

Algebraic expressions for effective potential characteristic parameters in heavy ion scattering

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Abstract. In this paper, we obtain reliable expressions to calculate the barrier and pocket positions of the real part of the effective phenomenological optical potential having Woods–Saxon form factor, for different partial waves. The comparison of the results obtained from these formulae, when compared with the numerical results obtained using Newton–Raphson iterative procedure are found to be quite accurate, with error less than 1%. We also obtain algebraic expressions for estimating l_{poc} , the angular momentum at which the potential pocket vanishes, the accuracy of which is tested with the exact calculations, using self-consistent iterative procedures. These and other expressions deduced in this paper provide simple and useful methods for calculating critical parameters of heavy ion effective potentials like barrier and pocket positions, curvatures at the barrier and pocket positions, l_{poc} and the grazing angular momentum l_g to carry out the analysis of heavy ion scattering.

Keywords. Heavy ion scattering; Coulomb barrier.

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1. Introduction

In the study of elastic scattering, fusion, etc, in heavy ion collisions, one generally uses the phenomenological optical model potential (OMP). This has the general form

$$U(r) = -U_R g(R, a, r) - i U_I g(R_I, a_I, r) + U_C(r)$$

where the most commonly used expression for $g(R, a, r)$ is the Woods–Saxon form factor given by

$$g(R, a, r) = \left[1 + \exp\left(\frac{r-R}{a}\right) \right]^{-1} \quad \text{with} \quad R = r_0(A_1^{1/3} + A_2^{1/3}). \quad (1.1)$$

The quantities r_0 and a are the radius and diffuseness parameters, U_R and U_I are the depths of the real and imaginary parts of the optical potential and A_1 and A_2 are the mass numbers of the colliding particles having corresponding proton numbers Z_1 and Z_2 . Similarly, one uses the following expression for the Coulomb potential:

$$U_C(r) = \frac{Z_1 Z_2 e^2}{2 R_C^3} (3 R_C^2 - r^2), \quad r < R_C \quad (1.2a)$$

$$= \frac{Z_1 Z_2 e^2}{r}, \quad r > R_C, \quad (1.2b)$$

where $R_C = r_c (A_1^{1/3} + A_2^{1/3})$ and r_c is the Coulomb radius parameter.

When this OMP is used in the radial Schrodinger equation, the effective potential governing the radial motion becomes

$$U_{\text{eff}}(l, r) = -U_R g(R, a, r) - iU_I g(R_I, a_I, r) + U_C(r) + \frac{l(l+1)\hbar^2}{2mr^2}, \quad (1.3)$$

where the last term is the centrifugal energy term and m is the reduced mass of the two colliding nuclei having masses m_1 and m_2 .

The effective potential $U_{\text{eff}}(l, r)$ in general is characterized by strong absorption, large Coulomb barrier and a potential pocket interior to the Coulomb barrier for most of the significant partial waves. The positions, heights and curvatures of the potential barrier are important in characterizing different aspects of the collision like nucleus-nucleus fusion around the Coulomb barrier [1], barrier region resonances, back angle oscillations generated by interference of barrier and internal regions [2–4] etc. [5] gives approximate methods to calculate some of these quantities. Similarly, the angular momentum $l = l_{\text{poc}}$ where the potential pocket vanishes and the grazing partial wave l_g such that $\text{Real}(U_{\text{eff}}(l_g, r)) = E$ where E is the centre of mass energy, are important in the study of heavy ion collisions.

For a global heavy ion potential having exponential form, Broglia and Winther [6] have given an approximate empirical expression for the position of the s -wave real potential barrier as

$$R_{B0} = 1.07 (A_1^{1/3} + A_2^{1/3}) + 2.72 \quad (1.4)$$

and starting from this, the barrier position R_{B_l} and pocket position R_{M_l} can be calculated using the Newton–Raphson iterative procedure. This procedure requires judicious choice of initial values to avoid numerical complications when R_{M_l} and R_{B_l} are close to each other.

