A STUDY OF EQUIVARIANCE PROBLEMS AND INVARIANTS RELATED TO GENERALIZATIONS OF BORSUK-ULAM THEOREM AND THE LEVEL OF SPACES

BASIL S. KOIKARA
DEPARTMENT OF MATHEMATICS
SCHOOL OF PHYSICAL SCIENCES

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I certify that the thesis entitled "A STUDY OF EQUIVARIANCE PROBLEMS AND INVARIANTS RELATED TO GENERALIZATIONS OF BORSUK ULAM THEOREM AND THE LEVEL OF SPACES", submitted by Mr. Basil S. Koikara for the Degree of Doctor of Philosophy of North-Eastern Hill University, Shillong, embodies the original work carried out by him under my supervision. He has been duly registered, and the thesis presented is worthy of being considered for the award of the Ph. D. degree. This work has not been submitted for any degree of any other university.

Himadri Kumar Mukerjee
Dr. Himadri Kr. Mukherjee
(Supervisor)
Reader, Dept. of Mathematics
NEHU, Shillong.

Date : 26th April 1993,
Place: Shillong.

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CONTENTS

INTRODUCTION

Given two topological spaces together with certain groups acting on them, let us consider the question : Does there exist a continuous map between them which preserves these group actions, namely an equivariant map ? This general question in specific situations gives rise to several problems. For example, the existence of an equivariant map gets related to questions on the existence of embeddings of manifolds or existence of cross sections.

One may consider two situations : (1) One considers the extent to which a given continuous map between specific spaces with group actions can be made equivariant. (2) One looks for restrictions on specific spaces with group actions, so that he may have an affirmative answer to the above question.

The generalisations of the Borsuk-Ulam Theorem attempts to answer questions of type (1), as is made specific below :

The Classical Borsuk-Ulam Theorem states that a continuous map of an n-sphere into the Euclidean n-space maps some pair of antipodal points into a single point [2]. In a general setting, Borsuk-Ulam Theorem can be formulated in terms of the following data :

(i) *A Domain* which may be a space with group action or more generally, a fibre-bundle with a fibrewise action of a group.

(ii) *A range* which can be a space with group action or more generally, a fibre-bundle with fibrewise action of a group

(iii) *A continuous map*, which may or may not be equivariant, from the given domain to the range.

One then studies the "size" of the union of points of the domain space on which the map is equivariant or of the union of points of the orbits of the domain space which are mapped to a single point by the given map.

Determination of certain invariants of spaces, namely level or coindex answers questions of type (2) as is made specific below:

Another important aspect of problems related to equivariance of maps is the level of spaces. The level of a topological space X with a fixed point free involution T is defined to be the number $\ell(T,X) = \min\{ n : \text{there exists a } \mathbb{Z}_2\text{-equivariant map } f : X \longrightarrow S^{n-1}\}$, where the space X and the sphere S^{n-1} are considered to have $\mathbb{Z}_2$-actions given by the involution T and the antipodal action respectively.

We now give below a brief historical survey of the progress of answers to the specific questions that have been answered in this thesis.

0.1 A BRIEF HISTORICAL SURVEY

While trying to study generalizations of the Borsuk Ulam Theorem, one gives suitable meaning to the word "size" referred to above. By looking at the various developments centred around this problem, one notices that it is the (co)homological dimension that one attempts to estimate to a sufficiently satisfactory limit. To do this estimation, various algebraic invariants have been introduced. To name a few : homology index of C.T. Yang [41], cohomology co-index of P.E. Conner and E.E Floyd [7], G-indices of Fadell-Husseini-Rabinowitz [16] and Jaworowski [22], characteristic polynomials [13], [33], certain ideals related to these polynomials, an ideal-valued index resulting from these [15], Nakaoka's invariant $\hat{I}(f)$ [31], the equivariant Lefschetz class $L(f)$ and the equivariant Euler class $e_G(f)$ of Necochea [34], and the self-intersection number $C(f)$ of Lin [25], [26] and Kahn [23], [24].

