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Some aspects of Kaluza–Klein cosmology

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Abstract. This article contains a brief account of Kaluza–Klein theory and its applications in cosmology from the very beginning.

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1. Motivation and historical development of Kaluza–Klein theory

There exist four fundamental forces in the Nature. These are gravitation, electromagnetism, weak and strong interactions. Unification of all these forces has been the ultimate goal of a theoretical physicist. Firstly, the unification of electricity and magnetism was obtained by Maxwell in his theory of electromagnetism in 1861–62. After the advent of special theory of relativity, three-space and time were unified into a four-dimensional space-time continuum by Minkowski. Unification of electromagnetism and weak interaction was obtained by Glashow, Weinberg and Salam. In 1961, Glashow [1] unified the weak and electromagnetic interactions using the gauge group $SU(2) \otimes U(1)$. Later on, Weinberg and Salam showed independently that weak gauge bosons can acquire mass without any loss of renormalizability [2]. This theory got experimental support in 1973 followed by discovery of the weak gauge bosons in 1983. Unification of strong, weak and electromagnetic interactions had been discussed by Georgi and Glashow, in 1974, based on the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ in the standard model of grand unified theories [3]. Various other gauge groups also have been used to obtain a satisfactory version of the grand unified theory.

In all these attempts, gravity (the oldest known interaction of the Nature) has been kept quite away, which is the source of dissatisfaction for a theoretical physicist. In 1914, Nordström [4] proposed to unify Maxwell's theory with his theory of gravity induced by scalar fields in a very imaginative way in the framework of 5-dimensional theory. In 1919, Kaluza proposed Einstein type theory of gravity in 5-dimensional space-time where 4-dimensional Einstein's gravity and electromagnetism are obtained using the cylindrical condition which implies independence of components of metric tensor from extra dimensional coordinates. In the beginning, Einstein did not like Kaluza's idea, but eventually he supported it and this work was published in 1921 [5]. In 1926, Klein found that Kaluza's results could be obtained using quantum mechanics also [6]. For a long time, Kaluza–Klein theory was almost idle barring few papers here and there. In the middle of 1970s, string theory [7] and supergravity [8] appeared in the arena of field theory with high hopes to unify gravity and other fundamental forces. It was found that the string theory also leads to a higher-dimensional geometry like Kaluza–Klein theory. So the importance of a higher-dimensional theory was realized in true sense. In this context, Kaluza–Klein theory became very much popular by the end of 1970s. In 1978, Cremmer, Julia and Scherk proposed 11-dimensional theory which could yield gravity and the standard model of grand unified theory using Kaluza's mechanism [9]. This model of 11-dimensional theory is called $N = 1$ supergravity. After this work, people started to think that the only way to do supergravity is via Kaluza–Klein theory. Later on, 10-dimensional theory for supergravity was also proposed to cure some problems of 11-dimensional supergravity , but it contained some other degrees of freedom of Yang–Mills multiplets in contrast to 11–dimensional theory. Currently, this theory has been abandoned, but Kaluza–Klein is still interesting in the context of the early universe at high energy level where gravity is supposed to be unified with all other interactions. In what follows, a brief account of latest developments of the theory and mathematical techniques are discussed.

2. Compactification

Our observable universe is 4-dimensional. So it seems difficult to accept the idea of higher dimensional geometry on a physical ground. A solution to this problem, namely 'spontaneous compactification', added a new dimension to the Kaluza–Klein theory. According to this idea, the higher dimensional geometry is supposed to be described as a product manifold $M^4 \otimes M^K$, where M^4 is the usual 4-dimensional space-time and M^K is a compact manifold [10–12]. At extremely high energy, the universe is supposed to be $(4 + K)$-dimensional, but below a certain energy level extra-dimensional manifold M^K gets compactified to extremely small size to be resolved. As a result, only M^4 is observed. Planck length is supposed to be a possible scale of compactification. The compact manifolds are homeomorphic (topologically equivalent) to a connected sum of tori or a sphere or a projective plane [13]. Physically, it means that M^K can be conveniently defôrmed to one of these manifolds. Thus, for all physical purposes, a compact manifold can be treated as a connected sum of tori or a sphere or a projective plane.

3.

(a) *Kaluza–Klein model of electromagnetism*

Kaluza had shown that electromagnetism could be obtained from 5-dimensional general relativity. For this purpose, he used the method of weak-field approximation. Later on, Klein showed that Kaluza's result could be obtained, in general also, without this approximation.

In Kaluza–Klein's theory, the manifold is taken as product of M^4 and a circle S^1. Circle is a compact manifold with initial and final points identified. So a field $f(x, y)$

(here x represents four coordinates on M^4 and y is a coordinate on S^1) satisfies the condition [2]

$$f(x,y) = f(x, y + 2\pi r), \tag{3.1}$$

where r is the radius of the circle. Fourier expansion for $f(x,y)$ can be written as

$$f(x,y) = \sum_{n=-\infty}^{\infty} f_n(x)\, \mathrm{e}^{in(y/r)},$$

which satisfies the condition given by (3.1). Using the quantum operator $-i\hbar(\partial/\partial y)$ on $f(x,y)$, eigenvalue of the momentum operator corresponding to y coordinate is obtained as $(n\hbar/r)$ for a particular n. So y component of momentum will be very high for sufficiently small r which means that only $n = 0$ mode can appear in the low energy physics. Due to this reason, Klein took all observed states independent of y which is analogous to Kaluza's cylindrical condition.

In Kaluza's model, the 5-dimensional line-element is taken as [14]

$$\mathrm{d}S^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu - \phi^2(x)[\kappa A_\mu\,\mathrm{d}x^\mu + \mathrm{d}y][\kappa A_\nu\,\mathrm{d}x^\nu + \mathrm{d}y]. \tag{3.2}$$

Here $g_{\mu\nu}$ are components of the metric tensor on M^4 with $g_{\mu 4} = g_{4\mu} = -\phi^2\kappa A_\mu$ and $g_{44} = -\phi^2$ as well as κ is a constant of length dimension keeping κA_μ dimensionless. To discuss general relativity with this line-element is a bit difficult due to the presence of components of electromagnetic field in the metric tensor components. So, for convenience, it is better to choose basis one-forms [15]

$$\hat{\theta}^\mu = \mathrm{d}x^\mu, \tag{3.3a}$$

$$\hat{\theta}^4 = \mathrm{d}y + \kappa A_\mu\,\mathrm{d}x^\mu. \tag{3.3b}$$

This type of basis is called horizontal lift basis (HLB). In HLB, components of the metric tensor, obtained from the line-element (3.2), are $\hat{g}_{\mu\nu}, \hat{g}_{\mu 4} = g_{4\mu} = 0$ and $\hat{g}_{\mu 4} = -\phi^2$. Basis vectors dual to bases $\hat{\theta}^{\hat{\mu}}$ $(\hat{\mu} = 0,1,2,3,4)$ are given as

$$\hat{e}_\mu = \frac{\partial}{\partial x^\mu} - \kappa A_\mu \frac{\partial}{\partial y}, \tag{3.4a}$$

$$\hat{e}_4 = \frac{\partial}{\partial y}. \tag{3.4b}$$

The dual bases $\hat{e}_{\hat{\mu}}$ obey the algebra

$$[\hat{e}_{\hat{\mu}}, \hat{e}_{\hat{\nu}}] = \hat{C}^{\hat{\lambda}}_{\hat{\mu}\hat{\nu}} \hat{e}_{\hat{\lambda}}, \tag{3.5}$$

where $[\hat{e}_{\hat{\mu}}, \hat{e}_{\hat{\nu}}]$ are commutators of $\hat{e}_{\hat{\mu}}$ and $\hat{e}_{\hat{\nu}}$. Using the definition of $\hat{e}_{\hat{\mu}}$, given by (3.4), one finds that structure constants

$$C^4_{\mu\nu} = C_{\mu\nu 4} = -\kappa F_{\mu\nu} \tag{3.6}$$

and all other structure constants vanish. It shows that bases $\hat{e}_{\hat{\mu}}$ are anholonomic [15] (all structure constants vanish for holonomic bases). The connection coefficients in anholonomic basis is defined as

$$\hat{\Gamma}^{\hat{\mu}}_{\hat{\nu}\hat{\lambda}} = \tfrac{1}{2}\hat{g}^{\hat{\mu}\hat{\alpha}}[\hat{e}_{\hat{\lambda}}(\hat{g}_{\hat{\alpha}\hat{\nu}}) + \hat{e}_{\hat{\nu}}(\hat{g}_{\hat{\alpha}\hat{\lambda}}) - \hat{e}_{\hat{\alpha}}(\hat{g}_{\hat{\lambda}\hat{\nu}}) + \hat{C}_{\hat{\alpha}\hat{\nu}\hat{\lambda}} + \hat{C}_{\hat{\alpha}\hat{\lambda}\hat{\nu}} - \hat{C}_{\hat{\lambda}\hat{\nu}\hat{\alpha}}]. \tag{3.7}$$

So non-vanishing connection coefficients for $\hat{\Gamma}^{\hat{\lambda}}_{\hat{\mu}\hat{\nu}}$ are

$$\hat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda}, \tag{3.8a}$$

$$\hat{\Gamma}^{\mu}_{\nu 4} = \hat{\Gamma}^{\mu}_{4\nu} = -\tfrac{1}{2}\kappa F^{\mu}_{\nu}, \tag{3.8b}$$

and

$$\hat{\Gamma}^{4}_{\nu\lambda} = -\hat{\Gamma}^{4}_{\lambda\nu} = \tfrac{1}{2}\kappa F_{\lambda\nu}. \tag{3.8c}$$

Components of the Riemann tensor are given as

$$\hat{R}^{\hat{\mu}}_{\hat{\nu}\hat{\alpha}\hat{\beta}} = \hat{e}_{\hat{\alpha}}(\hat{\Gamma}^{\hat{\mu}}_{\hat{\nu}\hat{\beta}}) - \hat{e}_{\hat{\beta}}(\hat{\Gamma}^{\hat{\mu}}_{\hat{\nu}\hat{\alpha}}) + \hat{\Gamma}^{\hat{\mu}}_{\hat{\sigma}\hat{\alpha}}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\nu}\hat{\beta}} - \hat{\Gamma}^{\hat{\mu}}_{\hat{\sigma}\hat{\beta}}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\nu}\hat{\alpha}} - \hat{\Gamma}^{\hat{\mu}}_{\hat{\nu}\hat{\sigma}}\hat{C}^{\hat{\sigma}}_{\hat{\alpha}\hat{\beta}} \tag{3.9}$$

with $\hat{R}_{\hat{\nu}\hat{\beta}} = \hat{R}^{\hat{\alpha}}_{\hat{\nu}\hat{\alpha}\hat{\beta}}$ and $\hat{R} = \hat{g}^{\hat{\nu}\hat{\beta}}\hat{R}_{\hat{\nu}\hat{\beta}}$.

Ricci scalar for the line-element, given by (3.2) is calculated to be

$$\hat{R} = R - \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} - \frac{2}{3\phi^2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}, \tag{3.10}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The volume element is

$$\sqrt{|\hat{g}|}\,\hat{\theta}^0 \wedge \hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \hat{\theta}^3 \wedge \hat{\theta}^4 = \sqrt{|\hat{g}|}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}y$$

as $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$.

The Einstein–Hilbert action in 5-dimensional Einstein's theory is written as

$$S = -\frac{1}{16\pi G_5}\int \mathrm{d}^4x\,\mathrm{d}y\phi\sqrt{|g|}\left[R - \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} - \frac{2}{3\phi^2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}\right]. \tag{3.11}$$

If the radius of the compact component of $M^4 \bigotimes S^1$ is r,

$$0 \le y \le 2\pi r. \tag{3.12}$$

Thus,

$$S = -\frac{1}{16\pi G_4}\int \mathrm{d}^4x\,\phi\sqrt{|g|}\left[R - \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} - \frac{2}{3\phi^2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}\right], \tag{3.13}$$

where $G_4 = G_5/2\pi r$ is the 4-dimensional gravitational constant. In the original work of Kaluza, $\phi(x)$ is taken as a constant.

(b) *Non-abelian Kaluza–Klein theories*

In 1954, Yang and Mills generalized local $U(1)$ invariance (which is an abelian theory) to local non-abelian theories. Around the same time, Shaw also did this kind of work but his contribution was not found interesting by physicists. The Yang–Mills theory was almost idle for a long span of time. In the early 1970s, this theory blossomed in the context of gauge theories, as gauge theorists found it a very convenient mathematical tool. In 1963, De-Witt proposed unification of Yang–Mills fields with gravitation under Kaluza–Klein scheme in a higher dimensional geometry [16]. A detailed discussion of this idea appeared in the work of Kerner [17]. $(4+N)$-dimensional Einstein–Hilbert action as well as derivation of gravitation, Yang–Mills fields and scalar field was discussed by Cho and Freund in 1975 [18].

In 1982, Salam and Strathdee raised a very important question 'What are the implications of extra dimensions in our observable universe?' In [19], they have discussed that through expansion of physical fields in normal modes associated symmetry groups in Kaluza–Klein theory make their appearance as massive multiplets and through components of the metric tensor these groups yield non-abelian fields. In what follows, a brief discussion on these aspects is given.

In the $(4+N)$-dimensional theory, coordinates are separated into $x^\mu (\mu = 0,1,2,3)$ of usual space-time and y^n of N-dimensional compact manifold. The components of the metric tensor are given as [20]

$$\begin{aligned} \bar{g}_{\mu\nu}(x,y) &= g_{\mu\nu}(x) + \tilde{g}_{mn}\,\xi^n_\alpha(y)\xi^m_\beta(y)A^\alpha_\mu(x)A^\beta_\nu(x), \\ \bar{g}_{\mu n}(x,y) &= \bar{g}_{n\mu}(x,y) = -\tilde{g}_{mn}\,\xi^m_\alpha(y)A^\alpha_\mu(x), \\ \tilde{g}_{mn}(x,y) &= \tilde{g}_{mn}(y), \end{aligned} \tag{3.14}$$

where $\xi^n_\alpha(y)$ are Killing vectors satisfying the algebra

$$\xi^m_\beta \partial_m \xi^n_\gamma - \xi^m_\gamma \partial_m \xi^n_\beta = f^\alpha_{\beta\gamma}\,\xi^n_\alpha \tag{3.15}$$

with $f^\alpha_{\beta\gamma}$ as structure constants.

The Einstein–Hilbert action in $(4+N)$-dimensional theory is written as

$$S_{(4+N)g} = -\frac{1}{16\pi G_{(4+N)}} \int \mathrm{d}^4x\,\mathrm{d}^ny \sqrt{|g_{(4+N)}|}\,R_{(4+N)} \tag{3.16a}$$

with

$$R_{(4+N)} = R_4 + R_N(y) + \tfrac{1}{4}\tilde{g}_{mn}\xi^n_\alpha(y)\xi^m_\beta(y)F^\alpha_{\mu\nu}(x)F^{\beta\mu\nu} \tag{3.16b}$$

which can be calculated in HLB as earlier. In 5-dimensional case, $N = 1$ and the compact manifold is S^1 for which Ricci scalar vanishes. In (3.16b),

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + f^\alpha_{\beta\gamma}A^\beta_\mu A^\gamma_\nu. \tag{3.17}$$

Here α, β and γ are gauge indices for which the degree of freedom is equal to the number of Killing vectors. The number of Killing vectors depend upon the Lie group on the compact manifold. For example, if the compact manifold is S^N, the symmetry group on S^N is $SO(N+1)$. The number of generators of $SO(N+1)$ is $[N(N+1)]/2$, which is the number of Killing vectors on S^N. Also the maximum number of possible Killing vectors on N-dimensional manifold is $[N(N+1)]/2$. But it is not possible to get the maximum number of Killing vectors for every compact N-dimensional manifold, because the number of Killing vectors depend on the symmetry group. The symmetry group on N-dimensional torus is $[U(1)]^N$. So possible number of Killing vectors is N, which is the number of generators on T^N. Thus T^N has less than maximum number of Killing vectors, whereas S^N accommodates maximum number of Killing vectors on a N-dimensional manifold. So S^N is maximally symmetric, but T^N is not.

4. Physical fields on $M^4 \bigotimes M^K$

Here we will consider compact manifolds T^K and S^K. Calculations on projective planes can be done in a similar manner.

(i) When $M^K = T^K$: T^K is K-times product of circles with different radii $r_1, r_2, \ldots, r_k$. So Fourier expansion of a physical field $\phi(x,y)$ can be written as

$$\phi(x,y) = \sum_{n_1\cdots n_k=-\infty}^{\infty} \phi_{n_1\cdots n_k}(x) \exp\left[i\left(\frac{n_1 y_1}{r_1} + \cdots + \frac{n_k y_k}{r_k}\right)\right] \tag{4.1}$$

and the line-element looks like

$$\mathrm{d}S^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu - (r_1^2\,\mathrm{d}\theta_1^2 + \cdots + r_k^2\,\mathrm{d}\theta_k^2), \tag{4.2}$$

where $0 \leq \theta_1, \ldots, \theta_k \leq 2\pi$. Even other boson fields can be treated in a similar way.

(ii) *When $M^K = S^K$*: In this case, harmonics are a bit complicated. A massless scalar field ϕ obeys the Klein–Gordon equation [21]

$$\Box_{(4+K)}\phi = \frac{1}{\sqrt{|g_{(4+K)}|}}\frac{\partial}{\partial x^M}\left(\sqrt{|g_{(4+K)}|}g^{MN}\frac{\partial}{\partial x^N}\right)\phi = 0, \tag{4.3}$$

where g_{MN} are components of the metric tensor on $M^4 \bigotimes M^K$. Equation (4.3) can be written as

$$(\Box_4 + \Box_K)\phi = 0, \tag{4.4}$$

where $\Box_4$ is defined on M^4 and $\Box_K$ on M^K.

