EXTENSIONS OF SOME RING THEORETIC CONCEPTS TO MODULES: A SURVEY

ABSTRACT

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ABSTRACT

A ring R is a *(von Neumann) regular* ring if for every $a \in R$ there exists $b \in R$ such that $aba = a$. This concept was extended to modules by Zelmanowitz, Ware and Elliger. Following Zelmanowitz, a module $_RM$ is regular if for every $m \in M$ there exists $f \in Hom_R(M, R)$ such that $(mf)m = m$. A ring R is an *anti-regular* ring if for every nonzero $a \in R$ there exists $b \in R$ such that $bab = b \neq 0$. A module $_RM$ is an *anti-regular* module if for every nonzero $m \in M$ there exists $f \in Hom_R(M, R)$ such that $f(mf) = f$. R is a *left V- ring*, if every simple left R-module is injective. The module analogue of this class of rings is called *co-semi-simple modules* or *V-modules*. A ring is *reduced* if it has has no non-zero nilpotent elements. Extension of this class of rings to modules was done by Lee and Zhou. A left R-module M is *reduced module* if for every $m \in M$ and every $a \in R$ such that $am = 0$ we have $Rm \cap aM = 0$. A ring R is called an *abelian* ring if all idempotent elements of R are central i.e $I(R) \subset C(R)$. A ring R is *semi-commutative* if whenever $a, b \in R$ satisfy $ab = 0$ we have $acb = 0$ for each $c \in R$(equivalently, $aRb = 0$). R is called an *Armendariz* ring if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0 \; \forall i, j$. These classes of rings have straightforward generalizations to modules.

We now record some interesting results collected by us in this dissertation.

Result. The Jacobson radical of regular, anti-regular and V-rings as well as their module generalizations vanishes.

1.1.12 Proposition If R is a regular ring then every principal left ideal of R is generated by an idempotent.

More generally, we have

1.1.19 Proposition If R is a regular ring then every finitely generated (f.g) right(left) ideal is generated by an idempotent element.

1.2.9 Proposition The following conditions are equivalent for an element a of a ring R.

(i) There exists $b \in R$ such that $ab \in \bar{I}(R)$.

(ii) There exists $b \in R$ such that $ba \in \bar{I}(R)$.

(iii) There exists $r, s \in R$ such that $ras \in \bar{I}(R)$.

(iv) The element a is anti-regular in R.

1.2.14 Proposition If R is a regular ring then it is anti-regular .

1.3.5 Proposition The following properties of a ring R are equivalent:

(1) R is a left V-ring

(2) For every left $R-$module M, $Rad(M) = 0$

(3) For every cyclic left $R-$module M, $Rad(M) = 0$

(4) Every left ideal of R is an intersection of some family of maximal left ideals of R.

Similar to Proposition 1.1.12, we have the following proposition in the case of modules.

1.4.11 Proposition Let M be a left R-module. Then M is Z-regular if and only if for each $m \in M$, Rm is a projective direct summand of M.

1.4.12 Proposition A cyclic Z-regular R-module is projective and is isomorphic to a left ideal of R generated by an idempotent.

The following is an analogue of Proposition 1.2.9.

1.5.13 Proposition Let M be a left R-module. Let $N = Hom_R(M, R)$ and $S = End_R(M)$. The following conditions are equivalent for an element

m of a left R-module M.

(i) There exists $f \in N$ such that $mf \in \bar{I}(R)$.

(ii) There exists $f \in N$ such that $[f, m] \in \bar{I}(S)$.

(iii) There exist $f \in N, \alpha \in S$ such that $[f, m\alpha] = [f, m]\alpha \in \bar{I}(S)$.

(iv) There exist $r \in R, f \in N$ such that $(rm)f = r(mf) \in \bar{I}(R)$.

(v) The element m is anti-regular in M.

(vi) Rm contains a non-zero, projective,direct summand of M.

1.6.5 Proposition Let M be a left $R-$module. Then the following conditions are equivalent

(1)M is a left V-module

(2) Every simple left $R-$module is $M-$injective

In Chapter 2, we record some results on reduced, abelian, semi-commutative, and Armendariz rings.

2.1.7 Proposition If R is a commutative and anti-regular ring then R is reduced.

2.2.10 Proposition Let R be a ring. The following conditions are equivalent:

(i) R is an abelian ring.

(ii) $R[x]$ is an abelian ring.

(iii) $R[[x]]$ is an abelian ring.

The following two propositions record relations between reduced, semi-commutative and abelian rings.

2.2.11 Proposition If R is reduced then R is abelian.

2.3.6 Proposition If R is semi-commutative ring then R is abelian.

For a multiplicatively closed subset S of $C(R)$, $S^{-1}R$ is a semi-commutative ring if R is semi-commutative. In the following special case, the converse is also true.

2.3.14 Proposition Let R be a ring and let S be a multiplicatively closed subset of R consisting of central non-zero divisors. Then R is semi-commutative if and only if $S^{-1}R$ is semi-commutative.

In this chapter we prove the following implications for a ring R.

Reduced $\Rightarrow$ Armendariz $\Rightarrow$ Abelian.

2.5.21 Proposition Let R be von Neumann regular ring. Then the following conditions are equivalent:

(i) R is Armendariz.

(ii) R is reduced.

(iii) If the product of two linear polynomials in R is zero, then the products of their coefficients are also zero.

3.1.3 Proposition The following conditions are equivalent for a left R-module M:

(1) ${}_RM$ is reduced.

(2)For any $m \in M$ and $a \in R$, the following conditions hold,

(a) $am = 0$ implies $aRm = 0$.

(b)$a^2m = 0$ implies $am = 0$.

3.2.12 Proposition Let ${}_RM$ be a module. Then ${}_RM$ is an abelian module if and only if the ring $S_M = End({}_RM)$ is abelian.

3.3.12 Proposition Let M be an R-module and $C(R)$ be the centre of R. Then the following conditions are equivalent.

(i) M is semi-commutative.

(ii) $S^{-1}M$ is a semi-commutative $S^{-1}R$-module for each multiplicatively closed subset S of $C(R)$.

(iii) M_P is a semi-commutative R_P-module for each $P \in Spec(C(R))$.

(iv) M_Q is a semi-commutative R_Q-module for each $Q \in Max(C(R))$.

3.4.13 Proposition Let M be an R-module and let $C(R)$ be the centre of R. Then the following conditions are equivalent.

(i) M is Armendariz.

(ii) $S^{-1}M$ is Armendariz $S^{-1}R$-module for each multiplicatively closed subset S of $C(R)$.

(iii) M_P is an Armendariz R_P-module for each $P \in Spec(C(R))$.

(iv) M_Q is an Armendariz R_Q-module for each $Q \in Max(C(R))$.

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DECLARATION

North-Eastern Hill University
March, 2008

I, Kh.Herachandra Singh, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

This dissertation is being submitted to the North-Eastern Hill University for the degree of Master of Philosophy in Mathematics.

Signature of the Candidate

Signature of the Head

Signature of the Supervisor

CERTIFICATE

I certify that the dissertation entitled **'EXTENSIONS OF SOME RING THEORETIC CONCEPTS TO MODULES: A SURVEY'** submitted by Mr. Khwairakpam Herachandra Singh in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any University before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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Khwairakpam Herachandra Singh

INTRODUCTION

Important ring theoretic concepts have been extended to modules by a number of algebraists. The resultant interplay of ring theory and module theory has often enriched both.

In this dissertation an attempt has been made to carry out a brief survey of various extensions to modules of some important ring theoretic concepts. In order to set up a proper foundation for this study properties of the rings in question have been collected in earlier sections and properties of their natural extensions to modules have been recorded in later sections. Rings whose extensions to modules have been studied are primarily the classes of (von Neumann) regular rings, anti-regular rings, V-rings, reduced rings, abelian rings,semi-commutative rings and Armendariz rings. These objects are of interest because there has been a great deal of significant research devoted to these rings and modules in the recent decades.

This dissertation consists of five chapters. Chapter 0 is devoted to fixing notation, clarifying our terminology and recording concepts and results which are used in later chapters.

Chapter 1 is devoted to a survey of some research work done in the areas of regular rings, anti-regular rings and V-rings and the corresponding classes of modules. The concept of a (von Neumann) regular ring has been studied by many authors since its introduction by John von Neumann in 1936. (See the vast bibliography of [G]). The class of anti-regular rings contains the class of regular rings. The class of V-rings, rings over which simple modules are injective, is named after Villamayor and was studied by Michler and Villamayor in [MV] and many other authors later.

Natural extensions of the regular ring concept to modules were carried out by Ware[W], Fieldhouse[F], Zelmanowitz[Z] and Elliger[E]. However, our main interest has been in the work of Zelmanowitz. The anti-regularity condition in rings and its extension to modules was studied in [CDR 1-4]. There is a natural extension of the V-ring concept to V-modules(also called 'cosemisimple modules'). In view of the interesting connections between regular rings and V-rings (see [SV]), it is natural to investigate relationships between V-modules and various classes of regular modules.

Chapter 2 is devoted to the study of reduced rings, abelian rings, semi-commutative rings, reversible rings and Armendariz rings. We shall now briefly introduce these concepts.

The notion of an Armendariz ring was introduced in [RC]. A ring R is *Armendariz* if given polynomials $f(x) = \Sigma_{i=0}^{i=m} a_i x^i$ and $g(x) = \Sigma_{j=0}^{j=n} b_j x^j$ with coefficients in R, the condition $f(x)g(x) = 0$ implies $a_i b_j = 0$ for every i and j. Basic properties and examples of Armendariz rings were recorded in [RC]

A number of authors have extended results from [RC]; see, for example, [AC],[H2],[HKR],[HLS],[KL1],[LW] and [LZ1].

A ring R is *reduced* if it has no nonzero nilpotent elements; R is *abelian* if every idempotent in R is central; R is *semi-commutative*(resp.*reversible*) if whenever elements a, b in R satisfy $ab = 0$, then $acb = 0$ for each element c of R (resp. $ba = 0$). Reduced rings are reversible, reversible rings are semi-commutative, and semi-commutative rings are abelian. The class of

Armendariz rings also lies between the classes of reduced rings and abelian rings, by [A], Lemma 1 and [HLS], Corollary 8. Semi-commutative rings need not be Armendariz by [RC, Example 3.2] and Armendariz rings need not be semi-commutative by [HLS], Example 14.

As mentioned in [RC, Remark 4.7] the Armendariz ring concept can be extended to power series rings (resp., polynomial modules, power series modules) to get the concept of ps-Armendariz rings (resp., Armendariz modules, ps-Armendariz modules). Extending the concept of a semi-commutative ring, a module is *semi-commutative* if it satisfies the following condition: whenever elements $a \in R$ and $m \in M$ satisfy $am = 0$, then $acm = 0$ for each element c of R. An extension of the concept of a reduced ring was introduced by Lee and Zhou [LZ2].The relationships of Armendariz modules and reduced modules with semi-commutative - and other classes of - modules have been studied in [AH], [BA], [BR] and [RB]. Chapter 3 has been devoted to a study of these module-theoretic ideas.

In the final Chapter 4, we suggest some questions which are expected to be useful for studying the interplay of these concepts further. It is planned to study them elsewhere.

In most proofs we have faithfully followed the papers where the research was published. Only in Section 2.6 some innovations in arguments may be found. New results may be expected while attempting solutions to problems in Chapter 4.

We conclude this dissertation by a bibliography.

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Chapter 0

Basic definitions and results

In this chapter we record some basic definitions, notation,terminology and well-known results which will be used throughout this dissertation. All our left-sided definitions and results have right-sided counterparts.

0.1 Rings

By a *ring* we mean an associative ring with an identity element.Subrings and ring homomorphisms are unitary. Sometimes we may consider algebraic systems which satisfy all axioms of a ring other than the existence of an identity element, so-called 'rings without identity' or 'rngs'. We may also use the term subrng in a similar manner.

Domain means a (possibly non-commutative) ring without zero-divisors.

The letter R denotes a ring and (unless the contrary is indicated by the context) D denotes a domain (often, a division ring).

0.1.1 Definition. A nonempty subset I of R is called a *left (right) ideal* of R if

$(i) a, b \in I \Rightarrow a - b \in I$ and

$(ii) a \in I \ x \in R \Rightarrow xa \in I (ax \in I)$

If I is both a left as well as a right ideal of R then it is called a *two-sided* ideal of R, or simply *an ideal* of R

0.1.2 Definition. Let a be an element of a ring R. By the *left annihilator* of a we mean the set $\{x \in R \mid xa = 0\}$ and by the *right annihilator* of a the set $\{x \in R \mid ax = 0\}$.

0.1.3 Notation. We denote by $l_R(a)$ the left annihilator and by $r_R(a)$ the right annihilator of a.Then $l_R(a)$ is a left ideal of R and $r_R(a)$ is a right ideal of R. When the ring is clear from the context we drop the letter R and write $l(a)$ for $l_R(a)$ and $r(a)$ for $r_R(a)$. When the left and right annihilators of an element a coincide (for example , in a commutative ring) we may write $\text{ann}(a)$ for $l(a) = r(a)$.

0.1.4 Definition. An element a of a ring R is called an *idempotent* element if $a^2 = a$.

0.1.5 Notation. We denote by $I(R)$ the set of all idempotent elements of a ring R and by $\bar{I}(R)$ the set of all non-zero idempotent elements.

0.1.6 Definition. An element a of a ring R is called a *nilpotent* element if $a^n = 0$ for some positive integer n.

0.1.7 Notation. We denote by *Nil(R)* the set of all nilpotent elements of R.

0.1.8 Definition. The *centralizer* of an element a in R is the set $\{x \in R \mid xa = ax\}$

0.1.9 Notation. The centralizer of an element a of R is denoted by $C(a)$. The centre of R (denoted by $C(R)$) is defined to be the subring $\{r \in R \mid rx = xr \ \forall x \in R\}$ of R.

0.1.10 Definition. A ring R is *semi-simple* if every left ideal of R is a direct summand of R (i.e., of the left R-module R).

0.1.11 Remark. It is known that semi-simplicity is a left-right symmetric condition.

0.1.12 Definition. A ring R is said to be *directly finite* (abbreviated as d.f) if whenever elements $a, b \in R$ satisfy $ab = 1$, we have $ba = 1$.

0.1.13 Notation. Let R be a ring then we denote the ring of polynomials over R by $R[x]$ and the ring of *Laurent polynomials* in x, coefficients in R by $R[x; x^{-1}]$ i.e $R[x; x^{-1}]$ consists of all formal sums $\sum_{i=k}^{n} a_i x^i$ (where $a_i \in R$ and $k, n \in \mathbb{Z}$) with obvious addition and multiplication.

0.1.14 Definition. A ring R is *left invariant* if every left ideal of R is two-sided.

0.1.15 Definition. A ring R is *reduced* if it has no nonzero nilpotent elements.

0.2 Modules

Let R be a ring. By a *left R-module* we mean a set M equipped with two binary operations (denoted below by $+$ and $.$) satisfying the following conditions:

(1) $(M, +)$ is an abelian group.

(2) There exists a map $. : R \times M \to M$, known as the external binary operation of R on M (or the scalar multiplication), such that the following conditions hold.

$(M1)\, r.(m_1 + m_2) = r.m_1 + r.m_2,$ for all $r \in R, m_1, m_2 \in M$

$(M2)\, (r_1 + r_2).m = r_1.m + r_2.m,$ for all $r_1, r_2 \in R, m \in M$

$(M3)\, (rs).m = r.(s.m),$ for all $r, s \in R, m \in M.$

$(M4)\, 1.m = m,$ for all $m \in M.$

0.2.1 Notation. The notation ${}_RM$ shall often be used to denote the left R-module M . We shall also simplify the notation $r.m$ to rm . Unless otherwise mentioned, by an R-module we mean a left R-module.

0.2.2 Remark. A right R-module M is similarly defined as an additive abelian group satisfying conditions dual to those for a left R-module, namely

$(M1)'\, (m_1 + m_2).r = m_1.r + m_2.r\ \forall m_1, m_2 \in M, r \in R$

$(M2)'\, m(r_1 + r_2) = m.r_1 + m.r_2\ \forall m \in M, r_1, r_2 \in R$

$(M3)'\, m.(rs) = (m.r).s\ \forall m \in M, r, s, \in R$

$(M4)'\, m.1 = m,\ \forall m \in M.$

0.2.3 Notation. M_R shall often denote the right R-module M. We shall again simplify $m.r$ (the notation for the scalar multiplication of $r \in R$ with $m \in M_R$) to mr.

0.2.4 Definition. A module M which is both a left and a right R-module and satisfies $(rm)r' = r(mr') \forall r, r' \in R$ and $m \in M$ is called a *left* $R-$, *right* R*-bimodule* (abbreviated to R-bimodule)

0.2.5 Definition. Let M, M' be left R-modules. A map $\theta : M \to M'$ is called an *R-module homomorphism or an R-homomorphism or an R-linear map* if

(1) $\theta(m_1 + m_2) = \theta(m_1) + \theta(m_2)\ \forall m_1, m_2 \in M$

(2) $\theta(rm) = r\theta(m)\ \forall r \in R, m \in M$

0.2.6 Definition. An R-module P is *projective* if given an exact sequence $\beta : M \to W \to 0$ (β is onto) and an R -homomorphism $g : P \to W$ there exists an R -homomorphism $f : P \to M$ such that $\beta \circ f = g$.

0.2.7 Definition. An R-module E is *injective* if for every monomorphism $\alpha : L \longrightarrow M$ and every $\phi : L \longrightarrow E$ there exists $\pi : M \longrightarrow E$ such that $\pi \circ \alpha = \phi$

0.2.8 Definition. Let M be a left $R-$module. A left R-module W is said to be *M-injective* if given an R-submodule M' of M, monomorphism $j : M' \hookrightarrow M$ and a homomorphism $\phi : M' \longrightarrow W$ there exists an R-homomorphism $\pi : M \longrightarrow W$ such that $\pi \circ j = \phi$

0.2.9 Remark. All injective modules are $M-$injective (for each R-module M).

0.2.10 Definition. Let R be a commutative ring and let M be a $R-$module. Then $R \times M$ is a ring under addition and multiplication defined by

$(r, m) + (s, n) = (r + s, m + n)$

$(r,m).(s,n) = (rs, rn + sm)$ where $r, s \in R$ and $m, n \in M$. We denote this ring by $R(+)M$ or $T(R, M)$. We call $T(R, M)$ the *Nagata* or *trivial extension* of R by M.

0.2.11 Notation. Let R be a commutative ring and $h : R \to R$ be a ring homomorphism. Let M be an R-module. We denote by $R(+)_h M$ the abelian group $R \times M$ which is given a ring structure via addition and multiplication defined by

$(r,m) + (s,n) = (r+s, m+n)$

$(r,m).(s,n) = (rs, h(r)n + sm)$ for $r, s \in R$ and $m, n \in M$.

We call $R(+)_h M$ the *twisted Nagata extension* of R by M.

0.2.12 Notation. Let I be an ideal of a ring R. Then $R \oplus (R/I)$ is a ring under multiplication defined as $(r, \bar{x})(s, \bar{y}) = (rs, \overline{rx + xs})$. We denote this ring by $R(+)(R/I)$.

0.2.13 Remark. We have the identifications $(R/I)[x] \cong R[x]/I[x]$ and $(R(+)M)[x] \cong R[x](+)M[x]$.

0.2.14 Definition. A left R-module M is said to be *semi-simple* if every submodule of M is a direct summand of M.

0.2.15 Notation. We use the notation $W \leq^{\oplus} M$ to mean that W is a direct summand of M.

0.2.16 Definition. Let M be a left R-module. Then the *radical* of M is the intersection of all maximal submodules of M. It is denoted by $Rad(M)$. The radical of a ring R, $Rad(R)$, is the intersection of all maximal left ideals of R.

0.2.17 Definition. A left R-module M is *directly finite* if whenever W is submodule of M such that (i) W is direct summand of M and (ii) $W \cong M$, then we have $W = M$.

0.2.18 Definition. An R-module is *torsionless* if it is a submodule of a direct product of copies of R.

0.2.19 Notation. In the study of regular, anti-regular and abelian modules and in some other places we write homomorphisms of modules on the side opposite of that of scalars. Following this convention, if $f : M \longrightarrow W$ is a homomorphism of left R-modules, the image of an element $m \in M$ under f is denoted by mf. Further, if $f : M \longrightarrow W$ and $g : W \longrightarrow P$ are R-homomorphisms, their composition - under which an element $m \in M$ is mapped onto $(mf)g$ - is denoted by $f \circ g$. This convention is not followed throughout; elsewhere the usual, *fuctional*, notation is used. We seek the reader's indulgence for the confusion that this may possibly cause.

Chapter 1

Regular rings,anti-regular rings, V-rings and corresponding generalizations to modules

In this chapter we study von Neumann regular, anti-regular and V-rings. These are considered in sections 1.1, 1.2 and 1.3. Regular rings are anti-regular but not conversely; examples are given in section 1.2. Every factor ring of a regular ring is regular but in the case of anti-regularity the analogue is false.As example of a non-anti-regular factor ring of an anti-regular ring is given in section 1.2. Generalizations of these notions to modules are also studied in this chapter.

1.1 Regular rings

In this section we study the basic examples and properties of von Neumann regular rings.We first begin with the following definition.

1.1.1 Definition. Let R be a ring. An element $a \in R$ is *(von Neumann) regular* if there exists $b \in R$ such that $aba = a$.

1.1.2 Notation. We denote the set of all regular elements of a ring by $Reg(R)$.

In general, product of two regular elements need not be regular. We give the following definition in case this condition holds.

1.1.3 Definition. A ring R has *property*(RC) if the subset $Reg(R)$ is closed under multiplication.

1.1.4 Proposition. *If R is commutative then R has property (RC).*

Proof. let a and b be two regular elements of R. Then there exists $c, d \in R$ such that $a = aca$ and $b = bdb$. Now $ab = aca.bdb = ab(cd)ab$, showing that ab is regular.

1.1.5 Definition. A ring is a *(von Neumann) regular* ring if every element is regular.

1.1.6 Remarks. (1)If R is regular then $Reg(R) = R$ which has property (RC).
(2) Let $\{R_i\}_{i \in I}$ be a family of rings and let $R = \Pi R_i$, then
$Reg(R) = \Pi Reg(R_i)$ under the natural identification.
(3) The ring ΠR_i has property (RC) if and only if each R_i has property (RC).

1.1.7 Examples. (1)0 and 1 are always regular in any ring R.

(2) Every element in a field is regular.(This holds since for $a \neq 0$, there exists $b \in R$ such that $ab = 1 \Rightarrow aba = a$)

1.1.8 Example. Every field(in general, division ring) is a regular ring.

1.1.9 Proposition. *A non-zero ring R is a regular domain if and only if it is a division ring.*

Proof. If R is regular then every element of R is regular . Let $a \neq 0 \in R$ then there exists $b \in R$ such that $aba = a \Rightarrow a(ba-1) = 0 \Rightarrow ba = 1$ (Since R is a domain). Again $(ab-1)a = 0 \Rightarrow ab - 1 = 0 \Rightarrow ab = 1$. Hence a has an inverse.Therefore R is a division ring. Again if R is a division ring then every non-zero elements $a \in R$ has an inverse.That means there exists $b \in R$ such that $ab = ba = 1 \Rightarrow aba = a \Rightarrow a$ is regular. Hence R is a regular ring and a domain. □

In proposition 1.1.10 and 1.1.11 we record some basic properties of regular elements.

