SOME PROBLEMS IN RING THEORY

SOME PROBLEMS IN POLYNOMIAL RINGS AND MONOID AND MODULE ANALOGUES OF RING THEORETIC RESULTS

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CERTIFICATE

I certify that the dissertation entitled "SOME PROBLEMS IN RING THEORY: SOME PROBLEMS IN POLYNOMIAL RINGS AND MONOID AND MODULE ANALOGUES OF RING THEORETIC RESULTS" submitted by Ms. Sima Chhawchharia in fulfilment of the requirements for the degree of Doctor of Philosophy is the outcome of a study undertaken by the candidate. I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities.

I certify that this dissertation is worthy of consideration by the examiners.

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S Chhawchharia
Sima Chhawchharia

Chapter 0.

Preliminaries

Chapter 1.

Some properties of semigroups and rings

Chapter 2.

Some results on polynomial rings and power series rings

Chapter 3.

A property of rings of functions.

INTRODUCTION

This dissertation is devoted to a study of certain topics in ring theory and in the theory of semigroups. The topics of interest to us are regular rings and semigroups, polynomial rings and rings of continuous real-valued functions on a topological space. Since these topics have very little in common, studies conducted in these areas have been placed in different chapters, namely, Chapters 1, 2 and 3 respectively.

The thesis consists of four chapters, i.e., Chapters 0, 1, 2 and 3. Chapter 0 is aimed at introducing terms and basic results which are of use in the thesis later. These have been placed in section 1 of this chapter. The remaining sections of Chapter 0 contain i)extensions of definitions in the context of rings to those in semigroups and ii) either easy extensions of results in rings to semigroups, or counter-examples to assertions which are analogues of known results in rings. With this approach in mind we study ideals (section 2), localisation (section 3) and von Neumann regularity (section 4) in rings and monoids.

The semigroup system has been studied in great detail in different contexts. Our interest in this system is in a study of the generalisation of some results in ring theory to those in semigroups and monoids. This approach has been taken in Chapter 0 and continued in Chapter 1. The paucity of structure in this system allows easy construction of examples which show that a number of results in rings have no analogues in monoids. In fact in many cases where the proof of a particular result in ring theory requires more than one binary operation, an example can be constructed to show that an analogue of the result fails in monoids. Proofs of results which hold in monoids have been recorded in some places for the sake of completeness.

Two examples have played a key role. These are the *bicyclic monoid* M_1 (Example 1.1.5) and the *left zero semigroup* S_l (Example 1.1.6). The (left-right) dual examples to them are (of course) counter-examples to the dual assertions.

It is well known that in the context of rings, the property of being without zero-divisors (or being without non-zero nilpotent elements), invariance, semi-commutativity and normality each imply direct finiteness.(See Chapter 1 for definitions of unfamiliar terms.) This is not so in the context of monoids. (M_1 is a monoid without zero-divisors which is not directly finite.)

Several other results connecting the above properties were obtained in the course of study of the generalisations of the properties mentioned above. For the sake of clarity, these results have been isolated and placed in sections 2 and 3 of Chapter 1. Section 3 also contains a few results on semi-centrality, and an example (Example 1.3.3D) to show the contrast between the situation in rings and in monoids. The questions of stability (under localisation) of the above properties of rings and monoids have been considered in section 4.

By combining results from [St], [G3], [R:74] and others, (an essentially) known theorem giving several equivalent conditions for the strong regularity of a ring R has been recorded (Theorem 1.5.2A). Many of the implications between the various conditions of this theorem do not have analogues in monoids. The strength of Example 1.1.7 (and its dual) has been exhibited in this section;it helps to settle many of the implications (see Chart 1.5.3B). In the final section we study the extendibility of a result of Savage on reflexive inverses in rings to monoids.Some results on unit-regular monoids are also recorded here.

Chapter 2 is devoted almost entirely to a study of certain conditions involving zero-divisors in polynomial rings. We define two new terms: "Armendariz rings" and "McCoy rings". The origin of the first term is Lemma 1 of [A:74] : Let R be a reduced ring and $f, g \in R[x]$ with $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j$. Then $f(x)g(x) = 0$ if and only if $a_i b_j = 0$ for all $0 \leq i \leq m, 0 \leq j \leq n$. While Armendariz proved his result for reduced rings, it was noticed that the condition $f(x)g(x) = 0 \Rightarrow a_i b_j = 0$ for each i and each j is satisfied in polynomial rings over many interesting classes of non-reduced rings. Rings which satisfy the above condition have been given the name 'Armendariz rings'.

The main results of this chapter are the following: If R is a commutative P.I.D and A is an ideal of R, then R/A is Armendariz (Theorem 2.2.2B); if R is a domain and A is an ideal of R such that R/A is Armendariz, then the ring $R(+)(R/A)$ is Armendariz (Theorem 2.2.3); if K is a field, and V a vector space over K, then the ring $K(+)V$ is Armendariz . That commutative rings need not be Armendariz has been demonstrated by Example 2.3.2.

Analogous to the definition of Armendariz rings we have the definition of McCoy rings. Commutative rings are McCoy rings; thus we have examples to show that McCoy rings need not be Armendariz. In this context relations between these rings have been studied and a few results on McCoy rings have been recorded in section 4 of this chapter.

In the final section the question of the stability of some of the classes being studied under the formation of the ring of polynomials has been considered. Relations between *-commutative, semi-commutative, normal and Armendariz rings have also been studied in this section. A few questions have been recorded and relevant results stated.

The final chapter is devoted to a study of rings of continuous functions aimed at answering the question of whether the total quotient ring of a commutative anti-regular ring is necessary regular. The basic definitions and results concerning anti-regular monoids and rings and their localisations are assembled in section 1 of this chapter. Results concerning the regularity of the ring $C(X)$ of continuous real-valued functions defined on a topological space X are well-known. In section 2 a necessary and sufficient condition for the anti-regularity of $C(X)$ has been given. This result is applied in section 3 to give an example which answers the above question in the negative.

References. Semigroups, monoids and other binary systems have been studied in [Br], [CP] and [P]. Bibliographies of these books contain references to a large number of original memoirs in the field. The journal 'Semigroup Forum' has been publishing papers in semigroups since 1970.

Because of various constraints like inaccessibility of references it has not been possible to carry out a thorough survey of the literature in the topics studied by us. It is possible that some results obtained by us are known to scholars working in the field. However in view of the fact that there is an interplay of semigroup theory, commutative algebra and non-commutative ring theory in our work, we have tried to make the exposition self-contained by giving necessary details.

CHAPTER 0

PRELIMINARIES

Chapter 0, section 0.

Introduction

This Chapter is aimed at defining terms and basic results which are of use in the later chapters of the thesis. These definitions and basic results have been placed in section 1. The following sections have recorded some extensions of definitions in the context of rings to those in semigroups and easy extensions of results (or counter-examples as the case may be) in rings to the theory of semigroups.

In section 2 we make an attempt to study ideals in semigroups and rings. Generalisations of a few ring theoretic results in the context of singular monoids have also been recorded.

The basic definitions and results in the context of localisation have been recorded in section 3. Section 4 contains the basic properties of regular monoids and rings.

Chapter 0, section 1.

Basic definitions and remarks.

In this section we define certain terms, namely nilpotent, idempotent and central elements, and some classes of semigroups and rings (namely, reduced and normal semigroups and rings). We also fix some related notation and record some results needed later.

0.1.1A. *Definitions.* A *semigroup* is a system consisting of a set S and an associative binary operation defined on the set. If S contains an element 0 satisfying the condition

$$x.0 = 0 = 0.x \qquad \forall x \in S$$

then S is called a *semigroup with zero*; such a "zero"element is (clearly) necessarily unique when it exists. An element x of a semigroup with zero is *nilpotent* if $x^n = 0$ for some positive integer n. A semigroup with zero is *reduced* if 0 is its only nilpotent element. An element e of a semigroup S is an *idempotent* element if $e^2 = e$.

0.1.1B. *Notation.* The set of all idempotent elements of a semigroup S is denoted by $I(S)$.

0.1.1C. *Convention.* Whenever terms in which the existence of 0 is necessary (for example reduced, semi-commutative) are mentioned it is assumed that the underlying semigroup is a semigroup "with zero".

0.1.1D. *Remark.* A semigroup is reduced if and only if the only element of the semigroup whose square is zero is the zero element.

0.1.1E. *Definition.* By the *centre* $C(S)$ of a semigroup S we mean the set

$$\{x \in S : xa = ax \qquad \text{for each element } a \in S\}.$$

Clearly, $C(S)$ is a subsemigroup of S.

0.1.1F. *Definition.* A semigroup S is *normal* if $I(S) \subseteq C(S)$.

0.1.2. *Definitions and Conventions.*

0.1.2A. *Definition.* A *monoid* is a semigroup with identity.

0.1.2B. *Definitions.* A monoid M is *reduced* (respectively, *normal*) if it is reduced (respectively, normal) as a semigroup.

0.1.2C. *Convention.* A convention similar to that adopted on 0.1.2B is followed throughout for other terms applicable to semigroups.

0.1.2D. *Notation.* The letter S generally denotes a semigroup and the letter M a monoid.

0.1.3. *Rings.* By a *ring* we mean an associative ring with identity. If the multiplicative monoid of the ring is reduced (respectively, normal) the ring is called a reduced (respectively, normal) ring; similar convention is followed for other terms applicable to monoids. (We remark that some authors use the term "abelian ring" for normal ring; see, e.g., Chapter 3 of [G3].)

0.1.4. *Definitions and Remarks.* Let S be a semigroup. and $A \subseteq S$; the set A is *a left ideal(S)* if $x \in S$ and $a \in A$ implies that $xa \in A$. *Right ideals(S)* are defined dually. Further *principal* left (right) ideals(S) are defined as usual. When there is no possibility of confusion, we shall drop S and simply talk about left/right ideals in semigroups. An *ideal(S)* of a semigroup S is a subset of S which is both a left as well as a right ideal.

Let M be a monoid, a an element of M. Then $Ma = \{xa : x \in M\}$ contains the element $a(= 1.a)$. and is a principal left ideal(S) of the monoid M .

When required, the notation *ideal(R)* is used for the familiar concept of an ideal of a ring.

0.1.5. *Bibliographical note.* The terms 'ring ideal' and 'semigroup ideal' have been used in section 8 of [K:63] for ideal(R) and ideal(S) respectively.

0.1.6. *Definition.* Let S be a semigroup.Consider $S^0 := S \cup \{0\}$ where the element 0 satisfies the relation $x.0 = 0 = 0.x$ for every $x \in S^0$. The system S^0, *the augmented semigroup of* S, is a semigroup with zero which is reduced in the sense of 0.1.1A. Consider the system $S^1 := S \cup \{1\}$ where the element 1 satisfies the relation $1.1 = 1$ and $1.x = x = x.1$ for every element x of the semigroup S. The system S^1, *the adjuncted semigroup of* S is a directly finite (in the sense of 1.1.1 below) monoid. (For convenience of notation, we may use notations like 0' and 1' in place of 0 and 1.)

0.1.7. *Annihilators.*

Definition. Let J be a subset of a semigroup S with zero.

By its *left annihilator* l_S(J) (or $l(J)$, if S is clear from the context) we mean the set

$$\{x \in S : xa = 0 \quad \text{for each } a \in J\ \};$$

l_S(J) is a left ideal of S. Dually we have *right annihilators.* We write $l(a)$ for $l(\{a\})$ and $r(a)$ for $r(\{a\})$. If $l(a) = r(a)$ (which happens in commutative as well as reduced semigroups -see 1.1.3A below) we shall use the notation $ann(a)$ for $l(a) = r(a)$; similar meaning is to be attached to $ann(J)$.

0.1.7B. *Remarks.* Clearly the above definitions make sense in rings. Further a number of arguments involving ring structure use the following facts:

(i) If $e \in I(R)$ then $1 - e \in I(R)$.

(ii) If $e \in I(R)$ then $l(e) = R(1 - e)$ and $r(e) = (1 - e)R$.

As a consequence of (ii) we get :

(iii) If $e \in I(R), e \neq 1$ then $l(e) \neq 0$.

While (i) and (ii) have clearly no analogues in monoids, an analogue of (iii) fails in monoids as can be seen by the following example.

0.1.7C. *Example.* If M is a totally ordered set with smallest element 0 and greatest element 1, then M is a commutative monoid under the definition $a \star b = min(a, b)$. The smallest element is the zero of this monoid. Clearly each $a \in M$ is an idempotent and for $a \in M, a \neq 0, ann(a) = \{0\}$.

0.1.7D. *Remark.* In view of Example 0.1.7C one has to exercise caution while extending to monoids results valid in rings which involve idempotents or annihilators. Examples of this will be found in this chapter and Chapters 1 and 3 of this thesis.

0.1.8. *Conventions.* Some results of Chapter 1 are numbered as X.Y.Z(R/m) or X.Y.Z(r/M) or X.Y.Z(r/∼m). We explain the conventions below.

(I) A result numbered X.Y.Z(R/m) indicates that *(i)* the result has been stated for rings and *(ii)* a monoid analogue of the result is valid. (The monoid analogue may or may not have been stated in an adjacent place. Proof(s) may or may not have been given.)

(II) A result numbered X.Y.Z(r/M) indicates that the result has been stated for monoids. A ring analogue of this result is automatically valid. (This analogue may or may not have been stated explicitly.)

(III) A result numbered X.Y.Z(r/~m) indicates that the statement holds only in rings and a counter-example shows that it is not valid in monoids . The paragraph number in which the example is given is also recorded .

0.1.9. *Remarks.* (1). All our left-sided concepts and results have right-sided counterparts. They have not been recorded explicitly.

If S is a semigroup (monoid) which serves as a counter-example for a certain assertion, the opposite semigroup of S, denoted by S^{op} (where the underlying set is the same as that of S , and the multiplication is given by $a \star b = ba$) is a counter-example for the dual assertion.

(2). While an effort has been made to avoid duplication of symbols, we have not been totally successful in this. For example the letter M has been used to denote a monoid in chapters 0 and 1 and has been used to denote an R-module in Chapter 2. Similarly the letter W has been used differently in various contexts. It is hoped that these usages will not create confusion.

Chapter 0, section 2.

Ideals in semigroups and rings.

In this section we recall some very basic results concerning ideals, maximal ideals and prime ideals. Concepts like "prime ideal", "semiprime ideal" are extended to monoids.

0.2.1. *Remark.* Note that a left ideal of a ring R is necessarily a left ideal(S) of the multiplicative semigroup of R. The converse is false; consider for example, the subset $2\mathbf{Z} \cup 3\mathbf{Z}$ of the ring $\mathbf{Z}$. This example also shows that an ideal of a commutative ring R which is maximal as an ideal(R) need not be maximal as an ideal(S).

0.2.2. *Remark.* Non-left invertible elements form the unique maximal left ideal(S) (which is also the largest among left ideals(S)) in any monoid M.

0.2.3. *Definitions.* A monoid M is a *monoid without (proper) divisors of zero* if whenever a, b are non-zero elements in M, then their product ab is also non-zero. Monoids augmented with zero are always without divisors of zero. A ring whose multiplicative monoid is without divisors of zero is a *domain*; thus domains need not be commutative.

In the next two paragraphs we shall adapt various standard definitions used in rings (see [M:71]) to monoids. McCoy has extended the work of Andrunakievic and Rjabuhin (see reference [1] in [M:71]). A more accessible reference is Chapter 4 of [M]; see also Part II of [P].

0.2.4. *Definitions.* *(i)* An ideal A of a monoid M is a *prime ideal* if it has the property that whenever ideals J and K satisfy the condition that $JK \subseteq A$, either $J \subseteq A$ or $K \subseteq A$.

(ii) A monoid M is a *prime monoid* if whenever J, K are non-zero ideals of M, then JK is also a non-zero ideal of M. (This is equivalent to the condition that 0 is a prime ideal in M.)