If λ is the eigenvalue of the operator $\Box_K$ for the scalar field ϕ,

$$\Box_K\phi = \lambda\phi. \tag{4.5}$$

To obtain λ, the set of all homogeneous harmonic polynomials $\hat{\phi}$ of degree l in Euclidean $(K+1)$-space are considered. A homogeneous polynomial of degree l can be written as

$$\hat{\phi}(x_1, \ldots, x_{(K+1)}) = c_{\alpha_1,\ldots,\alpha_l} x^{\alpha_1,\ldots,\alpha_l}, \tag{4.6}$$

where $c_{\alpha_1,\ldots,\alpha_l}$ are symmetric with each l taking $(K+1)$ values. Thus linearly independent $c_{\alpha_1,\ldots,\alpha_l}$ are ${}^{(N+l)}C_l$ in number. If $\hat{\phi}$ is a harmonic on $(K+1)$-dimensional Euclidean space

$$\nabla^2_{(K+1)}\hat{\phi} = 0, \tag{4.7a}$$

which yields

$$\delta^{\alpha_1\alpha_2} c_{\alpha_1,\alpha_2,\ldots,\alpha_l} = 0. \tag{4.7b}$$

The number of equations (4.7b) is ${}^{(N+l-2)}C_{(l-2)}$.

The $(K+1)$-dimensional Euclidean space is described by the line-element

$$\mathrm{d}S^2_{(K+1)} = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x^2_{(K+1)}. \tag{4.8}$$

The sphere S^K is embedded in $(K+1)$-dimensional Euclidean space. For the purpose of calculation of λ, it is convenient to write (4.8) in polar coordinates as

$$\mathrm{d}S^2_{(K+1)} = \mathrm{d}r^2 + r^2 k_{ij}\,\mathrm{d}\theta^i \mathrm{d}\theta^j, \tag{4.9}$$

where $0 \le \theta_1, \ldots, \theta_{(K-1)} \le \pi$ and $0 \le \theta_K \le 2\pi$. Now (4.7a) can be written as

$$\nabla^2_{(K+1)}\hat{\phi} = \left(r^{-K}\frac{\partial}{\partial r}\left(r^K\frac{\partial}{\partial r}\right) + r^{-2}\Box_K\right)\hat{\phi} = 0. \tag{4.10}$$

As $\hat{\phi}$ is homogeneous,

$$\hat{\phi} = r^l\phi. \tag{4.11}$$

So, from (4.10),

$$\Box_K\phi = -l(l+K-1)\phi. \tag{4.12}$$

Thus $\lambda = -l(l+K-1)$ with degeneracy,

$$d_S(K,l) = {}^{(K+l)}C_l - {}^{(K+l-2)}C_{(l-2)}, \tag{4.13}$$

where ${}^nC_r \equiv n!/(r!(n-r)!)$.

If ϕ is replaced by components of a vector, λ can be calculated as

$$\lambda = -[l^2 + (l-1)(K-1)], \quad l = 1, 2, \ldots, \tag{4.14}$$

with degeneracy

$$d_v(K,l) = (K+1)[{}^{(K+l)}C_l - {}^{(K+l)}C_{(l-2)}] - {}^{(K+l+1)}C_{(l+1)} + {}^{(K+l-3)}C_{(l-3)}. \tag{4.15}$$

If ϕ is replaced by divergenceless traceless rank-2 tensor components,

$$\lambda = -[l^2 + (K-1)(l-2)], \quad l = 2, 3, \ldots, \tag{4.16}$$

with degeneracy

$$\begin{aligned} d_T(K,l) = {}& \tfrac{1}{2}(K+1)(K+2)[{}^{(K+l)}C_l - {}^{(K+l-2)}C_{(l-2)}] \\ & - (K+1)[{}^{(K+l+1)}C_{(l+1)} - {}^{(K+l-3)}C_{(l-3)}]. \end{aligned} \tag{4.17}$$

If $T_{ij} = k_{ij}T$, λ can be calculated like scalars. The traceless tensor is of the form

$$Q_{ij}(z) = \nabla_i z_j + \nabla_j z_i - \frac{2}{K}k_{ij}(\nabla^l z_l). \tag{4.18}$$

Now two cases arise here:

(i) In case $\nabla^l z_l = 0, \lambda = -[l^2 + (K-1)l - (K+2)], l = 2, 3, \ldots,$ with degeneracy $d_v(K,l)$ given by eq. (4.15).

(ii) In case $z = \nabla\chi$ (χ is a scalar), $\lambda = -[l^2 + (N-1)l - 2N]$, $l = 2, 3, \ldots,$ with degeneracy $d_S(K,l)$ given by (4.13).

A spinor field ψ satisfies the higher-dimensional Dirac equation [22, 23]

$$[i\Gamma^M \nabla_M - m]\psi = 0, \tag{4.19}$$

where Γ^M are Dirac matrices satisfying the condition

$$\{\Gamma^M, \Gamma^N\} = 2g^{MN}. \tag{4.20}$$

Γ^M in (4.20) can be decomposed as (γ^μ, γ^m). Here

$$\gamma^\mu = e^\mu_a \tilde{\gamma}^a, \quad a = 0, 1, 2, 3 \tag{4.21a}$$

and

$$\gamma^m = e^m_{\acute{m}} \tilde{\gamma}^{\acute{m}}, \quad \acute{m} = 4, 5, \dots, 3 + K. \tag{4.21b}$$

Here $\tilde{\Gamma}^{\acute{M}} \equiv (\tilde{\gamma}^a, \tilde{\gamma}^{\acute{m}})$ are $(4 + K)$-dimensional Dirac matrices in flat space-time. In (4.21)

$$\tilde{\gamma}^a = *\gamma^a \otimes I, \tag{4.22a}$$

$$\tilde{\gamma}^{\acute{m}} = *\gamma^5 \otimes *\gamma^{\acute{m}}, \tag{4.22b}$$

where $*\gamma^a$ are standard 4×4 Dirac matrices and $*\gamma^{\acute{m}}$ are Dirac matrices on $M^K \cdot *\gamma^{\acute{m}}$ and I (identity matrix) are $2^{K/2} \times 2^{K/2}$ matrices (when K is even) and $2^{(K-1)/2} \times 2^{(K-1)/2}$ matrices (when K is odd). $*\gamma^5$ in (4.22b) is given as

$$*\gamma^5 = i*\gamma^0 *\gamma^1 *\gamma^2 *\gamma^3 \tag{4.23a}$$

and

$$(*\gamma^5)^2 = I. \tag{4.23b}$$

A spinor field $\psi(x, y)$ can be expanded as

$$\psi(x, y) = \sum_i \psi_i(x) \chi_i(y), \tag{4.24}$$

where $\chi_i(y)$ satisfy the equation

$$i\gamma^m \nabla_m \chi_i = -m_i *\gamma^5 \chi_i. \tag{4.25}$$

Equations (4.19) and (4.25) yield

$$[i\gamma^\mu \nabla_\mu - (m + m_i *\gamma^5)]\psi_i(x) = 0. \tag{4.26}$$

The Lagrangian density corresponding to the Dirac equation is given by (4.26)

$$L = \bar{\psi}_i(x)[i\gamma^\mu \nabla_\mu - (m + m_i *\gamma^5)]\psi_i(x). \tag{4.27}$$

For convenience, chiral transformations are used for $\psi_i(x)$, which is given as

$$\psi_i(x) \to \psi_i'(x) = \exp[i\alpha *\gamma^5]\psi_i(x), \tag{4.28}$$

where $\alpha = \arctan(m_i/m)$. Under these transformations

$$\bar{\psi}_i(x) i\gamma^\mu \nabla_\mu \psi_i(x)$$

remains invariant and the mass term acquires the canonical form

$$\bar{\psi}'_i(x)(m^2 + m_i^2)^{1/2}\psi'_i(x).$$

(For details of calculations, see Appendix A).

5. A realistic Kaluza–Klein model

Weak interaction and electromagnetism were unified under the symmetry group $SU(2)\otimes U(1)$. Later on, strong interaction was also brought under this scheme through the symmetry group $SU(3) \otimes SU(2) \otimes U(1)$. Popular name of this model is the standard model. With the recent discovery of top quark (truth quark) the standard model has become more interesting. In 1981, Witten suggested unification of gravity with the standard model under Kaluza–Klein scheme [24]. In this scheme, $SU(3) \otimes SU(2) \otimes U(1)$ is the symmetry group on a compact manifold. The space of lowest dimension with symmetry group $SU(3)$ is CP^2 (complex projective space). For details of CP^2, one can refer to Appendix B. S^2 is the lowest dimensional manifold with $SU(2)$ symmetry and S^1 is the group manifold for $U(1)$. Thus the compact manifold with $SU(3) \otimes SU(2) \otimes U(1)$ can be $CP^2 \otimes S^2 \otimes S^1$ which is a 7-dimensional real space. $CP^2 \otimes S^2 \otimes S^1$ is not a unique manifold with symmetry group of the standard model, but it is more convenient.

6. Cosmological models in Kaluza–Klein theory

It has been discussed above that, under Kaluza–Klein scheme, higher-dimensional space-time is taken as product of the usual 4-dimensional paracompact manifold and a compact manifold. It is also supposed that the internal space (the compact manifold is the internal space) is too small to be observed. In the early 1980s, the validity of this idea was discussed by many authors [25–28]. In this scheme, there is a large discrepancy between scales of the external space-time (observed 4-dimensional space-time) and the internal space. Now the question arises 'How is this large discrepancy caused?' To answer this question, two kinds of attempts were made. One is the method of quantum instability of the internal space for contraction by Appelquist and Chodos [29]. They computed one-loop effective potential for a scalar field in 5-dimensional space-time and found that it tends to minus infinity when the fifth dimension shrinks to zero. Later on, Rubin and Roth [30] studied finite temperature effect in the analysis done by Appelquist and Chodos and found two kinds of instabilities. Meanwhile, Tsokos [31] had shown that inclusion of 5-dimensional fermionic degree of freedom could save the one-loop effective potential from divergence to minus infinity. The other method uses a dynamical evolution of the higher-dimensional universe as suggested by Chodos and Detweiler [32]. This method is the cosmological dynamical reduction, which was found interesting because the expansion of external space-time and contraction of the internal space were explained simultaneously. Firstly, Chodos and Detweiler (CD) proposed Kasner type 5-dimensional solution of the field equations. The line-element for this model is

$$\mathrm{d}S^2 = \mathrm{d}t^2 - t^{1/2}[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2] - t^{-1/2}\rho^2\mathrm{d}\theta^2, \tag{6.1}$$

which can be generalized as

$$dS^2 = dt^2 - \sum_{i=1}^{3} t^{2p_i}(dx^i)^2 - \sum_{j=4}^{3+D} t^{2p_j}(dy^j)^2 \tag{6.2a}$$

with

$$\sum_{i=1}^{3} p_i + \sum_{j=4}^{3+D} p_j = 1 = \sum_{i=1}^{3} p_i^2 + \sum_{j=4}^{3+D} p_j^2. \tag{6.2b, c}$$

Later on, higher dimensional Robertson–Walker type solutions were obtained. The generalized $(1+d+D)$-dimensional Robertson–Walker line-element looks like [33–35]

$$dS^2 = dt^2 - R_d^2(t) g_{mn}\, dx^m dx^n - R_D^2(t) g_{MN}\, dy^M\, dy^N, \tag{6.3}$$

where $R_d(t)$ is the scale-factor associated with the external space and $R_D(t)$ is the scale-factor associated with the internal space. The topology of such a model is $R \otimes M^d \otimes B^D, g_{mn}(m, n = 1, 2, \ldots, d)$ are components of the metric tensor on B^D, which is the internal compact manifold. The non-vanishing components of Ricci tensor for the line-element, given by (6.3), are

$$R_{00} = -\left(d\frac{\ddot{R}_d}{R_d} + D\frac{\ddot{R}_D}{R_D}\right), \tag{6.4a}$$

$$R_{mn} = \left[\frac{2k_d}{R_d^2} + \frac{d}{dt}\left(\frac{\dot{R}_d}{R_d}\right) + \frac{\dot{R}_d}{R_d}\left(d\frac{\dot{R}_d}{R_d} + D\,\frac{\dot{R}_D}{R_D}\right)\right] g_{mn}, \tag{6.4b}$$

$$R_{MN} = \left[\frac{2k_D}{R_D^2} + \frac{d}{dt}\left(\frac{\dot{R}_D}{R_D}\right) + \frac{\dot{R}_D}{R_D}\left(d\frac{\dot{R}_d}{R_d} + D\frac{\dot{R}_D}{R_D}\right)\right] g_{MN}, \tag{6.4c}$$

where k_d is the spatial curvature of M^d and k_D is the spatial curvature of B^D. Sahadev has obtained solution of Einstein's equations for R_d and R_D taking different perfect fluids. In these models, R_d increases and R_D decreases with time. These solutions suffer from a problem that $R_D(t) = 0$ at a certain time, which is not correct because it implies that internal space vanishes at this particular time. It is true that internal space is too small to be observed, but it does not vanish. This problem is called 'crack of doom' singularity. In CD model [32], it occurs as $t \to \infty$. Matzner and Mezzacappa [36] have proposed 5-dimensional model, where this problem occurs at some time during the evolution of the model. Solutions obtained by Sahadev also, are not free from this problem. Physics requires $R_D(t)$ to stabilize asymptotically. Copeland and Toms (CT) realized the necessity of this constraint, but they found that all solutions are unstable if the higher dimensional manifold is a product of the flat 4-dimensional space-time and a sphere [37]. Actually CT did not solve $(4+D)$-dimensional Einstein equations, but they solved equations derived from 4-dimensional action for gravity. Gleisser and Taylor (GT) have obtained solutions of 6-dimensional Einstein's equations satisfying the above criterion [38, 39]. Rosenbaum *et al* [40] have suggested diagramatic solutions for 5-dimensional Kaluza–Klein cosmological model and have shown that by an appropriate choice of parameters 'crack of doom' singularity can be avoided. In [41, 42], some interesting solutions were obtained where $R_D(t)$ stabilizes asymptotically. In what follows, a brief account of these models is given.

Here the topology of the $(4+D)$-dimensional space-time is $R \otimes M^3 \otimes T^D$ with the line-element [38]

$$\mathrm{d}S^2 = \mathrm{d}t^2 - R_3^2(t)\left[\frac{\mathrm{d}r^2}{1-k_3 r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right] - R_D^2(t) \\ \times \delta_{mn}(\mathrm{d}y^m - \kappa A_\mu^m \,\mathrm{d}x^\mu)(\mathrm{d}y^m - \kappa A_\nu^m \,\mathrm{d}x^{nu}), \tag{6.5}$$

where k_3 is the spatial curvature of M^3 with possible values $+1, 0, -1$ for closed, flat and open models respectively. Here κ is a constant of length dimension like (3.2). The matter field considered in this model is a scalar field ϕ. The complete action is

$$S = \int \mathrm{d}^4x\,\mathrm{d}^Dy\sqrt{|g_{(4+D)}|} \\ \times \left[-\frac{1}{16\pi G_{(4+D)}}R_{(4+D)} + \frac{1}{2}\{g^{MN}(D_M\phi)^*(D_N\phi) - m^2\phi^*\phi\}\right], \tag{6.6}$$

where $G_{4+D} = G_4 V_D$ (G_4 is proportional to Newtonian gravitational constant which is equal to M_P^{-2}, M_P is Planck mass), $V_D = (2\pi)^2\rho_1\rho_2\cdots\rho_D$ ($\rho_1, \rho_2, \ldots, \rho_D$ are radii of circles which are components of T^D), $R_{4+D} = R_4 - \frac{1}{4}\kappa^2\delta_{mn}F_{\mu\nu}^m F^{n\mu\nu}, F_{\mu\nu}^m = \partial_\nu A_\mu^m - \partial_\mu A_\nu^m, D_\mu = \nabla_\mu + \kappa A_\mu^m \nabla_m, D_m = \nabla_m$ (∇_μ and ∇_m are covariant derivative in curved space-time) and $g_{(4+D)}$ is the determinant of the metric tensor, given by (6.5).

The conditions $2/(\sqrt{|g_{(4+D)}|})(\delta S/\delta g^{MN}) = 0$ imply Einstein's field equations

$$R_{MN} - \frac{1}{2}g_{MN}R_{4+D} = 8\pi G_{4+D}\left[T_{MN}^{(\phi)} + \frac{\kappa^2}{16\pi G}T_{MN}^{(F)}\right], \tag{6.7a}$$

where

$$T_{MN}^{(\phi)} = (\partial_M\phi)^*(\partial_N\phi) - \tfrac{1}{2}g_{MN}[g^{RS}(\partial_R\phi)^*(\partial_S\phi) - m^2\phi^*\phi] \tag{6.7b}$$

(neglecting terms containing gauge fields) and

$$T_{MN}^{(F)} = \delta_{mn}(F_{ML}^m F_N^{nL} - \tfrac{1}{4}g_{MN}F_{RS}^m F^{nRS}). \tag{6.7c}$$

Maxwell's field equations for F_{MN} and ϕ are obtained from the action given by (6.6) as

$$\left(1/\sqrt{|g_{(4+D)}|}\right)\partial_M\left[\sqrt{|g_{(4+D)}|}F^{MN}\right] = 0 \tag{6.8}$$

and

$$(\Box + m^2)\phi = 0. \tag{6.8a}$$

A_μ are required to live on $R \otimes M^3$ only [38], hence due to homogeneity of space-time F_{MN} are taken as

$$F^{MN} = \begin{cases} (\epsilon^{MN}/\sqrt{|g|})F(t) & \text{for } M = \mu,\ N = \nu \\ 0 & \text{otherwise} \end{cases}, \tag{6.9}$$

where ϵ^{MN} are components of Levi–Civita tensor. Connecting (6.8) and (6.9)

$$F(t) = A/(R_D)^D. \tag{6.10}$$

Further $\phi(x,y)$ is decomposed as

$$\phi(x,y) = \phi_0(t,0,0) + \bar{\phi}(t,x,y) \tag{6.11}$$

such that

$$(\Box + m^2)\phi_0 = 0 \tag{6.12a}$$

and

$$(\Box + m^2)\bar{\phi} = 0. \tag{6.12b}$$

$\bar{\phi}$ is a small fluctuation, so cosmic dynamics is supposed to be governed by ϕ_0 only. Hence $T_{MN}^{(\bar{\phi})}$ can be ignored compared to $T_{MN}^{(\phi_0)}$. The WKB solution of (6.12a) is written as

$$\phi_0 \simeq \frac{e^{imt}}{\sqrt{V_D V_m a^3 b^D}}. \tag{6.13}$$

Now the Einstein's field equations for the line-element (6.5) are [38]

$$3\frac{\ddot{R}_3}{R_3} + D\frac{\ddot{R}_D}{R_D} = 8\pi G_{4+D}\left[|\dot{\phi}_0|^2 - m^2|\phi_0|^2 + \frac{2D\kappa^2 A^2}{16\pi G_{4+D}(D+2)b^{2D}}\right], \tag{6.14a}$$

$$\frac{2k_3}{R_3^2 l^2} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{R}_3}{R_3}\right) + \frac{\dot{R}_3}{R_3}\left(3\frac{\dot{R}_3}{R_3} + D\frac{\dot{R}_D}{R_D}\right) = \frac{\kappa^2 D A^2}{(D+2)(R_D)^{2D}} - \frac{8\pi G_{4+D} m^2 \phi_0^2 t_P^2}{(D+2)}, \tag{6.14b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{R}_D}{R_D}\right) + \frac{\dot{R}_D}{R_D}\left(3\frac{\dot{R}_3}{R_3} + D\frac{\dot{R}_D}{R_D}\right) = \frac{\kappa^2 D(D+4)A^2}{2(D+2)(R_D)^{2D}} - \frac{8\pi G_{4+D} m^2 \phi_0^2 t_P^2}{(D+2)}, \tag{6.14c}$$

where l is the constant of unit magnitude and length dimension, which is used for the dimensional correction. To avoid the 'crack of doom' singularity, it is assumed that

$$R_D^2(t) = f^2 + [R_3(t)]^{-2}. \tag{6.15}$$

If $f^2 < [R_3(t)]^{-2}$, $[R_3(t)]^{-2} = 2t/lf + R_{03}^2$ is the solution in case $k_3 = -1$. Moreover, the integration constant R_{03} is given as

$$f^2 R_{03}^2 = \tfrac{1}{12}\left[-6(D-2) + \sqrt{36(D-2)^2 + 24(D-3)^2 - 24(D+3)}\right].$$

If $f^2 > [R_3(t)]^{-2}$,

$$[R_3(t)]^2 = \left(\sqrt{4\alpha\gamma - \beta^2}/2\gamma f^2\right)\sinh\left[(2t/t_P)\sqrt{\gamma/6} + \delta\right] - (\beta/2\gamma f^2),$$

where

$$\alpha = \frac{k_3^2 t_P^2 A^2 D^2 (D+1)(D+2)}{4f^{2D}},$$

$$\beta = -\left[\frac{k_3^2 t_P^2 A^2 D^2 (D+1)(D+2)}{4f^{2D}} - 6t_P^2 f^2\right],$$

$$\gamma = 2k_3^2 t_P^2 A^2 D^2 (D+1)(D+2) f^{2D},$$

$$f^2 R_3^2 > \frac{(D-3)}{3} \geq 1, \quad \text{as} \quad D \geq 6$$

and

$$2\pi G_4 < VR_3^2 R_D^2 m.$$

For details of these solutions, one can refer to [41]. Exact solutions of Einstein's field equations have been obtained in refs [23, 42].