1.1.10 Proposition. *If a is a regular element in a ring R then there exists $b \in R$ such that ab and ba are both idempotent elements.*

Proof. Since a is a regular element there exists $b \in R$ such that $aba = a$. We have to show that $ab, ba \in I(R)$ where $I(R)$ is the set of all idempotent elements in R. Now $(ab)^2 = ab.ab = (aba)b = ab$. Therefore $ab \in I(R)$. Similarly we can show that $ba \in I(R)$. □

1.1.11 Proposition. *If a is a regular element and if $aba = a$ for some $b \in R$ then $Ra = Rba$.*

Proof. We can easily see that

$$Rba \subseteq Ra. \tag{1}$$

Again, let us take $x \in Ra$ then $x = ca$ for some $c \in R$. Now from (1) $x = c(aba) = (ca)(ba) \in Ra$. Hence

$$Ra \subseteq Rba \tag{2}$$

From (1) & (2) we get that $Ra = Rba$. □

1.1.12 Proposition. *If R is a regular ring then every principal left ideal of R is generated by an idempotent.*

Proof. From Proposition 1.1.11 every principal left ideal Ra is of the form Rba for some $b \in R$ satisfying $a = aba$ and $ba \in I(R)$. Therefore $Ra = Re$ where $e = ba \in I(R)$. □

1.1.13 Proposition. *A direct product of regular rings is a regular ring.*

Proof. Let $\{ R_i \mid i \in I\}$ be a family of regular rings. Let $R = \prod_{i \in I} R_i$ and $a \in R$ then $a = (a_i)_{i \in I}$. Since R_i are regular for all $i \in I$ then there exists $b_i \in R_i$ such that $a_i = a_i b_i a_i$ for all $i \in I$. Now $(a_i)(b_i)(a_i) = (a_i b_i a_i) = (a_i)_{i \in I}$. Therefore there exists $b = (b_i) \in R$ such that $aba = a$. Hence $R = \prod_{i \in I} R_i$ is regular. □

1.1.14 Proposition. *Every factor ring of a regular ring is regular.*

Proof. Let I be two-sided ideal of a regular ring R. Let $a + I \in R/I$ for some $a \in R$. Since R is regular then there exists $b \in R$ such that $aba = a$. Now $(a + I)(b + I)(a + I) = (aba + I) = a + I$. Therefore $a + I$ is regular □

1.1.15 Remark. Subrings of a regular ring need not be regular as can be seen in the following example.

1.1.16 Example. As $\mathbb{Q}$ is a field so $\mathbb{Q}$ is a regular ring but its subring $\mathbb{Z}$ is not regular.

1.1.17 Proposition. *If R is a regular ring then its centre $C(R)$ is also regular.*

Proof. Recall that the centre $C(R)$ of a ring is defined as $C(R) = \{r \in R \mid rx = xr, \forall x \in R\}$. Let $a \in C(R)$. Since R is regular there exists $b \in R$ such that

$$aba = a \tag{$*$}$$

Let $e = ab = ba$. Then $e^2 = e$. Now, let us show that $e \in C(R)$. Let $x \in R$ .Then $ex = abx = bxa = bxaba = bxba^2$ (since $a \in C(R)$) and $xe = xba = a(xb) = aba(xb) = abxba = bxba^2$. Hence $xe = ex \; \forall x \in R$ implying that $e \in C(R)$. Again we can show that $eb \in C(R)$. Let $y \in R$ then

$$(eb)y = e(by) = (by)e = (by)ba = byba \tag{1}$$

and

$$y(eb) = e(yb) = ab(yb) = byba \tag{2}$$

From (1) and (2) we get $(eb)y = y(eb) \quad \forall y \in R$ Hence $eb \in C(R)$. Now $a = aba = ea = eaba = a(eb)a$. Thus $a = a(eb)a$ that is a is regular. Hence $C(R)$ is regular. □

1.1.18 Proposition. *If R is a regular ring and S is a multiplicatively closed subset of $C(R)$ then $S^{-1}R$ is regular.*

Proof. Let $a/s \in S^{-1}R$ be an element where $a \in R$ and $s \in S$. Since R is a regular ring then there exists $b \in R$ such that $aba = a$. Now $(a/s).(sb/1).(a/s) = (aba)/s = a/s$, showing that $S^{-1}R$ is regular. □

1.1.19 Proposition. *If R is a regular ring then every finitely generated (f.g) right(left) ideal is generated by an idempotent element.*

Proof. We prove this result for left ideals.
Let I be a f.g left ideal of R with as generators $a_1, a_2, ..., a_n$.
Then

$$I = Ra_1 + Ra_2 + ... + Ra_n. \tag{1}$$

We carry out induction on the number of generators of I.
When $n = 1, I = Ra_1$ where $a_1 \in R$. Then by Proposition 1.1.12, I is generated by an idempotent element. When $n = 2, I = Ra_1 + Ra_2$. Now

$$Ra_1 + Ra_2 = Ra_1 + Re \tag{2}$$

where $Ra_2 = Re,\ e \in I(R)$. Now there exists $f \in I(R)$ such that $Ra_1(1 - e) = Rf$. Therefore $f = ra_1(1 - e)$ for some $r \in R$. This implies that

$$fe = 0. \tag{3}$$

Now $Rf = Ra_1(1-e) \subseteq Ra_1 + Re \Rightarrow Rf + Re \subseteq (Ra_1 + Re) + Re = Ra_1 + Re$. Again let $x \in Ra_1 + Re \Rightarrow x = r_1a_1 + r_2e$ for some $r_1, r_2 \in R = r_1a_1 - r_1a_1e + r_1a_1e + r_2e = r_1a_1(1 - e) + (r_1a_1 + r_2)e \in Rf + Re$. Showing that $Ra_1 + Re \subseteq Rf + Re$. Hence $Ra_1 + Re = Rf + Re$, implying that

$$Ra_1 + Ra_2 = Rf + Re. \tag{4}$$

Now $(e+f-ef)^2 = (e+f-ef)(e+f-ef)$
$= e+ef-ef+fe+f-fef-efe-ef+efef$
$= e+f-ef$.(using (3))
Thus $(e+f-ef) \in I(R)$. Then let $y \in R(e+f-ef) \Rightarrow y = r(e+f-ef)$ for some $r \in R = (r-re)f + re \in Rf + Re$. Thus

$$R(e+f-ef) \subseteq Rf + Re. \tag{5}$$

Again let $z \in Rf + Re$. then $z = \alpha f + \beta e$ for some $\alpha, \beta \in R$. Now $z(e+f-ef) = (\alpha f + \beta e)(e+f-ef) = \alpha fe + \alpha f - \alpha fef + \beta e + \beta ef - \beta ef) = \alpha f + \beta e = z$. (using (3)) Thus $z \in R(e+f-ef) \Rightarrow Rf + Re \subseteq R(e+f-ef)$
From (5)we get $Rf + Re = R(e+f-ef)$
Hence by induction on the number of generators of I we can prove that every f.g left ideal is generated by an idempotent element. □

1.1.20 Proposition. *Suppose that* 0 *and* 1 *are the only idempotent elements of a nonzero ring R. Then the following conditions hold:*
(1) If a is a non-zero regular element of R then a is invertible.
(2) If R is a regular ring then R is division ring.

Proof. (1) Since a is a non-zero regular element of R then there exists $b \in R$ such that $aba = a$.
We know that $ab, ba \in I(R)$,
if $ab = 0 \Rightarrow aba = a = 0$, which is a contradiction, therefore $ab = 1$
Similarly, we can show that $ba = 1$. Hence a is invertible.
(2) follows from (1) □

1.1.21 Proposition. *A commutative ring R is von Neumann regular if and only if $I^2 = I$ for each ideal I of R.*

Proof. (⇒) Let $a \in I$. Since R is a regular ring then there exists $b \in R$ such that $a = aba \in I.I = I^2$. Therefore $I \subseteq I^2$. Since I is ideal then $I^2 \subseteq I$. So $I = I^2$.

(⇐) Let $a \in R$. Then $a \in Ra = (Ra)^2 = Ra.Ra = Ra^2$, [By hypothesis]. Therefore $\exists b \in R$ such that $a = ba^2$. Since R is commutative $a = aba$. Therefore, R is regular. □

1.1.22 Proposition. *If R is regular then $Rad(R) = 0$*

Proof. If $Rad(R) \neq 0$ then let us take x be a non-zero element in $Rad(R)$. Since R is regular there exists $y \in R$ such that $x = xyx \in RxRx$.

Therefore $Rx = RxRx$. Hence, there exists $z \in RxR$ such that $x = zx$. Since $x \in Rad(R)$ then $RxR \subseteq Rad(R)$. We know that $t \in Rad(R)$ if and only if $1 - st$ is a unit, for all $s \in R$. In particular $1 - z$ is a unit.

Hence $x(1 - z) = 0 \Rightarrow x = 0$. Therefore $Rad(R) = 0$. □

1.2 Anti-regular rings

In this section we study anti-regularity in rings. We begin with some definitions and general properties of anti-regular elements. We record examples of anti-regular rings which do not have anti-regular localizations or factor rings.

1.2.1 Definition. An element a of a ring R is said to be *left anti-regular* if there exists $b \neq 0$ of R such that $b = b^2a$.

1.2.2 Definition. Similarly we say a is *right anti-regular* if there exists $b \neq 0$ in R such that $b = ab^2$.

1.2.3 Definition. An element a of R is said to be *anti-regular* if there exists $b \neq 0$ in R such that $b = bab$.

1.2.4 Remark. All non-zero idempotent elements satisfy all three conditions of 1.2.1 - 1.2.3.

1.2.5 Definition. A ring R is said to be *anti-regular* if all non-zero elements are anti-regular elements.

1.2.6 Example. Division rings are anti-regular.

1.2.7 Proposition. *Let u and a be two elements of R. If a is an anti-regular and u is left invertible then ua is anti-regular.*

Proof. Since a is anti-regular then there exists $b \neq 0$ such that $b = bab$ and since u is left invertible $vu = 1$ for some $v \in R$. Now $bvu = b \Rightarrow bv \neq 0$. Further $bv.ua.bv = babv = bv \neq 0$. Therefore ua is anti-regular. □

1.2.8 Proposition. *If a is anti-regular and u is right invertible then au is anti-regular.*

Proof. Similar to that of the Proposition 1.2.7. □

1.2.9 Proposition. *The following conditions are equivalent for an element a of a ring R.*

(i) There exists $b \in R$ such that $ab \in \bar{I}(R)$.

(ii) There exists $b \in R$ such that $ba \in \bar{I}(R)$.

(iii) There exists $r, s \in R$ such that $ras \in \bar{I}(R)$.

(iv) The element a is anti-regular in R.

Proof. ((i)⇒(ii)) Since $ab \in \bar{I}(R)$ then $ab.ab = ab$. Let $c = bab$.
Then $ca.ca = (bab)a.(bab)a = bababa = (bab)a = ca$. Therefore there exists $c \in R$ such that $ca \in \bar{I}(R)$.
((ii) ⇒ (iii)) Since $ba \in \bar{I}(R)$ then $ba.ba = ba$
Let $r = b$ and $s = ba$.
Now $ras.ras = baba.baba = baba = ras$. Hence $ras \in \bar{I}(R)$.
((iii)⇒ (iv)) Since $ras \in \bar{I}(R)$ then we have $ras \neq 0$. Let $b = srasr$. If $b = 0$ then $ras = (ras)^3 = ras.ras.ras = ra(srasr)as = ra.b.as = 0$ which is contradiction. Therefore $b \neq 0$. Now $bab = srasr.a.srasr = s(ras)^3r = srasr = b$. Hence a is anti-regular.
((iv) ⇒(i)) Since a is anti-regular then there exists $0 \neq b \in R$ such that $bab = b$. It follows that $ab.ab = ab$. Hence there exists $b \in R$ such that $ab \in \bar{I}(R)$. □

Under every ring homomorphism, image of an anti-regular element need not be anti-regular. The following proposition gives a sufficient condition for this to happen.

1.2.10 Proposition. *Let $\phi : R \longrightarrow R'$ be a one-to-one ring homomorphism and let a be an anti-regular element of R then $\phi(a)$ is anti-regular in R'.*

Proof. Since a is an anti-regular we have $bab = b \neq 0$.
Now $\phi(bab) = \phi(b) \neq 0$, as ϕ is one-one. This implies that $\phi(b)\phi(a)\phi(b) = \phi(b) \Rightarrow \phi(a)$ is anti-regular. □

1.2.11 Remark. If a is left (right) invertible then a is anti-regular, left (right) anti-regular.

1.2.12 Proposition. *A left(right) anti-regular element cannot be a nilpotent element.*

Proof. Let a be a left anti-regular element.
Since a is left anti-regular then there exists $b \neq 0$ such that $b^2a = b$
Now $0 \neq b = bb^2aa = b^3a^2 = bb^3a^2a = b^4a^3 = \ldots\ldots\ldots\ldots = b^{n+1}a^n$ for each natural number n. Hence a cannot be a nilpotent element. □

Now we have the following corollary which follows from Proposition 1.2.12.

1.2.13 Corollary. *If R is a commutative anti-regular ring then R is reduced.*

The converse is not true. [e.g $\mathbb{Z}$ is reduced but not anti-regular.]

1.2.14 Proposition. (a) *If R is regular then it is anti-regular .*

(b) *If R is a commutative regular ring then R is reduced.*

Proof. **(a)** Let a be a non-zero element of R. Since R is regular there exists $b \in R$ such that $aba = a$.
Write $c = bab$ which is not zero, since $aca = ababa = aba = a \neq 0$. Now we

have $cac = (bab)a(bab) = bab(aba)b = babab = b(aba)b = bab = c \neq 0$. Hence a is anti-regular.
Thus every non-zero element in R is anti-regular and hence R is anti-regular.
(b) follows from Part**(a)** and Corollary 1.2.13. □

The converse of Proposition 1.2.14(a) is not true. This is shown through the following example.

1.2.15 Example. Let us consider the ring R of sequences of rational numbers which eventually take a constant integral value. We can easily verify that axioms of a ring hold in R under (pointwise) addition and multiplication of sequences. Thus we may write (assuming in the notation used below that for all $j \geq n+1$ we have $a_j = a$ an integer)
$R = \{(a_1, a_2, ..., a_n, a, a, a, ...) | a_i \in \mathbb{Q}, a \in \mathbb{Z}\}$. Let x be a non-zero element in R. Now $x = (a_i)$ where finitely many of them are rational and non-integral while the remaining entries are the same integer. Since x is not zero there exists some j such that $a_j \neq 0$. Now,let $y = (b_k)$ where $b_k = 1/a_j$ when $k = j$ and $b_k = 0$ for all $k \neq j$. Now $y \neq 0$ and we have $yxy = y$ showing that R is an anti-regular ring. Next we show that R is not a regular ring.
Let us consider the element $z = (2, 2, 2, 2, ...)$. If z is a regular element of R then there exists $t \in R$ such that $ztz = z$. We may assume that $t = (c_1, c_2, ..., c_n, c, c, c, ...)$ where c is an integer.
Since $ztz = z$ we have $2c2 = 2 \Rightarrow 2c = 1$ where $c \in \mathbb{Z}$ which is a contradiction. So the ring R is not a regular ring.

Some more examples of anti-regular, non-regular rings are as follows: the ring of continuous real-valued funtions defined on the space of rationals and the ring of bounded sequences of real numbers.

In the case of regular ring R we have seen that $S^{-1}R$ is regular for every multiplicatively closed subset S of $C(R)$. In the case of anti-regularity the analogue of this result is not true. This is shown below.

1.2.16 Example. We have seen above that the ring of all sequences of rational numbers which eventually take a constant integral value
i.e $R = \{(a_1, a_2, \ldots.., a_n, a, a, a\ldots.., a, \ldots.) \mid a_i \in \mathbb{Q}, a \in \mathbb{Z}\}$ is an anti-regular but not a regular ring .
The map $\theta : R \longrightarrow \mathbb{Z}$ defined by $\theta(a_1, a_2, \ldots.., a_n, a, a, a\ldots.., a, \ldots.) = a$ is an onto ring homomorphism. Consider $S = \{e_n \mid n \in \mathbb{N}\}$ where $e_{n_j} = 0$ for $1 \leq j \leq n-1$ and $e_{n_j} = 1$ for $j \geq n$. Then S is a multiplicatively closed subset of $C(R)$. Since $\theta(e_n) = 1$ for each $n \in \mathbb{N}$ there is an induced ring homomorphism $\phi : S^{-1}R \longrightarrow \mathbb{Z}$. It is easily verified that ϕ is an isomorphism. Thus $S^{-1}R$ is not anti-regular.
Therefore the localization of an anti-regular ring need not be anti-regular.

Now we have the following proposition for the localization of anti-regular rings.

1.2.17 Proposition. *Let R be a ring and S a multiplicatively closed subset of the centre of R. Assume that each element of S is a non-zero-divisor in R.*

(i) If r is anti-regular in R and $t \in S$ then r/t is anti-regular in $S^{-1}R$.

(ii) If R is an anti-regular ring then $S^{-1}R$ is also anti-regular ring.

Proof. (i) Since S does not contain any zero-divisors the natural homomorphism $\eta : R \longrightarrow S^{-1}R$ given by $\eta(r) = r/1$, is one-to-one. Then we know

that r anti-regular in R implies $\eta(r) = r/1$ is anti-regular in $S^{-1}R$. Again $1/t$ is invertible in $S^{-1}R$. By Proposition 1.2.7, r/t is anti-regular in $S^{-1}R$.
(ii) Let $r/t \in S^{-1}R$ be a non-zero element. Then $r \neq 0$ in R. Since R is anti-regular, the element r is anti-regular. Also $1/t$ is invertible in $S^{-1}R$.Therefore r/t is an anti-regular. Hence $S^{-1}R$ is an anti-regular ring. □

Factor rings of regular rings are regular but an analogue of this statement is not true for anti-regular rings as shown in the following example.

1.2.18 Example. Let us consider the ring R of all sequences of rational numbers which eventually take a constant integral value. It was noted in Example 1.2.16 that $\mathbb{Z}$ is a factor ring of R which is not anti-regular.

1.3 V-rings

In this section, we study simple propeties of V-rings and relation with other rings. We begin with the following definition.

1.3.1 Definition. A ring R is a left *V-ring* if every simple (left) R-module is injective.(Right V-rings are defined similarly. When R is commutative we talk of V-rings.)

1.3.2 Example. Every field is a V-ring.

1.3.3 Proposition. *Let D be a commutative non-zero domain. Then D is a V-ring if and only if D is a field.*

Proof. ($\Rightarrow$) Let μ be a maximal ideal of R.Then the R - module R/μ is simple. Since R is a V-ring therefore R/μ is injective as a module . If $\mu \neq 0$, let a be a nonzero element of μ. As D is a domain, we can define a D-linear map $\theta : Da \longrightarrow D/\mu$ by $\theta(xa) = \bar{x} = x + \mu$. As D/μ is D-injective, there exists $\bar{y} \in D/\mu$ satisfying $\bar{1} = \theta(a) = a\bar{y} = \bar{a}\bar{y}$. But then $1 - ay \in \mu \Rightarrow 1 \in \mu$, a contradiction. So D must be a field.
($\Leftarrow$) Trivial. □

1.3.4 Proposition. *Let R be a commutative ring. If R is regular then R is a V-ring.*

Let M be a simple R-module.Then $M = Rm$ for some nonzero element m of M i.e M is a cyclic module. Let A be an ideal of R and $f : A \rightarrow M$ be a R-homomorphism.
Case 1. if $f = 0$ then $g : R \rightarrow M$, defined by $g = 0$ extends f.
Case 2. if $f \neq 0$ then f is onto.

Therefore $A/K \simeq M$ where K is the kernel of f. Since M is cyclic then A/K is also cyclic.

Again R is regular then R/K also regular. Therefore A/K is generated by an idempotent element.

Therefore $A/K \leq^{\oplus} R/K \Rightarrow R/K = A/K \oplus B/K$ for some ideal B of R.

Now we want $g : R \to M$ such that $g\mid_A = f$,let us consider $p : R/K \to A/K$ such that isomorphism $r + K \longmapsto a + K$ uniquely, where $r = a + b$.

Therefore $R \xrightarrow{\eta} R/K \xrightarrow{p} A/K \xrightarrow{\tilde{f}} M$

$r \longmapsto r + K \longmapsto a + K \longmapsto f(a)$,then $g = \tilde{f}op o\eta$.

Let $a \in A$ then

$g(a) = (\eta)(a) = \tilde{f}op(a + K) = \tilde{f}(a + K) = f(a)$.Therefore $g\mid_A = f$.

Hence f can be extended to $g : R \to M$.

It follows that M is injective as an R-module.

Thus R is a V-ring □

1.3.5 Proposition. *The following properties of a ring R are equivalent:*

(1) R is a left V-ring

(2) For every left $R-$module M, $Rad(M) = 0$

(3) For every cyclic left $R-$module M, $Rad(M) = 0$

(4) Every left ideal of R is an intersection of some family of maximal left ideals of R

Proof. (1) $\Rightarrow$ (2) Let M be a left $R-$module. If x is a non zero element of M then by Zorn's lemma there is a submodule W of M which is maximal submodule among the submodules X of M with $x \notin X$. Let B denote the intersection of all submodules S of M with $S > W$

Then $x \in B$ and $W < B$.

Therefore $B/W \neq 0$.

Again B/W is simple. [For $B'/W \leq B/W \Rightarrow W \leq B' \leq B$. Two cases arise, Case 1: if $x \neq B'$ then $W = B' \Rightarrow B'/W = O$. Case 2: If $x \in B'$ then $B \leq B' \Rightarrow B = B' \Rightarrow B'/W = B/W$] .

Since R is a V-ring, B/W is injective. So $M/W = B/W \oplus K/W$ where K is a submodule of M. Since $x \notin K$ and W is a maximal submodule of M among all submodules which do not contain x, we have $K = W$. Now $M/W = B/W \oplus K/W = B/W \oplus \bar{0} \Rightarrow M/W \simeq B/W$ which is simple. Therefore M/W is simple, i.e W is a maximal submodule of M which does not contain x. Therefore $\bigcap_{0 \neq x \in M} W_x = 0 \Rightarrow \bigcap_{W_i \in Max(M)} W_i \subseteq \bigcap_{0 \neq x \in M} W_x = 0$ $\Rightarrow \bigcap_{W_i \in Max(M)} W_i = 0$. Hence $Rad(M) = 0$ i.e (2) is true.

((2) $\Rightarrow$ (3)) Trivial.

((3) $\Rightarrow$ (4)) Let I be a left ideal of R . Then R/I is cyclic $R-$module. Then by (3) we get $(\bigcap W_i)/I = \bigcap (W_i/I) = 0$ where W_i/I are maximal submodules of R/I. $\Rightarrow I = \bigcap W_i$, W_i are maximal left ideals of R. Hence (4)

((4) $\Rightarrow$ (1)) Let S be a simple R-module and I be a left ideal of R.

