Products of two vectors

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The *dot* and the *cross* products of **a** and **b** are defined, respectively, as[1]

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta, \quad (1)$$

$$\mathbf{a}\times\mathbf{b} = \hat{n}ab\sin\theta. \quad (2)$$

Here all notations have their usual meanings. Authors of various texts on vector algebra differ on the basis for adopting these definitions. Resnick and Halliday[1] believe that these operations are defined in this way because they prove useful in physics, otherwise one can have an alternative such as

$$\mathbf{a}\cdot\mathbf{b} = a^{1/3}b^{1/3}\tan\theta/2. \quad (3)$$

Similar views are held by Arfken.[2] However, others[3] do not state any reason explicitely; perhaps they agree with the argument (often given in the classroom) that Eqs. (1) and (2) simply define the two products and do not require any basis for this purpose. This note establishes that Eq. (1) and (2) are unique and could be formulated mathematically. For this purpose it examines the *net* product, $\mathbf{a}*\mathbf{b}$, defined as a quantity obtained through the normal process of algebraic multiplication of expressions defining the vectors. For example, if we define

$$\mathbf{a} = \hat{a}_{\parallel}a, \quad (4)$$

$$\mathbf{b} = \hat{a}_{\parallel}b\cos\theta + \hat{a}_{\perp}b\sin\theta, \quad (5)$$

then

$$\mathbf{a}*\mathbf{b} = \hat{a}_{\parallel}*\hat{a}_{\parallel}ab\cos\theta + \hat{a}_{\parallel}*\hat{a}_{\perp}ab\sin\theta. \quad (6)$$

where the unit vectors $\hat{a}_{\parallel}$ (parallel to **a**) and $\hat{a}_{\perp}$ (perpendicular to **a**) lie in the plane of **a** and **b**. Note, that the order of $\hat{a}_{\parallel}$ and $\hat{a}_{\perp}$ has been kept as it should appear through the process of multiplication because the precise operational nature of $\hat{a}_{\parallel}*\hat{a}_{\perp}$ is yet to be determined. In order to know $\mathbf{a}*\mathbf{b}$ we need to simply find $\hat{a}_{\parallel}*\hat{a}_{\parallel}$ and $\hat{a}_{\parallel}*\hat{a}_{\perp}$.

$\hat{a}_{\parallel}*\hat{a}_{\parallel}$: Consider a set of parallel vectors, $\{\mathbf{V}_i | \mathbf{V}_i = \hat{v}V_i\}$, where $\mathbf{V}_i$ is any vector of length $|V_i|$ pointing along a chosen direction defined by $\hat{v}$. Using simple argument and logic it can be shown[4] that in all frames connected through a rotation (proper or improper) $\{\mathbf{V}_i\}$ can be represented by the set of numbers $\{V_i\}$ with one to one correspondence in their elements if the problem does not involve any vector outside the set $\{\mathbf{V}_i\}$. Obviously in all frames connected through a rotation the *net* product of two parallel vectors $\mathbf{V}_r$ and $\mathbf{V}_s$ belonging to $\{\mathbf{V}_i\}$ would mean to be the product of V_r and V_s, i.e.,

$$\mathbf{V}_r*\mathbf{V}_s = V_rV_s \quad (7)$$

implying that

$$\hat{a}_{\parallel}*\hat{a}_{\parallel} = 1. \quad (8)$$

$\hat{a}_{\parallel}*\hat{a}_{\perp}$: Let **a** and **b** be the vectors in the x–y plane of a frame, then with reference to the situations shown in Fig. 1,we evaluate $\mathbf{a}*\mathbf{b}$ as

$$\mathbf{a}*\mathbf{b} = \hat{i}*\hat{i}ab\cos\theta + \hat{i}*\hat{j}ab\sin\theta$$
$$= ab\cos\theta + \hat{i}*\hat{j}ab\sin\theta \quad \text{[Fig. 1(A)]}, \quad (9)$$

where from Eq. (7), we have used $\hat{i}*\hat{i} = 1$. Similarly we have

$$\mathbf{a}*\mathbf{b} = ab\cos\theta + \hat{i}*\hat{j}ab\sin\theta \quad \text{[Fig. 1(B)]}, \quad (10)$$

$$\mathbf{a}*\mathbf{b} = ab\cos\theta - \hat{j}*\hat{i}ab\sin\theta \quad \text{[Fig. 1(C)]}. \quad (11)$$

