BORSUK-ULAM THEOREM AND DEVELOPMENTS-A SURVEY

BASIL S. KOIKARA
DEPARTMENT OF MATHEMATICS
SCHOOL OF PHYSICAL SCIENCES

Submitted in partial fulfilment of the requirement for the Degree of Master of Philosophy

To

NORTH-EASTERN HILL UNIVERSITY
SHILLONG
JUNE, 1990

CERTIFICATE

I certify that the dissertation entitled "Borsuk-Ulam Theorem and Developments - a Survey" submitted by Mr. Basil S. Koikara in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed, have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

Dr. H. Mukherjee

Supervisor,
Department of Mathematics,
North-Eastern Hill University,
Shillong

Shillong,
June 25, 1990.

ACKNOWLEDGEMENTS

This work was done under the guidance of Dr. H. Mukherjee. I wish to express my sincere thanks to him for his invaluable help and guidance in the preparation of this dissertation.

I am grateful to all the faculty members of the Department of Mathematics, North-Eastern Hill University, Shillong, especially those who had the responsibility of giving us M. Phil. courses and Seminars. Among these, I would like to mention Prof. S.S. Khare specially for all the help that he constantly gave me, also in spheres other than academical.

I want to express my gratitude to my research colleagues and to all other students of the Department of Mathematics who have always helped me. I would like to mention in a special way Mrs. Shyamali Dutta, Miss Beena George and Mr. Bipul Purkayasta who have always extended encouragement and helpful companionship without which any work becomes dull.

I am also grateful to all the office staff in the Department of Mathematics for their sincere co-operation.

Finally, but not the least in importance comes my wife Monica who has considered my work as her own and expressed her concern for its auspicious conclusion by giving me all possible help to complete it well.

Mr. Basil S. Koikara

CONTENTS

A Note on Symbols Used

Apart from the usual symbols used in this branch of Mathematics, we have had to use certain symbols to make typing easier.

Numbers enclosed in /̲ ̲7 indicate references from the bibliography given at the end of the work. The slash '/' has been used both as a symbol for quotienting (e.g., Z/G representing Z quotiented by G) and for restriction of maps (e.g., f/A representing the map f restricted to A). We hope that the reader will be able to distinguish between the usages according to the circumstance without too much difficulty.

Instead of using the usual symbols $\mathbb{R}$, $\mathbb{Z}$ etc., for the sets of real numbers and integers etc., we have represented them by $\underline{R}$, $\underline{Z}$ etc..

1.0 INTRODUCTION

If a balloon is deflated and placed on the floor, two antipodal points end up over the same point. This is an intuitively evident statement. Borsuk-Ulam theorem puts this fact on a sound mathematical footing. It therefore assumes the same importance, as a piece of fundamental work, as Jordan-Brower Separation Theorem (Jordan Curve Theorem in particular) or Brower's Fixed Point Theorem, which are both theorems having intuitively clear statements.

The Borsuk-Ulam Theorem was set forward by K. Borsuk in 1933 in the paper entitled "Drei Sätze über die n-dimensionale euclidische Späre" [3], which states that a continuous map of an n-sphere into the euclidean n-space maps some pair of antipodal points into a single point.

Ever since the publication of this result, mathematicians have tried to answer several questions that arise naturally from this theorem, and the process has continued to this day. What are some of these questions?

In order to answer this question, we should first have a closer look at Borsuk's 1933 statement. This can be restated as : A continuous map of an n-sphere into the euclidean n-space maps some pair of points symmetric under the antipodal action on S^n (or some orbit of the antipodal action on S^n) into a single point. One may then ask the following questions :

1.0.1 What is the nature and size (to be defined suitably) of the union of such orbits (which are mapped to a single point) ?

1.0.2 Can we replace S^n in the Borsuk-Ulam theorem by a space X together with some action (e.g., involution or a group action) and ask a question similar to 1.0.1 ?

1.0.3 As a particular case of 1.0.2 take $X = Y \times S^n$ or more generally an n-sphere bundle $X \longrightarrow Y$ with fibrewise action, and an equivariant (in case $\underline{R}^k$ also has an action) map $f: X \longrightarrow \underline{R}^k$. We could now ask a question more pointed than 1.0.2, viz., What is the size of that part of the union of orbits (as in 1.0.1 or 1.0.2) which is contributed by Y ?

1.0.4 Could now, the euclidean space (as the range space) in the Borsuk-Ulam's Theorem be replaced by a space which is only locally euclidean, i.e., manifolds (Riemannian, smooth, pl., topological), homology manifolds or Poincare duality spaces, and ask 1.0.1 ?

1.0.5 Consider a map $Y \times S^n \xrightarrow{1 \times f} Y \times \underline{R}^k$ or more generally, a fibre map of a sphere bundle over Y into a vector bundle over Y, where the spaces involved have some fibrewise action, and maps are equivariant. What answers can we obtain for 1.0.1 in this context ?

Several other questions may be raised, and we invite the reader to join us in raising them. One general question, however, emerges from these : "How far away are these Borsuk-Ulam sets (e.g., the zero sets) from becoming a manifold (which corresponds to the transversality principle) ?

1.1 This dissertation is the outcome of an attempt to assemble together some of the answers to these questions as obtained by mathematicians over the years. As the amount of work done in the last more than five decades on this topic is quite large, and as not every work fits into our line of enquiry, we have had to be selective. However, we have tried to pick up most of the significant works and tried to give a survey up to the present state of research on the topic.

We begin by mentioning two results which set the ball rolling in the direction of the developments we are concerned with :

1.1.1 Theorem : (Kakutani [26] Yamabe and Yujobo [38]). A continuous real-valued function on an n-sphere maps the terminals of some n+1 mutually perpendicular radii into a single value.

1.1.2 Theorem : (Dyson [11]). A continuous real-valued function on a 2-sphere maps four ends of some pair of orthogonal radii into a single value.

These results can be thought of as obtained by replacing the antipodal action on a sphere in the Borsuk-Ulam Theorem by more general group actions.

Guy Hirsch [18] was the first to take a step towards answering 1.0.2. He gave a sufficient condition under which S^n in the Borsuk-Ulam Theorem can be replaced by a space with an involution, but it is the work of C.T. Yang [39] which contains an answer to the question 1.0.2 and as a special case, the answer to 1.0.1 too. He has considered a T-space (a compact Hausdorff space together with a fixed point free involution T) and defined an index of such a space which in some sense gives the size of the T-space. Using this, he proves :

1.1.3 Theorem : (Yang [39]). If X is a T-space of index n and if $f : X \longrightarrow \underline{R}^k$ $(0 < k \leq n)$ is a map, then the set $A(f) = \{x \in X \ / \ f(x) = f(Tx)\}$, together with T is a T-space of index $\geq$ n-k.

Taking $X = S^n$ and T the antipodal action, the theorem answers 1.0.1. If k = n, then the theorem gives the classical Borsuk-Ulam result.