If $\alpha \in Hom_R(I, S)$ and $K = ker(\alpha)$. Since S is simple then $I/K \simeq S$ .Since $\alpha \neq 0, K \subset I$. Then by (4) there is a maximal left ideal μ of R such that $\mu \geq K$ but $K \not\leq I$. Since I/K is simple then $\mu \bigcap I = K$. Therefore $R = \mu + I$. Now $R \xrightarrow{\eta} R/\mu = (\mu + I)/\mu \overset{\phi}{\simeq} I/(\mu) = I/K \overset{\tilde{\alpha}}{\simeq} S$ where $\phi(a + M) = a + K$

Let $g = \tilde{\alpha} \circ \phi \circ \eta : R \to S$ Hence for any $a \in I$ $g(a) = (\tilde{\alpha} o \phi o \eta)(a) = (\tilde{\alpha} o \phi)(\eta)(a) = (\tilde{\alpha} o \phi)(a + M) = \tilde{\alpha}(\phi(a + M) = \tilde{\alpha}(a + K) = \alpha(a)$. Therefore α can be extended to $g : R \to S$. Hence R is a V-ring. □

1.3.6 Proposition. *Every factor ring of a left V-ring is a left V-ring.*

Proof. Let R be a left V-ring and I be a two-sided ideal of R. Let J/I be left ideal of R/I , then $J \leq R$. Since R is a left V-ring then by Proposition 1.3.5 we have $J = \bigcap W_i$ where W_i are maximal left ideals of R. Now $J/I = (\bigcap W_i)/I = \bigcap(W_i/I)$ where W_i/I are also maximal left ideals of R/I. Hence every left ideal of R/I is an intersection of maximal left ideals of R/I and hence R/I is a left V-ring. □

1.3.7 Corollary. *If R is a left V-ring then every left ideal I of R is idempotent.*

Proof. Suppose that I^2 is different from I then by Proposition 1.3.5 there is a maximal left ideal μ of R such that $I^2 \leq \mu$ but $I \not\leq \mu$. Hence $R = \mu + I$. Then $1 = m + x$ for some $m \in \mu$ and $x \in I$. Now $x = mx + x^2 \in \mu \Rightarrow x \in \mu \Rightarrow 1 \in \mu$ which is a contradiction. Therefore $I^2 = I$. Hence I is idempotent. □

1.3.8 Proposition. *If R is a left or right V-ring, then the centre $C(R)$ of R is a regular ring.*

Proof. Assume that R is a left V-ring. The proof in the right V-ring case is similar. Let $a \in C(R)$. By Corollary 1.3.7 $Ra = RaRa = aRa$ since $a \in C(R)$. Therefore there exists $b \in R$ such that $a = aba$. So a is a regular element in R. By a slight modification of the proof of Proposition 1.1.17 we can show that $b \in C(R)$. □

The following is a celebrated theorem of Kaplansky.

1.3.9 Proposition. *Let R be a commutative ring. Then R is a V-ring if and only if R is regular.*

Proof. ($\Rightarrow$) Let $a \in R$. Since R is a V-ring then by Proposition 1.3.7 $Ra = RaRa = aRa$(R is commutative). Hence there exists $b \in R$ such that

$a = aba$ So a is regular.

($\Leftarrow$) This implication was proved in Proposition 1.3.4. Using Proposition 1.3.5 We give a shorter proof. Let R be regular and M a cyclic R-module. Then $M \cong R/I$ for some ideal I of R. Since R is regular then R/I is also regular. Now we have $Rad(R/I) = 0$. Therefore the radical of the R-module M also vanishes. Hence $Rad(M) = 0$. □

1.3.10 Corollary. *If R is a (left or right) V-ring then $C(R)$ is a V-ring.*

1.4 Regular modules

In this section, we study regularity in modules. The definition of regular elements in modules in the sense of Zelmanowitz is given. The following notation is used in this section and the next.

1.4.1 Notation. Let M , P and W be left R-modules and $f : M \longrightarrow W$ be a module homomorphism.Then we denote the image of an element $m \in M$ under f by mf; if $g : W \longrightarrow P$ is a module homomorphism then the composition of f and g is denoted by $f \circ g$.

1.4.2 Definition. Let M be left R-module and $N = Hom_R(M, R)$ Then N is right R-module with the external operation defined as $m(fr) = (mf)r$ for any $m \in M$, $f \in N$ and $r \in R$.

1.4.3 Definition. An element m of a module M is said to be *Z-regular* (in M)if there exists $f \in N$ such that $(mf)m = m$. A subset K of M is Z-regular if every element of K is Z-regular (in M).

1.4.4 Example. $0 \in M$ is always a Z-regular element.

1.4.5 Definition. A module M is *Z-regular* if every element of M is Z-regular.

If there is no possibility of confusion we use the term regular for 'Z-regular'.

1.4.6 Proposition. *The ring R is a von Neumann regular ring if and only if R is a Z-regular R-module.*

Proof. ($\Rightarrow$) Let $a \in R$ be any element. Since R is von Neumann regular there exists $b \in R$ such that $aba = a$. Now let us consider the function $f_b : R \longrightarrow R$ defined by $(x)f_b = xb$, which is an R-homomorphism from ${}_RR$ to ${}_RR$. For this function f_b, we have $(af_b)a = (ab)a = aba = a$. Therefore there exists $f_b \in Hom_R(R, R)$ such that $(af_b)a = a$. Hence R is a Z-regular $R-$ module. ($\Leftarrow$) Let a be an element of R. Since R is Z-regular $R-$ module then there exists $f \in Hom_R(R, R)$ such that $(af)a = a$. Now if $(1)f = b$, then $(a)f = (a.1)f = a(1)f = ab$. Therefore $(af)a = a$ implies $aba = a$. Hence a is a regular element. Therefore R is a von Neumann regular ring. □

1.4.7 Corollary. *The following results are equivalent for an element a of a ring R.*

(i) a is Z-regular in ${}_RR$.

(ii) a is Z-regular in R_R.

(iii) a is von Neumann regular in the ring R.

1.4.8 Proposition. *Let M be a left R-module, $m \in M$ and $f \in Hom_R(M, R)$ such that $(mf)m - m$ is Z-regular. Then m is Z-regular.*

Proof. Since $(mf)m - m$ is Z-regular then ther exists $g \in Hom_R(M, R)$ such that $(mf)m - m = ((mf)m - m)g((mf)m - m)$. Now $m = (mf)m - ((mf)m - m)g((mf)m - m) = m\{f - g + g(mf) + f(mg) - f(mg)(mf)\}m$. Therefore, we get $h \in Hom_R(M, R)$ such that $m = (mh)m$, where $h = f - g + g(mf) + f(mg) - f(mg)(mf) \in Hom_R(M, R)$. Hence m is regular. □

1.4.9 Corollary. *Let $a \in R$ be an element. If $aba - a$ is a von Neumann regular element in R then a is a von Neumann regular element in R.*

Proof. Let us take $M =_R R$. Then this follows from Proposition 1.4.8. $\square$

1.4.10 Proposition. *Let M, W be left R-modules and $f : M \longrightarrow W$ be an R-homomorphism with kernel K. Assume that K is Z-regular in M. If mf is Z-regular in W for some $m \in M$ then m is Z-regular in M.*

Proof. Since mf is Z-regular in W then there exists $g \in Hom_R(W, R)$ such that $mf = (mf)g(mf)$. Let $h = f \circ g \in_R (M, R)$. Then we have $((mh)m - m)f = (m(f \circ g)m - m)f = (((mf)g)m - m)f = ((mf)g(mf) - mf = mf - mf = 0$. Therefore $(mh)m - m \in K$ which is Z-regular. Therefore, by Proposition 1.4.8 m is Z-regular in M. $\square$

1.4.11 Proposition. *Let M be a left R-module. Then M is Z-regular if and only if for each $m \in M$, Rm is a projective direct summand of M.*

Proof. ($\Rightarrow$) Since M is Z-regular then given $m \in M$ there exists $f \in Hom_R(M, R)$ such that

$$(mf)m = m \qquad (*)$$

Now consider the R-homomophism $\theta_m : R \longrightarrow M$ given by $(r)\theta_m = rm$. From (*) we get $(mf)\theta_m = (mf)m = m$. Let f_o be the restriction of f to Rm. Then $f_o \circ \theta_m = 1_{Rm}$; thus $f_o : Rm \longrightarrow R$ is split. Hence $Rm \leq^{\oplus} R$ which is free. Therefore Rm is projective. Next we have to show that Rm is direct summand of M. Consider $\phi = f \circ \theta_m : M \longrightarrow Rm$ given by $(n)\phi = (nf)\theta_m = (nf)m$. Here ϕ is R-linear onto and $(m)\phi = (mf)m = m$. For the inclusion map $j : Rm \longrightarrow M$ we have $j \circ \phi = 1_{Rm}$. Hence j is split. Therefore $Rm \leq^{\oplus} M$.

($\Leftarrow$) Let $m \in M$ and consider the map ~~map~~ $\theta : R \longrightarrow Rm$ defined by $(r)\theta = rm$. Then θ is an onto R-homomorphism. Since Rm is projective, there exists $g : Rm \longrightarrow R$ such that $g \circ \theta = 1_{Rm}$. Again, $Rm \leq^{\oplus} M$ so g can be extended to an R-homomorphism $g' : M \longrightarrow Rm$.

Note that $m = (m)1_{Rm} = (m)(g \circ \theta) = (mg)\theta = (mg')\theta = (mg')m$. This shows that m is Z-regular. $\square$

1.4.12 Proposition. *A cyclic Z-regular R-module is projective and is isomorphic to a left ideal of R generated by an idempotent.*

Proof. Let M be a cyclic Z-regular R-module. Then $M = Rm$ for some $m \in M$. By Proposition 1.4.11 M is projective. Since M is Z-regular then there exists $f \in Hom_R(M, R)$ such that $(mf)m = m$. Then we have $(mf)^2 = (mf)(mf) = ((mf)m)f = mf$. Hence $e = mf \in I(R)$.

Now let us define a map $\theta : Re \longrightarrow Rm$ given by $(re)\theta = rm$. It is well defined because if $r_1e = r_2e$ then $r_1(mf) = r_2(mf) \Rightarrow r_2(mf)m = r_2(mf)m \Rightarrow r_1m = r_2m$. We can easily show that θ is a R-module homomorphism.

Again if $(re)\theta = 0$ then $rm = 0 \Rightarrow (rm)f = 0 \Rightarrow r(mf) = 0 \Rightarrow r(mf) = 0 \Rightarrow re = 0$, therefore θ is 1-1 and onto.

Hence $Re \simeq Rm$, i.e M is projective and isomorphic to a left ideal of R generated by an idempotent element e. $\square$

1.4.13 Proposition. *If a left R-module M is Z-regular then Rad(M)* $= 0$

Proof. If $Rad(M) \neq 0$ then we can take a non-zero element $m \in Rad(M)$. Since M is regular, $Rm \leq^{\oplus} M$ i.e $Rm \oplus W = M$ for some submodule W of M . Again Rm has a maximal submodule (say) μ. Now let us consider $Rm \oplus W \xrightarrow{\pi} Rm \xrightarrow{\eta} Rm/\mu$. We have $Ker(\pi \circ \eta) = \{rm + n \mid rm + \mu = \mu\} =$

$\{rm+n \mid rm \in \mu\} = \mu \oplus W$ which is a maximal submodule of $Rm \oplus W = M$. As $m \in Rad(M)$ therefore $m \in \mu \oplus W$ i.e $m = m' + n$ for some $m' \in Rm$ and $n \in W$ which implies $m - m' = n \in Rm \cap W = 0$. Hence $m \in \mu$,which is a contradiction. Therefore $Rad(M) = 0$. □

1.4.14 Remarks. (i)If a left R-module M is semi-simple and projective then M is Z-regular.

(ii)Free modules over a von Neumann regular ring are Z-regular.

1.4.15 Proposition. *If R is a semi-simple ring then every left R-module M is Z-regular.*

Proof. Since R is semi-simple then M is also a semi-simple R-module. Let m be an element of M.Then $Rm \leq^{\oplus} M$. Next, let us show that Rm is projective. We have the natural onto R-homomorphism $\theta : R \longrightarrow Rm$. Consider the short sequence

$$0 \longrightarrow Ker\theta \xrightarrow{j} R \xrightarrow{\theta} Rm \longrightarrow 0$$

Since R is semi-simple $Ker\theta \leq^{\oplus} R$. So j, and therefore θ, both split. Thus Rm is isomorphic to a direct summand of R. Since R is free over itself then Rm is projective.

So by the Proposition 1.4.11 ${}_RM$ is Z-regular. □

1.4.16 Proposition. *Let R be a ring and M a left R-module. Let $a \in R$ and $m \in M$. Let us assume that there is an element $b \in R$ such that $a = aba$ and $ba \in C(R)$ where $C(R)$ is the centre of R.If m is Z-regular then am is Z-regular.*

Proof. Since m is Z-regular there exists $f \in Hom_R(M, R)$ such that $(mf)m = m$. Since $ba \in C(R)$ we have $am = (aba)(mf)m = (am)(fba)m = (am)(fb)(am)$. Therefore $am = (am)g(am)$ where $g = fb \in Hom_R(M, R)$.
Hence am is Z-regular in M. □

1.4.17 Proposition. *Let m be a Z-regular element of $_RM$ and t an element of S. Then m/t is a regular element of $_{S^{-1}R}S^{-1}M$.*

Proof. Since m is regular then there exists $f \in Hom_R(M, R)$ such that $m = (mf)m$. Let us consider $tf/1 \in Hom_{S^{-1}R}(S^{-1}M, S^{-1}R)$
(Note that $t \in S \subseteq C(R)$.) Now $((m/t)(tf/1)(m/t) = (mf)m/t = m/t$.
Hence m/t is regular in $_{S^{-1}R}S^{-1}M$. □

1.4.18 Corollary. *Let $_RM$ be a regular module.Then $S^{-1}M$ is a regular $S^{-1}R$-module for each multiplicatively closed subset S of $C(R)$.*

1.4.19 Result. Let $M = R$. Then we get a well-known result that if R is regular ring then so is $S^{-1}R$ for each multiplicatively closed subset S of $C(R)$.

1.4.20 Proposition. *Let M a left R-module. Consider the following conditions.*
(i) $_RM$ is regular.
(ii) $_{S^{-1}R}S^{-1}M$ is regular for each S.
(iii)$_{R_P}M_P$ is regular for each $P \in Spec(C(R))$.
(iv) $_{R_Q}M_Q$ is regular for each $Q \in Max(C(R))$.
Then (i)⇔(ii) ⇒(iii) ⇒(iv).

Proof. (i) ⇒(ii) is given in above proposition and (ii) ⇒(i) is trivial by taking $S = \{1\}$.The implication (ii)⇒(iii) and (iii) ⇒(iv) are clear. □

1.4.21 Proposition. *Let M be a left R-module. Then the following conditions are equivalent for a ring R.*

(i) R is semi-simple.

(ii)Every left R-module M is Z-regular.

Proof. Straight-forward application of Proposition 1.4.11 and 1.4.12. □

1.5 Anti-regular modules

Basic properties and examples of anti-regular modules are recorded in this section.

1.5.1 Definition. Let M be a left $R-$module. An element $m \in M$ is *anti-regular* if there exists a non-zero $f \in N = Hom_R(M, R)$ such that $f(mf) = f$.

1.5.2 Definition. A left $R-$module M is an *anti-regular* module if all non-zero elements of M are anti-regular.

1.5.3 Example. All vector spaces are anti-regular modules.

1.5.4 Proposition. *A ring R is an anti-regular ring if and only if ${}_RR$ is anti-regular module.*

Proof. ($\Rightarrow$) Let $a \in R$ be a non-zero element. Since R is an anti-regular ring then there exists $b \in R$ such that $bab = b \neq 0$. Now let us define a map $f_b : R \longrightarrow R$ given by $(x)f_b = xb$ which is a non-zero module endomorphism of ${}_RR$. For this endomorphism $(x)(f_b a f_b) = (x)f_b(af_b) = (xb)(ab) = x(bab) = xb = (x)f_b$ for all $x \in R \Rightarrow f_b(af_b) = f_b$).
Therefore ${}_RR$ is an anti-regular module.
($\Leftarrow$) Since ${}_RR$ is an anti-regular module then there exists non-zero homomorphism f such that $f(af) = f$.
Now $(1)(f(af) = (1)f \Rightarrow ((1)f)a((1)f) = (1)f$. Since f is non-zero then $(1)f \neq 0$. Let $(1)f = b$. Now we get a non-zero element $b \in R$ such that $bab = b$. Hence a is anti-regular.
Therefore R is an anti-regular ring. □

1.5.5 Result. If ${}_RM$ is Z-regular module then ${}_RM$ is an anti-regular module. But the converse does not hold.

Proof. The proof is similar to the case of ring. As anti-regular ring which is not regular (see Example 1.2.15) yields when regarded as a module over itself an example of an anti-regular module which is not Z-regular. □

1.5.6 Proposition. *Let M_1, M_2 be two left R-modules and $f : M_1 \longrightarrow M_2$ be R-homomorphism. If $(m)f$ is anti-regular in M_2 then m is anti-regular in M_1.*

Proof. Since $(m)f$ is anti-regular in M_2 then there exists a non-zero homomorphism $g : M_2 \longrightarrow R$ such that $g(mf)g = g$. Let $h = f \circ g : M_1 \longrightarrow R$. Then we have $h \neq 0$ and $h(mh) = (f \circ g)m(f \circ g) = f(g(mf)g) = f \circ g = h$, so $h(mh) = h$ and therefore m is anti-regular in M_1. □

1.5.7 Corollary. *Let W be a submodule of a module M and n be an element of W. If n is anti-regular as an element of M then n is also anti-regular as an element of W.*

Proof. Let us consider the inclusion map $j : W \longrightarrow M$. Then we can apply Proposition 1.5.6 to j. □

1.5.8 Remark. Submodules of anti-regular modules are anti-regular.(This is a consequence of Corollary 1.5.7.)

1.5.9 Proposition. *Let $\{M_i\}_{i \in I}$ be an arbitrary family of left R-modules. Let $P_o = \prod_{i \in I} M_i$ be the direct product and let $S_o = \oplus_{i \in I} M_i$ be the direct sum of this family.*
The following conditions are equivalent.

(i) Each M_i is anti-regular.

(ii) The module P_o is anti-regular.

(iii) The module S_o is anti-regular.

Proof. (i)⇒(ii) Let $m = (m_i)_{i \in I}$ be a non-zero element of P_o.Then there exists $i \in I$ such that $m_i \neq 0$. Consider the projection map $\pi_i : P_o \longrightarrow M_i$. Now $m\pi_i$ is non-zero and belongs to M_i which is an anti-regular module. By Proposition 1.5.6 m is anti-regular in P_o. Thus (ii) holds.
(ii)⇒(iii) and (iii) ⇒ (i) hold by Remark 1.5.8. □

$\bar{I}(R)$ denotes the set of all nonzero idempotents of R.

1.5.10 Proposition. *Let m be an anti-regular element of M. Then $m \notin Rad(M)$.*

Proof. Let m be an anti-regular element of a left R-module M. Then there exists $f \in Hom_R(M, R)$ such that $f(mf) = f \neq 0$ yielding $(mf)(mf) = mf \neq 0$ so that $mf \in \bar{I}(R)$. Therefore $mf \notin Rad(R)$. We know that if M_1, M_2 are left R-modules and $g : M_1 \longrightarrow M_2$ is an R-homomorphism then $(Rad(M_1))g \leq Rad(M_2)$. Since $mf \notin Rad(R)$ applying this observation to $f : M \longrightarrow R$, we get $m \notin Rad(M)$. □

1.5.11 Corollary. *If M is an anti-regular module then $Rad(M) = 0$*

1.5.12 Proposition. *Let M be a left R-module. Let $N = Hom_R(M, R)$ and $S = End_R(M)$. The following conditions are equivalent for an element m of a left R-module M.*

(i) There exists $f \in N$ such that $mf \in \bar{I}(R)$.

(ii) There exists $f \in N$ such that $[f, m] \in \bar{I}(S)$.

(iii) There exist $f \in N, \alpha \in S$ such that $[f, m\alpha] = [f, m]\alpha \in \bar{I}(S)$.

(iv) There exist $r \in R, f \in N$ such that $(rm)f = r(mf) \in \bar{I}(R)$.

(v) The element m is anti-regular in M.

(vi) Rm contains a non-zero, projective,direct summand of M.

Proof. ((i) $\Rightarrow$ (ii)) Let $e = mf \in \bar{I}(R)$. Then let us consider $[fe, m]^2 = [fe, m][fe, m] = [fe(m(fe)), m] = [fe(mf)e, m] = [fe, m]$. Again we know that $m[fe, m]f = mf.e.mf = e \neq 0$. Hence $[fe, m] \in \bar{I}(S)$.

((ii)$\Rightarrow$(iii)) It is given that $\exists f \in S$ such that $[f, m] \in I(S)$. Let $\alpha = 1_M \in S$ then $[f, m\alpha] = [f, m1_M] = [f, m] \in \bar{I}(S)$. Hence there exists $\alpha \in S$ such that $[f, m\alpha] \in \bar{I}(S)$.

((iii) $\Rightarrow$ (iv)) Suppose $f \in N, \alpha \in S$ are such that $[f, m]\alpha \in \bar{I}(S)$. Let $g = \alpha f \in N$ and $r = mg = (m\alpha)f \in R$.

Now $r^4 = m\alpha[f, m\alpha]^3 f = m\alpha[f, m\alpha]f = (m\alpha)f(m\alpha)f = r^2$. Again $[fr^2, m\alpha] = [fm\alpha[f, m\alpha]f, m\alpha] = [f, m\alpha][f, m\alpha][f, m\alpha] = [f, m\alpha] \neq 0$. Therefore $r^2 = r(mg) \in \bar{I}(R)$.

((iv) $\Rightarrow$(v)) Suppose $r \in R, f \in N$ such that $r(mf) \in \bar{I}(R)$.

Let $g = fr(mf)r \in N$. Now $r(mg)(mf) = r(mfr(mf)r)(mf)$
$= (r(mf))(r(mf))(r(mf)) = (r(mf))^3 = r(mf) \neq 0$. Therefore $g \neq 0$. Then $g(mg) = (fr(mf)r)(m(fr(mf)r)) = f(r(mf))^3 r = f(r(mf))r = fr(mf)r = g \neq 0$. Hence m is anti-regular.

(v) $\Rightarrow$(vi) Since m is anti-regular then there exists $f \in N$ such that $f(mf) = f \neq 0$. Then $(mf)(mf) = mf \neq 0$. Let $e = mf$ so that $f = fe$.

Now for any $x \in M$ we have $xf = (xf)e$. Therefore $Mf = Re = R(mf) = (Rm)f$. Let θ be the restriction of f. Since Re is R-projective, the map $\theta : Rm \longrightarrow Re$ is split. Therefore there exists $\psi : Re \longrightarrow Rm$ such that

$\psi \circ \theta = 1_{Re}$. Hence $(Re)\psi$ is a nonzero projective direct summand of Rm.

((vi) $\Rightarrow$(i))Let W be a non-zero projective direct summand of M contained in Rm. Let $\alpha : M \longrightarrow W$ be the projection map. Since $W \leq Rm$ and α is identity on W. Now we have $W = (Rm)\alpha = R(m\alpha)$. As W is non-zero and projective there exists $e \in \bar{I}(R)$ and an R-epimorphism $\beta : W \longrightarrow Re$ such that $(m\alpha)\beta = e$. Therefore the R-homomorphism $\alpha \circ \beta = f$ (say) satisfies that $mf \in \bar{I}(R)$.

□

1.6 V-modules

In this section we prove some properties of V-modules.

1.6.1 Definition. A left $R-$module M is a *V-module* if every submodule of M is an intersection of maximal submodules.

1.6.2 Remark. All V-rings are V-modules considered as modules over itself.

1.6.3 Example. All simple modules are V-modules.

1.6.4 Remark. A ring R is a V-ring if and only if the left R-module R is V-module.

1.6.5 Proposition. *Let M be a left $R-$module.Then the following conditions are equivalent*
(1)M is a V-module
(2) Every simple $R-$module is $M-$injective.