Here we observe that (i) $\hat{i}*\hat{j}$ is independent of **a**, **b**, and θ, (ii) situations in Figs. 1(A) and 1(B) are connected through a rotation of the frame about $\hat{k}$ by an angle $\varphi = -(\pi/2 - \theta)$, and (iii) in general, θ can have any value between π and $-\pi$. In view of these observations the same value of $\mathbf{a}*\mathbf{b}$ obtained for situations depicted in Figs. 1(A) and 1(B) reveals that $\hat{i}*\hat{j}$ is invariant under any rotation about $\hat{k}$. Furthermore, a comparison of Eqs. (9) and (11) shows that $\hat{i}*\hat{j}$
in an anticommuting operation, i.e.,

$$\hat{i}*\hat{j} = -\hat{j}*\hat{i}. \quad (12)$$

Comparing Eqs. (6) and (9) we note that these relations do not differ from each other except that $\hat{i}$ and $\hat{j}$ in Eq. (9), respectively, replace $\hat{a}_{\parallel}$ and $\hat{a}_{\perp}$ in Eq. (6). This leads to the conclusion that $\hat{a}_{\parallel}*\hat{a}_{\perp}$ must be invariant under any rotation about the axis (say $\hat{n}$) perpendicular to the plane of $\hat{a}_{\parallel}$ and $\hat{a}_{\perp}$, i.e., of **a** and **b**. In consequence of Eq. (12) we also have

$$\hat{a}_{\parallel}*\hat{a}_{\perp} = -\hat{a}_{\perp}*\hat{a}_{\parallel}. \quad (13)$$

These conclusions should remain valid in any frame of reference since the definitions of $\hat{a}_{\parallel}$ and $\hat{a}_{\perp}$ [hence the construction of Eq. (6)] are not affected by the location of **a** and

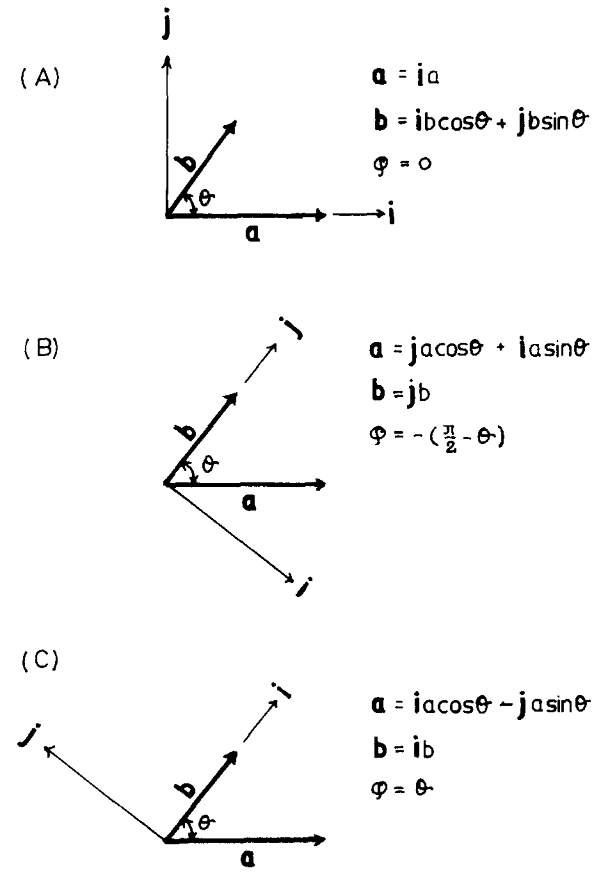

Fig. 1. Vectors **a** and **b** as they appear in the x–y plane of different frames of references rotated by an angle φ about $\hat{k}$. Anticlockwise rotation has been considered for positive φ.

b in the frame. However, it is clear that $\hat{n}$ would transform into a new axis $\hat{n}'$ under a general rotation of the frame according to

$$\hat{n}' = R^{-1}\hat{n}. \tag{14}$$

In view of these observations and the fact that a vector remains invariant under any rotation about the axis along itself and transforms according to a relation similar to Eq. (14) under a general rotation of the frame, $\hat{a}_{\parallel} * \hat{a}_{\perp}$ can be identified as a vector along $\hat{n}$, i.e.,

$$\hat{a}_{\parallel} * \hat{a}_{\perp} = \hat{n}K, \tag{15}$$

where K is a constant number. By appropriately defining the unit of $\hat{a}_{\parallel} * \hat{a}_{\perp}$ in relation with the units of $\hat{a}_{\parallel}$ and $\hat{a}_{\perp}$, K can always be chosen to be unity. Thus

$$\hat{a}_{\parallel} * \hat{a}_{\perp} = \hat{n}. \tag{16}$$

The information that $\hat{n}$ is perpendicular to the plane of **a** and **b**, does not uniquely decide the direction of $\hat{n}$. It is therefore decided conventionally in relation to the directions of **a** and **b**.