The last invariants attempt to measure the cohomology dimension of the Borsuk-Ulam set corresponding to a map of manifolds with group action. To be more specific, let $f : X \longrightarrow Y$ be a map of manifolds with $\mathbb{Z}_p$-action, p a prime, with the action of $\mathbb{Z}_p$ free on X. Let T and T' denote respectively, the action of $\mathbb{Z}_p$ on X and Y. The later generalisations of the Borsuk-Ulam Theorem attempt to study the cohomological dimension of the set $A(f) = \{x \in X / f(Tx) = T'f(x)\}$. In certain cases, for example, when X is a product of spheres and Y, a manifold, none of the invariants that are at present available is able to give any information about the dimension of A(f). In this work, we have used a deep analysis of the intrinsic nature of the problem, the natural properties and geometric interpretations of the Conner-Miller classes and the Bredon operation, to establish that the dimension of $A(f) \geq m+n-k$, where $X = S^m \times S^n$ and the dimension of Y is k. We state below the concerned result :

1.3.1 THEOREM

Let $f : S^m \times S^n \longrightarrow V^k$ ($k < m \leq n$) be a map of $S^m \times S^n$ into a manifold. Let $A(f) = \{(x,y) \in S^m \times S^n \mid f(-x,-y) = f(x,y)\}$. The cohom. dim $A(f) \geq m+n-k$.

We mentioned above the characteristic polynomials introduced by Dold and Nakaoka. They are used to generalize the Borsuk-Ulam Theorem to bundle maps of sphere bundles. In the present work, we have examined the possibility of generalising this to bundle maps of arbitrary manifold bundles. We have established that, as in the case when the manifold is a product of spheres, characteristic polynomials do not give satisfactory results. For example, when the manifold is $S^m \times S^n$, $(m \leq n)$, the method of characteristic polynomials gives the dimension of the Borsuk-Ulam set associated to a fibre map to be greater than or equal to $l+m-k$, where l is the dimension of the base space and k, the fibre dimension of the range. We refer to the results :

2.3.1 THEOREM

Let $\pi : E \longrightarrow B$ be an $S^m \times S^n$ bundle $(1 < m \leq n)$ with the fibrewise $\mathbb{Z}_2$-action $A\times A$, and $\pi' : E' \longrightarrow B$ be a vector bundle of dimension k, with fibrewise antipodal action, over a paracompact space B, such that the quotient bundle $\bar{\pi} : \bar{E} \longrightarrow B$ has the cohomology extension property. Let $f : E \longrightarrow E'$ be a fibre-preserving equivariant map. Put $Z = f^{-1}(0)$ and let $\bar{Z}$ denote the quotient by the action induced on Z. Now, if $q(t,s) \neq 0, \in H^*(B)[t,s]$ is a polynomial such that $q(t,s)|\ \bar{Z} \neq 0$, then $\exists$ polynomials $r_1(t,s)$, $r_2(t,s) \in H^*(B)[t,s]$ such that $q(t,s).W'(t) = r_1(t,s).W_1(t,s) + r_2(t,s).W_2(t,s)$.

2.3.2 COROLLARY : If the fibre dimension of E' is k, then $\deg(W'(t)) = k$, $\deg(W_1(t,s)) = m+1$ and $\deg W_2(t,s)) = 2n$. Then $q(t,s)|\bar{Z} \neq 0$, for all polynomials whose degree in t and s is less than $m-k+1$.

This means that the $H^*(B)$-homomorphism

$$\bigoplus_{i+j=0}^{m-k} H^*(B).\ t^i.s^j \longrightarrow H^*(\bar{Z})$$

$$t^i.s^j : \longrightarrow t^i.s^j|\bar{Z}$$

is a monomorphism. Hence if $k \leq m$,

$$\text{Cohom. dim. } \bar{Z} \geq \text{Cohom. dim. } B + m - k.$$

In particular, if B is a point, we have an equivariant map $f : S^m \times S^n \longrightarrow \mathbb{R}^k$ and if $k \leq m \leq n$, then Cohom. dim. $\bar{Z} \geq m-k$.