Similarly, based on a global potential, [7] gives an empirical expression for l_{poc} as

$$l_{\text{poc}} = \left[m (R_1 + R_2)^3 \left(4\pi\gamma R_{12} - \frac{Z_1 Z_2 e^2}{(R_1 + R_2)^2} \right) \right]^{1/2} \quad (1.5)$$

where $R_i = 1.233 A_i^{1/3} - 0.978 A_i^{-1/3}$, $R_{12} = R_1 R_2 / (R_1 + R_2)$ and

$$\gamma = 0.95 \left[1 - 1.8 \left(\frac{N_1 - Z_1}{A_1} \right) \left(\frac{N_2 - Z_2}{A_2} \right) \right],$$

N_1 and N_2 being the neutron numbers of the two nuclei.

However, we find that for the most commonly used OMP having Woods–Saxon form factor, there appears to be no simple formulae to calculate R_{B_l} and l_{poc} . This is important in view of the fact that l_{poc} calculated from (1.5) and R_{B0} from (1.4) can be significantly different for the OMP with Woods–Saxon form factor. In view of these, we develop reliable and accurate algebraic expressions for R_{B_l} , R_{M_l} and l_{poc} for the real part of the potential given by (1.3). We obtain these in the next section and in § 3, we compare the results obtained by these formulae with those obtained by the iterative procedures and demonstrate that the formulae are quite accurate, having less than 1% error. Section 4 gives the summary and conclusions.

2. Algebraic expressions for R_{B_l} , R_{M_l} and l_{poc}

For heavy ion systems, it is found that as l increases from 0 to l_{poc} , R_{B_l} shifts towards the interior region by about a fermi. Similarly, the pocket position shifts to the exterior region by about a fermi as l varies from 1 to l_{poc} . When $l = l_{poc}$, the barrier and pocket positions practically merge, generating a point of inflexion. We exploit this feature in deducing the algebraic formulae for R_{B_l} and R_{M_l} .

2.1 Expression for the barrier position R_{B_l}

Since R_{B_l} is in the vicinity of R_{B_0} , we can expand (1.1) around $r = R_0 = R_{B_0}$ for the barrier position calculation so that

$$g(R, a, r) = f(x) = \frac{1}{1 + Ce^x}, \quad (2.1)$$

where $C = \exp((R_0 - R)/a)$ and $x = (r - R_0)/a$.

Expanding $f(x)$ around $x = 0$, we get

$$f(x) = f(0) + x f_1(0) + \frac{x^2}{2!} f_2(0) + \frac{x^3}{3!} f_3(0) + \dots, \quad (2.2)$$

where

$$f(0) = \frac{1}{1 + C}, \quad f_1(0) = -\frac{C}{(1 + C)^2}$$

$$f_2(0) = \frac{C(C - 1)}{(1 + C)^3}, \quad f_3(0) = -C \left(\frac{(C - 2)^2 - 3}{(1 + C)^4} \right).$$

Here, $f_i(0)$ denotes the i th order derivative of $f(x)$ at $x = 0$.

The radius of convergence of series (2.2) is

$$|x_0| = \sqrt{\left(\frac{R_0 - R}{a} \right)^2 + \pi^2}$$

whereas the corresponding radius of convergence if one sets $C = 1$ is π (i.e. if the expansion is done at $r = R$). This means that, by expanding the original function around $r = R_0$, one increases the radius of convergence and hence approximating $f(x)$ by neglecting the fourth and higher order terms becomes reasonable.

We note that, in general, $R_{B_l} > R_C$. Hence, we use the following expansions for $1/r$ and $1/r^2$ for handling the Coulomb and centrifugal terms of the effective potential

$$\frac{1}{r} = \frac{1}{R_0} \left[1 - \left(\frac{a x}{R_0} \right) + \left(\frac{a x}{R_0} \right)^2 - \left(\frac{a x}{R_0} \right)^3 + \dots \right], \quad (2.3)$$

$$\frac{1}{r^2} = \frac{1}{R_0^2} \left[1 - 2 \left(\frac{a x}{R_0} \right) + 3 \left(\frac{a x}{R_0} \right)^2 - 4 \left(\frac{a x}{R_0} \right)^3 + \dots \right]. \quad (2.4)$$