(iii) An ideal A of a monoid M is a *semiprime ideal* if whenever J is an ideal in M and n is a positive integer, $J^n \subseteq A \Rightarrow J \subseteq A$.

(iv) A monoid M is a *semiprime* monoid if there are no non-zero nilpotent ideals in M. (This is equivalent to the condition that 0 is a semiprime ideal in M).

(v) An ideal A of a monoid M is *completely prime* if for elements a, b of M, $ab \in A$ implies that either $a \in A$ or $b \in A$.

(vi) An ideal A of a monoid M is *completely semiprime* if whenever $a \in M$ and n is a positive integer, $a^n \in A$, implies that $a \in A$.

0.2.5. *Definitions.* *(i)* A subset W of a monoid M is an *m-system* if whenever $c, d \in W$, there exists an element x of M such that $cxd \in W$.

(ii) A subset W of a monoid M is a *multiplicative system* if W is closed under multiplication.

If the multiplicative monoid of a ring is a prime monoid, then the ring must be a prime ring, for the elements of the ideal(S) generate the ideal(R). The converse also holds: i.e., if R is a prime ring then the multiplicative monoid of R is a prime monoid. Similar results hold for semiprime, completely prime and completely semiprime ideals in monoids.

We record a few obvious remarks below.

0.2.6. *Remark.* If P is a completely prime ideal in the sense of the above definition, then $M \backslash P$ is a multiplicative system and thus P is a prime ideal *in the sense of* Grimble ; *see p.40 and p.71 of* [CP].)

0.2.7. *Remark.* A monoid M is without divisors of zero if and only if 0 is a completely prime ideal in M.

0.2.8. *Remark.* A monoid M is reduced if and only if 0 is a completely semiprime ideal in M.

Some results on (non-commutative) reduced rings which have been proved by Renault [R:67] and the basic results concerning singular ideals of rings have straightforward generalisations to monoids. These results have been stated along with necessary definitions below. (For the vast and interesting theory on non-singular modules and rings we refer to [G1] and [G2] and the references given there.)

0.2.9. ***Proposition.*** Prime, reduced monoids are without divisors of zero. (see Proposition 2.1 in [R:67]; see also Lemma 1 and remarks following it in [M:71]; some related results are recorded in 1.1.3 below.)

0.2.10A. *Definitions.* A left ideal J of a monoid M is *essential in* M ($J \lhd M$) if its intersection with every non-zero left ideal of M is non-zero.

The monoid M is a *left non-singular monoid* if

$$Z_l(M) = \{a \in M : l(a) \lhd M\} = 0.$$

The monoid M is a *right non-singular monoid* if

$$Z_r(M) = \{a \in M : r(a) \lhd M\} = 0.$$

0.2.10B. *Remark.* $Z_l(M)$ and $Z_r(M)$ are ideals of M.

0.2.11A. *Proposition.* Reduced monoids are left and right non-singular.

0.2.11B. *Proposition.* Commutative non-singular monoids are reduced.

0.2.11C.*Examples.* Rings which are non-singular on the left but not on the right have been considered by Small [S:65], Goodearl [G1] and others. Here we remark that the monoid M of cardinality 4 considered in paragraph 1.1.4 also has the same property.

Some results recorded above are used in Section 1 of Chapter 3 for proving the non-singularity of (regular and) anti-regular monoids.

Chapter 0, section 3.

Localization in monoids and rings.

The concepts of rings and modules of fractions (also called localisations of rings and modules) have turned out to be of great interest in commutative algebra and algebraic geometry. As pointed out by Cohn [C:71] the definitions can be given in the setting of semigroups. In this section we record some basic concepts and results in the area of central localisations in monoids.

0.3.1. *Definition.* Let M be a monoid, T a central (i.e., such that $T \subset C(M)$) multiplicatively closed subset of M (containing identity, the empty product).

Consider $M \times T$. We say that $(m_1, t_1) \sim (m_2, t_2)$ holds if there exists an element $t \in T$ such that $tt_2m_1 = tt_1m_2$. It can be easily checked $\sim$ is an equivalence relation. On $T^{-1}M = M \times T/\sim$ we have the structure of a monoid defined by $(x/t).(y/s) = xy/ts$. Again it can be checked that the operation defined above is well-defined and associative. Any element of the type s/s is the identity element.

We now have the monoid homomorphism $\theta : M \longrightarrow T^{-1}M$ defined by $\theta(m) = m/1$. This map is one-one if the set T is contained in the set of non-zero-divisors of M.

(See also Theorem 1.2 of [C:71].)

0.3.2. *Remark.* Let A be a left ideal of M. Then $T^{-1}A$ is a left ideal of $T^{-1}M$. Conversely, if J is a left ideal of $T^{-1}M$ then J is of the type $T^{-1}I$ where I is a left ideal of M.

0.3.3. *Notation and remarks.*

*(i)*Let P be a prime ideal of a commutative monoid C. Then the set T $= C\backslash P$ is multiplicatively closed. If M is a monoid and $T = C(M)\backslash P$ we write M_P for $T^{-1}M$.

*(ii)*Notice that μ is a maximal ideal of a commutative monoid C, if and only if $C\backslash\mu$ consists of the units of C: Hence C necessarily has a unique maximal ideal. Thus the maximal spectrum $Max(C)$ (=the set of all maximal ideals) of C is a singleton set; however, we may use this notation to emphasise the analogy with rings.

(iii) If μ is the unique maximal ideal of a commutative monoid C, then μ must be a prime ideal, since the product of any two non-invertible elements is again non-invertible. Thus $Max(C) \subseteq Spec(C)$ the set of all prime ideals of C.

(iv) Let μ be the unique maximal ideal of the center $C(M)$ of a monoid M. Since $C(M)\backslash\mu$ consists of the units of $C(M)$ we have $M_\mu := (C(M)\backslash\mu)^{-1}M$ $= M$.

Chapter 0, section 4

Basic properties of regular monoids and rings.

Regular rings defined by von Neumann have been extensively studied by algebraists (see [G3]). Our main interest in Chapter 1 is going to be in the study of extendibility of various properties of a certain class of regular rings, namely, the so-called strongly regular rings, to monoids. Some preliminary results required by us for this study and also for applications in other chapters are collected here. We also show by an example (0.4.7B) that the invertibilty of non-zero-divisors in regular rings does not extend to regular monoids.

0.4.1. *Definitions.* Let M be a monoid. An element a of M is *regular* (in the sense of von Neumann) if there exists an element b in M satisfying the relation $a = aba$. The monoid M is *regular* if every element of M is regular.

0.4.2. *Remarks.* *(i)* All idempotent elements of a semigroup are regular. *(ii)* All left and right invertible elements of a monoid are regular.

0.4.3. *Proposition.* A regular non-zero-divisor in a ring is invertible.

0.4.4A. *Corollary 1.* In a regular ring every non-zero-divisor is invertible.

0.4.4B. *Corollary 2.* In a domain, every non-zero regular element is invertible.

0.4.4C. *Corollary 3.* A non-zero regular domain is a division ring.

Indeed 0.4.4A and 0.4.4C are analogues, respectively, of the elementary results 0.4.5A and 0.4.5B.

0.4.5A. *Proposition.* In a finite ring every non-zero-divisor is invertible.

0.4.5B. *Corollary.* A non-zero finite domain is a division ring (and therefore a field by a theorem of Wedderburn).

Following the usual conventions a prime ideal P of a commutative ring R will be different from R.

0.4.6A. *Proposition.* If R is a commutative regular ring and P a prime ideal of R, then P is maximal.

0.4.6B. *Proposition.* If R is a commutative ring of finite cardinality and P is a prime ideal of R, then P is maximal.

(Proposition 0.4.6A is a consequence of 0.4.4C and 0.4.6B of 0.4.5B.)
0.4.6C. *Bibliographical note* A study of rings satisfying the condition "every non-zero-divisor is invertible" addressed to the non-specialist has been carried out in [L:89].

Results 0.4.3 - 0.4.6. are not valid in monoids as can be seen from the example in 0.4.7B.

0.4.7A. *Remark.* If S is a regular semigroup then the semigroup S^0 and the monoid S^1 are also regular. (This principle is used to construct some examples which help exhibit the failure of the possibility of the extension of some results from rings to semigroups/monoids. Such an application is given in 0.4.7B.)

0.4.7B. *Example.* For a prime p let $M = \{\mathbb{Z}/p\mathbb{Z}\} \cup \{0'\}$ where the multiplication in the field {Z/pZ} is extended to M as follows.

$$0'.x = 0' = x.0' \qquad \text{for each element } x \in M$$

(Thus M is the augmented semigroup of Z/pZ in the sense of definition 0.1.6.)

If $x.y = 0'$ for some $x, y \in M$ it follows that either $x = 0'$ or $y = 0'$. The monoid M has p+1 elements.

(i) The monoid M is finite, commutative, regular and without zero-divisors in which the element 0 is a regular non-zero-divisor (indeed 0 is different from the zero element 0') . However 0 is not a unit in M. (Thus the natural analogues of 0.4.3, 0.4.4(A,B and C) and 0.4.5(A and B) fail in monoids.)

(ii) In this monoid the ideal {0'} is a prime ideal which is properly contained in the set $\{0\} \cup \{0'\}$ of non-units; Therefore {0'} is a prime ideal which is not maximal. (Thus the natural analogues of 0.4.6(A and B) fail in monoids.)

0.4.7C.*Krull Dimension.* The study of the dimension of a commutative ring (later called as the Krull dimension) was begun by Krull[K:38]; see also [Kr],[Ku] and [Se].This dimension is defined as the supremum of the lengths of all prime ideal chains. Thus the Krull dimension of a finite (or regular) commutative ring is zero, by 0.4.6. (See Lemma 5.6 of [St:68] for a result characterising commutative,zero-dimensional rings.)

The augmentation process which led to Example 0.4.7B can be indefinitely continued. Thus,a succession of " zeroes", $0_1, 0_2, ...0_n$ can be attached to {Z/pZ} to get a commutative, regular monoid of $p+n$ elements. In this monoid the chain of prime ideals

$$p_0 := \{0_n\} \subset p_1 := \{0_n, 0_{n-1}\} \ldots \subset p_n := \{0_n, 0_{n-1}, \ldots, 0_1, 0\}$$

has length n. Again this is different from the situation in commutative rings.

In the rest of the section we shall record a few results concerning regularity in monoids which are straightforward extensions of known results in rings.The proofs are sketched for the sake of completeness.

0.4.8. *Proposition.* Let a be an element of a monoid M. Then the following conditions are equivalent.

(i) The element a is a regular element.

(ii) The (principal left) ideal Ma is generated by an idempotent.

Proof. $(i) \Rightarrow (ii)$. Assume that $b \in M$ satisfies $aba = a$; then $ba \in I(M)$. Further $Ma = Maba \subseteq Mba \subseteq Ma$. Thus $Ma = Mba$.

$(ii) \Rightarrow (i)$. Let $Ma = Me$ where $e^2 = e$. Then $e = ba$ and $a = xe$ for some $b, x \in M$. We therefore have

$$aba = ae = xe^2 = xe = a$$

0.4.9. *Corollary.* Regular monoids are semiprime

Proof. Let $a \neq 0$ be an element of a regular monoid M. Now $Ma = Me$ for some non-zero idempotent e. Hence $e = e^n \in (Ma)^n$. Thus Ma is not nilpotent. Consider next a non-zero ideal A; then $A \supseteq Ma$ for some non-zero element $a \in M$. Thus the ideal A is not nilpotent.

Central localisations of regular rings and related rings have been studied in [AFS:74].

In the rest of this section we record analogues of results which holds in rings as well as modules.Indeed because of 0.3.3 some implications are trivially valid in monoids.

The letter T denotes (as in section 3) a subset of the centre $C(M)$ of a monoid M.

0.4.10. *Proposition.* Let a be an element of a monoid M. Then the folowing conditions are equivalent.

(i) The element a is regular in M.

(ii) For each T and for each $t \in T$ the element a/t is regular in $T^{-1}M$.

(iii) For each $\pi \in \mathrm{Spec}(C(M))$ the element $a/1$ is regular in M_π.

(iv) For each μ in $Max(C(M))$ the element $a/1$ is regular in M_μ.

(Indeed, $Max(C(M))$ is a singleton set.This condition has been formulated to emphasize the analogy with modules.)

Proof. $(i) \Rightarrow (ii)$. The equation $aba = a$ yields

$$(a/t)(tb/1)(a/t) = a/t.$$

The proofs of $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are trivial. For $(iv) \Rightarrow (i)$, use 0.3.3(iv).

0.4.11. *Corollary.* The following conditions are equivalent for a monoid M.

(i) M is a regular monoid.

(ii) For each T the monoid $T^{-1}M$ is regular.

(iii) For each $\pi \in \mathrm{Spec}\,(C(M))$, M_π is regular.

(iv) For each $\mu \in \mathrm{Max}(C(M)), M_\mu$ is regular.

CHAPTER 1

SOME PROPERTIES OF SEMIGROUPS AND RINGS

Chapter 1, section 0.

Introduction.

Semigroups are systems endowed with an associative binary operation which preserve the closure property and the associative law in the set. Many French authors use the term 'demigroup' for this system, while the Russian authors generally use the term 'associative system'. While we define a monoid to be a semigroup in which the existence of identity is assumed evident, Bourbaki has assumed the existence ofan identity in semigroups. The terminology used by us has become standard in English and German.

The above system seems to have been studied in great detail, in different contexts sometimes yielding interesting results whereas at other times no interesting results have been obtained. For example, the homomorphism theory of semigroups has been studied in great detail without any interesting results, while considerable use of the theory of semigroups has been found in the study of analysis and topology. An encyclopaedic reference to these applications is 'Functional Analysis and Semigroups'by Hille.)

The concept of rings in the study of algebra needs no introduction. The ring structure involves two binary operations which making it richer than the semigroup system. In this chapter of the dissertation, an attempt has been made to study generalisations of ring theoretic results to semigroups and monoids. (Similar questions concerning extensions of results from (regular) rings to (regular) monoids have been studied by a number of authors; see, e.g., Remark following Lemma 3.2 in [HM:89] .) An interesting point to note in this context is that, almost in all cases, when the proof of a particular result in ring theory requires both the binary operations, counter examples are available to show that the result does not extend to semigroups. An obvious example is Kaplansky's theorem for rings which goes as follows: If an element 'a' of a ring R has more than one right/left inverse, it must have infinitely many right/left inverses. (For a proof see Section 1, Exercise 3 in [AF].) This result does not hold in monoids as can be seen from the following example: Consider the set of all maps from the set of natural numbers to itself. This set is a monoid with respect to the composition of maps as the binary operation. Consider the element f_n of this monoid defined by $f_n(m) = 1$ for $m \leq n$; $f_n(m) = m - (n - 1)$ for $m \geq n + 1$; it can be seen that f_n has exactly n right inverses: let $h : \mathrm{N}\backslash\{1\} \longrightarrow \mathrm{N}$ be the map $h(m) = m+n-1$; for each i satisfying $1 \leq i \leq n$ consider the map g_i defined as follows: $g_i(1) = i$ and $g_i = h$ on the remaining elements of N. The set of right inverses of f_n is precisely the set $\{g_i\}_{1 \leq i \leq n}$.

Consider the following properties of rings. Invariant, reduced, normal. It is known that each of these properties implies direct finiteness. Moreover, for regular rings, these properties coincide. It is natural to ask if similar results are valid in monoids (regular monoids). In the course of doing so, many independent results connecting the above mentioned properties were obtained. For the sake of clarity these results have been isolated and placed in separate sections, namely, sections 2 and 3.

