STUDY OF SOME PROBLEMS IN DYNAMICAL OCEANOGRAPHY : A SURVEY

By

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CERTIFICATE

I certify that the dissertation entitled ***"Study of some problems in Dynamical Oceanography"*** *submitted by* ***Mrs. Smriti Choudhury*** *in partial fulfilment of the requirements for the degree of* ***Master of Philosophy*** *is the outcome of a study undertaken by the candidate. I certify that the sources from which ideas have been borrowed are duly referred to.*

The material in this dissertation has not been presented for the award of a degree in any university before.

The dissertation may be placed before the examiners for evaluation and necessary formalities.

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CHAPTER-I

INTRODUCTION

1. Dynamical Oceanography.

Oceanography is the study of the ocean in relation to the earth's total environment. The study covers various features of the ocean ranging from physical to biological aspects. The findings have immense practical implications.

In Dynamical Oceanography ocean waves and ocean currents form the subject matter of study. To explain these two oceanic phenomenon, such as waves and currents we apply the principles of fluid dyna mics along with those of thermodynamics. The starting point of oceanographic motions are determined by the systematic application of the fluid continuum equations of motion and conclusion derived are extensively used in the related field of oceanography.

1.1 Ocean Topography

The topography and structure of the oceans may be divided into three major parts. The Ist part includes the continental slopes and continental rises. The second is submarine canyons, deep-sea trenches and abyssal plain, midocean canyons and abyssal hills. The third part is the world wide ridge system and ridge valley.

The continental shelf which ringes the continents and slopes gently (gradient 1:1000) towards the ocean basin has a width from a few meters to several hundred miles and being mostly less than 600 ft deep. The continental slope extending from a marked shelf break point to a depth of 1 to 2 miles has a gradient 1:40. The continental rise, although it does not exist in all parts of the world, varies in width from a few miles to a few hundred miles, and has a gradual slope from 1:100 to 1:700, blending gently into the abyssal plain of the deep ocean. The continental margin is the transition zone between the continents and the ocean basins. The abyssal plains situate at the base of the continental rise are flat plains having a shallow gradient 1:1000. Associated with them are small oval shaped abyssal hills formed by continental sediment masses or igneous rock masses.

1.2. Ocean currents and waves

As preliminary to Dynamical oceanography, it is in place to mention as to the causes of ocean currents. There are two main generators of ocean currents. The first is wind, such as trade wind, cyclones etc. and the second is density difference.

The powerful flow of air which moves across the ocean surface in each hemisphere is known as Trade wind. In northern / southern hemisphere, the trade wind moves from N.E. to S.W./ S.E. to N.W. between the equator and 30^{o} north/south latitude. Each of these flows initiate the oceanic gyre of circulation in their respective hemisphere. North trades/south trades produces north equatorial/south equatorial current. Both of which move west parallel to each other on either side of the equator. These two currents are separated by the equatorial counter currents which runs to the east immediately north of the equator and is fed by the two equatorial currents and is a surface current.

The Coriolis force, and the shape and position of the land masses affect the direction currents take. The Coriolis force is the apparent force on particles moving relative the earth when observed from the earth. For unit mass, it is equal to $2V \times \Omega$ where V is the velocity moving relative to the earth and Ω is the earth's rotation vector. The vertical component is negligible compared with gravity and the horizontal component is V x f, where f = $2\Omega \sin \varphi$ (φ is the latitude) is the Coriolis parameter. It is observed that currents in the northern hemisphere veer right and those in the south veer left al l the objects moving over the surface of the earth de ate from their appointed path to the right/ left in the north/south hemisphere. This deviation of the currents and the deflection of the moving bodies from their appointed path occur because of the presence of the Coriolis force. The effect is more marked as the distance from the Equator increases, and results in the ocean currents circulating clockwise north of the Equator and anticlockwise south of it.

Strong and more well defined surface currents appear on the western sides of the ocean basin rather than on the eastern boundaries irrespective of the flow (northward or southward). The surface currents form huge circular rings in the ocean that move clockwise/anticlockwise in the northern/ southern hemisphere.

Body of ocean water can be thought of as a set of slabs, the top one driven by the wind and each driving the one below it by friction. At each stage, the speed of flow is reduced and in the northern hemisphere directed more to the right and in the southern hemisphere directed more to the left. This is due to the interaction between the Coriolis acceleration and the frictional drag existing between the vario's layers of water. This spirlling of current direction and decreasing velocity persists until friction become negligible. The layer in which this takes place is known as Ekman layer and the average flows in that layer is at right angle to the wind driving it.There is a vast system of deep water counter currents beneath the surface current. This is known as subsurface counter currents. Along with the surface currents, it also carries million tons of water in the opposite direction.

Temperature ans salinity changes the density of water, making it rise to the surface, or sink to the depths creating vertical circulatory currents. The thermocline is strongest in equatorial waters, where the contrast between the warned surface layer and the cold deeper water is greatest, at increasing latitudes it weakens, even vanishing in certain polar waters. In the upper oceans, from the thermocline-region to the surface are found the variety of inter-related motions, internal waves, turbulence and surface waves with whose dynamics we shall be concerned.

Ocean waves

Waves are the response made by the water as gravity or on a small scale surface tension tries to restore the surface to its original level when the wind disturbs it.

The size of the waves, both in length and in height increases with the length of time for which the wind has been blowing and the length of the water surface or 'fetch' over which it has been blowing.

When the wind stops blowing the waves do not disappear. Waves travel over hundreds of Kilometers of ocean. These waves running away from the wind which generated them are called 'swell'.

We may divide the types of waves into two groups:

(a) tidal waves (also called long waves in shallow water)

(b) surface waves (also called gravity waves).

Tidal waves possess the following distinctive features:

(i) wave length is large compared to the depth of water and so the surface tension may be neglected.

(ii) The vertical component of velocity and acceleration of the liquid particles are so small as to be negligible in comparison with the horizontal components. The horizontal component of velocity is the same for all particles across any vertical section of the bay or channels irrespective of their depths.

(iii) Water level at the mouth of the bay varies harmonically with time.

Gravity waves possess the following features.

(i) Wave length is much less than the depth of liquid.

(ii) The motion is irrotational as the flow arises initially out of conservative forces.

(iii) Vertical component of acceleration is not negligible.

(iv) Effect of disturbance is noticeable upto a finite depth only.

In order to study the above types of waves we make a fundamental assumption that the squares and the product of velocities and their derivatives are negligible, and the wave amplitude is small compared with the wave length.

In this dissertation we try to give an elementary survey of some important problems of waves in both rotating and nonrotating fluids. The discussion of the problems are based on the :

(i) Theory of linearised small amplitude irrotational wave motion in a nonrotating fluid.

(ii) The theory of linearised long wave motion wherein earth's rotation is taken into consideration.

The common theme of the problems is the estimation of the response of an inviscid unbounded ocean of constant depth and negligible compressibility to variable surface disturbances about a state of rest or of rest relative to the rotating earth, gravity being the only external body force. Referred to the frame of reference pertinent to any problem, the disturbances are either stationary or moving with constant or variable speeds, they are usually non-uniform and are periodic or oscillatory in time with a constant time period. Such disturbances can be related to natural meteorological causes or to artificial devices which are capable of altering a pre-existing atmosphere ocean blanace in a significant way.

The generation of long waves by time dependent and locally distributed atmospheric disturbances on the surface of rotating ocean has always been a prime subject of study for Oceanographers. It is well known that the free-wave response of a shallow, homogeneous, inviscid ocean of constant depth h to small perturbations about a state of rest relative to the rotating earth is in the form of Poincare - wave , provided the horizontal length scale is comparable with the Rossby radius of deformation, $(gh)^{1/2}/|f|$, the Coriolis parameter f being supposed constant. The transient barotropic response of the ocean is governed by the forced Klein-Gordon equation, which according to Gill [1] plays a great role in geophysical fluid dynamics.

Standard tools for dealing with the problems are Fourier transform, Laplace transform, special functions, Neumann's integral formula [6], contour integration etc. The solution of the forced Klein-Gordon equation in the form of an infinite integral can be obtained by constructing the appropriate Green's function as shown by Crease [17]. Because of the strongly oscillatory nature of the integrands, accurate numerical approximations are also not easily achieved and so one or the other methods of asymptotic evaluation are proven to be constantly needed to investigate the wave phenomenon.

Various asymptotic methods used are

(a) Stationary phase method [4].

If f(x) is oscillatory and large, and f(x) has a stationary point at x_0 i.e. $f'(x_0) = 0$, $0 < x < \infty$ and $f''(x_0) < 0$ then the integral $I(k) = \int_0^\infty \varphi(x)\, e^{i\nu f(x)}\, dx$

$$= \left\{ \frac{2\pi}{\nu f''(x_0)} \right\}^{1/2} \varphi(x_0)\, e^{i\nu f(x_0) - i\pi/4} + O(1/|\nu|)$$

If f(x is oscillatory and large, and f(x) has a stationary point at x_0 i.e. $f'(x_0) = 0$, $0 < x < \infty$ and $f''(x_0) > 0$ then the integral $I(k) = \int_0^\infty \varphi(x)\, e^{i\nu f(x)}\, dx$

$$= \left\{ -\frac{2\pi}{\nu f''(x_0)} \right\} \varphi(x_0)\, e^{i\nu f(x_0) + i\pi/4} + O(1/|\omega|)$$

(b) Lighthill's formula for real and distinct poles. [14]

(c) Bleistein's Method [13]

When the pole and the stationary point in the integrand coincide this method is used and is discussed in detail in various problems of the dissertation.

We now give the chapterwise arrangement of the dissertation.

Chapter II is set out to study the problems related to unsteady as well as the study-state wav e system in a nonrotating fluid due to moving or stationery oscillatory surface pressure.

Chapter III is devoted to study problems related to rotating fluid due to periodic surface pressure.

Chapter IV contains problems on storm surge and monsoon with reference to mathematical models developed for prediction of storm surge and monsoon.

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CHAPTER II

ASYMPTOTIC ANALYSIS OF SHALLOW WATER WAVES DUE TO AN OSCILLATORY SURFACE PRESSURE ON A NON-ROTATING FLUID

The purpose of this Chapter is to give an elementary survey of some problems of wave motion in non-rotating fluid.

The main objects of the problems are to determine unsteady as well as steady state (when it exists) wave system due to a surface pressure distribution of the form $F(x,y)e^{iwt}\,H(t)$, $H(t)$ being the Heaviside unit function.

The discussion of the problems is based on the linear theory of infinitesimal gravity waves by Wehausen and Laitone [2] . There the velocity potential integrals for the general initial value problem with an arbitary law of time dependence are given together with those for the steady state problems with harmonic pressures. The average rate of work is also mentioned. Choudhuri [3] applies the method of multiple Fourier transform to solve the general initial value problem with harmonic time dependence, assuming also a sudden initial rise of water level. The method is also applicable to the case of arbitrary time dependence to the pressure function. Asymptotic analysis of wave integrals obtained are carried out for large distances and times by the method of stationary phase. He also evaluates in exact terms the steady-state solutions found for a large class of function F(x,y) by direct passage to the limit $t \to \infty$.

A brief statement of the problem

Let x, y, z denote the rectangular cartesian co-ordinates taken in such a way that z = 0 and z = - h are the undisturbed free surface and the plane bottom of a horizontally unlimited mass of fluid of constant depth h.

A wave motion is excited in the ocean by a sudden surface displacement $\xi_0(x,y)$ at t = 0 together with a continuous surface pressure,

$$p_0(x,y,t) = F(x,y)\, e^{i\omega t} \quad , \quad t > 0 \qquad (1{\cdot}1)$$

With usual notations, the well known equations of motion and the boundary condition for the linearised theory are

$$\nabla^2 \varphi = 0 \qquad (1{\cdot}2)$$

$$\frac{\partial \varphi}{\partial t} = 0 \, , \quad \text{on} \quad z = -h \qquad (1{\cdot}3)$$

$$\frac{p_0}{\rho} = \left(\frac{\partial \varphi}{\partial t}\right)_{z=0} - g\zeta \qquad (1.4)$$

$$\frac{\partial \zeta}{\partial t} = -\left(\frac{\partial \varphi}{\partial z}\right)_{z=0}, \quad t > 0 \qquad (1.5)$$

$$\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial z} = \frac{1}{\rho}\frac{\partial p_0}{\partial t} \qquad z = 0 \qquad (1.6)$$

The prescribed initial conditions are

$$\left.\begin{aligned} \zeta(x, y, 0) &= \zeta_0(x, y) \\ \varphi_z(x, y, 0; 0) &= 0 \end{aligned}\right] \qquad (1.7)$$

The problem is to find a solution φ of (1.2) which satisfies (1.3), (1.6), and (1.7). ζ is then obtained from (1.4).

Solution

Equation (1.2) is transformed by using the double Fourier transform

$$\Phi(m_1, m_2, z; t) = \frac{1}{2\pi}\int\int_{-\infty}^{\infty} \varphi(x, y, z; t)\, e^{i(m_1 x + m_2 y)} \times dx\,dy$$

Solving the resulting differential equation subject to

(1.3)

$$\left.\begin{aligned} \phi &= C(m_1, m_2, t) \cosh m(z+h) \\ \text{where} \quad m &= (m_1^2 + m_2^2)^{1/2} \end{aligned}\right] \qquad (1\cdot 9)$$

Transforming the equation (1.6) and using (1.9), the resulting equation in C comes out to be

$$\frac{\partial^2 C}{\partial t^2} + \sigma^2 C = \frac{i\omega e^{i\omega t}}{\rho \cosh mh} \bar{F}(m_1, m_2) \qquad (1\cdot 10)$$

where

$$\sigma^2 = gm \tanh mh$$

and $\bar{F}(m_1, m_2)$ is transformed form of F(x,y). The solution of this equation under the conditions obtained by transforming (1.7), (1.4) and using (1.9) is

$$C = \sigma \cosh mh \left[\left\{ g\xi_0(m_1, m_2) + \frac{\bar{F}(m_1, m_2)}{\rho} \right\} \sin \sigma t + \frac{i\omega}{\rho} \bar{F}(m_1, m_2) \int_0^t e^{i\omega s} \sin \sigma(t-s)\, ds \right]$$

Inversion of (1.9) and (1.4) give finally

$$\varphi = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{\cosh m(z+h)}{\sigma \cosh mh}\, e^{-i(m_1 x + m_2 y)} \Big[g\zeta_0(m_1, m_2) \times \sin\sigma t + \rho^{-1} \bar{F}(m_1, m_2) \times \Big\{ \sin\sigma t + i\omega \int_0^t e^{i\omega s} \times \sin\sigma(t-s)\, ds \Big\} \Big] dm_1\, dm_2 \tag{1·11}$$

$$g\rho\zeta = -\bar{F}(x,y)\, e^{i\omega t} + \frac{1}{2\pi}\frac{\partial}{\partial t}\iint \sigma^{-1} e^{-i(m_1 x + m_2 y)} \Big[g\rho\zeta_0(m_1, m_2)\sin\sigma t + \bar{F}(m_1, m_2) \times \Big\{ \sin\sigma t + i\omega \int_0^t e^{i\omega s} \sin\sigma(t-s)\, ds \Big\} \Big] dm_1\, dm_2 \tag{1·12}$$

$$= -\bar{F}(x,y)\, e^{i\omega t} + \frac{g\rho}{2\pi}\frac{\partial}{\partial t}\iint_{-\infty}^{\infty} \zeta_0(m_1, m_2)\,\frac{\sin\sigma t}{\sigma} \times e^{-i(m_1 x + m_2 y)}\, dm_1\, dm_2 - \left(\frac{\partial^2}{\partial t^2} + i\omega\frac{\partial}{\partial t}\right) \cdot \frac{1}{2\pi}\iint_{-\infty}^{\infty} (\sigma^2 - \omega^2)^{-1} (\cos\sigma t - \cos\omega t)\, \bar{F}(m_1, m_2) \times e^{-i(m_1 x + m_2 y)}\, dm_1\, dm_2 \tag{1·13}$$

The various differentiations under the integral sign hold when disturbed surface area D is of finite extent.

Asymptotic value of ζ for large distances and times.

In the expression (1.12) passing to polar co-ordinates (m , ψ) and (s, θ) in the (m_1, m_2) and (x,y) plane respectively and replacing s by ts, then the part of these integrals containing ω is written first in the form

$$J_1 = \frac{\omega}{2\pi}\frac{\partial}{\partial t}\, t \int_0^1 e^{i\omega t(1-s)} \cdot J_2 \, ds$$

where

$$J_2 = \frac{1}{2}\int_0^{2\pi} d\psi \int_0^\infty \frac{m\,dm}{\sigma} \bar{F}(m\cos\psi, m\sin\psi) \times \left[e^{irP_1(m)} - e^{-irP_2(m)}\right],$$

$$P_j \equiv P_j(m, \psi; s) = \frac{ts}{r}\sqrt{gm\tanh mh} + (-1)^j m \times \cos(\psi - \theta), \quad j = 1, 2$$

$P_{j,m} = \frac{\partial P_j}{\partial m}$ is a monotone decreasing function in $0 < m < \infty$

Thus there is always a single +ve real root $m = \mu(\psi, s)$ of only either $P_1, m = 0$ or $P_2, m = 0$ according as $\cos(\psi - \theta)$ is $>$ or < 0 and $\frac{ts\sqrt{gh}}{r} > 1$

Applying the method of stationary phase [4] to the integral J_2,

$$J_2 \sim \frac{1}{2}\int_{\cos(\psi-\theta)>0}^{\psi} d\psi \cdot \frac{\mu}{\sigma(\mu)} \bar{F}(\mu\cos\psi, \mu\sin\psi) \times \left(\frac{2\pi}{-r P_{\mu\mu}(\mu,s)}\right)^{1/2} e^{in\varphi_1} + \frac{1}{2}\int_{\cos(\psi-\theta)<0}^{\psi} d\psi \times$$

$$\cdot \frac{\mu}{\sigma(\mu)} \bar{F}(\mu\cos\psi, \mu\sin\psi)\left(\frac{2\pi}{-r P_{\mu\mu}(\mu,s)}\right)^{1/2} e^{-in\varphi_2} \qquad (1\cdot 14)$$

where

$$\varphi_j(\psi, s) = \frac{ts}{r}\sqrt{g\mu\tanh\mu h} + (-1)^j \mu\cos(\psi-\theta) - \frac{\pi}{4r}$$

Therefore, $(1\cdot 14a)$

$$\varphi_{j,\psi} = 0 \quad \text{when} \quad \psi = \theta, \pi+\theta$$

$$|\varphi_{j,\psi\psi}|_{\psi=\theta,\pi+\theta} = \alpha(s),$$

and

$$\frac{ts\sqrt{g}}{2r}\left[\sqrt{\frac{\tanh\alpha h}{\alpha}} + \sqrt{\frac{\tanh\alpha h}{\alpha}}\, h\,\mathrm{sech}^2\alpha h\right] = 1$$

$$(1\cdot 15)$$

Applying the method of stationary phase again to (1.14)