In [35], Sahdev has assumed that when temperature drops below $1/R$ (R being the radius of compact manifold), extra-dimensional space ceases to be dynamically effective and 4-dimensional universe emerges. During this phase, extremely hot universe cools down to a minimum temperature $1/R$ and reheats until it becomes effectively $(1+3)$-dimensional. He has also examined horizon and entropy problems in this framework. As a result, Sahdev obtained an explanation to the horizon problem of the standard big-bang cosmology and suggests that this question should be rephrased in terms of entropy as 'Does the entropy falling within a causal horizon equals at any stage to the entropy of the observed universe ?' He has obtained that causal horizon is almost equal to $R_3(d = 3)$ and entropy within the causal horizon is a free parameter.

Some authors as Alvarez and Gavela have discussed heating effect of the internal space [43], which was more clarified by Bar and Brown [44]. Adapting this approach, largeness of the entropy in the early universe can be discussed [45]. Also Abbott, Barr and Ellis have discussed the possibility of large amount of inflation from Kaluza–Klein cosmological model [46].

7. Particle creation due to quantum effects

As mentioned above, dynamical behaviour of external and internal scale-factors were explored in [33–35] and later on by others also [36–42]. In these models, external manifold blows up and the internal one shrinks. While deriving these solutions of higher dimensional Einstein's field equations, perfect fluid energy-momentum tensor was used. Alvarez and Gavela [43] discussed entropy production due to shrinking of the internal manifold which was developed by Kolb *et al* [45]. According to these estimates, entropy production was far from satisfactory. Later on, Abbott *et al* [46] suggested that large number of dimensions (~ 40) of the manifold could be helpful to explain the entropy problem. In spite of all these attempts, satisfactory explanation for large entropy could not be obtained. So, study of quantum effects on Kaluza–Klein cosmologies was found necessary. In this approach, quantized matter field is considered in the background of classical gravity in the absence of fully renormalized quantum theory of gravity [47, 48]. Following this approach, it was expected that particle production due to quantum effects could explain sufficient amount of entropy generation. Another purpose of this study was to know the effect of particle creation on dimensional reduction [49–51]. This idea was motivated by the work of Zel'dovich [52], Zel'dovich and Starobinsky [53] as well as Hu and Parker [54] regarding isotropization of the universe in the convenient 4-dimensional theory.

In 1984, Maeda [55] found that isotropization process is rapid in 5-dimensional space-time like 4-dimensional case considered by Hu and Parker [54]. In the above mentioned paper, Maeda had not taken into account the closed behaviour of the internal space. The topology of the 5-dimensional manifold was $M^4 \otimes R^1$. Later on, he discussed the same problem for manifolds $M^4 \otimes S^1, M^4 \otimes T^D$ and $M^4 \otimes S^7$, where internal manifolds were compact. Some details of similar investigations for other topologies are given as under [47].

The line-element for $(1+d+D)$-dimensional space-time $(R^1 \otimes M^d \otimes M^D)$ can be taken as given in (6.3), where M^d and M^D are maximally symmetric spaces. Here the aim is to study the effect of particle creation due to quantum process. ϕ is the matter field which is massless and conformal. So, the Lagrangian for ϕ is given as

$$L_\phi = \tfrac{1}{2}\sqrt{|g_{(4+D)}|}\{g^{MN}(\partial_M\phi)(\partial_N\phi) - \xi R\phi^2\}, \tag{7.1a}$$

where

$$\xi = \frac{d+D-1}{4(d+D)} \tag{7.1b}$$

and R is the Ricci scalar for $(1+d+D)$-dimensional space-time. The field equation for ϕ is obtained from the Lagrangian given by (7.1) as

$$(\Box + \xi R)\phi = 0. \tag{7.2}$$

Components of the energy-momentum tensor for ϕ are given by

$$\begin{aligned} T_{MN} = {} & \partial_M\phi\partial_N\phi - \tfrac{1}{2}g_{MN}\partial^P\phi\partial_P\phi - \xi\phi^2(R_{MN} - \tfrac{1}{2}g_{MN}R) \\ & + \xi[\nabla_M\nabla_N\phi^2 - g_{MN}\nabla^P\nabla_P\phi^2]. \end{aligned} \tag{7.3}$$

In the case of a conformal field, trace of the energy-momentum tensor vanishes at the classical level, but an anomalous term appears at quantum level when dimension of the manifold is even. So, to avoid the trace anamoly at the quantum level also, Maeda took $(d+D)$ even and set $T = 0$ (see ch. 6 of [56] for further details).

Harmonics $h_l(x)$ and $H_L(y)$ can be introduced for M^d and M^D spaces respectively satisfying equations

$$\nabla_\mu\nabla^\mu h_l(x) = -k_l^2 h_l(x), [\mu, \nu = 0, 1, \ldots, (1+d)] \tag{7.4a}$$

and

$$\nabla_m\nabla^m H_L(y) = -k_L^2 H_L(y), [m = 2+d, \ldots, (1+d+D)]. \tag{7.4b}$$

These harmonics obey orthonormal conditions

$$\int d^dx\sqrt{|g_d|}h_l^*(x)h_{l'}(x) = \delta_{ll'} \tag{7.5a}$$

and

$$\int d^Dy\sqrt{|g_D|}H_L^*(y)H_{L'}(y) = \delta_{LL'}, \tag{7.5b}$$

where g_d is the determinant of $g_{\mu\nu}$ and g_D is the same for g_{mn}. ϕ can be expanded as

$$\phi = \tilde{R}(\eta)^{-(d+D-1)/2}\chi(\eta, x, y), \tag{7.6}$$

where

$$\tilde{R}(\eta) = [(R_d)^d (R_D)^D]^{1/(d+D)}$$

and

$$\eta = \int^t \frac{\mathrm{d}\tilde{t}}{\tilde{R}(\tilde{t})}.$$

Further $\chi(\eta, x, y)$ can be written as

$$\chi(\eta, x, y) = \sum_{l,L} [A_{l,L}\chi_{l,L}(\eta)h_l(x)H_L(y) + A^{\dagger}_{l,L}\chi^{*}_{l,L}(\eta)h^{*}_l(x)H^{*}_L(y)]. \tag{7.7}$$

Using ϕ, given by (7.6), equation (7.2) looks like

$$\chi'' + Q\chi - \tilde{R}^2(\nabla_\mu\nabla^\mu + \nabla_m\nabla^m)\chi - \xi\tilde{R}^2\left[\frac{R^{(d)}}{R_d^2} + \frac{R^{(D)}}{R_D^2}\right]\chi = 0, \tag{7.8}$$

where prime ($'$) denotes derivation with respect to η.

Further, connecting (7.7) and (7.8), one obtains

$$\chi'' + (\Omega_{lL}^2 + Q)\chi_{lL} = 0, \tag{7.9a}$$

where

$$Q = \frac{dD(d+D+1)}{4(d+D)^2}\left(\frac{R'_d}{R_d} - \frac{R'_D}{R_D}\right)^2 \tag{7.9b}$$

and

$$\Omega_{lL}^2 = \tilde{R}^2\left[\frac{\omega_l^2}{R_d^2} + \frac{\omega_L^2}{R_D^2}\right] \tag{7.9c}$$

with

$$\omega_l^2 = k_l^2 + d(d-1)k_d \tag{7.9d}$$

and

$$\omega_L^2 = K_L^2 + D(D-1)k_D \tag{7.9e}$$

as $R^{(d)} = d(d-1)k_d$ and $R^{(D)} = D(D-1)k_D$ because M^d and M^D are maximally symmetric spaces (homogeneous and isotropic space).

The scalar product of two spinless fields are given as

$$(\phi_1, \phi_2) = -i\int_\Sigma [\phi_1(x)\partial_M\phi_2^{*}(x) - \partial_M\phi_1^{*}(x)\phi_2(x)]n^M\sqrt{|g|}\,\mathrm{d}\Sigma, \tag{7.10}$$

where n^M is the future-directed vector given as $(1, 0, 0, \ldots, 0)$ orthogonal to space-like hyperspace Σ with $\mathrm{d}\Sigma$ as volume element which is equal to $\mathrm{d}^d x\,\mathrm{d}^D y$. With this definition, orthonormal conditions for ϕ_M and ϕ_N can be written as

$$(\phi_M, \phi_N) = \delta_{MN}, (\phi^{*}_M, \phi^{*}_N) = -\delta_{MN}, (\phi_M, \phi^{*}_M) = 0. \tag{7.11}$$

The conditions for canonical quantization are given as

$$[\phi(t,x,y),\phi(t,x',y')]=0,$$
$$[\pi(t,x,y),\pi(t,x',y')]=0,$$
$$[\phi(t,x,y),\pi(t,x',y')]=i\delta^d(x-x')\delta^D(y-y'), \tag{7.12}$$

where $\pi=\partial L/\partial(\partial_0\phi)$ and L is the Lagrangian density for ϕ.

The equal-time commutation relations (7.12) yield

$$[A_{lL},A_{l'L'}]=0=[A^\dagger_{lL},A^\dagger_{l'L'}] \tag{7.13a, b}$$

and

$$[A_{lL},A^\dagger_{l'L'}]=\delta_{ll'}\delta_{LL'} \tag{7.13c}$$

for creation and annihilation operators $A^\dagger_{lL}$ and A_{lL} respectively. The vacuum state $|0\rangle$ is defined as

$$A_{lL}|0\rangle=0 \tag{7.14}$$

for all l and L . The orthonormality condition is obtained as

$$\chi^{*\prime}_{lL}\chi'_{lL}-\chi^*_{lL}\chi'_{lL}=0. \tag{7.15}$$

Using (7.6), (7.8) and components of the metric tensor given by (6.3) in (7.3),

$$\begin{aligned}T^0_0=&\frac{1}{2}\tilde{R}^{-(d+D+1)}[\chi'^2-Q\chi^2+\tilde{R}^2\{R^2_d[(4\xi-1)(\nabla^\mu\chi)\nabla_\mu\chi\\&+4\xi\chi\nabla^\mu\nabla_\mu\chi-\xi R^{(d)}\chi^2]+R^2_D[(4\xi-1)(\nabla^m\chi)\nabla_m\chi\\&+4\xi\chi\nabla^m\nabla_m\chi-\xi R^{(D)}\chi^2]\}].\end{aligned} \tag{7.16}$$

The vacuum expectation value of T^0_0 yields the energy density as $\rho=\langle 0|T^0_0|0\rangle$ which is independent of spatial coordinates, so one can write

$$\rho=\frac{1}{V_dV_D}\int \mathrm{d}^dx\,\mathrm{d}^Dy\sqrt{|g|}\langle 0|T^0_0|0\rangle, \tag{7.17}$$

where $V_d=\int \mathrm{d}^dx\sqrt{|g|}$ and $V_D=\int \mathrm{d}^Dy\sqrt{|g|}$. Using (7.14) and (7.16) in (7.17), one can write

$$\rho=\frac{\tilde{R}^{-(d+D+1)}}{2V_dV_D}\sum_{l,L}[|\chi'_{lL}|^2+(\Omega^2_{lL}-Q)|\chi_{lL}|^2], \tag{7.18}$$

which includes ultraviolet divergence. It means that T_{MN} need regularization. The appropriate method for this case is the adiabatic regularization given by Hu and Parker [see Appendix C].

Evaluation of $\sum_l$ or $\sum_L$ depend on the topology of the space. If d-dimensional space is topologically open or flat ie, $k_d\le 0$,

$$\sum_l\to\frac{V_d}{(2\pi)^d}\int \mathrm{d}^dk.$$

Similarly for D-dimensional internal space

$$\sum_L \to \frac{V_D}{(2\pi)^D}\int \mathrm{d}^D k,$$

if $k_D \le 0$.

Looking at the expression of ρ given by (7.19), one finds that it is divergent for large modes l or L. Technically speaking, the equation for ρ contains ultraviolet divergence. So naturally it needs regularization for being physically meaningful. In the case of time-dependent space-time background which are not conformally flat, the only appropriate regularization method is adiabatic regularization. This method too has a problem that it cannot provide energy momentum explicitly as a functional of metric unless explicit form of the line-element is available. Under these circumstances, Maeda [47] has given crude approximation of higher-dimensional version of adiabatic regularization method given by Fulling *et al* [57], which is discussed below.

In this method, energy density ρ is divided into quantum part $\rho_{q(t)}$ and classical part $\rho_{c(t)}$. Quantum domain is given by the condition $\omega_{\underline{L}} = \tilde{R}^{-1}\Omega_{\underline{L}} \le r_H^{-1}(t)$, where $\underline{L} \equiv (l, L)$ and r_H is the horizon scale given as

$$r_H(t) = \tilde{R}(t)\int^t \frac{\mathrm{d}\tilde{t}}{\tilde{R}(\tilde{t})}.$$

The classical domain satisfies the condition $\omega_{\underline{L}} > r_H^{-1}(t)$. It can be realized below that for low frequency modes $\Omega_{\underline{L}}$, probability of creation of quantum particle is high. Here, by quantum domain, one means the domain in momentum space where quantum effects such as particle creation and vacuum polarization are dominant. Other regions in momentum space, where quantum effects are suppressed are called classical domain satisfying the above mentioned condition.

