Equivariant Ordinary (Co) homology Theory — A Survey

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TO

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CERTIFICATE

I certify that the dissertation entitled " Equivariant Ordinary (Co)homology Theory - a survey " submitted by Mrs. Syamali Datta ; in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate. I certify that sources from which ideas have been borrowed are duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities.

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PREFACE

The (co)homology functors are standard tools for dealing with topological problems and they have been quite effectively used to prove classical theorems like non-retractibility of disc into its boundary, the Brouwer fixed point theorem, nonexistence of vector fields on even dimensional spheres and so on. The following two results are notable in the sense that they demonstrate how effectively (co)homology theory reflects the geometric situations.

(I) Hopf- Whitney Extension theorem :

If Y is an $(n-1)$-connected space and $\dim (X,A) \leqslant n+1$ as CW-Complex pair, then the extension problem

$$\begin{array}{ccc} X & & \\ \uparrow & \searrow & \\ A & \longrightarrow & Y \end{array}$$

has a solution if and only if the algebraic problem.

$$\begin{array}{ccc} H^n(X;\pi_n(Y)) & \leftarrow & \\ \downarrow & & \\ H^n(A;\pi_n(Y)) & \longleftarrow & H^n(Y,\pi_n(Y)) \end{array}$$

has a solution.

(II) Hopf Classification Theorem.

If M^n is a closed, connected orientable manifold and S^n the unit n-sphere, then the homotopy classes of maps $[M,S^n]$, is bijective to $H^n(M^n ; \mathbb{Z}) \cong \mathbb{Z}$, and that degree characterises the homotopy class of maps from M to S^n. There are extensions of these results for finite CW-Complexes by Hurewicz using homology. The culmination of all these is the following :

(A) Classification theorem : If X is a CW-complex and $K(\pi, n)$ is the Eilenberg Macclane Space of type (π, n) then there is a one to one correspondence $[X, K(\pi,n)]$ —— $H^n(X; \pi)$.

After all these stories of success of (co)homology functors in Algebraic topology, as long as one ignores the problems involving internal symmetries of topological (or geometrical) objects, mathematicians naturally started asking in 1930's how much information can these functors give about the internal symmetries referred to above.

Although the study of internal symmetries of geometrical objects had already been initiated by Brouwer in 1919 with his famous existence theorems for fixed points of transformations of objects, the study of such questions using (co)homological tools was pioneered by P.A. Smith in 1935. Apart from mere existence theoremes of fixed points he also gave the (co)homological structure of fixed points of periodic transformations on Homology spheres.

It is the above work of P.A. Smith where so called equivariant (co)homology functors were first time introduced in algebraic topology in the study of the internal symmetries of geometric objects. Before we proceed further with the story of equivariant (co)homology let us introduce some formal terminology.

Let $\mathcal{C}$ be the category of topological spaces (or CW-complexes) and continuous maps. Let $\mathcal{C}^{Eq}$ be the category of triples (G, ρ, X) where $X \in ob\mathcal{C}$, G a topological group and $\rho : G \longrightarrow Aut_{\mathcal{C}}(X)$ a homomorphism of G into the group of $\mathcal{C}$-automorphisms of X. A morphism from (G, ρ, X) to (G', ρ', X') is a pair (h,f) where $h: G \longrightarrow G'$ is a group homomorphism, $f: X \longrightarrow X'$ is a morphism in $\mathcal{C}$, such that $f \circ \rho(g) = \rho'(h(g)) \cdot f$.

If we fix a topological group G, then we get the subcategory $\mathcal{C}_G$ of $\mathcal{C}^{Eq}$.

Given an object (G, ρ, X) of $\mathcal{C}_G$, one gets a continuous map : $G \times X \longrightarrow X$ defined by $(g,x) \longrightarrow \rho(g)(x)$, such that $(1,x) \longrightarrow x$ and $(g,(g', x)) = (gg', x)$. This is called an action of G on X. We denote $\rho(g)(x)$ simply by gx.

In this case, X is called a G-space.

Let $X^G = \{ x \in X \mid gx = x \ \forall g \in G \}$, this is called the fixed point set of the action of G.

Let $x \in X$, then $E_x = \{ gx \mid g \in G \}$ is called the orbit of x

The set $G_x = \{ g \in G \mid gx = x \}$ is called an isotropy subgroup of G. It is wellknown that $G/G_x \overset{\text{homeo}}{\approx} E_x \; \forall \; x \in X$.

The set $X/G = \{ E_x \mid x \in X \}$, is called the orbit space of X.

By orbit structure of the triple (G, ρ, X), we mean the structure of X^G, X/G etc.

Since, $Aut_{\mathcal{L}}(X)$ describes the internal symmetries of the object X and since the action of G's on X are intimately related to $Aut_{\mathcal{L}}(X)$, hence the terms action of G on X and internal symmetry of X are used interchangeably.

In the study of the internal symmetry of an object $X \in ob\mathcal{L}$, the basic questions which one tries to answer are :

Q. (i) What is the orbit structure of X ?

Q. (ii) What is the structure of $Hom_{Eq}((G, \rho, X), (G', \rho', X'))$?

With reference to these questions let us now come back to our survey of the evolution of Equivariant (co)homology theory. P.A. Smith considered (Z_p, ρ, X), where X is a homology sphere (that is periodic transformation on homology sphere) and studied the orbit structure of X. Using (co)homology groups $H^*_{\rho}(X)$ based on ρ- (co)chains (where ρ is some specific member of $\mathbb{Z}[Z_p]$) he determined the structure of X^G which again turned out to be a homology sphere. $H^*_{\rho}(X)$ may be called <u>Smith's Equivariant cohomology groups.</u>

Then it was Eilenberg who gave a similar definition of equivariant (co)homology groups $H^*_G(X)$ of a G-space X and proved that for a sufficiently nice based space X and a system $\mathcal{G}$ of local coefficients on X one has an isomorphism of $H_*(X ; \mathcal{G})$ (homology with local coefficients) with $H_*^{\pi_1(X)}(\tilde{X}, \mathcal{G}_0)$ where $\mathcal{G}_0$ is the fibre of over the base point, (see details about local coefficients in Steenrod [31] , Whitehead [25]), $\tilde{X}$ is the universal cover of X; and $\pi_1(X)$ is the fundamental group of X. This was a turning point in the study of manifolds, since using this notion of equivariant (co)homology, one can talk of fundamental class and Poincaré duality of nonorientable manifolds. e.g. $\mathbb{RP}^2$ (real projective plane) is nonorientable and that

$$H_0(\mathbb{RP}^2 ; \mathbb{Z}) = \mathbb{Z} , H_1(\mathbb{RP}^2; \mathbb{Z}) = \mathbb{Z}_2, H_2(\mathbb{RP}^2; \mathbb{Z}) = 0$$

but if we consider $\mathbb{Z}$ with nontrivial $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ action giving rise to a local coefficient system $\mathcal{L}$ (twisted integers), then

$$H_0(\mathbb{RP}^2; \mathcal{L}) \cong H_0^{\mathbb{Z}_2}(S^2; \mathbb{Z}) = \mathbb{Z}_2$$
$$H_1(\mathbb{RP}^2; \mathcal{L}) \cong H_1^{\mathbb{Z}_2}(S^2; \mathbb{Z}) = 0$$
$$H_2(\mathbb{RP}^2; \mathcal{L}) \cong H_2^{\mathbb{Z}_2}(S^2; \mathbb{Z}) = \mathbb{Z}$$

Note that S^2 is the universal covering space of $\mathbb{RP}^2$. These equivariant (co)homology theories were effectively used in defining Steenrod operations which in turn enriched the structure of (co)homology rings (see [16]). Smith's results gave rise to investigations which replaced homology spheres by more general spaces and $\mathbb{Z}_p$ by more general groups. For not so general groups, (see [8]) affirmative generalisations were obtained and the (co)homology theory used for such studies is a variant of the earlier equivariant (co)homology theories. This is defined by Borel. One associates with a G-space X , a fibre bundle $X \longrightarrow X_G \longrightarrow BG$, with fibre X, where BG is the classifying space of G, $X_G = (EG \times X)/G$, EG is the universal bundle over BG. Then define $H_*^G(X) = H_*(X_G)$. (co)homology structure of X^G, X/G can be determined in terms of $H_G^*(X)$. (see ch. I for all these results and [4] for details.)

The equivariant (co)homology theories considered so far gave sufficiently satisfactory answers to questions Q(i) above (although not completely satisfactorily see [8]). However the answer to the question Q(ii) involve the general equivariant obstruction theory for which the equivariant (co)homology theories developed so far were not adequate (for particular types of actions on particular types of spaces however one can develop an obstruction theory see chapter II and [5]).

In order to overcome this difficulty, Bredon introduced an equivariant (co)homology theory which extends Borel's definition and which is defined in terms of chain complexes which are functors on the orbit category of a given group G. This functor is defined in terms of ordinary chains of fixed points sets of the complex by the subgroups of G. (see chapter II and [5] for details). Using this (co)homology theory Bredon could develop an equivariant obstruction theory as mentioned above and succeed in proving an equivariant version of the classification theorem (A) mentioned above. This (co)homology theory satisfies seven Eilenberg Steanrod axioms, where, ofcourse, the dimension axiom has to be of a specific form involving G/H instead of points as in the nonequivariant case. Moreover he established that this is the only(co)homology theory satisfying the said axioms.

Bredon's cohomology groups are however defined only for finite groups G and G-complexes. S. Illman put this theory in the singular G-equivariant setting, where G can be any topological group, retaining all the good properties referred to above. Illman's theory restricts to Bredon theory for finite groups. (see details in chapter III and [10]) . All these equivariant (co)homology theories developed so far were greded over integers as in the nonequivariant case and served the purpose for which they were constructed. However as soon as one asks whether an equivariant version of Poincaré duality can be developed in terms of the available ordinary (co)homology theories, one encounters a difficulty which stems from the fact that the available equivariant integrally graded (co)homology theories does not satisfy stability (suspension isomorphism) with respect to arbitrary representations $V \in RO(G)$. (i.e. absence of an isomorphism of the kind : $H_G^*(X) \xrightarrow{\cong} H_G^{*+V}(S^V \wedge X)$, here S^V is the one point compactification of V). In order to remove this difficulty one should first of all define a RO(G) - graded (co)homology theory, and then it should satisfy the good properties needed to have equivariant Poincaré duality.

This kind of RO(G) - graded (co)homology theories were first introduced by G.Segal who defined RO(G)-graded K-theory. Similarly, RO(G)-graded cohomotopy theory were developed by Kosniowski. Finally, the RO(G)-graded ordinary cohomology theory has been developed by

May, Maclure, Steinbergh, Waner etc. extending Bredon's theory. The details are yet to be published.(see chapter IV and [12]).

In this dissertation we have elaborated upon the above survey and collected at one place all the equivariant (co)homology theories developed till date, which are otherwise scattered in the literature. In addition to this we have tried to pin point the relative merits and demerits of the various theories mentioned above, which will be helpful to the future workers on the subject.

We now give the chapterwise arrangement of the dissertation.

Chapter I is set out in the following fashion. We begin by giving preliminary definitions of G-complexes and the definition of an equivariant category in § 1.0. In § 1.1 we give Smith's definition of equivariant (co)homology groups and their applications in studying the structure of fixed point set. In § 1.2 we give the Eilenberg's definition of equivariant (co)homology and their relationship with (co)homology with local coefficients. § 1.3 contains Borel's definition of equivariant (co)homology groups and some of its important applications - in particular, its application in studying the orbit structure of spaces, generalising the Smith's results, as far as possible.

Chapter II is devoted to the Bredon's definition of equivariant (co)homology. In § 2.1 we introduce equivariant (co)homology theory and in § 2.2 we define suitable coefficient systems. Then in § 2.3 we give the Bredon's construction of equivariant cohomology groups. § 2.4 is concerned with the equivariant obstruction theory as developed by Bredon. In § 2.5 we describe, in brief, the equivariant obstruction cocycles for a free G-complex, using Eilenberg's equivariant cohomology as given by Tom Dieck.

Chapter III contains Illman's definition of equivariant cohomology. § 3.1 contains some computations of equivariant (co)homology groups, using only the existence of equivariant singular (co)homology theory. § 3.2 is devoted to the construction of singular (co)homology theory. § 3.3 is concerned with the construction of various notions like transfer homomorphism Kronecker index and cup product as defined by Illman in the equivariant setting. In § 3.4 we give a brief description of generalised cohomology theory.

Chapter IV § 4.1 begins with the definition of stable equivariant homotopy groups and the (co)homology theory given by this. Then in § 4.2 we enumerate, in brief, the necessity of having a RO(G) graded (co)homology theory. § 4.3 is devoted to some results, demonstrating the usefulness of having an RO(G) graded (co)homology and also a necessary condition of the existence of such an RO(G) graded (co)homology theory. Finally in § 4.4 we indicate a line of construction of the ordinary RO(G) graded (co)homology theory.

CONTENTS

CHAPTER I

Survey of Equivariant ordinary (Co)homology theory of P.A. Smith, S. Eilenberg and A. Borel.

Let $\mathcal{C}$ be the category whose objects are topological spaces (or cw-complexes) and whose morphisms are continuous maps. Let $\mathcal{C}^{Eq}$ be the category of triples (G, ρ, X) where $X \in ob\,\mathcal{C}$, G is a topological group and $\rho: G \to Aut_{\mathcal{C}}(X)$, a homomorphism of G into the group of $\mathcal{C}$-automorphisms of X. A morphism from (G, ρ, X) (G', ρ', X') is a pair (h, f) where $h: G \to G'$ is a group homomorphism, $f: X \to X'$ is a morphism in $\mathcal{C}$ and

$$f \cdot \rho(g) = \rho'(h(g)) \cdot f$$

If we fix a topological group G, then we get a subcategory $\mathcal{C}_G$ of $\mathcal{C}^{Eq}$.

We are interested in studying the orbit structure of objects of $\mathcal{C}_G$. The purpose of this chapter is to survey the methods of P.A. Smith and Borel to study these structures using suitable notions of (co)homology of these objects, which are called in the literature as equivariant (co)homology theories. Our emphasis will be more on the various ways in which these equivariant (co)homology theories were constructed by Smith, Eilenberg, Borel and others and to study how effectively they were used than the applications themselves.

The chapter is set out in the following fashion. In § 1.0 we will give the definitions of G-complexes and G-cw complexes. In § 1.1 we will briefly describe the methods of algebraic topology as used by P.A. Smith in getting an exact sequence which involved the introduction of certain equivariant (co)homology groups, and the interesting consequences of this exact sequence. In §1.2 we introduce the Eilenberg's definition of equivariant cohomology groups and indicate his result about the relationship of this equivariant (co)homology group with the ordinary (co) homology group with local coefficients. §1.3 will be devoted to the introduction of Borel's definition of an equivariant cohomology theory. §1.4 will be devoted to the applications of Borel's equivariant cohomology theory, and its relative merit over the earlier definition of equivariant (co) homology theories.

§1.0 G-complexes and G-CW complexes

Most of the time we will work in the category of G-complexes and G-CW complexes. We give below the definitions.

(1.0.1) Definition of a cell complex ([22])

A cell complex K on a Hausdorff space X is a Collection $K = \{ e^n_\alpha \mid n = 0, 1, 2, \ldots\ldots, \alpha \in J_n \}$ of subsets of X indexed by non negative integers n, and for each n, α running through some index set J_n. The set e^n_α is called a cell of dimension n. The cells must satisfy the conditions which are listed below:

The set $K^n = \{ e^r_\alpha \mid r \leq n, \alpha \in J_r \}$, $n \geq 0$ is called the n-skeleton of K. We take $K^{-1} = \emptyset$

Let $|K^n| = \bigcup_{\substack{r \leq n \\ \alpha \in J_r}} e^r_\alpha$, and for each cell e^n_α, let $\dot{e}^n_\alpha = e^n_\alpha \cap |K^{n-1}|$ = the boundary of e^n_α

$\overset{\circ}{e}{}^n_\alpha = e^n_\alpha - \dot{e}^n_\alpha$ = the interior of e^n_α

We require K to satisfy

(a) $X = \bigcup_{n,\alpha} e^n_\alpha = |K|$

(b) $\overset{\circ}{e}{}^n_\alpha \cap \overset{\circ}{e}{}^m_\beta \neq \phi \implies$ n = m and $\alpha = \beta$

(c) for each cell e^n_α, there is a map

$$f^n_\alpha : (D^n, S^{n-1}) \longrightarrow (e^n_\alpha, \dot{e}^n_\alpha),$$

which is surjective and maps $\overset{\circ}{D}{}^n$ homeomorphically onto $\overset{\circ}{e}{}^n_\alpha$. The maps f^n_α are called the characteristic maps of the cells e^n_α.

The cell e^m_β is called an immediate face of e^n_α if

$$\overset{\circ}{e}{}^m_\beta \cap e^n_\alpha \neq \phi .$$

(1.0.2) Definition of a CW-Complex ([22])

A CW-complex K on a space X is a cell complex K on X satisfying

(d) K is closure finite i.e. each cell has only a finite number of immediate faces.

(e) X has the weak topology induced by K ie. a subset $S \subset X$ is closed iff $S \cap e^n_\alpha$ is closed in e^n_α for each n, α .

(1.0.3) G- Complexes ([5])

Let G be a finite group. By a G-Complex we mean a CW-Complex K together with a given action of G on K by cellular maps such that for $g \in G$, $\{x \in K \mid g(x)=x\}$ is a sub-complex of K. This definition of a G-complex is due to Bredon ([5], PI-1)

(1.0.4) Definition of a G-cell complex ([24])

Let G be a topological group and X be a G-space. A cell complex K on X is said to be G-cell complex if (a) the orbit space X/G is a Hausdorff space.

