

# On Wave Solutions of Einstein's Field Equations of General Relativity Containing Zero-Rest-Mass Scalar Fields

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## Introduction

The study of relativistic field equations in the presence of scalar meson field has drawn the attention of many researchers. Brahmachary<sup>1</sup>, Bergmann and Leipnik<sup>2</sup> have investigated the spherically symmetric fields associated with zero-rest-mass. The static solutions for axially and spherically symmetric fields have been investigated by Buchdahl<sup>3</sup> who has also studied the physical aspects of these solutions. Janis, Newman and Winicour<sup>4</sup>, in an attempt to present an extension of Israel's<sup>5</sup> idea of singular event horizon, have considered the spherically symmetric solutions of the field equations of general relativity containing zero-rest-mass mason fields. Penney<sup>6</sup> and Gautreau<sup>7</sup> have extended the study to the case of axially symmetric fields and have found that the scalar field obeys a flat-space Laplaces equations such that a large class of solutions exists. Lal and Singh<sup>8,9</sup> have obtained exact cylindrical wave solutions of Einstein's field equations of general relativity in the presence of zero-rest-mass scalar fields and have further analyzed the non-singular nature of one of these solutions. Singh<sup>10</sup> and Patel<sup>11</sup> have investigated plane symmetric solutions of the field equations corresponding to zero-rest-mass scalar fields.

This paper is continuation of Lal and Ansari<sup>12</sup> in which we have investigated the plane wave-like solutions of the field equations of general relativity containing electromagnetic fields in a space-time whose geometry is described by the line element,

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 + 2Adzdt + 2Bdx dy, \dots \dots \dots (1)$$

where  $A = A(z, t)$  and  $B = B(x, y)$ .

The object of this investigation is to find out the plane - wave-like solutions of Einstein's field equations of general relativity in presence of zero-rest-mass scalar fields in

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the same space time (1).

The equations under consideration are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu} \dots \dots \dots (2)$$

$$g^{\mu\nu}\psi_{,\mu\nu} = 0, \dots \dots \dots (3)$$

where  $\psi$  is a scaler field having zero - rest - mass and  $T_{\mu\nu}$ , the energy momentum tensor of this field, is defined by

$$T_{\mu\nu} = \psi_{,\mu}\psi_{,\nu} - \frac{1}{2}g_{\mu\nu}(\psi_{,i}\psi_{,m}g^{im}), \dots \dots \dots (4)$$

Here and elsewhere a semicolon (;) indicates covariant differentiation with respect to Christoffel symbols  $\left\{ \begin{matrix} k \\ \mu \nu \end{matrix} \right\}$  and a comma (,) followed by an index  $\mu$  denotes partial differentiation with respect to  $x^\mu$ .

## Calculation of $G_{\mu\nu}$ and $T_{\mu\nu}$

The contravariant components of  $g_{\mu\nu}$  corresponding to the line-element (1) are given by

$$\begin{aligned} g^{11} &= g^{22} = \frac{g^{12}}{B} = \frac{-1}{1-B^2}, \\ -g^{33} &= g^{44} = \frac{g^{34}}{A} = \frac{-1}{1+A^2}, \end{aligned} \quad (5)$$

For the line-element (1) the components of Christoffel symbols of the second kind  $\left\{ \begin{matrix} k \\ \mu \nu \end{matrix} \right\}$  are

$$\begin{aligned} \left\{ \begin{matrix} 1 \\ 1 \ 1 \end{matrix} \right\} &= \frac{-BB_1}{1-B^2} & \left\{ \begin{matrix} 3 \\ 3 \ 3 \end{matrix} \right\} &= \frac{AA_2}{1+A^2} \\ \left\{ \begin{matrix} 2 \\ 1 \ 1 \end{matrix} \right\} &= \frac{-B_1}{1-B^2} & \left\{ \begin{matrix} 4 \\ 3 \ 3 \end{matrix} \right\} &= \frac{A_2}{1+A^2} \\ \left\{ \begin{matrix} 1 \\ 2 \ 2 \end{matrix} \right\} &= \frac{-B_2}{1-B^2} & \left\{ \begin{matrix} 3 \\ 4 \ 4 \end{matrix} \right\} &= \frac{-A_4}{1+A^2} \\ \left\{ \begin{matrix} 2 \\ 2 \ 2 \end{matrix} \right\} &= \frac{-BB_2}{1-B^2} & \left\{ \begin{matrix} 4 \\ 4 \ 4 \end{matrix} \right\} &= \frac{AA_4}{1+A^2} \end{aligned} \quad (6)$$

where  $B_1, B_2$ , etc stand for  $\frac{\partial B}{\partial x}, \frac{\partial B}{\partial y}$  respectively.