Using (7.18) as well as conditions for quantum(classical) domains, $\rho_{q(t)}$ and $\rho_{c(t)}$ can be estimated. For these estimations, solution of the differential equation (7.9) is needed. To get the required solution, method of variation of parameters can be used. To apply this method, a particular form of the complementary function is required, which is obtained here through WKB approximations. Thus solution of (7.9) can be written as

$$\chi_{lL} = (2\Omega_{lL})^{-1/2}\left[\alpha_{lL}\exp\left(-i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right) + \beta_{lL}\exp\left(i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right]. \qquad (7.19)$$

From (7.19), one obtains

$$\chi_{lL} = -i\left(\frac{\Omega_{lL}}{2}\right)^{1/2}\left[\alpha_{lL}\exp\left(-i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right) + \beta_{lL}\exp\left(i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right] \qquad (7.20\text{a})$$

imposing the condition

$$\begin{aligned} &2\Omega_{lL}\left[\alpha'_{lL}\exp\left(-i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right) + \beta'_{lL}\exp\left(i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right] \\ &\quad = \Omega'_{lL}\left[\alpha_{lL}\exp\left(-i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right) + \beta_{lL}\exp\left(i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right], \end{aligned} \qquad (7.20\text{b})$$

which is normally used in the method of variation of parameters. Connecting (7.9) and (7.20), one obtains

$$
\begin{aligned}
& i\left(\frac{\Omega_{lL}}{2}\right)^{1/2}\left[\alpha'_{lL}\exp\left(-i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)-\beta'_{lL}\exp\left(i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right] \\
&\quad = (2\Omega_{lL})^{-1/2}\left[\left(Q-\frac{i}{2}\Omega'_{lL}\right)\alpha_{lL}\exp\left(-i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right. \\
&\quad\left. +\left(2\Omega_{lL}^2+Q+\frac{i}{2}\Omega'_{lL}\right)\beta_{lL}\exp\left(i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right].
\end{aligned} \tag{7.21}
$$

Equations (7.20b) and (7.21) determine

$$\alpha'_{lL} = (\Omega_{lL})^{-1}\left[-\frac{i}{2}Q\alpha_{lL}+\frac{1}{2}(\Omega'_{lL}-iQ)\beta_{lL}\exp\left(2i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right] \tag{7.22a}$$

and

$$\beta'_{lL} = (\Omega_{lL})^{-1}\left[\frac{i}{2}Q\alpha_{lL}+\frac{1}{2}(\Omega'_{lL}+iQ)\alpha_{lL}\exp\left(-2i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\right]. \tag{7.22b}$$

In the quantum domain, ω_l and ω_L can be taken small. So there is no harm in setting

$$\exp\left(2i\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right)\simeq 1. \tag{7.23}$$

Using (7.23) in (7.22), one obtains first order ordinary differential equations for $(\alpha_{lL}+\beta_{lL})$ and $(\alpha_{lL}-\beta_{lL})$ yielding solutions

$$\alpha_{lL} = C_1\left[\Omega_{lL}^{1/2}-i\Omega_{lL}^{-1/2}\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right]+C_2\Omega_{lL}^{-1/2} \tag{7.24a}$$

and

$$\beta_{lL} = C_1\left[\Omega_{lL}^{1/2}+i\Omega_{lL}^{-1/2}\int_{\eta_0}^{\eta}\mathrm{d}\bar{\eta}\,\Omega_{lL}\right]-C_2\Omega_{lL}^{-1/2}, \tag{7.24b}$$

where C_1 and C_2 are complex constants. Connecting (7.15), (7.19) and (7.20), one obtains the condition

$$|\alpha_{lL}|^2-|\beta_{lL}|^2=1. \tag{7.25}$$

Equations (7.24) and (7.25) yield

$$C_1C_2^*+C_1^*C_2=\tfrac{1}{2}. \tag{7.26}$$

If the initial state is vacuum,

$$\alpha_{lL}(\eta_0)=1 \quad \text{and} \quad \beta_{lL}(\eta_0)=0.$$

Using these initial conditions in (7.24), one obtains

$$C_1 = \tfrac{1}{2}[\Omega_{lL}(\eta_0)]^{-1/2} \quad \text{and} \quad C_2 = \tfrac{1}{2}[\Omega_{lL}(\eta_0)]^{1/2}. \tag{7.27}$$

Now $\rho_{q(t)}$ is obtained as

$$\rho_{q(t)} = (2V_d V_D \tilde{R}^{(d+D+1)})^{-1} \sum_{l,L} \times \left[\frac{1}{2\Omega_{lL}(\eta_0)} \left\{ \Omega_{lL}^2 - Q + \left(\int_{\eta_0}^{\eta} Q \, \mathrm{d}\bar{\eta} \right)^2 \right\} + \frac{1}{2}\Omega_{lL}(\eta_0) \right], \tag{7.28}$$

using (7.19)–(7.27) in (7.18). Initially horizon scale will be very small. So the frequency will satisfy the condition $\omega_{\underline{L}} < r_H^{-1}(t)$ because $\omega_{\underline{L}} > r_H^{-1}(t)$ will lead to unphysical situation leading to $\omega_{\underline{L}}$ being almost infinite. Thus, initially one has quantum domain. So, it is reasonable to consider quantum particles not to exist initially. As a result,

$$\rho_{c(t_0)} = 0. \tag{7.29}$$

Quantum domain which satisfies the condition $\omega_{\underline{L}} < r_H^{-1}(t) = [\tilde{R}(t)\eta(t)]^{-1}$ (where $r_H(t) = \tilde{R}(t) \int^t \mathrm{d}\bar{t}/\tilde{R}(\bar{t}) = \tilde{R}(t)\eta(t)$) shrinks with time. So quantum particles will enter into the classical domain satisfying the condition $\omega_{\underline{L}} < r_H^{-1}(t)$ as horizon scale will increase with time ($r_H^{-1}(t)$ will decrease with time yielding $\omega_{\underline{L}} > r_H^{-1}(t)$ for some $\omega_{\underline{L}}$ satisfying $\omega_{\underline{L}} < r_H^{-1}(t)$ earlier). Thus one obtains energy density of quantum particles entering into classical domain in time interval $(\eta, \eta + \triangle\eta)$ given as

$$\delta\rho_q(t) = \rho_q(\underline{L}_m(\eta), \eta) - \rho_q(\underline{L}_m(\eta + \triangle\eta), \eta), \tag{7.30a}$$

$\underline{L}_m$ being the maximum quantum number which satisfies the condition $\omega_{\underline{L}} < r_H^{-1}(t)$. Maeda has assumed these classical particles to behave like the collision-dominated relativistic perfect fluid with isotropic pressure $P_c = \rho_c/(d + D)$. Conservation of energy momentum tensor leads to

$$\rho_c(\eta + \triangle\eta) = [R(\eta)/R(\eta + \triangle\eta)]^{1+d+D}[\rho_c(\eta) + \delta\rho_q(\eta)],$$

which can be written as

$$\begin{aligned}
\rho(\eta + \triangle\eta) &= [R(\eta)/R(\eta + \triangle\eta)]^{1+d+D}[\rho_c(\eta) + \delta\rho_q(\eta)] \\
&\quad + \rho_q(\underline{L}_m(\eta + \triangle\eta), (\eta + \triangle\eta)) \\
&= [R(\eta)/R(\eta + \triangle\eta)]^{1+d+D}\rho(\eta) + \{[R(\eta)/R(\eta + \triangle\eta)]^{1+d+D} \\
&\quad \times (\delta\rho_q(\eta) - \rho_q(\eta)) + \rho_q(\underline{L}_m(\eta + \triangle\eta), \eta) \\
&\quad + \triangle\eta \frac{\partial}{\partial\eta}\rho_q(\underline{L}_m(\eta + \triangle\eta), (\eta + \triangle\eta))\}_{\triangle\eta=0} \\
&= [R(\eta)/R(\eta + \triangle\eta)]^{1+d+D}\rho + \triangle\eta\tilde{R}^{-(1+d+D)} \\
&\quad \times \frac{\partial}{\partial\eta}\rho_q(L_m(\eta + \triangle\eta)\tilde{R}^{-(1+d+D)}(\eta + \triangle\eta))\}_{\triangle\eta=0}.
\end{aligned} \tag{7.30b}$$

Equation (7.30) leads to a very interesting result

$$(\rho\tilde{R}^{1+d+D})' = \frac{\partial}{\partial\eta}[\rho_q\tilde{R}^{1+d+D}], \tag{7.31}$$

on taking the limit $\triangle\eta \to 0$ and using the basic definition of derivatives. Equation (7.31) relates rate of creation of quantum particles to the total energy density.

The energy momentum tensor for the perfect fluid can be written as

$$T_{MN} = (\rho + p)u_M u_N - (\delta p_d + \delta' p_D)g_{MN}, \tag{7.32a}$$

where $u^M \equiv (1,0,0,\dots,0)$ are components of the velocity vector normalized to unity,

$$\delta = \begin{cases} 1 & \text{for } M,N = \mu,\nu = 0,1,\dots,d \\ 0 & \text{for } M,N = m,n = 1,\dots,D \end{cases} \tag{7.32b}$$

and

$$\delta' = \begin{cases} 0 & \text{for } M,N = \mu,\nu = 0,1,\dots,d \\ 1 & \text{for } M,N = m,n = 1,\dots,D. \end{cases} \tag{7.32c}$$

With the definition of energy-momentum tensor, given by (7.4), conservation equation $T^M_{N;M} = 0$ leads to

$$V^{-1}(\rho V)' + dp_d\frac{R'_d}{R_d} + Dp_D\frac{R'_D}{R_D} = 0, \tag{7.33}$$

where $V \equiv (R_d)^d(R_D)^D = \tilde{R}^{d+D}$. Trace of energy-momentum tensor is given as

$$T = \rho - dp_d - DP_D. \tag{7.34}$$

Using (7.31), eqs (7.33) and (7.34) yield

$$p_d = -\frac{(T-\rho)\beta'}{d(\beta'-\alpha')} + \frac{\tilde{R}^{-(1+d+D)}}{d(\beta'-\alpha')}\left[\frac{\partial}{\partial\eta}[\rho_q\tilde{R}^{1+d+D}]\right], \tag{7.35a}$$

and

$$p_D = \frac{(T-\rho)\beta'}{D(\beta'-\alpha')} - \frac{\tilde{R}^{-(1+d+D)}}{D(\beta'-\alpha')}\left[\frac{\partial}{\partial\eta}[\rho_q\tilde{R}^{1+d+D}]\right], \tag{7.35b}$$

where $\alpha' = R'_d/R_d$ and $\beta' = R'_D/R_D$. Thus, rate of creation of quantum particles contribute back reaction effect on the background.

In the background model $M^4 \otimes M^D$ (where M^4 is the conventional four-dimensional flat universe and M^D is the closed D-dimensional internal space), energy density of quantum particles can be calculated at a particular time $t = t_0$ from (7.28) as

$$\rho_{q(t_0)} = [16\pi^3 V_D\tilde{R}^{4+D}]^{-1}\sum_L\int d^3k\left[\frac{1}{2\Omega_L(\eta_0)}\{\Omega_L^2(\eta_0) - Q\} + \frac{1}{2}\int d^3k\Omega_L(\eta_0)\right]. \tag{7.36}$$

Here, $\sum_L \to (V_3/(2\pi)^3)\int d^3k$ has been used as $k_3 = 0$. Using the definition of Ω^2 from

(7.9) in (7.36), $\rho_{q(t_0)}$ is calculated as

$$\rho_{q(t_0)} = [16\pi^3 V_D \tilde{R}^{4+D}]^{-1} \left[(R_0/a_0)^2 \sum_L I_2 + (R_0/b_0)^2 \sum_L (\omega_L^2 I_0) - \frac{Q}{2} \sum_L I_0 \right], \tag{7.37a}$$

where $a_0 \equiv a(\eta_0),\ b_0 \equiv b(\eta_0),\ R_0 \equiv R(\eta_0)$,

$$I_0 \equiv \int \mathrm{d}^3k [\Omega_L(\eta_0)]^{-1} = 2\pi \frac{a_0}{R_0} \left[k_m \left(k_m^2 + \frac{a_0^2}{b_0^2}\omega_L^2 \right)^{1/2} - \frac{a_0^2}{b_0^2}\omega_L^2 \ln \left\{ \frac{k_m + [k_m^2 + (a_0\omega_0/b_0)^2]^{1/2}}{(a_0\omega_0/b_0)} \right\} \right] \tag{7.37b}$$

and

$$I_2 \equiv \int \mathrm{d}^3k [\Omega_L(\eta_0)]^{-1} k^2 = \frac{\pi a_0}{2R_0} \left[k_m [k_m^2 + (a_0\omega_0/b_0)^2]^{1/2} \left(2k_m^2 - 3\frac{a_0^2}{b_0^2}\omega_L^2 \right) + 3\frac{a_0^2}{b_0^2}\omega_L^2 \ln \left\{ \frac{k_m + [k_m^2 + (a_0\omega_0/b_0)^2]^{1/2}}{(a_0\omega_0/b_0)} \right\} \right]. \tag{7.37c}$$

Here $k_m = a\{r_H^{-2} - (\omega_L^2/b^2)\}^{1/2}$ can be attained as given below. Using the condition for quantum domain, one obtains

$$\frac{k^2}{a_0^2} + \frac{\omega_L^2}{b_0^2} \leq r_H^{-2}(\eta_0)$$

(as M^4 is flat), which yields

$$|k| \leq a_0 \{r_H^{-2}(\eta_0) - \omega_L^2/b^2)\}^{1/2}.$$

k_m is the maximum of $|k|$, so one gets the above mentioned value of the same which means

$$-k_m \leq k \leq k_m.$$

The adiabatic regularization method given by Fulling *et al* [57] is discussed in Appendix C. Using this method for the higher-dimensional case, regularized energy density can be written as

$$\begin{aligned}\rho \equiv -\langle 0_A | T^0_0 | 0_A \rangle = [2V_d V_D \tilde{R}^{(1+d+D)}]^{-1} \sum_l \sum_L \\ \times \left\{ |\chi_{lL}'|^2 + (\Omega^2 - Q)|\chi_{lL}|^2 - \Omega - \frac{1}{2\Omega}\left[\left(\frac{\Omega'}{2\Omega}\right)^2 - Q \right] \right. \\ \left. + \frac{1}{4\Omega}\left[\left(\frac{\Omega'}{2\Omega}\right)^2 \right] \epsilon_{2(2)} - \frac{\Omega'}{4\Omega}\epsilon'_{2(3)} - \frac{1}{4}\Omega^2(\epsilon_{2(2)})^2 - \frac{1}{2}Q\epsilon_{2(2)} \right\},\end{aligned} \tag{7.38}$$

where $\epsilon_{m(n)}$ is defined in Appendix C.

Assuming $d = 3$ and M^4 as a flat space-time

$$\rho_{q(t)} = [16\pi^3 V_D \tilde{R}^{(4+D)}]^{-1} \sum_L \int \mathrm{d}^3k \times \left\{ \frac{1}{2\Omega_0}\left[\Omega^2 - Q\left(\int_{\eta_0}^{\eta} Q\,\mathrm{d}\eta'\right)^2\right] + \frac{\Omega_0}{2} - \Omega + (Q/2\Omega)\right\}, \tag{7.39}$$

which is the regularized energy density in the quantum region.

8. Gravitational Casimir energy in Kaluza–Klein theories

As discussed above, one gets convinced that extra dimensions should be extremely small as these dimensions are not observed at low energy level. To understand it in the case of higher dimensional static models, one can make an analogy to the Casimir effect in electrodynamics which is a quantum effect. According to classical electrodynamics, there exists no force between two conducting uncharged infinite plates. However, results, obtained from quantum electrodynamics, are different. These results yield an attractive force between such conducting plates due to negative zero-point energy between them [56, 58, 59]. In Kaluza–Klein theories, similar situation is obtained between external and internal manifolds. For example, in the simplest case of 5-dimensional theory, $x^5 = 0$ and $x^5 = L_5$ where L_5 is the distance around the fifth dimension. Appelquist and Chodos have obtained the effective potential of the metric field up to one-loop quantum correction given as [29]

$$V_{\mathrm{eff}}(L_5) = -\frac{15\zeta(5)}{4\pi^2(L_5)^4},$$

where Riemann zeta function $\zeta(5) = 1.037\cdots$. The negative sign for V_{eff} shows that the effective potential is attractive causing contraction of the extra dimension. This result is reliable only when L_5 is greater than L_P as loop expansion is valid up to this scale only.

In what follows, the mathematical method for calculation of effective potential (Casimir energy) for gravity in $(d+D)$-dimensional space-time $M^d \otimes M^D$ is discussed [21]. The action is taken as

$$S[g] = -\frac{1}{16\pi G_{d+D}} \int \mathrm{d}^{d+D}x \sqrt{|g_{(4+D)}|}(R + 2\Lambda). \tag{8.1}$$

Under transformations $g_{MN} \to g_{MN} + h_{MN}$, one obtains

$$S[g+h] = S[g] + \delta S + \delta^2 S + \cdots, \tag{8.2}$$

where first variation is

$$\delta S[g] = -\frac{1}{16\pi G_{d+D}} \int \mathrm{d}^{d+D}x \sqrt{|g_{(4+D)}|}\, h_{MN} \times [-R^{MN} + \tfrac{1}{2} g^{MN}(R + 2\Lambda)], \quad (M, N = 0, 1, 2, \ldots, d+D) \tag{8.3}$$

and the second variation is

$$\delta^2 S[g] = -\frac{1}{16\pi G_{d+D}}\int \mathrm{d}^{d+D}x\sqrt{|g_{(4+D)}|}\,\{(\tfrac{1}{2}g^{MN}h - h^{MN})$$
$$\times[\tfrac{1}{2}\Box\, h_{MN} + \tfrac{1}{2}\nabla_M\nabla_N h - \tfrac{1}{2}g^{PQ}(\nabla_P\nabla_N h_{MQ} + \nabla_M\nabla_P h_{NQ})]$$
$$+ (2h^{MP}h^N_P - hh^{MN})R_{MN} + [\tfrac{1}{4}h^2 - \tfrac{1}{2}h^{RS}h_{RS}](R + 2\Lambda), \tag{8.4}$$

where h is the trace h_{MN}, $\Box = \nabla^N\nabla_N$ and indices are raised by g^{MN}. If g^{MN} are physical fields satisfying the gravitational field equations

$$\frac{\delta S}{\delta g^{MN}} = 0.$$

In the case of higher dimensional gravity, Killing symmetries (isometries) of the internal space provide gauge group of the theory. So, the gauge transformations are

$$x^M \to x^M + \epsilon^M(x), \tag{8.5}$$

where $\epsilon^M(x)$ are components of the Killing vector in the internal manifold M^D. Here $|\epsilon^M(x)| \ll 1$. As theory is covariant, $S, \delta S$ and $\delta^2 S$ are invariant under gauge transformations given by (8.4). Using Fourier transformations of h and h_{PQ}, one finds that the term

$$(\tfrac{1}{2}g^{MN}h - h^{MN})[\tfrac{1}{2}\nabla_M\nabla_N h - \tfrac{1}{2}g^{PQ}(\nabla_P\nabla_N h_{MQ} + \nabla_M\nabla_P h_{NQ})]$$

yields eigenstates with vanishing eigenvalues. To employ the method of Feynman path integrals, these terms should be removed. For doing so, a suitable gauge-fixing term should be added which is given as

$$S_{GF} = -\frac{1}{32\pi G_{d+D}}\int \mathrm{d}^{d+D}x\sqrt{|g_{(4+D)}|}\quad G_M(g)G_N(g)g^{MN}, \tag{8.6a}$$

where G_M is defined as

$$G_M = g^{PQ}\nabla_P g_{QM} - \tfrac{1}{4}\nabla_M(g^{PQ}g_{PQ}). \tag{8.6b}$$

Terms to be eliminated from $S[g+h]$ appears in the one-loop quantum term $\delta^2 S[g+h]$. So, one-loop term of S_{GF} is also required, which is given as

$$\delta^2 S_{GF} = \frac{1}{32\pi G_{d+D}}\int \mathrm{d}^{d+D}x\sqrt{|g_{(4+D)}|}\{(-\tfrac{1}{2}g^{MN}h + h^{MN})$$
$$\times[-\tfrac{1}{2}\nabla_M\nabla_N h + g^{PQ}\nabla_M\nabla_P h_{QN}]\}. \tag{8.7}$$