$J_2 \simeq \frac{\pi}{r\sigma(\alpha)}\left(-\frac{\alpha}{P_{\alpha\alpha}(\alpha,s)}\right)^{1/2} \bar{F}(\alpha\cos\theta, \alpha\sin\theta)\, e^{inP_1(\alpha,\theta;s)}$ + its complex conjugate when $\frac{ts\sqrt{gh}}{h} > 1$, $r \gg 1$ . . . . $(1\cdot 15a)$

and when $r < t < \sqrt{gh}$, the contribution of the first term of J_2 above to J_1 is

$$\frac{\omega}{2r}\frac{\partial}{\partial t}\left\{t\int_{r/t\sqrt{gh}} \frac{1}{\sigma(\alpha)}\left(-\frac{\alpha}{P_{\alpha\alpha}(\alpha,s)}\right)^{1/2} \bar{F}(\alpha\cos\theta, \alpha\sin\theta)\times\right.$$

$$e^{irW(\alpha,s)}\,ds \qquad \text{(1.16)}$$

where $W(\alpha, s) = \left[ts(g\alpha\tanh\alpha h)^{1/2} - \alpha r + \omega t(1-s)\right]/r$

$$W'(s) \equiv \frac{dW}{ds} = t\left[(g\alpha\tanh\alpha h)^{1/2} - \omega\right]/r$$

$$W''(s) = \frac{1}{s}\frac{d\alpha}{ds} > 0$$

Now $W'(s) = 0$ gives $\alpha\tanh\alpha h = \omega^2/g$ (1.16a)

This equation has a single positive real root $\alpha = \alpha_0(\omega)$ (say). By (1.15), the corresponding value $s = s_0$ is

$$s_0 = r/(ct)$$

where

$$C = (2g\omega\alpha_0)^{-1}\left[g\omega^2 + h(\alpha_0^2 g^2 - \omega^4)\right] \qquad \text{(1.17)}$$

The stationary point of $w(\alpha, s)$ belong to s-interval in (1.16) if

$$\left[r/t(gh)^{1/2}\right] < s_0 < 1$$

The right inequality gives $r < ct$,

If, however,

$$ct < r < t\sqrt{gh}$$

there is no stationary point in the s-interval concerned. A similar process shows that the corresponding phase function in the contribution of the second term of J_2 (1·15a) to J_1 has no stationary point in the same interval. For (1·16) Applying the method of stationary phase to J_1, the remaining parts of the double integral in (1.12), yields the term obtained by replacing $g\rho\zeta_0(\nu\cos\theta, \nu\sin\theta)$ by $\bar{F}(\nu\cos\theta, \nu\sin\theta)$

$$g\rho\zeta \sim \frac{1}{r}\left(-\frac{r}{P'_{\nu\nu}(\nu)}\right)^{1/2} \times Re\Big[\big\{g\rho\zeta_0(\nu\cos\theta, \nu\sin\theta) + \bar{F}(\nu\cos\theta, \nu\sin\theta)\big\} e^{irP_1(\nu,\theta)}\Big] + \left\{\begin{array}{ll} \left(\frac{\pi\alpha_0}{2r}\right)^{1/2}\left(\frac{\omega}{c}\right)\bar{F}(\alpha_0\cos\theta, \alpha_0\sin\theta)\, e^{i(\omega t - \alpha_0 r + 3\pi/4)} & \text{when } r < ct < t\sqrt{gh} \\ 0 & \text{when } ct < r < t\sqrt{gh} \end{array}\right. \quad (1\cdot 18)$$

$$g\rho\zeta \sim 0 \quad \text{when} \quad r > t\sqrt{gh} \quad (r \to \infty)$$

where ν is the real +ve root of (1·15), for $s = 1$, $r < t\sqrt{gh}$,

and where $P'_{\nu\nu}(\nu) = P'_{j,\nu\nu}(\nu, \theta)$, $j = 1,2$

<u>Value of ζ in the Steady state.</u>

In (1.13) a change of variable to polar co-ordinates (m, ψ) in the (m_1, m_2) plane is made and introducing the following function

$$f(x,y,m) = \int_0^{2\pi} \bar{F}(m\cos\psi , m\sin\psi) \times e^{-im(x\cos\psi + y\sin\psi)} d\psi$$

$$= \iint_D F(\alpha, \beta) J_0(mR)\, d\alpha\, d\beta \qquad (2\cdot 1)$$

$$R = \left[(x-\alpha)^2 + (y-\beta)^2 \right]^{1/2}$$

observing that,

$$f(x,y,m) = \check{O}\left(m^{-3/2}\right) \quad \text{as} \quad m \to \infty$$

Integrating by parts the second term on the R.H.S. of (1.13) and as $t \to \infty$, it tends to zero. Hence finally (1.13) reduces to

$$- F(x, y)\, e^{i\omega t} - \frac{1}{2\pi}\left(\frac{\partial^2}{\partial t^2} + i\omega \frac{\partial}{\partial t} \right) \int_0^{\infty} M(\sigma)(\sigma - \omega)^{-1} \times (\cos\sigma t - \cos\omega t)\, d\sigma, \qquad (2\cdot 2)$$

where

$$M(\sigma) = m f(x,y,m) \frac{dm}{d\sigma} \Big/ (\sigma + \omega) \qquad (2\cdot 3)$$

The part (0, $\omega/2$) of the σ -range of integration in (2.2) becomes

$$- \cos\omega t \int_0^{\omega/2} M(\sigma)(\sigma - \omega)^{-1}\, d\sigma \qquad (2\cdot 4a)$$

and the part ($\omega/2$, ∞) of the σ -range of integration by Bochner's theorem [2] and as t $\rightarrow \infty$ amounts to

$$- \cos \omega t \, PV \int_{\omega/2}^{\infty} M(\sigma)\,(\sigma-\omega)^{-1}\, d\sigma \;-\; \pi \sin\omega t\, M(\sigma) \tag{2.4b}$$

Applying (2.4a) and (2 .4b) in (2'2) and then (1.13) finally becomes as t $\rightarrow \infty$

$$g\rho\xi = - F(x,y)\, e^{i\omega t} + \frac{\omega^2 e^{i\omega t}}{2\pi g}\left[\frac{\pi i \alpha_0}{2}\, \frac{\sinh 2\alpha_0 h}{\sinh^2\alpha_0 + h\omega^2/g} \iint_D F(\alpha,\beta)\, J_0(\alpha_0 R)\, d\alpha\, d\beta - \iint_D F(\alpha,\beta)\, d\alpha\, d\beta \times \times PV\int_0^{\infty} \frac{z J_0(zR)}{z\tanh zh - \omega^2/g}\, dz \right] \tag{2.5}$$

To find the principal value in (2.5) we regard z as a complex variable and integrate the function $f(z) = z\, H_0^{(1)}(zR)/(z\tanh Zh - \omega^2/g)$.along a semicircle with centre at z = 0 and radius $R_n = (n + 1/2)\pi/h$ in the upper half plane. The contour is indented at z = 0 and $z = \alpha_0$ to account for a branch point and the real pole of f(z) at these points. Here $H_0^{(1)}$ is the Hankel function of the first kind of order zero.

Now by Cauchy's residue theorem

$$PV\int_0^{\infty} \frac{z J_0(zR)\, dz}{z\tanh zh - \omega^2/g} = 4g^2 \sum_{s=1}^{\infty} k_s^2 K_0(k_s R)/(h\omega^4 - g\omega^2 + hg^2 k_s^2) \quad \ldots\ldots (2.7)$$

where k's satisfies the equation $k \tanh kh = -\omega^2/g$

and f(z) has n purely imaginary poles ik_s, s = 1 to n.

$K_0(z)$ is the modified Bessel function of the third kind of order zero, which varies as $z^{-1/2} e^{-z}$ as $z \to \infty$

Therefore the series in (2.7) converges rapidly.

Observing that

$$J_0(x') = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix'\cos(\psi-\theta)}\, d\psi ,$$

$$\theta = \tan^{-1}\frac{y-\beta}{x-\alpha}$$

and

$$\frac{\sinh 2\alpha_0 h}{\sinh^2\alpha_0 + h\omega^2/g} = g/(c\omega)$$

c being defined by (1.17). (2.5) can be finally expressed as

$$g\rho\zeta = -F(x,y)e^{i\omega t} + \frac{1}{4}(\alpha_0\omega/c)\int_{-\pi}^{\pi}\bar{F}(\alpha_0\cos\psi, \alpha_0\sin\psi)$$
$$\times e^{i(\omega t + \pi/2 - \alpha_0 r\cos(\psi-\theta))}\, d\psi$$
$$- \frac{\omega^2 e^{i\omega t}}{2\pi g}\iint_D F(\alpha,\beta)\,d\alpha\, d\beta \times 4g^2 \times$$
$$\sum_{s=1}^{\infty} k_s^2 K_0(k_s R)/(h\omega^4 - g\omega^2 + hg^2k_s^2) \qquad (2.8)$$

Some characteristics of the Motion

In the unsteady state, the motion has the following characteristic according to (1.14).

(1) The disturbance is mainly confined within the circle

$c_1: r = t\sqrt{gh}$

(2) Inside c_1, the nature of this motion changes sharply across the circle $c_2 : r = ct$. Within c_2, there are both dispersive waves and dominant progressive waves represented by the two terms of the right of (1.18) taken in order. The diverging progressive wave is of length ($2\pi/\alpha_0$), freqnency $\omega/2\pi$ and a directionally variable amplitude diminishing as $r^{-1/2}$. Between c_2 and c_1 and possibly beyond c_1, the progressive waves disappear and dominant waves are dispersive in nature. Analytically, we have an example of the Stokes phenomenon in this change of character of the asymptotic expansions across the circle r = ct.

As $t \longrightarrow \infty$, the wave motion attains a steady state everywhere as given by (2.5). For large values of r, the steady state motion amounts to the same diverging progressive wave found within c_2 together with a stationary wave.

Thus asymptotic development obtained by Chaudhury exhibit a change of character of the waves about the circle $r = ct$ (i.e. where the stationary point coincide with the pole). This is also a non-uniform expansion in the parameter (r/ct) and valid only when (r/ct) is bounded away from unity. C.R. Mondal [5] uses Neumann's integral formula [6] to find the solution of the problem of waves generated by an initial surface pressure $F(x,y) \exp (i\omega t)$ on a nonviscous, horizontally unbounded liquid of finite depth h. In her paper an attempt has been made to determine an asymptotic expansion of the wave integral for large ($\omega^2 r/g$) which shall remain uniformly valid even when (r/ct) is near unity. To gain this objective a method due to Bleistein [13] is used and the final expression leads to the two limiting cases $r/ct \ll 1$ and $r/ct \gg 1$.

<u>A brief statement of the problem</u>

Let (r, θ, z) denote cylindrical co-ordinates taken in such a way that $z = 0$ and $z = -h$ are the undisturbed free surface and the plane bottom of a horizontally unbounded mass of liquid of constant depth h. A wave motion is generated in the ocean by the continued action of a surface pressure over a region D, starting from the moment $t = 0$. With usual notations the well known equations of motion and the boundary conditions on the linearised theory are:

$$\varphi_{rr} + r^{-1}\varphi_r + r^{-2}\varphi_{\theta\theta} + \varphi_{zz} = 0 \qquad (3\cdot 2)$$

$$\varphi_z = 0 \qquad z = -h \qquad (3\cdot 3)$$

$$\rho^{-1} p_0 = (\varphi_t)_{z=0} - g\zeta \qquad (3\cdot 4)$$

$$\zeta_t = -(\varphi_z)_{z=0}, \quad t > 0 \qquad (3\cdot 5)$$

$$\varphi_{tt} + g\varphi_z = \rho^{-1} p_0 t \qquad z = 0 \qquad (3\cdot 6)$$

The initial conditions are

$$\varphi(x, y, z; 0) = 0 \quad , \zeta(x, y; 0) = 0 \qquad (3\cdot 7)$$

Our problem is to find a solution φ of (3.2) which satisfy (3.3), (3.6) and (3.7). ζ is then obtained from (3.4).

Solution:

One can represent the function $G(r,\theta)$ by Neumann's integral formula [6]

$$2\pi G(r,\theta) = \int_0^\infty k\,dk \int_0^\infty \int_{-\pi}^{\pi} G(R,\Theta) \times J_0\left[k\left\{R^2 + r^2 - 2Rr\cos(\Theta - \theta)\right\}^{1/2}\right] R\,dR\,d\Theta \qquad (3\cdot 8)$$

Since the function

$$J_0\left[k\left\{R^2 + r^2 - 2Rr\cos(\Theta - \theta)\right\}^{1/2}\right]\cosh k(z+h)\,\mathrm{sech}\,kh;$$

is a Solution of Laplace's equation (3.2) which satisfies (3.3), the velocity potential φ can be expressed in the form

$$\varphi(r,\theta,z;t) = (2\pi\rho)^{-1}\int_0^\infty k\,dk\int_0^\infty\int_{-\pi}^{\pi} C_1(k,t)\cosh k(z+h)\,\mathrm{sech}\,kh$$
$$\times J_0\left[k\left\{R^2 + r^2 - 2Rr\cos(\Theta - \theta)\right\}^{1/2}\right]\times$$
$$G(R,\Theta)\,R\,dR\,d\Theta \qquad (3.9)$$

Substituting for φ in (3.6) and (3.7), the equation for c_1 comes out to be

$$C_{1tt} + \sigma^2 C_1 = i\omega\exp(i\omega t) \qquad (3.10)$$

Subject to the boundary conditions

$$C_1(k,0) = 0, \qquad C_{1t}(k,0) = 1, \qquad (3.11)$$
$$\sigma^2 = gk\tanh kh$$

The solution of (3.10) under (3.11) is

$$C_1 = \sigma^{-1}\sin\sigma t + i\omega\sigma^{-1}\int_0^t \sin\sigma(t-s)\exp(i\omega s)\,ds \cdots (3.12)$$

Following a transformation to cartesian coordinates

$$(r,\theta)\to(x,y), \quad (R,\Theta)\to(\alpha,\beta), \quad G(r,\theta)\to F(x,y)$$

the expression

$$J_0\left[k\left\{R^2+r^2-2Rr\cos(\Theta-\theta)\right\}^{1/2}\right]=J_0\left[k\left\{(x-\alpha)^2+(y-\beta)^2\right\}^{1/2}\right]$$

$$=(2\pi)^{-1}\int_0^{2\pi}\exp(-ikR'\cos\psi)\,d\psi=(2\pi)^{-1}\int_0^{2\pi}\exp\left\{-ikR'\cos(\psi-\gamma)\right\}d\psi$$

where

$$R'^2=(x-\alpha)^2+(y-\beta)^2 \qquad \gamma=\tan\left\{(y-\beta)/(x-\alpha)\right\}$$

using (3.9) and (3.12) in (3.4) and following the above transformation, (3.4) finally becomes

$$g\rho\zeta=-F(x,y)\exp(i\omega t)+(2\pi)^{-1}(\partial/\partial t)\int_0^{\infty}k\,dk\int_0^{2\pi}\sigma^{-1}\bar{F}(k\cos\psi,$$
$$k\sin\psi)\times\left\{\sin\sigma t+i\omega\int_0^t\sin\sigma(t-s)\exp(i\omega s)\,ds\right\}$$
$$\times\exp\left\{-ikr\cos(\psi-\theta)\right\}d\psi \qquad (3.16)$$

where

$$\bar{F}(k\cos\psi,k\sin\psi)=(2\pi)^{-1}\iint_{-\infty}^{\infty}F(\alpha,\beta)\exp\left\{ik(\alpha\cos\psi+\beta\sin\psi)\right\}d\alpha\,d\beta \qquad (3.17)$$

The expression (3.16) agrees with one obtained by Chaudhuri [3] if the initial rise of level assumed by him is neglected.

Uniform asymptotic expansion of ζ

The s-integration in (3.16) is first completed. The dominant part of the asymptotic value of ζ are determined from ψ -integral of (3.16).

writing

$$Q(\psi)=\cos(\psi-\theta)$$

The zeros of $Q'(\psi)$ are

$$\psi = \theta,\ \pi + \theta \qquad \text{when} \qquad 0 \le \theta \le \pi$$

and $$\psi = \theta,\ \theta - \pi \qquad \text{when} \qquad \pi < \theta < 2\pi$$

With $kr \gg 1$, the method of stationary phase [4] gives

$$i(8\pi r)^{1/2} g\rho\zeta = (\delta/\delta t)\int_k^\infty \Big(k^{1/2}\Big[\exp\{i(\pi/4 - kr)\}\bar{F}(k\cos\theta, k\sin\theta) + \exp\{-i(\pi/4 - kr)\}\bar{F}(-k\cos\theta, -k\sin\theta)\Big] \times \Big[(\sigma-\omega)^{-1}\{\exp(i\sigma t) - \exp(i\omega t)\} - (\sigma+\omega)^{-1} \times \{\exp(-i\sigma t) - \exp(i\omega t)\}\Big] + O(r'^{-1/2})\,dk \tag{3.19}$$

where $r' = r/h$

The part of k-integral of (3.19) of which the integral has both a stationary point and a pole is

$$I_{11} = \int_k^\infty (\sigma-\omega)^{-1} k^{1/2}\,\bar{F}(k\cos\theta, k\sin\theta) \times \exp\{ir(\sigma t/r - k) + i\pi/4\}\,dk \tag{3.20}$$

The stationary point $k = k_1(r/t)$ is the real positive root of the equation, $\sigma'(k)t/r - 1 = 0$ (3.21)

This root exists when $(t\sqrt{gh}/r) > 1$. The pole $k = k_0(a)$ is the real positive root of

$$k \tanh kh = \omega^2/g$$

when the stationary point coincides with the pole i.e. $k_1 = k_0$, from (3.21), it becomes

$$r/t = \sigma'(k_0) = (\sqrt{gh}/2)\left[(kh)^{-1/2}(\tanh kh)^{1/2} + (kh\coth kh)^{1/2}\operatorname{sech}^2 kh\right]_{k=k_0}$$

$$= (2k_0\omega g)^{-1}\left[g\omega^2 + h\left(g^2k_0^2 - \omega^4\right)\right] = C$$

For large r and t, the asymptotic expansion of I_{11} is obtained by Bleistein's method [13] which will remain valid when r/ct is near 1. For this purpose, a new variable z' defined by the relation

$$(\sigma t/r - k) - (\omega t/r - k_0) = -(2^{-1}z'^2 + az'), \qquad (3.24)$$

is used. Here $z' = 0$ corresponds to the pole $k = k_0$ and 'a' is a parameter set in for describing the distance $|k_1 - k_0|$ which is found out to be

$$a^2/2 = -\{\sigma''(k_1)t/2r\}(k_0 - k_1)^2$$

$$\therefore \quad a = \pm|k_0 - k_1|\left[\sigma''(k_1)t/r\right]^{1/2}$$

The positive sign of 'a' is fixed by requirement that

$\left(\frac{dk}{dz'}\right)_{z'=0}$ should be positive, so that

$$\operatorname{sgn} a = \operatorname{sgn}(k_0 - k_1).$$

Taking

$$(\sigma-\omega)^{-1} k^{1/2} \bar{F}(k\cos\theta, k\sin\theta)\frac{dk}{dz'} = (A/z') + \sum_{0}^{\infty} \beta_n (z'+a)^n$$

which gives

$$A = C^{-1} k_0^{1/2} \bar{F}(k_0\cos\theta, k_0\sin\theta),$$

and

$$\beta_0 = \Big[\{k_0^{1/2}/C(k_0-k_1)\} \times \bar{F}(k_0\cos\theta, k_0\sin\theta) + (k_1^{1/2}/\{\sigma(k_1)-\omega\}) \times \bar{F}(k_1\cos\theta, k_1\sin\theta)\Big] \times \Big[r/t\,|\sigma''(k_1)|\Big]^{1/2}$$

For large r, applying Watson's Lemma [16], the integral I_{11} finally becomes

$$I_{11} \sim \exp\{i(ra^2/2 + \omega t - k_0 r + \pi/4)\} \times \Big[A \int_{z'}^{\infty} z'^{-1} \exp\{-i(r/2)(z'+a)^2\}\,dz' + \beta_0 \int_{z'}^{\infty} \exp\{-i(r/2)(z'+a)^2\}\,dz'\Big]$$

$$\sim \exp\{i(ra^2/2 + \omega t - k_0 r + \pi/4)\} \times \Big[A \int_{-\infty}^{\infty} (z_1-a)^{-1} \exp(-irz_1^2/2)\,dz_1' + \beta_0 \int_{-\infty}^{\infty} \exp(-irz_1^2/2)\,dz_1\Big]$$

$$= -\pi(1+i) A \,\mathrm{sgn}\, a.\, \mathrm{CIS}(ra^2/2)\cdot \exp\{i(\omega t - k_0 r + \pi/4)\} + \beta_0 (\pi/r)^{1/2} (1-i) \exp\{i(ra^2/2 + \omega t - k_0 r + \pi/4)\} + O(r^{-1}) \tag{3.30}$$

where CIS(x) = C(x) + i s(x), C(x) and s(x) being Fresnel's integral. The two terms in (3.30) give an asymptotic expansion which is uniform when $|1 - (ct/r)| <$ constant.