(b) G acts cellularly ie. $e \in K \Rightarrow ge \in K \ \forall g \in G$

(c) every point x of an open cell e has the same isotropy subgroup denoted by He. In particular, each boundary point is fixed by He.

(d) If g is not contained in He, then ge is *disjoint from e*.

(e) *the topology of the subspace* $G\bar{e}$ is the identification topology determined by the induced G-characteristic map,

$$Gf^n (= \mu \circ (1d_G \times f^n)) : G \times \Delta^n \to G\bar{e} \subset X$$

where $\mu: G \times X \longrightarrow X$ is a fixed topological left G-action and $f^n: \Delta^n \rightarrow \bar{e}^n$ is a characteristic map of e^n.

(1.0.5) G-CW Complex ([24])

A G-Complex (X,K) is said to be G-finite complex, if it has only finite number of G-cells

The G-cell complex closure of a subset S of X, denoted by $G\mathcal{X}(S)$, is defined to be the G-orbit of X(S) which turns out to be the smallest G-invariant subcomplex which contains S. Here X(S) denotes the cell complex closure of S, which is defined to be the smallest sub complex which contains S.

(1.0.6) Definition

A G-cell complex (X,K) is called a G-CW complex, if it satisfies the following two conditions:

(G-c): G closure finiteness.

This means the G-cell complex closure of each cell $G\mathcal{X}(e)$ is a G-finite complex which is equivalent to the condition that the induced cell complex structure on the orbit space X/G is closure finite. In another way, the cell complex closure X(e) of each cell has intersection with only finite number of G-cells

(G-W): G- Weak topology.

That is, X has the identification topology with respect to the onto G-characteristic map of X,

$Gf_X : E_X = \coprod G \times \Delta_\lambda (\lambda \in \Lambda) \longrightarrow X$, where E_X is topologised as a topological disjoint union.

By condition (e) of (1.0.4) this topology coincides with the weak topology with respect to the closed covering $\{ \overline{Ge}, Ge \in GK \}$ of X and when X has this topology the orbit space X/G has the weak topology with respect to the characteristic maps $f/G \, (= \pi \circ f) : \Delta^n \rightarrow \overline{e}/G \subset X/G$.

(1.0.7) <u>REMARKS</u>

1. A G-cell complex is the G-equivariant version of a cell complex.
2. when G is a compact group and a G-space X is Haus-dorff then so is the orbit space X/G. If further X is Ist countable, then X is a G-CW complex if the induced cell complex structure on the orbit space X/G is a CW-complex structure. (For details see [24])
3. When G is a compact lie group then the condition (d) of (1.0.5) follows from the conditions (b) and (c) for a cell complex which is also a G-space (for details [29])

4. When G is a finite group, a G-CW complex itself must be a CW-complex.

Thus the definition of a G-CW Complex is reduced to :

a G-space which has a cw-structure satisfying the following conditions:

(i) G acts cellularly

(ii) for each $g \in G$, the g-stationary subspace $\{x \in X \mid gx = x\}$ forms a subcomplex.

This is the definition of a G-complex of G. Bredon (C.f (1.04)).

Hence a G-cw complex is a generalisation of the concept of a G-complex of Bredon.

§ 1.1 Smith theory, Smith sequence and its important Consequences

Let $G = Z_p$ (p a prime) and K be a regular (*) G-Complex. Let $L \subset K$ be an invariant sub-complex.

Consider the following elements of the group ring $Z_p G$:

$$\begin{cases} \sigma = 1 + g + g^2 + \cdots\cdots + g^{p-1} \\ \tau = 1 - g \end{cases}$$

(*) By a regular G-Complex here we mean a simplicial G-Complex such that the subcomplex formed by all invariant simplexes is closed. (see [6], [15]).

Then one can easily see that $\sigma\tau = 0 = \tau\sigma$

and $\tau^{p-1} = \sigma$

Then (see [19]) with the given G-pair (K,L), it is possible to associate, the Smith Cohomology groups [*] $H_\rho^*(K, L; Z_p)$, $\rho = \tau^i, 1 \le i \le p-1$, in such a way that there are natural Smith exact sequences (with coefficients in Z_p)

$$(1.1.0)\ldots\ldots \longrightarrow H^n_\rho(K,L) \xrightarrow{\rho^*} H^n(K,L) \xrightarrow{i^*} H^n_{\bar\rho}(K,L) \oplus H^n(K^G, L^G) \xrightarrow{\delta} H^{n+1}_\rho(K,L) \longrightarrow \ldots\ldots,$$ where

$$\bar\rho = \tau^{p-i}, \quad L^G = K^G \cap L, \quad K^G$$

denote the fixed point sets of L , K respectively.

These smith groups , $H^n_\rho(K,L)$, measure the nontriviality of the group action in the sense that they disappear as soon as the action becomes trivial.

Using these exact sequences one obtains the following results:

---

[*] $H_\rho^*(K,L;Z_p) \overset{\text{defn}}{=} H_\rho^*(Hom(\rho C(K,L;Z_p), Z_p))$, where $\rho C(K,L;Z_p)$ is the chain subcomplex of $C(K,L;Z_p)$ obtained by taking ρ translates of the chains of $C(K,L;Z_p)$.

(see [6], [15]).

(1.1.1) Rank formula

If $G = \mathbb{Z}_p$ (p, a prime) and K is a finite dimensional regular G-complex and $L \subset K$ is an invariant sub-complex then for any integer $n \geqslant 0$ and any $\rho = \tau^i$, $1 \leq i \leq p-1$.

$$\operatorname{rk} H^n_\rho(K, L) + \sum_{i \geqslant n} \operatorname{rk} H^i(K^G, L^G) \leq \sum_{i \geqslant n} \operatorname{rk} H^i(K, L)$$

where $\operatorname{rk} H^i(K)$ denotes the dimension of $H^i(K)$ as a vectorspace over $\mathbb{Z}_p$. Using these rank formula one can prove the next result.

(1.1.2) Euler characteristic formula

If $\operatorname{rk} H(K, L; \mathbb{Z}_p) < \infty$ and K is a finite dimensional regular G-complex, $L \subset K$ invariant sub complex, then

$$\chi(K, L) + (p-1)\chi(K^G, L^G) = p\chi(K/G, L/G)$$

In particular,

$$\chi(K^G, L^G) \equiv \chi(K, L) (\text{mod } p).$$

where $\chi(K, L) = \sum (-1)^i \operatorname{rk} H^i(K, L; \mathbb{Z}_p)$

using the above rank formula and Euler characteristic formula one obtains the following classical results due to P.A. Smith:

(1.1.3) Cohomology Sphere theorem

If G is a p-group (p a prime) and if K is a finite dimensional regular G-complex such that $K \underset{p}{\sim} S^n$ [that is

$$H^*(K; \mathbb{Z}_p) \approx H^*(S^n ; \mathbb{Z}_p)],\ \text{then}$$

$$K^G \underset{p}{\sim} S^r, \text{ for some } r, -1 \leq r \leq n$$

(where $r = -1$ means $K^G = \phi$).

(1.1.4) Classical Disc theorem

If G is a p-group (p, a prime) and K is a finite dimensional regular G-complex and $L \subset K$ an invariant sub-complex such that $(K,L) \underset{p}{\sim} (D^n, S^{n-1})$

i.e. $H^i(K,L; \mathbb{Z}_p) = 0$, for $i \neq n$

and $H^n(K,L; \mathbb{Z}_p) \approx \mathbb{Z}_p$

then $(K^G, L^G) \underset{p}{\sim} (D^r, S^{r-1})$, for some r, $0 \leq r \leq n$.

If p is odd, then $n-r$ is even.

§1.2 Eilenberg's definition of e uivariant homology and cohomology ([26] , [7])

Let X be a space on which a group Π acts.

If $\xi \in \Pi$ and $u : \Delta^q \to X$ is a singular q-simplex in X, we may define $\xi u : \Delta^q \to X$ by $\xi u(t) = \xi . u(t)$, $\forall\ t \in \Delta^q$.

The action of Π on singular simplexes is extended linearly to the action of Π on the singular chain complex $C_*(X)$ so that the latter group becomes a Π-module. Moreover, one shows that the boundary operator ∂ commutes with the action, so that $C_*(X)$ is a complex of Π-modules.

Let G be a right Π-module and Q(G,X) denote the subgroup $G \otimes C_*(X)$ generated by all elements of the form $g\xi \otimes c - g \otimes \xi c$, where $g \in G$, $\xi \in \Pi$ and $c \in C_*(X)$.

Form the complex $G \otimes_{\Pi} C_*(X) = \dfrac{G \otimes C_*(X)}{Q(G,X)}$

The boundary operator of $G \otimes C_*(X)$ maps $Q(G,X)$ into itself, so induces an endomorphism ∂ of $G \otimes_{\Pi} C_*(X)$. Thus $G \otimes C_*(X)$ is a chain complex.

(1.2.4) Definition: The homology groups,

$H_q(G \otimes_{\Pi} C_*(X))$ of the chain complex $G \otimes_{\Pi} C_*(X)$ are called the equivariant homology groups of X with coefficients in the Π-module G, $H_q^{\Pi}(X; G)$.

Similarly, one defines the equivariant cohomology groups.

Let G be a left Π-module. The group of operator homomorphisms $\mathrm{Hom}^{\Pi}(C_*(X), G)$ is a sub-complex of the cochain complex $\mathrm{Hom}(C_*(X), G)$.

(1.2. 2) <u>Definition</u> : The homology groups of $\mathrm{Hom}^{\Pi}(C_*(X),G)$, $H_q(\mathrm{Hom}^{\Pi}(C_*(X), G))$, are called the <u>equivariant Cohomology groups</u>, $H^q_{\Pi}(X;G)$.

If (X,x_0) is a path connected space, and if G is a local system of groups over X (which is a covariant functor from the fundamental groupoid to the category of groups) then $\pi_1(X,x_0)$ operates on $G(x_0)$. Conversely, whenever $\pi_1(X,x_0)$ operates on some group G_0 then there exists a local system of groups G such that $G(x_0) = G_0$ and the operation of $\pi_1(X,x_0)$ on $G(x_0)$ is the given action.

So, whenever X is a path connected space with an action of $\pi_1(X, x_0)$, and $\pi_1(X, x_0)$ also operates on a group G_0, one can construct Eilenberg's equivariant (co) homology groups $H^{\Pi}_*(X; Go)$ (or $H^*_{\Pi}(X; Go)$). One can also define $H_*(X; G)$ where G is the local coefficient groups determined by the operation of $\pi_1(X,x_0)$. These groups are related by the following theorem :

(1.2.3) Theorem (Eilenberg)

The homology groups $H_q(X;G)$ with respect to a system G of local coefficients on X, are isomorphic to the equivariant homology groups $H_q^{\pi_1(X)}(\widetilde{X}; G_0)$ of the universal covering space $\widetilde{X}$ of X with the fundamental group $\pi_1(X)$ operating on $\widetilde{X}$ as covering transformations. For cohomology one has an analogous theorem (due to Eilenberg).

(1.2.4) Theorem : With the same notations as above the cohomology groups $H^q(X;G)$ are isomorphic to the equivariant cohomology groups $H^q_{\pi_1(X)}(\widetilde{X} ; G_0)$.

Eilenberg's theorem has an important application when $X = K(\pi, 1)$ is an Eilenberg - MacLane Complex.

(1.2.5) Theorem If π is a group and G a system of local coefficients on $X = K(\pi, 1)$, then $H_q(K(\pi, 1) ; G) \approx Tor_q^{\pi}(G_0; Z)$

This is a generalisation to the local coefficients of the following theorem.

(1.2.6) Theorem : If X is an aspherical space with fundamental group π, then for an abelian group G,

$$H_q(X;G) \approx Tor_q^{Z(\pi)}(G;Z).$$

For cohomology one has analogous results.

(1.2.7) Theorem If π is a group and G is a system of local coefficients on $K(\pi, 1)$, then

$$H^q(K(\pi, 1) ; G) \approx Ext^q_{\pi}(Z, G_0).$$

§1.3 Bundle cohomology (Borel).

Let G be a compact lie group and X be a G-space. Let $p: E_G \longrightarrow B_G$ be the universal G-bundle. Then the total space X_G of the associated universal bundle with X as fibre may be regarded as the orbit space of $E_G \times X$ ie.

$$X_G = E_G \times X = (E_G \times X) / G.$$

where the G-action is given by

$$g. (e, x) = (eg^{-1} , gx).$$

(1.3.1) Definition The equivariant cohomology of the G-space X, is defined to be the ordinary cohomology of the total space X_G ie. $H_G^*(X) = H^*(X_G)$.

This cohomology was defined by A. Borel ([8], [6], [4]), in order to generalise Smith theory and get deeper results then those given by Smith's original method (mentioned in §1.1)

(1.3.2) Remark: It is easy to see that the Borel construction is functorial.

Given any G-map $f:X \to Y$, we get a continuous map $f_G : X_G \to Y_G$ which induces a homomorphism

$$f_G^* = f^* : H_G^*(Y) \longrightarrow H_G^*(X)$$ in bundle cohomology.

In fact, $H_G^*(-)$ is a contravariant functor on the category of G-pairs and their maps, which turns out to be a generalised cohomology theory i.e. satisfies all axioms of Eilenberg and Steenrod , except the dimension axiom.

(1.3.3) The coefficient groups are $H_G^*(pt) = H^*(B_G)$ = cohomology of the classifying space of G.

(1.3.4) <u>Remark</u> :

The equivariant cohomology of a G-space X, with coefficients is defined in a similar way ([20]).

(1.3.5) <u>Definition</u>

Let X be paracompact and let L be any $\mathbb{Z}$-module. The equivariant cohomology of the G-space X, with coefficients in L, denoted by $H_G^*(X; L)$, is defined to be the sheaf cohomology of the total space X_G with coefficients in the constant sheaf L on X_G

ie. $H_G^*(X;L) = H^*(X_G;L)$.

§1.4. <u>Applications of Borel's equivariant cohomology theory</u>

(1.4.1) Using Borel's definition, Steenrod defined power operation in cohomology ([16]) as follows:

Let K be a finite regular cell complex and K^n be the n-fold cartesian product. Let $S(n)$ be the symmetric group of n- elements acting as permutations of the factors of K^n. Let π be a subgroup of $S(n)$ and W be a π-free acyclic complex via the diagonal action.

Let $W \times_{\pi} K^n = (W \times K^n)/\pi$ and let j_{σ} be the composition:

$$K^n \hookrightarrow W \times K^n \longrightarrow W \times_{\pi} K^n$$

The map: $W \times_{\pi} K^n \longrightarrow W$ is a fibration with fibre K^n.

Let u be a q-cocycle on K with values in an abelian group G. G may be regarded as a complex with components

$$G_r = \begin{cases} 0, & r \neq 0 \\ G, & r = 0 \end{cases}$$

and $u: K \to G$ is a chain map of degree $-q$.

Let $G^n(q)$ be the $S(n)$- Complex defined as

$$G^n(q) = \begin{cases} 0, & q \neq 0 \\ G^n, & q = 0 \end{cases}$$

where we write $\underbrace{G \otimes G \otimes \cdots\cdots \otimes G}_{n \text{ factors}} = G^n$

Let $\alpha \in S(n)$ act on G^n as follows:

If q is even, α permutes the factors of G^n with no sign change. If q is odd, α act on G^n by the product of the sign of α and the permutation of the factors of G^n with no sign change.

Then $u^n : K^n \longrightarrow G^n(q)$ is an equivariant chain map of degree $-n\,q$.

Let $\varepsilon : W \rightarrow \mathbb{Z}$ be the augmentation on W. Then $\varepsilon \otimes 1 : W \otimes K^n \rightarrow K^n$ is an equivariant chain map (using the diagonal action on the domain)

So, the composition $W \otimes K^n \xrightarrow{\varepsilon \otimes 1} K^n \xrightarrow{u^n} G^n(q)$ is an equivariant chain map of degree $-nq$. Thus we have an equivariant nq- cocycle on $W \otimes K^n$, which we denote by $Pu \in C^{nq}_{\pi}(W \otimes K^n ; G^n(q))$.

It can be proved that

(i) If we vary u by a cohomology, then Pu varies by an equivariant cohomology, and

(ii) If u and v are cohomologous q-cocycles on K with values in G, then Pu and Pv are equivariantly cohomologous nq - cocycles on $W \otimes_{\pi} K^n$.

Thus P induces a map

$$P : H^{q}(K ; G) \longrightarrow H_{\pi}^{nq}(W \otimes K^{n} ; G^{n}(q)).$$

so that P is natural with respect to maps of the variable K, $P0 = 0$ and $j^{*}Pu = u \times u \times \cdots \times u$.

The reduced nth powers is defined by

$$(1 \times_{\pi} d)^{*} Pu \in H^{*}(W/\pi \times K)$$

where $d : K \rightarrow K^{n}$ is the diagonal, and $W \times_{\pi} K = (W \times K)/\pi$

$$1 \times_{\pi} d : W \times_{\pi} K \longrightarrow W \times_{\pi} K^{n}.$$

If we are working with a field of coefficients, we can expand in $H^{*}(W/\pi \times K)$ by the Kunneth theorem. The coefficients of the expansion of $(1 \times_{\pi} d)^{*} Pu$, which lies in $H^{*}(K)$ are the internal reduced powers.

(1.4.2) Borel used the cohomology theory developed by him to relate cohomology of a G-space with its fixed point sets in (c.f. Theorem 1.4.3) and its quotient spaces (c.f. Theorem 1.4.3). Let T^{1} be the one dimensional Torus.