P. Bacon /2/ extended Yang's work for involutions which are not necessarily fixed point free.

The answer to question 1.0.3 and also to 1.0.2 can be found in the work of Conner and Floyd /6/, who have developped the notion of a coindex of a space (which also in some sense gives the size of the space involved), and made use of Stiefel Whitney classes to prove the following results :

1.1.4 Theorem : /6/ The same statement as 1.1.3, replacing index by coindex.

1.1.5 Theorem : /6/ (i) Let $p: B \longrightarrow Y$ be an n-sphere bundle over a compact space Y, $f: B \longrightarrow \underline{R}^m$, $m \geqslant n+1$, a map, then the set $S = \{x \in Y \,/\, f(b) = f(-b)$ for some $b \in p^{-1}(x)\}$ supports every dual Whitney class $\overline{w}_p$, $p \geqslant m-n$.

(ii) If Y is a compact connected manifold, then result (i) implies that (cohomology)dim $S \geqslant \dim Y - p$, provided $\overline{w}_p \neq 0$ and $p \geqslant m-n$.

Two noteworthy features of this paper is that

(a) it proves a result which is the first step towards answering 1.0.4, viz.,

1.1.6 Theorem : Suppose f is a continuous map of S^n into a compact Riemannian n-manifold. Then for some x, either $f(x) = f(-x)$, or for some x, $f(x)$ and $f(-x)$ are not joined by a unique geodesic of shortest length.

(b) Techniques of this paper enables one to give a necessary condition for a smooth manifold to be immersed in another smooth manifold. cfr. 5.2.5

Jaworowski and Moszynski /21/ also prove results answering partially question 1.0.4 by taking $\underline{RP}^n$ as the range space. The works of Munkholm /30/ and Conner /5/, contain a more complete answer to 1.0.4. We mention here only one of these:

1.1.7 Theorem : (P.E. Conner /5/). Let $f: S^n \longrightarrow M^k$ be a continuous map, where M^k is a smooth manifold, $n \geqslant k$. Then $\dim(\{x \in S^n \,/\, f(x) = f(-x)\}) \geqslant n-k$, if $n > k$. If $n = k$ and $f^* : H^n(M^n;\underline{Z}_2) \longrightarrow H^n(S^n;\underline{Z}_2)$ is trivial, then $\{x \in S^n \ / \ f(x) = f(-x)\} \neq \emptyset$.

Answers to 1.0.5 were attempted by Jaworowski ([22], [23]), Nakaoka [31], Fadell and Husseini [11], and Dold [8]. A sample result is :

1.1.8 Theorem : (Dold [8]). Let $f : S(E) \longrightarrow E'$, be an odd fibre map of an

$$\begin{array}{ccc} S(E) & \longrightarrow & E' \\ & \searrow \ \swarrow & \\ & B & \end{array}$$

(m-1)-sphere bundle into an n-plane bundle over the same paracompact space B. Let $Z = \{x \in S(E) \ / \ f(x) = 0\}$ and $\overline{Z}$ be its quotient space under the antipodal action on the sphere. Then for every polynomial $q(t) \in H^*(B)[t]$ of degree less than $n-m$, $q(t)$ is supported in $\overline{Z}$ (in some sense made explicit in 7.1. cfr. 7.1.3).

The significance of Dold's work is first of all, that it opens up a way to attack the problems in more general settings, involving more general actions and more general bundles. Secondly, the work points a way for possible refinements of the results using K-theory or cobordism techniques. Another notable feature is that this approach connects the Borsuk-Ulam Theorem directly to immersion problems.

Several other more recent works dealing with the Borsuk-Ulam theorem have been published. Notable among them are the works of Fadell, Husseini and Rabinowitz ([15]), Hai Bao Duan [9], Jaworowski [24],[25] and Waclaw Marzantowicz [27].

As is always the case, answer to a question raises new questions and this dissertation is no exception to this. (For example, one could ask "What is the G-level of a space (i.e., the min($\{n \ / \ \exists$ an equivariant map f : Space $\longrightarrow S^{n-1}\}$)) ? Some work was done in this connection by Conner and Floyd [6], for $G = \underline{Z}_2$, who called the $\underline{Z}_2$-level, the coindex of a space.

A more recent work towards answering the question appeared in [35]. It is our hope that this survey will stimulate the reader to further enquiry and investigation.

1.2 In the survey work that we have undertaken, an attempt has been made to make each chapter self-sufficient, and hence any chapter can be read independantly of another. Apart from that, only a minimum of homotopy and homology theories has been taken for granted and most often, even elementary results have been quoted to aid smooth reading.

In the first chapter, the classical Borsuk-Ulam Theorem is stated and a proof is set forth using singular cohomology with coefficients in $\underline{Z}_2$. In the proof we use the lifting theorem (cfr. [34], 2.4.5).

The second chapter studies a generalisation of the Borsuk-Ulam Theorem by C.T. Yang to maps of T-spaces into euclidean spaces. Using Čech-Smith homology theory we define index-homomorphisms, and using these homomorphisms, we define the (homological) index of a T-space X. The main result of the chapter says that if f: $(X,T) \longrightarrow \underline{R}^k$ be a continuous map of a T-space X into euclidean k-space where $0 < k \leqslant \text{ind } X$, then $A(f) = \{x \in X \,/\, f(x) = f(Tx)\}$ is T-invariant and compact and ind $A(f) \geqslant$ ind X - k. We note that this result implies that the homological dimension of $A(f) \geqslant$ ind X - k. In conclusion, we prove that ind (S^n, A) where A is the antipodal map on S^n is equal to n, and hence that if f: $S^n \longrightarrow \underline{R}^k$, then dim $A(f) \geqslant n - k$, which is also a generalisation of the Borsuk-Ulam Theorem.

In the third chapter, we turn our attention to a proof of Yang's generalisation of Borsuk-Ulam Theorem put forward by P.E. Conner and E.E. Floyd. The proof uses the notion of an abstract Coindex which we define in the very first section of the chapter. In the second section, we set forth the proof of the theorem. The next two sections are devoted to studying a cohomology coindex defined using equivariant Alexander-Spanier cohomology, free acyclic resolution of a finite group G and certain cohomology classes which we shall define, and proving that this cohomology coindex is in fact an abstract Coindex.

In the next chapter we consider an equivariant map $f: B \longrightarrow \underline{R}^m$ where $p: B \longrightarrow X$ is an n-sphere bundle over a compact space X, $m \geq n+1$, and attempt to measure the set $f^{-1}(0)$. In the first section we define the Stiefel Whitney classes of a sphere bundle following Grothendieck. In the next section we introduce the dual of a Stiefel Whitney class, and obtain some preliminary results involving these classes. Using these results, we obtain in the next section, that the set $S = \{x \in X / f(b) = 0$ for some $b \in p^{-1}(x)\}$ supports every dual class $\bar{w}_p$, for $p \geq m-n$. From this result we obtain that if X is a compact connected manifold, $\dim (S) \geq \dim (X) - p$, provided that $\bar{w}_p \neq 0$, $p \geq m-n$. In the concluding section we prove a result independant of the above result, but included in this chapter because the proof of the result will use some of the results obtained in the chapter. We define the index, ind(X,T) of a space X equipped with a fixed point free involution T, and using it, prove that if $f: S^n \longrightarrow X$ is continuous, where X is a compact Riemannian n-manifold, then for some x, either f(x) = f(-x) or for some x, f(x) and f(-x) are not joined by a unique geodesic of shortest length.