The asymptotic values of the other pertinent k-integrals in (3.19) are

$$\text{(i)}\ \exp(i\omega t)\int_k^{\infty} k^{1/2}\,\bar{F}(k\cos\theta, k\sin\theta)\,(\sigma-\omega)^{-1}\exp(-ikr)\,dk_1$$
$$\sim -\pi i C^{-1} k_0^{1/2}\,\bar{F}(k_0\cos\theta, k_0\sin\theta)\exp\{i(\omega t-k_0 r)\}+O(r'^{-1}).$$

$$\text{(ii)}\ k^{1/2}(\sigma+\omega)^{-1}\,\bar{F}(-k\cos\theta, -k\sin\theta)\exp\{i(kr-\sigma t)\}dk$$
$$\sim\left[2\pi/t\,|\sigma''(k_1)|\right]^{1/2}\left[\sigma(k_1)+\omega\right]^{-1}k_1^{1/2}\,\bar{F}(-k_1\cos\theta, -k_1\sin\theta)$$
$$\times\exp\left[-i\left\{\sigma(k_1)t-k_1 r-\pi/4\right\}\right]\times H\left(t\sqrt{gh}/r-1\right)+O(r'^{-1}).$$
$$rk_1 \gg 1. \qquad (3'31)$$

With these values in (3.19) and replacing $ra^2/2$ by means of (3.24) with $z' = -a$, $k = k_1$ both before and after performing the operation by $\partial/\partial\xi$ the result obtained is

$$g\rho\zeta \sim \Big[-(\pi k_0/8r)^{1/2}(\omega/C)\left\{1-2^{1/2}\exp(-i\pi/4)\,\mathrm{sgn}\,a\cdot\right.$$
$$\left.\mathrm{CIS}(ra^2/2)\right\}\times\bar{F}(k_0\cos\theta, k_0\sin\theta)\exp\{i(\omega t-k_0 r$$
$$-\pi/4)\}+2^{-1}\sigma(k_1)\left\{rt\,|\sigma''(k_1)|\right\}^{-1/2}\times\Big\{k_1^{1/2}\left[\sigma_1(k_1)-\omega\right]^{-1}$$
$$\times\bar{F}(k_1\cos\theta_1, k_1\sin\theta_1)-\omega k_0^{1/2}\bar{F}(k_0\cos\theta, k_0\sin\theta)$$
$$\times\left[\sigma(k_1)\cdot C\cdot(k_1-k_0)\right]^{-1}\Big\}\times\exp\left\{i\left[\sigma(k_1)t-k_1 r\right]\right\}$$
$$+2^{-1}\left\{rt/\sigma''(k_1)|\right\}^{-1/2}k_1^{1/2}\,\sigma(k_1)\left\{\sigma(k_1)+\omega\right\}^{-1}$$
$$\times\bar{F}(-k_1\cos\theta, -k_1\sin\theta)\exp\left\{-i\left[\sigma(k_1)t\quad k_1 r\right]\right\}\Big]$$
$$\times H\left(t\sqrt{gh}/r-1\right)+O(r'^{-3/2}),\quad r'\to\infty,\ k_1 r\to\infty$$
$$(3\cdot 32)$$

limiting cases of (3.32)

(I) If $k_1/k_0 \gg 1$, then $\sigma''(k_1) \ll \sigma''(k_0)$ or

$r \ll ct < t(gh)^{1/2}$. Under this condition a is negative and $ra^2/2 \gg 1$.

The wave system as observed from (3.32) generally consists of dispersive wave accompanied by a forced progressive wave of velocity w/k_0. The latter constitutes the dominant part and possesses a verying amplitude expressed by a certain combination of Fresnel's integral.

(II) If $ko/k_1 \gg 1$, then $ct \ll r$ and for sufficiently small values of $c/\sqrt{gh}$ we see that the progressive wave disappear and the wave motion is wholly dispersive at places for which $ct \ll r$.

We will now discuss the problem of small amplitude surface waves due to an oscillatory pressure on an infinitely deep single fluid, initially at rest. This problem has been considered by several authors (Debnath, [8], Miles [9], Sen [10] either in two or three dimensions. The asymptotic behaviour of the unsteady waves presented by them does not remain uniformly valid through the transition zone. Mahanti [11] tries ro find a uniformly valid asympotic solution of the problem of unsteady waves resulting from the pressure $f(r)e^{iwt} \times H(t)$ where $r = (x^2 + z^2)^{1/2}$, $H(t)$ is Heavside unit function. He uses the method of Vander Waerden and stationary phase principle for large r and t.

Solution:

The integral solution for the surface displacement ζ [8] is

$$\zeta = \int_0^\infty k \bar{\zeta} J_0(kr)\,dk$$
$$= \frac{e^{i\pi/4}}{2^{3/2} g\rho (\pi R)^{1/2}} \left[I_1(t,-r) + I_2(-t,-r) \right] +$$
$$\frac{e^{-i\pi/4}}{2^{3/2} g\rho (\pi R)^{1/2}} \left[I_1(t,r) + I_2(-t,r) \right] \qquad (1)$$

$$I_1(t,-r) = \int_0^\infty \sqrt{k}\,\bar{f}(k) \frac{\sigma(\sigma+\omega)e^{i\omega t} - 2\sigma^2 e^{i\omega t}}{\sigma^2 - \omega^2} e^{-ikr} dk \qquad (2)$$

$$I_2(-t,r) = \int_0^\infty \sqrt{k}\,\sigma \bar{f}(k)(\sigma+\omega)^{-1} e^{-i(\sigma t - kr)} dk \quad \cdots (3)$$

where $\sigma = \sqrt{gk}$, ρ is the density of the fluid and a bar denotes the Hankel transform with respect to r. It is noted that the pole of the integrand in (2) is $k = \omega^2/g$ and the stationary point of the combination $\sigma t - kr$ is $k = gt^2/4r^2$.

Applying Vander Waerden's method for a uniform asymptotic expression of $I_1(t,-r)$ in the neighbourhood of $r = gt/2w$ which arises when the stationary point coincide with the poles.

writing

$$I_1(t,-r) = 2(g)^{-3/2} \oint_0^\infty \sigma^3 \bar{f}(\sigma^2/g)(\sigma-\omega)^{-1} \times e^{i(\sigma t - \sigma^2 r/g)} d\sigma - 4 e^{i\omega t}(g)^{-3/2} \times \oint_0^\infty \sigma^4 \bar{f}(\sigma^2/g)(\sigma^2-\omega^2)^{-1} e^{-i\sigma^2 r/g} d\sigma \tag{4}$$

where the path of integration passes below/ above according as $\omega \gtrless gt/2r$. The first integral of (4) above, after introducing the new variable $z = \sigma/\omega - gt/2r\omega$ assumes the form

$$\oint \left[\sigma^2 \bar{f}(\sigma^2/g)(\sigma-\omega)^{-1} \frac{d\sigma}{dz}\right] e^{i(gt^2/4r - r\omega^2 z^2/g)} dz \tag{5}$$

writing

$$\sigma^3 \bar{f}(\sigma^2/g)(\sigma-\omega)^{-1} \frac{d\sigma}{dz} = A(z-z_0)^{-1} + \sum_0^\infty B_n z^n \tag{6}$$

where $z = (z)_\sigma = \omega$

A and B_0 are found to be

$A = \omega^2 \bar{f}(\omega^2/g)$

$B_0 = \omega(gt/2r - \omega)^{-1}\left[g^3t^3/8r^3 \bar{f}(gt^2/4r) - \omega^3 \bar{f}(\omega^2/g)\right]$

For $\omega^2 r/g \gg 1$, application of Watson's Lemma to (5) and after (6) is substituted in it, we see that (5) is asymptotically equivalent to

$$-\pi\sqrt{2}\,A\,e^{i(\omega t - \omega^2 r/g + \pi/4)}\ \mathrm{cis}\left(\omega^2 r z_0^2/g\right)\ \mathrm{sgn}\, z_0$$

$$+\sqrt{\frac{\pi g}{r}}\ \frac{B_0}{\omega}\ e^{i(gt^2/4r - \pi/4)}$$

where

$$\mathrm{cis}(x) = C(x) + i\,S(x) = \int_0^{z} \frac{e^{it}}{\sqrt{\pi t}}\,dt$$

Again for $\omega^2 r/g \gg 1$, the second integral of (4) is equivalent to

$$-\frac{\pi i}{2}\,\omega^3\,\bar{f}\,(\omega^2/g)\ e^{-i\omega^2 r/g}$$

Hence the evaluation of $I_1(t, -r)$ is completed. The remaining integrals in (1) may be evaluated by the method of stationary phase [4] for $r \gg 1$. Therefore, the final expression for ζ from (1) can be written in the form

$$\zeta \sim \frac{i\sqrt{\pi}\,\omega^3 \bar{f}\,(\omega^2/g)}{\sqrt{2}\; g^{5/2}\; \rho\,\sqrt{r}} \left\{ 1 + i\,(1+i)\,\mathrm{Cis}\left(\frac{\omega^2 r}{g}\, z_0^{\,2}\right)\right\} \mathrm{sgn}\, z_0 x$$

$$e^{i(\omega t - \omega^2 r/g + \pi/4)} + \frac{g^3 t^3 / 8 r^3 \bar{f}\,(gt^2/4r^2) - \omega^3 \bar{f}\,(\omega^2/g)}{\sqrt{2}\,\rho\, g^2\, r\, (gt/2r - \omega}$$

$$\times\; e^{i(gt^2/4r)} + \frac{g t^3 \bar{f}\,(gt^2/4r^2)\; e^{-igt^2/4r}}{2^{7/2}\;\rho r^4\; (gt/2r + \omega)} +$$

$$O(r^{-3/2})\,,\; r \gg 1 \qquad (9)$$

This result is uniformly valid with respect to $gt/2r\omega$ in the neighbourhood of $gt/2r\omega = 1$, $r \longrightarrow \infty$. We further note that the dominant contribution to ζ is a progressive wave moving with velocity g/ω . In the limiting cases i.e. when $gt/2r\omega \gg 1$, the dominant contribution to ζ is nill and when $gt/2r\omega \ll 1$, the dominant contribution to ζ is

$$\frac{i\,\sqrt{2\pi}\;\omega^3 \bar{f}\,(\omega^2/g)}{g^{5/2}\;\rho\,\sqrt{r}}\; e^{i(\omega t - \omega^2 r/g + \pi/4)}$$

Lastly we will discuss the two dimensional initial-value problem of waves due to an oscillatory pressure moving uniformly on the surface of a fluid of depth d and density ρ_2 , overlying another fluid of depth h and density ρ_1 . This problem has also been considered by C.R. Mondal [15]. Here the integral representation of the displacements of both the free surface and the interface are obtained with the help of the integral transform method. To obtain the steady-state solution of the problem at large distance a theorem due to Lighthill [14] is applied.

Formulation of the problem

A heavy homogeneous fluid of density ρ_2 and of uniform depth d lies on another heavy homogeneous fluid of density ρ_1 ($\rho_1 > \rho_2$) and of uniform depth h-d, both fluids being of infinite horizontal extent, initially both the fluid are at rest and the surface of separation is a horizontal plane. Waves are generated both on the free-surface and on the interface by the action of a pressure distribution f(x) exp (iwt) which is suddenly applied on the free surface at the initial moment and is then made to move continually with an uniform velocity V.

For the two-dimensional motion in the xy-plane, a coordinate system is chosen with the origin on the undisturbed free surface, where y-axis is positive upwards and the x-axis is positive to the right. It is convenient to pose the problem in a moving co-ordinate system in which the origin moves with the velocity V along the positive x-direction, so that the applied pressure strip is fixed with respect to this system. Let $y = \eta(x,t)$ and $y = -d + \zeta(x,t)$ be the equations to the free surface and the surface of separation respectively at any subsequent time t. With usual notations, the linearised equations of wave motion and the other necessary conditions are

(i) $\nabla^2 \varphi_j = 0,\ x \in (-\infty, \infty),\ y \in [\cdot\ '\ 0] \qquad y \in [-h, -d]$

$$j = 2, 1 \tag{1}$$

(ii) $$\rho_2^{-1} p_2 + \varphi_{2t} - V\varphi_{2x} + g\eta = 0,\ -d \le y < 0,\ t > 0 \tag{2}$$

$$\eta_t - V\eta_x = \varphi_{2y},\quad y = 0,\ t > 0 \tag{3}$$

(iii) $$\rho_1^{-1}(p_1 - \rho_2 g d) + \varphi_{1t} - V\varphi_{1x} + g(y+d) = 0, \tag{4}$$

$$-h < y < -d,\ t > 0$$

where $\varphi_j(x,y,t)$ and $p_j(x,t)$, $j = 2,1$ are the velocity potentials and the pressure for the upper and lower fluid respectively.

(iv) The interfacial conditions are

$$\rho_2(\varphi_{2tt} - 2V\varphi_{2tx} + V^2\varphi_{2xx} + g\varphi_{2y})$$

$$= \rho_1(\varphi_{1tt} - 2V\varphi_{1tx} + V^2\varphi_{1xx} + g\varphi_{1y}),\ y = -d \cdots (5)$$

$$\varphi_{2y} = \varphi_{1y},\quad y = -d \tag{6}$$

(v) The initial conditions are

$$\left.\begin{array}{ll} \varphi_j = 0, & (j = 2, 1) \\ \eta = 0 = \zeta & \end{array}\right] \quad t = 0 \tag{7}$$

(vi) The condition at infinity are

$$\varphi_{jx} = 0, \quad \varphi_j = 0, \quad (j = 2, 1), \quad |x| \to \infty \tag{8}$$

(vii) The bottom condition is

$$\varphi_{1y} = 0 \qquad \text{on} \qquad y = -h \tag{9}$$

the suffixes 2 and 1 refer to upper and lower fluids respectively.

Solution:

The non-temporal part of the applied pressure f(x) is assumed to be a generalised function. The Fourier transform of φ with respect to x is denoted by $\bar{\varphi}$:

$$\bar{\varphi}(k, y; t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \varphi(x, y; t) \exp(-ikx)\, dx \tag{10}$$

Application of the usual Fourier transform method followed by a slight simplification enable us to write the following solution of the system (1) - (8).

$$\begin{aligned} g\bar{\eta} = \Big[& -i\sigma_1 A_1 \exp\{i(vk + \sigma_1)t\} + i\sigma_1 A_2 \exp\{i(vk - \sigma_1)t\} \\ & - i\sigma_2 A_3 \exp\{i(vk + \sigma_2)t\} + i\sigma_4 A_4 \exp\{i(vk - \sigma_2)t\} \\ & - \{i(\omega - vk)p + \rho_2^{-1}\bar{f}\} \exp(i\omega t) \Big] \end{aligned} \tag{11}$$

Where A_1, A_2, A_3, A_4 are constants and are given below

$$\begin{aligned} A_1 = & \frac{i\bar{f}}{2\rho_2\sigma_1(\sigma_1^2 - \sigma_2^2)} \left[\frac{\omega - kv}{\sigma_1 - \omega + kv} \left\{ (\omega - kv)^2 + R(k) \right\} \right] + \\ & + \frac{i\bar{f}}{2\rho_2\sigma_1(\sigma_1^2 - \sigma_2^2)} \left[\sigma_2^2 + (\omega - kv)(kv - \sigma_1) \right] + \frac{\bar{f}}{2\rho_2\sigma_1(\sigma_1^2 - \sigma_2^2)} \times \\ & \times \left[\omega(\omega - kv) + gk\,\text{th}\,kd \right], \end{aligned}$$

$$A_2 = \frac{i\bar{f}}{2\rho_2\sigma_1(\sigma_1^2-\sigma_2^2)}\left[\frac{(\omega-kv)}{\sigma_1+\omega-kv}\left\{(\omega-kv)^2+R(k)\right\}\right] - \frac{i\bar{f}}{2\rho_2\sigma_1(\sigma_1^2-\sigma_2^2)}$$

$$\times\left[\sigma_2^2+(\omega-kv)(kv+\sigma_1)\right] - \frac{\bar{f}}{2\rho_2\sigma_1(\sigma_1^2-\sigma_2^2)}\left[\omega(\omega-kv)+gk\,\mathrm{th}\,kd\right]$$

$$A_3 = -\frac{i\bar{f}}{2\rho_2\sigma_2(\sigma_1^2-\sigma_2^2)}\left[\frac{(\omega-kv)}{\sigma_2-\omega+kv}\left\{(\omega-kv)^2+R(k)\right\}\right] + \frac{i\bar{f}}{2\rho_2\sigma_2(\sigma_1^2-\sigma_2^2)}$$

$$\times\left[\sigma_1^2+(\omega-kv)(kv-\sigma_2)\right] + \frac{\bar{f}}{2\rho_2\sigma_2(\sigma_1^2-\sigma_2^2)}\left[\omega(\omega-kv)+gk\,\mathrm{th}\,kd\right] \tag{12}$$

$$A_4 = -\frac{if}{2\rho_2\sigma_2(\sigma_1^2-\sigma_2^2)}\left[\frac{(\omega-kv)}{\sigma_2-\omega+kv}\left\{(\omega-kv)^2+R(k)\right\}\right] - \frac{i\bar{f}}{2\rho_2\sigma_2(\sigma_1^2-\sigma_2^2)}$$

$$\times\left[\sigma_1^2+(\omega-kv)(kv+\sigma_2)\right] - \frac{\bar{f}}{2\rho_2\sigma_2(\sigma_1^2-\sigma_2^2)}\left[\omega(\omega-kv)+gk\,\mathrm{th}\,kd\right]$$

$$R(k) = \frac{(\epsilon-1)\,gk\,\mathrm{th}\,k(h-d)}{1+\epsilon\,\mathrm{th}\,kd\,\mathrm{th}\,k(h-d)} \qquad \text{and} \quad \epsilon = \rho_2/\rho_1$$

$$P(k) = i\rho_2^{-1}\left\{(\omega-kv)^2+R(k)\right\}\frac{(\omega-vk)\,\bar{f}(k)}{\left\{\sigma_1^2-(\omega-vk)^2\right\}\left\{\sigma_2^2-(\omega-vk)^2\right\}} \tag{13}$$

The quantities $\pm\sigma_1$, $\pm\sigma_2$ are the roots of the frequency equation

$$\sigma^4 - gk\,\frac{\mathrm{th}\,kd+\mathrm{th}\,k(h-d)}{1+\epsilon\,\mathrm{th}\,kd\,\mathrm{th}\,k(h-d)}\,\sigma^2 + \frac{(1-\epsilon)\,g^2k^2\,\mathrm{th}\,kd\,\mathrm{th}\,k(h-d)}{1+\epsilon\,\mathrm{th}\,kd\,\mathrm{th}\,k(h-d)} = 0$$

According to Wehausen [1] and Mahanti [11], there exist a constant h_0 ($0 < h_0 < d$) such that

$$\sigma_2^2 = gk\,\mathrm{th}\,kh_0 \tag{14}$$

$$\sigma_1^2 = gk\,\mathrm{th}\,k(h-h_0) \tag{15}$$

Applying Fourier inverse integral,

$$\eta = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \bar{\eta} \exp(ikx)\, dk$$

the surface displacement η can be obtained.