For convenience, we write the real part of $U_{\text{eff}}(l, r)$ as

$$V_{\text{eff}}(l, x) = \frac{2m}{\hbar^2} \left(-U_R g(R, a, r) + \frac{Z_1 Z_2 e^2}{r} + \frac{\lambda^2 \hbar^2}{2mr^2} \right), \quad (2.5)$$

where $\lambda^2 = l(l+1) \simeq (l + \frac{1}{2})^2$ for large l . Writing

$$\xi = \frac{Z_1 Z_2 e^2 m}{\hbar^2}, \quad V_R = \frac{2m}{\hbar^2} U_R$$

and using (2.1)–(2.5), the effective potential, after neglecting terms higher than x^3 becomes

$$V_{\text{eff}}(l, x) = - \left(\frac{4a^3 \lambda^2}{R_0^5} + \frac{2\xi a^3}{R_0^4} + \frac{V_R f_3(0)}{6} \right) x^3 + \left(\frac{3a^2 \lambda^2}{R_0^4} + \frac{2\xi a^2}{R_0^3} - \frac{V_R f_2(0)}{2} \right) x^2 - \left(\frac{2a \lambda^2}{R_0^3} + \frac{2\xi a}{R_0^2} + V_R f_1(0) \right) x + \left(\frac{\lambda^2}{R_0^2} + \frac{2\xi}{R_0} - V_R f(0) \right). \quad (2.6)$$

Thus

$$V'_{\text{eff}}(l, x) = a_B x^2 + b_B x + c_B,$$

where

$$a_B = - \left(\frac{12a^3 \lambda^2}{R_0^5} + \frac{6\xi a^3}{R_0^4} + \frac{V_R f_3(0)}{2} \right), \quad b_B = \frac{6a^2 \lambda^2}{R_0^4} + \frac{4\xi a^2}{R_0^3} - V_R f_2(0),$$

$$c_B = - \left(\frac{2a \lambda^2}{R_0^3} + \frac{2\xi a}{R_0^2} + V_R f_1(0) \right).$$

Next, we evaluate the appropriate zero of $V'_{\text{eff}}(l, x)$ (which is a quadratic function in x) to get x_{Bl} :

$$x_{\text{Bl}} = \frac{-b_B - (b_B^2 - 4a_B c_B)^{1/2}}{2a_B} \quad (2.7)$$

and

$$R_{\text{Bl}} = a x_{\text{Bl}} + R_0. \quad (2.8)$$

The other root of x_{Bl} gives values of R_{Bl} that increase with l and hence does not correspond to the barrier position. It does not correspond to the pocket position either as the true pocket position lies deeper inside. This root is therefore ignored. Equation (2.8) can be used to calculate R_{Bl} for all l 's from $l=0$ to l_{poc} .

2.2 Expression for the pocket position R_{MI}

It is found that for different l 's, the minimum of the potential occurs mostly in the region interior to $r = R$. We have found that if $f(x)$ is expanded around $R_0 = R - 3a$, minima calculations are quite accurate. The procedure is similar to that for the maxima except for the Coulomb part which can be written in the following exact

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form in x using (1.2a).

$$V_C(x) = \left(\frac{3\xi}{R_C} - \frac{\xi R_0^2}{R_C^3} \right) - \frac{2a\xi R_0}{R_C^3} x - \frac{\xi a^2}{R_C^3} x^2, \quad r < R_C.$$

In the neighbourhood of $x = 0$, i.e. $r = R - 3a$, we get the following approximate expression after ignoring higher terms:

$$\begin{aligned} V_{\text{eff}}(l, x) = & - \left(\frac{4a^3 \lambda^2}{R_0^5} + \frac{V_R f_3(0)}{6} \right) x^3 + \left(\frac{3a^2 \lambda^2}{R_0^4} - \frac{\xi a^2}{R_C^3} - \frac{V_R f_2(0)}{2} \right) x^2 \\ & - \left(\frac{2a \lambda^2}{R_0^3} + \frac{2\xi a R_0}{R_C^3} + V_R f_1(0) \right) x \\ & + \left(\frac{\lambda^2}{R_0^2} - \frac{\xi R_0^3}{R_C^3} + \frac{3\xi}{R_C} - V_R f(0) \right) + \dots \end{aligned} \quad (2.9)$$

and

$$V'_{\text{eff}}(l, x) = a_M x^2 + b_M x + c_M,$$

where

$$\begin{aligned} a_M = & - \left(\frac{12a^3 \lambda^2}{R_0^5} + \frac{V_R f_3(0)}{2} \right), \quad b_M = \frac{6a^2 \lambda^2}{R_0^4} - \frac{2\xi a^2}{R_C^3} - V_R f_2(0), \\ c_M = & - \left(\frac{2a \lambda^2}{R_0^3} + \frac{2a\xi R_0}{R_C^3} + V_R f_1(0) \right). \end{aligned}$$