Proof. ((1) $\Rightarrow$ (2)) Let S be a simple R-module and let W be a $R-$submodule of M. Let $\phi : W \rightarrow S$ be an R-homomorphism and $\alpha : W \hookrightarrow M$ the inclusion map. Now if $\phi = 0$ then we get a homomorphism $0 : M \rightarrow S$ such that $0o\alpha = \phi$. If $\phi \neq 0$ then ϕ is onto because S is simple.Therefore $W/ker\phi \backsimeq S \Rightarrow W/W'$ is simple. Write $W' = ker\phi \Rightarrow W'$ is maximal submodule of W. By (1) $W' = \bigcap_{\mu \in \widetilde{S}} \mu$ where $\widetilde{S} =$ is the set of all maximal submodules of M containing W'. We know that $W \neq W'$. Therefore there exixts a maximal submodule μ_0 of M such that $W \nsubseteq \mu_0$. Hence $\mu_0 + W = M$. Now $W' \subseteq \mu_o \cap W$, W' is maximal in W and $W \nsubseteq \mu_0$. Therefore $W' = \mu_0 \cap W$. Now let us consider,

$M \xrightarrow{\eta} M/\mu_0 = (\mu_0 + W)/\mu_0 \overset{\beta}{\simeq} W/(\mu_0 \cap W) = W/W' \overset{\tilde{\phi}}{\simeq} S.$

Let us take $f = \tilde{\phi} o \beta o \eta : M \to S$. Let $n \in W$. We have

$f(n) = (\tilde{\phi} o \beta o \eta)(n) = (\tilde{\phi} o \beta)\eta(n) = \tilde{\phi}(\beta(n + \mu_0) = \tilde{\phi}(n + W') = \phi(n)$.

Thus $f|_W = \phi$ i.e there exists $f : M \to S$ which is an $R-$homomorphism such that $\phi = f o \alpha$. Hence S is $M-$injective.

((2)$\Rightarrow$ (1)) First we prove that $Rad(M) = 0$. If it is not zero then let us choose a nonzero element $m \in Rad(M)$. As Rm is a finitely generated non-zero R-submodule of M , it has a maximal submodule, say, μ. Then the R-module Rm/μ is simple. Let us consider $\eta : Rm \longrightarrow Rm/\mu$ defined by $\eta(m) = m + \mu$ and the inclusion map $\alpha : Rm \hookrightarrow M$. Since $m \notin \mu$ therefore η is not zero homomorphism and Rm/μ is simple implies that η is onto. Since all simple R-modules are M-injective, therefore, for $\eta : Rm \longrightarrow Rm/\mu$ and $\alpha : Rm \hookrightarrow M$ there exists $\phi : M \longrightarrow Rm/\mu$ such that $\phi \circ \alpha = \eta$. Since ϕ is onto, $M/ker\phi \simeq Rm/\mu$ which is simple. So $ker\phi$ is a maximal R-submodule of M. Therefore $m \in ker\phi$, i.e $\phi(m) = \mu$. Now $\phi \circ \alpha = \eta$ implies $m + \mu = \eta(m) = \phi(\alpha(m)) = \phi(m) = \mu$ i.e $m \in \mu$ which is a contradiction. Therefore $m = 0$, hence $Rad(M) = 0$. Let K be a R-submodule of M. Then we will prove that every simple R-modules are M/K-injective. Let S be a simple R-module and W/K be a submodule of M/K. Consider the R-homomorphisms, $\theta : W/K \longrightarrow S$, $\alpha : W/K \longrightarrow M/K$ and the natural homomorphism $\eta : W \longrightarrow W/K$ then we get an R-homomorphism $\theta \circ \eta : W \longrightarrow S$. By (2) there exists $\phi : M \longrightarrow S$ such that $\phi \circ \beta = \theta \circ \eta$ where $\beta : W \longrightarrow M$ is the inclusion map. Now let us define $\psi : M/K \longrightarrow S$ such that $\psi(m + K) = \phi(m)$. It is well defined because if $m_1 + K = m_2 + K$ then $m_1 - m_2 \in K \subseteq W$. So $\phi(m_1) - \phi(m_2) = \phi(m_1 - m_2) = (\theta \circ \eta)(m_1 - m_2) =$

$\theta(\bar{0}) = 0$. Showing that $\psi(m_1 + K) = \psi(m_2 + K)$. We can easily verify that ψ is a homomorphism.

Let $n + K \in W/K$. Then $\psi(n + K) = \phi(n) = (\theta \circ \eta)(n) = \theta(n + K)$. Therefore, for every $\theta : W/K \longrightarrow S$ and $\alpha : W/K \longrightarrow M/K$ there exists $\psi : M/K \longrightarrow S$ such that $\theta \circ \alpha = \psi$. Hence every simple R-modules are M/K-injective.Then we get $0 = Rad(M/K) = \bigcap_i (W_i/K) = \bigcap_i (W_i)/K$, where W_i are maximal submodules of M containing K. Therefore $K = \bigcap_i W_i$, hence M is a V-module. □

Chapter 2

Reduced,abelian,semi-commutative,reversible and Armendariz rings

In this chapter, we record properties of reduced, abelian, semi-commutative, reversible and Armendariz rings. Apart from the various relations between these classes, we also record that some of these classes of these rings are stable under polynomial extensions. e.g polynomial rings over reduced, abelian and Armendariz rings are respectively reduced, abelian and Armendariz while those over semi-commutative and reversible rings do not inherit the some properties. References to these examples are included in Sections 2.3 and 2.4.

2.1 Reduced rings

In this section we have recorded basic properties of reduced rings and well-known results. We recall the following definition.

2.1.1 Definition. A ring R is a *reduced* ring if it has no non-zero nilpotent elements.

2.1.2 Examples. (1) Division rings are reduced.

(2)More generally all domains and in fact, direct products of domains are reduced.

(3)Let K be a field.It is easy to see that $M_n(K)$ is regular.[Details are given in Theorem 24, page 14 of [K]]. But $M_n(K)$ is not reduced.(When $n = 2$ we have E_{12} is a nonzero element of $M_2(K)$ and $(E_{12})^2 = 0$. Hence $M_2(K)$ is not reduced.

2.1.3 Proposition. *For a ring R,the following conditions are equivalent.*
(1) R is reduced.
(2) Whenever $a \in R$ satisfies $a^2 = 0$, then $a = 0$.

Proof. ((1) $\Rightarrow$(2)) This is trivial. ((2) $\Rightarrow$ (1)) Suppose that $a^n = 0$ for some $n \in \mathbb{N}$, $n \geq 2$, so that $2(n-1) = 2n - 2 \geq n$. This shows that $\{a^{n-1}\}^2 = a^{2(n-1)} = 0$. Hence by hypothesis, $a^{n-1} = 0$. Continuing in this way, we obtain $a = 0$. Hence R is reduced. □

The next proposition is easy to prove (and well-known).

2.1.4 Proposition. *R is reduced if and only if $R[x]$ is reduced.*

2.1.5 Proposition. *If a ring R is reduced then $l(a) = r(a)$ $\forall a \in R$.*

Proof. Let $x \in l(a)$. Then $xa = 0$. Now $(ax)^2 = ax.ax = 0$. As R is reduced, $ax = 0$ i.e $x \in r(a)$. This shows that $l(a) \subseteq r(a)$. Similarly, $r(a) \subseteq l(a)$. Hence $l(a) = r(a)\ \forall a \in R$ □

2.1.6 Proposition. *Let R be a reduced ring and a be an element in R. Let $I = l(a) = r(a)$, an ideal of R. Then R/I is a reduced ring and for each $a \notin I$ $\bar{a}$ is a non-zero-divisor in R/I.*

Proof. Let $\bar{x} \in R/I$ be an element such that $(\bar{x})^2 = \bar{0}$. Then $x^2 \in I = l(a) \Rightarrow x.xa = 0$. Since R is reduced we have $l(xa) = r(xa) \Rightarrow xax = 0 \Rightarrow xaxa = 0 \Rightarrow (xa)^2 = 0$ in R. Therefore $xa = 0 \Rightarrow x \in l(a) = I$. Hence $\bar{x} = \bar{0}$ in R/I. It follows by Proposition 2.1.3 that R/I is reduced.
Next let $a \notin I$. Now suppose that $\bar{a}.\bar{x} = \bar{0}$ in R/I. Then $ax \in l(a) \Rightarrow axa = 0 \Rightarrow (xa)^2 = 0 \Rightarrow xa = 0 \Rightarrow x \in l(a) = I \Rightarrow \bar{x} = \bar{0}$. Similarly $\bar{x}.\bar{a} = \bar{0} \Rightarrow \bar{x} = \bar{0}$. Hence $\bar{a}$ is a left and right non-zero divisor in R/I. □

2.1.7 Proposition. *If a ring R is reduced then R is directly finite.*

Proof. Let $a, b \in R$ such that $ab = 1$. Now $(ba)^2 = ba.ba = b(ab)a = b.1.a = ba \Rightarrow ba \in I(R)$. Let $e = ba$ and let x be any element of R. Let $n = ex - exe$. Then $n^2 = (ex - exe)^2 = exex - exexe - exeex + exeexe = exex - exexe - exex + exexe = 0$. Since R is reduced $n^2 = 0$ implies that $n = 0 \Rightarrow ex = exe$. Similarly $xe = exe$ by taking $m = xe - exe$. Hence $e = ba \in C(R)$. Now $ba = ba.1 = ba.ab = [(ba)a]b = [a(ba)]b = (ab)(ab) = 1.1 = 1$. Therefore $ba = 1$ whenever $ab = 1$. Hence R is directly finite. □

2.2 Abelian rings

In this section we study abelian rings.

2.2.1 Definition. A ring R is called an *abelian* ring if all idempotent elements of R are central i.e $I(R) \subset C(R)$.

2.2.2 Example. All commutative rings are abelian.

2.2.3 Proposition. *If each idempotent e of R commutes with all nilpotent elements in R, then R is an abelian ring.*

Proof. Let $e \in I(R)$ and $x \in R$. Let us take an element $n = ex - exe$. Now $n^2 = (ex - exe)^2 = (ex - exe)(ex - exe) = exex - exexe - exe.ex + exe.exe = exex - exexe - exex + exexe = 0$. Therefore n is nilpotent element. Then by given hypothesis we get $en = ne \Rightarrow e(ex - exe) = (ex - exe)e \Rightarrow ex - exe = exe - exe \Rightarrow ex = exe$. Similarly by taking $n = xe - exe$ we can show that $xe = exe$. Hence $ex = xe$; $\forall x \in R$
$\Rightarrow e \in C(R)$. Therefore, R is an abelian ring. □

2.2.4 Proposition. *If every idempotent e of R commutes with every other idempotent, then R is abelian.*

Proof. Let $e \in I(R)$ and $x \in R$. Then $n = ex - exe$ is nilpotent and $en = n, ne = 0$. Now let us consider $e' = e + n$. Then $e'^2 = (e+n)^2 = e^2 + en + ne + n^2 = e + n + 0 + 0 = e + n = e'$. Hence e' is idempotent and by given hypothesis we get $ee' = e'e \Rightarrow e + en = e + ne = e \Rightarrow en = 0 \Rightarrow e(ex - exe) = 0 \Rightarrow ex = exe$. Similarly, $xe = exe$. Hence $ex = xe$, for all $x \in R$.Therefore R is abelian. □

2.2.5 Proposition. *If R is abelian then R is directly finite.*

Proof. Let a, b be any two elements of R such that $ab = 1$. Now $(ba)^2 = ba.ba = b(ab)a = b.1.a = ba$. So $ba \in I(R)$. Since R is abelian, we have $ba = ba.1 = (ba)(ab) = [(ba)a]b = (ab)(ab) = 1.1 = 1$. Therefore if R is abelian then whenever $ab = 1 \Rightarrow ba = 1$ □

2.2.6 Proposition. *Let R be an abelian ring and $e, f \in I(R)$. If $Re = Rf$, we have $e = f$.*

Proof. By the given hypothesis, i.e $Re = Rf$, there exists $x \in R$ such that $e = xf$. Now $ef = xf.f = xf \Rightarrow e = ef$. Similarly, we can show that $f = fe$. Since R is abelian $ef = fe \Rightarrow e = f$. □

2.2.7 Remark. By symmetry, we see that if R is abelian and $e, f \in I(R)$ satisfy $eR = fR$, then $e = f$.

2.2.8 Proposition. *Let R be a ring. Suppose that $eR = Re$ for each idempotent e of R. Then R is abelian.*

Proof. Let $e \in I(R)$ and $x \in R$. Then by the given condition, there exists elements y and z in R such that

$$ex = ye \tag{1}$$

$$xe = ez \tag{2}$$

Now from (1) $exe = ye.e = ye^2 = ye = ex$. Similarly, from (2) $exe = e.ez = e^2z = ez = xe$.Therefore, $ex = exe = xe$, for all $x \in R$. Hence R is abelian. □

2.2.9 Remark. Subrings of abelian rings are abelian.

2.2.10 Proposition. *Let R be a ring. The following conditions are equivalent:*

(i) R is an abelian ring.

(ii) $R[x]$ is an abelian ring.

(iii) $R[[x]]$ is an abelian ring.

Proof. ((i) $\Rightarrow$(iii)) Let $f(x) \in R[[x]]$ be an idempotent element, where $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$

Since f is an idempotent element $f^2 = f$. Now we have the following system of equations:

$$a_0^2 = a_0 \qquad (0)$$

$$a_0a_1 + a_1a_0 = a_1 \qquad (1)$$

$$a_0a_2 + a_1a_1 + a_2a_0 = a_2 \qquad (2)$$

....

$$a_0a_n + a_1a_{n-1} + ... + a_na_0 = a_n \qquad (n)$$

....

From (0) we get a_0 is idempotent. Now multiplying (1) by a_0 on left side we get $a_0a_1 + a_0a_1a_0 = a_0a_1 \Rightarrow$. Since a_0 is central then we have $a_0a_1 = 0 \Rightarrow a_1 = 0$. Similarly, mutiply (2) by a_0 we can get $a_2 = 0$. Now let us assume that k is positive integer such that $a_i = 0$ for all $1 \leq i \leq k$. Then the equation (k+1) become $a_0a_{k+1} + a_{k+1}a_0 = a_{k+1}$. Multiplying on the left side by a_0 we get $a_{k+1} = 0$. Hence, by induction $a_i = 0 \ \ \forall i \geq 1$. Therefore

$f \in I(R)$. Since R is abelian $f \in C(R)$. $\Rightarrow f$ is in the centre of $R[[x]]$. Hence $R[[x]]$ is abelian.

((iii)$\Rightarrow$ (ii)) and ((ii)$\Rightarrow$(i)) are trivial because subrings of an abelian ring are abelian. □

2.2.11 Proposition. *If R is reduced then R is abelian.*

Proof. This follows from the proof of Proposition 2.1.5. □

2.2.12 Proposition. *Let R be an abelian ring and $e, f \in I(R)$. Then*

(1) Whenever $Re = Rf$, we have $e = f$

(2) Whenever $eR = fR$, we have $e = f$

Proof. (1) Since $Re = Rf$ then we have $e = xf$ for some $x \in R$. Therefore $e(1-f) = e - ef = e - xf.f = e - xf = e - e = 0 \Rightarrow e = ef$. Again $f = ye$ for some $y \in R$. Therefore in a similar way we get $f(1-e) = 0 \Rightarrow f = fe$. Since R is abelian then we have from above two results $e = ef = fe = f$.

(2) the proof is similar to that of (1). □

2.3 Semi-commutative rings

In this section, we record the definition, examples and basic properties of semi-commutative rings. Proposition 2.3.3(b) records an important property of regular, semi-commutative rings.

2.3.1 Definition. A ring R is *semi-commutative* if whenever $a, b \in R$ satisfy $ab = 0$ we have $acb = 0$ for each $c \in R$(equivalently, $aRb = 0$).

2.3.2 Example. All commutative rings are clearly semi-commutative.

2.3.3 Proposition. **(a)** *If R is a reduced ring then R is semi-commutative.* **(b)** *If R is a regular, semi-commutative ring then R is reduced.*

Proof. **(a)** Let $a, b \in R$ satisfy $ab = 0$. Now $(ba)^2 = ba.ba = b(ab)a = 0$. Since R is reduced, $(ba)^2 = 0 \Rightarrow ba = 0$. Let $c \in R$ be any element of R. Then $(acb)^2 = acb.acb = ac(ba)cb = 0 \Rightarrow$ (since R is reduced) $acb = 0$ $(\forall c \in R)$. Hence R is semi-commutative.

(b) Let $a \in R$ satisfy $a^2 = 0$. As R is regular there exists an element $b \in R$ such that $a = aba$. Again as R is semi-commutative $a^2 = 0$ yields $a = aba = 0$. Hence R is reduced. □

2.3.4 Proposition. *Let R be a reduced ring. Then*

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is a semi-commutative ring.

Proof. Let us consider two matrices
$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S.$

For simplicity of notation, we denote a matrix $\begin{pmatrix} x & y & z \\ 0 & x & u \\ 0 & 0 & x \end{pmatrix}$ by (x, y, z, u).

Then we can denote their addition and multiplication by
$A+B = (a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$ and
$\mathrm{AB} = (a_1, b_1, c_1, d_1).(a_2, b_2, c_2, d_2) = (a_1a_2, a_1b_2+b_1a_2, a_1c_2+b_1d_2+c_1a_2, a_1d_2+ d_1a_2)$ respectively.

Now, let us assume that $AB = (a_1, b_1, c_1, d_1).(a_2, b_2, c_2, d_2) = 0$, then we get the following system of equations;

$$a_1a_2 = 0 \tag{1}$$

$$a_1b_2 + b_1a_2 = 0 \tag{2}$$

$$a_1c_2 + b_1d_2 + c_1a_2 = 0 \tag{3}$$

$$a_1d_2 + d_1a_2 = 0 \tag{4}$$

We know that reduced rings are semi-commutative rings, therefore R is semi-commutative. This fact is used in the computations below.

From (1) we get $a_1Ra_2 = 0$. Again, multiplying (2) by a_2 from the right we get $a_1b_2a_2 + b_1a_1a_2 = 0 \Rightarrow b_1a_2a_2 = 0 \Rightarrow (b_1a_2)a_2 = 0 \Rightarrow (b_1a_2)(b_1a_2) = 0 \Rightarrow (b_1a_2)^2 = 0$ (since R is reduced) $b_1a_2 = 0 \Rightarrow b_1Ra_2 = 0$. Therefore, $a_1b_2 = 0 \Rightarrow a_1Rb_2 = 0$

Similarly from (4) we can get $d_1Ra_2 = 0$ and $a_1Rd_2 = 0$. By multiplying (3) by a_2 from right we obtain $c_1Ra_2 = 0$. Then it become

$$a_1c_2 + b_1d_2 = 0. \tag{5}$$

By multiplying (5) by a_1 we can prove that $a_1Rc_2 = 0$ and $b_1Rd_2 = 0$

Proceding in this way we get that for any matrix $C = (r, s, t, u)$ where $r, s, t, u \in R$

such that $ACB = (a_1, b_1, c_1, d_1)(r, s, t, u)(a_2, b_2, c_2, d_2) = (a_1ra_2, a_1rb_2 + a_1sa_2 + b_1ra_2, a_rc_2 + a_1sd_2 + b_1rd_2 + a_1ta_2 + b_1ua_2 + c_ra_2, a_1rd_2 + a_1ua_2 + d_1ra_2) = 0.$

Hence

$$ACB = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} r & s & t \\ 0 & r & u \\ 0 & 0 & r \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} = 0$$

for any $C = \begin{pmatrix} r & s & t \\ 0 & r & u \\ 0 & 0 & r \end{pmatrix} \in S$

Therefore S is a semi-commutative ring □

2.3.5 Proposition. *Let R be a reduced ring and define a new ring as follows:*

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \dots & a_{1n} \\ 0 & a & \dots & a_{2n} \\ 0 & 0 & \dots & a_{3n} \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

where n is a positive integer. Then R_n is not semi-commutative for $n \geqslant 4$

Proof. Let R be any ring and

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

Let us consider

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have $AB = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$

But for $C \in R_4$ where

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We get

$$ACB = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

Therefore R_4 is not semi-commutative. Since R_4 can be regarded as a subring of R_n for each $n \geq 4$. Hence R_n is not semi-commutative ring for $n \geqslant 4$ □

2.3.6 Proposition. *If R is a semi-commutative ring then R is an abelian.*

Proof. Let e be an idempotent element of R. Let $a = e, b = 1 - e$ then $ab = e(1-e) = 0$ Since R is semi-commutative then we have

$$axb = 0, \forall x \in R \Rightarrow ex(1-e) = 0 \Rightarrow ex = exe. \tag{1}$$

Similarly , we have

$$ba = (1-e)e = 0 \Rightarrow xe = exe \, \forall x \in R \tag{2}$$

From (1) and (2) we get $ex = xe, \forall x \in R$. Therefore $e \in C(R)$. Hence R is an abelian. □

2.3.7 Proposition. *Let R be a ring . Then the following conditions are equivalent:*

(i) R is semi-commutative.

(ii) The right annihilator of each element of R is an ideal of R.

(iii) The left annihilator of each element of R is an ideal of R.

Proof. ((i) $\Rightarrow$ (ii)) Let $a \in R$. Then we know that $r_R(a)$ is right ideal of R. Let $x \in r_R(a)$ so that $ax = 0$. Since R is semi-commutative then $arx = 0, \forall r \in R \Rightarrow rx \in r_R(a)$. Therefore $r_R(a)$ is left ideal of R. Hence $r_R(a)$ is an ideal of R.

((ii) $\Rightarrow$(iii)) Let $x \in l_R(a)$. Then $xa = 0 \Rightarrow a \in r_R(x)$. But $r_R(x)$ is an ideal of R. we have $ta \in r_R(x) \Rightarrow xta = 0$ for any $t \in R$. Hence we get that $xt \in l_R(a)$ for any $t \in R$. Hence $l_R(a)$ is an ideal of R.

((iii) $\Rightarrow$(i)) Let $a, b \in R$ such that $ab = 0$. Then we have $a \in l_R(b)$. Since $l_R(b)$ is an ideal of R then $at \in l_R(b) \Rightarrow atb = 0, \forall t \in R \Rightarrow aRb = 0$.

Hence R is semi-commutative. □

2.3.8 Proposition. *Let R be a ring and n any positive integer. If R is reduced then $R[x]/(x^n)$ is a semi-commutative ring, where (x^n) is the ideal generated by x^n.*

Proof. Let $T = R[x]/(x^n)$.

Case 1. If $n = 1$ then $T \cong R$ which is reduced. Therefore T is semi-commutative.

Case 2. If $n \geq 2$. let us put $u = x + (x^n)$.

Let $A = a_0+a_1u+a_2u^2+.....+a_{n-1}u^{n-1}, B = b_0+b_1u+b_2u^2+.....+b_{n-1}u^{n-1} \in T$ such that

$$AB = 0 \tag{$*$}$$

Now $a_ib_ju^{i+j} = 0 \;\; \forall i, j$ with $i + j \geq n$.

Therefore it is enough to prove that $i+j$ less than n. From (*) we have

$$a_0b_0 = 0 \tag{1}$$

$$a_0b_1 + a_1b_0 = 0 \tag{2}$$

$$a_0b_2 + a_1b_1 + a_2b_0 = 0 \tag{3}$$

$$..................$$

$$a_0b_{n-1} + a_1b_{n-2} + ... + a_{n-1}b_0 = 0. \tag{n}$$

Now we will use induction on $i+j$ and the condition that R is reduced in the following cases.