Adding (8.4) and (8.7), one obtains

$$\delta^2(S[g] + S_{GF}) = -\frac{1}{16\pi G_{d+D}}\int \mathrm{d}^{d+D}x\sqrt{|g_{(4+D)}|}$$
$$\times\{(\tfrac{1}{2}g^{MN}h - h^{MN})[\tfrac{1}{2}\Box h_{MN} - (2h^{MP}h^N_P - hh^{MN})]$$
$$\times R_{MN} - [\tfrac{1}{4}h^2 - \tfrac{1}{2}h^{RS}h_{RS}](R + 2\Lambda) + h^{MN}h^{PQ}R_{PMQN}\}$$
$$= -\frac{1}{64\pi G_{d+D}}\int \mathrm{d}^{d+D}x\mathrm{d}^{d+D}x'\sqrt{|g_{(4+D)}(x)|}$$
$$\times\sqrt{|g_{(4+D)}(x')|}\,h_{MN}\nabla^P\,\nabla^Q\zeta^{MN}h_{PQ}, \tag{8.8a}$$

where the operator $\nabla^P \nabla^Q \zeta^{MN}$ is given as

$$\nabla^P \nabla^Q \zeta^{MN} = [\tfrac{1}{2} g^{PQ} g^{MN} - g^{PM} g^{QN}]\Box' - 2(g^{PN} R^{MQ} - g^{PQ} R^{MN}) - R^{MNPQ} \tag{8.8b}$$

with

$$\Box' = \Box - R - 2\Lambda. \tag{8.8c}$$

Here gauge condition has not been used. Instead of it, gauge fixing term has been added. In such a case, Fadeev–Popov method is very convenient. Now it remains to calculate Fadeev–Popov ghost matrix $M^{(G)}$ defined as

$$\int \mathrm{d}^D x' \sqrt{|g|}\, M_Q^{(G)P}(x, x') \epsilon_P(x') = \delta_\epsilon G_Q. \tag{8.9}$$

Under gauge transformations, given by (8.5), change in g_{PQ} can be calculated as follows. It is known that

$$\begin{aligned} g'_{PQ}(x') &= \frac{\partial x^M}{\partial x'^P} \frac{\partial x^N}{\partial x'^Q} g_{MN}(x) \\ &= (\delta_P^M - \epsilon^M_{,P})(\delta_Q^N - \epsilon^N_{,Q})(g_{MN}(x') - \epsilon^R g_{MN,R}) \\ &= g_{PQ}(x') - \epsilon_{P,Q} - \epsilon_{Q,P} - \epsilon^R g_{PQ,R}, \end{aligned} \tag{8.10a}$$

where terms containing higher orders of ϵ are neglected. Equation (8.10a) yields

$$\delta_\epsilon g_{PQ} = \nabla_Q \epsilon_P + \nabla_P \epsilon_Q. \tag{8.10b}$$

The result, given by (8.10b), implies that

$$\delta_\epsilon G_M = \Box \epsilon_M + R_M^P \epsilon_P. \tag{8.11}$$

Now the effective action $\Gamma(g)$ is given by [60]

$$\exp(-\Gamma(g)) = \exp(-S_{cl}[g]) \int D[h_{MN}] \exp(-S_E) \det\left(\frac{\partial G_M}{\partial \epsilon_N}\right), \tag{8.12}$$

where

$$S_E = [\delta^2(S_{cl} + S_{GF})]_E = \frac{i}{64\pi G_{d+D}} \int \mathrm{d}^{d+D} x \sqrt{|g_{(4+D)}(x)|}\, h_{MN} \nabla^P \nabla^Q \zeta^{MN} h_{PQ} \tag{8.13a}$$

and

$$\det\left(\frac{\partial G_M}{\partial \epsilon_N}\right) = \det(M_M^{(G)N}) = \det(\Box \delta_M^N + R_M^N). \tag{8.13b}$$

Thus

$$\Gamma(g) = S[g] + \tfrac{1}{2} \ln \det \nabla^P \nabla^Q \zeta^{MN} - \ln \det M^{(G)}. \tag{8.14}$$

For evaluation of the integral in (8.12), one can refer to Appendix D.

Determinants in (8.14) can be evaluated on choosing a particular manifold. Chodos and Myers [21] have considered the manifold $M^d \otimes S^D$ (M^d is d-dimensional Minkowski space-time and S^D is D-dimensional sphere) with components of the metric tensor

$$g_{MN} = \eta_{MN} + r^2 \kappa_{MN}, \tag{8.15}$$

where

$$\eta_{MN} = \begin{cases} \eta_{mn} & \text{for } m,n = 0,1,\ldots,(d-1) \\ 0 & \text{for } i,j = d, d+1,\ldots,(d+D-1) \end{cases}$$

and

$$\kappa_{MN} = \begin{cases} 0 & \text{for } m,n = 0,1,\ldots,(d-1) \\ \kappa_{ij} & \text{for } i,j = d, d+1,\ldots,(d+D-1). \end{cases}$$

Here η_{ab} are Minkowski metric components and κ_{ij} are metric components on S^D.

In the manifold $M^d \otimes S^D, M^d$ has no curvature. Since S^D is maximally symmetric, curvature terms are given as

$$R^{MNPQ} = r^{-2}[\kappa^{MN}\kappa^{PQ} - \kappa^{MQ}\kappa^{NP}], \tag{8.16a}$$

$$R^{PQ} = r^{-2}(D-1)\kappa^{PQ}, \tag{8.16b}$$

$$R = r^{-2}D(D-1). \tag{8.16c}$$

If Ξ is the eigenvalue of the operator ζ^{MNPQ}, one can write

$$\int \mathrm{d}^D x' \sqrt{|g(x')|} \times \nabla^P \nabla^Q \zeta^{MN} h_{PQ} = \Xi h^{MN}. \tag{8.17}$$

h_{PQ} are components of a second rank symmetric tensor. So, in the language of differential forms, it can be treated as 1-form (p-form is a covariant anti-symmetric tensor). Now using Hodge's decomposition theorem (refer to Appendix E), h_{PQ} can be decomposed into exact form (longitudinal component), co-exact form (transverse component) and harmonic form. For S^D, Betti number b_0 and b_1 are equal to 1 and other Betti numbers vanish. So, harmonic components of h_{PQ} will vanish as $b_1 = 0$ (it is so because h_{PQ} are treated as 1-forms). On the basis of the above discussion, one can write

$$h_{PQ} = p_{PQ} + \Phi g_{PQ} + \chi \kappa_{PQ}, \tag{8.18a}$$

where p_{PQ}, being exact form, is the longitudinal component and $(\Phi g_{PQ} + \chi\kappa_{PQ})$, being co-exact form, is the transverse component. So

$$\Phi g_{PQ} + \chi \kappa_{PQ} = 0.$$

But Φ and χ are linearly independent, so

$$g^{PQ} p_{PQ} = 0 = \kappa^{PQ} p_{PQ}. \tag{8.18b}$$

Connecting (8.8), (8.17) and (8.18),

$$-\Box' p^{MN} - r^{-2}(D-1)[\kappa^{PN} g^{MQ} + \kappa^{QM} g^{NP}] p_{PQ} + 2r^{-2}\kappa^{MQ}\kappa^{NP} p_{PQ} = \Xi p^{MN}, \tag{8.19a}$$

$$\tfrac{1}{2}(D+d-2)\Box'\Phi+D(D-1)r^{-2}\Phi+\tfrac{1}{2}D\Box'\chi+D(D-1)r^{-2}\chi=\Xi\Phi, \qquad (8.19b)$$

$$\Box'\chi-(D-1)(D-4)r^{-2}\chi-(D-1)(D+d-4)r^{-2}\Phi=\Xi\chi. \qquad (8.19c)$$

Here $\Box$ is the Laplacian on $M^d \otimes S^D$, so

$$\Box = \Box_m + r^{-2}\Box_S, \qquad (8.20)$$

where $\Box_m$ is the Laplacian on M^d and $\Box_S$ is the same on S^D. Obviously, the operator $\Box_m$ has eigenvalues $-\kappa^2$. Moreover, coupled equations (8.19b) and (8.19c) yield

$$(D+d-2)\Box\Phi-2[(D+d-2)\Lambda+\Xi]\Phi$$
$$+[D(D-1)(D-2)r^{-2}+D\Xi]\chi=0, \qquad (8.21a)$$

$$\Box\chi-[2\Lambda+2(D-1)(D-2)r^{-2}+\Xi]$$
$$-[(D-1)(D+d-4)r^{-2}]\Phi=0. \qquad (8.21b)$$

Combining these equations, one obtains

$$(D+d-2)[\Box^2-\{2\Lambda+2(D-1)(D-2)r^{-2}+\Xi\}\Box]\Phi$$
$$-2[(D+d-2)\Lambda+\Xi][\Box-\{2\Lambda+2(D-1)(D-2)r^{-2}+\Xi\}]\Phi$$
$$+(D-1)(D+d-4)r^{-2}[(D-1)(D-2)r^{-2}+\Xi]\Phi=0. \qquad (8.22)$$

p^{MN} are scalars on S^D when $M=m$ and $N=n$. These are vector components when $M=i$ (internal index) and $N=n$ and tensor components when $M=i$ and $N=j$. Corresponding to different nature of p^{MN} in three different cases, (8.19a) can be written as

$$[-\Box+2\Lambda+D(D-1)r^{-2}-\Xi]p^{mn}=0, \qquad (8.23a)$$

$$[-\Box+2\Lambda+(D-1)^2r^{-2}-\Xi]p^{in}=0, \qquad (8.23b)$$

(using $\kappa^{in}=0$) and

$$[-\Box+2\Lambda+(D^2-3D+4)(D-1)r^{-2}-\Xi]p^{ij}=0. \qquad (8.23c)$$

Eigenvalue of the operator in (8.23a) is obtained using (4.12) as

$$\Xi=\kappa^2+2\Lambda+[l^2+(D-1)(l+D)]r^{-2}$$

with determinant given by

$$D_1=\prod_{\kappa}\prod_{l=0}^{\infty}[\{l^2+(D-1)(l+D)\}r^{-2}+2\Lambda+\kappa^2]^{(1/2)(d+1)dd_S(l)}, \qquad (8.24a)$$

where $d_S(l)$ is given by (4.13). The vector components p^{in} on S^D can be decomposed using Hodge's decomposition theorem as

$$p^{in}=W^{in}+\nabla^i\sigma^n,$$

where W^{in} is the co-exact form and $\nabla^i\sigma^n$ is an exact form (σ^n are scalars). The eigenvalue of the operator in (8.23b) corresponding to W^{in} is

$$\Xi = \kappa^2 + 2\Lambda + [l^2 + (D-1)(l+D-1) - 1]r^{-2}$$

with determinant

$$D_2 = \prod_{\kappa}\prod_{l=1}^{\infty}[\{l^2 + (D-1)(l+D-2)\}r^{-2} + 2\Lambda + \kappa^2]^{dd_v(l)}, \tag{8.24b}$$

where $d_v(l)$ is given by (4.15). Moreover,

$$\begin{aligned}\Box(\nabla^i\sigma^n) &= \nabla^i(\Box\sigma^n) + R^{il}\nabla_l\sigma^n\\ &= [-\kappa^2 - l(l+D-1)r^{-2}]\nabla^i\sigma^n + (D-1)r^{-2}\nabla^i\sigma^n\\ &= [-\kappa^2 - \{l^2 + l - 1)(D-1)r^{-2}\}]\nabla^i\sigma^n.\end{aligned}$$

Now Ξ for $\nabla^i\sigma^n$ is given as

$$\Xi = \kappa^2 + 2\Lambda + [l^2 + (D-1)(l+D-2)]r^{-2}.$$

So, the determinant is

$$D_3 = \prod_{\kappa}\prod_{l=1}^{\infty}[\{l^2 + (D-1)(l+D-2)\}r^{-2} + 2\Lambda + \kappa^2]^{dd_v(l)}. \tag{8.24c}$$

Three cases are possible for tensor components p^{ij}. Using (4.16) in (8.23c) for divergenceless tensor,

$$D_4 = \prod_{\kappa}\prod_{l=2}^{\infty}[\{l^2 + (D-1)(l+D-2)\}r^{-2} + 2\Lambda + \kappa^2]^{d_T(l)}. \tag{8.24d}$$

The other two cases yield determinants

$$D_5 = \prod_{\kappa}\prod_{l=2}^{\infty}[\{l^2 + (D-1)(l+D-3)\}r^{-2} + 2\Lambda + \kappa^2]^{d_v(l)}, \tag{8.24e}$$

$$D_6 = \prod_{\kappa}\prod_{l=2}^{\infty}[\{l^2 + (D-1)(l+D-4)\}r^{-2} + 2\Lambda + \kappa^2]^{d_S(l)}. \tag{8.24f}$$

The determinant of eigenvalues for the operator in (8.22) is given as

$$\begin{aligned}D_7 = \prod_{\kappa}\prod_{l=1}^{\infty}\{(D+d-2)[\{l^2 + (D-1)(l+D-2)\}r^{-2} + 2\Lambda + \kappa^2]\\ + 2(D-1)^2(D-2)(d-2)r^{-4}\}^{d_S(l)}.\end{aligned} \tag{8.24g}$$

The ghost determinant $\det M^{(G)}$ assumes a similar form with eigenvalue equation

$$(\Box\delta^N_M + R^N_M)\epsilon_N = \Xi_G\epsilon_N,$$

which can be rewritten as

$$(\Box\delta_M^N + (D-1)r^{-2}\kappa_M^N)\epsilon_N = \Xi_G\epsilon_N. \tag{8.25}$$

When $N = n, \epsilon_n$ is scalar on S^D. So, the determinant of eigenvalue

$$D_1^{(G)} = \prod_\kappa \prod_{l=0}^{\infty} [l(l+D-1)r^{-2} + \kappa^2]^{dds(l)}. \tag{8.26a}$$

When $N = i, \epsilon_i$ are components of a vector on S^D, which can be decomposed into exact and co-exact forms using Hodge's decomposition theorem as above. Now the determinant of eigenvalues are

$$D_2^{(G)} = \prod_\kappa \prod_{l=1}^{\infty} [\{l^2 + l(D-1) + (D-2)\}r^{-2} + \kappa^2]^{d_s(l)} \tag{8.26b}$$

and

$$D_3^{(G)} = \prod_\kappa \prod_{l=1}^{\infty} [\{l^2 + l(D-1)\}r^{-2} + \kappa^2]^{d_v(l)}. \tag{8.26c}$$

One can notice that all these determinants, given by (8.24) and (8.26), have a general form

$$D = \prod_\kappa \prod_{l=l_0}^{\infty} [\{l^2 + l(D-1) + (D-2) + c(x,D)\}r^{-2} + \kappa^2]^{d(l)}. \tag{8.27}$$

As a result

$$\ln D = -\frac{\mathrm{d}\zeta}{\mathrm{d}s}\bigg|_{s=0}. \tag{8.28}$$

Now $\zeta(s)$ is evaluated in the region where s is large and the result is analytically continued to $s = 0$. For evaluation of $\zeta(s)$, one can refer to Candelas and Weinberg [26] as well as Chodos and Myers [21].

As explained above, evaluation of these determinants yields one-loop quantum correction V_Q given as

$$V_Q = (2\pi r)^4\left[-\frac{15}{2}\left\{\sum_{n=1}^{\infty} \mathrm{e}^{(-2\pi nx)}\left(\frac{3}{2\pi^2 n^5} + \frac{3x}{\pi n^4} + \frac{2x^2}{n^3}\right) - \frac{1}{\pi^2}\zeta(5)\right\} + 4\pi^3 x^3\right]. \tag{8.29}$$

In 1990, Buchbinder *et al* have calculated 2-loop effective action on the background geometry $M^d \otimes T^D$ [61].

9. One-loop quantum correction and induced gravity

In 1983, using 5-dimensional space-time with topology $M^4 \otimes S^1$ and line-element, given by (3.2), for $\phi(x) = 1$, Toms [62] obtained induced gravitational action as well as

Maxwell's action through one-loop quantum correction to scalar and spinor fields. The action for the scalar field $\hat{\phi}$ is taken as

$$S_{\hat{\phi}} = \frac{1}{2}\int d^5x\sqrt{|\hat{g}|}[g^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} - m^2\hat{\phi}^2 - \xi\hat{R}_5\hat{\phi}^2], \tag{9.1}$$

where ξ is the non-minimal coupling constant, S^1 is a non-simply connected manifold. So $\hat{\phi}(x,y)$ can be twisted or untwisted field. Untwisted field is periodic in y and twisted field is anti-periodic in y [63, 64]. $\hat{\phi}(x,y)$ can be decomposed as

$$\hat{\phi}(x,y) = \sum_{n=-\infty}^{\infty}\hat{\phi}_{(n)}(x)\exp[2\pi(n+\alpha)y/L], \tag{9.2a}$$

where

$$\alpha = \begin{cases} 0 & \text{for untwisted field} \\ \frac{1}{2} & \text{for twisted field.} \end{cases} \tag{9.2b}$$

Connecting (9.1) and (9.2) and integrating over y

$$\begin{aligned} S_{\hat{\phi}} = \frac{1}{2}\sum_{-\infty}^{\infty}\int d^4x\sqrt{|g|}\,&[g^{\mu\nu}(D_\mu^{(n)}\hat{\phi}_{(n)})^*(D_\nu^{(n)}\hat{\phi}_{(n)}) \\ &- M_{(n)}^2\hat{\phi}_{(n)}^*\hat{\phi}_{(n)} - \xi R\hat{\phi}_{(n)}^*\hat{\phi}_{(n)} + \tfrac{1}{4}\xi\kappa^2F_{\mu\nu}F^{\mu\nu}\hat{\phi}_{(n)}^*\hat{\phi}_{(n)}], \end{aligned} \tag{9.3a}$$

where

$$\begin{aligned} D_\nu^{(n)}\hat{\phi}_{(n)} &= (\partial_\mu - iq_{(n)}A_\mu)\phi_{(n)}, \\ M_{(n)}^2 &= m^2 + 4\pi^2(n+\alpha)^2L^{-2}, \\ q_{(n)} &= 2\pi\kappa(n+\alpha)L^{-1}. \end{aligned} \tag{9.3b}$$

The one-loop effective action can be written, in this case, as

$$\Gamma^{(1)} = \frac{i}{2}\sum_{-\infty}^{\infty}\ln\det\triangle_{(n)}, \tag{9.4a}$$

where the operator

$$\triangle_{(n)} = g^{\mu\nu}D_\mu^{(n)}D_\nu^{(n)} + M_{(n)}^2 + \xi R - \tfrac{1}{4}\xi\kappa^2F_{\mu\nu}F^{\mu\nu}. \tag{9.4b}$$

Toms has computed $\Gamma^{(1)}$, using the heat-kernel method where kernel $K_{(n)}(s,x,x')$ is defined for the operator $\triangle_{(n)}$ [56]. As a result

$$\Gamma^{(1)} = -\frac{i}{2}\sum_{-\infty}^{\infty}\int d^4x\sqrt{|g|}\int_0^\infty\frac{ds}{s}\,\mathrm{tr}\,K_{(n)}(s,x,x'), \tag{9.5a}$$

where tr stands for the trace. The Schwinger DeWitt expansion of $\mathrm{tr}\,K_n(s,x,x')$ yields

$$\mathrm{tr}\,K_n(s,x,x') = i\mu^{4-N}(4\pi is)^{-N/2}\exp(-iM_n^2s)\sum_{p=0}^{\infty}(is)^pa_p(x), \tag{9.5b}$$

where μ is a parameter of mass dimension and N is a parameter for dimensional regularization such that one gets the actual 4-dimensional theory on taking the limit $N \to 4$. Connecting (9.5a) and (9.5b) one can find that the first three terms in the series yields divergence in the integral

$$\int_0^\infty \frac{ds}{s} i\mu^{4-N}(4\pi is)^{-N/2}\exp(-iM_n^2 s)$$
$$\times\ [a_0(x,x') + (is)a_1(x,x') + (is)^2 a_2(x,x') + \cdots]. \tag{9.5c}$$

All other terms are convergent for a 4-dimensional theory. So, the coefficients a_0, a_1 and a_2 are crucial. On taking the limit $x \to x'$, these coefficients are given as

$$a_0(x) = 1,$$
$$a_1(x) = (\tfrac{1}{6} - \xi)R + \tfrac{1}{4}\kappa^2\xi F^{\mu\nu}F_{\mu\nu},$$
$$a_2(x) = -\tfrac{1}{12}q_n^2 F^{\mu\nu}F_{\mu\nu} + \cdots. \tag{9.5d}$$

From (9.5), one obtains for N_0^+ untwisted fields

$$\Gamma^{(1)} = \frac{N_0^+}{4\pi^2}\int d^4x\sqrt{|g|}$$
$$\times\left[\frac{3\zeta(5)}{L^4} + \frac{\zeta(3)}{2L^2}\left(\frac{1}{6} - \xi\right)R + \left(\frac{1}{3} + \xi\right)\frac{\zeta(3)}{4L^2}\kappa^2 F^{\mu\nu}F_{\mu\nu} + \cdots\right]. \tag{9.6}$$

Other terms are not calculated here, because the purpose is to obtain induced gravity and induced Macwell's term.