Some equivalent conditions for a ring to be strongly regular may be stated in the form of a theorem as follows (of course 'left' can be changed to 'right' in conditions (i) and (ii) below):

Theorem. The following conditions are equivalent for a ring R

(i) Every element of R is left regular.

(ii) R is regular and left invariant.

(iii) R is regular and reduced.

(iv) R is regular and semi-commutative.

(v) R is regular and normal.

As will be seen in section 5 of this chapter, we do not have similar equivalences in monoids. While some implications hold between monoid analogues of these conditions, examples can be given to show that many other other assertions do not hold.

Chapter 1, section 1.

Definitions and examples

In this section we record the basic definitions, results and examples that are needed in the rest of the chapter.

1.1.1A. *Definition.* A monoid M is *directly finite* if whenever elements a, b of M satisfy the relation $ab = 1$, the relation $ba = 1$ also holds.

1.1.1B. *Example.* Finite monoids are directly finite.

The concepts 'semi-commutative' and '*-commutative' defined below extend commutativity . Our interest in these concepts is due to the fact (by Theorem 1.5.2B) that for regular rings both these conditions are equivalent to the condition of strong regularity.

1.1.2A. *Definition and Proposition.* A semigroup S with zero is *semi-commutative* if it satisfies the following equivalent conditions:

(i) Whenever $a, b \in S$ satisfy $ab = 0$, we have $axb = 0$ for each element $x \in S$.

(ii) For each $a \in S, l(a)$ is an ideal .

(iii) For each $a \in S, r(a)$ is an ideal.

1.1.2B. *Definition.* A semigroup S is **-commutative* if the condition $l(a) = r(a)$ holds for each element a of S. (This is equivalent to the condition $xy = 0 \Rightarrow yx = 0$ for elements x, y of S.)

1.1.2C.*Remarks.* *(i)* Consider a family $\{M_i\}_{i \in I}$ of monoids. Then each M_i is *-commutative(respectively, semi-commutative), if and only if the product monoid $\prod M_i$ is *-commutative (respectively, semi-commutative).

(ii) We can use (i) for constructing examples of monoids which are *-commutative, but which are neither reduced nor commutative.

1.1.3A.(r/∼m)*Proposition.* Consider the following conditions for a semigroup S with zero.

(i) S is reduced.

(ii) S is *-commutative.

(iii) S is semi-commutative.

(iv) S is semiprime.

Then $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(i) \Rightarrow (iv)$. If, further, S has an identity element then $(iii) + (iv) \Rightarrow (i)$

Proof. $(i) \Rightarrow (ii)$. Let $xy = 0$ in a reduced semigroup S. Then $(yx)^2 = yxyx = 0$ yields $yx = 0$.

$(ii) \Rightarrow (iii)$. Let $a \in S$. By definition (1.1.2B), $l(a) = r(a)$ and therefore $l(a)$ is an ideal. So S is semi-commutative by 1.1.2A.

$(i) \Rightarrow (iv)$. This is obvious (and well-known for rings).

$(iii) + (iv) \Rightarrow (i)$. Let $a \in S$ satisfy $a^2 = 0$. By the semicommutativity of S, we have $aSa = 0$. It follows that $SaSa = 0 \Rightarrow Sa = 0$ (since S is semi-prime). Since S has an identity, it follows that $a = 0$.By 0.1.1D, S is reduced.

1.1.3B(r/M).*Corollary.* Prime, semi-commutative monoids are reduced.

1.1.3C(R/m). *Remark.* If R is a reduced ring then R is semi-commutative

1.1.3D(r/∼m). *Remark.* If R is a semi-commutative ring then R is normal.

(For rings this result is well known; see e.g., Lemma 5 in §4 of [HT:79]. The next example shows that it does not extend to monoids.)

1.1.3E. *Example.* The augmented bicyclic monoid M_1 is a semi-commutative monoid which is not normal. Thus an analogue of 1.1.3D fails in monoids.

1.1.4.*Example.* Any commutative non-reduced semigroup is an example of a *-commutative semigroup which is not reduced. Further, the example below shows that semi-commutative does not imply *-commutative in semigroups (~~f~~or monoids). Consider the monoid defined as follows:

$$M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, u = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

where $\{0,1\} \in \mathbf{Z}_2$. It can be easily checked that $a^2 = a$, $u^2 = 0, au = 0$ and $ua = u$. Thus M is not *-commutative; but a direct computation shows that it is semi-commutative.

1.1.5.*Bibliographical note.* A class of semicommutative rings which are not *-commutative has been recorded by Shin (Example 5.1 in [Sh:73]); the least number of elements that an example in this class can have is 16. The monoid example given here is of interest since it has cardinality 4.

The basic properties of the following two examples are well-known. Our interest in them is in showing that certain results valid in rings do not extend to monoids. Thus a study of these examples recurs in this chapter.

1.1.6. *Example. The Bicyclic Monoid M_1.* Consider the set M of all maps from the set of natural numbers to itself. Let α, $\beta \in$ M be such that $\alpha\beta = 1$

but $\beta\alpha \neq 1$. Let $M_1 = \{\beta^i\alpha^j\}_{i,j\geq 0}$.Then M_1 is closed under multiplication. Also α, $\beta \in M_1$. So M_1 is not directly finite and not normal.

Consider $M_1 \cup 0 = M_1^0$. This is a reduced, regular monoid which is not normal. This will be referred to as the augmented bicyclic monoid.

1.1.7. *Example. The left zero semigroup S_l.* Let $n \geq 2$. Consider the set

$$S = \{x_1, x_2, ..., x_n\};$$

a semigroup structure can be defined on the set S as follows:

$$x_i \star x_j = x_i$$

The augmentation with zero of this semigroup in the sense of 0.1.6 (so that we have)

$$x_i \star 0 = 0 = 0 \star x_i$$

is *the left zero semigroup S_l* , of cardinality $n + 1$. It is left but not right invariant, regular and non-normal. Further, the adjunction of the augmented semigroup with 1, in the sense of 0.1.6 (so that we again have)

$$x_i \star 1 = x_i = 1 \star x_i$$

is *the left adjuncted semigroup M_l*, of cardinality $n + 2$. (Indeed, in the notation of 0.1.6 $M_l = S_l^1$.)

(Such a binary operation can also be defined for an infinite set S.)

The right zero semigroup S_r and *the right adjuncted semigroup M_r* are defined dually. Indeed they are the opposites of S_r and M_r in the sense of 0.1.9(1).

Chapter 1, section 2.

Direct finiteness and invariance.

In this section we record basic results on directly finite monoids and invariant monoids.

1.2.1. *Definition.* A monoid M is a *left invariant* monoid if every left ideal of M is an ideal.

1.2.2. *Definition.* A monoid is *invariant* if it is left and right invariant.

1.2.3A(r/M). *Proposition.* If M is a (left or right) invariant monoid, then M is directly finite.

Proof. Assume that M is left invariant and let $a, b \in M$ satisfy $ab = 1$. Then $M = aM \subset Ma \Rightarrow 1 = ca$ for some $c \in M$. Therefore

$$c = c.1 = c(ab) = (ca)b = 1.b = b$$

yielding $ba = 1$.

1.2.3B(R/m). ***Proposition.*** If R is a left or right invariant ring, then R is directly finite.

1.2.4. *Remark.* For rings 1.2.3B follows from the following results.

1.2.4A. Left (or right) invariant rings are normal.

1.2.4B. Normal rings are directly finite.

In Propositions 1.2.5 and 1.2.7 we shall give equivalent condtions for the direct finiteness of a monoid.

1.2.5(r/M). *Proposition.* A monoid M is directly finite if and only if whenever $a \in M$ satisfies $aM \neq M$, we have $Ma \neq M$.

Proof. ("Only if" part.) Let M be a directly finite monoid and assume $aM \neq M$ for some element a. Suppose, if possible, $Ma = M$. Then there exists $b \in M$ satisfying $ba = 1$. This implies $1 = ab \in aM \Rightarrow aM = M$. This contradiction shows that $Ma \neq M$.

1.2.6(r/M). *Proposition.* Let elements a, b of a monoid M satisfy $ab = 1$. The following conditions are equivalent.

(i) $ba = 1$.

(ii) $ba \in C(M)$.

Proof. The proof of $(i) \Rightarrow (ii)$ is trivial; we give the proof of $(ii) \Rightarrow (i)$. Let $ab = 1$. Then, by hypothesis, $ba \in C(M)$. Note that $ba = b(ab)a = (ba)(ba) = b.ba.a$ (since $ba \in C(M)$). This implies that $1 = (ab)^2 = a(ba)b = a(bbaa)b = (ab)ba(ab) = ba$.

The proof of the following result is along the lines of that of 1.2.6.

1.2.7(r/M). *Proposition.* Let M be a monoid. The following conditions are equivalent.

(i) The monoid M is directly finite.

(ii) If for elements $a, b \in M$, we have $ab = 1$, then $ba \in C(M)$.

The next result now follows as a corollary to Proposition 1.2.7.

1.2.8A(r/M). *Corollary.* Normal monoids are directly finite.

Proof. If $ab = 1$, then $ba = baba \in I(M) \subseteq C(M)$ (since M is normal).

1.2.8B(R/m). *Corollary.* Normal rings are directly finite.

The following two results are immediate consequences of 1.2.8B; see 1.1.3D.

1.2.9(r/$\sim$ m). *Corollary.* Reduced rings are directly finite.

1.2.10(r/$\sim$ m). *Corollary.* Semi-commutative rings are directly finite.

(For a monoid counter-example to 1.2.9 and 1.2.10, see 1.1.6.)

We now record a few examples which will help set the concept of directly finiteness in perspective.

1.2.11. *Example.* Directly finite monoids need not be invariant': indeed any "ring example" will do.

1.2.12. *Example.* The bicyclic monoid M_1 and the (augmented) bicyclic monoid M_1^0 are examples of monoids which are not directly finite, and so not invariant'.

1.2.13. *'Quasi-invariance' (in rings, modules and monoids).* The class of rings in which each maximal left ideal is an ideal has been considered by some authors. Brown [B:73] called such rings left quasi-duo.It was shown in [R:86] that an equivalence theorem well-known for commutative rings (namely, the equivalence of regularity, V-ring condition, SPI-condition, SF condition) extends to quasi-duo rings. The following result ($\equiv$ Proposition 4.4 of [R:86]) was used in the proof of this extension.

1.2.13A. *Proposition.* If R is left quasi-duo, then $R/Rad(R)$ is a reduced ring.

Using 1.2.13A we have the result:

1.2.13B. *Proposition.* Left (or right) quasi-duo rings are directly finite.

Consistent with the term "left invariant ring", it may be appropriate to use the term "left quasi-invariant ring" for a "left quasi-duo ring". This would also be in line with the use of the terms "invariant module" and "quasi-invariant module" (defined below).

Invariant modules, namely modules in which every submodule is fully invariant [C:75] generalize left invariant rings. Similarly, generalising the concept of a left quasi-duo ring, quasi-invariant modules can be defined as modules in which every maximal submodule is fully invariant. Proposition 1.2.13B can be extended to prove:

1.2.13C. *Proposition.* Finitely generated quasi-invariant modules have directly finite endomorphism rings (such modules are called directly finite).

When one comes to monoids, one notices that the condition "every maximal left ideal is two-sided" is indeed equivalent to direct finiteness of the monoid and thus the term "quasi-invariant monoid"is redundant.

Chapter 1, section 3.

Semi-commutativity and normality.

In this section we record some results concerning semi-commutative monoids and normal monoids. We begin by noting that the concepts of left and right semi-central idempotent elements in rings due to Birkenmeier [B:83] extend to monoids in a straightforward manner.

1.3.1. *Definition.* An idempotent e in a monoid M is right (left) semi-central if $eM = eMe(Me = eMe)$.

1.3.2. *Remarks.* (1). An idempotent e is both left and right semi-central if and only if it belongs to the centre .

(2). If an idempotent e satisfies $Me \subseteq eM$ then e is left semi-central .

(3). It follows from (2), dual of (2) and (1) that if e satisfies $Me = eM$ then $e \in C(M)$.

The following result is due to Birkenmeier (p.4, [B:83]).

1.3.3A(r/~m; see Example 1.3.3B.). *Proposition.* Let R be a ring in which every idempotent is right semi-central. Then R is a normal ring.

1.3.3B. *Example.* The adjuncted left zero semigroup M_r is an example of a monoid in which every element is an idempotent which is right semi-central, since

$$xM_r = \{x, 0\} = xM_rx \qquad \forall x \in M_r, x \neq 0.$$

In fact 0 and 1 are the only left semi-central (central) elements in M_r since $M_rx = M_r \backslash \{1\}$ whenever $x \neq 0, x \neq 1$. Further xM_rx is a proper subset of $M_r x$ for each $x \in M_r, x \neq 0, x \neq 1$. Clearly this monoid is not a normal monoid. Thus an analogue of 1.3.3A fails in monoids.

The following result is Exercise 5(b) on p.17 of [K].

1.3.3C.(r/~ m; see 1.3.3D.) *Proposition.* A ring R is normal if and only if all idempotents in R commute.

1.3.3D. *Example.* Consider the bicyclic monoid M_1 (see 1.1.6). A straightforward computation shows that $I(M_1) = \{\beta^i\alpha^i \mid i \geq 0\}$. The only central element in $I(M_1)$ is the identity element 1. (This holds since

$$\beta^i\alpha^i\beta \neq \beta\beta^i\alpha^i \qquad \text{whenever } i \neq 0.)$$

Thus this is an example of a non-normal monoid in which all idempotents commute.

1.3.4(r/M). *Proposition.* The following conditions are equivalent.

(i) M is a left invariant monoid.

(ii) $aM \subseteq Ma$ for every element a of M

Proof. Let $a \in M$. Since Ma is an ideal and $1 \in M, a \in Ma$, therefore $aM \subseteq Ma$. Conversely, let J be a left ideal. Let $z \in J$ and $y \in M$. By hypothesis, $zM \subseteq Mz$. Since $z \in J, zy \in zM$ which implies that $zy \in Mz$ which is a subset of J. Therefore J is an ideal.

(Recall (1.2.2) that by an invariant monoid we mean a left <u>and</u> right invariant monoid.)

1.3.5(r/M). *Corollary.* Invariant monoids are normal.

Proof. Let $e = e^2 \in M$. Since M is invariant, $eM \subseteq Me$ and so e is left semi-central. Also $Me \subseteq eM$ implies that e is right semi-central. It follows that e is central.

1.3.6A(r/M). *Proposition.* Left (or right) invariant monoids are semi-commutative.

Proof. Assume for the sake of definiteness that M is a right invariant monoid.Then for each $a \in M$ the right ideal $r(a)$ is an ideal and M is semi-commutative by 1.1.2A.

1.3.6B(R/m). *Proposition.* Left (or right) invariant rings are semi-commutative.

1.3.7. *Example.* Left zero semigroups are left invariant but are neither right invariant nor normal. This is in contrast to the situation in rings. (See 1.2.4A)

1.3.8. *Example.* The augmented bicyclic monoid is reduced , hence it is *-commutative and semi-commutative, but not directly finite and therefore non-normal, and non-invariant'.

1.3.9. *Remark.* By Example 1.1.6 reduced does not imply normal (in monoids); by any commutative non-reduced ring-example, normal does not imply reduced.

1.3.10. *Remark.* Again by Example 1.1.6 semi-commutative does not imply normal(in monoids); an example of a normal ring which is not semi-commutative has been furnished by Shin (Example 5.5 in [Sh:73]). We have used a modification of that example in another context in 1.4.5, where the details are recorded

Chapter 1, section 4.

Stability under localisation

In this section (as in section 3 of Chapter 0), T denotes a multiplicatively closed subset of the center $C(M)$ of a monoid M. Analogues of 0.4.11 and 0.4.12 can be proved for a number of concepts being studied in this chapter and chapter 0. We record a few examples and sketch some proofs.