The Ricci Tensor  $\bar{R}_{\mu\nu}$  is defined by

$$\left\{ \begin{matrix} s \\ \mu \ s \end{matrix} \right\}, \nu - \left\{ \begin{matrix} s \\ \mu \ \nu \end{matrix} \right\}, s + \left\{ \begin{matrix} s \\ t \ \nu \end{matrix} \right\} \left\{ \begin{matrix} t \\ \mu \ s \end{matrix} \right\} - \left\{ \begin{matrix} s \\ t \ s \end{matrix} \right\} \left\{ \begin{matrix} t \\ \mu \ \nu \end{matrix} \right\} \dots \dots \dots (7)$$

Using (6) and (7) the non-vanishing components of  $R_{\mu\nu}$  are given by

$$R_{11} = R_{22} = \frac{-R_{12}}{B} = X$$

$$-R_{33} = R_{44} = \frac{R_{34}}{A} = Y; \tag{8}$$

where

$$X \equiv \frac{B_{12}}{1-B^2} + \frac{BB_1B_2}{(1-B^2)^2},$$

$$Y \equiv \frac{A_{34}}{1-A^2} + \frac{AA_3A_4}{(1+A^2)^2}, \tag{9}$$

By a straight forward calculation the non-vanishing components of  $G_{\mu\nu}$  and  $T_{\mu\nu}$  have the following values:

$$G_{11} = G_{22} = \frac{G_{12}}{B} = Y, \tag{10}$$

$$-G_{33} = G_{44} = \frac{G_{34}}{A} = X, \text{ and}$$

$$\begin{aligned} T_{11} &= \psi_1^2 - \Phi & , & & T_{11} &= \psi_2^2 - \Phi \\ T_{12} &= T_{21} = \psi_1\psi_2 + B\phi & , & & T_{14} &= T_{41} = \psi_1\psi_4 \\ T_{13} &= T_{31} = \psi_1\psi_3 & , & & T_{24} &= T_{42} = \psi_2\psi_4 \\ T_{23} &= T_{32} = \psi_2\psi_3 & , & & T_{33} &= \psi_3^2 - \Phi \\ T_{34} &= T_{43} = \psi_3\psi_4 + A\phi & , & & T_{44} &= \psi_4^2 + \Phi \end{aligned} \tag{11}$$

where

$$\phi = (\psi_1^2 + 2B\psi_1\psi_2 + \psi_2^2)/2$$

$$(1 - B^2) - (\psi_3^2 + 2A\psi_3\psi_4 + \psi_4^2)/2(1+A^2).$$

**Solutions of Field Equations (2) and (3)**

Substituting the values of  $G_{\mu\nu}$  from (10) and  $T_{\mu\nu}$  from (11) into (2), we get

$$Y + 8\pi(\psi_1^2 - \Phi) = 0 \dots \dots \dots \tag{12}$$

$$Y + 8\pi(\psi_2^2 - \Phi) = 0 \dots \dots \dots \tag{13}$$

$$BY - 8\pi(\psi_1\psi_2 + B\phi) = 0 \dots \dots \dots \tag{14}$$

$$\psi_1\psi_3 = 0 \dots \dots \dots \tag{15}$$

$$\psi_1\psi_4 = 0 \dots \dots \dots \tag{16}$$

$$\psi_2\psi_3 = 0 \dots \dots \dots \tag{17}$$

$$\psi_2\psi_4 = 0 \dots \dots \dots \tag{18}$$

$$X - 8\pi(\psi_3^2 - \Phi) = 0 \dots \dots \dots \tag{19}$$

$$X + 8\pi(\psi_4^2 + \Phi) = 0 \dots \dots \dots \tag{20}$$

$$AX + 8\pi(\psi_3\psi_4 + A\phi) = 0 \dots \dots \dots \tag{21}$$

Equation (15)-(18) are satisfied when either

- (i)  $\psi_1 = \psi_2 = \psi_3 = \psi_4 = 0$  which is trivial ;

or (ii)  $\psi_3 = \psi_4 = 0$  that  $\frac{\partial\psi}{\partial z} = 0$ ,

or (iii)  $\psi_1 = \psi_2 = 0$ .

Taking case (ii) and (iii) when  $\frac{\partial\psi}{\partial z} = 0$ ,  $\psi$  becomes a function of  $x, y$  only, but this is not useful from the point of view of 'wave solutions'. Therefore, we take  $\psi_1 = \psi_2 = 0$  which reduces  $\psi$  to be a function of  $Z$  only. Using  $\psi_1 = \psi_2 = 0$  and  $\psi = \psi(z)$  in equations (12)-(21) solution is

$$A(x + y) - 8\pi\bar{\psi}^2 = 0, \dots \dots \dots \tag{22}$$

where a bar denotes partial differentiation with respect to  $z (= z - t)$ . Substituting the values of  $g^{\mu\nu}$  from (5) and  $\left\{ \begin{smallmatrix} i \\ \nu \end{smallmatrix} \right\}$  from (6), in the wave equation (3), we have

$$\left\{ \psi_{,11} + \psi_{,1} \frac{BB_1}{1-B^2} + \psi_{,2} \frac{B_1}{1-B^2} + 2B_{,12} + \psi_{,1} \frac{B_2}{1-B^2} + \psi_{,1} \frac{BB_2}{1-B^2} \right\} /$$

$$(1 - B^2) - \left\{ -\psi_{,33} + \psi_{,3} \frac{AA_3}{1-A^2} + \psi_{,4} \frac{A_3}{1+A^2} + 2A\psi_{,34} + \psi_{,44} + \psi_{,3} \frac{A_4}{1-A^2} \right.$$

$$\left. \psi_{,4} \frac{AA_4}{1-A^2} \right\} /$$

$$(1 + A^2) = 0 \tag{23}$$

Using  $\psi_1 = \psi_2 = 0$  and  $\Psi = \Psi(z)$  in equation (23), it reduces to

$$(1 - A)A_3 - (1 + A)A_4 + 2(1 + A^2)A\bar{\psi} = 0 \dots \dots \dots \tag{24}$$

Thus, the plane-wave-like solutions of the field equations (2) and (3) are composed of  $g^{\mu\nu}$  given by (1) satisfying the conditions (22) and (24).

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