From (1) and $b_0\times$ (2) we get $b_0a_1b_0 = 0 \Rightarrow a_1b_0a_1b_0 = 0 \Rightarrow a_1b_0 = 0$ (since R is reduced). Therefore we get $a_1b_0 = 0$ and

$$a_0b_1 = 0 \tag{2'}$$

Again from (1), (2′), $b_0\times$ (3) and $b_1\times$ (3) we obtain $0 = a_0b_2 = a_1b_1 = a_2b_0$ in a similar way. Inductively we may assume that $a_ib_j = 0$ for $i+j = 0, 1, 2, ..., n-2$.By using equation(n) and proceeding in this manner we finally have $0 = a_0b_{n-1} = a_1b_{n-2} = ... = a_{n-2}b_1 = a_{n-1}b_0$ Therefore we get $a_ib_j = 0$, for all $i+j = 0, 1, 2, ..., n-1$. Since R is reduced we have

$$a_icb_j = 0 \ \forall i, jc \in R. \tag{$**$}$$

Let $C = c_0 + c_1u + ... + c_{n-1}u^{n-1}$ be any element in T.

But from(**) we get $ACB = 0$. Hence T is semi-commutative. □

2.3.9 Remark. (i) If R is semi-commutative, the polynomial ring $R[x]$ need not be semi-commutative.[See Example 2 of [HLS].]

(ii) If $R[x]/(x^2)$ is semi-commutative,then R is semi-commutative.

(iii) If R is semi-commutative, then $R[x]/(x^2)$ need not semi-commutative. (Let H be the division ring of real quaternions and consider $R = H(+)H$. Then R is semi-commutative as H is reduced. Denote $\bar{x} = x + (x^2)$.

Now let us consider $A = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{x}$ and

$B = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{x} \in R[x]/(x^2).$

Then $AB = 0$.

Let $C = \begin{pmatrix} j & k \\ 0 & j \end{pmatrix} + \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \bar{x}.$

Now $ACB = \left\{ \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{x} \right\} \left\{ \begin{pmatrix} j & k \\ 0 & j \end{pmatrix} + \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \bar{x} \right\}$

$\left\{ \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{x} \right\}$

$= \left\{ \begin{pmatrix} 0 & ij \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \bar{x} + \begin{pmatrix} j & k \\ 0 & j \end{pmatrix} \bar{x} \right\} \left\{ \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{x} \right\}$

$= \left\{ \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} j & i+k \\ 0 & j \end{pmatrix} \bar{x} \right\} \left\{ \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bar{x} \right\}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \bar{x} + \begin{pmatrix} 0 & ji \\ 0 & 0 \end{pmatrix} \bar{x} + 0$

$$= \begin{pmatrix} 0 & -k \\ 0 & 0 \end{pmatrix} \bar{x} + \begin{pmatrix} 0 & -k \\ 0 & 0 \end{pmatrix} \bar{x}$$

$$= \begin{pmatrix} 0 & -2k \\ 0 & 0 \end{pmatrix} \bar{x} \neq 0.$$

Hence $R[x]/(x^2)$ is not semi-commutative.)

2.3.10 Corollary. *Let R be a semi-commutative ring and let $e, f \in I(R)$. Then*

(1) Whenever $Re = Rf$, we have $e = f$

(2) Whenever $eR = fR$, we have $e = f$

Proof. This follows from propositions 2.3.6 and 2.2.12. □

2.3.11 Proposition. *Let R be a ring and let e be a central idempotent element of R. Then R is semi-commutative if and only if eR and $(1-e)R$ are semi-commutative.*

Proof. ($\Rightarrow$) Let $x = ea, y = eb \in eR$ be any two elements which satisfy $xy = 0 \Rightarrow eaeb = 0$. Since R is semi-commutative then $eareb = 0, \ \ \forall r \in R \Rightarrow easeb = 0, \ \ \forall s \in eR$. Similarly $(1-e)R$ is semi-commutative.
($\Leftarrow$) Let us assume that eR and $(1-e)R$ are semi-commutative. Since e is a central idempotent, therefore we have $R \cong eR \times (1-e)R$. Hence R is semi-commutative. □

2.3.12 Corollary. *If R is semi-commutative then eRe and $(1-e)R(1-e)$ are semi-commutative.*

2.3.13 Proposition. *R is semi-commutative if and only if eRe is semi-commutative for all $e \in I(R)$.*

Proof. ($\Rightarrow$) Since R is semi-commutative subrngs ("subring without identity") of R are also semi-commutative.
($\Leftarrow$) Since eRe is semi-commutative for all $e \in I(R)$ then let us choose $e = 1$. We have $R = 1R1$. So R is semi-commutative. □

2.3.14 Proposition. *Let R be a ring and let S be a multiplicatively closed subset of R consisting of central non-zero divisors. Then R is semi-commutative if and only if $S^{-1}R$ is semi-commutative.*

Proof. ($\Rightarrow$) Let $\alpha = a/s, \beta = b/t \in S^{-1}R$ where $a, b \in R, s, t \in S$ such that $\alpha\beta = 0 \Rightarrow a/s.b/t = 0 \Rightarrow ab/st = 0 \Rightarrow ab = 0$ (since S contains only non-zero divisors).
Now $ab = 0$ in R and R is semi-commutative. Hence we have $ab = 0 \Rightarrow aub = 0$, for all $u \in R$. Let $\gamma = r/c \in S^{-1}R$ where $r \in R, c \in S$. Therefore $\alpha\gamma\beta = (a/s).(r/c).(b/t) = arb/sct = 0$ since $arb = 0$. Hence $S^{-1}R$ is semi-commutative.
($\Leftarrow$) This is trivial as a subring of a semi-commutative ring is again semi-commutative. □

2.3.15 Proposition. *Let R be a ring. Then $R[x]$ is semi-commutative if and only if $R[x; x^{-1}]$ is semi-commutative.*

Proof. ($\Rightarrow$) Let us consider $S = \{1, x, x^2, ...\}$ which is mutiplicatively closed subset of R. Therefore $R[x; x^{-1}] = S^{-1}R[x]$. Since $R[x]$ is semi-commutative then by Proposition 2.3.14 we have $R[x; x^{-1}]$ is semi-commutative.
($\Leftarrow$) Subring of a semi-commutative ring is also semi-commutative then $R[x]$ is semi-commutative as $R[x]$ is subring of $R[x; x^{-1}]$. □

2.4 Reversible rings

We begin this section with the following definition.

2.4.1 Definition. A ring R is called a *reversible* ring if whenever $a, b \in R$ satisfy $ab = 0$ then $ba = 0$ holds.

2.4.2 Example. All commutative rings are reversible.

2.4.3 Proposition. *Reversible rings are semi-commutative.*

Proof. Let R be a reversible ring and let a, b be two elements of R such that $ab = 0$. Then $ba = 0$. Now $bar = 0 \ \ \forall r \in R$. Since R is a reversible ring, therefore $arb = 0$, for all $r \in R$. Therefore $aRb = 0$. Hence R is semi-commutative. □

The converse of Proposition 2.4.3 is not true as shown in Example 2.4.4 below.

2.4.4 Example. Let R be a non-zero reduced ring. Then

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is a semi-commutative ring by the Proposition 2.3.4.
Let us consider

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in S\,.$$

Then we see that

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

But

$$BA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

So S is not reversible.

Let M be an R-bimodule. The trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$.

Again let us consider the following "formal matrix ring"

$$S = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M \right\}$$

S is a ring with usual matrix operations. We can show that $T(R, M) \cong S$. Now we can use this equivalent condition in later results.

2.4.5 Proposition. *Let R be a reduced ring. Then $T(R, R)$ is a reversible ring.*

Proof. Let us consider any two matrices

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in T(R, R) \text{ such that}$$

$$AB = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0$$

Then $ac = 0$ and $ad + bc = 0$.

Since R is reduced we have $ca = 0$

Therefore $cad + cbc = 0 \Rightarrow cbc = 0 \Rightarrow bc = 0$(as R is reduced)

Hence $cb = 0 = da$.

Therefore

$$\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = 0$$

Hence we get that whenever $A, B \in T(R, R)$ such that $AB = 0 \Rightarrow BA = 0$.
Therefore $T(R, R)$ is reversible if R is reduced. □

The trivial extension $T(R, R)$ of a reversible ring R need not be reversible which is shown in the following example.

2.4.6 Example. Let H be the division ring of real quaternions . As H is reduced the trivial extension $T(H, H)$ of H by H is a reversible ring. Let $R = T(H, H)$. Let S be the trivial extension of R by R i.e $S = R(+)R$.
Let us consider two elements $A, B \in S$ where

$$A = \begin{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{pmatrix}, B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

Now we have

$$AB = \begin{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = 0$$

However for $C \in S$ where $C = \begin{pmatrix} \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \end{pmatrix}$ we get

$$ACB = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0$$

Thus S is not semi-commutative. Hence by Proposition 2.4.3 it is not reversible.

2.4.7 Proposition. *The following conditions are equivalent for a ring R;*

(i) R is reversible.

(ii) $r_R(S) = l_R(S)$ for each $S \subseteq R$

(iii) For each $a \in R, l_R(a) = r_R(a)$.

(iv) $AB = 0$ implies $BA = 0$ for any two non -empty subsets A, B of R.

Proof. ((i)⇒(ii)) Let a be any element of R such that $aS = 0$. Since R is reversible $aS = 0 \Rightarrow Sa = 0$, and vice versa. Therefore (ii) is true.

((ii)⇒(iii)) is straightfoward.

((iii) ⇒(iv)) Let A, B be two non-emty subsets of R such that $AB = 0$. From this we got $ab = 0$; $\forall a \in A, b \in B$. By the given condition we got $ba = 0$. Now $BA = \{\sum b_j a_i \mid a_i \in A, b_j \in B\}$. But each term in this sum is zero. Therefore $BA = 0$

((iv)⇒(i)) This is trivial. □

2.4.8 Result. The class of reversible rings is closed under (1)subrings and (2)direct products.

Proof. (1) Let R be a reversible ring and S a subring of R. Let $a, b \in S$ such $ab = 0$. Since $S \subseteq R$ then $a, b \in R$ and R is reversible we get $ba = 0$. Hence S is reversible.

(2) Let $\{R_i\}_{i \in I}$ be a family of reversible rings and $R' = \prod_{i \in I} R_i$ be the direct product of this family. Let $a = (a_i)_{i \in I}, b = (b_i)_{i \in I} \in R'$ where $a_i, b_i \in R_i$, for all $i \in I$ such that $ab = 0 \Rightarrow (a_i b_i)_{i \in I} = 0 \Rightarrow a_i b_i = 0$, for all i. Since R_i is reversible for all i then we have $b_i a_i = 0$ for all i. Therefore $(a_i b_i)_{i \in I} = 0 \Rightarrow ba = 0$ in R' . Hence $R' = \prod_{i \in I} R_i$ is reversible. □

2.4.9 Proposition. *Suppose that R/I is a reversible ring for some ideal I of a ring R. If I is reduced (as a ring without identity) then R is reversible.*

Proof. Let $a, b \in R$ such that $ab = 0$. Then $ab \in I \Rightarrow ab + I = I \Rightarrow (a + I)(b + I) = I$. Since R/I is reversible,then $(b + I)(a + I) = I \Rightarrow ba + I = I \Rightarrow ba \in I$. Now $(ba)^2 = baba = 0$. Since I is reduced then we have $ba = 0$. Hence R is reversible. □

2.4.10 Proposition. *Let R be a ring and S a multiplicatively closed subset of R consisting of central elements. Assume that S contains only non-zero divisors. Then*

(i) eR and $(1 - e)R$ are reversible for some central idempotent e of R if and only if R is reversible.

(ii) R is reversible if and only if $S^{-1}R$ is reversible.

Proof. (i)($\Rightarrow$) Let $a, b \in R$ such that $ab = 0$. Then we have $eab = 0$ and $(1 - e)ab = 0$. Since e is a central and $eR, (1 - e)R$ are reversible then we get $eba = 0$ and $(1 - e)ba = 0$.

Now $ba = eba + (1 - e)ba = 0 \Rightarrow ba = 0$. Therefore R is reversible.

($\Leftarrow$) Let us assume that R is reversible. Let $x, y \in eR$ such that $xy = 0 \Rightarrow ea.eb = 0$ for some $a, b \in R$. Since R is reversible and e is a central idempotent, therefore we get $eb.ea = 0$. Hence eR is reversible. Similarly we can show that $(1 - e)R$ is also reversible.

(ii) Let α, β be two elements of $S^{-1}R$. Then $\alpha = a/s, \beta = b/t$ for some $a, b \in R$ and $s, t \in S$. such that $\alpha\beta = 0 \Rightarrow (a/s).(b/t) = 0 \Rightarrow ab/st = 0$ then $ab = 0$ (since S contains only non-zero-divisors). Since R is reversible then $ba = 0$. Therefore $\beta\alpha = (b/t).(a/s) = ba/st = 0$. Hence $S^{-1}R$ is reversible.

($\Leftarrow$) This is trivial because subrings of a reversible ring are reversible. □

2.4.11 Proposition. *Let R be a commutative domain and h be an injective endomorphism of R. Then the twisted Nagata extension of R by R namely $R(+)_h R$ is reversible.(See 0.2.11 for the definition of $R(+)_h R$)*

Proof. Let $(r_0, m_0), (r_1, m_1) \in R(+)_h R$ such that $(r_0, m_0)(r_1, m_1) = 0$
Now $(r_0, m_0)(r_1, m_1) = 0 \Rightarrow (r_0 r_1, h(r_0)m_1 + r_1 m_0) = 0$ we have

$$r_0 r_1 = 0 \qquad (*)$$

$$h(r_0)m_1 + r_1 m_0 = 0 \qquad (**)$$

Since R is a commutative domain then from (*)we get $r_0 = 0$ or $r_1 = 0$
Case 1. Suppose that $r_0 = 0$. Then from (**) $h(r_0)m_1 + r_1 m_0 = 0 \Rightarrow r_1 m_0 = 0 \Rightarrow r_1 = 0$ or $m_0 = 0$ hence we get $h(r_1)m_0 = 0$. Therefore , in this case $(r_1, m_1)(r_0, m_0) = (r_1 r_0, h(r_1)m_0 + r_0 m_1) = 0$
case 2. Suppose that $r_1 = 0$. From (**) $h(r_0)m_1 + r_1 m_0 = 0 \Rightarrow h(r_0)m_1 = 0 \Rightarrow h(r_0)$ or $m_1 = 0$. Since h is injective then we have $r_0 = 0$ or $m_1 = 0$. In this case also we get $(r_1, m_1)(r_0, m_0) = (r_1 r_0, h(r_1)m_0 + r_0 m_1) = 0$.
Hence $R(+)_h R$ is reversible. □

2.4.12 Remarks. (i) If a ring R is reversible then $R[x]$ need not be reversible. (An example is the ring in [KL1, Example 2.1])
(ii) Let R be a commutative reduced ring and let h be an injective endomorphism of R. Then the Nagata extension of R by R and h need not be reversible.
(Let D be a commutative domain of characteristic zero and consider
$R = D \times D$. Then R is a commutative reduced ring which is not a domain.
Let $h : R \longrightarrow R$ be defined by $h(r, s) = (s, r)$.

Then h is an automorphism of R.
Now $((0,1),(1,0)),((1,0),(0,1)) \in R(+)_h R$ then we have $((0,1),(1,0))((1,0),(0,1)) = (0, h((0,1))(0,1)+(1,0)(0,1) = 0$ but $((1,0),(0,1))((0,1),(0,1)) = (0, h((1,0))(0,1)+(0,1)(0,1) = (0,(0,2)) \neq 0$.
Hence $R(+)_h R$ is not reversible.)

We have *mentioned that the polynomial ring over a reversible ring* need not be reversible. A sufficient condition for this to happen is given in Proposition 2.5.34 below.

2.4.13 Proposition. *Let R be a ring. Then $R[x]$ is reversible if and only if $R[x, x^{-1}]$ is reversible.*

Proof. Suppose $R[x]$ is reversible. Let $S = \{1, x, x^2, ...\}$ which is a central multiplicatively closed subset consisting of non-zero-divisors of $R[x]$. We know that $R[x, x^{-1}] = S^{-1}R[x]$ which is reversible by Proposition 2.4.10. Conversely, if $R[x, x^{-1}]$ is reversible then $R[x]$ is reversible because subring of a reversible ring is reversible. □

We refer to [KL2] for the proof of the following result.

2.4.14 Proposition. *Let R be a ring and suppose that $C(R)$ contains an infinite subring whose non-zero elements are non-zero divisors in R. Then the following conditions are equivalent:*
(i) R is reversible.
(ii) $R[x]$ is reversible.
(iii)$R[x; x^{-1}]$ is reversible.

2.4.15 Proposition. *Let R be a ring and n any positive integer. If R is reduced then $R[x]/(x^n)$ is a reversible ring, where (x^n) is the ideal of $R[x]$ generated by x^n.*

Proof. Let $T = R[x]/(x^n)$.

Case 1. If $n = 1$ then $T \cong R$ which reversible. Therefore T is reversible.

Case 2. If $n = 2$ then $T = R[x]/(x^2)$, Let us put $u = x + (x^2)$, then $T = R[u]$. Let $f(u) = a_0 + a_1u, g(u) = b_0 + b_1u \in R[u]$ such that $f(u)g(u) = 0$ Then we have $a_0b_0 = 0, a_0b_1 + a_1b_0 = 0$. Since R is reduced then multiplying the second equation by b_0 we get $b_0a_0 = 0, b_0a_1 = 0, b_1a_0 = 0$ which implies that $g(u)f(u) = 0$. Hence T is reversible.

Case 3. If $n \geq 3$, let us put $u = x + (x^n)$. Let $A = a_0 + a_1u + a_2u^2 + ... + a_{n-1}u^{n-1}, B = b_0 + b_1u + b_2u^2 + ... + b_{n-1}u^{n-1} \in T$ such that

$$AB = 0 \qquad (*)$$

Now $a_ib_ju^{i+j} = 0 \ \forall i, j$ with $i + j \geq n$.

Therefore it is enough to prove for the case $i + j$ less than n. From (*) we have

$$a_0b_0 = 0 \qquad (1)$$

$$a_0b_1 + a_1b_0 = 0 \qquad (2)$$

$$a_0b_2 + a_1b_1 + a_2b_0 = 0 \qquad (3)$$

....

$$a_0b_{n-1} + a_1b_{n-2} + ... + a_{n-1}b_0 = 0 \qquad (n)$$

Now we will use induction on $i + j$ and the condition that R is reduced in the following cases.

From (1) and $b_0\times$ (2) we get $b_0a_1b_0 = 0 \Rightarrow a_1b_0a_1b_0 = 0 \Rightarrow a_1b_0 = 0$ (since R is reduced). Therefore we get
$a_1b_0 = 0$ and

$$a_0b_1 = 0 \qquad (2')$$

Again from (1), (2'), $b_0\times$ (3) and $b_1\times$ (3) we obtain $0 = a_0b_2 = a_1b_1 = a_2b_0$ in a similar way. Inductively we may assume that $a_ib_j = 0$ for $i + j = 0, 1, 2, ..., n-2$.By using eqn(n) and proceeding in this manner we finally have $0 = a_0b_{n-1} = a_1b_{n-2} = ... = a_{n-2}b_1 = a_{n-1}b_0$. Therefore we get $a_ib_j = 0$, for all $i+j = 0, 1, 2, ..., n-1$. Since R is reduced we have $b_ja_i = 0 \ \forall i, j$ with $i + j$ less than n. But we also have $b_ja_iu^{i+j} = 0$, for all i, j with $i + j \geq n$. Therefore $BA = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} b_ja_iu^{i+j} = 0$. Hence T is reversible ring. □

2.4.16 Remark. (i) If $R[x]/(x^n)$ is reversible, then R need not be reduced. (Choose R to be any commutative non-reduced ring.)
(ii) If $R[x]/(x^n)$ is reversible,then R is reversible.

2.5 Armendariz rings

In this section, we record the definition, basic properties and examples of Armendariz rings.

2.5.1 Definition. Let R be a ring. Then R is called an *Armendariz* ring if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0\ \forall i, j$

2.5.2 Example. If R is an integral domain then R is an Armendariz ring.

Proof. Since R is integral domain then $R[x]$ is also integral domain. Hence $f(x)g(x) = 0 \Rightarrow f(x) = o$ or $g(x) = 0 \Rightarrow a_i = 0$ or $b_j = 0\ \forall i, j$
$\Rightarrow a_i b_j = 0\ \forall i, j$
Thus, whenever $f(x), g(x) \in R[x]$ are such that $f(x)g(x) = 0$ we have $a_i b_j = 0\ \forall i, j$ □

2.5.3 Remark. Subrings of Armendariz rings are Armendariz.

2.5.4 Proposition. *Let $\{R_i\}_{i \in I}$ be family of Armendariz rings. Then $\prod R_i$ is Armendariz .*

Proof. Let $f(x), g(x) \in (\prod R_i)[x]$. such that $f(x)g(x) = 0$ where $f(x) = (f_i(x))_{i \in I}$ and $g(x) = (g_i(x))_{i \in I}$ and $f_i(x), g_i(x) \in R_i[x]\ \forall i$.
Let $f_i(x) = a_{io} + a_{i1}x + ... + a_{in}x^n$ and $g_i(x) = b_{io} + b_{i1}x + ... + b_{im}x^m$.
Now $f(x)g(x) = 0 \Rightarrow (f_i(x))_{i \in I}(g_i(x))_{i \in I} = 0 \Rightarrow (f_i(x)g_i(x))_{i \in I} = 0 \Rightarrow f_i(x)g_i(x) = 0\ \ \forall i$.
Since R_i is Armendariz then $f_i(x)g_i(x) = 0 \Rightarrow a_{ik}b_{il} = 0\ \forall k, l$. Therefore $\prod R_i$ is Armendariz. □

2.5.5 Remark. Direct sums of Armendariz rings are Armendariz (regarded as rings without identity)

2.5.6 Proposition. *If a ring R is reduced then R is Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{i=0}^{m} b_j x^j \in R[x]$ such that $f(x)g(x) = 0$. Then we get the following system of equations (assuming that $n \geq m$) :

$$a_0 b_0 = 0 \tag{1}$$

$$a_1 b_0 + a_0 b_1 = 0 \tag{2}$$

$$a_2 b_0 + a_1 b_1 + a_0 b_2 = 0 \tag{3}$$

....

$$a_{n-1} b_0 + a_{n-2} b_1 + \dots\dots\dots + a_{n-m-1} b_m = 0 \tag{$n-1$}$$

....

$$a_n b_m = 0 \tag{$n+m$}$$

From (1) $a_0 b_0 = 0 \Rightarrow (b_0 a_0)^2 = 0$. Since R is reduced, we get $b_0 a_0 = 0$. Therefore whenever $a_i b_j = 0$ we have $b_j a_i = 0$. Multiplying (2) by b_0 we get $b_0 a_1 b_0 = 0 \Rightarrow (a_1 b_0)^2 = 0 \Rightarrow a_1 b_0 = 0$. Similarly we can prove that $a_i b_0 = 0 \ \forall i$. Therefore, the above equations reduce to the form

$$a_0 b_1 = 0 \tag{$2'$}$$

$$a_1 b_1 + a_0 b_2 = 0 \tag{$3'$}$$

....

$$a_{n-2} b_1 + \dots\dots\dots + a_{n-m-1} b_m = 0 \tag{$(n-1)'$}$$

....

$$a_n b_m = 0 \qquad (n+m)'$$

Again from $(2)'$ we get $a_0b_1 = 0 \Rightarrow b_1a_0 = 0$. Therefore multiplying $(3)'$ by b_1 it reduced $b_1a_1b_1 = 0 \Rightarrow a_1b_1 = 0$. Continuing in this way we get $a_ib_1 = 0 \ \forall i$. By repeating in this way $\forall j$ we have $a_ib_j = 0 \ \forall i, j$. Hence R is Armendariz. □

2.5.7 Remark. (i)If R is a Boolean ring then R is Armendariz. (This holds since Boolean rings are reduced)

(ii) If R is a commutative regular ring then R is Armendariz.