The expression for x_{Ml} becomes

$$x_{Ml} = \frac{-b_M + (b_M^2 - 4a_M c_M)^{1/2}}{2a_M}, \quad (2.10)$$

$$R_{Ml} = a x_{Ml} + R_0, \quad (2.11)$$

where once again, the other root is neglected as this gives rise to values of R_{Ml} that decrease with increasing l . One may note here also that the radius of convergence for $f(x)$ in (2.9) is

$$|x_0| = \sqrt{9 + \pi^2} = 4.34.$$

This is significantly greater than π and supports the reliability of our approximation.

2.3 Calculation of l_{poc}

At angular momentum $l = l_{\text{poc}}$, the effective potential does not have a maxima or a minima but has a point of inflexion. It is found that this point is in the vicinity of $r = R$. An expansion for $f(x)$ appropriate for r in the vicinity of R is

$$f(x) = \frac{1}{2} \left(1 - \tanh \frac{x}{2} \right)$$

i.e.
$$f(x) = \frac{1}{2} \left[1 - \left(\frac{x}{2} - \frac{x^3}{24} + \dots \right) \right], \quad (2.12)$$

where x is now defined as $x = (r - R)/a$.

Using this, we get the following approximate expressions for $V'_{\text{eff}}(l, x)$ and $V''_{\text{eff}}(l, x)$.

For $r < R_C$,

$$V'_{\text{eff}}(l, x) = -\left(\frac{V_0}{16} + \frac{12a^3\lambda^2}{R^5}\right)x^2 + \left(\frac{6a^2\lambda^2}{R^4} - \frac{2\xi a^2}{R_C^3}\right)x + \left(\frac{V_0}{4} - \frac{2\xi a R}{R_C^2} - \frac{2a\lambda^2}{R^3}\right) \tag{2.13a}$$

and

$$V''_{\text{eff}}(l, x) = -\left(\frac{V_0}{8} + \frac{24a^3\lambda^2}{R^5}\right)x + \left(\frac{6a^2\lambda^2}{R^4} - \frac{2\xi a^2}{R_C^3}\right).$$

For $r > R_C$

$$V'_{\text{eff}}(l, x) = -\left(\frac{V_0}{16} + \frac{6\xi a^3}{R^4} + \frac{12a^3\lambda^2}{R^5}\right)x^2 + \left(\frac{4\xi a^2}{R^3} + \frac{6a^2\lambda^2}{R^4}\right)x + \left(\frac{V_0}{4} - \frac{2\xi a}{R^2} - \frac{2a\lambda^2}{R^3}\right) \tag{2.13b}$$

and

$$V''_{\text{eff}}(l, x) = -\left(\frac{V_0}{8} + \frac{12\xi a^3}{R^4} + \frac{24a^3\lambda^2}{R^5}\right)x + \left(\frac{4\xi a^2}{R^3} + \frac{6a^2\lambda^2}{R^4}\right).$$

If, for a given l near l_{poc} , R_{Bl} and R_{Ml} are present, by setting $V'_{\text{eff}}(l, x) = 0$, one can obtain their positions. However, when the roots of this equation in x become complex, we can say that the condition $l = l_{\text{poc}}$ has been reached. The critical condition for this is that the discriminants of the quadratic equations (2.13a) and (2.13b) vanish. The expression for λ_{poc} with this condition corresponding to (2.13a) and (2.13b) are

$$\lambda_{\text{poc}} = \left[\frac{\frac{4\xi^2 a^4}{R_C^6} + \frac{V_R^2}{16} - \frac{V_R \xi a R}{2R_C^3}}{\frac{120a^4 \xi}{R^4 R_C^3} + \frac{V_R a}{2R_C^3} - \frac{12V_R a^3}{R^5}} \right]^{1/2} \tag{2.14a}$$

and

$$\lambda_{\text{poc}} = \left[\frac{\frac{6\xi V_R a^3}{R^4} - \frac{32\xi^2 a^4}{R^6} + \frac{V_R^2}{16} - \frac{V_R \xi a}{2R^2}}{\frac{96a^4 \xi}{R^7} + \frac{V_R a}{2R^3} - \frac{12a^3 V_R}{R^5}} \right]^{1/2} \tag{2.14b}$$