(1.4.3) Theorem Let $G = T^{1}$ act on a finite dimensional space X and F be the fixed point set. Then

(i) $H_{G}^{*}(X - F, Q) \cong H^{*}((X - F)/G, Q)$

are torsion R-modules, where $R = H^{*}(B_{G})$

(ii) The kernel and cokernel of

$$\mathcal{H}_G^*(X, Q) \longrightarrow \mathcal{H}_G^*(F, Q) = R \otimes H^*(F, G)$$

are both torsion R-modules.

(for details see [9])

(1.4.4) <u>Theorem</u> : Let $G = T^1$ have a finite number of distinct isotropy groups on the compact space X. Let $L = \mathbb{Z}$ (resp. K_p, a field of characteristic p, prime or 0) Assume $\dim_L X < \infty$ and $X \sim_L S^n$

i.e. $H^*(X; L) \approx H^*(S^n; L)$

Let r be the integer s.t. $r \equiv n \pmod 2$ such that $F \sim S^r$

Then $H^i(X/G ; L) = L$ for $i = 0$

and $i = r+3, r+5, \cdots, n-1$

and isomorphism otherwise

(for details see [4] P.65)

(1.4.5) <u>Theorem</u> Let p be a prime (resp. 0) and let $G = \mathbb{Z}_p$ (resp. T^1), X is a G-space such that

$X \sim_p S_n$ and $\dim_p X < \infty$

Let r be such that $F \sim_p S_r$.

Then $H^i(X/G\,;K_p) = H^{i-r-1}(B_G\,;K_p)$
$r+2 \leq i \leq n$

and $H^i(X/G\,;K_p) = 0$, otherwise for $i \neq 0$ ([4], P64)

considering the action of $\mathbb{Z}_p^k$ or T^k on a Compact Space X, Borel obtained deeper results (see Theorem(1.4.6)), than those given by Smith's original method (namely cohomology sphere theorem §1.1). Here $T^k = \underbrace{T^1 \times \cdots \times T^1}_{k\text{-times}}$

(1.4.6) Theorem Let $G = T^k$, $L = \mathbb{Z}$ (K_p resp.) X is compact, $X \sim_L S^n$ and $\dim_L X < \infty$. Assume moreover that G has only a finite number of distinct isotropy groups on X. Then $X^G \sim_L S^r$ and n-r is even (see [4], P 63)

The basic tools in Borel's approach are the spectral sequences associated to the two maps $\pi_1 : X_G \to B_G$ and $\pi_2 : X_G \to X/G$. As for instance, the Serre spectral sequence of the fibre map $\pi_1 : X_G \to B_G$ is used in proving the following result.
(see [4] P.62)

(1.4.7) <u>Theorem</u> Let $G = T^{1}$ and $\dim_{0} X$ is finite i.e.

$$H_c^{n+1}(U;K_0) \text{ is finite for all } U \subset X.$$

Then for $a = 0,1$, $i \equiv a \bmod 2$ and $i > \dim_{0} X$, we have

$$\sum_{s \equiv a(2)} \dim H_c^s(X;K_0) \geq \sum_{s \equiv a(2)} \dim H_c^s(F;K_0) - \dim H_c^i(X_G;K_0).$$

where $H_c^i(X;K_0)$ denotes the ith cohomology group with compact support of X with coefficients in K_0.

(1.4.8) <u>Corollary</u> : There is equality iff the spectral sequence over K_0 of $X_G \longrightarrow B_G$ is trivial.

The Leray spectral sequence is used to prove theorem (1.4.3) which is the primitive form of the localisation theorem. If $\hat{R}$ denotes the quotient field of R and $S=R-\{0\}$ is taken as a multiplicative set in the ring R (in terms of the llcalisation concept) then $\hat{R} = S^{-1}R$ and one deduces from theorem (1.4.3) that

$$(1.4.9) \quad S^{-1}H_G^*(X;Q) \cong S^{-1}H_G^*(F;Q) \cong H^*(F;Q) \otimes_Q H^*(B_G;Q)$$

This isomorphism is a special case of the so called Atiyah-Segal-Borel type localisation theorem of Bundle cohomology (for further generalisation of localisation results refer to [19]).

CHAPTER II

Bredon's Equivariant (Co) homology Theory

The (Co) homologies defined by Eilenberg and Borel are readily computable and has many applications as we have seen in the last chapter. However, these definitions are not adequate for the development of equivariant obstruction theory. For G-CW complexes with free G-action Eilenberg's definition can be used to develop a G-obstruction theory. ([5])

G Bredon ([5]) introduced an equivariant ordinary (co) homology theory H_G^* (—), suitable for gendralising the classical obstruction theory in the general (free or non-free G-action) G-equivariant context. Using this theory he proved that

$$[X,Y]_G \xrightarrow{\text{bijection}} H_G^*(X; \widetilde{\omega}_*(Y))$$

where $\widetilde{\omega}_*$ (Y) is a suitably defined coefficient system, $[X,Y]_G$ is the set of equivariant homotopy classes of equivariant maps from X to Y.

This chapter is divided into 5 sections. In § 2.1 we shall axiomatically introduce equivariant (co) homology theory. In § 2.2 we define suitable coefficient systems and categories which are used in the construction of (co) homology groups.

In § 2.3 we shall give the Bredon's construction of cohomology groups. In § 2.4 we shall describe in brief the equivariant obstruction theory, as developed by Bredon. In § 2.5 we define equivariant obstruction cocycles for a free G-complex (G Compact Lie group) using Eilenberg's equivariant cohomology (as given in Tom Dieck [23]
For finite groups Bredon's definition of equivariant obstruction cocycles extends the above definition.

§ 2.1 Equivariant cohomology theory

Let G be a finite group, $\mathcal{G}$ be the category of G-Complexes (c.f.Chpt.I § 1.03) and continuous equivariant maps, and $\mathcal{G}^2$ be the category of pairs (K,L) of G-complexes ($L \subset K$, a G.Sub-complex)
'Abel' denotes the category of abelian groups. (see [32]).

(2.1.0) An equivariant (generalised) cohomology theory on the category $\mathcal{G}^2$ is a sequence of contravariant functors

$$\mathcal{H}^n : \mathcal{G}^2 \longrightarrow \text{Abel} \qquad (n \in \mathbb{Z})$$

together with natural transformations

$$\delta^n : \mathcal{H}^n(L,\phi) \longrightarrow \mathcal{H}^n(K,L)$$

such that the following three axioms are satisfied:

(1) If f_0, f_1 are equivariantly homotopic maps then $\mathcal{H}^n(f_0) = \mathcal{H}^n(f_1)$

(2) The inclusion : $(K, K \cap L) \hookrightarrow (K \cup L, L)$ induces an isomorphism

$$\mathcal{H}^n(K \cup L, L) \xrightarrow{\approx} \mathcal{H}^n(K, K \cap L)$$

(3) If $(K,L) \in \mathcal{C}_G^2$, then the sequence

$$\cdots \longrightarrow \mathcal{H}^n(K,L) \longrightarrow \mathcal{H}^n(K) \longrightarrow \mathcal{H}^n(L) \longrightarrow \mathcal{H}^{n+1}(K,L) \longrightarrow \cdots$$

is exact.

(2.1.1) The cohomology theory is called <u>equivariant classical Cohomology</u>, if it satisfies the additional dimension axiom :

(4) $H^n(G/H) = 0$ for $n \neq 0$ and all H.

(2.1.2) Similarly, one defines an <u>equivariant cohomology theory on the category</u> $\mathcal{C}_0$ of G-complexes with base point and base point preserving equivariant maps. (For details see [5] PP 1 - 7)

§2.2 Coefficient Systems for G

Before giving the construction of cohomology groups, we define the following coefficient systems and categories, which will be used in the construction.

(2.2.1) <u>Definition</u>.

<u>The category of canonical orbits of G, denoted by</u> $\mathcal{O}_G$, is defined to be the category whose objects are the left coset spaces G/H and whose morphisms are the equivariant maps. If $a \in G$ is such that $a^{-1}Ha \subset K$, then an equivariant map $\hat{a} : G/H \to G/K$ is given by $\hat{a}(xH) = xaK$.

(2.2.2) Definition

Generic coefficient systems for G is defined to be a contravariant functor $M: \Theta_G \longrightarrow$ Abel, Abel stands for the category of abelian groups.

The generic coefficient systems themselves form a category, Dgram (Θ_G^* , Abel), denoted by $\mathcal{C}_G$, which is an abelian category with projectives and injectives. (see [32]).

(2.2.3) Let $\mathcal{K}$ denote the the category whose objects are finite G-sub-complexes of a G-complex K and whose morphisms are as follows:

If L and L' are finite G-sub complexes of K, then hom (L,L') consists of all maps $g: L \to gL \hookrightarrow L'$ for $g \in G$.

(2.2.4) Definition

A local coefficient system on a G-complex K is a covariant functor from the category: $\mathcal{K} \to$ Abel .

(2.2.5) Let Θ be a contravariant functor $: \mathcal{K} \longrightarrow \Theta_G$ defined as follows:

For $L \subset K$, $\Theta(L) = G/G_L$, where

$G_L = \{ g \in G \mid g$ leaves L point wise fixed $\}$.

If $gL \subset L'$ and $f: L \to L'$ denotes the map induced by $g \in G$, then $\Theta(f) = \hat{g} : \Theta(L') \longrightarrow \Theta(L)$ is defined by $\hat{g}(g' G_{L'}) = g'g\, G_L$.

(2.2.6) Definition

Simple coefficient system .

If $M \in \mathcal{C}_G$ is a generic coefficient system then $M \ominus : \mathcal{K} \longrightarrow$ Abel is called a simple coefficient system on K .

§ 2.3 Bredon's Construction of Equivariant Cohomology groups.

Let $\mathcal{L}_0 : \mathcal{K} \longrightarrow$ Abel be a local coefficient system. Let us orient the cells of K in such a way that G preserves the orientation.

(2.3.1) The n-cochain group $C^n(K;\mathcal{L}_0)$ is defined to be the group of all functions f on n-cells of K with

$$f(\sigma) \in \mathcal{L}_0(\sigma) , \quad \sigma \text{ is an n-cell of } K.$$

Define homomorphism $\delta^n : C^n(K;\mathcal{L}_0) \longrightarrow C^{n+1}(K;\mathcal{L}_0)$

by. $$\delta^n(f)(\sigma) = \sum [\tau : \sigma] \mathcal{L}_0(\tau \to \sigma) f(\tau)$$

where $\mathcal{L}_0(\sigma) = \mathcal{L}_0(K(\sigma))$.

and for $K(\tau) \subset K(\sigma)$, $\mathcal{L}_0(\tau \to \sigma)$ denotes the morphism $\mathcal{L}_0(\text{inclusion} : K(\tau) \longrightarrow K(\sigma))$.

Also, whenever $[\tau : \sigma] \neq 0$, $K(\tau) \subset K(\sigma)$.

one verifies that $\delta^{n+1} \circ \delta^n = 0$.

(2.3.2) $\{C^n(K;\mathcal{L}), \delta^n\}$ is a cochain complex.

(2.3.3) Define an action of G on $C^n(K;\mathcal{L}_0)$, by

$$g(f)(\sigma) = \mathcal{L}_0(g)\, f(g^{-1}\sigma), \text{ for } g \in G \text{ and } f \in C^n(K;\mathcal{L}_0)$$

which gives $g(f)(g\sigma) = \mathcal{L}_0(g)(f\sigma) = g_*(f\sigma)$.

The automorphism $f \longmapsto g(f)$ of $C^*(K;\mathcal{L}_0)$ defines an action of G on $C^*(K;\mathcal{L}_0)$ by chain mappings, one verifies that the fixed point set $C^n(K;\mathcal{L}_0)^G$ consists precisely of the equivariant n-cochains $C^n_G(K;\mathcal{L}_0)$

(2.3.4) <u>Definition</u>

The equivariant cohomology group is defined by,

$$H^n_G(K;\mathcal{L}_0) = H^n(C^*(K;\mathcal{L}_0)^G).$$

(2.3.5) <u>Remark</u>

These cohomology groups are easily computable, the order of difficulty being roughly the same as for nonequivariant ordinary cohomology.

(2.3.6) <u>Remark</u>

One can similarly define the equivariant ordinary homology groups (see the next chapter).

(2.3.7) <u>Remark</u>

Bredon's cohomology theory includes Eilenberg's theory as a special case.

To prove this, another description of cochain was considered by Breden (see [5] pp I-20), which is described below:

(2.3.8) An element $\underline{C}_n(K;\mathbb{Z})$ belonging to the category $\mathcal{C}_G$ is defined by

$$\underline{C}_n(K;\mathbb{Z})(G/H) = C_n(K^H,\mathbb{Z})$$
$$\text{for } n = 0,1,2,\ldots$$

together with obvious values on morphisms of $\mathcal{O}_G$

$\{\underline{C}_n(K;\mathbb{Z})\}_{n=0,1,2,\ldots}$ form a chain complex.

(2.3.9) The homology of this chain complex i.e. $H_n(\underline{C}_n(K;\mathbb{Z}))$, denoted by, $\underline{H}_n(K;\mathbb{Z})$ is an element in the category $\mathcal{C}_G$, of the generic coefficient systems defined by.

$$\underline{H}_n(K;\mathbb{Z})(G/H) = H_n(K^H;\mathbb{Z}),$$ together with the obvious values on morphisms. With these definition one shows that there is an isomorphism:

(2.3.10) $C^n_G(K;M) \simeq \mathrm{Hom}(\underline{C}_n(K;\mathbb{Z}),M)$ [see §2.3.3]

For a G-module A, the coefficient system M, $M \in \mathcal{C}_G$ is defined as follows : $M : \mathcal{O}_G \longrightarrow Ab$ is a contravariant functor defined by

$$M(G/H) = A^H$$

and for $g \in G$, with $g^{-1}Hg \subset K$,

$$M(\hat{g}) : A^K \longrightarrow A^H$$

such that $x \longmapsto xg$, one can show that there is an isomorphism.

(2.3.11) $\operatorname{Hom}_{Z(G)}(C_n(K;Z), A) \approx$

$$\operatorname{Hom}(\underline{C}_n(K;Z), M)$$

From (2.3.10) and (2.3.11) we get the following isomorphism.

(2.3.12) $\operatorname{Hom}_{Z(G)}(C_n(K;Z), A) \approx C_G^n(K;M)$

Left hand side of (2.3.12) is, by definition (as defined by Eilenberg, see § 1.2), the equivariant cochain group with coefficients in the G-module A.

The isomorphism preserves the coboundary operators, so we may pass to homology and obtain the isomorphism of the corresponding homology groups.

§2.4 Equivariant obstruction theory

In this section we consider the problem of extending equivariantly an equivariant map and define equivariant obstruction cochain, deformation cochain etc.

[for details see [5] , PP II.1]

(2.4.1) Let (K, L) be a relative G-Complex and

Let $\varphi : K \longrightarrow Y$, be an equivariant map, where Y is a G-space. Assume Y^H to be nonempty, arcwise connected and n-simple for each subgroup H of G (where n is a positive integer).

Consider an $(n+1)$ cell σ of K with a characteristic map $f_\sigma : S^n \longrightarrow K^n$. Then $(\varphi . f_\sigma)(S^n) \subset Y^{G_\sigma}$, so that $\varphi . f_\sigma$ represents an element $\in \pi_n(Y^{G_\sigma})$. Denote this element by $c_\varphi(\sigma)$. This defines a cochain $c_\varphi \in C^{n+1}(K, L; \widehat{\omega}_n(Y))$, where $\widehat{\omega}_n(Y)$ is a generic coefficient system defined by

$$\widehat{\omega}_n(Y)(G/H) = \pi_n(Y^H, y_0)$$
$$\widehat{\omega}_n(Y)(\hat{g}) = g_{\#} : \pi_n(Y^{K'}, y_0) \to \pi_n(Y^H, y_0)$$

for $g \in G$ satisfying $g^{-1} H g \subset K'$.

$c_\varphi(g\sigma)$ is represented by $\varphi \cdot f_{g\sigma}$

and $\varphi \cdot f_{g\sigma} = \varphi g f_\sigma = g \varphi f_\sigma$, so that

$$c_\varphi(g\sigma) = g_{\#}(c_\varphi(\sigma)).$$

This gives $c_\varphi \in C_G^{n+1}(K, L; \widetilde{\omega}_n(Y))$.

(2.4.2) <u>Definition</u>

c_φ is called the obstruction (cochain) to extending $\varphi : K^n \cup L \to Y$

(2.4.3) The equivariant obstruction cochain has properties analogous to those of nonequivariant obstruction cochin, namely

(i) The obstruction cochiin is a cocycle.

(ii) The obstruction cochain c_φ is equal to zero iff φ can be extended equivariantly to $K^{n+1} \cup L$.

To study the obstructions to extending equivariant homotopies one defines the deformation cochain of two equivariant maps $\varphi, \theta : K^n \cup L \to Y$, with respect to an equivariant homotopy

$$F : \varphi|_{K^{n-1} \cup L} \simeq \theta|_{K^{n-1} \cup L}$$

This is denoted by $d_{\varphi, F, \theta}$

(2.4.4) <u>Definition</u>

The deformation cochain $d_{\varphi, F, \theta}$ is defined to be an element belonging to $C_G^n(K, L; \widetilde{\omega}_n(Y))$.

such that $d_{\varphi, F, \theta}(\sigma) = c_{\varphi \#_F \theta}(\sigma \times I)$

where $\varphi \#_F \theta : (K \times I)^n \cup L \times I$ is an equivariant map defined by $\begin{cases} \varphi \#_F \theta(x,0) = \varphi(x) \\ \varphi \#_F \theta(x,1) = \theta(x) \end{cases}$

and $\varphi \#_F \theta(x,t) = F(x,t)$.