(iii) An Armendariz ring need not be a regular ring. ($\mathbb{Z}$ is Armendariz but not regular)

2.5.8 Proposition. *Suppose R is an Armendariz ring. If $f_1, f_2, ..., f_n \in R[x]$are such that $f_1.f_2....f_n = 0$, then $a_1a_2....a_n = 0$ where a_i are coefficient of f_i for all i.*

Proof. Suppose $f_1f_2...f_n = 0$ and let a_i be any coefficient of f_i. Since R is Armendariz and $f_1, (f_2...f_n) \in R[x]$ such that $f_1(f_2...f_n) = 0$, so we have $a_1b = 0$ for any coefficient b of $f_2f_3...f_n$. Therefore $a_1f_2f_3...f_n = 0$. Thus $(a_1f_2)(f_3...f_n) = 0$. Since a_1a_2 is a coefficient of a_1f_2, and $a_1f_2, (f_3...f_n) \in R[x]$, therefore $(a_1a_2)c = 0$ for each coefficient c of $f_3f_4...f_n$. Hence $a_1a_2f_3...f_n = 0$. Similarly, we can get $(a_1a_2a_3)d = 0$, where d is coefficient of $f_4....f_n$. Continuing in this way we see that $a_1a_2...a_n = 0$. where a_i is a coefficients of f_i. Hence the result is proved □

2.5.9 Proposition. *A ring R is Armendariz if and only if $R[x]$ is Armendariz.*

Proof. ($\Rightarrow$) Assume that R is Armendariz and let $f(y), g(y) \in R[x][y]$ such that $fg = 0$.

Let us write $f(y) = f_0 + f_1y + ... + f_ny^n$ and $g(y) = g_0 + g_1y + ... + g_my^m$ where $f_i, g_j \in R[x]$. We have to show that $f_ig_j = 0\ \forall i, j$.

Let $k = degf_0 + ... + degf_n + degg_0 + ... + degg_m$ where degree of zero polynomial is taken to be 0. Now $f(x^k) = f_0 + f_1x^k + ... + f_nx^{kn}$ and $g(x^k) = g_0 + g_1x^k + ... + g_mx^{km}$ and the set of coefficient of $f_i's$ (resp. $g_j's$) equals the set of coefficients of $f(x^k)$ (resp. $g(x^k)$). Since $f(y)g(y) = 0$ and x commutes with elements of R, therefore $f(x^k)g(x^k) = 0$. Again, since R is Armendariz, therefore product of each co-efficient of $f_i's$ with co-efficients of $g_j's$ are zero. Therefore $f_ig_j = 0\ \ \forall i, j$. Hence $R[x]$ is Armendariz.

($\Leftarrow$) This is trivial because any subring of an Armendariz ring is again Armendariz. □

2.5.10 Proposition. *Let R be a ring. If there exist $u, v \in R$ such that $u^2 = 0 = v^2$ and $uv = vu \neq 0$ then R is not Armendariz.*

Proof. Let $f(x) = u - vx$ and $g(x) = u + vx$ polynomials over R. We have $f(x)g(x) = u^2 + (uv - vu)x + v^2x^2 = 0$ but $uv = vu \neq 0$.

Hence R is not Armendariz. □

2.5.11 Example. $T(\mathbb{Z}_4, \mathbb{Z}_4)$ is not Armendariz because we can find $u = (2, 0)$ and $v = (0, 1)$ which satisfy the condition in Proposition 2.5.10.

2.5.12 Result. For any ring R, $n \times n$ upper triagular matrix rings over R are not Armendariz, when $n \geq 2$.

[For if

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in R \right\},$$

then by taking

$$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x \text{ and } g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$$

we have $f(x)g(x) = 0$ but

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$$

showing that S is not Armendariz.]

2.5.13 Proposition. *Let R be a reduced ring. Then*

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is an Armendariz ring.

Proof. Let us consider two matrices

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S.$$

For simplicity of notation, we write a matrix $\begin{pmatrix} x & y & z \\ 0 & x & u \\ 0 & 0 & x \end{pmatrix}$ by (x, y, z, u).

Then we can denote their addition and multiplication by

$A+B = (a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$ and

$\text{AB} = (a_1, b_1, c_1, d_1).(a_2, b_2, c_2, d_2) = (a_1a_2, a_1b_2 + b_1a_2, a_1c_2 + b_1d_2 + c_1a_2, a_1d_2 + d_1a_2)$ respectively.

So every polynomial in $S[x]$ can be expressed in the form $(P_0(x), P_1(x), P_2(x), P_3(x))$ for some $P_i(x)'s$ in $R[x]$.

Let $f(x) = (f_0(x), f_1(x), f_2(x), f_3(x))$ and $g(x) = (g_0(x), g_1(x), g_2(x), g_3(x))$ be two elements in $S[x]$ such that $f(x)g(x) = 0$.

Now $f(x)g(x) = (f_0(x)g_0(x), f_0(x)g_1(x)+f_1(x)g_0(x), f_0(x)g_2(x)+f_1(x)g_3(x)+f_2(x)g_0(x), f_0(x)g_3(x) + f_3(x)g_0(x)) = 0$

Then we get the following system of equations:

$$f_0(x)g_0(x) = 0 \tag{1}$$

$$f_0(x)g_1(x) + f_1(x)g_0(x) = 0 \tag{2}$$

$$f_0(x)g_2(x) + f_1(x)g_3(x) + f_2(x)g_0(x) = 0 \tag{3}$$

$$f_0(x)g_3(x) + f_3(x)g_0(x) = 0. \tag{4}$$

Since R is reduced then $R[x]$ is also reduced. Multiplying (2) by $f_0(x)$ on the right side we get $f_0(x)g_1(x) = 0 \Rightarrow f_1(x)g_0(x) = 0$. Similarly multiplying (3) and (4) by $f_0(x)$ on the right side we get,

$f_0(x)g_3(x) = 0 \Rightarrow f_3(x)g_0(x) = 0$ and $f_0(x)g_2(x) = 0$ .Therefore (3) becomes

$$f_1(x)g_3(x) + f_2(x)g_0(x) = 0. \tag{5}$$

Again, multiplying (5) by $f_1(x)$ on the right side we get

$f_2(x)g_0(x) = 0$

Now let

$$f(x) = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} x^i \text{ and } g(x) = \sum_{j=0}^{m} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} x^j \in S[x]$$

where $f_0 = \sum_{i=0}^{n} a_i x^i, f_1 = \sum_{i=0}^{n} b_i x^i, f_2 = \sum_{i=0}^{n} c_i x^i, f_3 = \sum_{i=0}^{n} d_i x^i, g_0 = \sum_{i=0}^{m} a'_j x^j, g_1 = \sum_{i=0}^{m} b'_j x^j, g_2 = \sum_{i=0}^{m} c'_j x^j$ and $g_3 = \sum_{i=0}^{m} d'_j x^j$. Then we obtain that $a_i a'_j = 0, a_i b'_j = 0, b_i a'_j = 0, a_i c'_j = 0, b_i d'_j = 0, c_i a'_j = 0, a_i d'_j = 0$ and $d_i a'_j = 0 \ \forall i, j$.

Hence we get

$$\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} = 0 \ \forall i, j.$$

Therefore S is an Armendariz ring. □

2.5.14 Remark. Let R be a reduced ring and define a new ring as follows:

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \dots & a_{1n} \\ 0 & a & \dots & a_{2n} \\ 0 & 0 & \dots & a_{3n} \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

where n is a positive integer.

Then R_n is not Armendariz for $n \geqslant 4$, showing that Proposition 2.5.13 does not extend to the case $n \geqslant 4$.(For details the next paragraph can been seen.)

2.5.15 Example. We have

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

Let us consider the following two polynomials in $R[x]$:

$$f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$$

and

$$g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x.$$

We have $f(x)g(x) = 0$ but

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$$

Therefore R_4 is not Armendariz.

As R_4 is a subring of R_n for each $n \geq 5$, R_n is not Armendariz if $n \geq 5$.

2.5.16 Proposition. *Let R be a reduced ring. Then the trivial extension $T(R, R)$ is an Armendariz ring.*

Proof. We can easily show that $T(R, R) \cong U$ where

$$U = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

is a subring of R_4. By Proposition.2.5.13 R_3 is Armendariz and all its subrings are Armendariz. Therefore $T(R, R)$ is Armendariz. □

2.5.17 Proposition. *Let W be a non-zero ring and let $R = T(W, W)$, the trivial extension of W by W. Further, let S be $T(R, R)$ the trivial extension* *of R by R. Then S is not an Armendariz ring.*

Proof. Let W be a non-zero ring and let R and S be defined as above. Then

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in W \right\}$$

and S is isomorphic to

$$\left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \mid A, B \in R \right\} \text{ i.e}$$

$$S = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \mid A, B \in R \right\}.$$

$$\text{Let } f(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$$

and

$$g(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$$

be two polynomials in $S[x]$. Then we have $f(x)g(x) = 0$ but

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Therefore S is not Armendariz. □

2.5.18 Proposition. *Let R be an Armendariz ring. Then we have the following results:*

(i) If $ab = 0, ac^n b = 0$ for $a, b, c \in R$ and some integer $n \geq 1$, then $acb = 0$.

(ii) Let $a, b, c \in R$ satisfy $ab = 0$ and $c^n \in C(R)$ for some integer $n \geq 1$ then $acb = 0$.

Proof. Let $f(x) = a - acx$ and $g(x) = (1 + cx + ... + c^{n-1}x^{n-1})b \in R[x]$. Then we have

$f(x)g(x) = (a - acx)(1 + cx + ... + c^{n-1}x^{n-1})b = 0$ (since $ab = 0$ and $ac^n b = 0$). Since R is Armendariz, we get $acb = 0$.

(ii) Since $ab = 0$ then $c^n ab = 0$. Since $c^n \in C(R)$ therefore $ac^n b = 0$. By (i) we get $acb = 0$. □

2.5.19 Proposition. *Armendariz rings are abelian.*

Proof. Let $e \in I(R)$ be any idempotent element and $r \in R$ any element.

Let $a = e, b = 1-e$ and $c = er(1-e)$ then we have $ab = e(1-e) = 0 \Rightarrow ab = 0$ and $c^2 = er(1-e)er(1-e) = 0 \Rightarrow ac^2b = 0$.

Since R is Armendariz then by Proposition 2.5.18 we have

$acb = 0 \Rightarrow e.er(1-e)(1-e) = er(1-e) = 0 \Rightarrow er = ere$.

Similarly, again by taking $c = (1-e)re$ we can show that $re = ere$.

Hence we get $er = re \ \forall r \in R \Rightarrow e \in C(R)$. Therefore R is abelian. □

2.5.20 Remark. If R is abelian then R need not be Armendariz. [$T(\mathbb{Z}_4, \mathbb{Z}_4)$ is abelian but not Armendariz which is shown in Example 2.5.11.]

2.5.21 Proposition. *Let R be von Neumann regular ring. Then the following conditions are equivalent:*

(i) R is Armendariz.

(ii) R is reduced.

(iii) If the product of two linear polynomials in R is zero, then the products of their coefficients are also zero.

Proof. (ii) $\Rightarrow$(i) by Proposition 2.5.6.

(i) $\Rightarrow$(iii) is trivial by Definition 2.5.1.

(iii) $\Rightarrow$(ii) Let $e \in I(R)$ be any idempotent element and $r \in R$ be any arbitary element in R.

Let $f(x) = (1-e) + (1-e)rex$ and $g(x) = e - (1-e)rex$ be two linear polynomials in $R[x]$.

Now $f(x)g(x) = [(1-e) + (1-e)rex][e - (1-e)rex] = 0$.

Then by (iii) we get $1-e)(1-e)re = 0 \Rightarrow (1-e)re = 0 \Rightarrow re = ere$.

Similarly by taking $f(x) = e + er(1-e)x, g(x) = (1-e) - er(1-e)x$, we get by same argument $er = ere$.

Therefore $er = re$, for all $r \in R$.

Hence R is abelian.

This implies that

$$er(1-e) = 0, \quad \forall r \in R \tag{1}$$

Now let $x \in R$ such that $x^2 = 0$. Since R is von Neumann regular then there exists $y \in R$ such that $xyx = x$. Putting $e = xy, r = x$ in (1) we get

$er(1-e) = 0 \Rightarrow (xy)x(1-xy) = 0 \Rightarrow xyx - xyx^2y = 0 \Rightarrow xyx = 0 \Rightarrow x = 0.$

Hence R is reduced. □

2.5.22 Proposition. *A ring R is reduced if and only if the trivial extension $T(R,R)$ is Armendariz.*

Proof. ($\Rightarrow$) Let us assume that R is reduced. Then by Proposition 2.5.16, we have $T(R,R)$ is Armendariz.

($\Leftarrow$) Let us assume that $T(R,R)$ is Armendariz. If R is not reduced then ther exists $a \in R, a \neq 0$ such that $a^2 = 0$.

Let $u = (0,1)$ and $v = (a,0) \in T(R,R)$, then we have $u^2 = v^2 = 0$ but $uv = vu = (0,a) \neq 0$.

By Proposition 2.5.10 $T(R,R)$ is not Armendariz which is a contradiction.

Hence R is reduced. □

2.5.23 Remark. Let R be a reduced ring and I an ideal of R such that R/I is reduced. Then $T(R,R/I) = R(+)(R/I)$ is Armendariz.

2.5.24 Proposition. *Let R be a commutative P.I.D. and I an ideal of R. Then R/I is Armendariz.*

Proof. Since R is a P.I.D. and I is an ideal of R therefore $I = Ra$ for some $a \in R$. When $a \neq 0$ then $a = p_1^{\alpha_1} p_2^{\alpha_2}.......p_k^{\alpha_k}$ where $p_i's$ are distict prime elements in R. Then
$R/I = R/Ra = R/Rp_1^{\alpha_1} p_2^{\alpha_2}.......p_k^{\alpha_k} \cong R/Rp_1^{\alpha_1} \times R/Rp_2^{\alpha_2} \times .... \times R/Rp_k^{\alpha_k}$
(By the Chinese remainder theorem).
Now let us show that R/Rp^α is Armendariz when α is non-negative integer.Let $\overline{f(x)} = \sum_{i=0}^n \bar{a}_i x^i, \overline{g(x)} = \sum_{j=0}^m \bar{b}_j x^j \in R/Rp^\alpha$ such that $\overline{f(x)}.\overline{g(x)} = \overline{f(x)g(x)} = \bar{0}$.
Now $\overline{f(x)g(x)} = \bar{0} \Rightarrow p^\alpha \mid f(x)g(x)$.
Case 1: $\alpha = 1$
In this case Rp is maximal ideal of R and so R/Rp is field. Therefore R/Rp is Armendariz.
Case 2: $\alpha \geq 2$
Since p is prime then we have $p^\alpha \mid f(x)g(x) \Rightarrow f(x) = p^\beta f_1(x), g(x) = p^\gamma g_1(x)$, where p does not divide the g.c.d of the coefficients of $f_1(x)$ and $g_1(x)$. Therefore $\alpha \leq (\beta + \gamma) \Rightarrow p^\alpha \mid a_i b_j \quad \forall i, j$. Therefore we get $\overline{a_i b_j} = \bar{a}_i \bar{b}_j = \bar{0} \;\; \forall i, j$.
Hence R/Rp^α is Armendariz. Since direct products of Armendariz rings are Armendariz, therefore R/I is Armendariz. □

2.5.25 Remark. For each integer n , $\mathbb{Z}/n\mathbb{Z}$ is an Armendariz ring.

2.5.26 Proposition. *Let R be a domain, I an ideal of R. If R/I is an Armendariz ring then $R(+)(R/I)$ is Armendariz.*

Proof. Let $f(x), g(x)$ be elements in $R(+)(R/I)[x]$, where $f(x) = \sum_{i=0}^n (a_i, \bar{u}_i) x^i = (f_0(x), \overline{f_1(x)})$ and $g(x) = \sum_{j=0}^m (b_j, \bar{v}_j) x^j = (g_0(x), \overline{g_1(x)})$ which satisfy $f(x)g(x) =$

0. Now $f(x)g(x) = 0 \Rightarrow (f_0(x), \overline{f_1(x)})(g_0(x), \overline{g_1(x)}) = 0$. Therefore we get

$$f_0(x)g_0(x) = 0 \tag{1}$$

$$\overline{f_0(x)g_1(x) + g_0(x)f_1(x)} = \bar{0} \tag{2}$$

Since $R[x]$ is domain we have $f_0(x) = 0$ or $g_0(x) = 0$.

Case 1. If $f_0(x) = 0$ then from (2) we have $\overline{g_0(x)f_1(x)} = \bar{0}$ in $R/I[x]$. Since R/I is Armendariz then we get $\overline{b_j u_i} = \bar{0}$ for all i, j and $f_0(x) = 0 \Rightarrow a_i = 0$ for all i.

Therefore $(a_i, \bar{u}_i)(b_j, \bar{v}_j) = 0$ for all i, j.

Case 2. If $g_0(x) = 0$. Similar to the case 1. we get $(a_i, \bar{u}_i)(b_j, \bar{v}_j) = 0$ for all i, j.

Therefore in both cases we get $(a_i, \bar{u}_i)(b_j, \bar{v}_j) = 0 \;\; \forall i, j$.

Hence $R(+)(R/I)$ is Armendariz. □

2.5.27 Remarks. (1) If R is Armendariz and I is a two-sided ideal of R then R/I need not be Armendariz.

(proof: Let us consider $R = \mathbb{Z}[x]$ and $I = (4) + (x^2)$. Let $\overline{f(y)}, \overline{g(y)} \in R[y]/I[y]$ given by $\overline{f(y)} = \bar{2} + \bar{3}xy, \overline{g(y)} = \bar{2} - \bar{3}xy$. Then we have $\overline{f(y)g(y)} = 0$ but $\bar{2}\bar{3}x \neq 0$.)

(2) If R is abelian then R need not be Armendariz.

(Consider $R = \mathbb{Z}_4(+)\mathbb{Z}_4$ which is abelian (because it is commutative) but not Armendariz which is shown in Example 2.5.11.)

2.5.28 Proposition. *If R is semi-commutative and Armendariz then $R[x]$ is semi-commutative.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ such that $f(x)g(x) = 0$. Since R is Armendariz then $a_i b_j = 0, \forall i, j$. Let $h(x) = \sum_{k=0}^{l} c_k x^k$. again

R is semi-commutative then $a_i c_k b_j = 0, \forall i, j$ and k. Hence $f(x)h(x)g(x) = 0$, for every $h(x) \in R[x]$. Therefore $R[x]$ is semi-commutative. □

2.5.29 Proposition. *Let R be an abelian ring. Then the following conditions are equivalent:*

(i) R is an Armendariz.

(ii) eR and $(1-e)R$ are Armendariz for every idempotent element e of R.

(iii) eR and $(1-e)R$ Armendariz for some idempotent element e of R.

Proof. ((i) ⇒(ii))Let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in eR[x]$ such that $f(x)g(x) = 0$. Since R is Armendariz and $eR \subseteq R$ then $f(x), g(x) \in R[x]$. Therefore we have $a_i b_j = 0, \forall i, j$. Hence eR is Armendariz. Similarly we can prove that $(1-e)R$ is also Armendariz.

((ii) ⇒ (iii)) Trivial.

((iii) ⇒ (i)) Since R is abelian we know that $I(R) \subseteq C(R)$ .Therefore $R \cong eR \times (1-e)R$. Again direct products of Armendariz rings is also Armendariz. Hence R is Armendariz. □

2.5.30 Proposition. *Let R be an Armendariz ring. Then R is directly finite.*

Proof. This follows from Propositions 2.2.5 and 2.5.19. □

2.5.31 Proposition. *If K is a field, V a vector space over K and $h : K \to K$ a ring monomorphism then $K(+)_h V$ is Armendariz.*

Proof. The given map h induces a natural ring homomorphism $h : K[x] \to K[x]$ defined by $h(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} h(a_i) x^i$

We have torsion free polynomial module $V[x]$ over $K[x]$. We have already identified $K(+)_h V[x]$ with $K[x](+)_h V[x]$.

Now let $f(x), g(x) \in K(+)_h V[x]$ such that $f(x)g(x) = 0$. Let us write $f(x) = (f_0(x), f_1(x))$ and $g(x) = (g_0(x), g_1(x))$ where $f_0(x) = \sum_{i=0}^{m} a_i x^i, g_0(x) = \sum_{i=0}^{n} b_i x^i \in K[x]$ and $f_1(x) = \sum_{i=0}^{l} c_i x^i, g_1(x) = \sum_{i=0}^{r} d_i x^i \in V[x]$

Therefore $0 = f(x)g(x) = (f_0(x), f_1(x))(g_0(x), g_1(x))$

$= (f_0(x)g_0(x), h(f_0(x))g_1(x) + g_0(x)f_1(x))$. We get

$$f_0(x)g_0(x) = 0 \tag{1}$$

$$h(f_0(x))g_1(x) + g_0(x)f_1(x) = 0. \tag{2}$$

Since $K[x]$ is integral domain we have from (1) $f_0(x) = 0$ or $g_0(x) = 0$

Case 1. If $f_0(x) = 0$ then $h(f_0(x) = 0$ (h is a ring homomorphism). From (2) we get $h(f_0(x))g_1(x) + g_0(x)f_1(x)) = 0 \Rightarrow g_0(x)f_1(x) = 0$. Since $V[x]$ is torsion free $K[x]$-module then $g_0(x) = 0$ or $f_1(x) = 0$. Therefore in this case $f(x) = (0, f_1(x))$ and $g(x) = (0, g_1(x))$ or $f(x) = 0, g(x) = (g_0(x), g_1(x))$ we get from the first condition $(0, c_i)(0, d_j) = 0 \ \forall i, j$.

Case 2. If $g_0(x) = 0$ then from (2) we get

$h(f_0(x))g_1(x) + g_0(x)f_1(x)) = 0 \Rightarrow h(f_0(x))g_1(x) = 0$ Since $V[x]$ is torsion free $K[x]$-module we get

$$h(f_0(x)) = 0 \ \text{ or } \ g_1(x) = 0 \tag{3}$$

As h is one-one, either $f_0(x) = 0$ or $g_1(x) = 0$.

In the former case also we have $f(x) = (0, f_1(x))$ and $g(x) = (0, g_1(x))$ which is similar to case 1. In the latter case $g(x) = 0$.

Thus either of the cases $K(+)_h V$ is Armendariz. □

Example(i) below illustrates the fact that factor rings of an Armendariz ring need not be Armendariz.

2.5.32 Examples. (i) Let us consider ring $R = \mathbb{Z}(+)(\mathbb{Z}/8\mathbb{Z})$ which is Armendariz because $\mathbb{Z}$ is reduced and $\mathbb{Z}/8\mathbb{Z}$ is Armendariz. Again let us consider an ideal $I = 8\mathbb{Z}(+)0$ then $R/I \cong \mathbb{Z}/8\mathbb{Z}(+)\mathbb{Z}/8\mathbb{Z}$. we know that $\mathbb{Z}/8\mathbb{Z}(+)\mathbb{Z}/8\mathbb{Z}$ is not Armendariz because the square of the polynomial $f(x) = (\bar{4}, \bar{0}) + (\bar{4}, \bar{1})x \in \mathbb{Z}/8\mathbb{Z}(+)\mathbb{Z}/8\mathbb{Z}$ is zero but $(\bar{4}, \bar{0})(\bar{4}, \bar{1}) \neq 0$. Hence factor ring of an Armendariz ring need not be Armendariz.