One can also calculate l_{poc} by a different manner. We can find by a self-consistent calculation that both $V'_{\text{eff}}(l, x)$ and $V''_{\text{eff}}(l, x)$ vanish when $r = R_{\text{poc}}$ and $l = l_{\text{poc}}$. In this connection, we note that $g(R, a, r)$ varies most rapidly at $r = R$ and hence $g'(R, a, r)$ has a maxima at $r = R$. However, in the close vicinity of $r = R$, $g'(R, a, r)$ changes very rapidly, i.e. $r = R$ is a sharp maxima of $g'(R, a, r)$. And due to this, R_{poc} is very close to R .

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Hence, one may get the following first approximation to l_{poc} corresponding to (2.13a) and (2.13b) by setting $V'_{\text{eff}}(l_{\text{poc}}, R) = 0$ at $r = R$.

$$\lambda_{\text{poc}} = \sqrt{\frac{R^3}{2a} \left(\frac{V_R}{4} - \frac{2\xi a R}{R_C^3} \right)}, \quad (2.15a)$$

$$\lambda_{\text{poc}} = \sqrt{\frac{R^3}{2a} \left(\frac{V_R}{4} - \frac{2\xi a}{R^2} \right)}. \quad (2.15b)$$

Thus, we have three alternative procedures to estimate l_{poc} namely formulae (2.14a) and (2.14b), formulae (2.15a) and (2.15b), and the self consistent calculations. In table 2, we give the results of the calculations done using these procedures.

2.4 Some other features

Another advantage of the availability of algebraic expressions for R_{Bl} and R_{MI} is that using them, one can easily get the curvature of the effective potential α and β at the maxima and the minima, in the parabolic approximation [3]. For example, near the maxima and minima, one can write

$$\begin{aligned} U_{\text{eff}}(l, r) &= U_{\text{Bl}} - \frac{1}{2} \alpha (r - R_{\text{Bl}})^2, \quad \text{for maxima,} \\ &= U_{\text{MI}} + \frac{1}{2} \beta (r - R_{\text{MI}})^2, \quad \text{for minima,} \end{aligned}$$

where U_{Bl} and U_{MI} are the barrier and pocket heights.

It may be noted that the energy levels E of the potential pocket in the parabolic approximation are related to β by the formula

$$E = \hbar \omega_{\text{MI}}, \quad \omega_{\text{MI}} = \sqrt{\frac{\beta}{m}}.$$

Similarly, the parameter α can be related to the barrier top resonances. Due to this, the α and β factors have considerable significance in several cases of heavy ion collisions. It is clear that

$$\alpha = U''_{\text{eff}}(l, r)|_{r=R_{\text{Bl}}}, \quad \beta = U''_{\text{eff}}(l, r)|_{r=R_{\text{MI}}}$$

and these can be readily calculated using R_{Bl} and R_{MI} .

Hence a knowledge of R_{Bl} and R_{MI} readily helps us to determine additional features of the quantal states associated with the effective potential. For completeness, we also note that the grazing angular momentum λ_g at any centre of mass energy E above the Coulomb barrier, can be calculated [8] using the approximate formula, valid for Woods-Saxon type form factors (for the nuclear part of the effective potential), given below.

$$\left(1 + \frac{\lambda_g^2}{\eta^2} \right)^{1/2} = \frac{2}{V_{\text{CB}}} \left[E - \frac{1}{2} (V_{\text{CB}} + 2V_{\text{NB}}) + \frac{(E - E_B) V_{\text{NB}}}{E_B} \right],$$

where V_{CB} and V_{NB} are the Coulomb and nuclear potentials at R_{B0} , $E_B = V_{\text{CB}} + V_{\text{NB}}$ and the Coulomb parameter $\eta = Z_1 Z_2 e^2 / \hbar v$, v being the initial relative velocity.

