

where $x \in H^2(L_k^{2m-1}; \mathbb{Z})$ is a two dimensional generator, and $y \in H^{2m-1}(L_k^{2m-1}; \mathbb{Z})$ is a $2m-1$ dimensional generator.

\mathbb{Z}_p -COHOMOLOGY, FOR A PRIME p

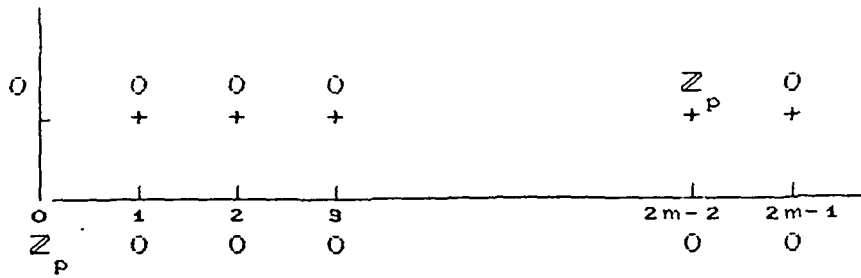
If $p|k$, then by (A), d_2^L is zero, so that the E_2 terms survive to infinity. Hence the E_∞ terms are as follows :

$$\begin{array}{c|cccccc} \mathbb{Z}_p & & 0 & \mathbb{Z}_p & 0 & & \mathbb{Z}_p & 0 \\ & & + & +^p & + & & +^p & + \\ \hline & 0 & 1 & 2 & 3 & & 2m-2 & 2m-1 \\ \mathbb{Z}_p & 0 & \mathbb{Z}_p & 0 & & & \mathbb{Z}_p & 0 \end{array}$$

zeroes elsewhere.

If $p \nmid k$, the filtration (B) gives that all the inclusions except the last are equalities and $E_\infty^{l,0} = F^{l,0} = \mathbb{Z}_p$. If $p|k$, (B) shows that all inclusions except the last but one are equalities. $E_\infty^{l,0} = F^{l,0} = 0$ and $E_\infty^{l-1,1} = F^{l-1,1} = \mathbb{Z}_p$. We thus obtain that when $p|k$, $H^*(L_k^{2m-1}) = \mathbb{Z}_p$, $0 \leq * \leq 2m-1$.

If $p \nmid k$, then the differentials d_2^L are isomorphisms. The E_∞ terms are as follows :



zeroes elsewhere.

An analysis of (B) immediately gives

$$H^l(L_k^{2m-1}) = \begin{cases} \mathbb{Z}_p, & l = 0, 2m-1 \\ 0, & \text{otherwise} \end{cases}$$

The \mathbb{Z}_p cohomology structure of L_k^{2m-1} is thus determined.

Ring Structure of $H^*(L_k^{2m-1}; \mathbb{Z}_p)$

Case 1 : $p|k$ and $p^2|k$

It is clear that the even dimensional cohomology groups $H^{2l}(L_k)$ are generated by x^l , where $x \in H^2(L_k^{2m-1}; \mathbb{Z}_p)$ is a two-dimensional generator. Let $y \in H^1(L_k^{2m-1}; \mathbb{Z}_p)$ be a one dimensional generator. Note that $H^1(L_k^{2m-1}) \cong E_\infty^{0,1}$, so that $y^2 \in E_\infty^{0,2} = 0$. Also $xy \neq 0$ as $ab \neq 0$.

$$\text{Hence } H^*(L_k^{2m-1}; \mathbb{Z}_p) = \frac{\mathbb{Z}_p[x, y]}{x^m, y^2}.$$

If $p = 2$, using the Gysin sequence of the S^0 - bundle $L_{k/2}^{2m-1} \longrightarrow L_k^{2m-1}$ we can easily show that in this case y is the first Stieffel-Whitney class of the line bundle

$$L_{k/2}^{2m-1} \times_{\mathbb{Z}_2} \mathbb{R} \longrightarrow L_k^{2m-1}.$$

Further, consider the exact sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

and the induced exact sequence

$$\longrightarrow H^0(L_k; \mathbb{Z}_p) \xrightarrow{\beta_p^*} H^1(L_k; \mathbb{Z}_p) \longrightarrow H^1(L_k; \mathbb{Z}_{p^2})$$

*

$$H^1(L_k; \mathbb{Z}_p) \xrightarrow{\beta_p^*} H^2(L_k; \mathbb{Z}_p) \longrightarrow \dots$$

Since $p^2 \nmid k$, $H^1(L_k; \mathbb{Z}_{p^2}) = \mathbb{Z}_{p^2}$. Hence the above sequence gives that $\beta_p^* = 0$.

Case 2 : $p \mid k$ but $p^2 \nmid k$.

In this case, the only difference we have from the above case is that using the above cohomology sequence, we get $H^1(L_k; \mathbb{Z}_{p^2}) = 0$. Hence β_p^* is one-to-one, so that $\beta_p^*(y) = x$. Further, if $p = 2$, then the Bockstein is Sq^1 and so, $\beta_p^*(y) = Sq^1(y) = y^2 = x$.

We can therefore gather the above results as :

$$H^*(L_k^{2m-1}) = \begin{cases} H^*(S^{2m-1}; \mathbb{Z}_p), & \text{if } p \nmid k \\ H^*(\mathbb{R}P^{2m-1}; \mathbb{Z}_2), & \text{if } p=2, 2 \mid k \text{ but } 2^2 \nmid k \\ \frac{\mathbb{Z}_p[x, y]}{x^m, y^2}, & \beta_p^*(y) = x, \text{ if } p \text{ is odd,} \\ & p \mid k, \text{ but } p^2 \nmid k, \text{ where } y \in \\ & H^1(L_k^{2m-1}; \mathbb{Z}_p), \quad x \in \\ & H^2(L_k^{2m-1}; \mathbb{Z}_p) \text{ and } \beta_p^* \text{ is the} \\ & \text{mod } p \text{ Bockstein.} \\ \frac{\mathbb{Z}_p[x, y]}{x^m, y^2}, & \text{if } p^2 \mid k, \text{ where } y \in \\ & H^1(L_k^{2m-1}; \mathbb{Z}_p), \quad x \in H^2(L_k^{2m-1}; \mathbb{Z}_p). \end{cases}$$

Moreover, When $p = 2$, y is the
first S-W class of the line

$$\text{bundle } L_{k/2}^{2m-1} \times \mathbb{Z}_2 \mathbb{R} \longrightarrow L_k^{2m-1}.$$

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