Chapter five studies maps $f: S^n \longrightarrow M^k$, $n \geq k$, into a smooth k-manifold M^k, which is not necessarily closed or compact. We look for results about $A(f) = \{x \in S^n / f(x) = f(-x)\}$. In the first section we define the twist of an n-plane bundle $p: \eta \longrightarrow X/T$, where (X,T) is a fixed point free involution, by the involution (X,T), and the universal whitney classes. Using this we obtain an expression for the Stiefel Whitney class of the twist. Further computations are made of Stiefel Whitney classes and in the next section we establish the main theorem which says that if $n > k$, $\dim A(f) \geq n-k$, and that if for $n = k$, $f^* : H^n(H^n; \underline{Z}_2) \longrightarrow H^n(S^n; \underline{Z}_2)$ is trivial, then $A(f) \neq \emptyset$. In the concluding section we see that the involution (S^n, A) can be replaced by a fixed point free involution (Σ^n, T) on an n-manifold Σ^n which is a homotopy n-sphere.

The sixth chapter deals with parametrized Borsuk-Ulam Theorems. We study fibre-preserving equivariant maps $f:SE \longrightarrow E'$, where $p: E \longrightarrow B$ and $p':E' \longrightarrow B$ are vector bundles of dimensions m and n respectively over a paracompact space B and $p/SE : SE \longrightarrow B$ is the induced sphere bundle over B. In the first section, using Čech-cohomology with coefficients in $\underline{Z}_2$, we define the Stiefel-Whitney polynomials $w(t)$ and $w'(t) \in H^*(B)[t]$, of the two vector bundles respectively. We let $Z = \{x \in SE / f(x) = 0\}$ and define the quotient spaces $\overline{SE}$ and $\overline{Z}$. The first theorem is now set forth : If $q(t) \in H^*(B)[t]$ is such that $q(t)/\overline{Z} = 0$, then $q(t).w'(t) = w(t).q'(t)$ for some polynomial $q'(t) \in H^*(B)[t]$. The theorem is clarified and particularised using three corollaries. The proof of the theorem is undertaken in the more general setting of G-sphere bundles and G-actions, for the case $G = \underline{Z}_2$. In the second section, the same theorem is proved, but now allowing fixed points in the action. In the last section of the chapter, we see some examples, and consider how the result obtained in the chapter may be used to measure obstructions to immersion of a manifold in another.

In the last and concluding chapter, we examine how the results of the previous chapter may be developed using generalised cohomology theories like K- theory and cobordism theory. In the first section of this chapter we do the preliminary spadework leading to, and the definition of characteristic classes in the context of generalised cohomology theories. The next section is devoted to an exposition of the rudiments of K-theory and cobordism theory. The third section studies spectra, associated homology and cohomology theories, and ring spectra. The section studies also the spectra for K-theory and cobordism theory, and examines how the ring spectrum structure can be extended to these spectra. In the final section we examine how characteristic classes can be defined for these generalised cohomology theories. We conclude by remarking that the results of the last chapter can be extended to the context of these generalised cohomology theories if we consider Čech cohomology constructed from the spectra in consideration. We further conjecture that the results hold even if we do not consider Čech cohomology. We also pose some problems and propose new avenues for exploration. It is true that the last chapter has turned out to be longer than expected. But the very general nature of the topics dealt with in this chapter has necessitated that.

2.0 THE CLASSICAL BORSUK-ULAM THEOREM

In the paper "Drei Satze uber die n-dimensionale euklidische Spare" published in 1933, K. Borsuk set forward theorems which are known now as theorems of Borsum-Ulam, [3]. The main theorem states that a map of an n-sphere into the euclidean n-space, maps some pair of antipodal points into the same point. A purely point-set/topological approach to the proof using triangulations to define the degree of a map is found in Dugundji ([10], XVI.6). Here, howe-ver, we shall look at a proof using singular cohomology theory ([34], 5.8.9).

2.1 Theorem : Let $f : S^n \longrightarrow \underline{R}^n$ be a continuous map and $A(f) = \{x \in S^n \,/\, f(x) = f(-x)\}$. Then $A(f) \neq \emptyset$.

For the proof we shall use the well-known result : For $n \geqslant 1$, $H^*(\underline{RP}^n; \underline{Z}_2)$ is a truncated polynomial algebra over $\underline{Z}_2$, generated by w_n (the characteristic class of $\underline{RP}^n$, $w_n \in H^1(\underline{RP}^n; \underline{Z}_2)$), of degree 1 and height n+1. We shall arrive at the proof after three lemmas.

2.1.1 Lemma: Let $n > m \geqslant 1$ and let $i: \underline{RP}^m \longrightarrow \underline{RP}^n$ be a linear imbedding. Then for $q \leqslant m$, $i^*: H^q(\underline{RP}^n; \underline{Z}_2) \cong H^q(\underline{RP}^m; \underline{Z}_2)$.

Proof :

$$\begin{array}{ccc} i^*(\xi) \cong S^m & & S^n \\ \Big\downarrow p_2 & & \Big\downarrow p_1 \\ \underline{RP}^m & \xrightarrow{\quad i \quad} & \underline{RP}^n \end{array}$$

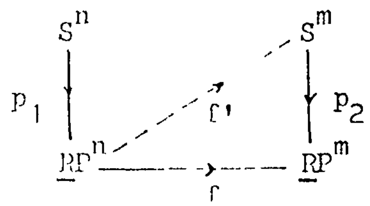
Consider the above situation where $\xi : S^n \xrightarrow{p_1} \underline{RP}^n$, is the O-sphere bundle over $\underline{RP}^n$. Then $i^*(\xi)$ is homeomorphic to the O-sphere bundle over $\underline{RP}^m$, i.e., the double covering $p_2: S^m \longrightarrow \underline{RP}^m$. By naturality of the characteristic class, $i^*(w_n) = w_m$. Further, by the result quoted above, $H^q(\underline{RP}^n;\underline{Z}_2)$ is generated by w_n^q, and $H^q(\underline{RP}^m;\underline{Z}_2)$ by w_m^q.

Now, $i^*(w_n^q) = (i^*w_n)^q = w_m^q$

hence, $i^* : H^q(\underline{RP}^n;\underline{Z}_2) \cong H^q(\underline{RP}^m;\underline{Z}_2)$ @

2.1.2 Lemma : Let $n > m \geq 1$ as above, and let $f: \underline{RP}^n \longrightarrow \underline{RP}^m$ be a map. Then, there exists a map $f': \underline{RP}^n \longrightarrow S^m$ such that $p_2 \circ f' = f$, where $p_2 : S^m \longrightarrow \underline{RP}^m$ is the double covering.