1.4.1. *Proposition.* The following conditions are equivalent for a monoid M.

(i) M is a prime monoid.

(ii) For each T consisting of non-zero-divisors of M the monoid $T^{-1}M$ is a prime monoid.

(iii) For each $\mu \in Max(C(M))$, the monoid M_μ is a prime monoid.

1.4.2. *Proposition.* The following conditions are equivalent for a monoid M.

(i) M is a reduced monoid.

(ii) For each T the monoid $T^{-1}M$ is reduced.

(iii) For each $\pi \in Spec(C(M))$, the monoid M_π is reduced.

(iv) For each $\mu \in Max(C(M))$, the monoid M_μ is reduced.

Proof of $(i) \Rightarrow (ii)$. Let a/t be a nilpotent element of $T^{-1}M$ so that $(a/t)^n = 0$ for some positive integer n. It follows that $ua^n = 0$ for some $u \in T$. This yields (as $u \in C(M)$) $(ua)^n = 0 \Rightarrow ua = 0$ since M is reduced.

1.4.3A. *Remark.* The remark about $Max(C(M))$ (which is a singleton set) made after the statement of Proposition 0.4.10 applies to the statements of Propositions 1.4.1 and 1.4.2 as well.

1.4.3B. *Remark.* Analogues of 1.4.2 hold with reduced replaced by '*-commutative' and 'semi-commutative' in each condition. The proofs are omitted.

1.4.4. *Remark.* Using 0.3.2 an analogue of 1.4.2. can be proved with 'reduced' replaced by 'left invariant'. The proof is again omitted.

1.4.5. *Remark.* It is not true that if M is a normal monoid, so is $T^{-1}M$. Consider the (multiplicative monoids) of the rings denoted by R in (i) and (ii) below.

(i) Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a+c, b \in 2\mathbf{Z} \right\}$$

In this ring 0 and I are the only idempotents and so it is normal. Now localise R with respect to the central multiplicatively closed set $T = \{2^n I\}$ where n varies over $\mathbf{N} \cup \{0\}$. Then

$$T^{-1}R = \left\{ \begin{pmatrix} (a/2^n) & (b/2^n) \\ 0 & (c/2^n) \end{pmatrix} \mid a+c, b \in 2\mathbf{Z}, n \in \mathbf{N} \cup \{0\} \right\}$$

The element

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in T^{-1}R$$

is an idempotent which is not central. (This example is a modification of an example due to Shin;its use in the context of localisation is believed to be new; see Remark 1.3.10.)

(ii) The integral group ring $R = \mathbf{Z}[G]$ of a finite group has 1 and 0 as the only idempotents. Hence it is normal. Examples can be given to show that its localisation $\mathbf{Q}[G]$ (which is a semi-simple ring by Maschke's theorem) can be non-normal.

Chapter 1, Section 5.

Regularity.

This section is devoted to a study of the extendibility of a characterisation of strongly regular rings (Theorem 1.5.2A) to monoids.

1.5.1A. *Definitions.* An element a of a semigroup S is *left regular* if there exists an element b in S such that $a = ba^2$. An element a is *right regular* if there exists an element b in S such that $a = a^2b$. Further we call a *strongly regular* if it is both right and left regular.

1.5.1B. *Remarks. (i)* In the definitions 1.5.1A above we follow [Az:59].

(ii) If a is left (right) regular and nilpotent then $a = 0$. (Let $a = ba^2$ for some b and let $a^n = 0$; it can be seen inductively that

$$a = ba^2 = baa = bba^2a = b^2a^3 = \cdots = b^{n-1}a^n = 0.)$$

(iii) If S is a semigroup in which every element is left (right) regular then it is necessarily reduced. (This follows from *(ii)* .)

(iv) If M is a monoid in which every element is left (right) regular then M is directly finite. (Assume for the sake of definiteness that every element of the monoid M is left regular and let $ab = 1$. Since a is left regular, there exists an element $c \in M$ such that $a = ca^2$. Then

$$ca = ca.ab = ca^2b = ab = 1.$$

It follows as in the proof of Proposition 1.2.3A that $ba = 1$.)

The following theorem defines a srongly regular ring as a ring satisfying a number of equivalent conditions.It has been obtained by combining results from [R:74], [St] (Proposition 12.3, Chapter 1), [G3] (Theorems 3.2 and 3.5) and other authors. The proof of these equivalences uses several results-for example, 1.1.3C, 1.1.3D, 1.3.6B. Some of these results (like 1.1.3D) do not extend to monoids. In view of this the question of extendibility of the implications between the conditions of Theorem 1.5.2A is of interest. This question has been studied in this section.

1.5.2A. *Theorem and Definition.* Let R be a ring. Then the following conditions are equivalent.

(i) Every element of R is left regular.

(i)' Every element of R is right regular.

(ii) Every element of R is strongly regular.

(iii) R is regular and left invariant.

(iii)' R is regular and right invariant.

(iv) R is regular and reduced.

(v) R is regular and semi-commutative.

(vi) R is regular and normal.

(vii) R is regular and if x, y are elements of R which satisfy $xyx = x$ then $xy = yx$.

A ring is ***strongly regular*** if it satisfies the equivalent conditions $(i) - (vii)$.

1.5.2B.*Notation.* The respective monoid analogues of the nine conditions considered in 1.5.2A are denoted by $M(i), M(i)'$ etc.

We begin the study of the question " which implications hold between the conditions $M(i)$ to $M(vii)$" with noting the equivalence of $M(iv)$ and $M(v)$.

1.5.2C. *Proposition.* A regular monoid M is reduced if and only if it is semi-commutative.

Proof. Both implications follow from 1.1.3A. For the "if" part note that regular monoids are semiprime by 0.4.9.

1.5.2D.*Remark.*The implication $(v) \Rightarrow (iv)$ of Theorem 1.5.2A is proved by noticing that $(v) \Rightarrow (vi)$ (a result which does not hold in monoids) and $(vi) \Rightarrow (iv)$. The purely multiplicative argument for the proof of $M(v) \Rightarrow M(iv)$ is believed to be new.

We now record monoid analogues of seven of the conditions stated in Theorem 1.5.2A. We omit conditions $M(ii)$ and $M(v)$ (Condition $M(ii)$ is equivalent to $M(i) + M(i)'$ and $M(v)$ is equivalent to $M(iv)$ by 1.5.2C.)

1.5.2E. Some conditions on a monoid M:

M(i) Every element of the monoid M is left regular.

M(i)' Every element of the monoid M is right regular.

M(iii) The monoid M is regular and left invariant.

M(iii)' The monoid M is regular and right invariant.

M(iv) The monoid M is regular and reduced.

M(vi) The monoid M is regular and normal.

M(vii) The monoid M is regular and if x, y are elements of M which satisfy $xyx = x$, then $xy = yx$.

1.5.2F. *Remark.* As has been seen in 1.3.3C if all the idempotent elements of a ring commute, then the ring is a normal ring. Combining regularity with the condition "all idempotents commute", we get another characterisation of strongly regular rings. As was seen in 1.1.6 and 1.3.3D a regular monoid in which all idempotents commute need not be normal.

In 1.5.3-1.5.5 we study which implications hold between these conditions. (We have been able to settle almost all the cases.)

1.5.3A. *Example.* The left zero semigroup S_l is an example of a reduced (and therefore semi-commutative), left invariant, regular semigroup in which every element is left and right regular; however it is non-normal, and not right invariant . Further it does not satisfy condition $M(vii)$ of the above theorem. Thus we have the following chart of non-implications as seen from this example.

1.5.3B. *Chart.*

Conditions satisfied by S_l	Conditions not satisfied by S_l
$M(i)$, $M(i)'$	
$M(iii)$	$M(iii)'$
$M(iv)$	$M(vi)$
	$M(vii)$

No condition in the first column of the chart implies any condition in the second column.

1.5.3C. *Remark.* The right zero semigroup S_r has dual properties and furnishes an example which gives a chart of non-implications dual to the above chart.

1.5.4A. *Example.* Consider the augmented bicyclic monoid. It is a reduced (and therefore semi-commutative), regular monoid. Every element of this monoid is left regular. However this monoid does not satisfy $M(i)'$ since the element β is not a right regular element. Further by Proposition 1.2.3 this monoid is neither left nor right invariant. Finally it can be easily checked that this monoid does not satisfy condition $M(vii)$. Thus we have a similar chart of non-implications.

1.5.4B. *Chart.*

Conditions satisfied by M_1	Conditions not satified by M_1
$M(i)$	$M(i)'$
$M(iii), M(iii)'$	
$M(iv)$	$M(vi)$
	$M(vii)$

Again no condition in the first column of the chart implies any condition in the second column.

1.5.5. *Remarks. (a)* By 1.3.6A left (or right) invariant monoids are semi-commutative and so regular left (or right) invariant monoids are regular and semi-commutative (therefore reduced). (Thus $M(iii)/M(iii)' \Rightarrow M(iv)$ hold.)

(b) Let M be regular and normal and $a \in M$. Then $Ma = Me$ for some $e = e^2$ by 0.4.8. Since M is normal e is central and the ideal Ma is a two sided ideal. Further if A is a left ideal on M then A (being a union of principal

left ideals)is also an ideal. Thus M is left (dually right) invariant. Also $a = aba \Rightarrow a = aab = a^2b$, since $ab \in I(M) \subseteq C(M)$ (by the normality of M). Therefore $M(vi) \Rightarrow M(i), M(vi) \Rightarrow M(i)', M(vi) \Rightarrow M(iii), M(vi) \Rightarrow M(iii)'$ hold.

(c) $M(i) \Rightarrow M(iv)$ and $M(i)' \Rightarrow M(iv)$ hold, by 1.5.1B (ii)

(d) $M(vi) \Rightarrow M(iv)$ follows from (b) and (c).

Similar conclusions follow from $M(vii)$.

1.5.6. *Remarks.* *(i)* In this paragraph by finite rings we mean rings of finite cardinality (and not directly finite rings).

(ii) Regular rings of finite cardinality are semisimple and isomorphic to $\prod_{i=1,\ldots,t} M_{n_i}(K_i)$ where $\{M_{n_i}(K_i)\}_{i=1,\ldots,t}$ are rings of matrices over finite fields. (This follows by combining the result "left noetherian regular rings are semisimple " with the Wedderburn-Artin theorem and the Wedderburn theorem on finite division rings. (Apparently this description of finite regular rings goes back to Dyer-Bennet's work in 1941 [D-B:41].)) Thus the smallest number of elements that a non-commutative regular ring can have is 2^4. In the case of semigroups, however, the left zero semigroup with two elements is a non-commutative finite regular semigroup. By adjuncting it with 1, we get an example of a non-commutative regular monoid of three elements.

(iii) As a special case of the result quoted in (ii) we see that finite strongly regular rings are precisely the finite products of finite fields. Thus R is finite and strongly regular $\Leftrightarrow$ R is finite, regular and commutative .

We do not have analogous results in monoids; as seen above the augmented left zero semigroup is an example of a finite semigroup in which $M(i)$ holds but which is not commutative.

Chapter 1, section 6

Generalised inverses and unit-regularity.

Regularity in monoids (0.4.1) can also be studied in the setting of generalised inverses. We begin with some definitions.

1.6.1. *Definitions.* An element b of a monoid M satisfying the relation $a = aba$ is a *1-inverse/generalised inverse of a.* By a *2-inverse* of a we mean an element b satisfying $bab = b$. A *reflexive inverse* of a is an element $x \in M$ which is a *1-2-inverse* i.e., both a 1-inverse and a 2-inverse of a, so that $a = axa$ and $x = xax$ hold. ([CP] has used the term "inverse" for a reflexive inverse)

While there is a vast theory of generalised inverses in matrices and in rings our main interest in the first half of this section is in the study of the extendibility of some results of Savage [Sa:80] on reflexive inverses in rings to monoids.

For the sake of completeness we reproduce Lemma 3.3 of [Sa:80]; see also Proposition 3.6 in [G3].

1.6.2. *Theorem.* The following conditions are equivalent for a regular element a of a ring R.

(i) a has a unique reflexive inverse.

(ii) There exists an element $x \in R$ such that $a = axa$ and both ax and xa are central idempotents.

(iii) If $a = aya$ then $ay = ya$.

(iv) If $a = aya = aza$, then $ay = az = za = ya$.

1.6.3A. *Remarks.* In this paragraph (and in 1.6.3B) we shall examine the extendability of 1.6.2 to monoids.

(a) The proof of $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$ in the lemma in [Sa:80] is purely multiplicative . Therefore the implications $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (iv)$, $(iv) \Rightarrow (i)$, $(ii) \Rightarrow (iv)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (i)$ extend to monoids.

(b) A study of the bicyclic monoid M_1 shows that certain implications of the above theorem do not extend to monoids:

In M_1, each element has a unique generalised inverse: the element $a = \beta^i\alpha^j$ has $y = \beta^j\alpha^i$ as the unique generalised inverse; further, y is also the (unique) reflexive inverse of a. If $i \neq 0$ or $j \neq 0$, either $ay = \beta^i\alpha^j\beta^j\alpha^i = \beta^i\alpha^i$ or $ya = \beta^j\alpha^i\beta^i\alpha^j = \beta^j\alpha^j$ is a non-central idempotent.(This shows that $(i) \Rightarrow (ii)$ is not valid in monoids; indeed condition (i)is satisfied by each element of the monoid M_1, but condition (ii) is satisfied only by the element 1)

Further (with the above notation) $a = aya$ but $ay \neq ya$ whenever $i \neq j$. (This shows that $(i) \Rightarrow (iii)$ and $(i) \Rightarrow (iv)$ are not valid in monoids.)

(c) We do not know whether the implications $(iii) \Rightarrow (ii)$ and $(iv) \Rightarrow (ii)$ hold in monoids.

1.6.3B. We record some results to provide a background for 1.6.3A.

Proposition. If an element a of a monoid M has a unique reflexive inverse, and if $a = aya = aza$, then $ay = az$ and $za = ya$.

Proof. Clearly yay and zaz are reflexive inverses of a. Therefore $yay = zaz$. It follows that $ay = (aya)y = a(yay) = a(zaz) = (aza)z = az$.

Corollary. In a commutative monoid $(i) \Rightarrow (iv)$.

Proposition 3.4 of [Sa:80] has been stated below in a form convenient to us.

1.6.4A(r/$\sim$ m). *Proposition.* Let R be a ring. Each element of R is left regular if and only if each element of R has a unique reflexive inverse.

Neither implication of 1.6.4A holds in monoids as seen below.

1.6.4B. *Remark.* An analogue of "if" part of this proposition does not hold in monoids, for the bicyclic monoid is a monoid in which every element has a unique reflexive inverse; it was seen in 1.5.4A that this monoid does not satisfy condition $M(i)$ 1.5.2E.

1.6.4C. *Remark.* An analogue of "only if" part of this proposition does not hold in monoids, for the left zero semigroup satisfies condition $M(i)$ in 1.5.2E, but in this case every element of the semigroup is a generalised inverse, and hence every element does not have a unique reflexive inverse.

In 1.6.5A we extend the definition of unit-regular (elements of) rings to monoids. The results 1.6.5B and 1.6.5C are immediate.

1.6.5A. *Definition.* An element a of a monoid M is *unit-regular* if there exists a unit $u \in M$ such that the relation $a = aua$ holds. A monoid M is *unit-regular* if each element of M is unit regular.

1.6.5B. Let W be the set of unit-regular elements of a monoid M. Then $I(M) \subset W$, and $U(M) \subset W$.

1.6.5C. Further, if $U(M) = \{1\}$ then $W = I(M)$.

1.6.5D. *Bibliographical notes.* Ehrlich ([E:68], [E:76]) introduced unit-regular rings and proved the unit-regularity of strongly regular rings as well as semisimple rings. Unit-regular rings have also been studied by Henriksen [H:73] and Goodearl (Chapter 4, [G3]).

