SOME PROBLEMS IN ALGEBRA - THE HOMOMORPHISM $\overset{*}{M} \otimes N \to$ HOM (M,N), SPLITTING OF RING EXTENSIONS, CRITERION FOR REGULARITY, HOMOLOGICAL DUALITY AND BNSI RINGS.

By

SIBNATH BOSE, M.Sc.

DEPARTMENT OF MATHEMATICS
SCHOOL OF PHYSICAL SCIENCES

SUBMITTED IN FULFILMENT OF THE REQUIREMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

To

NORTH-EASTERN HILL UNIVERSITY
SHILLONG-793001
INDIA

DECEMBER, 1987

Phone:
Grams : NEHU

North - Eastern Hill University

Bijni Complex
Bhagyakul, Shillong–793003 (Meghalaya)

Department of..Mathematics..

I certified that the thesis entitled "Some problems in Algebra - the homomorphism $M^* \otimes N \longrightarrow Hom\ (M,N)$, splitting of ring extensions, criterion for regularity, homological duality and BNSI rings " submitted by Mr. Sib Nath Bose for the Degree of Philosophy of North Eastern Hill University, Shillong embodies the original work carried out by him under my supervision. He has been duly registered and the thesis presented is worthy of being considered for the Award of the Ph.D. Degree. This work has not been submitted for any degree of any other University.

Date: 6.12.87

Place: Shillong

P. Jothilingam

(DR. P. JOTHILINGAM)

Supervisor

ACKNOWLEDGEMENTS

This work was carried out under the guidance of Dr. P. Jothilingam. I express my sincere thanks to him for his encouragement and help. I also thank him for giving regular seminars.

I am grateful to Prof. S.N. Dube, Head, Mathematics Department for his helping and encouraging attitude towards me. I am also grateful to Dr. P. Saikia, Dr. M.B. Rege and Dr. S.S. Khare not only for giving seminars but also for their constant and patient encouragement shown towards me. My association with the Department of Mathematics, N.E.H.U. will always remain a memorable event in my life.

I am indehted to my research colleagues Dr. N.J. Dev, Dr. A.B. Chakraborty, Mrs. S. Chowdhury, Miss I. Choudhury, Mrs. S. Dutta who always extended all possible help.

Mr. V.T. James has typed this dissertation in a very short period of time and I am thankful to him. Mr. Sangma and Mr. Thapa have helped me in cyclostyling this thesis and I am grateful to them.

Preface

The study of commutative algebra started with the works of famous authors like D. Hilbert, E. Noether, Macaulay and W. Krull. The latter progress in commutative algebra arises chiefly from quite different problems, issuing from Algebraic Geometry.

The concept of local rings was introduced by W. Krull in his paper " Dimensions Theorie in Stellen rigen". Krull conjectured a few problems in his paper. These were proved by C. Chevalley in " On the theory of Local rings " and by I.S. Cohen in his thesis " On the structure and ideal theory of complete local rings ". Authors like Zariski, Nagata proved further important results with the study of local properties of Algebraic varieties. These developments areembodied in the famous work of Nagata " Local rings ".

The second stage of development of the subject was ushered in by the lecture notes of J.P. Serre " Algebre - Localé Multiplicitiés", Classic papers of M. Auslander and D. Buchsbaum "Homological dimension in local rings" and "Maximal orders " by Auslander and Goldman. These authors introduced homological techniques into the subject. The proof of unique factorization in regular local rings by Auslander and Buchsbaum was an achievement signalling that these techniques have come to stay. Some more important results were proved by these authors by using homological techniques. Lichtenbaum's proof that $Tor_i^R(M,N) = (0)$ implies $Tor_j^R(M,N) = (0)$ for all $j \geq i$ for finitely generated modules over a regular local ring is another achievement of this period. The subsequent appearance of the thesis of C. Peskin and L. Szpiro " Dimension Projective finite et Cohomologie Locale" and the work of M. Hochster on big Cohen Macaulay modules have all pointed out the effectiveness of homological techniques in Commutative Algebra. The "Zero-divisor Conjecture" by I. Kaplansky, the "Tor - Conjecture" by M. Auslander, the Conjecture of H. Bass on Cohen - Macaulay rings are still open problems to mention a few.

This dissertation consists of six chapters.

Chapter I deals with preliminary definitions and results used in the dissertation.