Proof : The situation is as follows :



By the lifting criterion (cfr. [34], 2.4.5), if we prove that $f_\#(\pi(\underline{RP}^n)) = 0$, we are through.

First, let $m = 1$. Then $\pi(\underline{RP}^n) = \underline{Z}_2$ and $\pi(\underline{RP}^1) = \underline{Z}$.

Then as the only homomorphism from $\underline{Z}_2$ to $\underline{Z}$ is the zero homomorphism, we shall have $f_\#(\pi(\underline{RP}^n)) = 0$.

Now let $m > 1$.

Consider the map $f^* : H^1(\underline{RP}^m) \longrightarrow H^1(\underline{RP}^n)$

As $H^1(\underline{RP}^n) \cong \underline{Z}_2$, and its two elements as 0 and w_n, we shall have

$$f^*(w_m) = 0 \quad \text{or} \quad f^*(w_m) = w_n.$$

suppose $f^*(w_m) = w_n$

$$f^*(w_m^{m+1}) = w_n^{m+1}$$, as f^* is a module homomorphism.

$$\neq 0 \text{ as } n > m.$$

$$\Rightarrow \Leftarrow \text{ as } w_m^{m+1} = 0.$$

Conclude that $f^*(w_m) = 0$

Let $i : \mathbb{R}P^1 \longrightarrow \mathbb{R}P^n$ and $j : \mathbb{R}P^1 \longrightarrow \mathbb{R}P^m$ be the linear inclusion maps. Now, as $\mathbb{R}P^1 \approx S^1$ and as $\pi(\mathbb{R}P^n) \cong \pi(\mathbb{R}P^m) \cong \mathbb{Z}_2$, $[i]$ and $[j]$ can be considered to be the generators of $\pi(\mathbb{R}P^n)$ and $\pi(\mathbb{R}P^m)$ respectively.

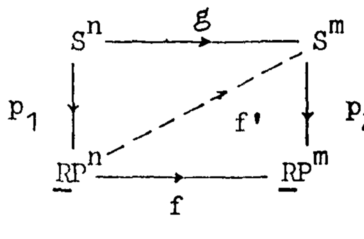
Now, since $f^*(w_m) = 0$, we shall have $i^*f^*(w_m) = 0$.
But by 2.1.1, $j^*(w_m) \neq 0$.
Hence, $(f \circ i)^* \neq j^* \Rightarrow f \circ i$ is not homotopic to j.
Now, $[j]$ is the non-zero element of $\pi(\mathbb{R}P^m) \cong \mathbb{Z}_2$. Hence $f \circ i$ must be null-homotopic.

i.e., $[f \circ i] = 0$. i.e., $f_\# [i] = 0$

$$f_\# (\pi(\mathbb{R}P^n)) = 0$$

2.1.3 Lemma : Let $n > m \geq 1$. Then there is no continuous map $g : S^n \longrightarrow S^m$ such that $g(-x) = -g(x)$ $\forall$ $x \in S^n$. (i.e., no continuous antipode preserving map.)

Proof : If such a map existed, then passing on to quotients, we would have a map $f : \mathbb{R}P^n \longrightarrow \mathbb{R}P^m$, making the following diagram commutative :-



By 2.1.2, f can be lifted to a map $f' : \mathbb{R}P^n \longrightarrow S^m$.
Then, $p_2 f' p_1 = f\, p_1 = p_2\, g$.
Hence, $f'\, p_1$ and g are both lifts of the same map $f\, p_1$.
Note that for any $x \in S^n$, either $g(x) = f'\, p_1\, (x)$

or $g(-x) = f'\, p_1\, (x)$

$= f'\, p_1\, (-x)$

so that g and $f' \, p_1$ agree at a point of S^n.

Hence by the unique lifting property, $g = f' \, p_1$.

But this is not possible, as for any $x \in S^n$, g maps x and $-x$ to different points, while p_1 and hence $f' \, p_1$ maps them to the same point.

Proof of Theorem 2.1

Suppose $\exists \, f : S^n \longrightarrow \underline{R}^n$ such that $A(f) = \emptyset$,

i.e., such that $f(x) \neq f(-x) \; \forall \; x \in S^n$.

Then, define $g : S^n \longrightarrow S^{n-1}$

$$\text{by } g(x) = \frac{f(-x) - f(x)}{|f(-x) - f(x)|}$$

which is both antipode preserving and continuous.

But this contradicts lemma 2.1.3 above.

3.0 MAPS FROM A T-SPACE TO A EUCLIDEAN SPACE - GENERALISATION OF THE BORSUK-ULAM THEOREM BY C.T. YANG

S. Kakutani /26/, H. Yamabe and Z. Yujobo /38/ and F.J. Dyson /11/ also studied similar phenomena for maps of spheres into euclidean spaces. The Kakutani-Yamabe-Yujobo theorem states that a continuous real-valued function on an n-sphere maps the terminals of some n+1 mutually orthogonal radii into a single value. Dyson's theorem states that a continuous real-valued function on a 2-sphere maps the four end points of some pair of orthogonal diameters into a single value. Chung-Tao Yang /39/ in papers presented to the American Mathematical Society in September and November, 1953, extended the Borsuk-Ulam theorem and the above-quoted results to the more general situation of maps of T-spaces into euclidean spaces. Here we shall give our attention only to the generalisation by Yang of the Borsuk-Ulam Theorem.

3.1 Theorem : Let (X,T) be a T-space and let f be a map of X into the euclidean k-space $\underline{R}^k$, $0 < k \leqslant \text{ind}X$. Then $A(f) = \{x \in X \;/\; f(x) = f(Tx)\}$ is T-invariant and compact and $\text{ind}\, A(f) \geqslant \text{ind}\, X - k$.

For the purposes of the proof, we shall outline briefly the salient features of the Čech-Smith homology theory. Using it, we shall define the index of a T-space and proceed to prove the theorem.

3.2 Čech-Smith Homology Theory

3.2.1 Preliminary Definitions, Remarks and Results

A T-space (X,T) is a pair, where X is a compact, Hausdorff space and T is a continuous fixed point free involution on X. A T-pair is a triple (X,A;T) where (X,T) is a T-space and A is a T-invariant closed subset of X.

A map of T-spaces, f: $(X,A;T) \longrightarrow (Y,B;T)$ is a map of X into Y which maps A into B and $fT = Tf$ (i.e., f is equivariant).

A T-pair (X,A;T) is simplicial if X is a finite euclidean simplicial complex whose simplices are permuted among themselves by T.

Let $\mathcal{A}$ denote the category of all T-pairs and all maps of such pairs and let $\mathcal{A}_1$ denote the category of all simplicial T-pairs and all simplicial maps of such pairs.