As noted in 1.6.5D we have

1.6.6A. *Proposition.* Strongly regular rings are unit-regular.

1.6.6B(r/~ m). *Corollary.* Commutative regular rings are unit-regular.

1.6.6C(r/~ m). *Corollary.* Reduced regular rings are unit-regular.

(The following example shows that analogues of 1.6.6B and 1.6.6C fail in monoids.)

1.6.6D. *Example.* Consider the monoid $M = \{\mathbf{Z}/m\mathbf{Z}\} \cup \{1'\}$, where m is a square free integer ≥ 3 and the multiplication in the ring $\mathbf{Z}/m\mathbf{Z}$ is extended to M by the process of adjunction (0.1.6). As noted in 0.4.7A M is a commutative and reduced regular monoid . Since $m \geq 3$ we have $I(M) \neq M$. Further $U(M) = \{1'\}$. Therefore by 1.6.5C we have $W = I(M) \neq M$ and M is not unit-regular.

CHAPTER 2

SOME RESULTS ON POLYNOMIAL RINGS AND POWER SERIES RINGS

Chapter 2, section 0.

Introduction

Let R be a domain (commutative or not) and R[x] its polynomial ring. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be elements of $R[x]$. (This notation for the coefficients of $f(x)$ and $g(x)$ is followed in the absence of explicit mention.) It is an elementary exercise to prove that if $f(x)g(x) = 0$, then $a_i b_j = 0$ for every i and j, since either $f(x)$ or $g(x) = 0$. (Of course the converse always holds.)

E.Armendariz ([A:74], Lemma 1) noted that the above result can be extended to the class of reduced rings, i.e., rings without non-zero nilpotent elements. In order to study additional classes of rings having this property, we introduce the concept of an *Armendariz ring* in Definition 2.1.1.

By a ring we mean an associative ring with identity. However the assumption of identity can be omitted in many places. Many remarks are thus valid in the context of "rngs" and subrngs (i.e., subrings which do not inherit the identity of the over ring). For defining left/right zero-divisors, we shall refer to ([J], p88).

In addition to reduced rings there are large classes of rings which are Armendariz. If R is a commutative P.I.D and A an ideal of R, then R/A

is Armendariz (Theorem 2.2.2). If K is a field and V is a vector space over K, then the ring $K(+)V$ - see 2.1.2 for definition - is an Armendariz ring (Corollary 2.2.9).

Analogous to the definition of Armendariz rings, we have the definition of McCoy rings (see 2.3.1. for definition). The relation between the above defined rings has been studied, and a few results on McCoy rings have been recorded in section 3 of this chapter.

To further study the relation between Armendariz rings with other classes of rings, the definitions of semi-commutative and normal rings have been recalled in section 4 , and results thus obtained have been recorded.

Chapter 2, section 1.

Preliminaries

In this section we define Armendariz rings. A construction due to Nagata has been recorded along with some variants.

2.1.1. *Definition.* A ring R is said to have the Armendariz property (or is an *Armendariz ring*) if whenever polynomials

$$f(x) = a_0 + a_1x + \dots + a_mx^m$$

$$g(x) = b_0 + b_1x + \dots + b_nx^n$$

of the ring $R[x]$ satisfy $f(x)g(x) = 0$, we have $a_ib_j = 0$ for every i and j.

For constructing examples of both Armendariz and non-Armendariz rings, we shall use the following principle of idealisation due to Nagata ([N],p.2.)

2.1.2. *A Construction.* Let R be a commutative ring and M an R-module. The R-module $R \oplus M$ acquires a ring structure where the product is defined by

$$(a,m)(b,n) = (ab, an + bm).$$

We shall use the notation $R(+)M$ for this ring. If M is not zero, this ring is not reduced since M can be identified with the ideal $0 \oplus M$ which has square zero. (It seems appropriate to call this ring as "R Nagata M".)

We shall also need the following variants of the construction in 2.1.2.

2.1.3. Let R be a commutative ring and $h : R \longrightarrow R$ a ring homomorphism . Let M be an R-module. On modifying the definition in 2.1.2

to

$$(a,m)(b,n) = (ab, h(a)n + bm)$$

we get a (non-commutative) ring structure on $R \oplus M$ which we shall denote by $R(+)_h M$.

2.1.4. Let R be a ring and A an ideal of R. The factor ring $\bar{R} = R/A$ has the natural structure of a left R-, right R-bimodule. Denote $\bar{a} = a + A \in \bar{R}$ for each $a \in R$. We use this structure to define a ring structure on $R \oplus (R/A)$ as follows:

$$(r, \bar{a})(s, \bar{b}) = (rs, \overline{rb + as}).$$

We denote this ring by $R(+)(R/A)$. Its properties are similar to those of $R(+)M$.

2.1.5. The polynomial rings $(R/A)[x]$, $\{R(+)M\}[x]$ and $\{R(+)(R/A)\}[x]$ will be identified (respectively) with the rings $R[x]/A[x]$, $R[x](+)M[x]$ and $R[x](+)(R[x]/A[x])$ in a natural manner.

Chapter 2, section 2.

Armendariz rings

We begin with some elementary remarks.The proofs of the $(i), (iii), (iv)$ and (v) are straightforward.

2.2.1. *Remarks.*

(i) Subrngs of Armendariz rings are Armendariz.

(ii) Examples can be given to show (2.3.3) that factor rings of Armendariz rings need not be Armendariz.

(iii) If $\{R_i\}_{i\in I}$ are Armendariz, then so is $\prod R_i$.

(iv) If R is an Armendariz ring, and T a multiplicatively closed subset of $C(R)$, then the ring $T^{-1}R$ is also Armendariz.

(v) If for each $p \in Spec(C(R))$ the ring R_p is Armendariz, then R is Armendariz.

(This can be proved using *(i)* and *(iii)*.)

2.2.2A. *Proposition.* For each integer n, $\mathbf{Z}/n\mathbf{Z}$ is an Armendariz ring which is not reduced whenever n is a natural number which is not square free.

Proof. We first consider the case $n = p^m, p$ a prime. Denote by

$$\overline{f(x)}, \overline{g(x)}$$

the cosets of

$$f(x), g(x)(mod p^m\mathbf{Z}[x]),$$

respectively. Assume

$$\overline{f(x)g(x)} = 0,$$

i.e., $p^m \mid f(x)g(x)$. Since p is a prime it follows that

$$f(x) = p^r f'(x)$$

and

$$g(x) = p^s g'(x)$$

for some f' and g' satisfying the conditions that the g.c.d. of the coefficients of f' (also of g') is not divisible by p. Clearly $r + s \geq m$. It follows that $a_i b_j$ $= 0$ for every i and j, showing that $\mathbf{Z}/p^m\mathbf{Z}$ is Armendariz.

Let n be a natural number. Then

$$n = p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}$$

where p_k's are primes. By the Chinese remainder theorem,

$$\mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}/p_1^{e_1}\mathbf{Z} \oplus \mathbf{Z}/p_2^{e_2}\mathbf{Z} \oplus \ldots \mathbf{Z}/p_t^{e_t}\mathbf{Z}.$$

Since each $\mathbf{Z}/p_k^{e_k}$ $\mathbf{Z}$ is Armendariz, it follows that $\mathbf{Z}/n\mathbf{Z}$ is Armendariz.

The following generalisation of 2.2.2A has a similar proof.

2.2.2B. *Theorem.* If R is a commutative P.I.D. and A an ideal of R, then R/A is Armendariz.

2.2.3. *Theorem.* Let R be a domain, A an ideal of R. Suppose R/A is Armendariz. Then $R(+)(R/A)$ is Armendariz. (See 2.1.4 for definition of $R(+)(R/A)$.)

Proof. Let $f(x), g(x)$ be elements of $\{R(+)(R/A)\}[x]$, where

$$f(x) = \sum_{i=0}^{m} (a_i, \overline{u_i}) x^i = (f_0(x), \overline{f_1(x)})$$

and

$$g(x) = \sum_{j=0}^{n} (b_j, \overline{v_j}) x^j = (g_0(x), \overline{g_1(x)}).$$

If $f(x)g(x) = 0$, we have

$$(f_0(x), \overline{f_1(x)})(g_0(x), \overline{g_1(x)}) = 0.$$

Thus we have the following sets of equations:

$$f_0(x) g_0(x) = 0 \qquad (I)$$

$$\overline{f_0(x) g_1(x) + f_1(x) g_0(x)} = 0 \qquad (II)$$

Case 1. $f_0(x) = 0$. Then (II) becomes $\overline{f_1(x) g_0(x)} = 0$ over R/A. Since R/A is Armendariz, it follows that $\overline{u_i}\overline{v_j} = 0$ for every i and j. Also $f_0(x) = 0$ implies that $a_i = 0$ for all i. We conclude that

$$(a_i, \overline{u_i})(b_j, \overline{v_j}) = (a_i b_j, \overline{a_i v_j + u_i b_j}) = 0$$

for every i and j.

Case 2. $g_0(x) = 0$. This case is similar to case 1.

As a special case of the above theorem, we have the following corollary.

2.2.4. *Corollary.* Z (+) (Z/nZ) is Armendariz for every integer n.

It follows from 2.2.3 that if R is a domain, then $R(+)R$ is Armendariz. This result can be extended to reduced rings. The following properties of these rings will be used.

(i) If a, b are elements of a reduced ring then $ab = 0$ if and only if $ba = 0$. (This holds by the extension of $(i) \Rightarrow (ii)$ of Proposition 1.1.3A to rings.)

(ii) Reduced rings are Armendariz. (Lemma 1 of [A:74].)

(iii) If R is reduced then so is the ring $R[x]$.

2.2.5. *Proposition.* Let R be a reduced ring. Then the ring $R(+)R$ is Armendariz.

Proof. Let $f(x) = (f_0(x), f_1(x))$, $g(x) = (g_0(x), g_1(x))$ be elements of $\{R(+)R\}[x]$ satisfying $f(x)g(x) = 0$. Write

$$f(x) = \sum_{i=0}^{m} (a_i, u_i)x^i,$$

$$g(x) = \sum_{j=0}^{n} (b_j, v_j)x^j,$$

with corresponding representations for $f_k(x)$, $g_k(x)$ (for $k = 0, 1$). Now we have

$$f_0(x)g_0(x) = 0 \qquad (A)$$

$$f_0(x)g_1(x) + f_1(x)g_0(x) = 0 \qquad (B)$$

Since $R[x]$ is reduced, (A) implies that

$$g_0(x)f_0(x) = 0 \qquad (C)$$

Multiplying equation (B) by $g_0(x)$ on the left and using (C) we get

$$g_0(x)f_1(x)g_0(x) = 0,$$

which implies that

$$(f_1(x)g_0(x))^2 = 0$$

and hence

$$f_1(x)g_0(x) = 0 \qquad (D)$$

This implies (on account of (B)) that

$$f_0(x)g_1(x) = 0 \qquad (E)$$

Now (A), (D) and (E) yield (since R is Armendariz) $a_ib_j = 0$, $a_iv_j = 0$, and $u_ib_j = 0$ for each i and j. It follows that

$$(a_i, u_i)(b_j, v_j) = (a_ib_j, a_iv_j + u_ib_j) = 0$$

for each i and j.

The following generalisation of 2.2.5 has a similar proof.

2.2.6. *Proposition.* Let R be a reduced ring and A an ideal of R such that R/A is reduced. Then $R(+)(R/A)$ is Armendariz.

2.2.7. *Remark.* Recall that a ring R is *strongly regular* if for each element $a \in R$, there exists an element b in R such that $a = a^2b$. A ring is strongly regular if and only if it is (von Neumann) regular and reduced. If R is a strongly regular ring, then for each ideal A of R, R/A is reduced. On applying 2.2.6. we get the following result:

If R is a strongly regular ring, then for each ideal A of R, the ring $R(+)(R/A)$ is Armendariz.

We now record a few more example of Armendariz rings.

2.2.8. *Proposition.* Let K be a field, $h : K \longrightarrow K$ a field monomorphism, and V a K-vector space. Then the ring $K(+)_h V$ is Armendariz.

Proof. The map h induces a natural ring homomorphism $h : K[x] \longrightarrow K[x]$. We identify $\{K(+)_h V\}[x]$ with $K[x](+)_h V[x]$ (see 2.1.3 for definitions).

Now let $f(x), g(x) \in \{K(+)_h V\}[x]$ satisfy $f(x)g(x) = 0$. Write $f(x)$ and $g(x)$ as $f(x) = (f_0(x), f_1(x))$ and $g(x) = (g_0(x), g_1(x))$ where $f_0(x), g_0(x) \in K[x]$ and $f_1(x), g_1(x)$ belong to the polynomial module $V[x]$. Then

$$f(x)g(x) = 0$$

$$\Rightarrow (f_0(x), f_1(x))(g_0(x), g_1(x)) = 0$$

$$\Rightarrow (f_0(x)g_0(x), h(f_0(x))g_1(x) + g_0(x)f_1(x)) = 0$$

$$\Rightarrow f_0(x)g_0(x) = 0; h(f_0(x))g_1(x) + g_0(x)f_1(x) = 0$$

Since the cases $f(x) = 0$ or $g(x) = 0$ are trivial we look at other cases.

Case 1. $f_0(x) = 0$ but $f_1(x) \neq 0$. Then $h(f_0(x)) = 0 \Rightarrow g_0(x)f_1(x) = 0$ which gives $g_0(x) = 0$ since $V[x]$ is $K[x]$-torsion free.

Case 2. $g_0(x) = 0$ but $g_1(x) \neq 0$. Then $h(f_0(x)g_1(x) = 0$. This implies that $h(f_0(x) = 0$ by an argument similar to that in case 1. Since h is a one-one map it follows that $f_0(x) = 0$. Therefore in either of the cases, $f(x), g(x)$ must be of the types $f(x) = (0, f_1(x))$, $g(x) = (0, g_1(x))$. It follows that $K(+)_h V$ is Armendariz.

2.2.9. *Corollary.* If K is a field and V a K-vector space, then $K(+)V$ is a commutative Armendariz ring which is not reduced if $V \neq 0$.

Proof. Let h be the identity map in Proposition 2.2.8.

Chapter 2, section 3.

Non-Armendariz Rings

In this section we record some example of rings which are not Armendariz.

2.3.1. *Remark.* Full matrix rings of degree ≥ 2 over any ring with identity are non-Armendariz. Consider the polynomials $f(x) = E_{12}x + E_{11}$, $g(x) = E_{11}x - E_{21}$. Then $f(x)g(x) = 0$ but $E_{11}E_{11} = E_{11} \neq 0$.

2.3.2. *Example.* Commutative rings need not be Armendariz.

Consider the polynomial $f(x) = (\bar{4}, \bar{0}) + (\bar{4}, \bar{1})x$ over the ring $\{Z_8(+)Z_8\}$. The square of this polynomial is zero but the product $(\bar{4}, \bar{0})(\bar{4}, \bar{1}) = (\bar{0}, \bar{4})$ is not zero.

2.3.3. *Remark.* The ring considered in 2.3.2 is a factor ring of an Armendariz ring, namely the ring of polynomials in sufficiently many variables over **Z**. It is also a factor ring of $\mathbf{Z}(+)\mathbf{Z}_8$ which is Armendariz by 2.2.4. Thus factor rings of Armendariz rings need not be Armendariz.

Chapter 2, section 4.

McCoy rings

In this section we define McCoy rings and obtain a few results concerning them. The motivation of this definition comes from some results of McCoy [M:42] .

2.4.1. *Definition.* A ring is a *left McCoy ring* if whenever $g(x)$ is a right zero-divisor in $R[x]$, there exists a non-zero element c in R such that $cg(x) = 0$. Right McCoy rings are defined dually. A ring is a *McCoy ring* if it is both left as well as right McCoy.