To find the integral representation for the displacement ζ we notice that $p_1 = p_2$ on the interface $y = -d + \zeta$. Therefore from equation (2) and (4)

$$\rho_2 (\varphi_{2t} - V\varphi_{2x} + g\zeta) = \rho_1 (\varphi_{1t} - V\varphi_{1x} + g\zeta) \text{ for } y = -d$$

Applying Fourier transform in this equation and after some simplification the result obtained is

$$\begin{aligned}\left\{ i(1-\epsilon) g^2 k \right\} \bar{\zeta} &= \left\{ \sigma_1^3 P_1(k) - gk\sigma_1 Q_1(k) \right\} \left[A_1 \exp\left\{ i(vk+\sigma_1)t \right\} \right. \\ &\quad \left. - A_2 \exp\left\{ i(vk-\sigma_1)t \right\} \right] + \left\{ \sigma_2^3 P_1(k) - gk\sigma_2 Q_1(k) \right\} \times \\ &\quad \left[A_3 \exp\left\{ i(vk+\sigma_2)t \right\} - A_4 \exp\left\{ i(vk-\sigma_2)t \right\} \right] \\ &\quad + \left\{ (\omega - kv)^3 P_1(k) - gk(\omega - vk) Q_1(k) \right\} p \exp(i\omega t) \\ &\quad - i\rho_2^{-1} \bar{f} (\omega - vk)^2 P_1(k) \exp(i\omega t) \qquad (16)\end{aligned}$$

Where

$$P_1(k) = c\, th\, k(h-d)\, ch\, kd + \epsilon\, sh\, kd$$

$$Q_1(k) = c\, th\, k(h-d)\, sh\, kd + \epsilon\, ch\, kd$$

The displacement ζ is obtained by the Fourier inverse transform formula

Asymptotic Analysis of η and ζ in the steady state ($t \to \infty$)

The dominant parts of the asymptotic values of η and ζ in the steady state are determined by the following result due to Lighthill [14]:

If f(k) has a simple pole at $k = \alpha$, then as $|x| \to \infty$

$$\int_a^b f(k)\, e^{ikx}\, dk = \pi i \operatorname{sgn} x . \text{(Residue of } f(k)e^{ikx} \text{ at } k = \alpha) + O(1/|x|), \text{ when } a < \alpha < b \quad \cdots\cdots (6)$$

We first express η in a suitable form. Equations (10) and (11) together with the expressions on (12) after transforming the range of integration to the positive k-axis give the following expression for η.

$$(2\pi)^{1/2} g \rho_2 \eta = \eta_1 + \eta_2 + \eta_3$$

$$\begin{aligned}\eta_1 = & - \exp(i\omega t)\int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1}(\omega - vk)^2\left\{(\omega - vk)^2 + R(k)\right\} \times \\ & \qquad \left\{\sigma_1^2 - (\omega - vk)^2\right\}^{-1} \times \bar{f}(k)\exp(ikx)\, dk \\ & - \exp(i\omega t)\int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1}(\omega + vk)^2\left\{(\omega + vk)^2 + R(-k)\right\} \times \\ & \qquad \left\{\sigma_1^2 - (\omega + vk)^2\right\}^{-1} \times \bar{f}(k)\exp(-ikx)\, dk \\ & - 1/2\int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1}(\omega - vk)\left\{(\omega - vk)^2 + R(k)\right\}(\sigma_1 + \omega - vk)^{-1} \times \\ & \qquad \bar{f}(k)\exp\left\{i(kx + \overline{vk - \sigma_1} \cdot t)\right\} dk +\end{aligned}$$

[contd]

$$+ 1/2 \int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1} (\omega - vk) \{ (\omega - vk)^2 + R(k) \} (\sigma_1 - \omega + vk)^{-1} \times$$
$$\bar{f}(k) \exp\{ i (kx + \overline{kv + \sigma_1 t}) \} dk +$$
$$+ 1/2 \int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1} (\omega + vk) \{ (\omega + vk)^2 + R(-k) \} (\sigma_1 - \omega - vk)^{-1} \times$$
$$\bar{f}(-k) \exp\{ -i (kx + \overline{vk - \sigma_1 t}) \} dk$$

$$- 1/2 \int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1} (\omega + vk) \{ (\omega + vk)^2 + R(-k) \} (\sigma_1 + \omega + vk)^{-1} \times$$
$$\bar{f}(-k) \exp\{ -i (kx + \overline{vk + \sigma_1 \cdot t}) \} dk$$

$$\eta_2 = \exp(i\omega t) \int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1} (\omega - vk)^2 \{ (\omega - vk)^2 + R(k) \} \times$$
$$\{ \sigma_2^2 - (\omega - vk)^2 \}^{-1} \times \bar{f}(k) \exp(ikx) dk +$$

$$+ \exp(i\omega t) \int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1} (\omega + vk)^2 \{ (\omega + vk)^2 + R(-k) \} \times$$
$$\{ \sigma_2^2 - (\omega + vk)^2 \}^{-1} \times \bar{f}(k) \exp(-ikx) dk +$$
$$+ 1/2 \int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1} (\omega - vk) \{ (\omega - vk)^2 + R(-k) \} \{ \sigma_2 + \omega - vk \}^{-1} \times$$
$$\times \bar{f}(k) \exp\{ i (kx + \overline{vk - \sigma_2 t}) \} dk$$
$$- 1/2 \int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1} (\omega - vk) \{ (\omega - vk)^2 + R(+k) \} \{ \sigma_2 \mp \omega - vk \}^{-1} \times$$
$$\times \bar{f}(k) \exp\{ i (kx + \overline{vk + \sigma_2 \cdot t}) \} dk$$

$$- 1/2 \int_0^\infty (\sigma_1^2 - \sigma_2^2)^{-1} (\omega + vk) \{ (\omega + vk)^2 + R(-k) \} \times$$
$$(\sigma_2 - \omega - vk)^{-1} \times \bar{f}(-k) \exp\{ -i (kx + \overline{vk - \sigma_2 \cdot t}) \} dk +$$

$$+ \frac{1}{2}\int_0^{\infty} (\sigma_1^2 - \sigma_2^2)^{-1} (\omega + vk) \{(\omega + vk)^2 + R(-k)\} \times$$

$$(\sigma_2 + \omega + vk)^{-1} \times \bar{f}(-k) \exp\{-i(kx + \overline{vk + \sigma_2} \cdot t)\} dk$$

$$\eta_3 = 1/2 \int_{-\infty}^{\infty} (\sigma_1^2 - \sigma_2^2)^{-1} \Big[\{\sigma_2^2 + (\omega - kv)(kv - \sigma_1)\} e^{i\sigma_1 t} +$$

$$+ \{\sigma_1^2 + (\omega - kv)(kv - \sigma_2)\} e^{i\sigma_2 t} +$$

$$+ \{\sigma_1^2 + (\omega - kv)(kv + \sigma_2)\} e^{-i\sigma_2 t} \Big] \times$$

$$\bar{f}(k) e^{ik(x + vt)} dk -$$

$$- i/2 \int_{-\infty}^{\infty} (\sigma_1^2 - \sigma_2^2)^{-1} \{ \omega(\omega - kv) + gk \operatorname{th} kd \} \times$$

$$\times (e^{i\sigma_1 t} + e^{-i\sigma_1 t} + e^{i\sigma_2 t} + e^{-i\sigma_2 t}) \times$$

$$\times \bar{F}(k) e^{ik(x + vt)} dk$$

$$- (2\pi)^{1/2} f(x) e^{i\omega t} .$$

The contribution of η_3 to the steady-state value of η will not be considered any further because the integrands therein have no poles. Now first we consider the integral in η_1. We see that the poles of the integrand in η_1 are the points where $\sigma_1 - \omega + vk = 0$, $\sigma_1 + \omega - vk = 0$, $\sigma_1 - \omega - vk = 0$.

These poles are located as the points of intersection of the curve $y = \sigma_1$ and the straight lines $y=Vk-w$ $y = -VK + w$, $y = VK + w$. We see that two distinct pole α_3 and α_4 are present for all values of w, V, k while the poles α_1 and α_2 may be distinct, may be coincident or may not be present at all. This depends upon whether the straight line $y = Vk + w$ intersects the curve $y = \sigma_1$ at two distinct points or touches it or remain outside the curve. When it is tangent to the curve at the point α_0 (say) then α_0 is obviously a double pole of the last two integrand in η_1.

The condition for the occurance of the double pole can be written as

$$\sigma_1'(k) = V, \quad \sigma_1(k) = Vk + \omega \tag{17}$$

where σ_1' denotes the differentiation of σ_1 with respect to k. It is a critical case. We characterise this case by $V = v_1^*$. Then it is clear that α_1, α_2 are distinct and real only when $V < v_1^*$.

The integrals in η_1 are denoted as I_{1n}, n = 1 to 6 taken in order and the integrals in η_2 as I_{2n}, n = 1 to 6. The asymptotic values of I_{11} and I_{12} as $|x| \to \infty$ can be written down in terms of the contribution from their poles.

$$I_{11} = (i\pi/2)\,\mathrm{sgn}\,x \left\{ \bar{f}(\alpha_3)\psi_1(\alpha_3)\exp(i\alpha_3 x) - \bar{f}(\alpha_4)\chi_1(\alpha_4) \times \right.$$
$$\left. \times \exp(i\omega t), \right. \qquad (18)$$

$$I_{12} = -(i\pi/2)\,\mathrm{sgn}\,x\; H(V_1^* - V)\left\{ \bar{f}(-\alpha_1)\psi_1(-\alpha_1)\exp(-i\alpha_1 x) + \right.$$
$$\left. + \bar{f}(-\alpha_2)\psi(-\alpha_2)\exp(-i\alpha_2 x)\right\} \times \exp(i\omega t), \qquad (19)$$

$$\psi_1(k) = (\sigma_1^2 - \sigma_2^2)^{-1}\left\{(\omega - Vk)^2 + R(k)\right\}(Vk - \omega)(\sigma_1' - V)^{-1}$$
$$\chi_1(k) = (\sigma_1^2 - \sigma_2^2)^{-1}\left\{(\omega - Vk)^2 + R(k)\right\}(Vk - \omega)(\sigma_1' + V)^{-1}$$

The integrand I_{13}, I_{14} and I_{15} will be evaluated for t The integral I_{13} is evaluated by substituting $m = Vk - \sigma_1$ To make the integrand single-valued within the range of integration, we divide the range of integration $(0, \infty)$ over k into two sub-ranges $(0, \alpha_0')$ and (α_0', ∞). It is easily found that $m(\alpha_0') = -M$, where M is a fixed positive number. Therefore I_{13} reduces to

$$I_{13} = 1/2\left(-\int_{-M}^{0} + \int_{-M}^{\infty}\right)\left[(\sigma_1^2 - \sigma_2^2)^{-1}(\sigma_1' - V)^{-1}(\omega - kV)\left\{(\omega - kV)^2 + R(k)\right\} \times \right.$$
$$\left. \times \bar{f}(k)\exp(ikx)\right](m - \omega)^{-1}\exp(imt)\,dm.$$

The quantities within the square bracket being expressed in terms of m. Since the pole $k = \alpha_3$ occurs in the second integral, from relation (b)

$$I_{13} \simeq -(i\pi/2)\,\bar{f}(\alpha_3)\psi(\alpha_3)\exp\left\{i(\omega t + \alpha_3 x)\right\}, \quad t \to \infty \qquad (20)$$

To evaluate the integral I_{14}, a substitution $m = Vk + \sigma_1$ is made. Then using the relation (b)

$$I_{14} \simeq -(i\pi/2)\,\bar{f}(\alpha_4)\chi_1(\alpha_4)\exp\left\{i(\omega t + \alpha_4 x)\right\}, \quad t \to \infty \qquad (21)$$

The integral I_{15} is evaluated by the same substitution and by the same sub-division of the range of integration as I_{13}. Observing that the pole $k = \alpha_1$ occurs in the second integral and the pole $k = \alpha_2$ occurs in the first integral. By using (b),

$$I_{15} \simeq -(i\pi/2) H(V_1^* - V)\left[\bar{f}(-\alpha_1)\psi_1(-\alpha_1)\exp(-i\alpha_1 x) - \bar{f}(-\alpha_2)\times \psi_1(-\alpha_2)\exp(-i\alpha_2 x)\right]\exp(i\omega t) \quad \underset{t\to\infty}{\cdots} \quad (22)$$

The integrand of I_{16} having no positive pole will not contribute to the asymptotic value of η_1.

The evaluation of η_2 follows silimarly; the correspondence of symbols and relation with the preceding case being as follows:

$$\sigma_1 \rightarrow \sigma_2, \quad \alpha_j \longrightarrow \beta_j, \quad j = 0, 1, 2, 3, 4;$$

$$V_1^* \rightarrow V_2^*, \quad (\sigma_1^2 - \sigma_2^2)(\psi_1, \chi_1) \longrightarrow (\sigma_1^2 - \sigma_2^2)(\psi_2, \chi_2).$$

The results (18) to (22) and the corresponding values of the integrals in η_2 enable us to write η in the following form

$$(2\pi)^{1/2} g \rho_2 \eta \simeq i\pi H(V_1^* - V)\psi(-\alpha_2)\bar{f}(-\alpha_2)\exp\{i(\omega t - \alpha_2 x)\}$$
$$- i\pi H(V_2^* - V)\psi_2(-\beta_2)\bar{f}(-\beta_2)\exp\{i(\omega t - \beta_2 x)\},$$
$$x \longrightarrow \infty \dots\dots (22)$$

$$(2\pi)^{1/2} g\rho_2\eta \simeq -i\pi H(V_1^* - V)\Psi_1(-\alpha_1)\bar{f}(-\alpha_1)\exp\{i(\omega t - \alpha_1 x)\}$$
$$+ i\pi\Psi_1(\alpha_3)\bar{f}(\alpha_3)\exp\{i(\omega t + \alpha_3 x)\} -$$
$$i\pi\chi_1(\alpha_4)\bar{f}(\alpha_4)\exp\{i(\omega t + \alpha_4 x)\} +$$
$$i\pi H(V_2^* - V)\Psi_2(-\beta_1)\bar{f}(-\beta_1)\exp\{i(\omega t - \beta_1 x)\} -$$
$$i\pi\Psi_2(\beta_3)\bar{f}(\beta_3)\exp\{i(\omega t + \beta_3 x)\} +$$
$$i\pi\chi_2(\beta_4)\bar{f}(\beta_4)\exp\{i(\omega t + \beta_4 x)\}, \; x \to -\infty \quad (22)$$

Proceeding in the same way as for η, we obtain the asymptotic values of ζ as follows:

$$(2\pi)^{1/2}(1-\epsilon)\rho_2 g\zeta \simeq i\pi H(V_1^* - V) M_1(-\alpha_2)\Psi_1(-\alpha_2)\bar{f}(-\alpha_2)\exp\{i(\omega t - \alpha_2 x)\}$$
$$- i\pi H(V_2^* - V) M_2(-\beta_2)\Psi_2(-\beta_2)\bar{f}(-\beta_2)\times$$
$$\times \exp\{i(\omega t - \beta_2 x)\}, \; x \to \infty \quad (23)$$

$$(2\pi)^{1/2}(1-\epsilon)\rho_2 g\zeta \simeq i\pi M_1(\alpha_3)\Psi_1(\alpha_3)\bar{f}(\alpha_3)\exp\{i(\omega t + \alpha_3 x)\} -$$
$$i\pi M_1(\alpha_4)\chi_1(\alpha_4)\bar{f}(\alpha_4)\exp\{i(\omega t + \alpha_4 x)\} -$$
$$i\pi H(V_1^* - V) M_1(-\alpha_1)\Psi_1(-\alpha_1)\bar{f}(-\alpha_1)\exp\{i(\omega t - \alpha_1 x)\} -$$
$$i\pi H(V_2^* - V) M_2(-\beta_1)\Psi_2(-\beta_1)\bar{f}(-\beta_1)\exp\{i(\omega t - \beta_1 x)\}$$
$$- i\pi M_2(\beta_3)\Psi_2(\beta_3)\bar{f}(\beta_3)\exp\{i(\omega t + \beta_3 x)\} +$$
$$i\pi M_2(\beta_4)\chi_2(\beta_4)\bar{f}(\beta_4)\exp\{i(\omega t + \beta_4 x)\},$$
$$x \to -\infty \quad (23)$$

$$M_1(k) = \frac{\operatorname{sh} k(h-h_0-d) + (1-\epsilon)\operatorname{sh} k(h-d)\operatorname{ch} k(h-h_0-d)}{\operatorname{sh} k(h-d)\operatorname{ch} kh_0}$$

$$M_2(k) = \frac{\operatorname{sh} k(h-h_0) + (1-\epsilon)\operatorname{sh} k(h-d)\operatorname{ch} k(d-h_0)}{\operatorname{sh} k(h-d)\operatorname{ch} kh_0}$$

Critical case

Here we consider the cases in which the integrands have double poles. When $V = V_1^*$, let $\alpha_1 = \alpha_2 = \alpha_0$ (say), it is easily seen that $\eta(x,t)$ and $\zeta(x,t)$ both become singular on the critical curves.