From Eqs. (6), (8), and (16) we have

$$\mathbf{a}*\mathbf{b} = ab\cos\theta + \hat{n}ab\sin\theta. \tag{17}$$

Thus the *net* product of two vectors turns out to be the sum[5] of a scalar quantity $ab\cos\theta$ and a vector quantity, $\hat{n}ab\sin\theta$. Such a sum may have some significance as a mathematical entity but does not carry any sense as a physical quantity. However, if both terms are considered separately, they may have some significance in physics too. Therefore, we can separately name and define these quantities as

$$\textit{dot}\ \text{product:}\quad \mathbf{a}\cdot\mathbf{b} = ab\cos\theta, \tag{18}$$

$$\textit{cross}\ \text{product:}\quad \mathbf{a}\times\mathbf{b} = \hat{n}ab\sin\theta. \tag{19}$$

It may be noted that in its definition, the *net* product **a*****b** is not much different from the normal algebraic product of two quantities; whatsoever difference exists in its evaluation is a technical necessity. Equations (1) and (2) are, therefore, unique in the sense that they, respectively, account fully for the scalar and vector components of the *net* product [cf. Eq. (17)]. Thus it is clear that we cannot have any alternative of these operations which serve the same purpose. It also becomes clear that the physical observations are not essential for quoting them as the bases for defining these operations the way they are defined; in fact this note provides to these definitions a necessary mathematical background.

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[1] R. Resnick and D. Halliday, *Physics* (Wiley Eastern, New Delhi, 1976), Chap. 2.

[2] G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1970), Chap. 1.

[3] See, for example, (i) G. D. Oates in *Handbook of Applied Mathematics* edited by C. E. Pearson (Van Nostand-Reinhold, New York, 1974), Chap. 3; (ii) N. M. Queen, *Vector Analysis* (McGraw-Hill, London, 1967) and several other workers.

[4] Y. S. Jain, Technical Report No. 1/1981, Department of Physics, North-Eastern Hill University, Shillong-3, India (unpublished).

[5] $\mathbf{a}*\mathbf{b} = ab\cos\theta + \hat{n}ab\sin\theta$ should not be sensed as a simple sum of two quantities. This point may be understood by examining the transformation properties of **a*****b** particularly under rotation–reflection operations. For example, under inversion/(reflection through the plane of **a** and **b**), **a*****b** transforms to $ab\cos\theta - \hat{n}ab\sin\theta$. Using $\mathbf{a} = \hat{\imath}a_x + \hat{\jmath}a_y + \hat{k}a_z$ and $\mathbf{b} = \hat{\imath}b_x + \hat{\jmath}b_y + \hat{k}b_z$ one may easily have $\mathbf{a}*\mathbf{b} = a_xb_x + a_yb_y + a_zb_z + \hat{\imath}(a_yb_z - a_zb_y) + \hat{\jmath}(a_zb_x - a_xb_z) + \hat{k}(a_xb_y - a_yb_x)$. Under reflection through, say $x-y$ plane, it will transform to $\mathbf{a}*\mathbf{b} = a_xb_x + a_yb_y + a_zb_z + \hat{\imath}(a_yb_z - a_zb_y) + \hat{\jmath}(a_zb_x - a_xb_z) - \hat{k}(a_xb_y - a_yb_x)$.

Addendum: "Trouble with the method of images"

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The recent paper by Newcomb[1] on the difficulties involved in summing a series associated with the method of images is very interesting for the detailed discussions it contains. As a supplement to this paper, I wish to bring to the readers' attention several papers and comments which have previously appeared in this Journal.[2–8]

[1] W. A. Newcomb, Am. J. Phys. **50**, 601–607 (1982).

[2] C. Y. Fong and C. Kittel, Am. J. Phys. **35**, 1091–1092 (1967).

[3] J. Pumplin, Am. J. Phys. **37**, 737–739 (1969).

[4] J. J. G. Scanio, Am. J. Phys. **41**, 415–418 (1973).

[5] B. G. Dick, Am. J. Phys. **41**, 1289–1290 (1973).

[6] M. Zahn, Am. J. Phys. **44**, 1132–1134 (1976).

[7] J. Pleines and S. Mahajan, Am. J. Phys. **45**, 868–869 (1977).

[8] G. Simon, Am. J. Phys. **47**, 566 (1979).

Erratum: "Einstein coefficients, cross sections, *f* values, dipol moments, and all that" [Am. J. Phys. 50, 982 (1982)]

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The numerator of the first factor in Eq. (25) should be $\gamma_{cl}/2\pi$. Consequently, the numerical values of $\sigma_a(\omega = \omega_{21})$ and $g(\omega_{21})$ in Sec. VIII should each be reduced by a factor of 2. In the caption for Table I, the correct equations are $B_{12} = (g_2/g_1)B_{21}$, $f_{12} = -(g_2/g_1)f_{21}$. I thank A. E. Siegman for pointing out the factor of 2 error.