2.4.2. *Remark.* It was proved by McCoy [M:42] that commutative rings have the above property; for an inductive proof of this result see [S:54]; see also [Fo:43]. If T is a ring with identity, the matrix ring $M_2(T)$ is neither left nor right McCoy. (There do not exist nonzero matrices satisfying $Cg(x) = 0$ and $f(x)D = 0$ for the polynomials considered in Remark 2.3.1.)

2.4.3. *Remark.* Let R be an Armendariz ring and assume that $g(x)$ is a right zero-divisor in $R[x]$. Then there exists a non-zero polynomial $f(x) \in R[x]$ satisfying $f(x)g(x) = 0$. Since R is Armendariz, $a_ib_j = 0$ for each i and each j. Since $f(x) \neq 0, a_t \neq 0$ for some t. Then $a_tg(x) = 0$. Thus R is left (similarly right) McCoy. The converse is not true; commutative rings are McCoy, as noted in 2.4.2 , but we have examples of commutative non-Armendariz rings.

The following example shows that McCoyness is not a left-right symmetric property.

2.4.4. *Example.* Let R be a non-zero ring with identity. Then the upper traingular matrix ring $UT_2(R)$ is an example of a ring which is left McCoy, not right McCoy. Consider the polynomials $f(x) = E_{12}x + E_{11}$; $g(x) = E_{12}x - E_{22}$. Then $f(x)g(x) = 0$. Now let $r \in UT_2(R)$ be such that $f(x)r = 0$; then r must be of the type $aE_{11} + bE_{12} + cE_{22}$. Since $f(x)r = 0$ we have $cE_{12}x + aE_{11} + bE_{12} = 0$. This implies that a, b, c all vanish, hence $r = 0$ (notice that $E_{22}f(x) = 0$ while $E_{22} \neq 0$).

Dually we have examples of rings which are right McCoy but not left McCoy.

An analogous definition of McCoy rings for power series rings is as follows:

2.4.5. *Definition.* A ring R is a *left McCoy(PS) ring* if the following holds. Whenever $g(x)$ is a right zero-divisor in the power series ring $R[[x]]$, there exists a non-zero element $r \in R$ satisfying $rg(x) = 0$. Right McCoy(PS) rings are defined dually. A ring R is a *McCoy(PS) ring* if it is both left as well as right McCoy(PS).

2.4.6. *Remark.* McCoy(PS) rings are McCoy since $R[x] \subseteq R[[x]]$ (as a subring). The converse is however not true, for there exists a commutative non-Noetherian ring which is McCoy but not McCoy(PS).For an example see Example 3 of [Fi:71].

2.4.7. *Bibliographical Note.* A different definition of McCoy rings has been given in [F:91].

Chapter 2, section 5.

Stability questions.

Recall the following definition (see 1.1.2A, where the concept was defined for semigroups).

2.5.1. *Definition.* A ring R is *semi-commutative* if it satisfies the following condition : if elements a, b in R satisfy $ab = 0$, then $acb = 0$ for each element c of R.

2.5.2. *Remarks and questions.* The class of commutative rings and the class of reduced rings are contained in the class of semi-commutative rings. Both these (smaller) classes are trivially stable under the formation of polynomial rings. Recall that a ring is called *normal* if every idempotent in R is central; semi-commutative rings are normal (Remark 1.1.3D). Against this background we consider the following "stability "assertions:

(i) R normal $\Rightarrow R[x]$ normal;

(ii) R semi-commutative $\Rightarrow R[x]$ semi-commutative; and

(iii) R *-commutative $\Rightarrow R[x]$ *-commutative.

(iv) R Armendariz $\Rightarrow R[x]$ Armendariz.

The conditions '*-commutative' and semi-commutative' have been studied in the context of monoids and rings in §1.1 above.

We remark that (i) is Corollary 2.5.8 below. We do not know whether $(ii), (iii)$ and (iv) are true. In view of these questions the following results may be of some interest.

2.5.3. *Proposition.* If R is a semi-commutative ring which is Armendariz, then $R[x]$ is semi-commutative.

Proof. Let $f(x), g(x)$ be polynomials in $R[x]$ satisfying $f(x)g(x) = 0$. Let

$$h(x) = \sum_{k=0}^{t} c_k x^k \in R[x]$$

Since R is Armendariz and $f(x)g(x) = 0$,

$$a_i b_j = 0$$

for each i and j. Since R is semi-commutative

$$a_i c_k b_j = 0$$

for each i, j and k. Hence

$$f(x)h(x)g(x) = 0$$

This proves that $R[x]$ is semi-commutative.

2.5.4. *Proposition.* Let R be a ring which is *-commutative and Armendariz. Then the ring $R[x]$ is *-commutative.

Proof. With the usual notation,

$$f(x)g(x) = 0$$

$$\Rightarrow a_i b_j = 0 \qquad \text{for each } i \text{ and } j \text{ (since } R \text{ is Armendariz)}$$

$$\Rightarrow b_j a_i = 0 \qquad \text{for each } j \text{ and } i \text{ (since } R \text{ is *-commutative)}$$

$$\Rightarrow g(x)f(x) = 0$$

showing the *-commutativity of $R[x]$.

An analysis of the proof of Proposition 2.2.5 shows that the condition 'R is reduced' can be replaced by the condition 'R is *-commutative' in Proposition 2.2.6. Indeed the implication (i) ⇒ (ii) of the following result extends Proposition 2.2.6.

2.5.5. *Proposition.* Let R be a ring and A an ideal of R such that R/A is a reduced ring. Then the following conditions are equivalent.

(i) R is *-commutative and Armendariz.

(ii) $R[x]$ is *-commutative and $R(+)(R/A)$ is Armendariz.

Proof. $(i) \Rightarrow (ii)$. The *-commutativity of $R[x]$ follows from Proposition 2.5.4; the proof of the fact that $R(+)(R/A)$ is Armendariz has been indicated in the remarks above.

$(ii) \Rightarrow (i)$. This is trivial since both *-commutativity and Armendarizness are inherited by subrings.

2.5.6. *Examples.* Proposition 2.5.5 can be applied to give some more examples of Armendariz rings. Let m, n be positive integers such that the square free integer $n \mid m$. Then $\mathbf{Z}/n\mathbf{Z}$ is a reduced factor ring of the commutative Armendariz ring $\mathbf{Z}/m\mathbf{Z}$. Applying 2.5.5 we get that $\mathbf{Z}/m\mathbf{Z}(+)\mathbf{Z}/n\mathbf{Z}$ is Armendariz by Proposition 2.2.2A.

The following result was proved for the smaller class of reduced rings by Armendariz. It may be known in the case of commutative rings but we could not find a suitable reference.

2.5.7. *Proposition.* Let R be a normal ring and $R[x]$ its ring of polynomials. If $f(x) \in R[x]$ is such that $(f(x))^2 = f(x)$, then $f(x) \in R$.

Proof. Let

$$(f(x))^2 = f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$$

Then we have

$$(f(x))^2 = (a_0 + a_1 x + \dots + a_n x^n)(a_0 + a_1 x + \dots + a_n x^n)$$

$$= a_0^2 + (a_0 a_1 + a_1 a_0)x + (a_0 a_2 + a_1^2 + a_2 a_0)x^2 + \dots + a_n^2 x^{2n}$$

$$= a_0 + a_1 x + \cdots + a_n x^n \qquad (I)$$

We shall now equate coefficients of powers of x in (I). First we have

$$a_0^2 = a_0 \qquad (II)$$

which implies

$$a_0 \in C(R) \qquad (III)$$

the centre of R (since R is normal). Therefore

$$a_0 a_i = a_i a_0 \quad \forall i, 1 \leq i \leq n.$$

Hence,

$$a_0 a_1 + a_1 a_0 = a_1$$

which yields,

$$2a_0 a_1 = a_1 \qquad (IV)$$

Multiplying (IV) by a_0 we get

$$2a_0^2 a_1 = a_0 a_1,$$

Therefore we have, using (II) and (IV)

$$2a_0a_1 = a_0a_1 \Rightarrow a_0a_1 = 0 \Rightarrow a_1 = 0.$$

Next we have

$$2a_0a_2 + a_1^2 = a_2 \qquad (V)$$

It follows from the above results that

$$2a_0a_2 + a_1^2 = a_2 \Rightarrow 2a_0a_2 = a_2$$

$$\Rightarrow 2a_0^2a_2 = a_0a_2 \Rightarrow 2a_0a_2 = a_0a_2 \Rightarrow a_0a_2 = 0,$$

and so

$$a_2 = 2a_0a_2 = 0.$$

Continuing in this way, we get that

$$a_i = 0 \quad \forall i \geq 1.$$

Therefore $f(x) \in R$.

2.5.8. *Corollary.* If R is a normal ring, then the ring of polynomials $R[x]$ is also a normal ring.

Chapter 2, section 6

Armendariz modules and McCoy modules

In this section we shall extend the definitions of Armendariz rings and McCoy rings to modules. Initially a study of these extensions was to be included here. However due to the tentativeness of the results obtained during such a study we only record the relevant definitions below.

2.6.1. *The module of polynomials* ${}_{R[x]}M[x]$. Let M be a left R-module. The elements of $M[x]$ are formal sums of the form $\sum_{j=0}^{l} a_j x^j, l \geq 0, l \in \mathrm{N} \cup \{0\}$ and $a_j \in M$. Addition is defined by adding the corresponding coefficients. Next let $f(x) = \sum_{i=0}^{k} \alpha_i x^i \in R[x]$ and $g(x) = \sum_{j=0}^{l} a_j x^j \in M[x]$. Then

$$(\sum_{i=0}^{k} \alpha_i x^i)(\sum_{j=0}^{l} a_j x^j) = \sum_{\mu=0}^{k+l} c_\mu x^\mu \qquad (I)$$

where

$$c_\mu = \sum_{i+j=\mu} \alpha_i a_j \quad \text{for each } \mu$$

The right hand side of (I) is clearly an element of $M[x]$ as M is a left R-module. This gives a left $R[x]$-module structure on $M[x]$. Any non-zero element $g(x)$ of $M[x]$ can be written uniquely as $\sum_{j=k}^{l} a_j x^j$ with $l \geq k \geq 0, a_i \in M$, and $a_k \neq 0, a_l \neq 0$. The term a_k is referred to as the *initial coefficient* and a_l the *final coefficient*.

2.6.2. *Definition.* A left R-module M is an *Armendariz module* if the following holds: whenever elements $f(x) \in R[x], g(x) \in M[x]$ satisfy $f(x)g(x) = 0$, (with notation as in 2.6.1), $\alpha_i a_j = 0$ for each i and each j.

2.6.3. *Definition.* A left R-module M is a *McCoy module* if whenever elements $f(x) \in R[x], g(x) \in M[x]$ satisfy $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $rg(x) = 0$.

2.6.4. *Remark.* When $M = R$, Definitions 2.6.2 and 2.6.3 coincide with the definitions of Armendariz ring (2.1.1) and left McCoy ring (2.4.1) respectively.

2.6.5. *Remark.* It was pointed out in 2.4.3 that Armendariz rings are left McCoy. An analogous proof shows that Armendariz modules are McCoy modules.

CHAPTER 3

A PROPERTY OF RINGS OF FUNCTIONS

Chapter 3, section 0.

Introduction.

Consider the following question which was asked in [CDR:91]. (The relevant concepts have been defined in §3.1 below.)

Question A. If in a commutative anti-regular ring, every non-zero-divisor is a unit, is it necessarily regular? Equivalently, is the total quotient ring of a commutative anti-regular ring necessarily regular?

An example in the class of rings of continuous functions answers question A in the negative.

Basic results concerning anti-regular monoids and rings, as well as basic facts about rings of continuous functions are collected in §1.

The main result of next section characterises anti-regularity of the ring $C(X)$ of real-valued continuous functions on a topological space X through a topological condition on X.

The example which answers Question A in the negative is furnished in the final section.

Chapter 3, section 1.

Preliminaries

As mentioned in the introduction, our interest in this chapter is in total quotient rings of anti-regular rings of continuous real-valued functions defined on topological spaces. However for the sake of completeness (and in the spirit of the approach adopted in Chapter 1) we shall record a few definitions and results in the most general setting possible.

3.1.1. *Anti-regularity in monoids and rings.*

3.1.1A. *Basic definitions.* Let M be a monoid. Consider the following conditions for an element $a \in M$.

(i) $a \neq 0$.

(ii) There exists a non-zero element $b \in M$ such that $bab = b$. (i.e., in the terminology of 1.6.1. b is a 2-inverse of a.)

Clearly *(ii)* $\Rightarrow$ *(i)* (always). If condition *(ii)* holds, then we call the element a *anti-regular.* If (*i*) $\Rightarrow$ (*ii*) in M then the monoid M is called *anti-regular.* (Both these definitions extend to semigroups.)

Following the usual conventions, a ring is *anti-regular* if its multiplicative monoid is anti-regular.

3.1.1B *Proposition.* Let a be a non-zero regular element of a semigroup S. Then a is anti-regular.

Proof. Let $aba = a$. Then $b_1 = bab$ satisfies

$$b_1 a b_1 = (bab)a(bab) = bab = b_1$$

Further, $ab_1a = a(bab)a = a \neq 0$ shows that $b_1 \neq 0$.

3.1.1C. *Corollary.* Regular semigroups (monoids, rings) are anti-regular.

0.4.8
3.1.1D. *(cf. Proposition 0.4.9.) Proposition.* Let a be an element of a monoid M. Then the following conditions are equivalent.

(i) The element a is anti-regular.

(ii) The (principal left) ideal Ma contains a non-zero idempotent.

Proof. $(i) \Rightarrow (ii)$. Assume that $b \in M$ satisfies $bab = b \neq 0$. Then $ba \in I(M)$, $ba \neq 0$ and $ba \in Ma$.

$(ii) \Rightarrow (i)$. Let e be a non-zero idempotent element in Ma. Therefore $e = ba$ for some $b \in M$. Write $c := bab$. Then we have (since $ba = (ba)^2 = (ba)^3$)

$$ca = baba = ba = e \neq 0 \Rightarrow c \neq 0$$

Next

$$cac = babab ab = bab = c \neq 0 .$$

0.4.9
3.1.1E. *(cf. Corollary 0.4.10.) Corollary.* Anti-regular monoids are semiprime.

0.4.9
Proof. Similar to that of Corollary 0.4.10.

3.1.1F.*Corollary.*Semi-commutative,anti-regular monoids are reduced.

Proof. Apply Proposition 1.1.3.

For basic results on non-singularity in monoids we refer to §0.3

3.1.1G. *Corollary.* Let a be an anti-regular element of a monoid M. Then a cannot belong to either the left singular ideal or the right singular ideal of M.

Proof. Let a be an anti-regular element of M. If possible, let $a \in Z_l(M)$. By 3.1.1D, the left ideal Ma contains a non-zero idempotent e. Since $e \in Z_l(M)$ (ideal!) $l(e) \lhd M$ (as a left ideal). Therefore $l(e) \cap Me \neq 0$. So let $ze \in l(e), ze \neq 0$. Then $ze = zee = 0$, a contradiction!

3.1.1H. *Corollary.* Anti-regular monoids are left and right non-singular.

3.1.1I. *Corollary.* Regular monoids are left and right non-singular.

3.1.1J.*Remarks.* A systematic study of anti-regularity in monoids is not attempted here.However it is worth recording that natural analogues of some basic results on anti-regular rings fail in monoids. Indeed we have:

(i) Non-zero anti-regular domains are division rings, and

(ii) Noetherian anti-regular rings are regular(equivalently,semisimple).