For the N_0^- twisted fields ($\alpha = \frac{1}{2}$),

$$\Gamma^{(1)} = -\frac{N_0^-}{32\pi^2}\int d^4x\sqrt{|g|}$$
$$\times\left[\frac{45\zeta(5)}{2L^4} + \frac{3\zeta(3)}{L^2}\left(\frac{1}{6} - \xi\right)R + \left(\frac{1}{3} + \xi\right)\frac{3\zeta(3)}{4L^2}\kappa^2 F^{\mu\nu}F_{\mu\nu} + \cdots\right]. \tag{9.7}$$

For calculation of these terms, one can refer to Appendix F.

Including $N_{1/2}^+$ untwisted Dirac spinors and $N_{1/2}^-$ twisted spinors also in the theory, the total one-loop effective action

$$\Gamma^{(1)} = \frac{1}{8\pi^2}\int d^4x\sqrt{|g|}\left[-\frac{45\zeta(5)}{8L^4}\left\{(N_0^- - 4N_{1/2}^-) - \frac{16}{15}(N_0^+ - 4N_{1/2}^+)\right\}\right.$$
$$+\frac{\zeta(3)}{3L^2}\left\{\left(N_{1/2}^+ - \frac{3}{4}N_{1/2}^-\right)\right\} + 3\left(\frac{1}{6} - \xi\right)\left(N_0^+ - \frac{3}{4}N_0^-\right)R - \frac{5\zeta(3)}{24L^2}\kappa^2$$
$$\left.\times\left\{\left(N_{1/2}^+ - \frac{3}{4}N_{1/2}^-\right) - \frac{6}{5}\left(\frac{1}{3} + \xi\right)\left(N_0^+ - \frac{3}{4}N_0^-\right)\right\}F^{\mu\nu}F_{\mu\nu} + \cdots\right] \tag{9.8}$$

This result shows that the induced cosmological constant can vanish provided that $N_0^+ = 4N_{1/2}^+$ and $N_0^- = 4N_{1/2}^-$.

In 1984, Candelas and Weinberg [26] have calculated induced cosmological constant, induced gravity and fine-structure constant using the background geometry of $(4+N)$-dimensional space-time with topology $M^4 \otimes B^N$. Moreover they have calculated one-loop effective potential and discussed stability problem.

Yoshimura [51] has computed the one-loop quantum effective action at finite temperature in the background geometry of higher-dimensional Robertson–Walker type model with topology $R^1 \otimes S^{d_1} \otimes S^{d_2}$. The line-element is taken as

$$dS^2 = d\tau^2 + a_1^2(\tau)\gamma_{ab}^{(1)}dx^a dx^b + a_2^2(\tau)\gamma_{mn}^{(2)}dy^m dy^n, \tag{9.9}$$

where $\tau = it$ (t is the cosmic time), the scale factor a_i is periodic i.e. $a_i(\tau+\beta) = a_i(\tau)$ with $\beta = T^{-1}$. Here scale factors are slowly varying to maintain thermal equillibrium up to a good approximation.

The effective 4-dimensional gravitational constant G_{eff} is obtained here as

$$G_{\text{eff}}^{-1} \simeq G_4^{-1} - \frac{5}{6}\pi^{-(d_2+2)/2} d_2(d_2+1)\Gamma\left(\frac{d_2+2}{2}\right)T^{d_2+2}, \tag{9.10}$$

where G_4 is the Newtonian gravitational constant. It is interesting to note that $G_{\text{eff}} < 0$ above the critical temperature given as

$$T_{\text{cr}} \simeq \left[\frac{6}{5}\pi^{+(d_2+2)/2}\left\{d_2(d_2+1)\Gamma\left(\frac{d_2+2}{2}\right)G_4\right\}^{1/(d_2+2)}\right]. \tag{9.11}$$

Physically, it means that gravity becomes repulsive when $T > T_{\text{cr}}$. In natural units $(\hbar = c = 1)G_4 \simeq M_{\text{P}}^{-2}$($M_{\text{P}}$ is the Planck mass), so T_{cr} will be much below Planck energy if d_2 is large.

Assuming infinitely many cycles of expansion and contraction of the universe, Yoshimura has discussed that temperature might have raised above the critical temperature during contraction and due to anti-gravity effect of $G_{\text{eff}} < 0$, the present expansion might have started. He has also suggested that a huge amount of entropy might have produced due to many cycles of the universe possibly solving horizon and flatness problems of the standard cosmology.

In 1985, Gleiser and Taylor [38] have obtained time-dependence of coupling constants in the six-dimensional cosmological models with topology $M^4 \otimes S^2$. Thus, first time dynamical model was used for these kinds of investigations.

In [41], a gauge-dependent model with topology $M^4 \otimes T^D$ has been considered and solutions of higher-dimensional Einstein's equations have been obtained with energy momentum tensor of scalar as well as Maxwell's fields. Using the background geometry of these solutions, time-dependence of gravitational constant, cosmological constant and fine-structure constants are obtained through dimensional reduction and one-loop quantum correction to scalar fields. Some mathematical details of this approach is given as follows.

The line-element for the underlying model is given by (6.5) with solution of Einstein's equations, given by (6.15). Using the horizontal lift basis, discussed in § 3 in the case of 5-dimensional theory, the action for pure gravity can be written as

$$S_g = \frac{1}{16\pi G_{4+D}} \int d^4x\, d^Dy \sqrt{|g|}(R_4 - \tfrac{1}{4}\kappa^2 \delta_{mn} F^m_{\mu\nu} F^{n\mu\nu}), \tag{9.12}$$

where Ricci scalar vanishes on T^D.

Components of the metric tensor g_{MN} on the manifold $M^4 \otimes T^D$ can be written as

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & -R_D^2(t)\delta_{mn} \end{pmatrix}, \tag{9.13}$$

where $g_{\mu\nu} \equiv \mathrm{diag}\,(1, -R_3^2, -R_3^2, -R_3^2)$ on M^4 and g_{MN} are metric tensor components on T^D. Now conformal transformation yields

$$g_{MN} \to R_D^2(t)\, \bar{g}_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} & 0 \\ 0 & -\delta_{mn} \end{pmatrix}, \tag{9.14}$$

where $\bar{g}_{\mu\nu} \equiv \mathrm{diag}\,(1, -R_D^{-2}R_3^2, -R_D^{-2}R_3^2, -R_D^{-2}R_3^2)$.

On using conformal transformations, given by (4.2), in (4.1) double divergence terms for b will appear. Using Gauss's divergence theorem on such terms, one can realize that these terms do not contribute to the theory. As a result, one obtains 4-dimensional action for gravity alongwith Maxwell's term as

$$S_g^{(4)} = -\frac{(2\pi)^2\rho_1\rho_2\cdots\rho_D}{16\pi G_{4+D}} \int d^4x \sqrt{|\tilde{g}_4|} R_D^{D-2} \times \left[\tilde{R}_4 - \frac{D}{4}\kappa^2\kappa^D \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - (D+2)(D+3)\left(\frac{\dot{R}_D}{R_D}\right)^2\right], \tag{9.15}$$

on integrating over y.

Conformal transformations, given by (9.14), were used for mathematical convenience. So, to come back to the original system another conformal transformation is done to undo the earlier one which are given as

$$\tilde{g}_{\mu\nu} \to R_D^{-2}(t) g_{\mu\nu}. \tag{9.16}$$

Using these transformations, $S_g^{(4)}$ (given by (9.15)) is written as

$$S_g^{(4)} = -\frac{1}{16\pi G_4} \int d^4x (R_3)^3 (R_D)^D \left[R_4 - \frac{D}{4}\kappa^2 F_{\mu\nu}F^{\mu\nu} - D(D-1)\left(\frac{\dot{R}_D}{R_D}\right)^2\right], \tag{9.17}$$

where $G_4 \equiv G_{4+D}/(2\pi)^2 \rho_1\, \rho_2 \cdots \rho_D$.

The action for scalar field $\hat{\phi}$ is given by (6.6). Further it is decomposed into homogeneous part $\hat{\phi}_0$ and inhomogeneous part $\hat{\phi}$ as given by (6.11). For convenience, one can write

$$\hat{\phi}_0 = [(2\pi)^2 \rho_1\rho_2\cdots\rho_D (R_D)^D]^{(-1/2)} \tilde{\hat{\phi}}_0 \tag{9.18a}$$

and

$$\tilde{\hat{\phi}} = [(2\pi)^2 \rho_1 \rho_2 \cdots \rho_D (R_D)^D]^{(-1/2)} \sum_{n_1 n_2 \cdots n_D = -\infty}^{\infty} \tilde{\hat{\phi}}_{(n)} \exp\left[i \sum_{j=1}^{D} \{2\pi(n_j + \alpha) y_j / \rho_j\} \right], \tag{9.18b}$$

where α is given by (9.2b) and $\tilde{\hat{\phi}}_{(n)} \equiv \tilde{\hat{\phi}}_{n_1 n_2 \cdots n_D}$. Connecting (6.6) and (9.18) and integrating over y, it is obtained that

$$S(\phi) = S^{(4)}(\tilde{\hat{\phi}}_0) + S^{(4)}(\tilde{\hat{\phi}}_n), \tag{9.19a}$$

where

$$S^{(4)}(\tilde{\hat{\phi}}_0) = \frac{1}{2} \int d^4x (R_3)^3 \tilde{\hat{\phi}}_0 \left[-\Box_4 + m^2 + \frac{3D}{2}\left(\frac{\dot{R}_3}{R_3}\right)\left(\frac{\dot{R}_D}{R_D}\right) + D\left(\frac{\dot{R}_D}{R_D}\right)^2 + \frac{D}{2}\frac{d}{dt}\left(\frac{\dot{R}_D}{R_D}\right) \right] \tilde{\hat{\phi}}_0 \tag{9.19b}$$

and

$$S^{(4)}(\tilde{\hat{\phi}}_{(n)}) = \frac{1}{2} \sum_{n_1 n_2 \cdots n_D = -\infty}^{\infty} \int d^4x (R_3)^3 \times [g^{\mu\nu} (D_\mu \tilde{\hat{\phi}}_{(N)})^* (D_\mu \tilde{\hat{\phi}}_{(N)}) - M_{(n)}^2 \tilde{\hat{\phi}}_{(N)}^* \tilde{\hat{\phi}}_{(N)}], \tag{9.19c}$$

where

$$M_{(n)}^2 = \left(\frac{2\pi}{R_D}\right)^2 \sum_{j=1}^{D} \{2\pi(n_j + \alpha)/\rho_j\}^2 + m^2 + \frac{3D}{2}\left(\frac{\dot{R}_3}{R_3}\right)\left(\frac{\dot{R}_D}{R_D}\right) + D\left(\frac{\dot{R}_D}{R_D}\right)^2 + \frac{D}{2}\frac{d}{dt}\left(\frac{\dot{R}_D}{R_D}\right) \tag{9.19d}$$

and

$$D_\mu \tilde{\hat{\phi}}_{(n)} = \nabla_\mu \tilde{\hat{\phi}}_{(n)} + i q_{(n)} A_\mu \tilde{\hat{\phi}}_{(n)} \tag{9.19e}$$

with

$$q_{(n)} = \kappa M \sum_{j=1}^{D} n_j, \quad M = \frac{2\pi}{\rho_H}$$

and

$$\rho_H = \sum_{j=1}^{D} n_j \Big/ \sum_{j=1}^{D} (n_j/\rho_j).$$

Recognizing κM as e, one finds $q_{(n)}$ as charge of scalar field $\tilde{\hat{\phi}}_{(n)}$ being integral multiple of e.

In curved spaces, operator regularization method is very convenient, because it leads to finite results [65, 66]. Using this method, one-loop quantum correction to $\tilde{\hat{\phi}}_{(n)}$ is computed and the series is summed up. Up to adiabatic order 4, the effective action with one-loop quantum correction is given as

$$\Gamma = S(\tilde{\hat{\phi}}) + \sum_{n_1 n_2 \cdots n_D = -\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s}\Bigg[\left(\frac{\mu^2}{M^2_{(n)}}\right)^s \int \mathrm{d}^4x (R_3)^2 \times \Bigg\{ \frac{M^4_{(n)}}{(s-1)(s-2)} + \frac{M^2_{(n)}}{6(s-1)} R_4 + \left(\frac{1}{30}\Box_{(4)} R_{(4)} + \frac{1}{180} R^{\mu\nu\alpha\beta}_{(4)} R_{(4)\mu\nu\alpha\beta} - \frac{1}{180} R^{\mu\nu}_{(4)} R_{(4)\mu\nu} + \frac{1}{72} R^2_{(4)} + \frac{D}{12} F_{\mu\nu}F^{\mu\nu}\right)\Bigg\}\Bigg]\Bigg|_{s=0}$$

which reduces to

$$\Gamma = S(\tilde{\hat{\phi}}) + \int \mathrm{d}^4x (R_3)^2 \Bigg[\ln(\mu^2/m^2) \Bigg\{ \frac{m^4}{2} - \frac{m^2}{6} R_{(4)} + \frac{1}{30}\Box_{(4)} R_{(4)} + \frac{1}{180} R^{\mu\nu\alpha\beta}_{(4)} R_{(4)\mu\nu\alpha\beta} - \frac{1}{180} R^{\mu\nu}_{(4)} R_{(4)\mu\nu} + \frac{1}{72} R^2_{(4)} + \frac{D}{12}\kappa^2 F_{\mu\nu} F^{\mu\nu}\Bigg\} + \left\{\frac{3m^4}{4} - \frac{m^2}{6} R_{(4)}\right\}\Bigg]. \tag{9.20}$$

For summation of series, one can refer to Appendix F.

Equations (9.17) and (9.20) yield the effective Maxwell's term as

$$\int \mathrm{d}^4x (R_3)^2 \left[\frac{D\kappa^2 (R_D)^D}{64\pi G_4} + \frac{D\kappa^2}{12}\ln(\mu^2/m^2)\right] F_{\mu\nu}F^{\mu\nu}.$$

Using normalization of electromagnetic fields, one obtains

$$\left[\frac{D\kappa^2 (R_D)^D}{64\pi G_4} + \frac{D\kappa^2}{12}\ln(\mu^2/m^2)\right] = 1, \tag{9.21}$$

as the standard Maxwell's term in the action is

$$\frac{1}{4}\int \mathrm{d}^4x \sqrt{|g|} F_{\mu\nu}F^{\mu\nu}.$$

Since $\kappa = e/M = e\rho_H/2\pi$, (9.21) implies that

$$\frac{e^2}{4\pi} = \left[\frac{D\rho_H^2}{\pi}\left\{\frac{(R_D)^D}{16\pi G_4} + \frac{1}{3}\ln(\mu^2/m^2)\right\}\right]^{-1}. \tag{9.22}$$

Equations (9.17) and (9.20) yield the effective action for 4-dimensional gravity as

$$S_g^{(4)} = -\int d^4x (R_3)^3 \left[\left\{ \frac{(R_D)^D}{16\pi G_4} + \frac{m^2}{6} + \frac{m^2}{6} \ln(\mu^2/m^2) \right\} R_4 - \left\{ D(D-1) \left(\frac{\dot{R}_D}{R_D} \right)^2 + \frac{3m^4}{4} + \frac{m^4}{2} \ln(\mu^2/m^2) \right\} \right], \tag{9.23}$$

which implies that

$$\frac{1}{16\pi G_{\text{eff}}} = \frac{(R_D)^D}{16\pi G_4} + \frac{m^2}{6} + \frac{m^2}{6} \ln(\mu^2/m^2) \tag{9.24a}$$

and

$$\frac{\Lambda_{\text{eff}}}{8\pi G_{\text{eff}}} = \left\{ D(D-1) \left(\frac{\dot{R}_D}{R_D} \right)^2 + \frac{3m^4}{4} + \frac{m^4}{2} \ln(\mu^2/m^2) \right\}. \tag{9.24b}$$

Renormalization at $\mu^2 = m^2$ leads to the following results from (9.22) and (9.24) given as

$$\frac{e^2}{4\pi} = \frac{16\pi^2 G_4}{D\rho_H^2 (R_D)^D}, \tag{9.25a}$$

$$\frac{1}{16\pi G_{\text{eff}}} = \frac{(R_D)^D}{16\pi G_4} + \frac{m^2}{6} \tag{9.25b}$$

and

$$\frac{\Lambda_{\text{eff}}}{8\pi G_{\text{eff}}} = D(D-1) \left(\frac{\dot{R}_D}{R_D} \right)^2 + \frac{3m^4}{4}. \tag{9.25c}$$

Using solutions of Einstein's field equations, given by (6.15), one obtains time-dependence of $e^2/4\pi, G_{\text{eff}}$ and Λ_{eff}. It is interesting to note that when $t \to \infty$, time-dependence of these terms ceases to be effective. In [42], time-dependence of fundamental constants has been discussed in singularity-free 5-dimensional Kaluza–Klein cosmological model. In this paper, heat-kernel method has been used for calculation of one-loop correction. In [67, 68], $(1+1)$-dimensional as well as $(1+2)$-dimensional cosmological models have been obtained using these methods.