Let (X,A;T) be a simplicial T-pair. Since T is a simplicial map of (X,A) into itself, it induces a chain map of the chains of (X,A) into themselves. Denote the chain map also by T. (All through this discussion, the coefficient group will be assumed to be $\underline{Z}_2$, the additive group of integers modulo 2).

A p-chain c is called a T-invariant p-chain or a (T,p)-chain if $T(c) = c$. All the (T,p)-chains of (X,A) form a group denoted by $C_p(X,A;T)$.

3.2.2 <u>Result</u> : A p-chain c is a (T,p)-chain $\Leftrightarrow$ $c = d + T(d)$ for some p-chain d. (cfr. [33])

The boundary operator ∂ maps $C_p(X,A;T)$ into $C_{p-1}(X,A;T)$.

Let

$$Z_p(X,A;T) = \{ c \in C_p(X,A;T) \;/\; \partial c = 0 \}$$

$$B_p(X,A;T) = \partial C_{p+1}(X,A;T)$$

$$H_p(X,A;T) = Z_p(X,A;T) / B_p(X,A;T).$$

The elements of the above groups are respectively called (T,p)-cycles, (T,p)-boundaries and pth T-homology classes of (X,A).

A simplicial map $f : (X,A;T) \longrightarrow (Y,B;T)$ defines a chain mapping $f: C_p(X,A;T) \longrightarrow C_p(Y,B;T)$, and induces a homology homomorphism $f_* : H_p(X,A;T) \longrightarrow H_p(Y,B;T)$.

For a T-pair (X,A;T) there is, as in singular homology theory, a boundary operator $\partial_* : H_p(X,A;T) \longrightarrow H_{p-1}(A,T)$

First, let (X,A;T) be a proper simplicial T-pair (A simplicial T-pair (X,A;T) is said to be proper if for each vertex u of X, the star of u does not intersect the star of T(u).) and let (X',A') be the simplicial pair obtained from (X,A) by identifying every point of X with its T-image. Then the natural map $\Upsilon : (X,A) \longrightarrow (X',A')$ is simplicial and for every simplex σ of X (or A), $\Upsilon(\sigma)$ is the simplex of X' (or of A') obtained by identifying σ with $T(\sigma)$.

Denote by $C_p(X',A')$ the group of p-chains of (X',A').
Note that there is a chain map $\Upsilon_\# : C_p(X,A;T) \longrightarrow C_p(X',A')$

defined by $\Upsilon_\#(d+ Td) = \Upsilon(d) = \Upsilon(Td)$.

Thus we define a homomorphism $\tilde{\Upsilon} : H_p(X,A;T) \longrightarrow H_p(X',A')$.
Note that the homology on the right is singular homology.

3.2.3 <u>Proposition</u> : $\tilde{\Upsilon}$ is an isomorphism of $H_p(X,A;T)$ onto $H_p(X',A')$ commutative with f_* and ∂_*.

<u>Proof</u> : This follows immediately as $\Upsilon_\#$ is itself one-one and onto and commutativity follows upon the definition of f_* and ∂_*.

Now since singular homology theory satisfies the Eilenberg-Steenrod axioms, we have :

3.2.4 By 3.2.3, the system H_* = $\{H_p, f_*, \partial_*\}$ is a homology theory on $\mathcal{Q}_1$.

Now, let $(X,A;T)$ be an arbitrary T-pair. By a covering of $(X,A;T)$ we mean a pair = (λ_1, λ_2) where λ_1 is a finite open covering of X and λ_2 is a subset of λ_1 which satisfy the following :-

(i) The members of λ_i are permuted among themselves by T, i = 1,2.

(ii) λ_2 covers A.

and

(iii) whenever $U \in \lambda_1$, the star of U (i.e., the union of all the members of λ_1 that intersect U) does not intersect the star of T(U).

Then the nerve of each covering λ of $(X,A;T)$ is a proper simplicial T-pair. We shall denote it by $(X_\lambda, A_\lambda; T)$. The collection Λ of all the coverings of $(X,A;T)$ is directed by $>$, where $\mu > \lambda$ means μ refines λ. If $\lambda, \mu \in \Lambda$ and $\mu > \lambda$, then there is a projection

$$\pi_{\lambda\mu} : (X_\mu, A_\mu; T) \longrightarrow (X_\lambda, A_\lambda; T)$$

If $\pi_{\lambda\mu}$ and $\pi'_{\lambda\mu}$ are two such projections, then they are contiguous and hence we have a unique homomorphism

$$\pi^*_{\lambda\mu} : H_p(X_\mu, A_\mu; T) \longrightarrow H_p(X_\lambda, A_\lambda; T).$$

The groups $H_p(X_\lambda, A_\lambda; T)$ and the homomorphisms $\pi^*_{\lambda\mu}$ form an inverse system.

3.2.5 The p^{th} T-homology group of (X,A) is defined to be the inverse limit of this system, i.e.,

$$H_p(X,A;T) \overset{def}{=} \varprojlim H_p(X_\lambda, A_\lambda; T)$$

If $(X,A;T)$ is a proper simplicial T-pair, then $(X,A;T)$ itself can be considered as the nerve of one of its coverings, λ say, and the natural homomorphism $H_p(X,A;T) \longrightarrow H_p(X_\lambda, A_\lambda; T)$ will be an isomorphism. Thus $H_p(X,A;T)$ is well-defined.

A map $f : (X,A;T) \longrightarrow (Y,B;T)$ of T-pairs induces a homomorphism

$$f_* : H_p(X,A;T) \longrightarrow H_p(Y,B;T)$$

and for any T-pair $(X,A;T)$ there is a boundary operator

$$\partial_* : H_p(X,A;T) \longrightarrow H_{p-1}(A,T).$$

Now, given any T-pair $(X,A;T)$, denote by (X',A'), the compact pair obtained from (X,A) by identifying every point with its T-image. Let the p^{th} Čech homology group of (X',A') be denoted by $\check{H}_p(X',A')$.

3.2.6 There is a natural isomorphism

$$\tilde{\Upsilon} : H_p(X,A;T) \xrightarrow{\cong} \check{H}_p(X',A')$$

commutative with f_* and ∂_*.

Now, since the Čech homology theory satisfies the Eilenberg-Steenrod axioms and the continuity axiom, we have

3.2.7 The system $H_* = \{H_p, f_*, \partial_*\}$ defined above is a homology theory on the category $\mathcal{a}$ and satisfies the continuity axiom. We shall call it the Čech-Smith homology theory.

3.3 Index of a T-Space

3.3.1 Let (X,T) be a simplicial T-space. Define homomorphisms

$$\nu : Z_p(X,T) \longrightarrow \underline{Z}_2 \quad \text{by recurrence,}$$

as follows :-

$$\nu(z) = \begin{cases} \text{In } c & , \text{ if } p = 0 \\ \nu(\partial c) & , \text{ if } p > 0 \end{cases}$$

where $z \in Z_p(X,T)$ is given by $c + Tc$,

and for $d = \sum g_j \sigma_i \in C_0(X;G)$, In d is defined by In $d = \sum g_i$

(G is the coefficient group).