(The first result above is immediate from the definitions, while (*ii*) follows as a corollary of Theorem 2.8 of $[CDR : 90]_2$)

However, analogues of both these results fail in monoids as can be seen from the following example:

Consider the element

$$a = (\bar{2}, \bar{1}) \in \mathrm{Z}/4\mathrm{Z} \times \mathrm{Z}/2\mathrm{Z}.$$

We clearly have $a^n = a^2$ for each integer $n \geq 2$. It follows that the monoid

$$M_4 = \{0, a, a^2, 1\}$$

is a finite, commutative, anti-regular monoid without divisors of zero. (Since $a^2 = a^2.a.a^2 \neq 0$ the element a is anti-regular.) The elements a and a^2 are not invertible in M_4. Further, this monoid is not regular, since there is no element b for which $aba = a$ holds.

3.1.1K. *Bibliographical notes.* Rings R in which for each $a \notin J(R)$ (the Jacobson radical of R) is anti-regular (called as I_0-rings in [N:75]) and related classes of I-rings and Zorn rings have been studied by Nicholson, Jacobson, Levitzki, Kaplansky and others; see [N:75] for detailed references.

The non-singularity of anti-regular modules and rings has been proved in $[CDR : 90]_1$ (Corollary 3.3); for a related result see Proposition 1.27(A) of [G3]; for a proof of the non-singularity of (Zelmanowitz) regular modules see [Z:72]. The purpose of 3.1.1F and 3.1.1G is to point out that the additive structure of a ring R is not needed and the ring results extend to monoids.

3.1.2 *Anti-regularity under localisation.*

3.1.2A. *Proposition.* Let $\phi : M \longrightarrow M'$ be a one-one monoid homomorphism. If a is anti-regular in M then $\phi(a)$ is anti-regular in M'.

Proof. The equation $bab = b \neq 0$ yields (since ϕ is one-one)

$$\phi(b)\phi(a)\phi(b) = \phi(bab) = \phi(b) \neq 0$$

showing that $\phi(b)$ is a non-zero 2-inverse of $\phi(a)$ in M'.

3.1.2B. *Proposition.* Let M be a monoid (ring) and T a central multiplicatively closed subset of M such that every element of T is a non-zero-divisor in M. Let a be an anti-regular element in M. Then for each $t \in T$, a/t is anti-regular in $T^{-1}M$.

Proof. This can be proved in the same manner as 3.1.2A. (Note that (by 0.3.1) the homomorphism $M \longrightarrow T^{-1}M$ is one-one and $1/t$ is a central element in $T^{-1}M$.)

3.1.2C. *Corollary.* Let M and T as in the Proposition. If M is anti-regular then so is $T^{-1}M$.

3.1.2D. *Example.* An example of a commutative anti-regular ring R and a multiplicatively closed set T such that $T^{-1}R = \mathbf{Z}$ has been given in [CDR:91]. Thus 3.1.2C does not extend to arbitrary multiplicatively closed sets.

Note that by 0.4.12R, the above ring cannot be regular. For other examples of anti-regular, non-regular rings see $[CDR:90]_1$

3.1.3. *Definition.* Let R be a commutative ring, S_0 the set of all non-zero-divisors of R. Clearly, S_0 is a multiplicatively closed subset of R. By the *total quotient ring (T.Q.R) of R*, we mean the localisation $S_0^{-1}R$ of R with respect to the set S_0.

3.1.4. *Remark.* As a special case of 3.1.2D, the total quotient ring of a commutative anti-regular ring is anti-regular.

In paragraphs 3.1.5 to 3.1.7 we recall basic definitions and results in the theory of rings of continuous functions. For unexplained concepts and results we refer to [FGL] and [GJ]

3.1.5. *Definition.* Let X be a topological space. The set $C(X)$ of all continuous, real-valued functions on X can be made into a ring called *the* *ring of continuous functions* by providing an algebraic structure on the set. Addition and multiplication are defined by the formulas

$$(f+g)(x) = f(x) + g(x),$$

and

$$(fg)(x) = f(x)g(x).$$

The zero element is the constant funtion **0**, and the unity element is the constant function **1**. The additive inverse $-f$ of f is characterised by the formula

$$(-f)(x) = -f(x).$$

The multiplicative inverse f^{-1} (which exists when the function f does not vanish anywhere) is characterised by the formula

$$f^{-1}(x) = 1/f(x)$$

3.1.6. *Definition.* Consider the subsets of X of the form

$$f^{\leftarrow}(r) = \{x \in X : f(x) = r\} \qquad f \in C(X), r \in \mathbf{R}$$

The set $f^{\leftarrow}(0)$is called the *zero-set* of f. We shall denote this set by $\mathbf{Z}(f)$ or for clarity by $\mathbf{Z}_X(f)$:

$$\mathbf{Z}(f) = \mathbf{Z}_X(f) = \{x \in X : f(x) = 0\} \qquad (f \in C(X))$$

3.1.7. *Definition.* Let $f \in C(X)$. Then by 3.1.3. $\mathbf{Z}(f)$ is the zero set of f. The ***cozero set of*** f (denoted by $\mathrm{coz}(f)$), is the complement of the zero set (i.e., of the form $X - \mathbf{Z}(f)$).

Chapter 3, section 2.

A characterisation.

In this section we give a necessary and sufficient condition for the anti-regularity of the ring $C(X)$.

Proposition 3.2.1 is a part of Exercise 4J of [GJ]. They call a topological space satisfying the conditions of (3.2.1) (and several other equivalent conditions) a *P-space.*

3.2.1. *Proposition.* The ring $C(X)$ is regular if and only if every zero-set is open; equivalently every cozero set is closed (and therefore, open-and-closed).

In the following proposition we give an analogous necessary and sufficient condition for the anti-regularity of $C(X)$. (Since the characteristic functions of clopen sets are idempotents in $C(X)$, the motivation of this result can be found in Proposition 3.1.1D; indeed 3.1.1D can be used in its proof. However for the sake of clarity we give a direct argument.)

3.2.2. *Proposition.* Let X be a topological space. The ring $C(X)$ is anti-regular if and only if every non-empty cozero set contains a non-empty open-and-closed (clopen) subset.

Proof. Let $C(X)$ be anti-regular and W a non-empty cozero set, so that $W = \text{coz}(f)$ for some $f \in C(X)$. Since $f \neq 0$, there exists $g \in C(X), g \neq 0$ such that $gfg = g$. Now $e = gf$ is an idempotent in $C(X)$ and hence is the characteristic function of coz(e). Since e is non-zero and continuous, coz(e) is a non-empty clopen subset of W.

To prove the converse, let f be a non-zero element of $C(X)$. By hypothesis coz(f) contains a non-empty subset B. We define a map $g : X \longrightarrow \mathbf{R}$ as follows. If $b \in B$, we have $f(b) \neq 0$ and we set $g(b) = 1/f(b)$; and if $b \notin B$, we set $g(b) = 0$. Since B is non-empty and clopen g is a non-zero continuous map from X to $\mathbf{R}$. Clearly $gfg = g$. This proves the anti-regularity of $C(X)$.

3.2.3. *Corollary.* Let X be a topological space, whose topology has a base consisting of clopen sets. Then $C(X)$ is an anti-regular ring.

Proof. Coz(f) is open for every $f \in C(X)$.

Chapter 3, section 3.

An example.

The results of section 2 will be applied in two distinct situations in this section.

3.3.1. *Remark.* Let X be a topological space satisfying the hypothesis of Corollary (3.2.3) which is not a P-space. Then $C(X)$ is an anti-regular ring which is not regular. An example is the space $\mathbf{Q}$ of rationals, since the zero-set of the inclusion map $j : \mathbf{Q} \longrightarrow \mathbf{R}$ is not open in $\mathbf{Q}$, being a singleton set (see 3.2.1).

However, for any metric space X, by §§2.6 and 3.3 of [FGL], the total quotient ring of $C(X)$ is regular. Question Λ cannot therefore be settled by considering $C(X)$ for a metric space X.

3.3.2. *Example.* The one-point compactification of an uncountable discrete space Δ is denoted by $\Delta *$; thus $\Delta * = \Delta \cup \{\infty\}$. It is easily seen that the topology on $\Delta *$ has a base B consisting of clopen sets, namely $B = B_1 \cup B_2$ where $B_1 = \{\{x\} : x \in X\}$ and $B_2 =$ all open sets containing the point at infinity. It follows from 3.2.3 that $C(\Delta *)$ is anti-regular. However as recorded in Beispiel 11.6 of Storrer [St:68], this is a non-regular ring in which every non-zero-divisor is a unit. Thus question Λ has a negative answer. (Notice that $\Delta *$ is non-metrizable, as was needed; see 3.2.4.)

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Armendariz Rings

By

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$$\text{(3)}\quad \frac{\boldsymbol{e}(2\,{}^t\eta'_{\{k,l\}}\eta''_m)\vartheta[U\circ(V_2\cup\{k,l\})]^2\vartheta[U\circ(V_2\cup\{m\})]^2}{a_k-a_l}$$

$$=\frac{\boldsymbol{e}(2\,{}^t\eta'_{\{l,m\}}\eta''_k)\vartheta[U\circ(V_2\cup\{l,m\})]^2\vartheta[U\circ(V_2\cup\{k\})]^2}{a_l-a_m}$$

$$=\frac{\boldsymbol{e}(2\,{}^t\eta'_{\{m,k\}}\eta''_l)\vartheta[U\circ(V_2\cup\{m,k\})]^2\vartheta[U\circ(V_2\cup\{l\})]^2}{a_m-a_k}.$$

In fact, the equality (3) is a combination of (1) and (3.0), and (3.0) is straightforward from the equality $\boldsymbol{e}(2({}^t\eta'_k\eta''_l-{}^t\eta'_l\eta''_k))=-1$, and the likes.

Corollary 4. *For* $V_2\subset B-\{k_1,k_2,k_3,k_4\}$ *with* $\# V_2=g-1$, *we put* $\langle k_i,k_j\rangle=\boldsymbol{e}(2\,{}^t\eta'_{k_i}\eta''_{k_j})$ $\vartheta[U\circ(V_2\cup\{k_j,k_j\}]^2$. *Then we have*

$$\text{(4)}\qquad \frac{\langle k_1,k_3\rangle}{\langle k_1,k_4\rangle}:\frac{\langle k_2,k_3\rangle}{\langle k_2,k_4\rangle}=\frac{a_{k_1}-a_{k_3}}{a_{k_1}-a_{k_4}}:\frac{a_{k_2}-a_{k_3}}{a_{k_2}-a_{k_4}}$$

Note 4.1. In the formula(1), $\vartheta[U\circ(V_2\cup\{m,\infty\})]^2$ may be written more naturally than $\vartheta[U\circ(V_2\cup\{m\})]^2$, and the formula (1) is a special case of (4).

Note 4.2. Thus we have struck the branch point ∞ out in the formula (4), and this formula is also valid for hyperelliptic curves having no branch point at infinity.

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Armendariz Rings

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1. Introduction. Let R be a domain (commutative or not) and $R[x]$ its polynomial ring. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be elements of $R[x]$. (This notation for the coefficients of $f(x)$ and $g(x)$ will be followed in the absence of explicit mention.) It is an elementary exercise to prove that if $f(x)g(x) = 0$, then $a_i b_j = 0$ for every i and j, since either $f(x) = 0$ or $g(x) = 0$. (Of course the converse always holds.)

E. Armendariz ([1], Lemma 1) noted that the above result can be extended to the class of reduced rings, i.e., rings without non-zero nilpotent elements. In order to study additional classes of rings having this property we introduce the following definition.

1.1. Definition. A ring R is said to have the Armendariz property (or is an *Armendariz ring*) if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)\,g(x) = 0$, we have $a_i b_j = 0$ for every i and j.

By a ring we mean an associative ring with identity. However, the assumption of the existence of identity can be omitted in many places. Many remarks are thus valid in the context of "rings" and subrings (i.e., subrings which may not inherit the identity of the over-ring). For defining left/right zero-divisors, we shall refer to ([4], p. 88).

In addition to reduced rings, there are large classes of rings which are Armendariz. If R is a commutative P.I.D and A an ideal of R, then R/A is Armendariz (Theorem 2.2). If K is a field and V is a vector space over K, then the ring K $(+)$ V (see 1.2 for notation) is an Armendariz ring (Corollary 2.9).

For constructing examples of both Armendariz rings and non-Armendariz rings, we shall use the following principle of idealisation due to Nagata ([6], p.2).

1.2. Let R be a commutative ring and M an R-module. The R-module $R \oplus M$ acquires a ring structure where the product is defined by

$$(a, m)(b, n) = (ab, an + bm).$$

We shall use the notation $R\ (+)\ M$ for this ring. If M is not zero, this ring is not reduced, since M can be identified with the ideal $0 \oplus M$ which has square zero. (It seems appropriate to call this ring as "R Nagata M").

We shall also need the following variants of the construction in 1.2.

1.3. Let R be a commutative ring and h : $R \to R$ a ring homomorphism. Let M be an R-module. On modifying the definition in 1.2 to

$$(a, m)(b, n) = (ab, h(a)n + bm),$$

we get a (non-commutative) ring structure on $R \oplus M$ which we shall denote by $R\ (+)_h M$.

1.4. Let R be a ring and A an ideal of R. The factor ring $\bar{R} = R/A$ has the natural structure of a left R-, right R- bimodule. Denote $\bar{a} = a + A \in \bar{R}$ for each $a \in R$. We use this structure to define a ring structure on $R \oplus (R/A)$ as follows:

$$(r, \bar{a})(r', \bar{a}') = (rr', \overline{ra' + ar'}).$$

We denote this ring by $R\ (+)\ R/A$. Its properties are similar to those of $R\ (+)\ M$.

2. Rings which have the Armendariz property. It is easy to see that subrings of Armendariz rings are also Armendariz. However, factor rings need not be so (see 3.3). If $\{R_i\}_{i \in I}$ are Armendariz, so is ΠR_i. We begin with examples of familiar non-reduced rings which are Armendariz.

2.1. Proposition. For each integer n, $\boldsymbol{Z}/n\boldsymbol{Z}$ is an Armendariz ring, which is not reduced whenever n is a natural number which is not square free.

Proof. We first consider the case $n = p^m$, p a prime. Denote by $\overline{f(x)}$, $\overline{g(x)}$ the cosets of $f(x)$, $g(x)$ (mod $p^m \boldsymbol{Z}[x]$), respectively. Assume $\overline{f(x)g(x)} = 0$, i.e. $p^m \mid f(x)g(x)$. Since p is a prime, it follows that $f(x) = p^r f'(x)$ and $g(x) = p^s g'(x)$ for some f' and g' satisfying the conditions that the $g.\ c.\ d.$ of the coefficients of f' (also of g') is not divisible by p. Clearly $r + s \geq m$. It follows that $\bar{a}_i\bar{b}_j = 0$ for every i and j, showing that $\boldsymbol{Z}/p^m\boldsymbol{Z}$ is Armendariz.

Let n be a natural number. Then $n = p_1^{e_1}p_2^{e_2}\cdots p_i^{e_i}$ where p_k's are primes. By the Chinese remainder theorem,

$$\boldsymbol{Z}/n\boldsymbol{Z} \cong \boldsymbol{Z}/p_1^{e_1}\boldsymbol{Z} \oplus \boldsymbol{Z}/p_2^{e_2}\boldsymbol{Z} \oplus \ldots \oplus \boldsymbol{Z}/p_i^{e_i}\boldsymbol{Z}.$$

Since each $\boldsymbol{Z}/p_k^{e_k}\boldsymbol{Z}$ is Armendariz, it follows that $\boldsymbol{Z}/n\boldsymbol{Z}$ is Armendariz.

The following generalisation of 2.1 has a similar proof.

2.2. Theorem. If R is a commutative P.I.D and A an ideal of R, then R/A is Armendariz.

2.3. Theorem. Let R be a domain, A an ideal of R. Suppose R/A is Armendariz. Then R (+) R/A is Armendariz. (See 1.4 for definition of R (+) R/A.)