We now proceed to use the Bredon operation and Conner-Miller classes, to extend the generalisations of Dold and Nakaoka, to bundle maps of arbitrary manifold bundles. We see, in fact, that this method in several cases, gives better estimates of the dimension of the Borsuk-Ulam set than the method of characteristic polynomials. For example, using this method, in the case mentioned above, we get the dimension of the Borsuk-Ulam set to be greater than or equal to $m+n+1-k$. The relevant results are

3.4.1 THEOREM

Let (X, π_1, M, K_1) and (Y, π_2, M, K_2) be G-bundles where M, K_1 and K_2 are closed differentiable manifolds of dimensions m, n_1, n_2 respectively, and such that there exist cohomology extensions of the fibres for both bundles. Let $f: X \longrightarrow Y$ be a fibre-preserving map and $A(f) = \{x \in X \mid f(T^i x) = T^i(f(x))$, $i = 0,1,2,\ldots,p-1$, $T^i \in G\}$. Define $g : X \longrightarrow X \oplus Y$ by $g(x) = (x, f(x))$. Let $\bar{g}$ denote the restriction of g to a fibre and let $\hat{Q} : H^k(K_1 \times K_2) \longrightarrow H^{pk}(K_1 \times_G K_2)$ be the Bredon operation. Then if $\hat{Q}(\bar{g}_!(1))$ is non-zero, then $A(f)$ is non-empty and $\dim A(f) \geq m + n_1 - (p-1)n_2$.

3.5.1 COROLLARY

Let $f: X \longrightarrow Y$ be a fibre-preserving map from an n-sphere bundle with G action given by complex multiplication by pth roots of unity on S^n (n odd, when p odd), to a V^k- bundle with trivial G action, where V^k is a closed k-dimensional manifold, over the same closed m-dimensional manifold M.

Let $A(f) = \{x \in X \mid f(T^{\iota}x) = f(x),\ \iota = 0,1,2,\ldots,p-1\}$. Then if $(p-1)k < n$, $\dim A(f) \geq m+n-(p-1)k$.

3.5.2 <u>COROLLARY</u> :

Let $f\colon X \longrightarrow Y$ be a fibre-preserving map from an $S^m \times S^n$-bundle with G action induced by the diagonal action of complex multiplication by the pth roots of unity on $S^m \times S^n$ ($m \leq n$, m and n odd when p odd), to a V^k-bundle with trivial G action, where V^k is a closed k-dimensional manifold, over the same closed ι-dimensional manifold M^{ι}. Let $A(f) = \{(x,y) \in S^m \times S^n \mid f(T^{\iota}(x,y)) = f(x,y),\ \iota = 0,1,2,\ldots,p-1\}$. Now, if $(p-1)k < m \leq n$, then $\dim A(f) \geq \iota + m + n - (p-1)k$.

In earlier topological literature, the number $\ell-1$ was used under the name "co-index" (cfr. [7], [8]). In [8], Conner and Floyd computed the level of the real projective space $\mathbb{R}P^{2m-1}$ for certain lower dimensions ($\mathbb{R}P^{2m-1}$ here is considered to have a fixed point free involution induced by complex multiplication by i). A. Pfister and S. Stolz in [35] examine upper and lower bounds for the level of the Real Projective Space and the Complex Projective Space. Later Stolz in [39] computed fully, the level of the Real Projective Space $\mathbb{R}P^{2m-1}$. Other results on the level of spaces, in particular for Stieffel manifolds, are discussed by Dai and Lam in [12].

We have taken inspiration from the works of A. Pfister and S. Stolz and undertaken the computation of the level of the

Lens Space $L_k^{2m-1} \cong S^{2m-1}/\mathbb{Z}_k$. Using complex K-theory and some number theoretic arguments we establish a lower bound for the level of the Lens Space. We are also able to compute the level lf L_k^{2m-1}, when k is odd. Using cohomotopy Euler classes and K-theoretic codegree of vector bundles, we are able to refine the lower bound obtained earlier, when k is even. To establish an upper bound, we use a "vanishing line" for the E_2 term of Adams' Spectral Sequence. We find that if $2|k$, but $2^2 \nmid k$, the computation of the upper and lower bounds and a little analysis, enables us to fix the level of the Lens space L_k^{2m-1}, k even. The results are as follows :

4.2.2 PROPOSITION.

(i) The additive order of $\sigma_2 \in \mathbb{Z}[\sigma_2]/(1-(1+\sigma_2)^2, \sigma_2^{\,t})$ is 2^{t-1}. (cfr. [35]).