Let $z = c + Tc = c' + Tc'$ be two representations of z,

$c = \sum g_i \sigma_i$; $c' = \sum g_j \sigma'_j$.

Then, $c+Tc = c' + Tc' \Rightarrow \sum g_i \sigma_i + \sum g_i T \sigma_i$

$$= \sum g_j \sigma'_j + \sum g_j T \sigma'_j .$$

But $\{\sigma_i\}$ are free generators. So the above equality implies that

$\forall$ i, $\sigma_i = \sigma'_j$ for some j or

$= T\sigma'_j$ for some j.

i.e., $g_i = g_j$ for some j, by the uniqueness of representation of c's.

so $\sum g_i = \sum g_j$.

Hence In c = In c'

Thus ν is well-defined when p = 0.

Further, if $z = \partial(d + Td)$ for some $d + Td \in C_1(X,T)$

then $\nu(z) = \text{In}(\partial d) = 0$

Hence ν annihilates $B_0(X,T)$.

i.e., ν is well-defined and annihilates $B_p(X,T)$ when p = 0.

By induction, for $p > 0$ assume that ν is well-defined on $Z_{p-1}(X,T)$ and that ν annihilates $B_{p-1}(X,T)$.

Now choose $z = c + Tc \in Z_p(X,T)$.

If $c' + Tc'$ is another representation of z, then we may write

$z = c_1 + c_2 + T(c_1 + c_2)$ such that $c = c_1 + c_2$ and $c' = c_1 + Tc_2$.

Now, $\nu(\partial c) - \nu(\partial c') = \nu(\partial c_1) + \nu(\partial c_2) - \nu(\partial c_1) - \nu(\partial Tc_2)$

$= \nu(\partial c_2) - \nu(\partial Tc_2)$

$= 0$ by induction hypothesis as ∂c_2 and ∂Tc_2 represent the same cycle.

Therefore, $\nu(\partial c) = \nu(\partial c')$ and hence ν is well-defined on $Z_p(X,T)$.

Further, if $z \in B_p(X,T)$, say $z = \partial d + \partial Td$ for $d + Td \in B_{p+1}(X,T)$,

then, $\nu(z) = \nu(\partial \partial d)$ (or $= \nu(\text{In } \partial d)$ if $p = 1$)

$= 0$

Hence ν annihilates $B_p(X,T)$.

Thus we are able to define a homomorphism

3.3.2 $$\nu : H_p(X,T) \longrightarrow \underline{Z}_2$$

defined by $\nu([z]) = \nu(z)$, $z \in Z_p(X,T)$

called the <u>index homomorphism</u>.

3.3.3 <u>Proposition</u> : If $f : (X,T) \longrightarrow (Y,T)$ is a simplicial map, for $[z] \in H_p(X,T)$, $\nu(f_*([z])) = \nu([z])$.

<u>Proof</u> : we are to prove that $\nu(f(z)) = \nu(z)$, $z \in Z_p(X,T)$, by the definition of ν, 3.3.2 and as $f_*[z] = [f(z)]$.

We proceed by induction.

For $p = 0$, we have $\text{In } f(z) = \text{In } z$ and we are through.

Assume the result for $p-1$, $p > 0$.

Let $z = c + Tc \in Z_p(X,T)$.

Then $\nu(z) = \nu(\partial c)$.

Also $\nu(f(z)) = \nu(\partial f(c)) = \nu(f(\partial c)) = \nu(\partial c)$ by induction hypothesis and we are through.

3.3.4 Now let (X,T) be an arbitrary T-space, and let $[z] = \{[z_\lambda]\}$ be an element of $H_p(X,T)$. Since (X_λ, T) is simplicial, $\gamma([z_\lambda])$ is defined. By the commutativity property of the inverse limit homomorphisms and 3.3.3, we shall have that $\gamma([z_\lambda])$ is independant of λ, so that

$$\gamma : H_p(X,T) \longrightarrow \underline{Z}_2$$

given by $$\gamma(\{[z_\lambda]\}) = \gamma([z_\lambda])$$

is a well-defined homomorphism called the index homomorphism.

3.3.5 By 3.3.3 and 3.3.4, we shall have that if $f: (X,T) \longrightarrow (Y,T)$ be a map of T-spaces, then for $[z] \in H_p(X,T)$, $\gamma(f_*([z])) = \gamma([z])$.

3.3.6 Proposition : Let (X,T) be a T-space and let F be a closed subset of X, with $F \cup T(F) = X$. Let $A = F \cap T(F)$. Then there is a homomorphism $\Delta : H_p(X,T) \longrightarrow H_{p-1}(A,T)$, $p > 0$

so that $$\gamma([z]) = \gamma(\Delta[z]).$$

Proof : First of all, suppose that $(X,A;T)$ is simplicial and let $[z] \in H_p(X,T)$. Let $z = c + Tc$ be a representative of $[z]$, such that the support of c is contained in F.

Then $\partial(z) = 0 \Rightarrow \partial c + \partial Tc = 0$

$$\Rightarrow \partial c = \partial Tc$$

$$\Rightarrow \partial c \in Z_{p-1}(A,T).$$

Further, we have that the $p-1^{th}$ homology class of (A,T) containing ∂c depends only on $[z]$.

Thus we are enabled to define a homomorphism

$$\Delta : H_p(X,T) \longrightarrow H_{p-1}(A,T)$$

by $\Delta([z]) = [\partial c]$.

Now, $\gamma([z]) = \gamma(z)$ by 3.3.2

$$= \gamma(\partial c) \qquad \text{by 3.3.1}$$
$$= \gamma([\partial c]) \qquad \text{again by 3.3.2}$$
$$= \gamma(\Delta[z]).$$

Hence the simplicial case is established.

The non-simplicial case may be established by the standard procedure of taking limits. @

3.3.7 <u>Corollary</u> : If $\gamma(H_n(X,T)) = \underline{Z}_2$, then $\gamma(H_p(X,T)) = \underline{Z}_2$ for $0 \leqslant p \leqslant n$.

<u>Proof</u> : Taking $F = X$ in 3.3.6, we get a homomorphism

$$\Delta : H_p(X,T) \longrightarrow H_{p-1}(X,T)$$

which makes the diagram

$$\begin{array}{ccc} H_n(X,T) & \xrightarrow{\ \Delta\ } & H_{n-1}(X,T) \\ & \searrow^{\gamma} \quad {}^{\gamma}\swarrow & \\ & \underline{Z}_2 & \end{array}$$

commutative.

Now, since $\gamma(H_n(X,T)) = \underline{Z}_2$, for $0 \in \underline{Z}_2$, $\exists [z] \in H_n(X,T)$ such that $\gamma([z]) = 0 = \gamma(\Delta[z])$.

Similarly for $1 \in \underline{Z}_2$.