The expressions deduced in this section help in readily calculating R_{Bl} , R_{Ml} , l_{poc} , l_p , ω_{Ml} and ω_{Bl} for OMP with Woods–Saxon form factors. However, it is necessary to verify the accuracy of the formulae for R_{Bl} , R_{Ml} and l_{poc} by comparing with the exact calculation of these quantities. This is described in the next section.

3. Results and discussion

In order to verify the accuracy of our formulae for R_{Bl} and l_{poc} , we have used a number of optical potentials corresponding to systems of different heavy ion pairs. The OMP parameters are listed in table 1. In figures 1 and 2, we give a typical set of results for the variation of R_{Bl} and R_{Ml} as a function of l obtained from our expressions (2.7), (2.8) and (2.10), (2.11) respectively and compare the same with the corresponding exact results obtained using Newton–Raphson iterative method. It is seen that our formula (2.7), (2.8), and (2.10), (2.11) give R_{Bl} and R_{Ml} quite accurately and in general, the error is less than 1% and in many cases, the accuracy is still better. Typical results are given in figures 1 and 2. We have verified that results similar to those indicated in figures 1 and 2 are obtained for all the heavy ion systems listed in table 1. The results shown in figures 1 and 2 include both shallow ($^{40}\text{Ca} + ^{40,44}\text{Ca}$) and deep potential wells ($^{16}\text{O} + ^{208}\text{Pb}$).

In table 2, we list the l_{poc} values obtained from the different procedures. Column A corresponds to the result obtained from the self-consistent calculation of $V'_{\text{eff}}(l, x) = 0$ and $V''_{\text{eff}}(l, x) = 0$ using the cubic polynomial approximation (2.12) to $V_{\text{eff}}(l, x)$. Columns

Table 1. Optical model parameters used in the calculations.

System	a (fm)	r_0 (fm)	U_R (MeV)	r_c (fm)	References
$^{16}\text{O}_8 + ^{144}\text{Nd}_{60}$	0.57	1.34	20.0	1.25	[9]
$^{16}\text{O}_8 + ^{148}\text{Sm}_{62}$	0.57	1.34	20.0	1.25	[10, 11]
$^{16}\text{O}_8 + ^{152}\text{Sm}_{62}$	0.57	1.34	20.0	1.25	[10, 11]
$^{16}\text{O}_8 + ^{154}\text{Sm}_{62}$	0.57	1.34	20.0	1.25	[10, 11]
$^{16}\text{O}_8 + ^{208}\text{Pb}_{82}$	0.50	1.25	100.0	1.25	[12, 11]
$^{32}\text{S}_{16} + ^{58}\text{Ni}_{28}$	0.65	1.20	50.0	1.20	[13]
$^{32}\text{S}_{16} + ^{64}\text{Ni}_{28}$	0.65	1.20	50.0	1.20	[13]
$^{37}\text{Cl}_{17} + ^{123}\text{Sb}_{51}$	0.51	1.25	41.8	1.25	[9]
$^{40}\text{Ar}_{18} + ^{122}\text{Sn}_{50}$	0.51	1.25	41.8	1.25	[14, 11]
$^{40}\text{Ca}_{20} + ^{40}\text{Ca}_{20}$	0.43	1.35	35.0	1.35	[15, 11]
$^{40}\text{Ca}_{20} + ^{44}\text{Ca}_{20}$	0.43	1.35	35.0	1.35	[15, 11]
$^{58}\text{Ni}_{28} + ^{124}\text{Sn}_{50}$	0.294	1.26	58.1	1.26	[17, 11]
$^{64}\text{Ni}_{28} + ^{118}\text{Sn}_{50}$	0.294	1.26	58.1	1.26	[17, 11]
$^{64}\text{Ni}_{28} + ^{96}\text{Zr}_{40}$	0.55	1.25	40.0	1.25	[9]
$^{80}\text{Se}_{34} + ^{80}\text{Se}_{34}$	0.55	1.25	40.0	1.25	[9]
$^{81}\text{Br}_{35} + ^{90}\text{Zr}_{40}$	0.43	1.35	35.0	1.35	[11]
$^{81}\text{Br}_{35} + ^{94}\text{Zr}_{40}$	0.43	1.35	35.0	1.35	[11]
$^{81}\text{Br}_{35} + ^{96}\text{Mo}_{42}$	0.43	1.35	35.0	1.35	[11]
$^{81}\text{Br}_{35} + ^{104}\text{Ru}_{44}$	0.43	1.35	35.0	1.35	[11]
$^{58}\text{Ni}_{28} + ^{18}\text{O}_8$	0.50	1.22	90.0	1.22	[18]
$^{58}\text{Ni}_{28} + ^{58}\text{Ni}_{28}$	0.55	1.20	40.0	1.25	[16, 11]
$^{58}\text{Ni}_{28} + ^{64}\text{Ni}_{28}$	0.55	1.20	40.0	1.25	[16, 11]