An important special case is that in which φ and θ agree on $K^{n-1} \cup L$ and the homotopy F is stationary.

(2.4.5) If $\varphi|_{K^{n-1} \cup L} = \theta|_{K^{n-1} \cup L}$ and the homotopy F is stationary the deformation cochain $d_{\varphi, F, \theta}$ is abbreviated to $d_{\varphi, \theta}$.

The deformation cochain is not a cocycle, however, one has a useful coboundary formula.

(2.4.6) Theorem:

The coboundary of the deformation cochain is given by $\delta d_{\varphi, F, \theta} = c_\theta - c_\varphi$. One has also the following theorem.

(2.4.7) Theorem

For a given equivariant map $\varphi : K^n \cup L \longrightarrow Y$ and $d \in C^n_G(K, L; \tilde{\omega}_n(Y))$,

there is an equivariant map $\theta : K^n \cup L \longrightarrow Y$ coinciding with φ on $K^{n-1} \cup L$ such that $d_{\varphi,\theta} = d$.

This theorem is used in proving the following important result on equivariant extension of maps.

(2.4.8) Theorem.

The cohomology class of the obstruction cochain is equal to zero iff $\varphi|_{K^{n-1}\cup L}$ can be extended equivariantly to $K^{n+1}\cup L$.

Suppose we are given an equivariant map $f:L \longrightarrow Y$ and we wish to determine whether f can be extended over K. We have seen that f can be extended over K^n iff the obstruction cochain $c^n_{f_{n-1}} = 0$

But there may be different extensions of f over K^n, one proves the following theorem.

(2.4.9) Theorem.

$f: L \longrightarrow Y$ has an equivariant extension to $K^n \cup L$, if the following conditions are satisfied, viz;

(i) Y^H is nonempty, arcwise connected and r-simple for all r and for all $H \subset G$.

(ii) $H^{r+1}_{G}(K, L; \tilde{\omega}_r(Y)) = 0$, for all $r < n$

This theorem has the following corollary.

(2.4.10) Corollary.

If all the conditions of theorem (2.4.9) are satisfied and also one has the following condition.

(iii) $H^r_G(K, L; \tilde{\omega}_r(Y)) = 0$ for all $r > n$ then any two such equivariant extensions are cohomologous.

(2.4.11) Definition

The uniquely defined cohomology class

$$[c_{f_n}] \in H_G^{n+1}(K, L; \widetilde{\omega}_n(Y))$$

is called the primary obstruction to extending f and is denoted by $\gamma^{n+1}(f)$.

One can prove the following extension theorem analogous to the extension theorem in nonequivariant extension theory.

(2.4.12) Extension Theorem

If $H_G^{r+1}(K, L; \widetilde{\omega}_r(Y)) = 0$, for $n < r < \dim(K-L)$

then an equivariant map $f: L \to Y$ has an extension to K iff $\gamma^{n+1}(f) = 0$ under the condition (i), (ii) and (iii) of the Theorem (2.4.9) and the corollary (2.4.10); when the primary obstruction $\gamma^{n+1}(f)$ is defined.

As an application of the extension theorem one has the following homotopy theorem.

(2.4.13) Homotopy theorem

Any two equivariant maps f, g: $K \to Y$ with $f|_L = g|_L$ are equivariantly homotopic iff $\omega^n(f, g) = 0$ provided the following conditions are satisfied:

(i) Y^H is r-simple for all $H \subset G$ and for all r

(ii) $H_G^r(K, L; \widetilde{\omega}_r(Y)) = 0$, $r < n$ and $n < r \leq \dim(K-L)$,

(iii) $H_G^{r-1}(K, L; \widetilde{\omega}_r(Y)) = 0$, $r < n$

where, $\omega^n(f, g) = \lambda^{-1}(\gamma^{n+1}(f \# g))$.

and $\lambda : H_G^n(K, L; \widetilde{\omega}_n(Y)) \xrightarrow{\approx} H_G^{n+1}(K \times I, K \times \partial I \cup (L \times I); \widetilde{\omega}_n(Y))$

is an isomorphism.

We have the following classification theorem analogous to Eilenberg classification theorem for the ordinary (non-equivariant) extension of a map.

(2.4.14) <u>Classification theorem</u>

Let us assume the conditions.

(i) Y^H is r- simple for all r and for all $H \subset G$.

(ii) $\begin{cases} H_G^r(K, L; \widetilde{\omega}_r(Y)) = 0 = H_G^{r-1}(K, L; \widetilde{\omega}_r(Y)), & r < n \\ H_G^r(K, L; \widetilde{\omega}_r(Y)) = 0 = H_G^{r+1}(K, L; \widetilde{\omega}_r(Y)), & r > n \end{cases}$

If $f: K \to Y$ is an equivariant map then the equivariant homotopy classes (relative to L) of maps $g: K \to Y$ (with $g/L = f/L$) are in one to one correspondence with the elements of $H^n(K, L; \widetilde{\omega}_n(Y))$.

(2.4.15) Corollary.

Taking L = ϕ , the corresponding classification theorem assests that for such a space Y, $[K,Y]_G \longleftrightarrow \mathcal{H}^n_G(K;\widetilde{\omega}_n(Y))$.

(2.4.16) Eilenberg - Maclane G-space

Using the existence of projectives in the category $\mathcal{C}_G$, one constructs G-complexes of type $(\widetilde{\omega}, n)$ for all $n \geqslant 1$ and for all $\widetilde{\omega} \in \mathcal{C}_G$

(for details see [5] .).

As an important consequence of the classification theorem one has the following result.

(2.4.17) Theorem

The Bredon cohomology groups $H^n_G(\text{---}; \widetilde{\omega})$ are representable for all $\widetilde{\omega} \in \mathcal{C}_G$

A space (Y, y_0) with

$$\widetilde{\omega}_q(Y, y_0) = \begin{cases} 0, & q \neq n \\ \widetilde{\omega}, & q = n \end{cases}$$

being the classifying space.

(2.4.18) Remark

We can give a description of $\nu^{n+1}(f)$ in terms of the standard operations of homology theory and a suitable universal class, called the characteristic class.

(2.4.19) Definition

The characteristic class is defined by $\chi^n(f) \in H^n_G(K; \widetilde{\omega}_n(Y))$,

such that $\gamma^{n+1}(f) = \delta^*(\chi^n(f))$.
where, $\delta^* : H^n_G(L; \widetilde{\omega}_n(Y)) \longrightarrow H^{n+1}_G(K, L; \widetilde{\omega}_n(Y))$
is the coboundary.

(2.4.20) Definition

The characteristic class of a G-Complex Y is defined by the characteristic class $\chi^n(1d) \in H^n_G(Y; \widetilde{\omega}_n(Y))$, where

$1d : Y \longrightarrow Y$ is the Identity map.
This is denoted by $\chi^n(Y)$; for any equivariant map $f: K \longrightarrow Y$, one has $\chi^n(f) = f^*(\chi^n(Y))$;
one can show that there is a natural isomorphism.

(2.4.21) $H^n_G(Y; \widetilde{\omega}_n(Y)) \approx \mathrm{Hom}(\widetilde{\omega}_n(Y), \widetilde{\omega}_n(Y))$

Under this isomorphism $\chi^n(Y)$ corresponds to the Identity homomorphism. This result allows the computation of the cgaracteristic class.

§ 2.5 Equivariant obstruction Theory for free actions of Compact Lie groups.

We have seen in the previous sections that for a finite group G and a general action of G (on a G-complex) the obstruction cocycles are represented in a cohomology theory with special coefficient systems and that this is not describable in the setting of Eilenberg-Borel equivariant cohomology theory.

In this section, we will see, however, that for free actions of Compact Lie groups one can develop an equivariant obstruction theory in the setting of Eilenberg definition. For finite group this comes as a particular case of Bredon's description.

Let (X,A) be a relative G-complex with free G-action on $X \setminus A$ i.e. X_n is obtained from X_{n-1} by attaching n-cells, $G \times D^n$, of type G.

Let G_o denote the component of G containing e., the identity element
The G-action on X_n induces a G-action on $H_n(X_n , X_{n-1})$ which factors over G/G_o by homotopy invariance of homology. Thus $H_n(X_n, X_{n-1})$ becomes a $\mathbb{Z}(G/G_o)$ module .

The filtration (X_n) leads to a cellular chain complex, $C_*(X,A)$, of $\mathbb{Z}(G/G_0)$ modules, namely,

(2.5.1) $\cdots \to H_{n+1}(X_{n+1}, X_n) \to H_n(X_n, X_{n-1}) \to \cdots$

consider a path connected, n - simple ($n \geq 1$) G-space Y.

If $\pi_1(Y, y)$ acts trivially on $\pi_n(Y,y)$ then the canonical map $\pi_n(Y,y) \to [S^n, Y]$ from pointed to free homotopy classes is bijective. The action of G on Y thus induces a well defined action of G/G_0 on $\pi_n(Y)$ making $\pi_n(Y)$ a $\mathbb{Z}(G/G_0)$ - module.

(2.5.2) Definition

The cochain complex $C^*_G(X, A; \pi_n(Y))$

$$= \mathrm{Hom}_{\mathbb{Z}(G/G_0)}(C_*(X,A), \pi_n(Y)),$$

yields cohomology groups which are denoted by $\mathfrak{H}^*_G(X,A; \pi_n(Y))$.

One shows that these groups are isomorphic to the groups $\mathfrak{H}^*_{G/G_0}(X/G_0, A/G_0; \pi_n(Y))$.

If the $\mathbb{Z}(G/G_0)$ module $\pi_n(Y)$ is interpreted as a local coefficient system $\widetilde{M}$ on $X/G \setminus A/G$, then the groups $\mathfrak{H}^n_{G/G_0}(X/G_0, A/G_0; \pi_n(Y))$ can be thought of as cohomology $H^n(X/G, A/G ; \widetilde{M})$ with local coefficients.

(2.5.3) Definition of the obstruction cochain

Consider $[h] \in [X_n, Y]_G$

Let $\phi = (\phi_j) : \coprod G \times (D^{n+1}, S^n) \longrightarrow (X_{n+1}, X_n)$

be a characteristic map, and $e_j \in C_{n+1}(X, A)$ be the basis element corresponding to ϕ_j.

Then $c^{n+1}(h)(e_j) \in [S^n, Y]$ is the homotopy class represented by $h\phi_j$.

From the exact sequence of the pair (X_{n+1}, X_n) it follows immediately that $c^{n+1}(h) = 0$ if h is extendable over X_{n+1}.

(2.5.4) Remarks on obstruction theory in the equivariant setting.

If we consider (K,L) to be a relatively free G-CW Complex i.e. if G acts freely on $K \setminus L$, then for every (n+1) cell of K away from L, the group

$$G_\sigma = \{ g \in G \mid g \text{ leaves } \sigma \text{ point wise fixed} \} = \{e\}.$$

Let Y be a n-simple G-space then in the notation of Bredon

$$c_\varphi(\sigma) \in \pi_n(Y^{G_\sigma}) = \pi_n(Y^{\{e\}}) = \pi_n(Y).$$

So obstruction cochains are elements

$$c_\varphi \in C_G^{n+1}(K, L; \pi_n(Y)).$$

Thus Tom Dieck's definition of equivariant obstruction cochains come as a particular case of Bredon's equivariant obstruction cochain.

CHAPTER III

Illman's definition of equivariant singular (co)homology.

In chapter II, we have discussed the equivariant cohomology theory and the equivariant obstruction theory as developed by G.Brdon ([5]). Bredon defined G-equivariant cohomology theory on a G-complex, where G is discrete. The definition of Bredon was generalised by Illman ([10]) to cover G-complexes, where G is an arbitrary topological group. This was done by replacing the category Θ_G, as used by Bredon, by a category $\tilde{\Theta}_G$, whose objects are G-spaces of the form G/H, H belonging to some orbit type family $\mathcal{F}$ for G (see § 3.1.1.) and G-homotopy classes of G-maps. This (co)homology theory satisfies all the seven equivariant Eilenberg-Steenrod axioms.

Remark : For discrete G, the two categories coincide.

Notions such as transfer homomorphism in equivariant singular (co)homology with coefficients in a commutative ring coefficient system, and cup products were also defined by Illman ([10]).

The chapter is set out in the following fashion. In § 3.1 we give some computations which uses only the axioms of the equivariant homology and cohomology (singular) theory. § 3.2 will be devoted to a brief outline of the Illman's construction of the singular (co)homology theory. In § 3.3 we describe briefly various notions like transfer homomorphism, kronecker index and capproduct as defined by Illman in the equivariant setting. In § 3.4 we describe briefly the generalised equivariant cohomology theory and show how any generalised theory is connected by a spectral sequence to the Bredon's theory of chapter II.

§3.1 The Existence theorems for equivariant singular (co)homology theory.

Let G be an arbitrary topological group, R a ring with unity, and that all R-modules be unitary.

(3.1.1) Definition:

A family $\mathcal{F}$ of subgroups of G is called an orbit type family for G, if $H \in \mathcal{F}$ and H' is conjugate to H then $H' \in \mathcal{F}$.

(3.1.2) Definition

A covariant (resp. contravariant) coefficient system k (resp. m), for $\mathcal{F}$ over the ring R is a covariant (resp. contravariant) functor from the category $\mathcal{O}_G$ of G-spaces of the form G/H where $H \in \mathcal{F}$, and G-homotopy classes of G-maps, to the category of left (resp. right) R-modules.

(3.1.3) Existence Theorem for homology

There exists an equivariant homology theory H_*^G (—; k), a covariant functor, defined on the category of G-pairs and G-maps and having values in the category of left R-modules, which satisfy all seven equivariant E-S axioms and which has the given coefficient system k as coefficients.

(3.1.4) Existence theorem for cohomology

There exists an equivariant cohomology theory H_G^* (—; m), a contravariant functor, defined on the category of G-pairs and G-maps and having values in the category of right R-modules, which satisfy all seven equivariant E-S axioms and which has the given coefficient system m as coefficients.

(3.1.5) Example (Computation using only E-S axioms).

Let $G = S^1$ be the circle group and $\mathcal{X} = S^2$ be the two dimensional sphere. Assume that S^1 acts on S^2 by the standard rotation leaving the north and south poles fixed, and acting freely elsewhere.

Let X_1 and X_2 denote the northern and southern hemispheres respectively and $X_0 = X_1 \cap X_2$ the equator.

Let $\mathfrak{F}$ be the orbit type family such that both $G \in \mathfrak{F}$ and $\{e\} \in \mathfrak{F}$. Let R be a ring with Identity element and k be a covariant coefficient system for $\mathfrak{F}$ over R. As a consequence of the E-S axioms one has the following exact Mayer - Vietoris sequence

$$0 \longleftarrow H_0^G(X,k) \xleftarrow{j_{1*} + j_{2*}} H_0^G(X_1;k) \oplus H_0^G(X_2;k) \longleftarrow H_0^G(X_0;k)$$
$$\xleftarrow{\partial} H_1(X;k) \longleftarrow 0$$

Since both X_1 and X_2 are G-homotopy equivalent to a point and $X_0 \cong G$ as G-spaces, it follows that the above exact sequence equals

$$0 \longleftarrow H_0^G(X;k) \longleftarrow k(G/G) \oplus k(G/G) \xleftarrow{(p_*,-p_*)} k(G/\{e\})$$
$$\longleftarrow H_1^G(X,k) \longleftarrow 0$$

where $p_* : k(G/\{e\}) \longrightarrow k(G/G)$,

is induced by the G-map $p : G \longrightarrow G/G$.

Thus $$H_0^G(X,k) \cong \frac{(k(G/G) \oplus k(G/G))}{\{(p_*(a), -p_*(a)) \mid a \in k(G}$$

$$H_1^G(X,k) \cong \ker (p_* : k(G) \longrightarrow k(G/G))$$

$$H_m^G(X,k) = 0 \quad , \text{ for } m \neq 0,1 .$$

Let us take the orbit type family $\mathcal{F}$ to be the family of all closed subgroups of $G = S^1$ and let R be the ring of integers.

Define a covariant coefficient system k as follows:

$$k(G/H) = \begin{cases} \mathbb{Z} & , H \neq G \\ \mathbb{Z}_2 & , H = G \end{cases}$$

Let $p: G/H \longrightarrow G/G$, where $H \neq G$ induce the natural projection $\mathbb{Z} \longrightarrow \mathbb{Z}_2$ and let all other induced homomorphisms on k be the identity on $\mathbb{Z}$. Then ,

$$H_0^G(X, k) \cong \mathbb{Z}_2$$

$$H_1^G(X;k) \cong \mathbb{Z}$$

$$H_n^G(X;k) = 0 \quad , \text{ for } n \neq 0,1 \quad (\text{see introduction})$$

(3.1.6) The equivariant singular cohomology groups of the G-space X (defined in §3.1.5) can similarly be computed. (using only the E-S axioms). For a contravariant coefficient system m, the cohomology groups are as follows:

$$H^0_G(X;m) \cong \ker(p^*(\pi_1 - \pi_2): m(G/G) \oplus m(G/G) \longrightarrow m(G/\{e\}).$$

$$H^1_G(X;m) \cong m(G/\{e\})/\operatorname{Im} p^*(\pi_1 - \pi_2)$$

$$H^q_G(X;m) = 0 \quad , \text{ for } q \neq 0,1$$

where $\pi_i : m(G/G) \oplus m(G/G) \longrightarrow m(G/G)$, denotes the projection on the ith factor i = 1,2.

§ 3.2 Construction of equivariant singular homology and cohomology groups.