For the first component η_1 of η, it is clear that the values of I_{11}, I_{13} and I_{14} remain unchanged, while each of the integrands of I_{12} and I_{15} has a double pole at α_0 and hence these integrals are to be calculated. This can be done with the help of Lighthill's result (b).

$$I_{12} \simeq -i\pi \operatorname{sgn} x \left[\frac{\sigma_1}{\sigma_1''} (\sigma_1^2 - \sigma_2^2)^{-1} \left\{ (\omega + vk)^2 + R(-k) \right\} \right]_{k=\alpha_0} \times$$
$$\times \bar{f}(-\alpha_0) \exp\left\{ i(\omega t - \alpha_0 x) \right\}, \quad |x| \to \infty, \quad V = V_1^* \tag{24}$$

The dominant contribution to the integral I_{15} comes from the neighbourhood of the point $k = \alpha_0$. The integral I_{15} can be written as

$$I_{15} = 1/2\Big(\int_{\alpha_0-\epsilon'}^{\alpha_0} + \int_{\alpha_0}^{\alpha_0+\epsilon'}\Big)\Big[(\sigma_1^2-\sigma_2^2)^{-1}(\omega+vk)\{(\omega+vk)^2+R(-k)\}\times$$

$$\bar{f}(-k)\Big](\sigma_1-\omega-vk)^{-1}\exp\{i(-kx+\overline{\sigma_1-vk}\,.t)\}\,dk;\ (0<\epsilon'<1) \qquad \ldots\ldots (25)$$

Substituting $m = Vk + \omega - \sigma_1$, near $k = \alpha_0$, then

$$m = -1/2(k-\alpha_0)^2\,\sigma_1''(\alpha_0)$$

Therefore

$$m = 0 \quad \text{for} \quad k = \alpha_0$$

$$m = \epsilon_1 \text{ (say)} \quad \text{for} \quad k = \alpha_0 - \epsilon' \text{ and } k = \alpha_0 + \epsilon'$$

ϵ_1, being a positive number.

Substituting these in (25)

$$I_{15} = \int_0^{\epsilon_1}\frac{(\omega+vk)\{(\omega+vk)^2+R(-k)\}}{(\sigma_1^2-\sigma_2^2)(\sigma_1-\omega-vk)}\times$$

$$\times\,\bar{f}(-k)\exp\{i(-kx+\overline{\sigma_1-vk}\,.t)\}\,dk$$

$$= 2^{-1/2}\Big[\frac{\sigma_1}{\sqrt{\sigma_1''}}\ \frac{\{(\omega+vk)^2+R(-k)\}\bar{f}(-k)}{\sigma_1^2-\sigma_2^2}\Big]_{k=\alpha_0}\exp\{i(\omega t-\alpha_0 x)\}\times$$

$$\int_0^{\epsilon_1} m^{-3/2}\exp(-imt)\,dm.$$

Replacing the upper limit of the integral by ∞, its asymptotic value is found out.

$$I_{15} \simeq 2^{-1/2}(-3/2)!\,t^{1/2}\Big[\frac{\sigma_1}{(\sigma_1'')^{1/2}}(\sigma_1^2-\sigma_2^2)^{-1}\{(\omega+vk)^2+R(-k)\}\Big]_{k=\alpha_0}\times$$

$$\times\,\bar{f}(-\alpha_0)\exp\{i(\omega t-\alpha_0 x+\pi/4)\},\ t\to\infty,\ v=v_1^* \qquad \ldots\ldots (26)$$

Similarly the two integrals I_{22} and I_{25} are evaluated in terms of their contributions from the double pole β_0 ,

$$I_{22} = \pi i \operatorname{sgn} x \left[\frac{\sigma_2}{\sigma_2''} (\sigma_1^2 - \sigma_2^2)^{-1} \left\{ (\omega + vk)^2 + R(-k) \right\} \right]_{k=\beta_0} \times$$

$$\bar{f}(-\beta_0) \exp\{ i(\omega t - \beta_0 x) \}, \quad |x| \to \infty, \quad V = V_2^* \tag{27}$$

$$I_{25} = 2^{-1/2} (-3/2)! \, t^{1/2} \left[\frac{\sigma_2}{(\sigma_2'')^{1/2}} (\sigma_1^2 - \sigma_2^2)^{-1} \left\{ (\omega + vk)^2 + R(-k) \right\} \right]_{k=\beta_0} \times$$

$$\times \bar{f}(-\beta_0) \exp\{ i(\omega t - \beta_0 x) \}, \quad |x| \to \infty, \quad V = V_2^* \tag{28}$$

It is seen that the solution for $\eta(x,t)$ becomes singular on the critical curves. For its contribution from the integrals I_{12}, I_{15}, I_{22} and I_{25} as given in equations (24), (26), (27) and (28) respectively become asymptotically infinitely large. Similar analysis will show that the solution for $\zeta(x,t)$ also becomes singular on the critical curves.

6. Physical Discussions

When $V \neq V_j^*$ ($j = 1,2$) we see from (22) and (23) that the steady-state wave system consists of several simple progressive waves. The numbers and distributions of the surface and interfacial waves relative to the pressure system follow the same rule and so we mention the characteristic of surface waves only. From (22), we see that there are three possible distributions of surface waves.

(i) If V is less than both V_1^* and V_2^*, the total number of the waves is eight, two on the front side and six on the back side.

(ii) If V lies between V_1^* and V_2^*, there are six waves, one on the front side and five on the back side.

(iii) If V exceeds both V_1^* and V_2^*, there are four waves, all of which are on the backside.

The waves on the front side are of lengths $2\pi(\alpha_2^{-1}, \beta_2^{-1})$ and they move with constant velocities $\omega(\alpha_2^{-1}, \beta_2^{-1})$ along the positive x-direction. The waves on the back side are of lengths $2\pi(\alpha_1^{-1}, \beta_1^{-1}, \alpha_3^{-1}, \alpha_4^{-1}, \beta_3^{-1}, \beta_4^{-1})$ and these move with constant velocities $\omega(\alpha_1^{-1}, \beta_1^{-1}, \alpha_3^{-1}, \alpha_4^{-1}, \beta_3^{-1}, \beta_4^{-1})$, the first two in the positive x-direction and the last four in the negative x-direction. The amplitudes of all these waves are independent of x and t, these however, depend on the parameter

$$V' = V(gh)^{-1}, \qquad \omega' = \omega(h/g)^{1/2}, \qquad d' = d/h$$

and $\varepsilon = \rho_2/\rho_1$.

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CHAPTER III

DISPERSIVE LONG WAVES DUE TO ATMOSPHERIC DISTURBANCES ON A ROTATING OCEAN.

In this chapter we are concerned with the transient barotropic response of a shallow, homogeneous inviscid ocean of constant depth h to a horizontal wind forcing which is time dependent as well as varying in space. Such a response (In fact, response to many other forcing mechanisms such as atmospheric pressure variations, precipitation, evaporation differences, tide producing forces etc.) is governed by the forced shallow water equation or the forced Klein-Gordon equation [1]. It is also not difficult to obtain the solution of this equation in the form of infinite integrals by constructing the appropriate Green's function, as shown by Crease [17] who uses the Morse - Feshbach solution to analyse the propagation of long wave due to some particular steady atmospheric disturbances acting on a sea surface. This approach is based on formulation of the so called radiation condition at infinity. Next we will discuss an initial value problems of shallow water waves where the solution is obtained by the method of integral transform. The wave integral is then subjected to an asymptotic analysis which follows essentially a method due to Bleistein [13] and is uniform across the line produced by the coalescing of the pole and the stationery point of the wave spectrum.

We shall discuss about the following two problems:

(i) Propagation of long waves due to atmospheric disturbances on a rotating sea. (Crease) [17].

(ii) Shallow water waves on a rotating ocean due to an oscillatory surface pressure (Mondal) [18].

Problem (i)

Introduction: Long waves in shallow water in a nonrotating system are not dispersive but in a rotating system they are. Here we will study the effect of this dispersion on the wave elevation and velocity. Bottom friction is neglected and nonlinear terms in the equations of motion are also neglected.

A stationery force of constant amplitude is suddenly applied over one half of an infinite sea $-\infty < x < \infty$ at time $t = 0$ and thereafter. It is found that the interval between the arrival of a surge at a point and its first maximum there decreases with increasing distance of the point from the edge of the generating area, but the maximum diminishes in amplitude correspondingly. In a nonrotating system the amplitude would increase without limit, while in a rotating system it is found that a steady-state amplitude is reached by a series of oscillations approximating in period to the inertia period.

The longitudinal and transverse velocities are also found out.

The equations of motion.

Neglecting the nonlinear terms, the fundamental equations of motion are

$$\frac{\partial u^*}{\partial t} - \gamma v^* = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u^*}{\partial z^2} \qquad (1)$$

$$\frac{\partial v^*}{\partial t} + \gamma u^* = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 v^*}{\partial z^2} \qquad (2)$$

and the hydrostatic pressure equation is

$$p = p_a + g\rho(\zeta - z) \qquad (3)$$

where u^*, v^* are the horizontal velocities in the x,y direction and ζ is the elevation of the surface above the mean level. p is the pressure in the sea at depth z and p_a is the air pressure on the sea-surface. $\gamma = 2\omega \sin\varphi$, where w is the angular velocity of the earth and φ, the north latitude, is γ is known as the Coriolis parameter,

ν is the kinematic viscosity.

Equation (3) is substituted in (1) and (2) and are integrated from surface to the bottom to obtain

$$\frac{\partial u}{\partial t} - \gamma v = -g\frac{\partial \zeta}{\partial x} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} - \left[\frac{\nu}{h}\frac{\partial u}{\partial z}\right]_{z=-h}^{z=0}$$

$$\frac{\partial v}{\partial t} + \gamma u = -g\frac{\partial \zeta}{\partial y} - \frac{1}{\rho}\frac{\partial p_a}{\partial y} + \left[\frac{\nu}{h}\frac{\partial v}{\partial z}\right]_{z=-h}^{z=0}$$

where u and v are depth mean horizontal velocities.

$$\tau_{xo} = \rho\nu\left(\frac{\partial u}{\partial z}\right) \quad \text{and} \quad \tau_{yo} = \rho\nu\left(\frac{\partial v}{\partial z}\right) \text{at } z=0$$

are the components of wind stress at the surface.

$$\tau_{xb} = \rho\nu\left(\frac{\partial u}{\partial z}\right) \qquad \tau_{yb} = \rho\nu\left(\frac{\partial v}{\partial z}\right) \text{at } z=-h$$

are the components of bottom stress and are neglected. $\frac{\partial p_a}{\partial x}$ and $\frac{\partial p_a}{\partial y}$ are the components of air pressure gradient over the surface.

Since the air pressure gradient and wind stress are assumed to act only in the x-direction and the conditions are assumed constant in the y-direction, the equations of motion takes the form.

$$\frac{\partial u}{\partial t} - \gamma v = -g\frac{\partial \zeta}{\partial x} - gE(x,t) \qquad (4)$$

$$\frac{\partial v}{\partial t} + \gamma u = 0 \qquad (5)$$

where $E(x,t) = \frac{1}{\rho g}\left(\frac{\partial p_a}{\partial x} - \frac{\tau_x}{h}\right)$ is a known function of x,t.

The equation of continuity becomes

$$\frac{\partial u}{\partial x} = -\frac{1}{h}\frac{\partial \zeta}{\partial t} \quad \text{(6)}$$

The velocities u,v can now be eliminated from (4), (5) and (6) and the differential equation for ζ is found to be.

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{gh}\left(\frac{\partial^2}{\partial t^2} + \gamma^2\right)\right] = -\frac{\partial^2 E}{\partial x \partial t} \quad \text{(7)}$$

Equation (7) when integrated with respect to t, gives

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{gh}\left(\frac{\partial^2}{\partial t^2} + \gamma^2\right)\right] = -\frac{\partial E}{\partial x} \quad \text{(8)}$$

without the inhomogeneous term, the equation (8) is known as the Klein-Gordon equation. Crease has found the solution of (8) in terms of a Green function which satisfies the equation

$$\frac{\partial^2}{\partial x^2} G(xt \mid x_0 t_0) - \frac{1}{gh}\left(\frac{\partial^2}{\partial t^2} + V^2\right) G(xt \mid x_0 t_0)$$
$$= -4\pi\, \delta(x - x_0)\, \delta(t - t_0) \quad \text{(9)}$$

where δ (y) is the Dirac delta function and G is to represent an outgoing wave for large $|x|$, t.

The solutio;n of (9) in three dimension is well known (Morsë and Feshbach 1953, [19]) which is in one dimension takes the form:

$$G(xt\,|\,x_0 t_0) = 2\pi (gh)^{1/2}\, J_0\left\{ \gamma \left[(t-t_0)^2 - \frac{(x-x_0)^2}{gh} \right]^{1/2} \right\} \cdot$$
$$\times\, H\left[(t-t_0) - \frac{|x-x_0|}{(gh)^{1/2}} \right]$$

where H(y) is the Heaviside unit function and $J_0(y)$ is a Bessel function of the first kind of order zero. The expression of ζ in terms of the Green's function is

$$4\pi\zeta = 2\pi (gh)^{1/2} \int_0^{t+\epsilon} dt_0 \int_{-\infty}^{\infty} \frac{\partial E}{\partial x_0} \times J_0\left\{ \gamma \times \right.$$
$$\left. \left[(t-t_0)^2 - \frac{(x-x_0)^2}{gh} \right]^{1/2} \right\} \times H\left[(t-t_0) - \frac{|x-x_0|}{(gh)^{1/2}} \right] \cdots\cdots (11)$$

where ϵ is a small positive number and $\frac{\partial \zeta}{\partial t} = 0$ at $t = 0$.

Particular solutions:

To study the solution (11) Crease has taken two particular forms of E(x,t):

Case 1 : if $E(x,t) = -AH(t)\, H(-x)$ .... (12)

The equation (11) after substituting (12) becomes

$$2\zeta = (gh)^{1/2} A \int_{-\epsilon}^{t} H(t-\tau)\, J_0\left\{ \gamma \left[\tau^2 - \frac{x^2}{gh} \right]^{1/2} H\left[\tau - \frac{|x|}{(gh)^{1/2}} \right] d\tau \right. \cdots\cdots (13)$$

where $t - t_0 = \tau$

Using the nondimensional parameters

$$\gamma\tau = \beta \;, \quad \gamma t = b \;, \quad \frac{\gamma x}{(gh)^{1/2}} = a$$

the equation (13) can be expressed as

$$\frac{2\gamma}{A(gh)^{1/2}}\zeta = \int_{-\gamma\epsilon}^{b} H(b-\beta)\, J_0 \left|(\beta^2 - a^2)^{1/2}\right| H(\beta - |a|)\, d\beta$$

or more concisely

$$\frac{2\gamma}{A(gh)^{1/2}}\zeta = \int_{|a|}^{\max |a|,\, b} J_0(\beta^2 - a^2)^{1/2}\, d\beta$$

which is the basic solution of the problem.

Case II

If $E(x,t) = -AH(t-T)\, H(X-x)$ ... (16)

then $$\frac{2\gamma}{A(gh)^{1/2}}\zeta = \int_{|a_0|}^{\max |a_0|,\, b_0} J_0(\beta^2 - a^2)^{1/2}\, d\beta \quad \cdots \quad (17)$$

where $a_0 = [\gamma(x-X)/(gh)^{1/2}]$ and $b_0 = \gamma(t-T)$

If the system are nonrotating the waves are not dispersive and all the waves propagate with the same velocity '$\sqrt{gh}$'. The force such as (16) extending over a semi-infinite range $-\infty < x < 0$ would set up a surge of ever increasing height . In the case of rotating system, the waves will be dispersive. From equation (8), it follows that a wave frequency has a wave number given by $k^2 = (\sigma^2 - \gamma^2)/gh$

and a wave velocity

$$c^2 = \frac{\gamma^2}{k^2} + gh$$

and the group velocity $\quad C = ghk/(ghk^2 + \gamma^2)^{1/2}$

$$\therefore \quad Cc = gh$$

Thus it follows that the wave velocity has a minimum of $(gh)^{1/2}$ and the group velocity a maximum. The shortest waves will be observed at a given point first, although they have the smallest wave velocity.

The period of oscillations of ζ

The maximum and minimum of ζ at a given place occur [equation (17)] when $J_0(b^2-a^2) = 0$. Let odd order zero of $J_0(x)$ be denoted successively by j_{0}, n.
Therefore the maximum of ζ occur when

$$b_n^2 = j_{0,n}^2 + a^2 \qquad [j_{0,1} = 2{\cdot}40, \; j_{0,2} = 8{\cdot}65]$$

$$b_n \simeq j_{0,n} + \frac{a^2}{2j_{0,n}} \qquad [a \ll j_{0,n}] \qquad (18)$$

$$\simeq a + \frac{j_{0,n}^2}{2a} \qquad [j_{0,n} \ll a]$$

If T_n (a) defines the local period at a as the time between two successive maxima n, and n+1, and T_I is the inertia period $2\pi/\gamma$, and $$\frac{T_n(a)}{T_I} \simeq 1 - \frac{a^2}{2 j_{0,n}\, j_{0,n+1}} \qquad (a \ll j_{0,n})$$

(19)

$$\frac{T_n(a)}{T_I} \simeq \frac{j_{0,n+1} + j_{0,n}}{2a} \qquad a \gg j_{0,n+1}$$

where

(20)

$$j_{0,n+1} - j_{0,n} \simeq 2\pi$$

Equation (19) and (20) shows that for points close to the edge of the generating area the local period of the surge is slightly less than the inertia period and approaching it rapidly, while at points a long distance away, the local period is at first much smaller than the inertia period and increases linearly for the first few oscillations.

The time T_0 elapsed from the arrival of the first disturbance at point 'a' to the first maximum of ζ is given by

$$\nu T_0 \simeq (j_{0,1} + a^2)^{1/2} - a$$

$$\simeq j_{0,1} + a\left(\frac{a}{2 j_{0,1}} - 1\right) \qquad a \ll j_{0,1}$$

$$\simeq \frac{j_{0,1}}{2a} \qquad a \gg j_{0,1}$$

From the graphic representation of νT_0 against a, it is made clear that the internal between the arrival of the surge at 'a' and its first maximum there decreases with increasing distance a from the edge of the generating area.