(ii) The additive order of $(1+\sigma_{2k})^k-1 \in \mathbb{Z}[\sigma_{2k}]/(1-(1+\sigma_{2k})^{2k}, \sigma_{2k}^{\,m})$ is $2^{\left[\frac{m-2}{2^r}\right]+1}$, where $k = 2^r.a$, $a \geq 1$, odd, $r \geq 0$.

In particular when k is odd, the additive order of $(1+\sigma_{2k})^k-1$ is 2^{m-1}.

4.2.4 THEOREM.

$\ell(m) \geq 2\left[\frac{m-2}{2^r}\right] + 3$, where $k = 2^r.a$, $a \geq 1$, odd, $r \geq 0$.

In particular $\ell(m) \geq 2m - 1$, if k is odd.

4.3.2 THEOREM

$\ell(m) = 2m$, where $\ell(m)$ denotes the level of the Lens space L_k^{2m-1}, k odd.

5.3.6 THEOREM

Let $\ell(m)$ denote the level of the Lens Space L_k^{2m-1}, $k = 2^r p$, p odd and $r > 1$. Also let $m-2 = 2^r a + b$, $a \geq 0$ and $0 \leq b \leq 2^r - 1$. Then,

$$\begin{aligned} \ell(m) \;&\geq\; 2a + 5 + c, \quad c < 2\left[\frac{m-1}{r+1} - a\right], \ c \text{ even} \\ &\qquad \text{when } 2m-(2a+2+c) \equiv 6 \bmod 8 \\ &\geq\; 2a + 4 + c, \quad c < 2\left[\frac{m-1}{r+1} - a\right] - 1, \ c \text{ odd}, \\ &\qquad \text{when } b = 2^r - 1 \\ &\geq\; 2a + 3 + c, \quad c < 2\left[\frac{m-1}{r+1} - a\right], \ c \text{ even} \\ &\qquad \text{in all other cases.} \end{aligned}$$

6.3.4 THEOREM

Let $\ell(m)$ be as above. Then,

$$\begin{aligned} \ell(m) &\leq m + 1 \quad \text{when } m \equiv 0,2 \bmod 8 \\ &\leq m + 2 \quad \text{when } m \equiv 1,3,4,5,7 \bmod 8 \\ &\leq m + 3 \quad \text{when } m \equiv 6 \bmod 8. \end{aligned}$$

6.4.1 THEOREM

Let $\ell(m)$ denote the level of the Lens space L_k^{2m-1}, $k = 2p$, p odd. Then,

$$\begin{aligned} \ell(m) &= m+1 \quad \text{if } m \equiv 0,2 \bmod 8 \\ &= m+2 \quad \text{if } m \equiv 1,3,4,5,7 \bmod 8 \\ &= m+3 \quad \text{if } m \equiv 6 \bmod 8. \end{aligned}$$

0.2 LAYOUT OF THE THESIS

We have tried to make each chapter self-sufficient, without repetition, of course, from chapter to chapter. The first three chapters deal with the Borsuk-Ulam Problem and the last three address themselves to the computation of the level of the Lens Space. We also have an Appendix. We have numbered most of the definitions, results, Lemmas, Propositions and Theorems. In the numbers that we have used, the first number refers to the chapter, the second to the section and the third to the definition, result, Lemma, Proposition or Theorem as may be the case.

In the first chapter we start with the definitions of Steenrod's external cohomology operation P, the Bredon operation Q, generalized mod p Conner-Miller classes and mod p Bredon-Hattori classes leading to the definition of the Self-intersection number. We also state without proof certain results needed for the remaining work of the chapter. We then show that the existing invariants fail to give any information about the Borsuk-Ulam set for a map from a product of spheres to a closed manifold, with respect to the product antipodal action on the product of spheres and trivial action on the closed manifold. We finally prove the Borsuk-Ulam Theorem for a product of spheres. We conclude with a few remarks.