(ii) Let R be any ring with identity . Let R' = full matrix ring of degree ≥ 2 over R. Then R' is not Armendariz. [Consider $f(x) = E_{11} + E_{12}x, g(x) = -E_{21} + E_{11}x$. Then we have $f(x)g(x) = 0$ but $E_{11}E_{11} \neq 0$.]

2.5.33 Proposition. *Let R be an Armendariz ring and I an ideal of R. Then $R/l_R(I)$ is Armendariz.*

Proof. Let $\overline{f(x)} = \sum_{i=0}^{m} \bar{a}_i x^i, \overline{g(x)} = \sum_{j=0}^{n} \bar{b}_j x^j \in R/l_R(I)[x]$ which satisfy $\overline{f(x)}.\overline{g(x)} = \bar{0}$ where $\bar{a}_i = a_i + l_R(I), \bar{b}_j = b_j + l_R(I)\ \ \forall i, j$.

Now $\overline{f(x)g(x)} = \bar{0} \Rightarrow \sum_{i+j=k} a_i b_j \in l_R(I)[x]$ where $0 \leq k \leq n+m, 0 \leq i \leq m$ and $0 \leq j \leq n$. Since $l_R(I)$ is the left annihilator of I in R we get

$\sum_{i+j=k} a_i b_j u = 0, \ \ \forall u \in I$

This implies that $(a_0 + a_1 + \ldots.. + a_m x^m)(b_0 u + b_1 u x + \ldots.. + b_n u x^n) = 0 \in R[x]$. Since R is Armendariz we have $a_i b_j u = 0, \ \ \forall i, j$ and $\forall u \in I \Rightarrow a_i b_j \in I \Rightarrow \overline{a_i b_j} = 0\ \ \forall i, j$. Hence $R/l_R(I)$ is an Armendariz ring. □

2.5.34 Proposition. *Let R be an Armendariz ring. Then the following conditions are equivalent:*

(i) R is reversible.

(ii) $R[x]$ is reversible.

(iii)$R[x; x^{-1}]$ is reversible.

Proof. ((i) $\Rightarrow$(ii)) Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ such that $f(x)g(x) = 0$. Since R is Armendariz then we have $a_i b_j = 0 \;\; \forall i, j$. But R is reversible. Hence $b_j a_i = 0, \quad$ for all i, j. Therefore $g(x)f(x) = 0$. Hence $R[x]$ is reversible.

((ii)$\Rightarrow$(iii)) This follows from Proposition 2.4.13..

((iii)$\Rightarrow$(i)) This is trivial because a subring of a reversible ring is reversible.

$\square$

2.5.35 Proposition. *Let R be an Armendariz ring and let $C(R)$ be the centre of R. If I is an ideal of R such that $I \subseteq Nil(R)$ then $C(R)+I$ is both an Armendariz and semi-commutative ring.*

Proof. Let $x, y \in C(R) + I$ then $x = a + n, y = b + m$ for some $a, b \in C(R)$ and $n, m \in I$. Now, $x - y = (a - b) + (n - m) \in C(R) + I$ (since $C(R)$ and I are both subrings of R). Therefore $C(R) + I$ is also a subring of R. Hence $C(R) + I$ is Armendariz.

Again let $a+n, b+m \in C(R) + I$ such that $(a+n)(b+m) = 0$ then for any $r \in C(R)$ we have

$$(a + n)r(b + m) = r(a + n)(b + m) = 0 \qquad (*)$$

and for any $s \in I$, $\exists k \in \mathbb{N}$ such that $s^k = 0$. Then we have $(a+n)s^k(b+m) = 0$. Now $(a + n)(1 - s^k x^k)(b + m) = 0$

$\Rightarrow (a + n)(1 - sx)(1 + sx + s^2x^2 + ... + s^{k-1}x^{k-1})(b + m) = 0$

$\Rightarrow ((a + n) - (a + n)sx)((b + m) + s(b + m)x + s^2(b + m)x^2 + ... + s^{k-1}(b + m)x^{k-1})(b + m) = 0$. Since R is Armendariz then we have

$$(a + n)s(b + m) = 0 \qquad (**)$$

From (*) and (**) we get $(a+n)(r+s)(b+m)=0, \ \forall r+s \in C(R)+I$.
Hence $C(R)+I$ is semi-commutative. $\square$

2.6 Regularity and other conditions

In this section, we record some results connecting regularity with some other conditions studied earlier in this dissertation.

The following results extends Propositions 2.3.3(b) and 2.5.21.

2.6.1 Proposition. *Let R be a regular ring. Then the following conditions are equivalent:*

(i) R is left (or right) invariant.

(ii) R is reduced.

(iii) R is reversible.

(iv) R is semi-commutative.

(v) R is Armendariz.

(vi) R is abelian.

Proof. Some of these implications hold without the regularity assumption. These are as follows.

((i) $\Rightarrow$ (iv)) By Proposition 3.3.7.

((ii) $\Rightarrow$ (iii)) is a consequence of Proposition 2.1.5.

((ii) $\Rightarrow$ (v)) by Proposition 2.5.6.

((iii) $\Rightarrow$ (iv)) by Proposition 2.4.3.

((iv) $\Rightarrow$ (vi)) by Proposition 2.3.6.

((v) $\Rightarrow$ (vi)) by Proposition 2.5.19.

Regularity condition is required for ((vi) $\Rightarrow$ (ii) $\Rightarrow$ (i)). We prove these below.

((ii) $\Rightarrow$ (i)) Since R is reduced, R is abelian. Let $a \in R$. As R is regular there exists $b \in R$ such that $aba = a$. Write $e = ba \in I(R) \subset C(R)$. Also $Ra = Re$, showing that Ra is an ideal. Hence if A is left ideal, since $A = \sum_{a \in A} Ra$, A is necessarily an ideal. Thus R is left invariant.

((vi) $\Rightarrow$ (ii)) Let a be a non-zero element of R. We know that $a = aba$ for some $b \in R$. Hence $e = ba$ satisfies $e^2 = e \in I(R) \subset C(R)$ (since R is abelian). Hence $a = aba = ba^2$, implying that a is not nilpotent. Hence R is reduced.

It has been already shown that regular imply anti-regular. In view of this it is natural to ask if any of the implications considered in the Proposition 2.6.1. also hold in the presents of anti-regularity. Indeed we have:

2.6.2 Proposition. *Let R be an anti-regular ring. Then R is reduced if and only if R is abelian.*

Proof. The proof is similar to that of the implication (vi) $\Rightarrow$ (ii) of Proposition 2.6.1. □

The following result connects regular rings with V-rings.

2.6.3 Proposition. *Let R be a regular ring satisfying the equivalent conditions (i) to (vi) of Proposition 2.6.1. Then R is a left and right V-ring.*

Proof. As seen above , R must satisfy the condition " for each a in R there exists $b \in R$ such that $a = a^2b$." Now let M be a cyclic left R-module. Then $M \cong R/I$ for some left ideal I of R. As R is left invariant, I is an ideal of R and R/I has the structure of a ring. As R is regular so is R/I. It follows that regarding R/I as a ring, $Rad(R/I) = 0$. Hence $Rad_R(R/I) = 0$ where R/I is regarded as a left R-module. So $Rad(M) = 0$. Hence (by Proposition 1.3.5) R is a left V-ring. □

It is natural to ask which of these results extend to module-theoretic generalizations studied in the next chapter, either for all modules or for cyclic modules. A result giving an affirmative answer is Proposition 3.3.18. Other questions are also worth considering.

Chapter 3

Some module-theoretic extensions

In this chapter, a survey of some research done in the area of module-theoretic extensions of some concepts studied in Chapter 2 is carried out.

3.1 Reduced modules

We begin this section with the following definition.

3.1.1 Definition. A left R-module M is *reduced module* if for every $m \in M$ and every $a \in R$ such that $am = 0$ we have $Rm \cap aM = 0$.

3.1.2 Proposition. *R is a reduced ring if and only if ${}_RR$ is a reduced module.*

Proof. ($\Rightarrow$) Let $a, b \in R$ such that $ab = 0$. Let $x \in Ra \cap bR$, then $x = ra = bs$ for some $r, s \in R$. Now $(ra)^2 = ra.ra = ra.bs = 0$. Since R is a reduced ring we have $ra = 0$. Therefore $x = 0 \Rightarrow Ra \cap bR = 0$. Hence ${}_RR$ is a reduced module.

($\Leftarrow$) Let $a \in R$ such that $a^2 = 0$. Since ${}_RR$ is reduced module, then $Ra \cap aR = 0 \Rightarrow a = 0$. Hence R is reduced ring. $\square$

3.1.3 Proposition. *The following conditions are equivalent for a left R-module M:*

(1) ${}_RM$ is reduced.

(2)For any $m \in M$ and $a \in R$, the following conditions hold:

(a) $am = 0$ implies $aRm = 0$.

(b) $a^2m = 0$ implies $am = 0$.

Proof. (1)$\Rightarrow$ (2) Let $a \in R$ and $m \in M$ such that

(a) $am = 0$. Now for any $r \in R$, $arm \in Rm \cap aM$. Since M is reduced, $Rm \cap aM = 0$ and so $arm = 0, \forall r \in R$ i.e $aRm = 0$.

(b) $a^2m = 0$ then $a(am) = 0$. Since M is a reduced module, $Ram \cap aM = 0$ yielding $am = 0$.

(2) $\Rightarrow$ (1) Let $a \in R, m \in M$ such that $am = 0$. Let $x \in Rm \cap aM$, then $x = rm = am'$ for some $r \in R, m' \in M$. By 2(a) we have $aRm = 0$ i.e $asm = 0, \forall s \in R$ Therefore $a^2m' = arm = 0 \Rightarrow a^2m' = 0$. But by 2(b), this again gives $x = am' = 0$, i.e $Rm \cap aM = 0$ whenever $am = 0$. Therefore M is a reduced R-module. $\square$

3.1.4 Proposition. *Every submodule of a reduced module is reduced.*

Proof. Let W be a submodule of a reduced R-module M. Let $a \in R$ and $n \in W$ satisfy $an = 0$. Since $W \subseteq M$ and M is reduced then $an = 0 \Rightarrow Rn \cap aM = 0$. But $Rn \cap aW \subseteq Rn \cap aM = 0 \Rightarrow Rn \cap aW = 0$. Hence W is reduced. $\square$

3.1.5 Remark. If I is a left ideal of a reduced ring R, then ${}_RI$ is a reduced R-module.

3.1.6 Proposition. *Every direct products of reduced R-modules is a reduced R-module.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of reduced R-modules. Let $M = \prod_{i\in I} M_i$ be their product. Let $m \in M, a \in R$ satisfy $am = 0$. Then $m = (m_i)_{i\in I}$ where $m_i \in M_i, \forall i \Rightarrow am_i = 0, \forall i$. Since M_i is reduced R-module for every $i \in I$, then $Rm_i \cap aM_i = 0, \forall i \Rightarrow Rm \cap aM = 0$. Hence $M = \prod_{i\in I} M_i$ is a reduced R-module. □

3.1.7 Remark. If M_t is a reduced R_t-module for each $t \in I$, then $\prod_t M_t$ is a reduced $\prod_t R_t$-module.

3.1.8 Proposition. *Let R be a subring of a ring S with $1_S \in R$, such that ${}_RM \subseteq_S L$. If ${}_SL$ is reduced then ${}_RM$ is also reduced.*

Proof. Let $a \in R, m \in M$ satisfy $am = 0$. Since R and M are subsets of S and L respectively, $am = 0$ in S. Again L is a reduced S-module. Hence $Sm \cap aL = 0$. But $Rm \cap aM \subseteq Sm \cap aL = 0 \Rightarrow Rm \cap aM = 0$.
Hence ${}_RM$ is also reduced. □

3.1.9 Proposition. *Let M be a reduced R-module. If $f(x)m(x) = 0$ where $f(x) = \sum_{i=0}^{\infty} a_ix^i \in R[[x]], m(x) = \sum_{j=0}^{\infty} m_jx^j \in M[[x]]$, then $a_im_j = 0$, for all i, j.*

Proof. From $f(x)m(x) = 0$, we get a system of equations:

$$a_0m_0 = 0 \qquad (0)$$

$$a_0m_1 + a_1m_0 = 0 \tag{1}$$

$$a_0m_2 + a_1m_1 + a_2m_0 = 0 \tag{2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_0m_k + a_1m_{k-1} + \dots\dots\dots + a_km_0 = 0 \tag{k}$$

$$\dots\dots\dots\dots\dots\dots\dots$$

so on.

Since M is reduced then from (0) we get $a_0Rm_0 = 0$.

Now multiplying (1)on the left side by a_0 , we get $a_0^2m_1 = 0 \Rightarrow a_0m_1 = 0 \Rightarrow a_0Rm_1 = 0$. Let us assume that $a_0m_j = 0, \forall j \leq k-1$.. Then multiplying on the left side of *(k)* by a_0 we get $a_0^2m_k = 0$. Since M is reduced we get $a_0m_k = 0$. Hence by induction $a_0m_j = 0, \forall j$. Similarly we can show that for each i , $a_im_j = 0$. Therefore $a_im_j = 0, \forall i, j$. □

3.1.10 Proposition. *The following conditions are equivalent for a left R-module M:*

(i) ${}_RM$ *is reduced.*

(ii) ${}_{R[x]}M[x]$ *is reduced.*

(iii) ${}_{R[[x]]}M[[x]]$ *is reduced.*

Proof. ((i)⇒(iii)) Let $f(x) = \sum_{i=0}^{\infty} a_ix^i \in R[[x]], m(x) = \sum_{j=0}^{\infty} m_jx^j \in M[[x]]$, satisfy $f(x)m(x) = 0$. Let

$$f(x)m'(x) = g(x)m(x) \tag{$*$}$$

for some $g(x) = \sum_{i=0}^{\infty} b_ix^i \in R[[x]]$ and $m'(x) \sum_{j=0}^{\infty} m'_jx^j \in M[[x]]$.

Now by Proposition 3.1.8 we have $f(x)R[[x]]m(x) = 0$. Multiplying from left side of (*)by $f(x)$ we get

$$f^2(x)m'(x) = 0 \tag{**}$$

Now we get the system of equations:

$$a_0^2 m'_0 = 0 \tag{0}$$

$$a_0(a_0 m'_1 + a_1 m'_0) + a_1 m'_0 = 0 \tag{1}$$

$$a_0(a_0 m'_2 + a_1 m'_1 + a_2 m'_0) + a_1(a_0 m'_1 + a_1 m'_0) + a_2 m'_0 = 0 \tag{2}$$

....

so on.

Since M is reduced we get from (0) $a_0 m'_0 = 0 \Rightarrow a_0 R m'_0 = 0$. Multiplying from left side of (1) by a_0 we get $a_0^2 m'_1 = 0 \Rightarrow a_0 m'_1 = 0 \Rightarrow a_0 R m'_1 = 0 \Rightarrow a_1 m'_0 = 0$. Similarly we can show that from (2) $a_1 m'_1 = 0 = a_2 m'_0 = a_0 m'_2$. Therefore $a_i m'_j = 0, \forall i, j \Rightarrow f(x)m'(x) = 0$. From (*) we get $f(x)m'(x) = g(x)m(x) = 0$. Hence $R[[x]]m(x) \cap f(x)M[[x]] = 0$ whenever $f(x)m(x) = 0$. Therefore ${}_{R[[x]]}M[[x]]$ is reduced.

((iii)⇒(ii))and ((ii)⇒(i)) These follow from Proposition 3.1.4. □

3.2 Abelian modules

In order to study abelianness in modules, we need the following basic definitions.

3.2.1 Definition. Let ${}_RM$ be a module and W a submodule of M. We say W is *fully invariant* if for each $f \in End({}_RM)$ the submodule Wf is contained in W.

3.2.2 Remark. M and 0 are fully invariant submodules of M.

3.2.3 Definition. A module M is an *invariant* module if every submodule of M is fully invariant.

3.2.4 Example. $\mathbb{Z}$ is invariant as a $\mathbb{Z}$-module.

3.2.5 Proposition. *A vector space ${}_FV$ is not invariant if $dim(V) \geq 2$.*

Proof. Let β be a basis of ${}_FV$. We have $n(\beta) \geq 2$. Let $x_1, x_2 \in \beta$ be two distinct elements. Consider the subspace $W = Fx_1$ of V. Now let us the define linear transformation $f : V \longrightarrow V$, given by

$$(x)f = \begin{cases} x_2 & \text{if } x = x_1 \\ 0 & \text{otherwise} \end{cases}$$

and extended linearly. Under this linear transformation we get $Wf = (Fx_1)f = F(x_1)f = Fx_2 \not\subseteq Fx_1 = W$. Therefore W is not fully invariant. Hence ${}_FV$ is not an invariant module. □

3.2.6 Definition. Let M be a left R-module. Then M is an *abelian* module if every direct summand of M is fully invariant in M.

3.2.7 Examples. All invariant modules are trivially abelian. By Proposition 3.2.5, vector spaces of dimension ≥ 2 are never abelian modules.

3.2.8 Remarks. (1) Let R be a ring and let $M = R$. Define for $a \in R$, $g_a : R \to R$ by $g_a(x) = xa$. It is easy to see that $End(_RR) = \{g_a \mid a \in R\}$. Consider a submodule N of M (i.e N is a left ideal of R). Then N is fully invariant in M if and only if $Ng \leq N, \forall g \in End(_RR)$ i.e if and only if $Ng_a \leq N$, $\forall a \in R$ if and only if $Na \leq N, \forall a \in R$, i.e if and only if N is a two sided ideal of R.

(2) Let R be a ring. Then $_RR$ is an invariant module if and only if every left ideal is an ideal(i.e, by definition, if and only if R is a left invariant ring).

3.2.9 Proposition. *Let R be a ring. Then R is an abelian ring if and only if $_RR$ is an abelian module.*

Proof. ($\Rightarrow$) Let $N \leq^{\oplus} R$. Then $N = Re$, for some $e \in I(R)$.
Now we have to show that N is fully invariant in $_RR$. By Remark 3.2.8(1) we have to show that N is an ideal.Let $n \in N$. Then $n = ae$ for some $a \in R$. Now for any $x \in R$, $nx = (ae)x = a(ex)$. Since R is an abelian $e \in C(R)$. Therefore $nx = a(ex) = a(xe) = (ax)e \in Re = N$. Hence N is an ideal. Therefore N is fully invariant, i.e $_RR$ is abelian module.
($\Leftarrow$) Let $e \in I(R)$. We have to show that $e \in C(R)$. Now we have $Re \leq^{\oplus} R$. Since $_RR$ is an abelian module then Re is fully invariant in $_RR$. Let $a \in R$ be any element. By Remark 3.2.8(1) we get $(Re)x \leq Re, \forall x \in R$. In particular $(Re)a \leq Re \Rightarrow ea = be$, for some $b \in R$. Multiplying on both sides by $1 - e$ we have

$$ea(1-e) = be(1-e) = 0 \Rightarrow ea = eae \qquad (*)$$

Similarly by choosing the idempotent $(1-e)$ in the place of e we get

$$ae = eae \qquad (**)$$

From (*) and (**) we get $ea = ae, \forall a \in R \Rightarrow e \in C(R)$. Hence R is an abelian ring. $\square$

3.2.10 Proposition. *Let $M = N \oplus K$. Then there exists a unique map $f : M \longrightarrow M$ such that $f^2 = f, f|_N = 1_N$ and $f|_K = 0$.*

Proof. We shall use the fact that each $m \in M$ can be uniquely represented as $m = x + x'$ with $x \in N$ and $x' \in K$. We define $f : M \longrightarrow M$ by $(x+x')f = x$. Therefore $f^2 = f, f|_N = 1_N$ and $f|_K = 0$. If there exists g which satisfies all the given conditions then $(x + x')f = x = (x + x')g, \forall x + x' \in M$. Hence $f = g$. $\square$

3.2.11 Proposition. *Let ${}_RM$ be a module. Then ${}_RM$ is an abelian module if and only if the ring $S_M = End({}_RM)$ is abelian.*

Proof. $(\Rightarrow)$ Let $f \in I(S_M), g \in S_M$. Since M is abelian , $M(fg) = (Mf)g \leq Mf$. Again $f^2 = f$, we have, for all $m \in M$ $(m)fg = m'f$ for some $m' \in M$. Now $(m)fgf = (m')f^2 = (m')f = (m)fg, \forall m \in M$.
Therefore for every $f \in I(S_M)$ we have

$$fgf = fg \qquad (*)$$

Therefore $(1-f)g(1-f) = (1-f)g$ i.e $g - fg - gf + fgf = (g - fg)(1-f) = g - fg$. Therefore

$$fgf = gf \qquad (**)$$

From (*) and (**) we have $fg = gf$. Hence S_M is an abelian ring.

($\Leftarrow$) Let $N \leq \oplus M$. Then $\exists N' \leq M$ such that $M = N \oplus N'$. By Proposition 3.2.10 we have $f \in S_M$ such that $f^2 = f, f|_N = 1_N, f|'_N = 0$. Since S_M is abelian we have $fg = gf, \forall g \in S_M$. Now $Ng = (Nf)g = N(fg) = N(gf)(Ng)f \leq Mf = N \Rightarrow Ng \leq N$. Therefore N is fully invariant. Hence M is an abelian module. $\square$

3.3 Semi-commutative modules

This section is devoted to the study of basic properties and examples of semi-commutative modules.

3.3.1 Definition. A module M over a ring R is said to be *semi-commutative* if whenever $r \in R$ and $m \in M$ satisfy $rm = 0$ then $rsm = 0$ for each s of R.

3.3.2 Remarks. (i) A ring R is semi-commutative if and only if the module ${}_RR$ is semi-commutative.

(ii) If R is a commutative ring then every R-module M is semi-commutative.

3.3.3 Proposition. *Let $\theta : R \longrightarrow R'$ be a ring homomorphism and let M be an R'-module. Regard M as a left R-module via θ. Then we have :*

(i) if M is a semi-commutative R'-module , then M is semi-commutative R-module;

(ii) if θ is onto, then the converse of the statements in (i) hold;

(iii) if R' is a semi-commutative ring, then R' is semi-commutative as a left R-module.

Proof. (i) Let us take $r, s \in R, m \in M$ and assume $rm = 0$. Then $\theta(r)m = 0 \Rightarrow (rs)m = \theta(rs)m = \theta(r)\theta(s)m = 0$ since the left R'-module M is semi-commutative.

(ii)Similar to the proof of (i).

(iii) It is given that R' is semi-commutative and so R' is semi-commutative as R'-module. Hence by (i) R' is semi-commutative as a left R-module. □

3.3.4 Proposition. *The class of semi-commutative R-modules is closed under (i) direct sums (ii) direct products and (iii) submodules.*

3.3.5 Proposition. *Let R be a ring and I a left ideal of R. If the cyclic left R-module R/I is semi-commutative, then I is an ideal of R.*

Proof. Let $a \in I, r \in R$. Then we have $a\bar{1} = a(1+I) = a + I = I = \bar{0}$ in R/I. Since R/I is semi-commutative then we have $ar\bar{1} = \bar{0} \Rightarrow ar \in I$. Hence I is ideal of R. □

3.3.6 Proposition. *The following conditions are equivalent for a ring R.*

(i) R is left invariant.

(ii) Every left R-module is semi-commutative.

(iii) Every cyclic left R-module is semi-commutative.