<u>The velocity in the direction of propagation:</u>

From (4), (5) and (6), the differential equation for u is found to be

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{gh}\left(\frac{\partial^2}{\partial t^2} + V^2\right)u = \frac{1}{h}\frac{\partial E}{\partial t} \qquad (21)$$

The solution of (21) in terms of Green's function which satisfies the equation (9)

$$4\pi u = -2\pi (gh)^{1/2}\int_0^{t+\epsilon} dt_0 \int_{-\infty}^{\infty} \frac{\partial E}{\partial t_0}\, J_0\left\{V\left[(t-t_0)^2 - \frac{(x-x_0)^2}{gh}\right]^{1/2}\right\} \times H\left[(t-t_0) - \frac{|x-x_0|}{(gh)^{1/2}}\right] dx_0$$

Now, the result obtained by using

$$\frac{\partial E}{\partial t} = -A\,\delta(t)\,H(-x)$$

and $\left[V(x-x_0)\right]/(gh)^{1/2} = \alpha$ is

$$\frac{2V}{gA}u = \int_a^b J_0\,(b^2-\alpha^2)^{1/2}\,H(b-|\alpha|)\,d\alpha \qquad (23)$$

Considering the case $a > 0$, then

$$\frac{2V}{gA}u = H(b-a)\int_a^b J_0\,(b^2-\alpha^2)^{1/2}\,d\alpha \qquad (24\,a)$$

and when $a < 0$

$$\frac{2V}{gA}u = \int_{-b}^{b} J_0\,(b^2-\alpha^2)^{1/2}\,d\alpha - H(b-|a|)\int_{-b}^{a} J_0\,(b^2-\alpha^2)^{1/2}\,d\alpha$$

$$= 2\sin b - H(b-|a|)\int_{|a|}^{b} J_0\,(b^2-\alpha^2)^{1/2}\,d\alpha \qquad (24b)$$

So that, for

$$b<|a| \qquad \frac{2V}{gA}u = 2\sin b \qquad (25)$$

The velocity transverse to the direction of propagation.

The equation (5) on integration becomes

$$\frac{\partial v}{\partial b} = -u$$

Now from equation (23)

$$\frac{2\gamma v}{gA} = -\int_0^b d\beta \int_a^{\infty} J_0\left(\beta^2-\alpha^2\right)^{1/2} H\left(\beta-|\alpha|\right) d\alpha \qquad (25)$$

if $a > 0$, then it follows from equation (25) after some manupulation of the integral

$$\frac{2\gamma v}{gA} = -\left[1-\cos b - \int_0^a d\alpha \int_a^b J_0\left(\beta^2-\alpha^2\right)^{1/2} d\beta\right] H(b-a) \qquad (26)$$

If $a < 0$, two separate cases are considered.

(i) $b < |a|$, the equation (25) becomes

$$\frac{2\gamma v}{gA} = \int_0^b d\beta \int_{-\beta}^{\beta} J_0\left(\beta^2-\alpha^2\right)^{1/2} d\alpha$$

$$= -2(1-\cos b) \qquad (27)$$

(ii) $b > |a|$

$$\frac{2\gamma v}{gA} = -2-2\cos b - \int_{|a|}^{b} d\beta \int_{|a|}^{\beta} J_0\left(\beta^2-\alpha^2\right)^{1/2} d\alpha \qquad (28)$$

The last integral is similar to that occuring in equation (26) with $|a|$ instead of a.

Physical conclusions

Some physical interpretation are made from the solutions of ζ, u, and v. When a force is applied over one half of an infinite sea, it is observed from equations (25), (28) that a motion is set up in the generating region in the direction of applied force and following immediately from this a transverse velocity is set up to balance the rotational forces arising from the original motion.

The effect of discontinuity in the applied force at the boundary of the generating area is transmitted to other parts of the sea, with maximum group velocity equal to $(gh)^{1/2}$. From equations (15), (24a), (24b), (26), (28) it is shown that, not until sufficient time has elapsed for this effect to be propagated to any particular point does any change take place in the state of motion (that is, no motion outside the generating area and inertia type motion within the generating area). When this time has elapsed the water acquires a superimposed motion which by the continuity condition must lead to an increase in water level.

The conditions when the time becomes very great ($b \to \infty$), the limiting values of ζ u,v are found to be (Erdelyi 1951) [22].

$$\left.\begin{aligned}
\frac{2\gamma}{A(gh)^{1/2}}\zeta &\longrightarrow e^{-|a|} \\
\frac{2\gamma}{Ag}u &\longrightarrow \sin b \\
\frac{2\gamma}{Ag}v &\longrightarrow \cos b - e^{-a} \qquad (a>0) \\
&\longrightarrow \cos b + e^{-|a|} - 2 \quad (a<0)
\end{aligned}\right] \quad (32)$$

If the system were nonrotating, the application of a force given by (12) would lead to an ever increasing surface elevation. (all wave lengths travelling with velocity $(gh)^{1/2}$) but owing to dispersion in the rotating system a steady elevation is reached. The pressure gradient resulting from the steady slope is balanced by the geostropic force.(due to the term $-e^{-|a|}\,a/|a|$ in the transverse velocity).The term -2 in the transverse velocity is just that needed to create a coriolis force to balance the applied force. Thus the forcing function $E(x,t)$ is balanced at large times by a transverse current and not by a surface elevation.

Problem - (II)

The propagation of long waves in a shallow rotating ocean due to steady atmospheric disturbances is discussed from the notable work of Crease. (Problem I). Recently Debnath and Kulchar [20] attempted a study of the unsteady plane long waves produced by an oscillatory horizontal surface stress distribution on a homogeneous rotating ocean of constant depth h. The asymptotic expression for the wave height η obtained by them is however non-uniform in the neighbourhood of the critical line as determined by the coalescing of the pole of the wave spectrum of η and the stationary point of its phase. This results in the presence of infinite wave amplitudes in the asymptotic expansions obtained by them for the unsteady state, moreover the steady state solution presented therein turns out to be singular when the coriolis parameter $f = 2\Omega \sin \varphi$ be comes equal to the circular frequency ω of the forced oscillation.

C.R. Mondal [18] has giv en a uniform asymptotic analysis of the problems of this type for large distances and times. The solution is found to be non-singular for $\omega = f$ and $|x| = ct$. Here the integral transform method is used to obtain the solution more easily. In the last part of the problem, an attempt is also made to show that the wave elevation reduces to corresponding basic solutions derived by Crease [17] of wind stresses of constant strength (w = 0).

Formulation of the problem.

Long waves are generated in a shallow horizontally unbounded ocean of constant depth h and density by the action of a time periodic wind stress which suddenly begin to act on the surface at time $t = 0$.

The linearised equations of motions referred to horizontal axis ox and oy rotating with earth are

$$\left.\begin{aligned} u_x + v_y &= -h^{-1}\eta_t \\ u_t - fv &= -g\eta_x + \tau^x/\rho h \\ v_t + fu &= -g\eta_y + \tau^y/\rho h \end{aligned}\right] \quad (1) - (3)$$

Here u,v, η are respectively velocity components and the surface elevation above the mean level. (τ^x, τ^y) are the wind stress components in the x- and y- directions. $f = 2\Omega \sin\varphi$ is the coriolis parameter supposed constant.

Eliminating u,v from (1) - (3), yield the following equation for η:

$$(c^2\nabla^2 - D)\eta_t = Q(x, y, t) \quad \cdot\cdot \qquad (4)$$

$$c^2 = gh, \quad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2, \quad D \equiv \partial^2/\partial t^2 + f^2$$

$$Q(x, y; t) = \rho^{-1}\left\{ \tau^x_{xt} + \tau^y_{yt} + f(\tau^y_x - \tau^x_y)\right\}$$

It is assumed that the wind stress is time-periodic and unidirectional, parallel to the x-axis and that it is independent of y. Therefore, $\tau^y = 0$,

$\tau^x = F(x)\exp(i\omega t)\,H(t)$, where

H(t) = Heaviside unit function.

Equation (4), therefore reduces to

$$\left[\frac{\partial^2}{\partial x^2} - c^{-2}(\partial^2/\partial t^2 + f^2)\right]\eta = \rho^{-1}c^{-2}(dF/dx)\exp(i\omega t) \qquad (6)$$

The constant of integration being supposed zero. The initial conditions and the condition at infinity for (6) are respectively

$$\eta(x, 0) = 0 \quad ; \quad \dot{\eta}(x, 0) = 0 \qquad (7)$$

$|\eta|$ = a bounded quantity as $|x| \longrightarrow \infty$

Solution.

Introducing the Fourier exponential transform $\bar{\eta}(k,t)$ of

$$\bar{\eta}(k,t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \eta(x,t) \exp(-ikx)\, dx$$

The equation (6) transforms to

$$\ddot{\bar{\eta}} + m^2 \bar{\eta} = -ik\rho^{-1} \bar{F}(k) \exp(i\omega t), \quad \cdots \quad (8)$$

where

$$m^2 = (k^2c^2 + f^2)^{1/2}$$

The general solution of (8) is

$$\bar{\eta} = \bar{\eta}_0(k) \cos(mt + \epsilon) - ik(\rho m)^{-1} \bar{F}(k) \int_0^t \sin m(t-s) \times$$

$$\times\, e(i\omega s)\, ds .$$

Now by (7), we see that $\bar{\eta}_0(k) = 0$. Evaluating the s-integral and using the Fourier inversion theorem, the result obtained is

$$\eta(x,t) = (2\pi)^{-1/2} i\rho^{-1} \int_{-\infty}^{\infty} \left\{ \frac{m \cos mt + i\omega \sin mt - m \exp(i\omega t)}{m(m^2 - \omega^2)} \right\} \times$$

$$\times\, k \bar{F}(k) \exp(ikx)\, dk \quad \cdots\cdots (9)$$

<u>Asymptotic analysis of η.</u>

Observing that the integrand in $\eta(x,t)$ is non-singular for all k, and is likely to be convergent with sufficient restrictions on $\bar{F}(k)$, η can be written as the sum of two integrals possessing Cauchy principal values

$$\eta = (8\pi)^{-1/2}\, i \rho^{-1} (\eta_1 + \eta_2) \cdots \quad (10)$$

where

$$\eta_1 = -2 \exp(i\omega t) \int_{-\infty}^{\infty} \frac{k \bar{F}(k)}{m^2 - \omega^2} \exp(ikx)\, dk \quad (11)$$

$$\eta_2 = \int_{-\infty}^{\infty} \left\{ \frac{\exp(imt)}{m-\omega} + \frac{\exp(-imt)}{m+\omega} \right\} \times \frac{k \bar{F}(k)}{m} \exp(ikx)\, dk \cdots \quad (12)$$

The poles of the integrand in η_1 are given by $k = \pm(w^2 - f^2)^{1/2}$. These are real and distinct iff $\omega > f$. Denoting these poles by $\pm k_0$, and using Lighthill's formula [14], the following asymptotic estimate of η_1 is obtained for large $|x|$:

$$\eta_1 = -\pi i c^{-2} \operatorname{sgn} x \cdot \exp(i\omega t) \times \left[\bar{F}(k_0) \exp(ik_0 x) + \bar{F}(-k_0) \exp(-ik_0 x) \right] \times H(1 - f/\omega) + O(1/|x|) \cdots (13)$$

As η_2 possess both poles and stationary points and according to their distribution η_2 can be written as

$$\eta_2 = I_1 + I_2 + J_1 + J_2$$

where

$$I_{1,2} = \int_0^{\infty} \frac{(\pm k)\, \bar{F}(\pm k)}{m(m-\omega)} \exp\{i(mt \pm kx)\}\, dk \quad (15)-(16)$$

$$J_{1,2} = \int_0^\infty \frac{(\mp)\ \bar{F}(\mp k)}{m(m+\omega)} \exp\left\{-i(mt \pm kx)\right\} dk \qquad (17)-(18)$$

The integrals for $J_{1,2}$ possess only stationary points. Hence the asymptotic expansion of J_1 and J_2 for large $|x|$ and t is obtained by the method of stationary phase:

$$J_1 \simeq -\left(\frac{2\pi f}{c^3}\right)^{1/2} \frac{|x|\,\bar{F}(-k_1)\exp\left[-i\left\{fc^{-1}(c^2t^2-|x|^2)+\pi/4\right\}\right]}{(c^2t^2-|x|^2)^{1/4}\left\{fct+\omega(c^2t^2-|x|^2)^{1/2}\right\}}$$

$$+\ O(1/|x|), \quad -ct < x < 0 \quad ----(21)$$

$$J_1 \simeq O(1/|x|), \quad \ldots\ldots \quad x < -ct,\ x > 0 \quad (22)$$

$$J_2 \simeq \left(\frac{2\pi f}{c^3}\right)^{1/2} \frac{|x|\,\bar{F}(k_1)\exp\left[-i\left\{fc^{-1}(c^2t^2-|x|^2)^{1/2}\right\}+i\pi/4\right]}{(c^2t^2-|x|^2)^{1/4}\left\{fct+\omega(c^2t^2-|x|^2)^{1/2}\right\}}$$

$$+\ O(1/|x|) \qquad 0 < x < ct \quad \cdots\ (23)$$

$$J_2 \simeq O(1/|x|) \quad \ldots\ldots\ldots \quad x > ct,\ x < 0 \quad (24)$$

The integrands of $I_{1,2}$ both possess a simple pole $k = k_0$ iff $\omega > f$,

$$k_0 = c^{-1}(\omega^2 - f^2)^{1/2}$$

and if $0 < |x| < ct$, for phases in each of the integrands, there is a stationery point $k = k_1$ in I_2 and J_2, for $x > 0$, and in I_1 and J_1 for $x < 0$:

$$k_1 = \frac{|x|\,f}{c\,(c^2t^2 - |x|^2)^{1/2}}$$

The asymptotic values of I_1 and I_2 differ according as their integrands possess or do not possess poles i.e their integrands possess or do not possess poles i.e. according as $\omega > f$ or $\omega < f$.

Case I $\omega < f$.

I_1 and I_2 have stationary points only. Consequently the method of stationary phase [4] gives.

$$I_1 \simeq \left(\frac{2\pi f}{c^3}\right)^{1/2} \frac{|x| \bar{F}(k_1) \exp\left[i\left\{fc^{-1}(c^2t^2-|x|^2)^{1/2}+\pi/4\right\}\right]}{(c^2t^2-|x|^2)^{1/4}\left\{fct-\omega(c^2t^2-|x|^2)^{1/2}\right\}}$$

$$+ O(1/|x|), \quad \text{when} \quad -ct < x < 0, \, |x| \to \infty \cdots (25)$$

$$I_1 \simeq O(1/|x|), \quad \text{when} \quad x < -ct, \, x > 0, \, |x| \to \infty \qquad (25)$$

$$I_2 \simeq -\left(\frac{2\pi f}{c^3}\right)^{1/2} \frac{|x| \bar{F}(-k_1) \exp\left[i\left\{fc^{-1}(c^2t^2-|x|^2)^{1/2}+\pi/4\right\}\right]}{(c^2t^2-|x|^2)^{1/4}\left\{fct-\omega(c^2t^2-|x|^2)^{1/2}\right\}}$$

$$+ O(1/|x|), \quad 0 < x < ct, \quad |x| \to \infty$$

$$I_2 \simeq O(1/|x|), \text{when} \quad x > ct, \, x < 0, \quad |x| \to \infty \qquad (25)$$

Case II. $\omega > f$, $0 < |x| < ct$.

In this case, the integrands in I_1 and I_2 each have a pole together with a stationary point respectively for $x < 0$ and $x > 0$. Hence Bleistein's method is used to estimate the asymptotic of both I_1 and I_2 which will remain uniform even when the pole in either integrands coincides with the corresponding stationary point.

Writing $I_{1,2}$ as I and $k = (f/c)$ sinhz the equation (15-16) reduces to

$$I = c^{-1}\int_0^\infty \frac{(\pm k)\,[\bar{F}(\pm k)]}{f\cosh z - \omega} \times$$
$$\exp\left\{ i f c^{-1} |x| (ct|x|^{-1}\cosh z - \sinh z)\right\} dz. \tag{26}$$

The points $z = z_0$, $z = z_1$ are taken to corresponds to the pole $k = k_0$ and $k = k_1$ respectively.

Hence

$$z_0 = \ln\left[f^{-1}\left\{\omega + (\omega^2 - f^2)^{1/2}\right\}\right]$$

$$z_1 = (1/2)\ln\left\{(ct + |x|)/(ct - |x|)\right\}$$

When the stationary point z_1 coincide with the pole z_0,

$$|x|/t = (c/\omega)(\omega^2 - f^2)^{1/2} = c_0$$

A new variable s defined b y the relation

$$(ct|x|^{-1}\cosh z - \sinh z) - (ct|x|^{-1}\cosh z_0 - \sinh z_0) = 2^{-1}s^2 + bs \tag{31}$$

is introduced. $s = 0$ corresponds to the pole z_0 and b is a parameter set in for describing the distance $|z_1 - z_0|$. The requirement that the derivative

$$\frac{dz}{ds} = \frac{s+b}{ct\,|x|^{-1}\sinh z - \cosh z}$$

is finite and nonzero, makes $s+b = 0$ when $z = z_1$. Now by L'Hospital's rule,

$$\left(\frac{dz}{ds}\right)^2_{s=-b} = \frac{1}{(ct\,|x|^{-1}\cosh z - \sinh z)_{z=z_1}}$$

Putting $s = -b$, $z = z_1$ in (31),

$$b = \pm |x|^{-1}\,|z_0 - z_1|\,(c^2t^2 - |x|^2)^{1/2} \qquad (32)$$

The sign of b is fixed by the requirement that $\left(\frac{dz}{ds}\right)_{s=0}$ be positive which gives

$$b = |x|^{-1}(z_0 - z_1)(c^2t^2 - |x|^2)^{1/2}$$

In a new form of (26),

$$I = c^{-1}\exp\left[i\left\{\omega t - |x|c^{-1}(\omega^2 - f^2)^{1/2}\right\}\right] \times$$
$$\int_{s=S}^{\infty}\left[\frac{(\pm k)\,\bar{F}(\pm k)}{f\cos z - \omega}\,\frac{dz}{ds}\right]\exp\left\{ifc^{-1}|x|\,(2^{-1}s^2 + bs)\right\}ds \qquad (33)$$

where $S > 0$ is such that $Sx \gg 1$, and the stationary point of the phase, if any, lies in (S, ∞).

Now, it is assumed that the expansion

$$\left\{ \frac{(\pm k)\left[\bar{F}(\pm k)\right]}{f\cosh z - \omega} \frac{dz}{ds} \right\} = s^{-1} A + \sum_{0}^{\infty} B_n (s+b)^n ,$$

where A and B_n ($n = 0,1,2...$) are constant.