The construction of equivariant singular (co)homology groups as given by Illman is very much analogous to that of ordinary (co)homology.

(3.2.1) Definition: An equivariant singular n-simplex in a G-space X is a G-map $T: \Delta_n \times G/H \longrightarrow X$ for some subgroup H of G, where Δ_n is the standard n-simplex. we write $t(T) = H$ and call it the type of T.

The ith face of T is defined as the equivariant singular (n-1) simplex

$$T^i = T \circ (e_n^i \times \mathrm{id}) : \Delta_{n-1} \times G/H \longrightarrow X$$

where $e_n^i : \Delta_{n-1} \longrightarrow \Delta_n$ is a map defined by $e_n^i(x_0, x_1, \ldots, x_{n-1}) = (x_0, x_1, x_2, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$.

for $i = 0, 1, 2, 3, \ldots, n$.

(3.2.2) Let Z_T denote the infinite cyclic group on the generator T, where T is an equivariant singular n- Simplex with type $t(T) \in \mathcal{H}$

$Z_T \otimes k(G/t(T))$ denotes the tensor product over the integers.

The left R-modules $\sum_{t(T)\in\mathfrak{F}} \oplus Z_T \otimes k(G/t(T))$,
where, the direct sum $\oplus$ is taken over all equivariant simplexes of fixed type $t(T)$ and the direct sum '$\sum$' is over all T with $t(T) \in \mathfrak{F}$, are denoted by $\hat{C}_n^G(X;k)$, k is a covariant coefficient system.

Define homomorphisms, $\hat{\partial}_n : \hat{C}_n^G(X;k) \longrightarrow \hat{C}_{n-1}^G(X;k)$

by
$$\hat{\partial}_n(T\otimes a) = \begin{cases} \sum_{i=0}^{n} (-1)^i T^{(i)} \otimes a, & \text{for } n>0 \\ 0, & \text{for } n \leq 0. \end{cases}$$
where $a \in k(G/t(T))$.

then $\hat{\partial}_{n-1} \circ \hat{\partial}_n = 0$, so that
$$\hat{S}^G(X;k) = \{\hat{C}_n^G(X;k), \hat{\partial}_n\}$$ is a chain complex.

(3.2.3) Let $S_n \subset \hat{C}_n^G(X;k)$ denote the set of all elements of $\hat{C}_n^G(X;k)$ that have atmost one coordinate $\neq 0$. Now one defines a relation '$\sim$' in the set S_n.

Let $h: \Delta_n \times G/K \longrightarrow \Delta_n \times G/K'$ be a G-map which covers $\mathrm{Id}: \Delta_n \longrightarrow \Delta_n$ for every $x \in \Delta_n$, there is a G-map $h_x: G/K \longrightarrow G/K'$ defined by $h_x(gK) = pr.h(x, gK)$.

where, $pr: \Delta_n \times G/K' \longrightarrow G/K'$

is projection on the second factor. Then one can show that the G-maps $h_{\mathfrak{X}}$ and $h_{\mathfrak{Y}}$ are G-homotopic.

Thus if $K, K' \in \mathcal{J}$, it follows that

$$(h_{\mathfrak{X}})_* = (h_{\mathfrak{Y}})_* : k(G/K) \longrightarrow k(G/K'),$$

that is the G-map h induces in this way a unique homomorphism from $k(G/K)$ to $k(G/K')$. We denote this homomorphism by

$$h_* : k(G/K) \longrightarrow k(G/K').$$

(3.2.4) We define a relation '$\sim$' in $\mathcal{S}_n$ in the following way. Consider two arbitrary elements $T \otimes a$, $T' \otimes a'$ in $\mathcal{S}_n$

where $T: \Delta_n \times G/K \longrightarrow X$ and

T' $\Delta_n \times G/K' \longrightarrow X$ are equivariant singular n-simplexes belonging to $\mathcal{J}$ in X, and

$a \in k(G/K)$, $a' \in k(G/K')$.

we define

$T \otimes a \sim T' \otimes a'$ iff there exists a G-map

$h : \Delta_n \times G/K \longrightarrow \Delta_n \times G/K'$ covering

$1d : \Delta_n \to \Delta_n$ such that $T = T' \circ h$ and $h_*(a) = a'$.

Let $\tilde{C}^G_n(X,k) \subset \hat{C}^G_n(X;k)$ be the submodule of $\hat{C}^G_n(X;k)$ consisting of all elements of the form $\sum_{i=1}^{\infty}(T_i \otimes a_i \sim T'_i \otimes a'_i)$, '$\sim$' stands for difference here.

where $T_i \otimes a_i \sim T_i' \otimes a_i'$

or $T_i' \otimes a_i' \sim T_i \otimes a_i$

for $i = 1, 2, 3 \ldots\ldots s$

one observes that the boundary homomorphisms

$$\hat{\partial}_n : \hat{C}_n^G(X;k) \longrightarrow \hat{C}_{n-1}^G(X;k)$$

restricts to $\bar{\partial}_n : \bar{C}_n^G(X;k) \longrightarrow \bar{C}_{n-1}^G(X;k)$.

and thus induces a boundary homomorphism $\partial_n : C_n^G(X;k) \longrightarrow C_{n-}^G$

where $C_n^G(X;k) = \hat{C}_n^G(X;k)/\bar{C}_n^G(X;k)$.

one verifies that $\partial_{n-1} \circ \partial_n = 0$, so that

$S^G(X,k) = \{C_n^G(X;k), \partial_n\}$ is a chain complex.

(3.2.6) Definition.

The homology of the chain complex $S^G(X,k)$ gives the equivariant singular homology groups of X with coefficients in k.

Considering a G-pair (X,A) and using the earlier notations and results one defines a chain complex $S^G(X,A;k) = \{C_n^G(X;k)/C_{n}^{(}$

and obtains a short exact sequence of chain complexes

$$0 \to S^G(A;k) \longrightarrow S^G(X;k) \longrightarrow S^G(X,A;k) \to 0$$

(3.2.7) Definition: The nth homology module of the chain complex $S^G(X, A; k)$ is defined to be the nth homology module $\mathcal{H}_n^G(X, A; k)$.

(3.2.8) Remark The homology sequence of the G-pair is obtained in the standard way.

The other axioms viz. the equivariant homotopy, excision and the dimension axiom are proved by imitating the corresponding proofs for ordinary singular homology.

(3.2.9) Remark.

The equivariant singular cohomology theory with coefficient system 'm' is constructed by a procedure dual to the one described above. The details are available in Illman (see [10] pp 17). (See also remark (3.2.14)).

(3.2.10) Remark

The coefficient system, which is contravariant for cohomology and covariant for homology, as used in the Bredon - Illman equivariant theories can be unified using the notions of a Mackey functor as defined below: (for details see [23]).

(3.2.11) Definition

Let G be a finite group and let G-set be the category of finite G-sets and G-maps.

A bifunctor $M = (M^*, M_*)$: G-set $\longrightarrow$ $\mathbb{Z}$- Mod consists of a contravariant functor M^*: G-set $\longrightarrow$ $\mathbb{Z}$ Mod and a covariant functor M_*: G-set $\longrightarrow$ $\mathbb{Z}$- Mod with $M_*(S) = M^*(S)$, for finite G-sets S. If $f: S \to T$ is a morphism, then $M^*(f) = f^*$ and $M_*(f) = f_*$

(3.2.12) Definition

A bifunctor is called a Mackey functor if it has the following properties:

for each pull back diagram in G-set

$$\begin{array}{ccc} U & \xrightarrow{F} & S \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle h} \\ T & \xrightarrow[f]{} & Y \end{array} \qquad \text{we have } F_* H^* = h^* f_*$$

The two embeddings $S \to S + T \leftarrow T$ into the disjoint union define an isomorphism

$$M^*(S+T) \cong M^*(S) \oplus M^*(T)$$

The use of Mackey functor makes it possible to present $H_*(X,A;M)$ and $H^*(X,A;M)$ in the usual dual fashion as one does in the ordinary non-equivariant cohomologies. Moreover, each of $H_*(-,-;M)$ and $H^*(-,-;M)$ can be extended to Mackey functors such that $H_0(-,-;M)$ is the covariant part of M and $H^0(-,-;M)$ is the contravariant part of M.

(3.2.13) Remark

For actions of a discrete group G, the category $\tilde{\mathcal{O}}_G$ of Illman's definition (§ 3.1) coincides with that of Bredon's definition. (§ 2.2).

If G is discrete, then G/H etc. are also discrete and there cannot be any nontrivial G-homotopy : G/H x I ⟶ G/K so that every G-homotopy class contains a single element and thus the category $\tilde{\mathcal{O}}_G$ consists of G-spaces of the form G/H and G-maps.

(3.2.14) The cochain groups Hom $(\underline{C}_n(K;\mathbb{Z}), M)$ of Bredon's theory (discribed in § 2.3.5) are isomorphic to the cochain groups $C^n_G(K;M)$ of Illman's theory (given in § 3.2).

We define a map.

$$\theta : \mathrm{Hom}(\underline{C}_n(K,\mathbb{Z}), M) \longrightarrow C^n_G(K,M)$$

in the following way :

Let $f \in \mathrm{Hom}(\underline{C}_n(K,\mathbb{Z}), M)$.

Then $\theta(f) \in \hat{C}^n_G(K;M)$ defined by

$$\theta(f)(T) = f(G/H)(T_H) \in M(G/H).$$

where $T : \Delta^n \times G/H \longrightarrow K$ is an equivariant singular n-Simplex and $T_H : \Delta^n \longrightarrow K^H$ is defined by

$$T_H(x) = T(x, eH).$$

Now we show that $\theta(f) \in C^n_G(K,M)$

Let $T : \Delta^n \times G/H \longrightarrow K$ and $T' : \Delta^n \times G/L \longrightarrow K$ be two equivariant singular n-simplexes and $h : \Delta^n \times G/H \longrightarrow \Delta^n \times G/L$ be a G-map which covers $\mathrm{Id} : \Delta^n \longrightarrow \Delta^n$, such that $T = T'h$.

To show that $h^* \bar{f}(T') = \bar{f}(T) = f(G/H)(T_H) \in M(G/H)$. Using the fact that f is a natural transformation, we see that the following diagram commutes

$$\begin{array}{ccc} C_n(K)(G/H) = C_n(K^H) & \xrightarrow{f(G/H)} & M(G/H) \\ \uparrow h_{\#} & & \uparrow h^* \\ C_n(K)(G/L) = C_n(K^L) & \xrightarrow[f(G/L)]{} & M(G/L) \end{array}$$

Thus (3.2.15) $h^* \bar{f}(T') \overset{\text{defn}}{=} h^* f(G/L)(T'_L) = f(G/H)\, h_{\#}(T'_L)$

where $T'_L : \Delta^n \longrightarrow K^L$ is defined by $T'_L(X) = T'(x, eL)$

$\forall\, t \in \Delta^n$, $\exists$ a G-map $h ; G/H \longrightarrow G/L$.

Now, let $h : G/H \longrightarrow G/L$ be a G-map, then there is an element $g \in G$ with $g^{-1} H g \subset L$

Such that $h(xH) = xgL$. Corresponding to this $g \in G$ a G map from $K^L \longrightarrow K^H$ is given by $a \longmapsto ga$.

Thus $h_{\#} : C^n(K^L) \to C^n(K^H)$ θ is defined by

$h_{\#}(T_L') = gT_L'$

From (3.2.15) we get

$$h^* \bar{f}(T') = f(G/H)gT_L'$$
$$= f(G/H)\ T_H$$

[For, $T'h(t,eL) = T'(t,gL)$
and $T'h = T$, so $T(t,eH) = T'(t,gL)$
i.e. $T_H(t) = gT_L'$]

Now, we define a map Φ as follows :

$$\Phi . \ C^n_G(K;M) \longrightarrow Hom(C_n(K,Z),M)$$

Let $\bar{f} \in C^n_G(K,M)$

then $\Phi(f) \in Hom(C_n(K;Z),M)$ is

ddfined by $\Phi(f)(G/H)(\sigma) = \bar{f}(\sigma \circ \pi_1 \circ h)$

where $\sigma . \Delta^n \to K^H$ is a singular n-simplex, $H \subset G_\sigma$,

$h . \Delta^n \times G/H \xrightarrow{1 \times \bar{h}} \Delta^n \times G/G_\sigma$ is induced by the G-map : $G/H \xrightarrow{\bar{h}_1} G/G_\sigma$ given by $gH \to gG_\sigma$ and $\pi_1 : \Delta^n \times G/G_\sigma \longrightarrow \Delta^n$ is the projection on the first factor.

It is easy to see that Φ and θ are inverses of each other. Thus there is an isomorphism $C^n_G(K;M) \longleftrightarrow Hom(C_n(K;Z),M)$ given by $\bar{f} \longleftrightarrow f$.

This isomorphism clearly preserves the coboundary operators. Thus we may pass to homology and obtain the isomorphism

$$H^n_G(K;M) \approx H_n(Hom(C_n(K;Z),M)$$

§ 3.3 The properties like functoriality in the transformation group, existence of transfer homomorphisms for both singular homology and cohomology were established by Illman. (For details see [10] pp 43-79).

(3.3.1) The transfer homomorphism

Let $P \subset G$ be a closed subgroup of G such that the right coset spaces $P \setminus G$ consists of s elements, that is

$$P \setminus G = \{Pg_1, Pg_2, \ldots\ldots, Pg_s\}.$$

Since P is closed in G, each point in $P \setminus G$ is closed in $P \setminus G$. $P \setminus G$ has the quotient topology from the projection map $\Pi : G \to P \setminus G$. It follows that $P \setminus G$ has the discrete topology.

(3.3.2) A G-map $\beta : G/H \to G/H'$, where H, H' are arbitrary subgroups of G, is said to be of 'type P', if $\beta(eH) = p_0 H'$, where $p_0 \in P$.

In this case, we have $H \subset p_0 H' p_0^{-1}$ and hence

$P \cap H \subset p_0 (P \cap H') p_0^{-1}$. Thus we can define a P-map

$$\beta! : P/P \cap H \longrightarrow P/P \cap H'$$

by the condition

$$\beta_{!}(e(P\cap H)) = p_0(P\cap H');$$

we have $\beta_{!}(p(P\cap H)) = pp_0(P\cap H')$,

where $p \in P$.

The P-map $\beta_{!}$ is independent of the choice of the element $p_0 \in P$ and depends only on the G-map β of 'type P'.

For, if $\beta(eH) = p_1 H'$, $p_1 \in P$

then $p_1^{-1} p_0 \in P \cap H'$

and hence, $p_0(P\cap H') = p_1(P\cap H')$.

(3.3.3) Any G-map $\beta : G/H \longrightarrow G/H'$ of 'type P' determines a P-map

$$\beta_{!} : P/_{P\cap H} \longrightarrow P/_{P\cap H'}$$

Let $\mathfrak{H}$ be an orbit type family for G and $\mathfrak{H}'$ be an orbit type family for P, such that if $H \in \mathfrak{H}$, then $H \cap P \in \mathfrak{H}'$.

Let k' and k be 'covariant coefficient systems for $\mathfrak{H}'$ and $\mathfrak{H}$, respectively, over the ring R. Let $\Lambda' k \longrightarrow k'$ be a natural transformation of transfer type with respect to the inclusion $P \hookrightarrow G$, this means that $\forall\, H \in \mathfrak{H}$ (i) we have a homomorphism of left R-modules

$\Lambda : k(G/H) \longrightarrow k'(P/P\cap H)$

such that if $\beta : G/H \longrightarrow G/K$ is a G-map of 'type P', then the diagram below commutes

$$\begin{array}{ccc} k(G/H) & \xrightarrow{\Lambda} & k'(P/P\cap H) \\ \beta_* \downarrow & & \downarrow (\beta^!)_* \\ k(G/K) & \longrightarrow & k'(P/P\cap K) \end{array}$$

(3.3.4) Let (Y, B) be a G-pair and (Y', B') denote the P-pair obtained by restricting the G-action to the subgroup P. One constructs a transfer homomorphism in the following way.

For each element $Pg \in P\backslash G$, an induced chain map $(Pg)_\#$,

$$(Pg)_\# : \hat{S}^G(Y, B, k) \longrightarrow S^P(Y', B', k')$$

is defined and for $g \in G$,

a map $(g)_\# : \hat{C}^G_n(Y, B, k) \longrightarrow \hat{C}^P_n(Y', B'; k')$

is defined as follows.

Let $T: \Delta_n \times G/K \rightarrow Y$ be an equivariant singular n-simplex belonging to $\mathfrak{F}$ in Y.

Consider the composite map

$$\Delta_n \times P/P\cap gKg^{-1} \xrightarrow{\eta} \Delta_n \times G/gKg^{-1} \xrightarrow{[g]} \Delta_n \times G/K \rightarrow Y$$

where $\eta(x; p(P\cap gKg^{-1})) = (x, p(gKg^{-1}))$

$p \in P \subset G$

and $[g]$ is the G-map, which is a G-homeomorphism, determined by the condition $[g](x, e(gKg^{-1})) = (x, gK)$.

The map $T[g]\eta : \Delta_n \times P/P \cap gKg^{-1} \longrightarrow Y$

when considered as a map into the P-space Y' , is a P-equivariant singular n-Simplex belonging to $\tilde{\eta}$ in Y'.