In Chapter II, we prove three theorems. In Theorem 2.1, we prove a result, a special case of which says that if M and N are two reflexive modules of finite projective dimensions over a Gorenstein local ring such that Hom (M,N) is a third module of syzygies, then the natural homomorphism $M^* \otimes N \longrightarrow$ Hom (M,N) is an isomorphism. This extends the result in [13]. In Theorem 2.2 we give a criterion for a module M over a regular local ring to have projective dimension less than or equal to an integer n. This extends the usual criterion for the projectivity of a module. In Theorem 2.3, we prove that over a 1-dimensional Gorenstein local ring R if M,N are finitely generated R-modules such that Hom (M,N) is nonzdro free than both M^* and N are free. This is a generalization of a result of W. Vasconceles [29 , Theorem 3.1].

If $R \subset S$ is an extension of rings (not necessarily commutative) making S into a projective R-module then R is a direct summand of S if and only if S is faithfully projective. This is a result of Cartzen and others [11 , Theorem - 1] . The authors also give an example of a ring extension in which S is R-projective but not faithfully projective. The example involved noncommutative rings. In Chapter III, we show such a situation cannot happen if R is commutative i.e. if $R \subset S$ is an extension of rings making S projective as R-module then R is a direct summand of S and S is faithfully projective. This is Theorem 3.1 of this Chapter. In Bourbaki [10 , Ex 5.4, p-176] this is mentioned as an exercise when S is finitely generated as R-module.

In Chapter IV, we give a criterion for a noetherian local ring to be regular. This involves homological conditions on prime ideals of small height as compared to Hilbert-Serre Theorem which says that R is regular if and only if the maximal ideal of R has finite homological dimension. More precisely, suppose $pd\,\mathfrak{a} < \infty$ for every unmixed ideal $\mathfrak{a}$ of height atmost 2, then R is regular. W. Bruns [9] had earlier shown that if pd $\mathfrak{a} < \infty$ for every ideal $\mathfrak{a}$ all of whose associated prime ideals have depth atmost 2, then R is regular. But our proposition is a direct improvement of M. Auslander's criterion given in [4, Theorem B, Corollary 5]. The proof also includes some results of independent interest giving criterion for a domain to be U.F.D.

In chapter - V, we give an application of homological duality to generalized M-regular sequences. The notion of M-sequence is generalized in[14] as follows: a sequence $P_1, P_2, ----, P_n$ of nonfree modules is said to be an M-sequence if $\mathrm{Tor}_1^R (M \otimes P_1 \otimes \ldots \otimes P_{i-1}, P_i) = (0)$ for $1 \leq i \leq n$. The sum Σ Pd P_i is defined to be the length of the M-sequence $P_1, P_2, .. P_n$. We shall apply Strebel's homological duality ([28] , §3, Theorem 13) to prove Theorem 5.1 on generalized M-sequences which states that if R is a regular local ring and M a nonzero R-module and $\{P_1, P_2 .---. P_n\}$ is an M-sequence of nonfree perfect modules of length $\sum_{i=1}^{n} d_i$ $\mathfrak{a}_i$ = annP_i, d_i = Pd P_i; for $1 \leq i \leq n$. Then for every i, $1 \leq i \leq n$, $\exists$ d_i elements $x_1^{(i)}, x_2^{(i)}, -- x_{d_i}^{(i)}$ in $\mathfrak{a}_i$ such that the sequence of $\sum d_i$ elements $\{x_j^{(i)}\}$, $1 \leq i \leq n$, $1 \leq j \leq d_i$ form an M-sequence in the usual sense.

In Chapter VI, i.e. the last Chapter we take R to be a non-commutative local ring i.e. R is a ring with jacobson radical $\mathfrak{a}$ such that $R/\mathfrak{a}$ (= K) is a division ring. Mark Ramaras introduced the notion of BNSI rings in [24] . We have generalized this notion to non-commutative rings. A non-commutative local ring R is called a left BNSI ring if for every nonfree left R-module M, the sequence $\{ \beta_i(M)\}_{i \geq 1}$ is strictly increasing where $\beta_i(M) = \dim_K \mathrm{Tor}_i^R(K, M)$ is called the ith Bettinumber of M. If R is a left and right BNSI-ring with jacobson radical $\mathfrak{a}$ nilpotent and M is an indecompossable left R-module such that $\mathrm{Ext}^1_R(M, R) = (0)$ then M is free. This is Theorem 6.6 of this Chapter which generalizes a theorem in [24] .

CONTENTS

CHAPTER - I	PRELIMINARIES	1-12
CHAPTER - II	THE HOMOMORPHISM $M^* \otimes N \longrightarrow Hom(M,N)$	13-22
CHAPTER - III	SPLITTING OF RING EXTENSIONS	23-27
CHAPTER - IV	CRITERION FOR REGULARITY	28-32
CHAPTER - V	HOMOLOGICAL DUALITY	33-39
CHAPTER - VI	BNSI RINGS	40-46