The values of A and B_0 are found to be

$$A \equiv A_{\pm} = c^{-1} \bar{F}(\pm k_0)$$

$$B_0 \equiv B_{0\pm} = \pm \left[\frac{\bar{F}(\pm k_0)}{bc} + \frac{f\,|x|^{3/2}\,\bar{F}(\pm k_1)}{c(c^2t^2 - |x|^2)^{1/4}\left\{fct - \omega(c^2t^2 - |x|^2)^{1/2}\right\}} \right]$$

Application of Watson's Lemma [16], and replacing the range of integration by $-\infty < x < \infty$, the equation (33) now becomes for large $|x|$,

$$I \sim c^{-1} \exp\left[i\left\{ \omega t - |x| c^{-1} (\omega^2 - f^2)^{1/2} - (2c)^{-1} f b^2 |x| \right\}\right] \times$$

$$A \int_{-\infty}^{\infty} \frac{1}{s' - b} \exp\left\{ i (2c)^{-1} f\,|x|\,s'^2 \right\} ds' +$$

$$B_0 \int_{-\infty}^{\infty} \exp\left\{ (2c)^{-1} f\,|x|\,s'^2 \right\} ds' \Big] + O(1/|x|)$$

On evaluating the integrals

$$I \sim c^{-1} \pi (1-i) A \,\mathrm{sgn}\, b \,\mathrm{CiS} \left(\frac{f b^2 |x|}{2c}\right) \times$$
$$\exp\left[i\left\{\omega t - c^{-1}(\omega^2 - f^2)^{1/2} |x|\right\}\right] +$$
$$c^{-1}(1+i) B_0 (\pi c/f|x|)^{1/2} \times \exp\left[i\left\{\omega t - c^{-1}(\omega^2 - f^2)^{1/2} \times\right.\right.$$
$$\left.\left.|x| - (2c)^{-1} f b^2 |x|\right\}\right] + O(1/|x|), \quad |x| \to \infty \tag{37}$$

where $\mathrm{CiS}(\omega) = C(\omega) - iS(\omega) = \int_0^{\omega} \frac{\exp(-it)}{(2\pi t)^{1/2}}\, dt$,

$C(\omega)$ and $S(\omega)$ being the Fresnel's integral [12]

From equation (37), the values of I_1 and I_2 can be easily obtained.

<u>Case III.</u> $\omega > f$, x outside $[-ct, 0]$, $[0, ct]$

In these two cases I_1 and I_2 have poles only, consequently, Lighthill's formula [14] gives

$$I_1 \simeq i\pi c^{-2} \,\mathrm{sgn}\, x\, \bar{F}(k_0) \exp\left[i\left\{\omega t + c^{-1}(\omega^2 - f^2)^{1/2} x\right\}\right]$$
$$+ O(1/|x|), \quad x < -ct, \quad x > 0, \quad |x| \to \infty$$

$$I_2 \simeq i\pi c^{-2} \,\mathrm{sgn}\, x\, \bar{F}(-k_0) \exp\left[i\left\{\omega t - c^{-1}(\omega^2 - f^2)^{1/2} x\right\}\right]$$
$$+ O(1/|x|), \quad x > ct, \quad x < 0, \quad |x| \to \infty$$

Collecting the results systematically derived in different cases, the final expressions for η in various cases are found to be

Case I. $\qquad \omega < f, \quad 0 < |x| < ct.$

$$(i)\ \eta \simeq \frac{i}{2}\left(\frac{f}{\rho^2 c^3}\right)^{1/2} \frac{|x|}{(c^2t^2 - |x|^2)^{1/4}} \times$$

$$\left[\frac{\bar{F}(k_1)\exp[-i\{fc^{-1}(c^2t^2-|x|^2)^{1/2}+\pi/4\}]}{fct+\omega(c^2t^2-|x|^2)^{1/2}} - \frac{\bar{F}(-k_1)\exp[i\{fc^{-1}(c^2t^2-|x|^2)^{1/2}+\pi/4\}]}{fct-\omega(c^2t^2-|x|^2)^{1/2}}\right] + O(1/|x|)$$

$$0 < x < ct \qquad (38)$$

$$(ii)\ \eta \simeq \frac{i}{2}\left(\frac{f}{\rho^2 c^3}\right)^{1/2} \frac{|x|}{(c^2t^2 - |x|^2)^{1/4}} \times$$

$$\left[\frac{\bar{F}(k_1)\exp[i\{fc^{-1}(c^2t^2-|x|^2)^{1/2}+\pi/4]}{fct-\omega(c^2t^2-|x|^2)^{1/2}} - \frac{\bar{F}(-k_1)\exp[i\{fc^{-1}(c^2t^2-|x|^2)^{1/2}+\pi/4}{fct+\omega(c^2t^2-|x|^2)^{1/2}}\right] + O(1/|x|)$$

$$-ct < x < 0 \qquad (39)$$

Case II $\qquad \omega > f, \quad 0 < |x| < ct$

(i)

$$\eta \simeq \frac{i\sqrt{\pi}\,\bar{F}(-k_0)}{4\rho c^2}\left[2\,\mathrm{sgn}\,b\cdot \mathrm{cis}\left(\frac{f b^2 |x|}{2c}\right) + (1-i)\right]\times$$

$$\exp\left[i\left\{\omega t - c^{-1}(\omega^2 - f^2)^{1/2} x - \pi/4\right\}\right] + \frac{1}{2(f\rho^2 c^3)^{1/2}}\times$$

$$\times\left[\frac{\bar{F}(-k_0)}{b|x|^{1/2}} + \frac{f\bar{F}(-k_1)|x|}{(c^2t^2 - |x|^2)^{1/4}\left\{fct - \omega(c^2t^2 - |x|^2)^{1/2}\right\}}\right]$$

$$\times\exp\left[i\left\{fc^{-1}(c^2t^2 - |x|^2)^{1/2} - \pi/4\right\}\right] + \frac{if\bar{F}(k_1)|x|}{2(f\rho^2 c^3)^{1/2}}\times$$

$$\frac{\exp\left[-i\left\{fc^{-1}(c^2t^2 - |x|^2)^{1/2} + \pi/4\right\}\right]}{(c^2t^2 - |x|^2)^{1/4}\left\{fct + \omega(c^2t^2 - |x|^2)^{1/2}\right\}} + O(1/x)$$

when $\quad 0 < x < ct \qquad\qquad (40)$

(ii)

$$\eta \simeq -\frac{i\sqrt{\pi}\,\bar{F}(k_0)}{4\rho c^2}\left[(1-i) + 2\,\mathrm{sgn}\,b\cdot \mathrm{cis}\left(\frac{f b^2|x|}{2c}\right)\right]\times$$

$$\exp\left[i\left\{\omega t + c^{-1}(\omega^2 - f^2)^{1/2} x - \pi/4\right\}\right] - \frac{1}{2(f\rho^2 c^3)^{1/2}}\times$$

$$\left[\frac{\bar{F}(k_0)}{b|x|^{1/2}} + \frac{f\bar{F}(k_1)|x|}{(c^2t^2 - |x|^2)^{1/4}\left\{fct - \omega(c^2t^2 - |x|^2)^{1/2}\right\}}\right]\times$$

$$\times\exp\left[i\left\{fc^{-1}(c^2t^2 - |x|^2)^{1/2} - \pi/4\right\}\right] - i\frac{f\bar{F}(-k_1)|x|}{(2f\rho^2 c^3)^{1/2}}\times$$

$$\times\frac{\exp\left[-i\left\{fc^{-1}(c^2t^2 - |x|^2)^{1/2} + \pi/4\right\}\right]}{(c^2t^2 - |x|^2)^{1/4}\left\{fct + \omega(c^2t^2 - |x|^2)^{1/2}\right\}} + O(1/x)$$

when $\; -ct < x < 0 \qquad\qquad (41)$

Case III. when $\omega \gtrless f$, x outside $[-ct, ct]$

$$\eta \simeq O(1/|x|), \qquad \text{when} \quad x < -ct, \; x > ct$$

Limiting cases

(a) If $|x| << c_0 t$, then $k_1 << k_0$, $b > 0$ and $\mathrm{CiS}(2^{-1} b^2 |x|) \sim 2^{-1/2} e^{-i\pi/4}$ [22], with these values in (38) - (41), η can be obtained as

$$\eta \simeq O\left((ct)^{-3/2} |x|\right) \qquad \text{when} \quad \omega < f, \; -ct < x < ct,$$

$$\eta \simeq (\pi/2)^{1/2} (\rho c^2)^{-1} \bar{F}(-k_0) \exp\left[i\left\{\omega t - c^{-1}(\omega^2 - f^2)^{1/2} x\right\}\right],$$

$$\text{when} \quad \omega > f, \; 0 < x < ct. \qquad (42)$$

$$\eta \simeq (\pi/2)^{1/2} (\rho c^2)^{-1} \exp\left[i\left\{\omega t + c^{-1}(\omega^2 - f^2)^{1/2} x\right\}\right] \bar{F}(k_0)$$

$$\text{when} \quad \omega > f, \; -ct < x < 0 \quad \cdots\cdots\cdots (42)$$

(b) If $c_0 t << |x| << ct$. Then $k_0 << k_1$, $b < 0$, and we have from (38) - (41).

$$\eta \simeq -(f \rho^2 c^3)^{-1/2} (|x|/ct)(c^2t^2 - |x|^2)^{-1/4} \times$$
$$\mathrm{Im}\left[\bar{F}(k_1) \exp\left\{-i\left(fc^{-1}\sqrt{c^2t^2 - |x|^2} + \pi/4\right)\right\}\right]$$
$$\text{when} \quad \omega < f, \; 0 < x < ct.$$

$$\eta \simeq -(f\rho^2c^3)^{-1/2}(|x|/ct)(c^2t^2-|x|^2)^{-1/4} \times$$
$$\mathrm{Im}\left[\bar{F}(k_1)\exp\left\{i\left(fc^{-1}\sqrt{c^2t^2-|x|^2}+\pi/4\right)\right\}\right]$$

when $\omega < f,\ -ct < x < 0$

$$\eta \simeq (f\rho^2c^3)^{-1/2}(|x|/ct)(c^2t^2-|x|^2)^{-1/4} \times$$
$$\mathrm{Re}\left[\bar{F}(k_1)\exp\left\{-i\left(fc^{-1}\sqrt{c^2t^2-|x|^2}-\pi/4\right)\right\}\right]$$

when $\omega > f,\ 0 < x < ct.$

$$\eta \simeq -(f\rho^2c^3)^{-1/2}(|x|/ct)(c^2t^2-|x|^2)^{-1/4} \times$$
$$\mathrm{Re}\left[\bar{F}(k_1)\exp\left\{i\left(fc^{-1}\sqrt{c^2t^2-|x|^2}-\pi/4\right)\right\}\right]$$

when $\omega > f,\ -ct < x < 0$

The limiting case (b) shows specially the progressive waves disappear and the wave motion is wholly dispersive at places for which $c_0 t \ll |x|$ for all $\omega \neq f$.

<u>An illustrative case</u>

$$p_0(x,t) = (p/2a)\cos\omega t, \quad |x| < a$$
$$= 0 \quad |x| > a$$
$$\bar{F}(k_0) = (2\pi)^{-1/2} \cdot x \cdot p\left\{\sin(k_0 a)/(k_0 a)\right\}$$

then from the limiting case (a), equation (42) becomes

$$\frac{2\eta\rho c^2}{p} = \frac{\sin(k_0 a)}{k_0 a}\cos\left\{\omega t - \frac{1}{c}(\omega^2-f^2)^{1/2}x\right\}$$

the characteristic of this type of wave motion is graphically illustrated in (Fig.1) by plotting (1) $2\eta\rho g h/p$ against x' $[x' = 10^6 \times x]$

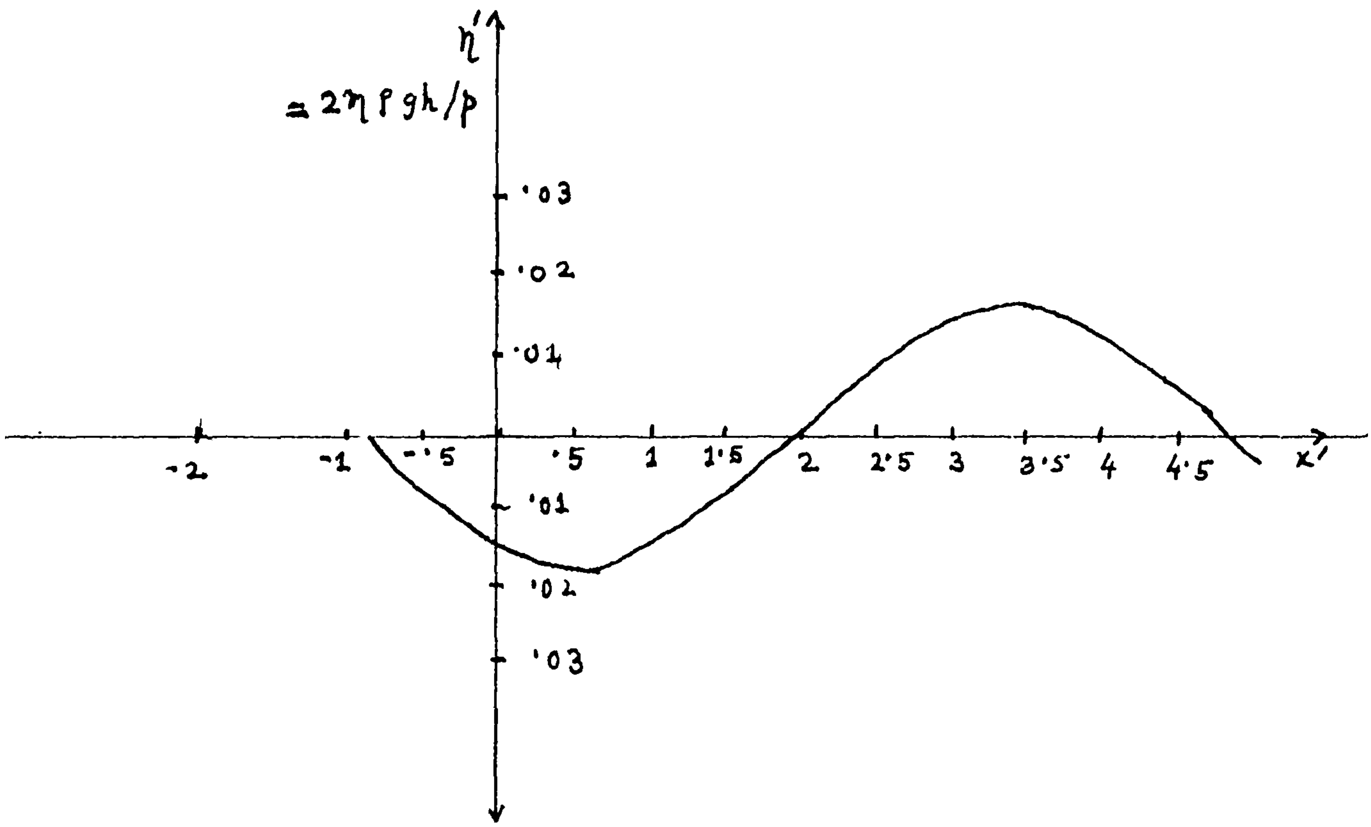

Fig. 1. Variation of η' with x'.

For $a = 0{\cdot}25$, $h = 1$, $g = 32\,\mathrm{ft}/\mathrm{sec}^2$, $\varphi = 15^\circ = \pi^c/12$

$f = 2\omega \sin\varphi$, $\omega = 7{\cdot}3 \times 10^{-5}\ \mathrm{sec}^{-1}$

Physical conclusions:

Several characteristics of the unsteady wind forced motion can be easily made from (38) - (41). First, we note that the disturbance outside the region $(-ct, ct)$ is insignificant. Within this region, the wave system differs in character according as the forcing frequency f is greater or less than the inertial frequency f. For $\omega < f$, the waves are entirely dispersive in nature, while, for $\omega > f$, the waves system consists of dispersive waves superposed on two dominant oppositely moving progressiv e waves having different and variable amplitudes expressed by certain combinations of Fresnel integrals, if $\bar{F}(k_0) = 0$ in (40) and (41), the progressive waves disappear and there are only dispersive waves at places in $-ct < x < ct$. Within the region of validity, the expression (38) - (41) are all seen to be nonsingular for $\omega = f$, for the dispersive waves in (38) and (39), and for progressive wave components, this non-singularity holds for all ω .

It is also observed that the properties of one dimensional wave motion are mostly similar to those of the corresponding problem with circular symmetry obtained by C.R. Mondal [21] . One notable distinction, however is that, there the wave has a spatially damped amplitude, this damping factor $(r^{-1/2})$ is absent in one dimensional problem.

Crease's solution.

(i) If $F(x) = -A_1 H(t) H(-x)$, $F'(x) = A H(t) \delta(-x)$

$$\bar{F}(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} - A_1 H(t) H(-x) e^{-ikx} dx .$$

$$= -(2\pi)^{-1/2} A_1 H(t) \cdot \frac{1}{ik} \qquad (1)$$

where $\delta(x)$ is Dirac's delta function. The expression

for η may be written as

$$\eta = (2\pi)^{-1/2} (i/\rho) \exp(i\omega t) \times \int_0^t \exp(-i\omega s) ds \times$$
$$\times \int_{-\infty}^{\infty} m^{-1} k \bar{F}(k) \sin(ms) \exp(ikx) dk \qquad (2)$$

Substituting (1) into (2) we get

$$\eta = -(\pi \rho c)^{-1} A_1 \exp(i\omega t) \times \int_0^t \exp(-i\omega s) ds \times \int_0^{\infty} \frac{A_1 H(t) \cos kx \sin(ms)}{\{k^2 + (fc^{-1})^2\}^{1/2}} dk$$

$$= -(2\rho c)^{-1} A_1 \exp(i\omega t) \int_0^t H(t-s) \cdot \exp(-i\omega s) \times$$
$$\times J_0 \left(f \sqrt{s^2 - x^2 c^{-2}} \right) H\left(s - \frac{|x|}{c}\right) ds$$

[[26], P. 26]

for $\omega = 0$, and $fs = \beta$, $fx/c = a$, $ft = b$

$$\eta = -(2\rho c)^{-1} A_1 \int_{|fc^{-1}x|}^{\max(ft, |fc^{-1}x|)} J_0 \sqrt{\beta^2 - a^2} \, d\beta$$

which is Crease's basic solution of the problem of wind stresses of constant strength which starts to blow uniformly at t = 0 over the region $x \not< 0$ in the direction of x increasing.

Case (ii)

(ii) $$F(x) = H(t)\, H(vt - x)$$
$$F'(x) = -H(t)\, \delta(vt - x)$$

$$\bar{F}(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} H(t)\, H(vt - x)\, e^{-ikx}\, dx$$
$$= - \frac{H(t)\, e^{-ikvt}}{(2\pi)^{1/2}\, ik} \qquad (1)$$

where $\delta(x)$ is Dirac's delta function. The expression for η may be written as

$$\eta = -(2\pi)^{-1/2}\, (i/\rho)\, \exp(i\omega t) \times \int_0^t \exp(-i\omega s)\, ds \times$$
$$\times \int_{-\infty}^{\infty} m^{-1}\, k\, \bar{F}(k)\, \sin(ms)\, \exp(ikx)\, dk \qquad (2)$$

Substituting (1) into (2) we get

$$\eta = (\pi \rho c)^{-1} \exp(i\omega t) \int_0^t \exp(-i\omega s)\, ds \times$$
$$\times \int_0^{\infty} \frac{H(t)\, \sin(ms)\, \cos k(x - vt)}{\{k^2 + (fc^{-1})^2\}^{1/2}}\, dk$$

[[26], p. 26]

$$\eta = (2\rho c)^{-1} \exp(i\omega t) \int_0^t H(t-s) \exp(-i\omega s) \times$$

$$J_0\left(f \sqrt{s^2 - \left(\frac{x - vt}{c}\right)^2}\right) H\left(s - \frac{|x - vt|}{c}\right) ds$$

which corresponds to Crease's solution (47) of the problem of wind stresses created at t = 0 and moving with a velocity V . [17]

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CHAPTER IV

MONSOON AND SURGE

Monsoon and surge are two important natural phenomena that affect human life and property in the coastal area. Rainfall due to monsoon is connected with the production of foodgrains and generations of hydro-electric power. It may also be responsible for flood and causes devastations equal to surges.