Hence, $\gamma(H_{n-1}(X,T)) = \underline{Z}_2$.

Proceeding downwards we get the required result.

3.3.8 <u>Definition : Index of a T-Space</u>

For any T-space (X,T), there is an integer n such that $\gamma(H_p(X,T)) = \underline{Z}_2$ for $0 \leqslant p \leqslant n$ and $\gamma(H_p(X,T)) = 0$ for $p > n$.

<u>Proof</u> : By corollary 3.3.7, it suffices to prove that $\gamma(H_p(X,T)) = 0$ for some p.

Let λ be a covering of (X,T) and let p-1 be the dimension of (X_λ, T).

Then $H_p(X_\lambda, T) = 0$.

Now, let $[z] \in H_p(X,T)$. Then $\vartheta([z]) = \vartheta([z_\lambda]) = 0$

so that $H_p(X,T) = 0$.

This integer n is defined as the index of (X,T).

3.4 Generalised Borsuk-Ulam Theorem

Now, we are ready to restate theorem 3.1 and proceed to prove it.

Let (X,T) be a T-space and let f be a map of X into the euclidean k-space $\underline{R}^k$, $0 < k \leqslant \text{ind } X$. Then $\Lambda(f) = \{x \in X \mid f(x) = f(Tx)\}$ is T-invariant and compact and ind $\Lambda(f) \geqslant \text{ind } X - k$. Further, $\Lambda(f)$ is of homological dimension $\geqslant \text{ind } X - k$.

Proof : Let $f = (f_1, f_2, \ldots\ldots, f_k)$ where each f_j is a continuous real-valued function on X.

Let $\Lambda_o(f) = X$

$$\Lambda_j(f) = \{x \in X \mid f_i(x) = f_i(Tx),\ i = 1,2,\ldots,j\}$$

$$F_j = \{x \in \Lambda_{j-1}(f) \mid f_j(x) \leqslant f_j(Tx)\}$$

$$j = 1,2,\ldots\ldots,k.$$

Then $\Lambda(f) = \Lambda_k(f)$.

Clearly, each F_j is a closed subset of $\Lambda_{j-1}(f)$ and

$$\Lambda_{j-1}(f) = F_j \cup T(F_j)$$

and

$$\Lambda_j(f) = F_j \cap T(F_j)$$

Let j = 1.

$F_1 = \{x \in X \mid f_1(x) \leqslant f_1(Tx)\}$

and $\Lambda_1(f) = F_1 \cap T(F_1)$.

Then by proposition 3.3.6, we have that

$$\text{ind } A_1(f) \geq \text{ind } X - 1.$$

By induction, we shall obtain ind $A_k(f) \geq$ ind $X - k$.

Now, let us put ind $X = n$. Let $A'_k(f)$ be the compact Hausdorff space obtained from $A_k(f)$ by identifying every point with its T-image. Then, by 3.2.6, we have $H_{n-k}(A_k(f), T) \cong H_{n-k}(A'_k(f)) \neq 0$.
Hence dim $A'_k(f) \geq n - k$.

But as the natural map of $A_k(f)$ into $A'_k(f)$ is a local homeomorphism, dim $A_k(f) \geq n - k$. @

3.4.1 Corollary : Let (X,T) be a T-space of index n. Then a map from X into $\underline{R}^n$ maps some involution pair into a single point.

Proof follows directly from 3.4

3.4.2 Corollary : Given any map $f : S^n \longrightarrow \underline{R}^k$, the set $\{x \in S^n / f(x) = f(Tx)\}$ is of dimension $\geq n - k$, where T is the antipodal map on S^n.
Proof: For the proof, it suffices to show that ind $S^n = n$. This we shall do in the following lemma.

3.4.3 Lemma : $\gamma : H_p(S^n,T) \cong \underline{Z}_2$, $p = 0,1,2,\ldots,n$, and hence (S^n,T) is of index n.

Proof of lemma :

Note that since by 3.2.3, $H_p(S^n;T) \cong H_p(\underline{RP}^n)$ we have that

$$H_p(S^n,T) \cong \underline{Z}_2 \quad \text{if } 0 \leq p \leq n$$
$$= 0 \quad \text{if } p > n.$$

Hence by 3.3.7 we only have to prove that $\gamma(H_n(S^n,T)) = \underline{Z}_2$

We shall prove it by induction.

Since S^o consists of two points, from the definition 3.3.1, it follows that $\gamma(H_o(S^o,t)) \cong \underline{Z}_2$.

Now, assume the result for n-1.

Let S^n be the unit sphere in the euclidean (n+1)-space, let T be the antipodal map. Let $F = E^n_+$ be the subset of S^n consisting of the points whose $(n+1)^{th}$ co-ordinates are non-negative. And $T(F) = E^n_-$.

Now, $E^n_+ \cap E^n_- = S^{n-1} = F \cap T(F)$.

Then the homomorphism Δ constructed in 3.3.6 is an isomorphism onto.

$$\begin{aligned} \text{Hence,} \quad \gamma(H_n(S^n,T)) &= \gamma(H_{n-1}(F \cap T(F), T)) \\ &= \gamma(H_{n-1}(S^{n-1}, T)) \\ &= \underline{Z}_2, \text{ by the induction hypothesis.} \end{aligned}$$

3.5 Remarks

Putting k = n in 3.4.2, and using 3.4.1, we get the Borsuk-Ulam Theorem 2.1

3.4 and 3.4.1 are clearly both generalisations of the Borsuk-Ulam Theorem to T-spaces and maps to euclidean spaces. 3.4.2 gives more information on the set $A(f) = \{x \in S^n \ / \ f(x) = f(-x)\}$ for a map f of S^n into the euclidean k-space, than does theorem 2.1.

4.0 ABSTRACT COINDEX AND THE BORSUK-ULAM THEOREM

In this chapter, we are not interested as much in looking at a further generalisation of the Borsuk-Ulam Theorem, as in looking at the set $A(f)$ we defined in chapter three, from a new angle. To this end, we follow P.E. Conner and E.E. Floyd [6] and first of all define the abstract Coindex of a T-space (X,T) and proceed to prove the generalisation of the Borsuk-Ulam Theorem as found in Chang [39] (cfr. theorem 3.1) using this definition. After that we proceed to define the cohomology coindex of a T-space using fundamental classes. We next prove that the cohomology coindex is a Coindex, i.e., that it satisfies the postulates of the definition of abstract Coindex.

4.1 Abstract Coindex

Let J be a collection of T-spaces. J is said to be hereditary if and only if

(i) Whenever $(X,T) \in J$ and $A \subset X$ is closed and T-invariant, then $(A,T/A) \in J$.

(ii) $(S^0,A) \in J$, where A is the antipodal map of S^0.

4.1.1 Definition : A Coindex function on a hereditary collection J of T-spaces assigns to each $(X,T) \in J$, $X \neq \emptyset$, a non-negative integer or ∞, and satisfies

(i) If $m : (X,T) \longrightarrow (Y,T')$ is an equivariant map between two elements of J, then Coindex $X \leqslant$ Coindex Y.