Now, one sets

$$(g)_{\#}(T \otimes a) = T[g]\eta \otimes \Lambda [g]^{-1}_{*}(a)$$

where $a \in k(G/K)$ and

$$[g]_{*} : k(G/gKg^{-1}) \longrightarrow k(G/K)$$

is the isomorphism determined by the G-homeomorphism $[g]$, and

$$\Lambda : k(G/gKg^{-1}) \longrightarrow k'(P/P \cap gKg^{-1})$$

This defines the homomorphism

$$(g)_{\#} : \hat{C}^{G}_{n}(Y, B; k) \longrightarrow \hat{C}^{P}_{n}(Y', B', k').$$

$(g)_{\#}$ commutes with the boundary homomorphism. Also if $p \in P$, then

$$\pi(pg)_{\#} = \pi(g)_{\#} : \hat{C}^{G}_{n}(Y, B; k) \longrightarrow C^{P}(Y', B'; k')$$

where

$$\pi : \hat{C}^{P}_{n}(Y', B'; k') \longrightarrow C^{P}_{n}(Y', B'; k')$$

denotes the natural projection. Thus for each element $Pg \in P \backslash G$, one defined an induced homomorphism

$$(Pg)_{\#} : \hat{C}^{G}_{n}(Y, B; k) \longrightarrow C^{P}_{n}(Y', B', k')$$

by

$$(Pg)_{\#} = \pi(\bar{g})_{\#}$$

, where $\bar{g}$ is any

representative for the right coset Pg. i.e. $\bar{g} \in Pg$.
The homomorphisms $(Pg)_{\#} : S^G(Y,B;k) \to S^P(Y',B';k')$ are chain maps

we define $\tilde{\tau}_{\#} : S^G(Y,B;k) \longrightarrow S^P(Y',B';k')$

to be the chain map $\tilde{\tau}_{\#} = \sum_{i=1}^{s} (Pg_i)_{\#}$

$\tilde{\tau}_{\#}$ induces a chain map $\tau_{\#} : S^G(Y,B;k) \to S^P(Y,B;k')$

which commutes with the chain map induced by a G-map $f : (X, A) \to (Y,B)$, and its corresponding P-map $f' : (X', A') \longrightarrow (Y',B')$, and in turn induces the homomorphism

$$(\tau', \wedge) : H_n^G(Y,B;k) \longrightarrow H_n^P(Y',B';k') .$$

Thus one has the following theorem:

(3.3.5) __Theorem:__ Let P be a closed subgroup of G such that $P \backslash G$ is finite. Let the covariant coefficient systems k' and k for P and G respectively, be as above, and let $\wedge : k \longrightarrow k'$ be a natural transformation of transfer type with respect to $P \hookrightarrow G$. For any G-pair (Y, B) we have transfer homomorphisms

$$(\tau', \wedge) : H_n^G(Y,B;k) \longrightarrow H_n^P(Y',B';k')$$

for every n.

These transfer homomorphisms commute with boundary homomorphism and with homomorphisms induced by G-maps.

One now studies the composite of the transfer homomorphism $(\tau^!, \wedge)$ and the homomorphism $(i, \Phi)_*$ induced by the inclusion $i : P \hookrightarrow G$,

Let $\Phi : k' \longrightarrow k$ be a natural transformation w.r.t. $i : P \hookrightarrow G$ i.e., for each $H' \in \mathcal{H}'$, there is a homomorphism. $\Phi : k'(P/H') \to k(G/i(H'))$ such that id $K' \in \mathcal{C}'$ and $\alpha : P/H' \longrightarrow P/K'$ is a P-map with

$$\alpha(eH') = p_c K' \quad \text{and} \quad \alpha(pH') = pp_c K',$$

then $\qquad \Phi\alpha_* = i(\alpha)\Phi,$

where $\quad i(\alpha) : G/i(P') \longrightarrow G/i(K')$ is given by

$$i(\alpha)(e(i(H'))) = i(p_c)\, i(K').$$

Thus $\qquad i(\alpha)(g\, i(H')) = g\, i(p_0)\, i(K').$

Let $\theta : k \longrightarrow k'$ be a natural transformation with respect to the Identity homomorphism id : $G \longrightarrow G$. With the notations, as above, one has the following theorem.

(3.3.6) Theorem :

If for every $H \in \tilde{\mathcal{H}}$, we assume that

$$\rho_* \circ (\bar{\Phi} \cdot \Lambda) = \Theta \quad , \text{ where}$$

$\rho : G/P \cap H \longrightarrow G/H$ is the natural projection, then the composite homomorphism :

$$H_n^G(Y, B; k) \xrightarrow{(\tau^!, \Lambda)} H_n^P(Y', B', k') \xrightarrow{(i, \bar{\Phi})_*} H_n^G(Y, B; k)$$

equals $s\,\Theta$, s is the number of elements in the right coset space $P \backslash G$.

(3.3.7) Corollary

If Θ = id, the composite equals multiplication by s.

(3.3.8) The transfer homomorphism in cohomology is dual to the construction in homology.

[for details See [10] pp 61]

Let P, G, s, $\tilde{\mathcal{H}}$, $\tilde{\mathcal{H}}'$ etc. are as above and m' , m be the contravariant coefficient systems for $\tilde{\mathcal{H}}'$ and $\tilde{\mathcal{H}}$, respectively over the ring R.

Let Ω : m' $\longrightarrow$ m be a natural transformation of transfer type with respect to the inclusion P $\hookrightarrow$ G, then one constructs transfer homomorphisms

(3.3.9) $(\tau^!, \Omega) : H^n_P(Y', B'; m') \longrightarrow H^n_G(Y, B, m)$

Now, let $\psi : m \to m'$ be a natural transformation with respect to $i : P \hookrightarrow G$, $\theta : m \to m$ be a homomorphism from the contravariant coefficient system m onto itself.

We have the following theorem.

(3.3.10) Theorem:

If for every $H \in \tilde{\mathcal{F}}$, we assume that $(\Omega \circ \psi) p_* = \theta$ where $p : G/P \cap H \longrightarrow G/H$ denotes the natural projection, then we have for any G-pair (Y, B) and for every integer n, the composite homomorphism

$$H^n_G(Y, B; m) \xrightarrow{(i, \psi)^*} H^n_P(Y', B'; m') \xrightarrow{(\tau^!, \Omega)} H^n_G(Y, B; m)$$

equals θ_*

(3.3.11) Corollary

If θ = 1d, this composite equals multiplication by $\mathcal{S}$.

For equivariant singular cohomology with coefficients in a commutative ring coefficient system, Illman defined a kronecker Index and also a cup product, which is shown to be commutative.

We shall now give in brief the definitions of the Kronecker Index and the cup product.

(3.3.12) Definition of Kronecker Index.

A pairing w of k and m, covariant and contravariant coefficient systems for $\mathcal{F}$ over R, consists of a homomorphism of R-modules

$$\omega : m(G/H) \oplus k(G/H) \longrightarrow R \quad \forall\ H \in \mathcal{F}$$

such that if $\alpha : G/H \longrightarrow G/K$, $K \in \mathcal{F}$

be a G-map and $b \in m(G/K)$

$a \in k(G/K)$

then $\omega(b \oplus_R \alpha_*(a)) = \omega(\alpha^*(b) \otimes_R a)$.

Now let $\hat{c} \in \hat{C}^n_G(X;m)$ and $\hat{\sigma} \in \hat{C}^G_n(X;k)$

(3.3.13) Definition: The kronecker index of $\hat{c}$ and $\hat{\sigma}$, denoted by, $\langle \hat{c}, \hat{\sigma} \rangle \in R$, is defined as follows:

$$\langle \hat{c}, \hat{\sigma} \rangle = \omega\left(\sum_{i=1}^{q} \hat{c}(T_i) \otimes_R a_i\right), \text{ where}$$

$$\hat{\sigma} = \sum_{i=1}^{q} T_i \otimes a_i$$

It is a well defined homomorphism of R-modules

$$\langle\ ,\ \rangle : \hat{C}^n_G(X;m) \otimes_R \hat{C}^G_n(X;k) \longrightarrow R .$$

For $c \in \hat{C}^n_G(X, m)$, $\sigma \in C^G_n(X,k)$

one shows that the definition $\langle c, \sigma \rangle = \langle c, \hat{\sigma} \rangle$,
where $\hat{\sigma} \in \hat{C}^G_n(X;k)$ is any representative for σ,
gives a well-defined homomorphism

$$\langle\ ,\ \rangle : C^n_G(X, m) \otimes_R C^G_n(X, k) \longrightarrow R .$$

For a G-pair (X,A), the homomorphism in (3.3.11) induces a pairing on the cohomology level, namely

(3.3.14) $\langle\ ,\ \rangle : H^n_G(X, A; m) \otimes H^G_n(X, A; k) \longrightarrow R$

which is a homomorphism of R-modules and we call it the kronecker index.

(3.3.15) <u>Definition of cup product</u>

Here we consider a commutative ring coefficient system m.
Let T : $\Delta^{n+p} \times G/K \longrightarrow X$, $K \in \mathfrak{F}$

be an equivariant (n+p)- Singular simplex.

Let α_n and β_n denote respectively the front n-face and back n-face of $\Delta^{n+p} \times G/K$

where $\alpha_n((x_0, \dots, x_n), gK) = ((x_0, \dots, x_n, 0, \dots, 0), gK)$

and $\beta_n((x_0, \dots, x_n), gK) = ((0, 0, \dots, 0, x_0, \dots, x_n), gK)$

(3.3.16) Definition

For $\hat{c} \in \hat{C}^{n}_{G}(X;m)$

and $\hat{c}_1 \in \hat{C}^{p}_{G}(X;m)$

the cup product $\hat{c} \cup \hat{c}_1 \in \hat{C}^{n+p}_{G}(X;m)$

is defined by ;

$$\hat{c} \cup \hat{c}_1 (T) = (\hat{c}(T\alpha_n))(\hat{c}_1(T\beta_n)) \in m(G/K)$$

(3.3.17) Remark

For $c \in C^{n}_{G}(X;m)$ and $c_1 \in C^{p}_{G}(X;m)$

one shows that the cup product

$$c \cup c_1 \in C^{n+p}_{G}(X;m)$$

(3.3.18) Remark

For a G-pair (X,A), and for $c \in C^{n}_{G}(X,A;m)$

and $c_1 \in C^{p}_{G}(X,A,m)$

we define the cup product

$$c \cup c_1 \in C^{n+b}_{G}(X,A;m).$$

This defines a homomorphism of R-modules

(3.3.19) $\cup : C^{n}_{G}(X,A,m) \otimes_R C^{p}_{G}(X,A;m) \to C^{n+b}_{G}(X,A,m)$

One establishes the following formula by standard calculations

(3.3.20) $\delta(c \cup c_1) = (\delta c) \cup c_1 + (-1)^n c \cup (\delta c_1)$.

that is, the homomorphisms '$\cup$' form a cochain map.

Thus one gets a cup product on the cohomology level.

(3.3.21) <u>Remark</u>

The cup product is commutative in the sense that if

$y \in H^n_G(X, A; m)$ and $y_1 \in H^p_G(X, A; m)$, then

$$y \cup y_1 = (-1)^{np} (y_1 \cup y).$$

(3.3.22) <u>Remark</u>

We also have the notions of cup product and transfer in the Eilenberg - Borel setting of equivariant cohomology theory, which are given below.

Let (π, G, K) and (π', G', K'). be two algebraic triples. Then one has a triple $(\pi \times \pi', G \otimes G', K \otimes K')$ and there is a map

$$: C^*_\pi(K, G) \otimes C^*_{\pi'}(K'; G') \to C^*_{\pi \times \pi'}(K \otimes K', G \otimes G')$$

which gives a <u>cross product</u> or the external product.

(3.3.23) $H^*_\pi(K; G) \otimes H^*_{\pi'}(K'; G') \longrightarrow H^*_{\pi \times \pi'}(K \otimes K', G \otimes G')$

One can define a diagonal map

$$d : (\pi, G \otimes G', K) \longrightarrow (\pi \times \pi, G \otimes G', K \times K)$$

where π acts on $G \otimes G'$ in the first triple by the diagonal action.

Hence, one has a map

$$d^* : H^*_{\pi \times \pi}(K \times K, G \otimes G') \longrightarrow H^*_{\pi}(K; G \otimes G')$$

Combining with the external product given in (3.3.23) one has the cup product pairing

(3.3.24) $$H^*_{\pi}(K, G) \otimes H^*_{\pi'}(K'; G') \longrightarrow H^*_{\pi}(K; G \otimes G').$$

The transfer homomorphism in the Eilenberg-Borel setting is described as follows:

Let P be a subgroup of finite index in π and let $K = C_*(X)$ be a π-complex (c.f § 1.2) Let G be a π-module and $C^*_{\pi}(C_*(X), G) = Hom^{\pi}(C_*(X), G)$.

be the cochain complex. One has the inclusion

$$i : C^*_{\pi}(K; G) \longrightarrow C^*_{P}(K; G)$$

inducing a map $i^* : H^*_{\pi}(K; G) \longrightarrow H^*_{P}(K; G)$.

(3.3.25) The transfer homomorphism

$$\tau : C^*_{P}(K; G) \longrightarrow C^*_{\pi}(K; G)$$

is defined as follows:

Let $u \in C_p^*(K;G)$ and $c \in C_*(X) = K$

then $\tau u(c) = \sum_i u(\alpha_i^{-1} c)$

where $\{\alpha_i\}$ is a set of left coset representatives of P in Π.

K is a Π-complex so $\alpha_i^{-1} c \in$ K. One can show that the definition of τ is independent of the choice of reprsentatives and τ is a chain map.

τ induces a map

$$\tau : H_P^*(K;G) \longrightarrow H_\Pi^*(K;G)$$

which is natural for equivariant maps of Π-complexes K.

(3.3.26) If $[\Pi : P]$ denotes the index of P in Π, that is, the number of elements in the Set $\{\alpha_i\}$, then the composition

$$C_\Pi^*(K;G) \xrightarrow{i} C_P^*(K;G) \longrightarrow C_\Pi^*(K;G)$$

is multiplied by $[\Pi : P]$.

Therefore the definition of transfer by Illmann extends that of Borel's.

§ 3.4 Generalised equivariant Cohomology theory.

A generalised equivariant cohomology theory can be constructed using G-spectra (c.f [5] IV.1)

(3.4.1) Definition A G-spectrum is a collection

$\underline{\underline{Y}} = \{Y_n \mid n \in \mathbb{Z}\}$ of G-spaces, together with equivariant maps $\varepsilon_n : SY_n \longrightarrow Y_{n+1}$.

Let $\underline{\underline{Y}}$ be a G-spectrum and X be a G-space and let $[S^{k-n}X, Y_k]_G$ denote the set of G-homotopy classes of G-maps.

Consider the homomorphisms η_k, defined by the composite

$$[S^{k-n}X ; Y_k]_G \xrightarrow{\text{suspension } S} [S^{k-n+1}X ; SY_k]_G \xrightarrow{\varepsilon_k^{\#}} [S^{k-n+1}X ; Y_{k+1}]_G .$$

The groups $[S^{k-n}X, Y_k]_G$, together with the homomorphisms η_k form a direct system.

(3.4.2) Definition

The nth equivariant cohomology group is defined as

$$\tilde{H}^n_G(X;Y) = \varinjlim_k [S^{k-n}X ; Y_k]_G = \varinjlim_k [S^k X, Y_{n+k}]_G$$

Now, $[S^k X, Y_{n+k}]_G \cong [X, \Omega^k Y_{n+k}]_G$.

If X and Y are G-spaces, let F(X,Y) denote the space of all (base point preserving) maps from X to Y in the compact open topology. F(X,Y) is a G-space with the following G-action.

If f: X $\longrightarrow$Y and $g \in G$, we get $g(f)(x) = g(f(g^{-1}x))$

The set $F(X,Y)^G$ of stationary points of G on F(X,Y) is just the set of equivariant maps from X to Y.

Let us write $F(X,Y)^G = E(X,Y)$. Let X be locally compact.

Let $\underline{\underline{E}}(X,Y)$ be a spectrum consisting of the spaces $E(X,Y_n)$

Then from (3.4.2) we have

$$(3.4.3) \quad \pi_{-n}(\underline{\underline{E}}(X,\underline{\underline{Y}})) = \varinjlim_k \pi_{k-n}(E(X;Y_k)).$$

If $A \subset X$ is invariant under G, then for any G-space W, there is the following exact sequence

$$(3.4.4) \quad [X \cup CA\,;W]_G \longrightarrow [X;W]_G \longrightarrow [A;W]_G.$$

where CA is the mapping cone of A. If (X,A) is a pair of G-Complexes then $X \cup CA$ has the same equivariant homotopy type as does X/A.

Taking $W = \Omega^k Y_{n+k}$ and passing to the limit over k, we obtain the following exact sequence on the category of G-Complexes with base point.

$$(3.4.5) \quad \widetilde{H}^n_G(X/A\,;\underline{\underline{Y}}) \longrightarrow \widetilde{H}^n_G(X;\underline{\underline{Y}}) \longrightarrow \widetilde{H}^n_G(A;\underline{\underline{Y}}).$$

Since, there is a natural isomorphism

$S^{k-n}X \simeq S^{k-(n+1)}SX$, we obtain a natutal isomorphism

$$\sigma_k : [S^{k-r}X, Y_k]_G \xrightarrow{\approx} [S^{k-(n+1)}SX ; Y_k]_G$$

These commute with η_k, hence define a natural isomorphism

(3.4.6) $$s^* : \tilde{H}^n_G(X ; \underline{Y}) \longrightarrow \tilde{H}^{n+1}_G(SX ; \underline{Y})$$

Thus $\tilde{H}^*_G(X, \underline{Y})$ defines an equivariant cohomology theory on $\mathcal{G}_0$ (the category of G-Complexes with base point).

(3.4.7) <u>Remark</u> In Chapter 4, we will give a RO(G) stable version of the above concept.