Storm surges are the outcome of cyclonic winds blowing over a large surface of water, which is bounded by a shallow basin. The driving force of the wind leads to the accumulation of water on the shore line which, in turn, results in a sudden and substantial rise in sea level. They are known to cause devasting floods along the coastal area. Mathematical models have been prepared for the prediction of storm surges and coastal inundation associated with several cyclones in the Bay of Bengal and the Arabian Sea.

Wind blows in opposite directions over certain regions of the tropics in summer and winter. For instance, over the Arabian sea, the wind at low levels blows from the south-west in summer and from the north-east in the winter. Such a seasonal reversal in the direction of the wind is traditionally known as monsoon. The word 'monsoon' is derived from the Arabic word 'mausam' for a season. Thus the essence of monsoon is seasonality.

Nowadays special emphasis is given on Mathematical Modelling in Atmospheric sciences, Oceanic science and Physiological fluid dynamics. Mathematical models help the predictibility of monsoon and storm surges. These predictions are highly valuable in terms of security and economic activity of men in the coastal area.

Advance information is urgently needed on three aspects of monsoon, namely,

(1) the likely dates of onset and withdrawal of monsoon,

(2) periods of heavy and sufficient rainfall during the monsoon season,

(3) the total quantum of monsoon rainfall.

Monsoon modelling is therefore one of the prime objective of continued research in India and other coastal areas.

This chapter is set out to study briefly,

(1) Mathematical models developed for the prediction of storm surges and coastal inundation associated with the severe cyclones in the Bay of Bengal and the Arabian sea.

(2) Monsoon and Monsoon modelling.

<u>Model-I</u>

We shall study here a mathematical model by P.K. Das, (1981), [23] showing how a basin responds to a sudddn impulse of the kind provided by a tropical cyclone. Here, the scaling of shallow water equations in terms of the speed of a cyclone moving over a basin and another velocity related to basic characteristics is reported. For a basin of iniform depth, an analytical solution has been obtained for the zero order equations. The solution suggests that, in the initial stage, the divergence of the wind stress is more important than the curl of the wind stress. In the subsequent stages, the curl become more important. It is also shown that the basin does not respond to a cyclone until a certain time has elapsed. The adjustment of time depends on the storm speed and is inversely proportional to the square root of the basin depth.

Basic equations

The shallow water equations are given below:

$$u_t + (u u_x + v u_y) - f v = -\frac{\partial}{\partial x}\left(g\zeta + \frac{p_a}{\rho}\right) + \frac{1}{\zeta + h}\left(\frac{\Delta F}{\rho}\right) \qquad (1)$$

$$v_t + (u v_x + v v_y) + f u = -\frac{\partial}{\partial y}\left(g\zeta + \frac{p_a}{\rho}\right) + \frac{1}{\zeta + h}\left(\frac{\Delta G}{\rho}\right) \qquad (2)$$

$$\zeta_t + [(\zeta + h) u]_x + [(\zeta + h) v]_y = 0 \qquad (3)$$

where f is the Coriolis parameter and p_0, ρ, ζ are atmospheric pressure, density of water and amplitude of the surge. u, v are depth averageed velocities in rectangular cartesian coordinates (oxyz) where the x and y axes point to the east and north respectively and z is the vertical axis.
F_x is the eastward component of the wind stress at the surface.
F_b is the sea bed friction. G_s, G_b northward component of surface stress (G_s) and bottom stress (G_b).

$F = F_s - F_b$

$G = G_s - G_b$

h = depth of water.

Suffixes have been used to denote partial derivatives. These equations are useful for examining the response of a basin. For scaling, the following assumptions are taken.

L = Characteristic length (500 km)

T = Characteristic time (24 hr)

Z = 2 m = mean surge amplitude

H = 1- m = mean depth of water

V_s= average speed of storm propagation (6 ms^{-1})

C = Characteristic velocity of surge.

The vertical velocity at the free surface (W) is of the order of (Z/T), but from the equation of continuity

$$W/H \sim C/L$$

whence $C \sim (L/H)(Z/T)$.

Thus, the magnitude of C is 1-2 ms^{-1}, which is about a third of the storm speed (V_s).

The speed of the storm may be used to define Froude Number (F_r).

$$F_r = V_s/(gH)^{1/2}$$

Considering average values, $F_r \simeq 0{\cdot}6$. The equations (1) - (3) are made non-dimensional by using the nondimensional quantities.

$$f_* = fT$$
$$u_*, v_* = u, v/C$$
$$x_*, y_* = x, y/L$$
$$p_* = p_a/\rho g H$$
$$\Delta F_* = \Delta F/\rho u_*^2$$
$$\zeta_* = \zeta/z$$
$$h_* = h/H$$
$$t_* = t/T \qquad (4)$$
$$\Delta G_* = \Delta G/\rho u_*^2$$

For strong winds, the magnitude of the friction velocity u_* is approximately 10^{-3} $m^2 s^{-2}$.

On droping the *, the nondimensional equations are

$$u_t + \epsilon(uu_x + vu_y) - fv = -\frac{1}{F_r^2}\left[\zeta + \frac{V_s}{C}P\right]_x + \left(\frac{u_*^2 T}{CH}\right)\frac{\Delta F}{h+\epsilon\zeta}, \quad (5)$$

$$v_t + \epsilon(uv_x + vu_y) + fu = -\frac{1}{F_r^2}\left[\zeta + \frac{V_s}{C}P\right]_y + \left(\frac{u_*^2 T}{CH}\right)\frac{\Delta G}{h+\epsilon\zeta}, \quad (6)$$

$$\zeta_t + [(\epsilon\zeta + h)u]_x + [(\epsilon\zeta + h)v]_y = 0, \quad (7)$$

where $\epsilon = Z/H = CT/L$ (8)

Zero-Order Equations

The dependent variable in (5), (6) and (7) when expanded in powers of ϵ, systems of increasingly higher order may be derived. The zero-order equations, when $\epsilon \ll 1$, are

$$u_t - fv = -\frac{1}{F_r^2}\left[\zeta + \frac{V_s}{C}P\right] + \left(\frac{u_*^2 T}{CH}\right)\frac{\Delta F}{h}, \quad (9)$$

$$v_t + fu = -\frac{1}{F_r^2}\left[\zeta + \frac{V_s}{C}P\right] + \left(\frac{u_*^2 T}{CH}\right)\frac{\Delta G}{h}, \quad (10)$$

$$\zeta_t + (hu)_x + (hv)_y = 0 \quad (11)$$

Numerical solutions of the zero-order equations have been obtained (Das et al 1975) for the northern sector of Bay of Bengal. An idealized storm with specified distribution of pressure and winds, which moves along a straight track with constant speed is assumed. This hits a given sector of coast. By changing the storm characteristics, it is possible to relate the surge amplitude with the intensity and speed of the storm. It is seen that with same storm characteristic, the response may be different for two basins if their bottom topography is different. The correct formulation of the sea-bed friction and wind stress and boundary conditions on the sea boundary are some of the difficulties encountered in numerical models. Tide-surge interactions are important but difficult to estimate in shallow seas where ϵ is of the order of unity. The tidal cycle, often imposed as an initial state, cannot be determined with much accuracy due to lack of observations on the open sea. Despite these limitations, the zero order equations provide a useful base for modelling a storm surge.

Basin response to winds:

The zero-order equations could be used to estimate how quickly a basin of depth (h) will respond to the cyclonic winds of a tropical cyclone.

Eliminating u, v from (5), (6) and (7) and by computing the vorticity and divergence of the water, the result obtained is

$$\zeta_{ttt} + \left[f^2 - \frac{h}{F_r^2}\nabla^2\right]\zeta_t = \frac{h}{F_r^2}\left(\frac{V_s}{C}\right)\nabla^2 P_t + \left(\frac{u_*^2 T}{CH}\right) \times \left[\nabla \cdot \left(\frac{\nabla \tau}{\rho}\right)\right]_t + f\left(\frac{u_*^2 T}{CH}\right)\left[k \cdot \nabla \times \frac{\nabla \tau}{\rho}\right] \qquad (12)$$

where

$$\nabla \tau = i \nabla F + j \nabla G$$

and i, j, k are unit vectors in cartesian co-ordinates. From equation (12) we can conclude that the surge is generated by three forcing functions:

(I) the pressure tendency in the field of cyclone.

(II) the rate of change of divergence of $\nabla\tau$ and

(III) the curl of differential friction. As sea-bed friction is a small fraction of the wind stress, the 2nd and 3rd forcing terms in (12) represents,

(1) the rate of change of divergence and (II) the curl of the wind stress. So we see that it is the only curl of the wind stress which has no time dependency.

Now, Laplace transform is applied to equation (12) and noting that the Laplace transform of ζ can be represented as

$$\hat{\zeta} = \int_0^\infty \zeta \exp(-st)\, dt$$

and similar expression for the transform of P and $\nabla\tau$.

A distinction can be made between two situations:

(a) small values of time (t) when $s \rightarrow \infty$, and

(b) large time (t) when $s \rightarrow 0$

For case (a), the relevant equation for $\hat{\zeta}$ is found to be

$$\nabla^2 \hat{\zeta} - \left(\frac{F_r^2 s^2}{h}\right) \hat{\zeta} = - \phi(x, y, s) \qquad (14)$$

where $\phi(x,y,s) = \frac{F_r^2}{h}(s\zeta_0) + \left(\frac{V_s}{C}\right) \nabla^2 \hat{p} + \frac{F_r^2}{h}\left(\frac{u_*^2 T}{CH}\right) \nabla \cdot \left(\frac{\Delta \hat{\tau}}{\rho}\right);$

ζ_0 denotes the value of ζ at $t = 0$

For (b),

$$\nabla^2 \hat{\zeta} - \left(\frac{F_r^2 f^2}{h}\right) \hat{\zeta} = \frac{1}{s}\left[\phi_1 + \phi_2 - \phi_3\right] \qquad (15)$$

where

$$\phi_1 = \nabla^2 \zeta_0 + \left(\frac{V_s}{C}\right) \nabla^2 P_0$$

$$\phi_2 = \left(\frac{F_r^2}{h}\right)\left(\frac{u_*^2 T}{CH}\right) k \cdot \nabla \times \left[\frac{\Delta \hat{\tau}}{\rho} + \left(\frac{\Delta \tau}{\rho}\right)_0\right],$$

$$\phi_3 = f^2 \zeta_0 + \zeta_0''$$

A comparison between (14) and (15) suggests that for small value of t ie. at the initial stage, the divergence of the wind stress plays the dominent part and the rate of change of cyclone pressure is more important. At longer times the curl of the wind stress becomes more important as the gradients of the initial values of ζ and P appear to assert themselves, while in the beginning only ζ_0 is important.

These inferences are also physically reasonable, because the initial response of a basin would be to drive away the water from the disturbance source. This is measured by the divergence of the wind stress. At later stage, rational character of the wind will become more important. This is the curl of the wind stress.

We may also express the solution of (14) in terms Green's function when one considers infinite domain. If R is the distance between an observation point (x,y) and a source point (x_0, y_0), then,

$$R = \left[(x-x_0)^2 + (y-y_0)^2\right]^{1/2} \qquad (16)$$

For small R,

$$\hat{\zeta} = \iint_{-\infty}^{\infty} K_0\left[F_{ns} R/\sqrt{h}\right] \phi(x_0, y_0)\, dx_0\, dy_0 ,$$

where $\left[K_0 F_{ns} R/\sqrt{h}\right]$ represents a modified Bessel function of the second kind which also Laplace transform of $\left[t^2 - F_n^2 R^2/h\right]$ for $t > F_n R/\sqrt{h}$ (Morse and Feshback (1953).

As $\hat{\zeta}$ is the Laplace transform of ζ , we see that the basin does not respond until a time ($F_r R/\sqrt{h}$) has elapsed. This time depends on the storm speed and inversely proportional to $\sqrt{h}$ where h is the depth of the basin.

Model 2

Here we shall discuss about a numerical model by M.P. Singh [24] for the prediction of storm surge.

Formulation of the model

The basic hydrodynamic equations of continuity and momentum for the dynamical process in the sea form the basis of the formulation of the numerical model. The vertically integrated form of these equations are given by

$$\frac{\partial \xi}{\partial t} + \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial x}(u\bar{u}) + \frac{\partial}{\partial y}(v\bar{u}) - f\bar{v} = -g(\xi+h)\frac{\partial \xi}{\partial x} - \frac{1}{\rho}(\xi+h)\frac{\partial p_a}{\partial x} + \frac{F_s}{\rho} - \frac{C_f\,\bar{u}}{(\xi+h)}(u^2+v^2)^{1/2}$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial}{\partial x}(u\bar{v}) + \frac{\partial}{\partial y}(v\bar{v}) + f\bar{u} = -g(\xi+h)\frac{\partial \xi}{\partial y} - \frac{1}{\rho}(\xi+h)\frac{\partial p_a}{\partial y} + \frac{G_s}{\rho} - \frac{C_f\,\bar{v}}{(\xi+h)}(u^2+v^2)^{1/2}$$

where u,v are depth averaged velocity components in the direction of x and y respectively. ξ , h and ρ are the sea surface elevation, depth of the basin and the density of the sea water.

Then

$\bar{u}, \bar{v} = \left[(\xi + h) u, (\xi + h) v \right]$ are two new prognostic variables.

$\xi + h$ = Total depth of the basin.

(F_s, G_s) are wind stress components.

$C_f = 2.6 \times 10^{-2}$, an empirical bottom friction co-efficient.

The above equations are solved numerically subject to the following initial and boundary conditions:

$\xi = u = v = 0$ everywhere for $t \leq 0$.

At the coast, $u \cos\alpha + v \sin\alpha = 0$ for all $t \geq 0$, where α denotes the inclination of the outward drawn normal to the X-axis. Normal currents across the the open sea boundaries may be prescribed by a radiation type of condition.

The model has been used successfully in simulating the surges generated by the devastating 1975 Porbandar, 1977 Andhra and 1982 Orissa and Saurastra cyclones. It is noticed that the model predicted maximum surge is in fairly good agreement with the observed surge at two tidal stations, viz., Paradip and Dharma.

Like surge, monsoon is also a matter of great concern to the people living in the monsoonal regions. Sulochana Gadgil [25] has made a detailed study of "Fluid dynamics of the monsoon". The monsoon is by no means restricted to the Indian region and the surrounding oceans.

Ramage's (1971) delineation of the monsoonal regions of the world using a criterion based on the seasonality of winds shows the monsoon to be a planetary scale phenomenon. The seasonality of the wind is but one aspect of the variation of the tropical circulation. Two scales appear prominently in the variation: (1) The synoptic scale of 2 - 5 days associated with coherent structures of a few hundred kilometers in the horizontal extent, called tropical disturbances and (II) the longer period of fluctuation (with a time-scale of two weeks or more) between the active spells and weak spells associated with the planetary scale.

The synoptic-scale disturbances organise the cumulus clouds. This intense tropical disturbances are typhoons or hurricane. These synoptic-scale disturbances are embedded in the planetary scale. The interactions among these three scales havd been studied by application of fluid dynamics.

Planetary-scale monsoon:

The major features of the longitudinally and temporally averaged tropical circulation at the surface of the earth are:

(1) The trade winds blowing from the subtropical high ppressure belts towards the equatorial region in the north-easterly (south easterly) direction in the northern (southern) hemisphere.

(II) A zone of low pressure in the equatorial region called the equatorial trough or the Intertropical Convergence Zone (ITCZ) in which the trades from the two hemisphere converge. Although the pressure field is axisymmetric, the trades have a westward component of velocity due to action of the Coriolis force which is significant for these planetary scales. The air converging in the ITCZ rises and moves poleward in the upper troposphere. The vertical circulation comprises ascent over ITCZ and descent everywhere else. This meridional cell is called the Hadley cell. The updrift in the ITCZ which occurs throughout the troposphere is associated with intense moist convection.

The location of the ITCZ varies with season in association with the variation of the sea-level trough. The amplitude of the seasonal variation is maximum in the monsoonal regions. The zonal component changes from easterly to westerly upon crossing the equator because of the changes in sign of the local vertical component of the rotation of the earth.

Poleward of the trough, the notth easterly trades prevails. Thus a seasonal reversal in the direction of the wind occurs over the region lying between two latitudes corresponding to the extreme locations of the equatorial trough.

<u>Hadley cell:</u> a simple model.

The ultimate source of energy for the atmospheric motion is the radiation from the sum. Earth's atmosphere may be considered to be transparent to the solar radiation. The atmosphere is heated at its bottom boundary ie. at the surface of the earth. The heat balance at the surface of the earth determines the nature of the boundary condition. Large heat capacity of the ocean implies a prescription of the surface temprature with latitude. Again since the heat capacity of the land is negligible, heat flux is taken into consideration again as a function of latitude. The Hadley cell is then the axisymmetric response of a rotating planetary atmosphere to longitudinally varying heating from below.

This problem of side ways convection has been studied by analytical and numerical models, in the linear as well as nonlinear regime. The vertical transport of heat and momentum by turbulence can be assumed to parametrized in terms of eddy co-efficient. The important results emerging from these models are (1) the rising limits of the convective cell is located at the latitude at which the specified surface temperature or heat flux is maximum. (II) When the Rayleigh numbers are high, the ascending limb is much narrower than the descending one. (III) The circulation being more or less restricted to a depth of about one kilometer from the surface for parameters appropriate to the earth's atmosphere. The circulation extends throughout the troposphere only when a mid-tropospheric heat source representing the latent heat released by the convective clouds associated with the ITCZ is included in the model.

Centre for Advanced studies in Atmospheric and Fluid Sciences at I.I.T., Delhi has succeeded in the development of a variable grid General Circulation Model and has been termed as Monsoon General Circulation Model (MGCM) [24]. The unique feature of the model is that it allows higher concentrations of grid points in the Indian Monsoon area and thus allow a finer description of the monsoon structure while retaining the main features of the global general circulation.

The model equations are governed by the well known laws of conservation of momentum, mass and energy. This model is written in σ-co-ordinates (pressure normalised by surface pressure, p/p_0). It has the advantage of incorporating the lower boundary condition in an easier way as the earth's surface becomes the co-ordinates surface. In the horizontal the resolution is variable which gives the finer resolution over the monsoon area.

MGCM was successfully run on real time basis using May 25, 1985 data as the initial input to capture the characteristic large scale monsoon circulation for the current onset of monsoon.

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