Appendix A

Dimensional reduction of Dirac spinors

In a space-time with topology $M^{d-D} \otimes K^D$ where K^D is a compact manifold with space-like coordinates y^a and M^{d-D} is the $(d-D)$-dimensional Minkowskian space-time with coordinates x^μ. Dirac lagrangian for a spinor $\psi(x,y)$ is given as

$$L_\psi = \tfrac{1}{2}\bar{\psi}[i(\gamma^\mu \nabla_\mu + \gamma^a \nabla_a) + m_d]\psi + \text{h.c.} \tag{A1}$$

ψ can be expanded as [21, 22, 66]

$$\psi(x,y) = \sum_i \psi_M(x)\psi_{iA}(y), \tag{A2}$$

where $A = 1, \ldots, \bar{A}$ and $M = 1, \ldots, \bar{M}$ such that $\bar{A}\bar{M} = 2^{[d/2]}$. The appropriate choice for the d-dimensional Dirac matrices $\gamma^{\hat{\mu}} \equiv (\gamma^\mu, \gamma^a)$ are given by [69]

$$\gamma^\mu = \Gamma^\mu \otimes S_{\bar{A}}, \quad \gamma^a = \Gamma^a \otimes S_{\bar{M}} \tag{A3}$$

with $S_{\bar{A}}S_{\bar{A}} = I_{\bar{A}}$, $S_{\bar{M}}S_{\bar{M}} = I_{\bar{M}}$, $S^\dagger_{\bar{A}} = S_{\bar{A}}$, $S^\dagger_{\bar{M}} = S_{\bar{M}}$ and $\{S_{\bar{A}}, \Gamma^a\} = 0$, $[S_{\bar{M}}, \Gamma^\mu] = 0$ or $[S_{\bar{A}}, \Gamma^a] = 0$, $\{S_{\bar{M}}, \Gamma^\mu\} = 0$. If $(d-D)$ is even, one can choose $\bar{A} = 2^{[d-D]/2}$, $\bar{M} = 2^{[D]/2}$, $S_{\bar{A}} = I_{\bar{A}}$, $S_{\bar{M}} = \eta\Gamma^0\Gamma^1\cdots\Gamma^{d-D-1}$. If $(d-D)$ is odd , Γ^μ are $2^{[d-D-1]/2}$ in number and $S_{\bar{M}}$ is the last Γ^μ i.e. Γ^{d-D-1} which implies that

$$\begin{aligned} S_{\bar{M}} &= \eta\Gamma^0\Gamma^1\cdots\Gamma^{d-D-1} = \eta\Gamma^0\Gamma^1\cdots\Gamma^{d-D-2}S_{\bar{M}} \\ &= \eta\Gamma^0\Gamma^1\cdots\Gamma^{d-D-2}\eta\Gamma^0\Gamma^1\cdots\Gamma^{d-D-2} = I_{\bar{M}}. \end{aligned}$$

Thus $S_{\bar{M}}$ commutes with all Γ^μ in case $(d-D)$ is odd. If D is odd, $S_{\bar{A}} = I_{\bar{A}}$ with $\bar{A} = 2^{[D-1]/2}$.

In (A2) , $\psi_{iA}(y)$ satisfy the equation

$$\Gamma^a\nabla_a\psi_{iA}(y) = M\psi_{iA}(y), \tag{A4}$$

where M is the mass matrix . Connecting (A1), (A2) and (A4)

$$L_\psi = \tfrac{1}{2}\bar{\psi}_M[i\gamma^\mu\nabla_\mu + (m_d + iMS_{\bar{M}})]\psi_M + \text{h.c.} \tag{A5}$$

It is difficult to handle the mass term obtained here. So, for convenience, a transformation is used which is given as

$$\psi_M \to \psi'_M = \exp[i\alpha S_{\bar{M}}]\psi_M. \tag{A6}$$

Also,

$$\begin{aligned} \exp[i\alpha S_{\bar{M}}] &= 1 + i\alpha S_{\bar{M}} + \frac{(i\alpha S_{\bar{M}})^2}{2!} + \frac{(i\alpha S_{\bar{M}})^3}{3!} + \cdots \\ &= \cos\alpha + iS_{\bar{M}}\sin\alpha. \end{aligned} \tag{A7}$$

From (A6), one obtains

$$\bar{\psi}_M = \psi^\dagger_M\Gamma^0 = \bar{\psi}'_M(\cos\alpha - iS_{\bar{M}}\sin\alpha). \tag{A8}$$

Now

$$\begin{aligned} &\bar{\psi}_M[i\Gamma^\mu\nabla_\mu + (m_d + iMS_{\bar{M}})]\psi_M \\ &\quad = \bar{\psi}'_M(\cos\alpha - iS_{\bar{M}}\sin\alpha)(\cos\alpha + iS_{\bar{M}}\sin\alpha)\Gamma^\mu\nabla_\mu\psi'_M \\ &\qquad + \bar{\psi}'_M[(m_d\cos 2\alpha + M\sin 2\alpha) + iS_M(M\cos 2\alpha - m_d\sin 2\alpha)]\psi'_M \\ &\quad = \bar{\psi}'_M[\Gamma^\mu\nabla_\mu + ((m_d)^2 + M^2)^{1/2}]\psi'_M, \end{aligned} \tag{A9}$$

as $\tan 2\alpha = M/m_d$.

Appendix B

Complex projective space

If $z^k, k = 1, 2, \ldots, N+1$ are complex numbers and λ is another complex number such that

$$(z^1, z^2, \ldots, z^{N+1}) = (\lambda z^1, \lambda z^2, \ldots, \lambda z^{N+1}),$$

the resulting space is the complex projective space CP^N. It is a simply connected compact manifold. In case $N = 1$,

$$(z^1, z^2) = (\lambda z^1, \lambda z^2),$$

which gives CP^1 or S^2. We have

$$(z^1, z^2, z^3) = (\lambda z^1, \lambda z^2, \lambda z^3),$$

in case $N = 2$, which gives CP^2. The coordinates on CP^2 are z^1/z^3 and z^2/z^3. Thus $(z^1/z^3,\ z^1/z^3)$ cover all points on CP^2 barring points for which $z^3 = 0$, such regions are homeomorphic to R^4 (deformable to R^4). Points, for which, $z^3 = 0$ may be regarded as points at infinity. Such points belong to CP^1. If

$$|z^1|^2 + |z^2|^2 + |z^3|^2 = \frac{6}{\lambda},$$

the line-element on CP^2 is given as

$$\mathrm{d}S^2 = \left[1 + \frac{\lambda}{6}\left(|\xi^1|^2 + |\xi^2|^2\right)\right]^{-2} \left[|\mathrm{d}\xi^1|^2 + |\mathrm{d}\xi^1|^2 + \frac{\lambda}{6}|\mathrm{d}(\xi^1\xi^{-2})|^2 - \frac{\lambda}{6}\,\mathrm{d}(|\xi^1|^2)\mathrm{d}(|\xi^2|^2)\right],$$

where $\xi^1 = z^1/z^3$ and $\xi^2 = z^2/z^3$.

Appendix C

Adiabatic regularization

Adiabatic regularization method was developed by Parker *et al* for regularization in Robertson–Walker space-times,

$$\mathrm{d}S^2 = \mathrm{d}t^2 - a_1^2(t)\mathrm{d}x^2 - a_2^2(t)\mathrm{d}y^2 - a_3^2(t)\mathrm{d}z^2. \tag{C1}$$

The matter field is provided by neutral scalar field ϕ with the lagrangian

$$L = \tfrac{1}{2}\sqrt{|g|}\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{6}R\phi^2 - m^2\phi^2\}, \tag{C2}$$

which yields the field equation for ϕ in the background geometry, given by (C1), as

$$\ddot{\phi} + (\dot{V}/V)\dot{\phi} - \frac{1}{a_1^2}\frac{\partial^2}{\partial x^2}\phi - \frac{1}{a_2^2}\frac{\partial^2}{\partial y^2}\phi - \frac{1}{a_3^2}\frac{\partial^2}{\partial z^2}\phi + \left(\frac{1}{6}R + m^2\right)\phi = 0, \tag{C3}$$

where $V = \sqrt{|g|} = a_1 a_2 a_3$ and dot denotes partial differentiation with respect to time t. The equal-time canonical commutation relations are

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0, \tag{C4a}$$

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta(\mathbf{x} - \mathbf{x}'), \tag{C4b}$$

where $\pi = \partial L/\partial\dot{\phi} = V\dot{\phi}$.

Components of the energy-momentum tensor are given by

$$\begin{aligned} T_{\mu\nu} = {} & \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}g^{\lambda\sigma}\partial_\mu\phi\partial_\nu\,\phi + \tfrac{1}{2}g_{\mu\nu}m^2\phi^2 - \tfrac{1}{6}\,\nabla_\mu\partial_\nu(\phi^2) \\ & + \tfrac{1}{6}g_{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\partial_\sigma(\phi^2) - \tfrac{1}{6}(\phi^2)(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R). \end{aligned} \tag{C5}$$

Equation (C5) yields

$$\begin{aligned} T^0_0 = V^{-2/3}\Bigg[& \frac{1}{3}\,V^{-2/3}(\partial_\eta\chi)^2 + \frac{1}{6}\sum_{i=1}^{3} a_i^{-2}(\partial_i\chi)^2 \\ & - \frac{1}{3}\sum_{i=1}^{3} a_i^{-2}\chi\partial_i^2\chi\,\frac{1}{2}\,(m^2 - V^{-2/3}Q)\chi^2\Bigg] \end{aligned} \tag{C6a}$$

and

$$\begin{aligned} -T^i_i = V^{-2/3}\Bigg[& \frac{1}{6}\,V^{-2/3}(\partial_\eta\chi)^2 + \frac{1}{3}\,V^{-2/3}\left(\frac{a_i'}{a_i} - \frac{1}{3}\frac{V'}{V}\right)\chi\partial_\eta\chi \\ & + \frac{2}{3}\,a_i^{-2}(\partial_i\chi)^2 - \frac{1}{6}\sum_{j=1}^{3} a_j^{-2}(\partial_j\chi)^2 - \frac{1}{3}\,a_i^{-2}\chi\partial_i^2\chi \\ & \times\frac{1}{6}\left\{-m^2 + V^{-2/3}\partial_\eta\left(\frac{a_i'}{a_i} - \frac{1}{3}\frac{V'}{V}\right) - V^{-2/3}Q\right\}\chi^2\Bigg], \end{aligned} \tag{C6b}$$

where $\chi = V^{1/3}\phi, \eta = \mathrm{int}'\,V^{-1/3}\mathrm{d}t'$ and prime ($\prime$) denotes differentiation w.r.t. η and

$$Q = \frac{1}{12}\sum_{i<j}\left(\frac{a_i'}{a_i} - \frac{a_j'}{a_j}\right)^2. \tag{C6c}$$

The field equation (C3) is written as

$$\partial_\eta^2\chi - V^{2/3}\sum_{i=1}^{3} a_i^{-2}\chi\partial_i^2\chi + (V^{2/3}m^2 + Q)\chi = 0. \tag{C7}$$

Equation (C7) is a linear equation, so its solution can be written as

$$\chi = (2\pi)^{-3/2}\int \mathrm{d}^3k[A_\mathbf{k}\chi_\mathbf{k}(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} + A^\dagger_\mathbf{k}\chi^*_\mathbf{k}(\eta)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{C8}$$

where $\chi_\mathbf{k}(\eta)$ satisfies the equation

$$\chi''_\mathbf{k} + (\Omega^2_\mathbf{k} + Q)\chi_\mathbf{k} = 0 \tag{C9a}$$

with

$$\Omega_{\mathbf{k}}^2 = V^{2/3}\omega_{\mathbf{k}}^2 = V^{2/3}\left(\sum \frac{k_i^2}{a_i^2} + m^2\right). \tag{C.9b}$$

If

$$\chi_{\mathbf{k}}^{*\prime}\chi_{\mathbf{k}} - \chi_{\mathbf{k}}^{*}\chi_{\mathbf{k}}' = i, \tag{C10}$$

using (C4)

$$[A_{\mathbf{k}}, A_{\mathbf{k}'}] = [A_{\mathbf{k}}^{\dagger}, A_{\mathbf{k}'}^{\dagger}] = 0, \tag{C.11a}$$

$$[A_{\mathbf{k}}, A_{\mathbf{k}'}^{\dagger}] = \delta(\mathbf{k} - \mathbf{k}'). \tag{C.11b}$$

The vacuum state $|0_A\rangle$ is defined as

$$A_{\mathbf{k}}|0_A\rangle = 0. \tag{C12}$$

Connecting (C6), (C8) and (C9), one obtains

$$\rho_0 \equiv \langle 0_A|T_0^0|0_A\rangle = (16\pi^2)^{-1}V^{-4/3}\int \mathrm{d}^3k[|\chi_{\mathbf{k}}'|^2 + (\Omega_{\mathbf{k}}^2 + Q)|\chi_{\mathbf{k}}|^2] \tag{C13a}$$

and

$$\begin{aligned}(P_i)_0 &\equiv -\langle 0_A|T_i^i|0_A\rangle \\ &= \frac{1}{3}(16\pi^2)^{-1}V^{-4/3}\int \mathrm{d}^3k\left[|\chi_{\mathbf{k}}'|^2 + \left(\frac{a_i'}{a_i} - \frac{1}{3}\frac{V'}{V}\right)|\chi_{\mathbf{k}}|^2\right. \\ &\quad \left. + \left[6V^{2/3}a_i^2k_i^2 + \partial_\eta\left(\frac{a_i'}{a_i} - \frac{1}{3}\frac{V'}{V}\right) - (\Omega_{\mathbf{k}}^2 + Q)\right]|\chi_{\mathbf{k}}|^2\right].\end{aligned} \tag{C13b}$$

One can easily find that ρ_0 and $(P_i)_0$ are divergent. There exist two methods to obtain suitable finite observables: (1) regularization and (2) renormalization. Regularization means replacement of divergent quantities by well-defined expressions in a manner consistent with the physical basis of the theory. In renormalization, infinities are either absorbed in physical constants such as charge and mass or cancelled by counter-terms.

Adiabatic regularization is a subtraction scheme. The essential point in this method is the identification of contributions of the vacuum state. But it is difficult to define physical vacuum in curved space-time which really corresponds to 'no particle state'. The reason is change in gravitational field which feeds energy to perturbed mode of scalar fields. In Robertson–Walker type space-times, one has special advantage of having a priviledged class of observers, called co-moving observers, who see the universe precisely isotropic. This is why this regularization method is applicable to Robertson–Walker type space-times only as four velocity for co-moving observers are given as (1,0,0,0). Still there remains one problem which is time-dependence. To get time-independent physical vacuum state (which is a must to get 'no particle state'), the universe should expand adiabatically.

To ensure adiabaticity, Parker and Fulling [54] introduced a parameter T in the scale factors $a_i(t)(i = 1, 2, 3)$ replacing t by t/T. In the limit of large T, $a_i(t/T)$ and its derivatives will necessarily be slowly varying functions of t, because $\partial/\partial t \to T^{-1}(\partial/\partial(t/T))$. Thus T^{-1} and its powers appear in every equation. However, original equations can be obtained by putting $T = 1$. The limit $T \to \infty$ yields the static case.

This procedure requires two important conditions.

(i) The operators $A_{\rm k}$ and $A_{\rm k}^{\dagger}$ must be annihilation and creation operators in the adiabatic limit (arbitrarily slow time variation of components of the metric tensor). Moreover, identification of physical particles should be valid up to T^4.
(ii) Regularization of infinite quantities like ρ_0 and $(P_i)_0$ should be done using mode by mode subtraction in the integrands. Expanding the integrand in powers of T^{-1}, three leading terms are sufficient to subtract to get non-divergent quantities. Thus, adiabatic regularization is a misnomer of subtraction scheme.