Proof. Let $f(x)$, $g(x)$ be elements of $\{R$ (+) $R/A\}[x]$, where

$$f(x) = \sum_{i=0}^{m} (a_i, \bar{u}_i)x^i = (f_0(x), \overline{f_1(x)}) \quad \text{and}$$

$$g(x) = \sum_{j=0}^{n} (b_j, \bar{v}_j)x^j = (g_0(x), \overline{g_1(x)}).$$

If $f(x)g(x) = 0$, we have $(f_0(x), \overline{f_1(x)})(g_0(x), \overline{g_1(x)}) = 0$. Thus we have the following equations:

$$\begin{cases} f_0(x)g_0(x) = 0 & \text{(I)} \\ \overline{f_0(x)g_1(x) + f_1(x)g_0(x)} = 0 & \text{(II)} \end{cases}$$

Case 1. $f_0(x) = 0$. Then (II) becomes $\overline{f_1(x)g_0(x)} = 0$ over R/A. Since R/A is Armendariz, it follows that $\overline{u_ib_j} = 0$ for every i and j. Also $f_0(x) = 0$ implies that $a_i = 0$ for all i. We conclude that $(a_i, \bar{u}_i)(b_j, \bar{v}_j) = (a_ib_j, \overline{a_iv_j + u_ib_j}) = 0$ for every i and j.

Case 2. $g_0(x) = 0$. This case is similar to case 1.

As a special case of the above proposition, we have the following corollary.

2.4. Corollary. $\boldsymbol{Z}$ (+) $\boldsymbol{Z}/n\boldsymbol{Z}$ is Armendariz for each integer n.

It follows from 2.3 that if R is a domain then R (+) R is Armendariz. This result can be extended to reduced rings. The following properties of these rings will be used: *i*) If a, b are elements of a reduced ring then $ab = 0$ if and only if $ba = 0$. ii) Reduced rings are Armendariz. iii) If R is reduced, then so is the ring $R[x]$. We shall also identify $\{R$ (+) $R\}[x]$ with the ring $R[x]$ (+) $R[x]$ in a natural manner.

2.5. Proposition. Let R be a reduced ring. Then the ring R (+) R is Armendariz.

Proof. Let $f(x) = (f_0(x), f_1(x))$, $g(x) = (g_0(x), g_1(x))$ be elements of $\{R$ (+) $R\}[x]$ satisfying $f(x)g(x) = 0$.

Write $f(x) = \sum_{i=0}^{m} (a_i, u_i)x^i$, and $g(x) = \sum_{j=0}^{n} (b_j, v_j)x^j$, with corresponding representations for $f_k(x)$, $g_k(x)$ (for $k = 0,1$).

Now we have

(A) $f_0(x)g_0(x) = 0$.

(B) $f_0(x)g_1(x) + f_1(x)g_0(x) = 0$.

Since $R[x]$ is reduced, (A) implies

(C) $g_0(x)f_0(x) = 0$.

Multiplying equation (B) by $g_0(x)$ on the left and using (C) we get $g_0(x)f_1(x)g_0(x) = 0$. This implies $(f_1(x)g_0(x))^2 = 0$ and so (since $R[x]$ is reduced)

(D) $f_1(x)g_0(x) = 0$.

This implies (on account of (B)) that

(E) $f_0(x)g_1(x) = 0$.

Now (A), (D) and (E) yield (since R is Armendariz)

$a_ib_j = 0$, $a_iv_j = 0$ and $u_ib_j = 0$ for each i and j. It follows that

$(a_i, u_i)(b_j, v_j) = (a_ib_j, a_iv_j + u_ib_j) = 0$ for each i and j.

The following generalisation of 2.5 has a similar proof.

2.6. Proposition. Let R be a reduced ring and A an ideal of R such that R/A is reduced. Then R (+) R/A is Armendariz.

2.7. Remark. Recall that a ring R is *strongly regular* ([3], §4) if for each element a in R, there exists an element b in R such that $a = a^2b$. A ring is strongly regular. if and only if it is (von Neumann) regular and reduced. If R is a strongly regular ring, then for each ideal A of R R/A is strongly regular and reduced. On applying 2.6 we get the following result: if R is a strongly regular ring, then for each ideal A of R, the ring R (+) R/A is Armendariz.

We conclude this section with a few more examples of Armendariz rings.

2.8. Proposition. Let K be a field, $h: K \rightarrow K$ a field monomorphism, and V a K-vector space. Then the ring $K\,(+)_h\,V$ is Armendariz.

Proof. The map h induces a natural ring homomorphism $h: K[x] \rightarrow K[x]$. We have the torsion free "polynomial module" $V[x]$ over $K[x]$. We identify $\{K\,(+)_h\,V\}[x]$ with $K[x]$

$(+)_h V[x]$. (See 1.3 for definitions).

Now let $f(x), g(x) \in \{K (+)_h V\}[x]$ satisfy $f(x)g(x) = 0$. Write $f(x)$ and $g(x)$ as $f(x) = (f_0(x), f_1(x))$ and $g(x) = (g_0(x), g_1(x))$, where $f_0(x), g_0(x) \in K[x]$ and $f_1(x), g_1(x)$ belong to the polynomial module $V[x]$.

Then $f(x)g(x) = 0 \Rightarrow (f_0(x), f_1(x))(g_0(x), g_1(x)) = 0$

$\Rightarrow (f_0(x)g_0(x), h(f_0(x))g_i(x) + g_0(x)f_i(x)) = 0$

$$\Rightarrow \begin{cases} f_0(x)g_0(x) = 0 \text{ and} \\ h(f_0(x))g_1(x) + g_0(x)f_1(x) = 0. \end{cases}$$

Since the cases $f(x) = 0$ or $g(x) = 0$ are trivial, we look at other cases.

Case 1. $f_0(x) = 0$ but $f_1(x) \neq 0$. Then $h(f_0(x)) = 0 \Rightarrow g_0(x)f_0(x) = 0$ which gives $g_0(x) = 0$ since $V[x]$ is $K[x]$-torsion free.

Case 2. $g_0(x) = 0$ but $g_1(x) \neq 0$. Then $h(f_0(x))g_1(x) = 0$. This implies that $h(f_0(x)) = 0$ by an argument similar to that in Case 1. Since h is a one-one map it follows that $f_0(x) = 0$. Therefore in either of the cases $f(x), g(x)$ must be of the types $f(x) = (0, f_1(x)), g(x) = (0, g_1(x))$. If follows that $K (+)_h V$ is Armendariz.

2.9. Corollary. If K is a field and V a K-vector space, then $K (+) V$ is a commutative Armendariz ring which is not reduced if $V \neq 0$.

Proof. Let h be the identity map in Proposition 2.8.

3. Rings which do not have the Armendariz property. In this section we shall give a few examples of rings which are not Armendariz.

3.1. Remark. Full matrix rings of degree ≥ 2 over any ring with identify are non-Armendariz. Consider the polynomials $f(x) = E_{12}x + E_{11}$, $g(x) = E_{11}x - E_{21}$. Then $f(x)g(x) = 0$ but $E_{11}E_{11} = E_{11} \neq 0$.

3.2. Example. Commutative rings need not be Armendariz. Consider the polynomial $f(x) = (\bar{4}, \bar{0}) + (\bar{4}, \bar{1})x$ over the ring $(\mathbf{Z}/8\mathbf{Z} (+) \mathbf{Z}/8\mathbf{Z})$. The square of this polynomial is zero but the product $(\bar{4}, \bar{0})(\bar{4}, \bar{1}) = (\bar{0}, \bar{4})$ is not zero.

3.3. Remark. The ring considered in 3.2 is a factor ring of an Armendariz ring, namely the ring of polynomials in many variables over $\mathbf{Z}$. It is also a factor ring of $\mathbf{Z} (+) \mathbf{Z}/8\mathbf{Z}$ which is Armendariz by 2.4. Thus factor rings of Armendariz rings need not be Armendariz.

4. Other classes of rings. In this section we shall record a few results which connect Armendariz rings to some other classes of rings. We introduce the following definition.

4.1. Definition. A ring R is *a left McCoy ring* if whenever $g(x)$ is a right zero-divisro in $R[x]$ there exists a non-zero element c in R such that $cg(x) = 0$. Right McCoy rings are defined dually. A ring is *a McCoy ring* if it is both left as well as right McCoy.

4.2. Remark. It was proved by McCoy [5] that commutative rings have the above property; for an inductive proof of this result see [7]; see also [2]. If T is a ring with identity, the matrix ring $M_2(T)$ is neither left nor right McCoy. (There do not exist nonzero matrices C, D satisfying $Cg(x) = 0$ and $f(x)D = 0$ for the polynomials considered in Remark 3.1.)

4.3. Remark. Let R be an Armendariz ring and assume that $g(x)$ is a right zero-divisor in $R[x]$. Then there exists a non-zero polynomial $f(x) \in R[x]$ such that $f(x)g(x) = 0$. Since R is Armendariz, $a_ib_j = 0$ for each i and j. Since $f(x) \neq 0$, $a_t \neq 0$ for some t; clearly $a_tg(x) = 0$. Thus R is left (similarly right) McCoy. This shows that Armendariz rings are McCoy. The converse is not true: commutative rings are McCoy, as noted in 4.2, but we have examples of commutative non-Armendariz rings.

4.4. Definition ([3], §4). A ring R is *semi-commutative* if it satisfies the following condition: whenever elements a, b in R satisfy $ab = 0$, then $acb = 0$ for each element c of R.

4.5. Remarks and questions. The class of commutative rings and the class of reduced rings are contained in the class of semi-commutative rings. Both these (smaller) classes are trivially stable under the formation of polynomial rings.

A ring R is called *normal* if every idempotent in R is central: semi-commutative rings are normal ([3], Lemma 5). Against this background consider the following "stability" assertions:

(i) R normal $\Rightarrow R[x]$ normal;

(ii) R semi-commutative $\Rightarrow R[x]$ semi-commutative; and

(iii) R Armendariz $\Rightarrow R[x]$ Armendariz.

We remark that (i) easily follows from an extension of ([1], Corollary 1) to normal rings. (It may be a known result but we have not seen a proof of (i) in the literature).

We do not know whether (ii) and (iii) are true. In view of these questions, the following proposition may be of some interest.

4.6. Proposition. If R is a semi-commutative ring which is Armendariz, then $R[x]$ is semi-commutative.

Proof. Let $f(x)$, $g(x)$ be polynomials in $R[x]$ satisfying $f(x)g(x)=0$. Let $h(x)=\sum_{k=0}^{l} c_k x^k \in R[x]$. Since R is Armenderiz and $f(x)\,g(x)=0$, $a_i b_j=0$ for each i and j. Since R is semi-commutative $a_i c_k b_j=0$ for each i, j and k. Hence $f(x)h(x)g(x)=0$. This proves that $R[x]$ is semi-commutative.

4.7. Remark. The concepts introduced and studied in this note have extensions in the context of modules, graded rings and graded modules. Related concepts can also be defined for power series rings. These generalisations will be carried out elsewhere.

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An Extension of Sturm's Theorem to Two Dimensions

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1. Introduction and notation. Let $f(x, y) \in \boldsymbol{R}[x, y]$ be a square free polynomial with real coefficients, namely $f(x, y)$ is decomposed into the irreducible factors whose multiplicities are only one. Let C be the set of points $(x, y) \in \boldsymbol{R}^2$ such that $f(x, y) = 0$. Until now, only the following primitive method has been used to draw the curve C by computer, within a given rectangle R. We decompose R into many small rectangles D and obtain $C \cap R$ by gathering $C \cap D$. $C \cap D$ is found as follows.

Let D be the set $\{(x, y) \in \boldsymbol{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$, and put $P_1 = (a, c)$, $P_2 = (b, c)$, $P_3 = (b, d)$ and $P_4 = (a, d)$. For example, if $f(P_1)f(P_2) < 0$, $f(P_3)f(P_4) < 0$ then we can find approximately a point P_5 in $C \cap \overline{P_1P_2}$ and a point P_6 in $C \cap \overline{P_3P_4}$. Then the line $\overline{P_5P_6}$ can be considered approximately as $C \cap D$.

But the above method has next two problems.

(1) Even if $f(P_1)f(P_2) > 0$, it is possible that $C \cap \overline{P_1P_2} \neq \emptyset$.

(2) Even if $C \cap$ (the boundary of D) $= \emptyset$, it is possible that $C \cap$ (the interior of D) $\neq \emptyset$.

In this paper, we would like to propose a more reliable method which permit us to liberate from these incertainties.

Let ∂D be the boundary of D and D^i be the interior of D. Then $C \cap D$ is the direct union of $C \cap \partial D$ and $C \cap D^i$. The search for $C \cap D$ is made separately in two cases: the first case for $C \cap \partial D$ and the second case for $C \cap D^i$.

2. First case. This case can be treated as the equation $f = 0$ is restricted to a boundary line. Then we can use Sturm's theorem.

The Sturm sequence associated with the (one-variable) polynomial $f(x)$ is a sequence of polynomials with $f_0(x), f_1(x), \ldots, f_k(x)$ defined by the following equations:

$$f_0(x) = f(x), f_1(x) = f'(x),$$
$$f_i(x) = -\text{ remainder } (f_{i-2}(x), f_{i-1}(x))$$

where remainder means the remainder from the division of the former by the latter.

Let $(a_1, \ldots, a_s)$ be a sequence of real numbers and $(a'_1, \ldots, a'_t)$ be the subsequence of all non-zero numbers. Then $var(a_1, \ldots, a_s)$, the number of sign variations, is the number of i, $1 \leq i < t$, such that $a'_i a'_{i+1} < 0$.

Theorem (Sturm). *Let $f(x)$ be a square free polynomial. When $gcd(f(x), f'(x)) = f_k(x)$, the number of real roots of $f(x)$ in the interval $a < x \leq b$ is*

$$var(f_0(a), f_1(a), \cdots, f_k(a)) -$$
$$var(f_0(b), f_1(b), \cdots, f_k(b)).$$

Let D be the set $\{(x, y) \in \boldsymbol{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$. Using Sturm's theorem we can determine whether $f(x, c) = 0$ has a root in the interval $[a, b]$ of not. Thus we can determine whether $C \cap \partial D \neq \emptyset$ or not, and if $C \cap \partial D \neq \emptyset$, find this set approximately in considering from divisions of ∂D.

3. Second case. When $C \cap \partial D = \emptyset$ then we can find $C \cap D^i$ in the following manner.

If $C \cap D^i \neq \emptyset$, then there is a point (x_0, y_0) such that $(x_0, y_0) \in C \cap D^i$, but if $(x, y) \in C \cap D^i$, then $y \leq y_0$. Such a point (x_0, y_0) will be called a *maximal point* (of $C \cap D^i$ with respect to y). We write $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ and show $f_x(x_0, y_0) = 0$ for a maximal point (x_0, y_0). If $f_x(x_0, y_0) \neq 0$ then using implicit function theorem, there exists a function $g(y)$ near y_0 such that $f(g(y), y) = 0$ and (x_0, y_0) cannot be a maximal point. Therefore we have $f_x(x_0, y_0) = 0$.

As $f(x, y)$ is square free, we have $gcd(f(x, y), f_x(x, y)) = 1$ in $\boldsymbol{R}(y)[x]$. Using Euclidean algorithm we can find $g(x, y), h(x, y) \in \boldsymbol{R}[x, y]$, $F(y) \in \boldsymbol{R}[y]$ such that

(3) $f(x, y)g(x, y) + f_x(x, y)h(x, y) = F(y)$

If $f(x, y) = 0$, $f_x(x, y) = 0$, then $F(y)$ must be zero. Using Sturm's theorem, we can count correctly the number of roots $F(y) = 0$ in the interval $[c, d]$ and we can calculate approximately all roots in this interval. Therefore we can calculate