Proof. ((i)⇒(ii)) Let M be a left R-module. Suppose that $a \in R$ and $m \in M$ such that $am = 0$. Since Ra is two sided we have $aR \subseteq Ra$. Let $r \in R$ be any element, then $ar \in aR \subseteq Ra \Rightarrow ar \in Ra \Rightarrow arm \in Ram = 0 \Rightarrow arm = 0$, for all $r \in R$. Hence M is semi-commutative.

((ii)⇒(iii)) Trivial.

((iii)⇒(i)) This follows from Proposition 3.3.5. □

3.3.7 Proposition. *The following conditions are equivalent.*

(i) R is a semi-commutative ring.

(ii) Every torsionless R-module is semi-commutative.

(iii) Every submodule of a free R-module is semi-commutative.

(iv) There exists a faithful R-module which is semi-commutative .

Proof. ((i) ⇒(ii)) Let M be a torsionless R-module. Then M is a submodule of a direct product of copies of R. Now submodules of semi-commutative modules are semi-commutative,and by (i) direct products of copies of R are semi-commutative.

Hence M is semi-commutative.

((ii)⇒(iii)) Let W be a submodule of a free R-module M. Since M is a free R-module then M is a submodule of a direct product of copies of R. Therefore W is torsionless. Hence W is semi-commutative.

((iii)⇒(iv)) ${}_RR$ is a free R-module which is faithful. By (iii) ${}_RR$ is a faithful R-module which is semi-commutative.

((iv)⇒(i)) Let M be a faithful R-module. Consider the module homomorphism $\theta : R \longrightarrow \prod M$ defined by $(r)\theta = (rm)_{m \in M}$. Now $(r)\theta = 0 \Rightarrow rm = 0, \forall m \in M \Rightarrow r \in ann(M) = 0$. Therefore θ is one one. Hence R is a semi-commutative ring. □

3.3.8 Proposition. *A module M is semi-commutative if and only if every cyclic submodule of M is semi-commutative.*

Proof. (⇒) Since submodules of a semi-commutative module are semi-commutative this is trivial.

(⇐) Let $a \in R, m \in M$ be two elements which satisfy $am = 0$. Since $m \in M$ we have $m \in Rm$ which is cyclic. By the given condition we have $abm = 0, \forall b \in R$. Hence M is semi-commutative. □

The following result is Corollary 11.4 of Chapter 1 of [S].

3.3.9 Proposition. *Let M be a flat R-module and*

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

be an exact sequence with F free over R. For each finite family $u_1, u_2, \ldots\ldots, u_m$ of elements of K there exists an R-homomorphism $v : F \longrightarrow K$ such that

$(u_i)v = u_i$ *for each* i.

3.3.10 Proposition. *Flat modules over semi-commutative rings are semi-commutative .*

Proof. Let R be a semi-commutative ring and M a flat R-module. Let

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

be an exact sequence with F free over R. We denote $\bar{y} = y + K$ in M for any element $y \in F$.Let $m \in M$ and $a \in R$ such that $am = 0 \Rightarrow a\bar{x} = 0$ where $m = x + K$ for some $x \in F$. Now $a\bar{x} = 0 \Rightarrow ax \in K$. Then by Proposition 3.3.9 there exists homomorphism $v : F \longrightarrow K$ such that $(ax)v = ax$. Let $w = (x)v - x$ in F. Now $aw = a(x)v - ax = (ax)v - ax = 0$ in F. Since F is free, it is semi-commutative as an R-module.Hence we have $arw = 0, \forall r \in R \Rightarrow ar(x)v - arx = 0 \Rightarrow arx \in K$, for all $r \in R$. Therefore $ar\bar{x} = 0 \Rightarrow arm = 0$. Hence M is semi-commutative. □

3.3.11 Proposition. *Let R be a ring and S be a multiplicatively closed subset of $C(R)$. Let M be S-torsion free. Then the R-module M is semi-commutative if and only if the $S^{-1}R$-module $S^{-1}M$ is semi-commutative.*

Proof. ($\Rightarrow$) Let $a/s \in S^{-1}R, m/t \in S^{-1}M$ satisfy $(a/s).(m/t) = 0 \Rightarrow am/st = 0 \Rightarrow u(am) = 0$ for some $u \in S$. Since M is $S-$torsion free, we have $am = 0$. Simce M is semi-commutative, then $arm = 0, \forall r \in R$. Now $(a/s).(r/v).(m/t) = arm/svt = 0, \forall r/v \in S^{-1}R$. Hence $S^{-1}M$ is semi-commutative as an $S^{-1}R$-module.

($\Leftarrow$) Trivial. □

3.3.12 Proposition. *Let M be an R-module and $C(R)$ be the centre of R. Then the following conditions are equivalent.*

(i) M is semi-commutative.

(ii) $S^{-1}M$ is a semi-commutative $S^{-1}R$-module for each multiplicatively closed subset S of $C(R)$.

(iii) M_P is a semi-commutative R_P-module for each $P \in Spec(C(R))$.

(iv) M_Q is a semi-commutative R_Q-module for each $Q \in Max(C(R))$.

Proof. ((i)⇒(ii)) Let $m/s \in S^{-1}M, a/t \in S^{-1}R$ be two elements which satisfy $(a/t).(m/s) = 0$. Since $am/ts = 0$, then there exists $u \in S$ such that $uam = 0 \Rightarrow (ua)m = 0$. Since M is semi-commutative then $(ua)rm = 0$, for all $r \in R \Rightarrow (a/t).(r/1).(m/s) = 0/1$. Let r/v be any element $S^{-1}R$. Then $(a/t).(r/v).(m/s) = (1/v).(a/t).(r/1).(m/s) = 0$. Hence $S^{-1}M$ is semi-commutative.

((ii)⇒(iii)) and ((iii) ⇒ (iv)) are trivial.

((iv)⇒(i)) We write $C := C(R)$. Let $m \in M, a \in R$ be two elements and let $b \in R$ such that $am = 0$. Then $(a/1).(m/1) = 0$ in M_Q for all $Q \in Max(C)$. Since M_Q is semi-commutative, $(a/1).(b/s).(m/1) = 0$. Hence there exists $u_Q \in C - Q$ such that $u_Q(abm) = 0$. Let A be the ideal of C generated by the elements u_Q. Since $u_Q \in A$ we have $A \not\subseteq Q$, for all $Q \in Max(C)$. Hence $A = C$, necessarily. As $A(abm) = 0$, we have $abm = 0$. □

3.3.13 Proposition. *Let D be a domain and let M be a torsion free D-module. Then M is semi-commutative.*

Proof. Let $a \in D, m \in M$ be two elements which satisfy $am = 0$. Since M is a torsion free module we have $a = 0$ or $m = 0$. In both the cases we get

$abm = 0, \forall b \in R$. Hence M is semi-commutative. $\square$

3.3.14 Proposition. *Let R be a semi-commutative ring and let M be a free R-module. Then M is semi-commutative.*

Proof. Since M is free then we have $M \cong \oplus R$. As R is semi-commutative, direct sum of copies of R is also semi-commutative. Hence M is a semi-commutative R-module. $\square$

3.3.15 Remarks. (i) If M is a free R-module, M need not be semi-commutative. (Let us take $R = M_n(\mathbb{Z}), M = R$ for $n \geq 2$, then M is R-free but M is not semi-commutative.)

(ii) If R is reduced and M is a free R-module, then M is semi-commutative.

3.3.16 Proposition. *Let D be a (not necessarily commutative) integral domain and let M be a D-bimodule. The ring $D(+)M$ is a semi-commutative ring if and only if M is a semi-commutative D-module on both sides.*

Proof. ($\Rightarrow$) Let $a \in D, m \in M$ be two elements which satisfy $am = 0$. then we have $(a,0)(0,m) = (0,am) = (0,0) = 0$. Since $D(+)M$ is semi-commutative then for all $(b,n) \in D(+)M$ we have $(0,0) = (a,0)(b,n)(0,m) = (ab,an)(0,m) = (0,abm)$ which implies $abm = 0$ for all $b \in D$. Hence M is semi-commutative.

($\Leftarrow$) Let $(a,m),(b,n) \in D(+)M$ be two elements which satisfy $(a,m)(b,n) = (0,0) \Rightarrow (ab, am + mb) = (0,0)$. Now we get the system of equations

$$ab = 0 \tag{1}$$

$$an + mb = 0 \tag{2}$$

Since D is a domain, $ab = 0 \Rightarrow a = 0$ or $b = 0$

Case 1. If $a = 0$, then from (2) we get $mb = 0$. Since M is semi-commutative then $mcb = 0$, for all $c \in D$.

Therefore for any $(c, l) \in D(+)M$ we have

$(a, m)(c, l)(b, n) = (0, m)(c, l)(b, n) = (0, mc)(b, n) = (0, mcb) = (0, 0)$.

Case 2. If $b = 0$. As in Case 1 we can get $(a, m)(c, l)(b, n) = (0, 0)$. Thus we see that the ring $D(+)M$ is semi-commutative. □

3.3.17 Remark. If $R(+)M$ is a semi-commutative ring then the ring R as well as the modules $_RM$ and M_R are semi-commutative.

The following is the module-theoretic analogue of Proposition 2.3.3.

3.3.18 Proposition. **(a)** *If M is a reduced module then M is semi-commutative.*
(b) *If M is a regular, semi-commutative module then M is reduced.*

Proof. **(a)** is a consequence of Proposition 3.1.3.
(b) Suppose that $a^2m = 0$ for $a \in R$ and $m \in M$. Since M is regular there exists $f \in Hom_R(M, R)$ such that $arm = [(arm)f]arm$. As M is semi-commutative from $a^2m = 0$ we get

$$arm = a[(rm)f]arm = 0$$

Using Proposition 3.1.3 we deduce that M must be reduced. □

3.4 Armendariz modules

In this section, we extend the concept of Armendariz rings to modules. We begin with the following definition.

3.4.1 Definition. Let M be a module over a ring R. Let $M[x]$ be the corresponding polynomial module over $R[x]$. The module M is an *Armendariz* module if whenever polynomials $f(x) = \Sigma a_i x^i \in R[x], g(x) = \Sigma m_j x^j \in M[x]$ satisfy $f(x)g(x) = 0$ we have $a_i m_j = 0$ for every i and j.

3.4.2 Remark. A ring R is Armendariz if and only if the module ${}_RR$ is Armendariz.

3.4.3 Proposition. *Let M be an (R, R)-bimodule. Then $R(+)M$ is an Armendariz ring if and only if the following conditions are satisfied:*
(i) R is an Armendariz ring.
(ii)M is Armendariz as a left and a right R-module.
(iii) If $f(x)g(x) = 0$ in $R[x]$, then $f(x)M[x] \bigcap M[x]g(x) = 0$.

Proof. ($\Rightarrow$) we know that

$$R(+)M \cong \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M \right\}.$$

It is given that $R(+)M$ is Armendariz.

(i) Since $R \cong \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \mid r \in R \right\} \subseteq R(+)M$

and subrings of an Armendariz ring are Armendarizs. Therefore R is Armendariz.

(ii) Let $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{n} a_j x^j \in R[x]$ satisfies $f(x)m(x) = 0$. Now we have

$$\left\{ \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} x + \ldots\ldots + \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix} x^n \right\}$$

$$\left\{ \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & m_1 \\ 0 & 0 \end{pmatrix} x + \ldots\ldots + \begin{pmatrix} 0 & m_k \\ 0 & 0 \end{pmatrix} x^k \right\}$$

$$= \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} 0 & m(x) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f(x)m(x) \\ 0 & 0 \end{pmatrix} = 0.$$

Since $R(+)M$ is Armendariz we have

$$\begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 0 & m_j \\ 0 & 0 \end{pmatrix} = 0, \forall i, j \Rightarrow a_i m_j = 0.$$

Therefore M is Armendariz.

(iii) Since $R(+)M$ is Armendariz, then by above proposition $(R(+)M)[x] = R[x](+)M[x]$ is also Armendariz. Now let us assume that $f(x)g(x) = 0$. If $f(x)M[x] \bigcap M[x]g(x) \neq 0$. Then consider $f(x)m(x) = m'(x)g(x) \neq 0$ where $m(x), m'(x) \in M[x]$.

Now we have

$$\left\{\begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} - \begin{pmatrix} 0 & m'(x) \\ 0 & 0 \end{pmatrix} y\right\} . \left\{\begin{pmatrix} g(x) & 0 \\ 0 & g(x) \end{pmatrix} + \begin{pmatrix} 0 & m(x) \\ 0 & 0 \end{pmatrix} y\right\}$$

$= 0.$

But $\begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} 0 & m(x) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f(x)m(x) \\ 0 & 0 \end{pmatrix} \neq 0.$

which is a contradiction.

Hence $f(x)M[x] \bigcap M[x]g(x) = 0$.

($\Leftarrow$) Let $\alpha(x), \beta(x) \in (R(+)M)[x]$ satisfy $\alpha(x).\beta(x) = 0$. where

$$\alpha(x) = \begin{pmatrix} a_0 & m_0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & m_1 \\ 0 & a_1 \end{pmatrix} x + ... + \begin{pmatrix} a_n & m_n \\ 0 & a_n \end{pmatrix} x^n$$

and

$$\beta(x) = \begin{pmatrix} b_0 & l_0 \\ 0 & b_0 \end{pmatrix} + \begin{pmatrix} b_1 & l_1 \\ 0 & b_1 \end{pmatrix} x + ... + \begin{pmatrix} b_k & l_k \\ 0 & b_k \end{pmatrix} x^k.$$

Let

$f(x) = a_0 + a_1x + ... + a_nx^n, g(x) = b_0 + b_1x + ... + b_kx^k,$

$m(x) = m_0 + m_1x + ... + m_nx^n, l(x) = l_0 + l_1x + ... + l_kx^k.$

Then we have $f(x), g(x) \in R[x]$ and $m(x), l(x) \in M[x]$.

Now $\alpha(x).\beta(x) = 0$

$$\Rightarrow \begin{pmatrix} f(x) & m(x) \\ 0 & f(x) \end{pmatrix} . \begin{pmatrix} g(x) & l(x) \\ 0 & g(x) \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} f(x)g(x) & f(x)l(x)+m(x)g(x) \\ 0 & f(x)g(x) \end{pmatrix} = 0.$$

Therefore we get $f(x)g(x) = 0$ and $f(x)l(x) + m(x)g(x) = 0$. By (i) R is Armendariz. Hence we have $f(x)g(x) = 0 \Rightarrow a_ib_j = 0, \forall i, j$. Again by (iii)we have

$f(x)l(x) = -m(x)g(x) \in f(x)M[x] \bigcap M[x]g(x) = 0$

$\Rightarrow f(x)l(x) = m(x)g(x) = 0.$

By (ii) we get $a_il_j = 0 = m_ib_j, \forall i, j.$

Hence $\begin{pmatrix} a_i & m_i \\ 0 & a_i \end{pmatrix} \begin{pmatrix} b_j & l_j \\ 0 & b_j \end{pmatrix} = 0, \ \forall i, j.$

So $R(+)M$ is an Armendariz ring.

□

3.4.4 Corollary. *Let D be a (not necessarily commutative) domain and let M be an (D, D)-bimodule. Then $D(+)M$ is an Armendariz ring if and only if M is an Armendariz left and right D-module.*

3.4.5 Proposition. *Let $\theta : R \longrightarrow R'$ be a ring homomorphism and let M be an R'-module. Regard M as a left R-module via θ. Then we have :*
(i) if M is an Armendariz R'-module then M is an Armendariz R-module;
(ii) if θ is onto, then the converse of the statement in (i) holds;
(iii) if R' is an Armendariz ring, then R' is Armendariz as a left R-module.

Proof. Similar to the proof in semi-commutative case. □

3.4.6 Proposition. *The class of Armendariz R-modules is closed under (i) direct products (ii) submodules and (therefore) (iii) direct sums.*

3.4.7 Proposition. *The following conditions are equivalent.*

(i) R is an Armendariz ring.

(ii) Every torsionless R-module is Armendariz.

(iii) Every submodule of a free R-module is Armendariz.

(iv) There exists a faithful R-module which is Armendariz.

Proof. Similar to that of Proposition 3.3.7. □

3.4.8 Proposition. *A module M is Armendariz if and only if every finitely generated submodule of M is Armendariz.*

Proof. Similar to the proof in the semi-commutative case. □

3.4.9 Proposition. *Let D be a commutative domain and M a D-module. The module M is Armendariz if and only if its torsion submodule $T(M)$ is Armendariz.*

Proof. ($\Rightarrow$) This is trivial because submodules of an Armendariz module are Armendariz.

($\Leftarrow$) Let $f(x) = \sum_{i=0}^{k} a_i x^i \in D[x], g(x) = \sum_{j=0}^{n} m_j x^j \in M[x]$ wich satisfy

$$f(x)g(x) = 0 \qquad (*)$$

From (*) we get the system of equations

$$a_0 m_0 = 0 \qquad (1)$$

$$a_0m_1 + a_1m_0 = 0 \quad (2)$$

$$a_0m_2 + a_1m_1 + a_2m_0 = 0 \quad (3)$$

....

$$a_km_n = 0. \quad (k+n+1)$$

Let us assume that $a_0 \neq 0$. Since D is commutative then from (1) and (2) by multiplying a_0 we get $a_0^2m_1 = 0$.Thus a_0^2 annihilates both m_0 and m_1. Again from (3) we get $a_0^3m_2 = 0$. Proceeding in this way we get $g(x) = \sum_{j=0}^{n} m_jx^j \in T(M)[x]$. Since $T(M)$ is Armendariz as a D-module, we get $a_im_j = 0, \ \forall i, j$. Hence M is Armendariz. □

3.4.10 Remark. If M is a torsion free module over a commutative domain then M is Armendariz as a D-module.

3.4.11 Proposition. *Flat modules over Armendariz rings are Armendariz.*

Proof. The proof is similar to that in the case of semi-commutative modules. □

3.4.12 Proposition. *Let R be a ring and let S be a multiplicatively closed subset of $C(R)$. Let M be S-torsion free. Then the R-module M is Armendariz if and only if the $S^{-1}R$-module $S^{-1}M$ is Armendariz.*

Proof. The proof is similar to that in the case of semi-commutative modules. □

3.4.13 Proposition. *Let M be an R-module and let $C(R)$ be the centre of R. Then the following conditions are equivalent.*
(i) M is Armendariz.

(ii) $S^{-1}M$ *is Armendariz* $S^{-1}R$*-module for each multiplicatively closed subset* S *of* $C(R)$.

(iii) M_P *is an Armendariz* R_P*-module for each* $P \in Spec(C(R))$.

(iv) M_Q *is an Armendariz* R_Q*-module for each* $Q \in Max(C(R))$.

Proof. The proof is similar to that in the case of semi-commutative modules. □

3.4.14 Remark. It is a consequence of Proposition 3.1.9 that reduced modules are Armendariz.

Chapter 4

Some Questions

In this chapter, we include some questions which arise out of this survey. We have been unable to find their answers.

1. Consider the conditions

(1) ${}_RM$ is reduced.

(2)Whenever $m \in M$ and $a \in R$, the following hold:

(a) $am = 0$ implies $aRm = 0$.

(b)$a^2m = 0$ implies $am = 0$.

By Proposition 3.1.3, (1) and (2) are equivalent. If R is a commutative non-reduced ring then ${}_RR$ satisfies 2(a) but does not satisfy (1).

Question 1. Does 2(b) imply (1)?

2. Definition. Let M be a left R-module. An element $m \in M$ is $\star$ *-regular* if there exists $f \in Hom(M, R)$ such that $(mf)tm = tm, \forall t \in R$. A left R-module M is $\star$ *-regular* if every element of M is $\star$ -regular.

Proposition A. Let M be a left R-module. If M is semi-commutative and regular then M is $\star$ -regular.

Proof. Let $m \in M$. Since M is regular then there exists $f \in Hom_R(M, R)$ such that $(mf)m = m$.

Now $(mf)m - m = 0$ implies $(mf - 1)m = 0$. Again we have M is semi-commutative, therefore $(mf - 1)rm = 0, \forall r \in R$ i.e $(mf)rm = rm, \forall r \in R$. Hence M is $\star$ -regular. □

Question 2. We do not know whether the converse of Proposition A holds; equivalently, whether $\star$-regular modules are necessarily semi-commutative or not.

3. Definition. Let P be a property of rings. A ring R is *completely* P if R/I has property P for all ideals I of R.

For example, we call a ring R completely reduced if R/I is reduced for all ideals I of R.

Proposition B. Let R be a commutative ring. If R is a completely reduced, commutative ring then R is regular.

Proof. Let $a \in R$. Since R is commutative we can consider the factor ring R/Ra^2. Now we have $\bar{a}^2 = \bar{0}$ in R/Ra^2. Hence we have $\bar{a} = \bar{0} \Rightarrow a \in Ra^2 \Rightarrow a = ba^2$, for some $b \in R$. Since R is commutative we get $a = aba$, for some $b \in R$. Hence R is regular. □

Remark C. If R is commutative and regular then for every ideal I of R, R/I is commutative and regular. Therefore R is completely reduced. Thus the converse of Proposition B holds.

Now consider the following conditions for a ring R:

(i) R is a regular ring.

(ii) R is a completely regular ring.

(iii) R is an anti-regular ring.

(iv) R is a completely anti-regular ring.

(v) R is a reduced ring.

(vi) R is a completely reduced ring.

Note that (ii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (v) are clearly trivial. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) hold by Proposition 1.1.14 and 1.2.14.; hence (ii) $\Rightarrow$ (iv) clearly holds; (v) $\not\Rightarrow$ (vi) (consider $\mathbb{Z}$), (iii) $\not\Rightarrow$ (i) (see Example 1.2.15) , (iii) $\not\Rightarrow$ (iv)(see Example 1.2.18).

Against this background we ask:

Question 3. If R is completely anti-regular then is R necessarily regular? i.e does the converse of (i) $\Rightarrow$ (iv) hold ?

In this context we note that in the commutative case, Question 3 has an affirmative answer (Proof : R commutative and completely anti-regular implies R is commutative and completely reduced (by Corollary 1.2.13) which implies R is regular by Proposition B.)

4. Consider the short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0 \qquad (*)$$

where M,M_1 and M_2 are R-modules. If (*) is split then M is regular (resp. anti-regular) if and only if both M_1 and M_2 are regular(resp.anti-regular), since $M \cong M_1 \oplus M_2$.

Question 4. Is it always the case that when M_1, M_2 are both regular(resp. anti-regular) the module M is regular(resp. anti-regular)?

5. Definition A ring R is *ps-Armendariz* if it satisfies the following condition: Suppose that for $f(x), g(x) \in R[[x]]$ we have $f(x).g(x) = 0$. Then for each coefficient a_i of $f(x)$ and b_j of $g(x)$ we have $a_ib_j = 0$.

Proposition D. If a left R-module M is ps-Armendariz then M is a semi-commutative module.

Proof. Let us consider $a \in R$ and $m \in M$ such that $am = 0$. Now for any $r \in R$ we get

$$(a - arx)(m + rmx + r^2mx^2 + r^3mx^3 + \ldots\ldots\ldots\ldots\ldots\ldots) = 0$$

Since M is ps-Armendariz we have $arm = 0$. Therefore M is semi-commutative. □

Remark E. If M is ps-Armendariz and regular then M is semi-commutative (by Proposition D) and regular. Hence M is reduced.

In the context of Propositions 2.6.1, 3.1.9 and 3.3.18 and Remark E we have the following question.

Question 5. If M is a regular and Armendariz module is M necessarily reduced?

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(iv) First position in debating comp-01, conducted by SMSU Shillong.

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(vi) Third position in State Level debating comp –96.

(vii) Participated in NBHM sponsored 'Mathematical Olympiad Camp.' organized by deptt. of Math. NEHU.