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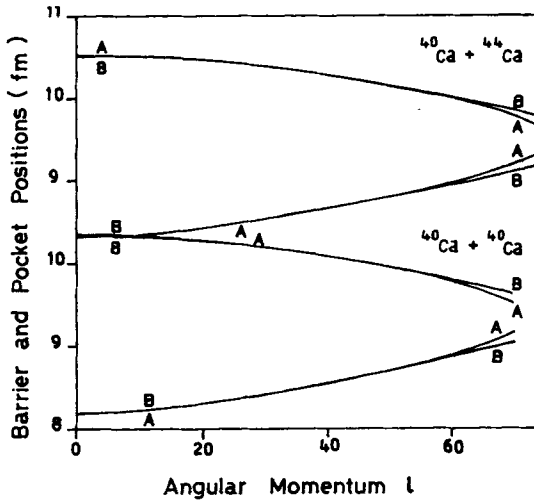


Figure 1. Variation of barrier position (R_{Bl}) and pocket position (R_{Ml}) as a function of l for systems $^{40}\text{Ca} + ^{40,44}\text{Ca}$, obtained from algebraic expressions given in the text and these curves are denoted by symbol B. Symbol A denotes the corresponding results obtained using the Newton-Raphson iterative procedure. Upper pair of curves denote barrier positions and lower pair the pocket positions.

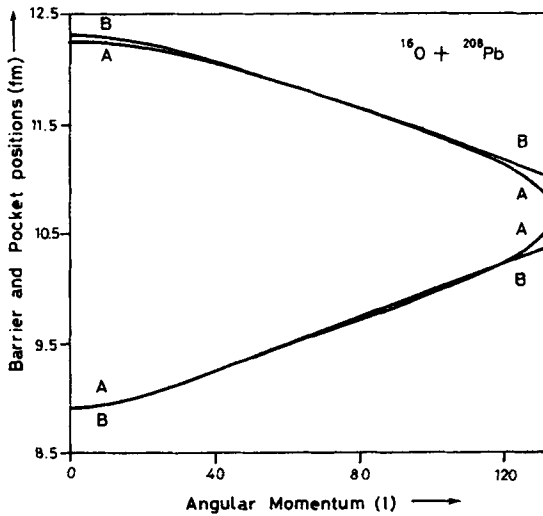


Figure 2. Same as figure 1 for system ($^{16}\text{O} + ^{208}\text{Pb}$).

B and C are the results for l_{poc} obtained from (2.15b) and (2.14b) respectively. The results listed under column D estimates l_{poc} setting the condition that at $l = l_{\text{poc}}$ the difference between $V_{\text{eff}}(l, x)$ at its maximum and minimum values is less than 0.1 MeV and using Newton-Raphson procedures to calculate the maximum and minimum. We have noticed that this procedure can give rise to numerical problems of convergence near

Table 2. Comparison of l_{poc} values from various formulations. The l_{poc} values under columns (A) and (D) are obtained from the self-consistent calculation and Newton–Raphson method respectively. The l_{poc} values under columns (B) and (C) are obtained from (2.15b) and (2.14b) respectively.