The second chaper attempts to parametrize the Borsuk-Ulam Theorem for a product of spheres using the method of characteristic polynomials. We first of all compute the cohomology structure of $S^m \times S^n/T$ and basing on this, define the characteristic polynomials. We then have the Theorem, and the Corollary which give the parametrized Borsuk-Ulam

Theorem for a product of spheres. We conclude by noting that the estimate for the dimension of the Borsuk-Ulam set obtained by using this theorem is weaker than that obtained in Chapter One.

Chapter Three proceeds logically from Chapters One and Two. Since the method of Chapter One gives a better estimate of the dimension of the Borsuk-Ulam set in the case under consideration, than the method of Chapter two, we seek to extend the use of the invariant of Chapter One, to fibre-preserving maps of arbitrary manifold bundles over the same closed manifold. To enable us to use the method of Chapter One in the context of bundles, we first of all define parametrized Reduced power operation and Bredon operation on a fibre bundle. We proceed then to state and prove certain lemmas on the (co)homology of fibre-preserving maps of fibre bundles, needed later in the chapter to prove the parametrized Borsuk-Ulam Theorem for fibre-preserving maps of arbitrary manifold bundles, with fibrewise $\mathbb{Z}_p$-action. As applications we have a parametrized $\mathbb{Z}_p$-Borsuk-Ulam Theorem for fibre-preserving maps from a sphere bundle to an arbitrary manifold bundle over the same closed manifold, and also a parametrized $\mathbb{Z}_p$-Borsuk-Ulam Theorem for fibre-preserving maps from an $S^m \times S^n$-bundle to an arbitrary manifold bundle over the same closed manifold.

In Chapter Four, Five and Six, we address the problem of the level of spaces. In Chapter Four we use the complex

K-theory structure of the Lens Space and some Number Theoretic arguments to establish a lower bound for the level of Lens Spaces. We further use some results of Pfister and Stolz [35] adapted to the present situation and certain results of Conner and Floyd [7], to establish the level of the Lens Space L_k^{2m-1} when k is odd.

In Chapter five, we first formulate the problem of determining the level of a space, in terms of existence of a nowhere vanishing section for a certain vector bundle, an obstruction for which is the Cohomotopy Euler class. We proceed to establish preliminaries on Cohomotopy Euler Class and Codegree of vector bundles, and using these, we refine the lower bound for the level of L_k^{2m-1}, established in Chapter Four in certain cases, when k is even.

Chapter Six is devoted to establishing an upperbound for the level of L_k^{2m-1}, k even. We use Adams' spectral sequence, to do this. First of all, we present the necessary preliminaries on Adams' spectral sequence and certain results of Adams [1], and Stolz [39]. We proceed, using the $\mathbb{Z}_p$ cohomology structure of the Lens Space, to establish an upper bound for the level of the Lens Space when k is even. A little elementary analysis using the results of Chapters Four and Five enables us to determine the level of L_k^{2m-1} when $2|k$ but $2^2 \nmid k$.

At the end, we present an Appendix which presents the details of the computation of the integral and $\mathbb{Z}_p$-cohomology structure of the Lens Space. We have used the $\mathbb{Z}_p$-cohomology structure in the preceding work. We did the detailed computation because the necessary details were not available in a compact form anywhere.

ADDENDUM

After the preparation of the present thesis was completed, it was brought to our notice by S. Stolz that some of our results could be obtained by using obstruction theory as follows : Consider the natural projection map $\pi : L_k^{2m-1} \longrightarrow L_{kp}^{2m-1}$. If p is an odd integer, then π is a $\mathbb{Z}_{(2)}$-equivalence, i.e., it induces an isomorphism on $\pi_1(\ ) \otimes \mathbb{Z}_{(2)}$, where $\mathbb{Z}_{(2)}$ stands for integers localised at 2. By obstruction theory, it follows that the level of L_{kp}^{2m-1} is the same as the level of L_k^{2m-1}. We are thankful to Stolz for bringing this to our notice.

We have also come to learn that some of the results of Chapter Four is included in a result of Vicks (Bull AMS 75(1969) p 1017, Cor 3.3). Our proof is, however, more elementary. As it is also of independant interest and does not use any obstruction theory, we decided to retain it in the thesis.