To carry out the programme of adiabatic regularization, higher order WKB approximation is needed which was obtained by Parker [57] through iterative process. Later on, Chakravorty [70] obtained WKB solution of the (C9a) to all orders in an explicit form. According to Chakravorty's result positive frequency solution of (C9a) can be written as

$$\chi_{\rm k}(\eta) = [2V^{1/3}(\eta)W_{\rm k}(\eta)]^{-1/2}\exp\left[-i\int^{\eta} V^{1/3}(\eta')W_{\rm k}(\eta'){\rm d}\eta'\right], \qquad \text{(C14a)}$$

where

$$W_{\rm k}(\eta) = V^{-1/3}[Y(1+\epsilon_2)(1+\epsilon_4)]^{1/2} \qquad \text{(C14b)}$$

with

$$\epsilon_2 = -Y^{-3/4}\partial_\eta(Y^{-1/2}\partial_\eta Y^{1/4}) \qquad \text{(C14c)}$$

and

$$\epsilon_4 = -Y^{-1/2}(1+\epsilon_2)^{-3/4}\partial_\eta\{[Y(1+\epsilon_2)]^{-1/2}\partial_\eta[(1+\epsilon_2)^{1/4}]\}, \qquad \text{(C14d)}$$

Y being given as

$$Y = \Omega_{\rm k}^2 + Q. \qquad \text{(C14e)}$$

Using definition of $\Omega_{\rm k}^2$, given by (C9b) and Y by (C14e) in (C14b) we have

$$W_{\rm k} = \omega_{\rm k}[1 + \epsilon_2 + Q\Omega_{\rm k}^{-2} + \epsilon_2 Q\Omega_{\rm k}^{-2} + \epsilon_4 + \epsilon_4 Q\Omega_{\rm k}^{-2} + \epsilon_2\epsilon_4 + Q\Omega_{\rm k}^{-2} + \cdots]. \qquad \text{(C15)}$$

The definition of Q, given by (C6c) yields

$$Q = O(T^{-2}). \qquad \text{(C16)}$$

Using these definitions

$$\epsilon_2 = 1 + O(T^{-2}) + O(T^{-4}) + \cdots,$$

$$\epsilon_4 = O(T^{-4}) + O(T^{-6}) + \cdots.$$

So, up to order 4 of T^{-1}

$$W_{\mathbf{k}} = \omega_{\mathbf{k}}[1 + \epsilon_2 + \epsilon_4]^{1/2}, \tag{C17a}$$

where

$$\epsilon_2 = \underline{\epsilon}_2 + Q\Omega^{-2} \tag{C17b}$$

and

$$\epsilon_4 = \underline{\epsilon}_4 + \underline{\epsilon}_2 Q\Omega^{-2}. \tag{C17c}$$

Defining $\epsilon_{n(m)}$ as a term of ϵ_n which is of the order T^{-m}, one obtains up to order T^{-4},

$$|\chi|^2 = (2\Omega)^{-1}\left\{1 - \frac{1}{2}\,\epsilon_{2(2)} + \frac{3}{8}\,[\epsilon_{2(2)}]^2 - \frac{1}{2}\,\epsilon_{2(4)} - \frac{1}{2}\,\epsilon_{4(4)}\right\}, \tag{C18a}$$

$$|\chi'|^2 = (2\Omega)^{-1}\left\{\Omega^2 + \left[\frac{1}{2}\,\Omega^2 + \frac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2\right] + \frac{1}{2}\,\Omega^2\epsilon_{2(4)} + \frac{1}{2}\,\Omega^2\epsilon_{4(4)} - \frac{1}{8}\,\Omega^2[\epsilon_{2(2)}]^2 - \frac{1}{8}\left(\frac{\Omega'}{\Omega}\right)^2\epsilon_{2(2)} + \frac{1}{4}\left(\frac{\Omega'}{\Omega}\right)\epsilon'_{2(3)}\right\}, \tag{C18b}$$

$$\partial_\eta|\chi|^2 = -(2\Omega)^{-1}\left\{\left(\frac{\Omega'}{\Omega}\right) + \frac{1}{2}\left[\epsilon'_{2(3)} - \left(\frac{\Omega'}{\Omega}\right)\epsilon_{3(2)}\right]\right\}. \tag{C18c}$$

Using results, given by (C18), in (C13) divergent terms in ρ_0 and $(P_i)_0$ are obtained as

$$(\rho_0)_{\text{div.}} = (32\pi^3)^{-1}V^{-4/3}\int \mathrm{d}^3k\Omega^{-1}\left\{2\Omega^2 + \left[\frac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2 - Q\right] - \frac{1}{8}\left(\frac{\Omega'}{\Omega}\right)^2\epsilon_{2(3)} + \frac{1}{4}\left(\frac{\Omega'}{\Omega}\right)\epsilon'_{2(3)} + \frac{1}{4}\Omega^2\epsilon^2_{2(3)} + \frac{1}{2}Q\epsilon_{2(2)}\right\}, \tag{C19a}$$

$$\begin{aligned}(P_{i0})_{\text{div.}} = (96\pi^3)^{-1}V^{-4/3}\int \mathrm{d}^3k\Omega^{-1}\Bigg\{&6V^{2/3}\left(\frac{k_i}{a_i}\right)^2 + \left[\epsilon_{2(2)}\left(\Omega^2 - 3V^{2/3}\left(\frac{k_i}{a_i}\right)^2\right.\right.\\ &+\left[\frac{1}{4}\left(\frac{\Omega'}{\Omega}\right) - \left(\frac{\Omega'}{\Omega}\right)\left(\frac{a'_i}{a_i} - \frac{V'}{3V}\right) + \partial_\eta\left(\frac{a'_i}{a_i} - \frac{V'}{3V}\right) - Q\right]\\ &+\Omega^2\left(\epsilon_{2(4)} + \epsilon_{4(4)} - \frac{1}{2}\epsilon^2_{2(2)}\right) - 3V^{2/3}\left(\frac{k_i}{a_i}\right)^2\left(\epsilon_{2(4)} + \epsilon_{4(4)} - \frac{3}{4}\,\epsilon^2_{2(2)}\right)\\ &-\frac{1}{2}\,\epsilon_{2(2)}\left[\left(\frac{\Omega'}{2\Omega}\right)^2 - \frac{\Omega'}{\Omega}\left(\frac{a'_i}{a_i} - \frac{V'}{3V}\right) + \partial_\eta\left(\frac{a'_i}{a_i} - \frac{V'}{3V}\right) - Q\right]\\ &-\frac{1}{2}\epsilon'_{2(3)}\left(\frac{a'_i}{a_i} - \frac{V'}{3V} - \frac{\Omega'}{2\Omega}\right)\Bigg\}.\end{aligned} \tag{C19b}$$

Regularized $(\rho_0)_{\text{reg}}$ and $(P_{i0})_{\text{reg}}$ are obtained by subtracting these divergent terms from ρ_0 and $(P_i)_0$ respectively i.e.

$$(\rho_0)_{\text{reg}} = \rho_0 - (\rho_0)_{\text{div.}}$$

and

$$(P_{i0})_{\rm reg} = (P_i)_0 - (P_{i0})_{\rm div.}.$$

Appendix D

Path integration

The Gaussian integral $\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2}$ can be evaluated as given below. Suppose that

$$I = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2}$$

Now

$$I^2 = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-y^2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}y\, \mathrm{e}^{-(x^2+y^2)}.$$

Using $x = r\cos\theta$ and $y = r\sin\theta$

$$I^2 = \int_0^{\infty} \int_0^{2\pi} r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{e}^{-r^2} = \pi,$$

which implies that

$$I = \sqrt{\pi}. \tag{D1}$$

Using this idea [57]

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-Q(x)} = \left(\frac{2\pi}{a}\right)^{1/2} \exp(b^2/2a), \tag{D2}$$

where

$$Q(x) \equiv \frac{1}{2}\, ax^2 - bx = -\frac{b^2}{2a} + \frac{1}{2}\, a\left(x - \frac{b}{a}\right)^2.$$

Taking the form of $Q(x)$ for n variables $\{x_1, x_2, \ldots, x_n\}$ as

$$\bar{Q}(x) = \frac{1}{2} \sum_{i,j=1}^{n} x^i A_{ij} x^j - \sum_{i=1}^{n} b_i x^i,$$

the generalized form of the result (D2) can be obtained. In the matrix form $\bar{Q}(x)$ can be written as

$$\bar{Q}(x) = \tfrac{1}{2} x^T A x - b^T x = \tfrac{1}{2}(x - x_0)^T A (x - x_0) - \tfrac{1}{2}\, b^T A^{-1} b, \tag{D3}$$

where A is a $n \times n$ non-singular matrix with elements A_{ij}, $x_0 = A^{-1}b, x^T$ is the transpose of x,

$$x \equiv \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$b \equiv \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Now

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \mathrm{e}^{-\bar{Q}(x)} = (2\pi)^{n/2} (\det A)^{-1/2} \exp(\tfrac{1}{2} b^T A^{-1} b), \tag{D4}$$

where $\det A = a_1 a_2 \cdots a_n, a_i (i = 1, 2, \ldots, n)$ being eigenvalues of the matrix A.

The effective action Γ is calculated as

$$\exp(i\Gamma/\hbar) = N \int (D\phi) \exp(i/\hbar) \left[S_0[\phi] - i \int \mathrm{d}^4 x_E \phi(x_E) J(x_E) \right], \tag{D5}$$

where N is the normalization constant, $D\phi = (2\pi)^{-n/2} \mathrm{d}\phi_1(x_E) \times \mathrm{d}\phi_2(x_E) \cdots \mathrm{d}\phi_n(x_E)$ and

$$\begin{aligned} S_0[\phi] &= \frac{1}{2} \int \mathrm{d}^4 x \sqrt{|g|} \, [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2] \\ &= -\frac{1}{2} \int \mathrm{d}^4 x \sqrt{|g|} \, \phi(x) [\Box^2 - m^2] \phi(x) \\ &= \frac{i}{2} \int \mathrm{d}^4 x_E \sqrt{|g|} \, \phi(x_E) [-\Box^2 + m^2] \phi(x_E) \end{aligned}$$

(suffix E shows Euclideanized form of the variable after Wick's rotation) and

$$\phi(x) \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}.$$

To employ the above method of integration, Euclideanization is needed. The operator $(-\Box^2 + m^2)$ on $\phi(x_E)$ yields eigenvalues corresponding to every element of the column matrix ϕ forming a $n \times n$ diagonal matrix. So, from (D5)

$$\exp(i\Gamma/\hbar) = N(\det A)^{-1/2} \exp \frac{1}{2\hbar} [J^T(-\Box^2 + m^2) J],$$

which implies that

$$\begin{aligned} \frac{i\Gamma}{\hbar} &= \frac{1}{2\hbar} [J^T(-\Box^2 + m^2) J] + \ln [N(\det A)^{-1/2}] \\ &= \frac{1}{2\hbar} [J^T(-\Box_E^2 + m^2) J] - \frac{N}{2} \operatorname{tr} \ln A. \end{aligned} \tag{D6}$$

Appendix E

Cohomology, Betti numbers and Hodge's decomposition theorem

It is possible to distinguish between 2-dimensional non-homeomorphic spaces (two spaces are called homeomorphic if both are deformable into each other without cutting. Technically speaking, two spaces S_1 and S_2 are homeomorphic if there exists a map $f : S_1 \to S_2$ such that (1) f is one-one and onto (2) f and f^{-1} are continuous) in an intuitive manner. But this method is not effective to distinguish between higher-dimensional non-homeomorphic spaces. Homology and cohomology groups serve this purpose [71].

p-dimensional subspaces of an n-dimensional space X forming abelian groups C_p with an infinite sequence

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_0} 0$$

are called to constitute a chain complex denoted by C^*. Here ∂_p are differentials or boundary operators such that $\partial_{p-1}\partial_p = 0$ for $p \geq 1$. Moreover ∂_p is a map such that

$$\partial_p(C_p^1 + C_p^2) = (\partial_p C_p^1) + (\partial_p C_p^2).$$

The set $\mathrm{Ker}\,\partial_p \equiv \{c \in C_p : \partial_p c = 0\}$ forms a subgroup in C_p called a group of p-dimensional cycles. The set $\mathrm{Im}\,\partial_{p+1} \equiv \{c \in C_p : c = \partial_{p+1} u\}$ forming subgroup in C_p is called a group of p-dimensional boundaries. The factor group $\mathrm{Ker}\,\partial_p/\mathrm{Im}\,\partial_p$ is called p-homology group of C^*. Thus p-homology group contains p-cycles which are not boundaries. Dual space of p-homology group is called p-cohomology group. It is also called pth de Rahm cohomology denoted as $H^p(X)$.

A general space X is not necessarily a Hausdorff space (A space is Hausdorff if open sets around any two distinct points are disjoint). If X is not Hausdorff, it may contain elements with torsion as well as torsion-free elements. pth Betti number is the dimension of pth cohomology group of torsion-free elements, because only torsion-free elements form a linear space (vector space) as linearity is lost in presence of elements with torsion.

Manifold is a space which is locally Euclidean and Hausdorff. So, all elements of a manifold are torsion-free. If manifold is real and differentiable (analytical), p-forms can be defined on it. Using differentiable p-forms, pth de Rahm cohomology group $H^p(M)$ of a real manifold M^n can be defined as the quotient space of the real vector space of closed p-forms which are not exact.

A p-form ω is closed if $\mathrm{d}\omega = 0$ and it is called exact if $\omega = \mathrm{d}\alpha$, where α is $(p-1)$-form. Similarly, a p-form is called co-closed if $\delta\omega = 0$ and co-exact, if $\omega = \delta\beta$, where β is $(p+1)$-form.

The operator d is defined as

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i,$$

where f is 0-form (scalar) and $\mathrm{d}f$ is 1-form. Thus operator d raises the form by 1 i.e. $d : F^p(M) \to F^{(p+1)}(M)$, where $F^p(M)$ is the space of p-forms. The Hodge operator is given as $* : F^p(M) \to F^{(n-p)}(M), 0 \leq p \leq n$. If α is a p-form given as

$$\alpha = \alpha_{i_1 \cdots i_p} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_p},$$
$$*\alpha = \alpha_{i_1 \cdots i_p} \mathrm{d}x^{i_{p+1}} \wedge \mathrm{d}x^{i_{p+2}} \wedge \cdots \wedge \mathrm{d}x^{i_n}.$$

The operator δ is given by the map $\delta : F^p(M) \to F^{(p-1)}(M)$. Thus, $\delta = - * d *$. If f is a 0-form on M^3,

$$\begin{aligned}
\mathrm{d}f &= \frac{\partial f}{\partial x^1}\mathrm{d}x^1 + \frac{\partial f}{\partial x^2}\mathrm{d}x^2 + \frac{\partial f}{\partial x^3}\mathrm{d}x^3 \\
\mathrm{d}f &= \frac{\partial f}{\partial x^1}\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\partial f}{\partial x^2}\mathrm{d}x^3 \wedge \mathrm{d}x^1 + \frac{\partial f}{\partial x^3}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
\mathrm{d} * \mathrm{d}f &= \frac{\partial(* \mathrm{d}f)}{\partial x^1}\mathrm{d}x^1 + \frac{\partial(* \mathrm{d}f)}{\partial x^2}\mathrm{d}x^2 + \frac{\partial(* \mathrm{d}f)}{\partial x^3}\mathrm{d}x^3 \\
&= \left[\frac{\partial^2 f}{\partial(x^1)^2} + \frac{\partial^2 f}{\partial(x^2)^2} + \frac{\partial^2 f}{\partial(x^3)^2}\right]\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3,
\end{aligned}$$

as $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$. Now,

$$-\delta \mathrm{d}f = * \mathrm{d} * \mathrm{d}f = \left[\frac{\partial^2 f}{\partial(x^1)^2} + \frac{\partial^2 f}{\partial(x^2)^2} + \frac{\partial^2 f}{\partial(x^3)^2}\right],$$

which is a 0-form.

A p-form is an anti-symmetric covariant tensor. A p-form is called harmonic if $\Box\omega = 0$ (where $\Box = \mathrm{d}\delta + \delta\mathrm{d}$) which implies that $\mathrm{d}\omega = 0$ and $\delta\omega = 0$. Moreover, $\mathrm{d}\omega = 0$ and $\delta\omega = 0$ imply $\Box\omega = 0$. Each cohomology class on a compact Riemannian manifold $H^p(M)$ contains a unique harmonic representative i.e.

$$H^p(M) \equiv \{\omega \in F^p(M) : \Box\omega = 0\}.$$

To be more specific, $H^p(M)$ does not contain exact and co-exact forms.

Using these results, Hodge proposed a theorem of p-forms which states that on a compact orientable Riemannian manifold M^n, $F^p(M)$ of differentiable p-forms can be decomposed into harmonic and non-harmonic forms as [72]

$$\begin{aligned}
F^p(M) &= \Box F^p(M) \oplus H^p(M) \\
&= (\mathrm{d}\delta + \delta\mathrm{d})F^p(M) \oplus H^p(M) \\
&= \mathrm{d}F^{p-1}(M) \oplus \delta F^{p+1}(M) \oplus H^p(M).
\end{aligned}$$

Here $0 \leq p \leq n$ (p is an integer) and $H^p(M)$ is finite dimensional. In other words, if ω is a p-form on a compact orientable Riemannian manifold, it has a unique representation

$$\omega = \mathrm{d}\alpha + \delta\beta + \gamma,$$

where α is a $(p-1)$-form, β is a $(p+1)$-form and γ is a harmonic p-form. Here, it is clear that $\mathrm{d}\alpha$ is an exact form, $\delta\beta$ is a co-exact form.

As discussed above, pth Betti number b_p is the dimension of finite dimensional vector space $H^p(M)$. So, if b_p is zero on M^n, p-forms can be the sum of exact and co-exact forms only.

In the language of a physicist, exact 1-form is longitudinal component and co-exact 1-form is transverse component of a vector. In tensorial notation, a co-exact 1-form component A_μ can be recognized if it satisfies $A_\mu^{;\mu} = 0$ and exact 1-form component as $A_\mu = \partial_\mu f (f$ is a scalar).

Appendix F

Riemann zeta function

Using binomial expansion, one can write

$$(1-x)^{-1} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$. This kind of expansion is not possible, when $|x| \geq 1$, because in this case $\sum_{n=0}^{\infty} x^n$ is divergent. The above expansion yields

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{1}{p^s}\right)^n, \tag{F1}$$

where $s > 1$ and p is a prime number. It should be noted that 1 is not a prime number. Multiplying (F1) for all prime numbers, one obtains

$$\begin{aligned}\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} &= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right)\cdots \\ &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s}, \ s > 1.\end{aligned} \tag{F2}$$

This identity was found by Euler who had taken s as a real number, which sets a link between the series $\sum_{n=1}^{\infty} (1/n^s)$ and prime numbers. Riemann discussed that deepest features of prime numbers can be realized taking s as a complex number with $\mathrm{Re}\, s > 1$. So, when s is complex, $\sum_{n=1}^{\infty} (1/n^s)$ is called Riemann Zeta function written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}\, s > 1. \tag{F3}$$

The series is convergent for $\mathrm{Re}\, s > 1$ only. Integral representation of this function is given as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t \frac{t^{s-1}}{e^t - 1}, \tag{F4}$$

showing that $\zeta(s)$ can be analytically continued and it is single valued for all s except $s = 1$. A consequence of (D4) is given as

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s),$$

$$\zeta(s-1) = 2^{1-s}\pi^{-s}\cos(1/2\pi s)\Gamma(s)\zeta(s). \tag{F5}$$

Another important result of analytical continuation of this function is given as

$$\lim_{\epsilon\to 0}\zeta(-m+\epsilon) = (-1)^m\frac{B_{m+1}}{m+1}, \tag{F6}$$

where m is a natural number, B_n's are Bernoulli numbers, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0\cdots$. Thus,

$$\zeta(0) = -\frac{1}{2},\quad \zeta(-2n) = 0,\quad \zeta(1-2n) = -\frac{B_{2n}}{2n}$$

$$\zeta(2n) = -\frac{(2\pi^{2n})}{2(2n)!}B_{2n}\quad (n = 1,2,3,\ldots). \tag{F7}$$

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