System	l_{poc} A	l_{poc} B	l_{poc} C	l_{poc} D
$^{16}\text{O}_8 + ^{144}\text{Nd}_{60}$	31	30	31	29
$^{16}\text{O}_8 + ^{148}\text{Sm}_{62}$	30	29	30	28
$^{16}\text{O}_8 + ^{152}\text{Sm}_{62}$	31	30	31	29
$^{16}\text{O}_8 + ^{154}\text{Sm}_{62}$	31	31	32	30
$^{16}\text{O}_8 + ^{208}\text{Pb}_{82}$	133	131	135	132
$^{32}\text{S}_{16} + ^{58}\text{Ni}_{28}$	57	55	58	56
$^{32}\text{S}_{16} + ^{64}\text{Ni}_{28}$	60	58	62	59
$^{37}\text{Cl}_{17} + ^{123}\text{Sb}_{51}$	83	82	84	81
$^{40}\text{Ar}_{18} + ^{122}\text{Sn}_{50}$	86	84	86	84
$^{40}\text{Ca}_{20} + ^{40}\text{Ca}_{20}$	72	71	73	71
$^{40}\text{Ca}_{20} + ^{44}\text{Ca}_{20}$	76	75	77	75
$^{58}\text{Ni}_{28} + ^{124}\text{Sn}_{50}$	209	208	210	207
$^{64}\text{Ni}_{28} + ^{118}\text{Sn}_{50}$	217	216	217	214
$^{64}\text{Ni}_{28} + ^{96}\text{Zr}_{40}$	70	68	70	67
$^{80}\text{Se}_{34} + ^{80}\text{Se}_{34}$	69	67	70	66
$^{81}\text{Br}_{35} + ^{90}\text{Zr}_{40}$	103	102	103	99
$^{81}\text{Br}_{35} + ^{94}\text{Zr}_{40}$	107	106	107	103
$^{81}\text{Br}_{35} + ^{96}\text{Mo}_{42}$	103	102	103	99
$^{81}\text{Br}_{35} + ^{104}\text{Ru}_{44}$	104	103	104	100
$^{58}\text{Ni}_{28} + ^{18}\text{O}_8$	82	80	84	79
$^{58}\text{Ni}_{28} + ^{58}\text{Ni}_{28}$	54	53	55	51
$^{58}\text{Ni}_{28} + ^{64}\text{Ni}_{28}$	59	58	60	55

$l = l_{\text{poc}}$ due to the closeness of the maximum and minimum. For the purpose of comparison, we may treat results in column A obtained from self consistent calculations of R_{poc} and l_{poc} as exact results. We find that the result from algebraic expressions corresponding to columns B and C give very good estimates of l_{poc} and difference of ± 1 in most cases originate due to rounding off to the nearest integer. Further we note that the simple formula (2.15b) is a very good estimate of l_{poc} for OMP with Woods–Saxon form factor. The results from (2.14a) and (2.15a) also give substantially similar results except for the cases of OMP where $R_C < R$. Hence, we give only the numerical results obtained from (2.14b) and (2.15b) and the self-consistent calculation.

It may be noted that the cubic polynomial expression for $V_{\text{eff}}(l, x)$ around the barrier region given by (2.6) is quite accurate (error $< 1\%$) near the barrier region $(R_{\text{B0}} - s) < r < (R_{\text{B0}} + s)$, $s \leq 1$ fm. Similarly, $V_{\text{eff}}(l, x)$ approximation by the cubic polynomial (2.9) around $r = (R - 3a)$ is also a reliable expression in the region $(R - 3a - s) < r < (R - 3a + s)$, where $s \leq 1$ fm. Hence, one may use appropriate generalization of this approach for complex OMP to calculate algebraically the turning points of the effective potential in the WKB analysis of the heavy ion collisions. However, to have accurate results, one should use complex optical potentials. Further, in our procedure, the expression for turning points close to the barrier can be obtained

as a solution of a cubic equation which are straightforward to calculate but algebraically lengthy. Hence, we have not listed the expressions here.

4. Summary

We have obtained algebraic expressions for calculating the barrier and pocket positions for all partial waves of the real effective OMP's with Woods–Saxon form factor and have verified their accuracy in a number of cases for heavy ion systems. We have also obtained simple algebraic expressions for calculating l_{poc} where the effective potential vanishes. This gives a very good estimate of l_{poc} obtained from the self-consistent calculation. The algebraic expression for R_{Bl} , R_{Ml} and $V_{\text{eff}}(l, x)$ in the barrier and pocket regions are useful in calculating easily the curvature parameters ω_{Bl} and ω_{Ml} near the barrier and pocket positions in the parabolic approximation. The approximate expressions for $V_{\text{eff}}(l, x)$ are also useful in the calculation of the turning points in the WKB analysis of heavy ion collisions.

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