§ 3.5 <u>The spectral sequence for an equivariant cohomology theory</u>

Let $\{{}_G H^*, \delta^*\}$ be any equivariant cohomology theory and let K be a G-Complex of dimension $N < \infty$ If K is not finite, then we shall assume that ${}_G H^*$ satisfies the following axiom:

(3.5.1) If S is a discrete G-set with orbits S_α, then

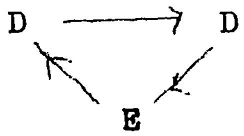
$$\prod i_\alpha^* : {}_G H^n(S) \longrightarrow \prod {}_G H^n(S_\alpha)$$

is an isomorphism, where $i_\alpha : S_\alpha \longrightarrow S$ is the inclusion.

Let $\{K_p\}$ be a sequence of G-sub complexes such that $K_p = K^p$, the p-skeleton of K.

Put
$$\begin{cases} E^{p,q} = \mathcal{H}^{p+q}(K^p, K^{p-1}) \\ D^{p,q} = \mathcal{H}^{p+q-1}(K^{p-1}) \end{cases}$$

Then the exact cohomology sequence of the pair (K^P, K^{P-1}) provides an exact couple:



The differential d_1 is the composition

$$E_1^{p,q} = \mathcal{H}^{p+q}(K^p, K^{p-1}) \longrightarrow \mathcal{H}^{p+q}(K^p) \xrightarrow{\ \delta\ } \mathcal{H}^{p+q+1}(K^{p+1}, K^p) = E_1^{p+1,q}$$

given by This exact couple and the spectral sequence converges to the graded group associated with the filtration

$$J^{p,q} = \ker\{\mathcal{H}^{p+q}(K) \longrightarrow \mathcal{H}^{p+q}(K^{p-1})\}$$

of $\quad J^{p+q} = \mathcal{H}^{p+q}(K)$

$$E_1^{p,q} = \mathcal{H}^{p+q}(K^p, K^{p-1}) \approx \mathcal{H}^{p+q}(K^p/K^{p-1})$$

Since $\quad K^p/K^{p-1} \approx S^p C_p^+$

where, $S^p C_p^+$ is the pth suspension of the discrete G-set C_p^+ , C_p is the set of all p-cells of K, '+' denotes an external base point.

Thus, $\quad E_1^{p,q} \approx \mathcal{H}^{p+q}(S^p C_p^+) \approx \tilde{H}^q(C_p^+) \approx \mathcal{H}^q(C_p)$

Let $h^q \in \mathcal{C}_G$ denote the coefficient system defined by $h^q(G/H) = \mathcal{H}^q(G/H)$; for any equivariant map $f: G/H \longrightarrow G/K$

$$h^q(f) : \mathcal{H}^q(G/K) \longrightarrow \mathcal{H}^q(G/H)$$

Then $h^q(G/H) = \mathcal{H}^q(G/H) = \tilde{\mathcal{H}}^q((G/H)^+)$.

One has an isomorphism

(3.5.2) $\alpha : \tilde{\mathcal{H}}^q(C_p^+) \longrightarrow C_G^p(K; h^q)$, on to Bredon's equivariant chain groups,

and verifies that under this isomorphism the differential $d_1 : E_1^{p,q} \longrightarrow E_1^{p+1,q}$ becomes, upto sign, the coboundary.

Thus we have:

(3.5.3) $E_2^{p,q} \approx H_G^p(K; h^q)$, the Bredon's equivariant cohomology group.

(for details see [5] pp IV.10)

The spectral sequence converges (when dim K $< \infty$) to the graded group associated with the above filtration of $\mathcal{H}^{p+q}(K)$.

(3.5.4) For any equivariant cohomology theory $\{\mathcal{H}^*, \delta^*\}$ and a G-Complex K with dimension N $< \infty$, there is a spectral sequence with $E_2^{p,q} \approx H_G^p(K; h^q)$ which converges to the graded group associated with some filtration of $\mathcal{H}^{p+q}(K)$.

(3.5.5) <u>The classical uniqueness Theórem</u>

Suppose $\mathcal{H}^*$ is an equivariant cohomology theory satisfying the dimension axiom.

Let $h \in \mathcal{C}_G$ denote the coefficient system defined by $h(G/H) = \mathcal{H}^0(G/H)$. Let K be a finite dimensional G-complex. (If K is infinite, we assume that the condition (3.5.$\ast$) is satisfied.)

In this case, the spectral sequence degenerates for $r \geqslant 2$.

In fact, $E_2^{p,q} = \begin{cases} H_G^p(K;h) & , q = 0 \\ 0 & , q \neq 0 \end{cases}$.

It follows that $\mathcal{H}^p(K) \approx H_G^p(K;h)$

and the isomorphism is shown to be natural.

Thus this is the only equivariant cohomology theory having coefficients h. For general $h \in \mathcal{C}_G$, h is the coefficient system of the cohomology theory $H_G^*(K;h)$, that is, there is a natural isomorphism $h(G/H) \approx H_G^0(G/H;h)$.

CHAPTER IV

RO(G) graded equivariant (co)homology.

The advent of equivariant stable homotopy (Segal etc.) gave the motivation for the development of equivariant cohomology theories which are stable in this general setting.

Equivariant K-theory and equivariant cohomotopy theories provided such examples, using suitable G-spectra. The ordinary cohomology theories described in the last chapters do not come with generalised stable character. There is, therefore, a need to develop such a theory.

The construction of such a theory is under way [30]. The purpose of this chapter is to give a brief report on the various facets of the RO(G) graded ordinary cohomology theory.

We give a brief explanation of the term 'stable' described above; indicate the usefulness of such a theory; and the status of its calculation.

In §1 we give the definition of stable equivariant homotopy groups and the (co)homology theory given by this.

In §2 we give the necessity of having a RO(G) graded (co)homology theory (Poincaré duality).

§3 is devoted to some results demonstrating the usefulness of having an RO(G) graded (co)homology (conner conjecture) and also a necessary condition (in terms of transfer), of the existence of an RO(G) graded (co)homology theory.

In § 4 we indicate a line of construction of the ordinary RO(G) - graded (co)homology theory.

§ 4.1 Stable equivariant homotopy groups.

(4.1.1) Definition ([23])

Let G be a compact Lie group. Consider the directed category of complex G-modules $U, V, W, \ldots\ldots$ We write $V < W$ if there exists U and an isomorphism $U \oplus V \cong W$. Let $S^V = V \cup \{\infty\}$ be the one point compactification of V with ∞ as base point.

Let X and Y be pointed G-spaces and let $[X,Y]^0_G$ denote the set of G-homotopic pointed G-maps: $X \to Y$. Suppose $V < W$ and choose an isomorphism $\varphi : U \oplus V \cong W$.

To each object V of the category, we associate the set $[S^V \wedge X; S^V \wedge Y]^0_G$ and to any morphism $i : V \to W$ associate the suspension map $b_{W,V}$ which is the composite

$$[S^V \wedge X, S^V \wedge Y]^0_G \xrightarrow{(1)} [S^U \wedge S^V \wedge X, S^U \wedge S^V \wedge Y]^0_G \xrightarrow[\cong]{(2)} [S^{U \oplus V} \wedge X, S^{U \oplus V} \wedge Y]^0_G \xrightarrow[\cong]{(3)} [S^W \wedge X, S^W \wedge Y]^0_G$$

where (1) is the suspension with $id(S^U)$. (2) is induced by the canonical homeomorphism $S^U \wedge S^V \longrightarrow S^{U \oplus V}$ and (3) is induced by the isomorphism $\varphi : U \oplus V \cong W$. One can show that $b_{W,V}$ is independent of the choice of φ and if $U < V < W$, then $b_{W,V} b_{V,U} = b_{W,U}$.

The direct limit of this system is defined to be the set of stable equivariant maps, $\{X,Y\}_G$.

Thus $\{X,Y\}_G = \varinjlim_V [S^V \wedge X, S^V \wedge Y]^0_G$.

$\{X,Y\}_G$ has the structure of an abelian group. This group is denoted by $\omega_0^G(X,Y)$.

Let us define,

$$\omega_n^G(X,Y) = \omega_0^G(S^n X;Y), \; n \geq 0$$
$$= \varinjlim_V [S^n \wedge S^V \wedge X, S^V \wedge Y]$$

$\omega_n^G(-,-)$ are functors (for morphisms $(X,Y) \xrightarrow{(f_X, f_Y)} (X',Y')$ $\omega_n^G(f_X, f_Y)$ is defined in the natural way) contravariant in the first coordinate and covariant in the second coordinate. These functors will be used to define homology and cohomology theories in what follows.

We now give the defininition of a (stable) spectrum ([23]).

(4.1.3) A spectrum consists of the following data:

(a) A pointed G-space $X(U)$ one for each complex G-module U.

(b) A pointed G-map $\varphi_{W,V} : S^U \wedge X(V) \longrightarrow X(W)$. whenever $W \cong U \oplus V$

(c) A pointed G-homotopy equivalence $h(W,V) : X(V) \longrightarrow X(W)$, wherever V and W are isomorphic. $h(W,V)$ satisfies the following axioms.

(i) $h(W_3, W_2)\, h(W_2, W_1) = h(W_3, W_1)$.

So $X(W)$ depends only on the isomorphism class of W.

(ii) If U and U' are isomorphic and $k : S^U \longrightarrow S^{U'}$ is the map induced by an isomorphism, then the following diagrams are commutative upto pointed G-homotopy.

(I)

$$\begin{array}{ccc}
S^U \wedge X(V_1) & \xrightarrow{\varphi_{W_1,V_1}} & X(W_1) \\
\downarrow{\scriptstyle k \wedge h(V_2,V_1)} & & \downarrow{\scriptstyle h(W_2,W_1)} \\
S^{U'} \wedge X(V_2) & \xrightarrow[\varphi_{W_2,V_2}]{} & X(W_2)
\end{array}$$

(II)

$$\begin{array}{ccc}
S^U \wedge S^V \wedge X(W) & \xrightarrow{1d(S^U) \wedge \varphi_{V\oplus W,W}} & S^U \wedge X(V \oplus W) \\
\cong \downarrow & & \downarrow{\scriptstyle \varphi_{U\oplus V\oplus W,\, V\oplus W}} \\
S^{U\oplus V} \wedge X(W) & \xrightarrow{\varphi_{U\oplus V\oplus W,W}} & X(U\oplus V\oplus W)
\end{array}$$

We will now construct (co)homology theory given by a stable G-spectrum $\mathcal{X} = (X(U), \varphi_{U,V})$

Let X and Y be two pointed G-spaces. Consider the suspension map as the composite

$$[S^V \wedge X, X(V) \wedge Y]^0_G \xrightarrow{(1)} [S^U \wedge S^V \wedge X, S^U \wedge X(V) \wedge Y]^0_G$$
$$\xrightarrow{(2)} [S^{U \oplus V} \wedge X, X(W) \wedge Y]^0_G \xrightarrow{(3)} [S^W \wedge X, X(W) \wedge Y]^0_G ,$$

where the map (1) is suspension with $id(S^U)$, the map (2) is induced by the canonical homeomorphism $S^U \wedge S^V \cong S^{U \oplus V}$ and (3) by an isomorphism $U \oplus V \cong W$.

With these maps $\sigma_{W,V}$ the G-homotopy sets of pointed G-maps $[S^V \wedge X; X(V) \wedge Y]^0_G$ form a direct system.

(4.1.4) We denote $\mathcal{X}^G_0(X ; Y) = \varinjlim [S^V \wedge X; X(V) \wedge Y]^0_G$

$\mathcal{X}^G_0(X;Y)$ carries a natural structure of an abelian group. It is contravariant in X and covariant in Y.

(4.1.5) We define.

$$\mathcal{X}^G_n(X;Y) = \mathcal{X}^G_0(S^n X; Y), \quad n \geq 0$$
$$= \varinjlim_V [S^n \wedge S^V \wedge X, X(V) \wedge Y]^0_G$$

$\mathcal{X}^G_*(- , -)$ satisfy the following axioms of an equivariant (co)homology theory, unstable if $* \in Z$ and stable if $* \in RO(G)$ (see (4.1.8)).

First we recall the equivariant (co)homology theory as studied so far.

Let G-Com^0 denote the category of pointed spaces which admit the structure of a pointed G-Complex.

Let 'Abel' be the category of Abelian groups.

(4.1.6) An unstable equivariant homology theory consists of a sequence $\{h_n \mid n \in \mathbb{Z}\}$ of covariant functors $h_n : G\text{-}Com^0 \longrightarrow$ Abel and a sequence of natural transformations

$$\sigma_n : h_n(X) \longrightarrow h_{n+1}(SX),\ n \in \mathbb{Z}\ (\text{where } SX = S^1 \wedge X)$$

such that the following holds:

(1) h_n is homotopy invariant.

(2) σ_n is an isomorphism (suspension isomorphism)

(3) for any map $f : X \longrightarrow Y$ in G-Com^0, the sequence

$$h_n(X) \xrightarrow{h_n(f)} h_n(Y) \xrightarrow{h_n(C_f)} h_n(C_f)$$

is exact.

We now give the axioms of an equivariant (stable) (co)homology theory. ([25])

(4.1.7) A stable equivariant (co)homology theory, graded over an abelian group A, consists of the following data:

(a) A family $\{h_\alpha \mid \alpha \in \mathcal{A}\}$ of covariant functors

$$h_\alpha : G\text{-}\mathcal{C}om^0 \longrightarrow \text{Abel}.$$

(b) A homomorphism : $R(G) \to \mathcal{A}$ from the (additive group of the) complex representation ring $R(G)$ into $\mathcal{A}$

(c) A homomorphism : $\mathbb{Z} \longrightarrow \mathcal{A}$

(d) for each complex representation V, a family of natural transformations

$$\sigma^V : h_\alpha(X) \longrightarrow h_{\alpha+V}(S^V \wedge X), \alpha \in \mathcal{A}$$

(e) A family of natural transformations

$$\sigma_\alpha : h_\alpha(X) \longrightarrow h_{\alpha+1}(S^1 \wedge X), \alpha \in \mathcal{A}$$

These data satisfy the following axioms:

(i) for each $\alpha \in \mathcal{A}$ $(h_{\alpha+n}(-) \mid n \in \mathbb{Z})$ and $(\sigma_{\alpha+n} \mid n \in \mathbb{Z})$ form an unstable homology theory

(ii) σ^V is an isomorphism (suspension by V)

(stability)

(iii) The diagram below is commutative

$$\begin{array}{ccc}
h_\alpha(X) & \xrightarrow{\sigma^V} & h_{\alpha+V}(S^V \wedge X) \\
\downarrow \sigma^{U \oplus V} & & \searrow \sigma^U \\
h_{\alpha+U+V}(S^{U \oplus V} \wedge X) & \xrightarrow{\cong} & h_{\alpha+U+V}(S^U \wedge S^V \wedge X)
\end{array}$$

(iv) The diagram below is commutative

$$\begin{array}{ccc} h_\alpha(X) & \xrightarrow{\sigma_\alpha} & h_{\alpha+1}(S^1\wedge X) \\ \downarrow \sigma^V & & \downarrow \sigma^V \\ h_{\alpha+V}(S^V\wedge X) & & h_{\alpha+1+V}(S^V\wedge S^1\wedge X) \\ & \searrow \sigma_\alpha \qquad \swarrow T_* & \\ & h_{\alpha+V+1}(S^1\wedge S^V\wedge X) & \end{array}$$

T_* is induced by the twisting map

$$T:\ S^V\wedge S^1 \longrightarrow S^1\wedge S^V$$

(v) If $U \underset{\text{iso}}{\cong} V$, the following diagram is commutative

$$\begin{array}{ccc} & \overset{\sigma^U}{\nearrow} & h_{\alpha+U}(S^U\wedge X) \\ h_\alpha(X) & & \downarrow \cong \circledast \\ & \underset{\sigma^V}{\searrow} & h_{\alpha+V}(S^V\wedge X) \end{array}$$

The isomorphism (*) is induced by an isomorphism $U \longrightarrow V$ and the corresponding homeomorphism $S^U \longrightarrow S^V$.

(4.1.8) Remarks

1. One may think of σ^V as defining a natural isomorphism from the unstable homology theory $(h_{\alpha+n}(-) \mid n\in\mathbb{Z})$ $(\sigma_{\alpha+n} \mid n\in\mathbb{Z})$ into the unstable homology theory $(h_{\alpha+V+n}(-) \mid n\in\mathbb{Z})$, $(T_*\sigma_{\alpha+V+n} \mid n\in\mathbb{Z})$.

2. Equivariant unstable or stable cohomology theories are defined in the obvious dual way.

3. Equivariant stable (co)homology theory for a pair (X,Y), where (X,Y) carries the structure of a G-complex and a subcomplex is defined as $h_\alpha(X,Y) = \tilde{h}_\alpha(X/Y)$

[The homology theory of pointed spaces is called a reduced homology theory and is denoted by $\tilde{h}(X)$].

4. If we take A = RO(G), the real representation ring of G, then the corresponding (co)homology theory is called an RO(G)-graded (co) homology.

An RO(G)-graded (co)homology theory corresponding to a stable spectrum (K(V), Qu,v) is defined by

$$\mathcal{X}^G_V(X;Y) = \mathcal{X}^G_0(S^V X, Y).$$

[See § 4.1.4 for the definitions of $\mathcal{X}^G_0(-,-)$]

(4.1.10) <u>Remarks:</u>

1. The definition of an unstable equivariant (co)homology theory given by an unstable G-spectra has been described in Chapter III.

2. If we take the G-spectrum BU(V) we get stable equivariant K-theory.

3. If we take the G-spectrum $S(V)$ (the sphere spectrum) we get stable cohomotopy theory.

4. The stable G-spectrum for ordinary (co)homology theory is still in the process of construction and we only give a sketch of it and the usefulness of the cohomology theory given by it, in the rest of the chapter.

§ 4.2. Poincaré duality in Equivariant Category

In this section we will describe the necessity of having a RO(G)-graded ordinary (singular) (co)homology theory.

(4.2.1) The necessity of having a RO(G)-graded singular (co)homology theory is best described by the following problem:

To develop an equivariant analogue of Poincaré duality theorem (14) which states that

If M is compact and oriented then $H^i M \cong H_{n-i} M$.

To obtain an equivariant analogue, we need the notion of an orientation of a real G-vector bundle ξ, such as the tangent bundle of a smooth G-manifold.

Let $T\xi$ be the Thom-complex of ξ. $T\xi$ is obtained from the total space by one point compactification of fibres and identification of the resulting points at ∞.

If the base space of ξ is a point then ξ is a real G-representation V and $T\xi$ is its one point compactification S^V. If the base space is an orbit G/H, then $T\xi = G^+ \wedge_H S^W$ for an H-representation W.

Let k_G^* be a ring valued cohomology theory on G-spaces. If Y is an H-space then by Borel's definition of cohomology $\tilde{k}_H^*(Y) = \tilde{k}_G^*(Y_H) = \tilde{k}_G^*(G^+ \wedge_H Y)$ gives the associated cohomology theory on Y.

$$\tilde{k}_G^*(T\xi) = \tilde{k}_G^*(G^+ \wedge_H S^W) = \tilde{k}_H^*(S^W).$$

Now, a k_G^*- orientation of ξ is a cohomology class u, where $u \in k_G^*(T\xi)$, whose restrictions to Thom Complexes of base orbits G/H are 'generators'.

This definition is meaningful only when $k_H^*(S^W)$ is free with one generator over $k_H^*(S^0)$. For general $\mathbb{Z}$-graded cohomology theories, this is not true. So Bredon's theory developed in the earlier chapters, which is an integrally graded cohomology theory, is not adequate to develop Poincaré duality.

The RO(G)-graded cohomology theories come with suspension isomorphisms

$$\tilde{k}_G^{\alpha}(X) \cong \tilde{k}_G^{\alpha+V}(\Sigma^V X), \quad \text{for } \alpha \in RO(G)$$

and G-representations V, where $\Sigma^V X = X \wedge S^V$,

as we have seen in the last section.

More generally, one has an associated homology theory and there are isomorphisms

$$\tilde{k}^{\alpha}_{G}(G^{+} \wedge_{H} \Sigma^{W} Y) \cong \tilde{k}^{\alpha+W}_{H}(Y)$$

and $$\tilde{k}_{\alpha}^{G}(G^{+} \wedge_{H} \Sigma^{W} Y) \cong \tilde{k}^{H}_{\alpha-W-T}(Y)$$

for H-spaces Y and H-representation W, where T is the tangent H-representation of G/H at eH.

For such theories, the notion of a k_G^*- orientation of a real G-bundle ξ makes good sense, because

$$\tilde{k}_{H}^{\alpha+W}(S^W) \cong \tilde{k}^{\alpha}_{H}(S^0),$$

the Stable equivariant suspension isomorphism

So, $\tilde{k}^{\alpha+W}(S^W)$ is free on one generator over $\tilde{k}_H^{\alpha}(S^0)$.

Thus, a k_G^*-orientation of the tangent bundle of a smooth G-manifold implies Poincaré duality exactly as in the nonequivariant case.

§ 4.3 Usefulness of RO(G)-graded ordinary cohomology and conditions of its existence

RO(G)-graded ordinary cohomology has proved to be quite useful in giving easy proofs of deep results which were earlier proved using different and lengthy techniques, for example the following theorem has an easy proof using RO(G)-graded ordinary cohomology theory. (See [13])

(4.3.1) Theorem Let X be an ordinary G-space and H be a closed subgroup of G; $\pi : X/H \longrightarrow X/G$ be the natural projection. Then for any coefficient group R, there exists a natural transfer homomorphism

$$\tau : H^n(X/H\,;R) \longrightarrow H^n(X/G\,;R),\ n \geq 0$$

such that $\tau . \pi^*$ is multiplication by the Euler characteristic $\chi(G/H)$. It has the following consequence.

(4.3.2) Connor Conjecture

Let X be finite dimensional and have finitely many orbit types.

Then $\tilde{H}^*(X/G, R) = 0$ if $\tilde{H}^*(X;R) = 0$

(4.3.3) Remarks

(i) Theorems (4.3.1) and (4.3.2) were first proved by

Oliver using Čech cohomology and totally different techniques.

(ii) If H has finite index in G, a definition of a transfer was given by Illman (See § 3.3.1)

Now, we give a necessary (and sufficient under certain conditions) condition (in terms of transfer) for the existence of ordinary RO(G)-graded cohomology theory.

Let H be a closed subgroup of G, embed G/H as a G-subspace of a G-representation V.

Let T be the tangent H-space of G/H at eH and write $V = T \oplus T^{\perp}$ as an H-space.

The Normal G-bundle of the embedding $G/H \hookrightarrow V$ is the projection : $G \times_H T^{\perp} \longrightarrow G/H$.

and $G \times_H T^{\perp}$ may be embedded as a normal tube in V.

Applying Pontryagin - Thom Construction, one obtains a G-map

$$(4.3.4)\quad t : S^V \longrightarrow G^+ \wedge_H S^{T^{\perp}} \subset G^+ \wedge_H S^V \simeq (G/H)^+ \wedge S^V.$$

Let $\pi : (G/H)^+ \wedge S^V \longrightarrow S^V$ be the projection.

Hopf's theorem implies that the degree of the composite map

$$\pi . t : S^V \longrightarrow S^V$$ is the Euler characteristic $\chi(G/H)$.

Consider the Bredon cohomology $\tilde{H}^*_G(- ; M)$ with coefficient system M. For $H \subset G$ there is a homomorphism $M(i) : M(G/G) \longrightarrow M(G/H)$, where $i : G/H \hookrightarrow G/G$ is the inclusion.

Now,
$$\begin{cases} M(G/H) = \tilde{H}^0_G((G/H)^+, M) \\ M(G/G) = \tilde{H}^0_G(S^0; M) \end{cases}$$

Suppose that Bredon cohomology extends to an $RO(G)$-graded cohomology theory.

Then $\tilde{H}^0_G((G/H)^+, M) \cong \tilde{H}^V_G((G/H)^+ \wedge S^V, M)$

and $\tilde{H}^0_G(S^0; M) \cong \tilde{H}^V_G(S^V; M)$.

The G-map t: $S^V \longrightarrow (G/H)^+ \wedge S^V$

induces a homomorphism

$$t^* : \tilde{H}^V_G((G/H)^+ \wedge S^V; M) \longrightarrow \tilde{H}^V_G(S^V, M),$$

that is, a homomorphism

(4.3.5) $t^* : M(G/H) \longrightarrow M(G/G)$

Similar construction shows that:

(4.3.6) <u>Theorem</u>. If the Bredon cohomology extends to an RO(G)-graded cohomology then the coefficient system M admits a transfer homomorphism

$$\tau : M(G/K) \longrightarrow M(G/H), \text{ when } K \subset H \subset G$$

such that if $i : G/K \hookrightarrow G/H$ is the inclusion map then

$\tau \circ M(i) : M(G/H) \longrightarrow M(G/H)$ is the multiplication by $\chi(H/K)$.

This condition is also sufficient for the extension of $H^*_G(-;M)$ to an RO(G) graded cohomology theory (for suitable M).
(for details see [13]).

§ 4.4 <u>Equivariant RO(G)- graded singular (co)homology</u>

Refer to the RO(G)-graded cohomology theory defined in terms of stable G-spectra in §4.1. The purpose of this section is to indicate the specific stable G-spectrum which will give rise to the singular RO(G)-graded (co)homology. At the end of the section we will give a little more explicit definition of RO(G)-graded (co)homology for cw(V)-complexes as given by Waner ([25]) .

(4.4.1) To fix notations let us indicate the stable - equivariant categories we will work with and recall the definition of Bredon's equivariant cohomology in this general setting.

Let $\mathcal{U} = \oplus V_i^{\infty}$, where V_i^{∞} is the sum of countably many copies of V_i and $\{V_i\}$ runs through a set of representatives for the irreducible real representations of G.

A G-spectrum $\underline{\underline{E}}$ is a collection of based G-spaces EV indexed on the finite dimensional invariant subspaces V of $\mathcal{U}$ together with G-homeomorphisms

$$EV \cong \Omega^{W} E(V+W), \quad \text{for } V \text{ orthogonal to } W.$$

Maps $\underline{\underline{E}} \longrightarrow \underline{\underline{F}}$ are collections of G-maps $EV \longrightarrow FV$ compatible with the given homeomorphisms. Homotopies are families of maps parametrised by the unit interval. A map is a weak equivalence if each fixed point map: $(EV)^H \longrightarrow (FV)^H$ is a weak equivalence.

(4.4.2)
The stable category HS_G is obtained from the homotopy category of G-spectra by adjoining formal inverses to the weak equivalences. There is a notion of a G-CW spectrum, and HS_G is equivalent to the homotopy category of G-CW spectra and cellular maps.

(See [12])

(4.4.3) Let $\mathcal{O}$ denote the full sub-category of HS_G with objects $G/H^+ \wedge S$.

(4.4.4) for a G-spectrum Y and an integer n, define a Mackey functor $M = \pi_n Y$ on the category $\mathcal{O}$ such that

$$\pi_n Y (G/H^+ \wedge S) = [G/H^+ \wedge S^n, Y]_G$$

where $S^n = \Sigma^n S$.

(4.4.5) Definition

Consider a G-CW spectrum Y with nth skeleton Y^n.
Define $C_n Y = \pi_n (Y^n/Y^{n-1})$.

Y^n/Y^{n-1} is a wedge of G-spectra of the form $G/H^+ \wedge S^n$

Then $C_* Y$ is a complex in the abelian category of Mackey functors.

(4.4.6) Definition

The cellular cochain complex is defined by

$$C^*(Y;M) = \mathrm{Hom}_{\mathcal{O}} (C_* Y , M)$$

Passing to homology one obtains a $\mathbb{Z}$ -graded cohomology theory $H^*(Y;M)$ on G spectra, with coefficients in M.

(4.4.7) The 0^{th} term of the cohomology theory $H^*(Y;M)$, that is, $H^0(Y;M)$ is represented by an Eilenberg MacLane spectrum $K(M, O)$.

The RO(G) graded theory, determined by $K(M,0)$ extends the $\mathbb{Z}$-graded cellular theory $H^*(Y;K)$ to an RO(G)-graded Bredon cohomology with coefficients in M.

(4.4.8) Remarks

1. We have a dual construction of RO(G)-graded homology theories $H_*(Y;N)$ with coefficients in covariant functors $N : \mathcal{O} \longrightarrow Ab$.

2. For finite G, $\mathcal{O}$ is self-dual and then the two kinds of coefficient systems are equivalent.

3. For general G, a quite different kind of Eilenberg-Mac Lane spectrum $K(N,0)$ represents these homology theories.

We now make little more explicit the definition of RO(G) -graded (co)homology of a G-CW(V) Complex .

(4.4.9) Definition:

Let $\mathfrak{U}$ be the orthogonal G-module, $(RG)^{\infty}$, where RG is the real group algebra endowed with its natural inner product. We write $V < \mathfrak{U}$, if V is a finite dimensional G-invariant submodule of $\mathfrak{U}$.

If $V < \mathfrak{U}$, then a G-CW(V) complex is a G-space X with a given decomposition $X = \operatorname{Colim} X^n$ such that

(i) $X^o = \coprod_{\nu} G/H_{\nu}$ is a disjoint union of G-orbits, where V is a trivial H_{ν}-module for each ν.

(ii) X^n is obtained from X^{n-1} by attaching cells of the form $G \times_H D(V-m)$, where H is such that V has a trivial m-dimensional summand as an H-module, and where n = dim V-m D(W) denotes the unit disc in $W < \mathfrak{U}$.

(4.4.10) Definition

Let $\bar{\theta}$ denote the category whose objects are the G-spaces G/H for $H \subset G$ and whose morphisms are given by

$$\bar{\theta}(G/H, G/K) = \operatorname*{Colim}_{V < \mathfrak{U}} [G/H^+ \wedge S^V ; G/K^+ \wedge S^V]_G$$

where the subscript + denotes the addition of a disjoint basepoint.

(4.4.11) Definition

Let X be a G-CW(V) Complex, then a differentially graded contravariant system is given by

$$\bar{C}_{V+n}(X)(G/H) = [\Sigma^{V+W} G/H^{+}, \Sigma^{W-n} X^{v+n}/X^{v+n-1}]_G ,$$

where $v =$ dim V,

and W is large enough to contain a trivial n-dimensional summand.

The following property of contravariant and covariant functors has been used to construct cochain and chain complexes.

(4.4.12) If $\bar{T}$ and $\bar{S}$ are contravariant additive functors

$\bar{T} : \bar{\Theta} \longrightarrow Ab$, $\bar{S} : \bar{\Theta} \longrightarrow Ab$, and

$\underline{T} : \bar{\Theta} \longrightarrow Ab$ is a covariant additive functor

then one may form the abelian groups Hom $(\bar{T}, \bar{S})$, [where

Hom$_{\bar{\Theta}}(\bar{T}, \bar{S})$ is the set of all natural transformations from $\bar{T} \longrightarrow \bar{S}$ in the category $\bar{\Theta}$]

and $\bar{T} \otimes_{\bar{\Theta}} \underline{T} = \sum_{H \subset G} \bar{T}(G/H) \otimes \underline{T}(G/H)^{\sim}$

[where we define $f^* t \otimes t' \sim t \otimes f_* t'$

for $t \in \bar{T}(G/K)$, $t' \in \underline{T}(G/H)$)

and for a morphism $f : G/H \to G/K$ in the category $\bar{\theta}$.

(4.4.13) <u>Definition</u>

Using result on (4.4.12) and considering the contravariant coefficient system $\bar{C}_{V+n}(X)$ (defined in (4.4.11)) The cochain and chain complexes are defined respectively by

$$C^*(X,\bar{T}) = \mathrm{Hom}_{\bar{\theta}}(\bar{C}_{V+*}(X), \bar{T})$$

$$C(X,\underline{T}) = \bar{C}_{V+*}(X) \otimes_{\bar{\theta}} \underline{T}$$

(4.4.14) <u>Definitions</u>

By passage to homology of the cochain complex $C^*(X,\bar{T})$ one defines the RO(G) graded cohomology $H_G^{V+*}(X;\bar{T})$.

Similarly, homology of the chain complex $C_*(X;\underline{T})$ defines the RO(G)-graded homology $H^G_{V+*}(X;\underline{T})$.

(4.4.15) We give below a list of basic properties of RO(G)-graded singular cohomology ([25])

(i) 'Dimension axiom'

$$H_G^0(G/H, \bar{T}) \cong \bar{T}(G/H) \quad \text{for each } H \subset G$$

$$H_G^n(G/H ; \bar{T}) = 0 \quad \text{if} \quad n \neq 0.$$

(2) $H_G^{V+n}(G/H ; \bar{T}) = H_G^{V+n}(G/H , \bar{T}) = 0$ if $n>0$

(3) $H_G^{\nu}(G \times_K X ; \bar{T}) \cong H_K^{\nu|K}(X ; \bar{T}|K)$.

if $K \subset G$ and $\bar{T}|K$ is $\bar{T}_0$ regarded as a coefficient system for K-orbits. Here $\nu|K$ means $\nu|K \in R(K)$.

(4) 'Suspension Isomorphism'

$$\bar{H}_G^{\nu}(X; \bar{T}) \overset{\sigma}{\cong} \bar{H}_G^{\nu+V}(\Sigma^V X ; \bar{T}),$$

where the reduced cohomology of a based G-space X is given by the natural construction $H_G((X, *); \bar{T})$ for pairs.

(5) $H_G^*(X; \bar{T})$ has a natural module structure over $A(G)$. ($A(G)$ is the Burnside ring of G).

(4.4.16) Remarks

The details of all the above constructions and their properties will appear in [30] .

Remark

The RO(G) graded cohomology theory developed above (in §4.4.10) is related to Bredon cohomology as follows: Let $\mathcal{G}$ denote the category whose objects are those of $\bar{\Theta}$ and whose morphisms are the G-maps ; $G/H \longrightarrow G/K$. A contravariant system $\bar{T} : \bar{\Theta} \longrightarrow Ab$, is a Bredon contravariant system $\bar{T}|_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{A}b$ (in the sense of §2.2)

via the inclusion $\mathcal{G} \hookrightarrow \bar{\Theta}$.

If a Bredon system $\bar{T} : \mathcal{G} \rightarrow \mathcal{A}b$, extends to a contravariant (Mackey) system $\bar{T}' : \bar{\Theta} \longrightarrow Ab$, then the Bredon cohomology with $\bar{T}$ coefficients agrees with $H^n_G(X; \bar{T}')$ for $n \in \mathbb{Z}$ upto natural isomorphism.

(4.4.17) Remark: As far as the computations of the RO(G) graded cohomology rings are concerned we have the following information (see [25])

Let $\underline{A}$ be the coefficient system given by

$\underline{A}(G/H) = A(H)$, ($A(H)$ Burnside ring of H).

$H^*_G(X; \underline{A})$ is the equivariant analogue of the ordinary integral cohomology.

The computation of the RO(G)-graded cohomology ring $H^*_G(pt, \underline{A})$ for an arbitrary G is wide open.

For G a finite group of order p (a prime) Stong([21]) has done the computations.

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