(ii) If $X = A \cup B$, where $(X,T) \in J$ and $A,B \subset X$ are closed and T-invariant, then Coindex $X \leqslant$ Coindex A + Coindex B + 1

(iii) Coind $S^0 = 0$.

4.1.2 Definition: Continuity of the Coindex Function

We say that Coindex is continuous on J if whenever $(X,T) \in J$ and $A \subset X$ is closed and invariant, then there exists an open neighbourhood C of A such that $\overline{C}$ is T-invariant and Coindex A = Coindex $\overline{C}$.

4.2 Yang's Generalisation of Borsuk-Ulam Theorem

Let (X,T) be a T-space and let $f: X \longrightarrow \underline{R}^k$ be a map of X into euclidean k-space. Let $\Lambda(f) \subset X$ denote the set $\{x \in X \,/\, f(x) = f(Tx)\}$. Then $\Lambda(f)$ is a closed invariant subset of X and Coindex $\Lambda(f) \geq$ Coindex X - k, for any continuous Coindex with Coindex $S^n = n$, for all n.

Proof :

Inductively, define $\Lambda_j(f)$ as follows :

Let $f = (f_1, f_2, \ldots, f_k)$ where each f_j, $j = 1,2,\ldots,k$ are continuous real-valued functions.

Define $A_o(f) = X$

$$\Lambda_j(f) = \{x \in \Lambda_{j-1}(f) \,/\, f_j(x) = f_j(Tx)\}, \quad j = 1,2,\ldots k.$$

Then $\Lambda_k(f) = \Lambda(f)$.

Clearly $\Lambda(f)$ is closed and T-invariant.

We shall prove the theorem by induction on j.

We have $\Lambda_1(f) = \{x \in X \,/\, f_1(x) = f_1(Tx)\}$

By continuity of Coindex, as $\Lambda_1(f)$ is closed and T-invariant, $\exists$ an open neighbourhood C of $\Lambda_1(f)$ such that $\overline{C}$ is T-invariant and Coindex $\overline{C}$ = Coindex $\Lambda_1(f)$.

Let $B = X - \overline{C}$

Then $\overline{B}$ is closed, T-invariant and $\overline{B} \cap A_1(f) = \emptyset$.

Further, $X = \overline{B} \cup \overline{C}$.

Now, since $\overline{B} \cap A_1(f) = \emptyset$, if $x \in \overline{B}$, $f_1(x) \neq f_1(Tx)$.

Thus, we may define an equivariant map

$$\overline{B} \longrightarrow S^0$$

by
$$x \longmapsto \frac{f_1(x) - f_1(Tx)}{\| f_1(x) - f_1(Tx) \|}$$

so that Coindex $\overline{B} \leqslant$ Coindex $S^0 = 0$, by definition.

Now, Coindex $X \leqslant$ Coindex $\overline{C}$ + Coindex $\overline{B}$ + 1 by 4.1.1 (ii)

$\Rightarrow$ Coindex $X \leqslant$ Coindex $A_1(f) + 1$

$\Rightarrow$ Coindex $A_1(f) \geqslant$ Coindex $X - 1$.

By induction, we have Coindex $A_k(f) \geqslant$ Coindex $X - k$

4.3 Cohomology Coindex

4.3.1 Our aim now is to exhibit a concrete example of a Coindex function. This, we do using fundamental classes. If T is a fixed point free involution on a paracompact space X, we have the fundamental class $c \in H^1(X/T; \underline{Z}_2)$. (H denotes here, the Alexander-Wallace-Spanier cohomology). For example, we may take c to be the Stieffel Whitney class $w_1 \in H^1(X/T; \underline{Z}_2)$ of the 0-sphere bundle $\nu : X \longrightarrow X/T$ ([29], 12.3). We may also define c in terms of the Smith-Gysin sequence of $\nu : X \longrightarrow X/T$ with coefficients in $\underline{Z}_2$:

The two-fold covering $\nu : X \longrightarrow X/T$ gives rise to a line bundle $p : E \longrightarrow X/T$. Let $\dot{E}$ be the subspace of E consisting of the non-zero vectors of E. Then the exact sequence for the pair $(E, \dot{E})$

gives the following diagram and the consequent sequence at the bottom. We call it the Smith-Gysin sequence.

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H^n(E,\dot{E}) & \xrightarrow{j^*} & H^n(E) & \longrightarrow & H^n(\dot{E}) & \xrightarrow{\partial^*} & H^{n+1}(E,\dot{E}) & \longrightarrow \cdots \\
 & \cong \Big\uparrow \phi^* & & \cong \Big\uparrow p^* & & \cong \Big\uparrow r & & \cong \Big\uparrow \phi^* & \\
\cdots \longrightarrow & H^{n-1}(X/T) & \xrightarrow[\delta^*]{} & H^n(X/T) & \xrightarrow[\nu^*]{} & H^n(X) & \xrightarrow[\tau^*]{} & H^n(X/T) & \longrightarrow \cdots
\end{array}
$$

Now consider the following section of this sequence :

$$H^0(X/T) \xrightarrow{\delta^*} H^1(X/T) \xrightarrow{\nu^*} H^1(X) \xrightarrow{\tau^*} H^1(X/T) \longrightarrow \cdots$$

Let $1 \in H^0(X/T)$ denote the unit class in $H^0(X/T)$.

Then put $c = \delta^*(1)$.

In general, $\delta^* : H^n(X/T) \longrightarrow H^{n+1}(X/T)$

is given by $\delta^*(\alpha) = \alpha . c$

Thus we have the powers $c^m \in H^m(X/T)$.

We also have that $c^m.c^n = c^{m+n}$.

Further if $m : X \longrightarrow Y$ is equivariant, then $m^* : H^1(Y/T) \longrightarrow H^1(X/T)$ map $c(Y)$ onto $c(X)$. In particular we have that if the mth power $c^m(X)$ of $c(X)$ is non-zero, then $c^m(Y)$ is also non-zero.

These are all well-known results. In this section however, we would like to define fundamental classes in a more general setting : for group action by any finite group G and for any principal ideal domain.

Let a finite group G act freely on a paracompact space X. Let L be a P.I.D. and L(G) denote the Group-Ring (i.e., the ring of all finite linear combinations of the form $\sum l_g.g$). Let J be an L(G)-module. We use here, the equivariant cohomology (as outlined in [12]).

An equivariant n-cochain, accordingly, is an Alexander-Spanier n-cochain $\emptyset$ ([3], 6.3) which assigns to each (n+1)-tuple $(x_o, x_1, \dots, x_n)$ of points in X, an element $\emptyset(x_o, x_1, \dots, x_n)$ of J such that $\emptyset(gx_o, gx_1, \dots, gx_n) = g.\ \emptyset(x_o, x_1, \dots, x_n)$, $\forall g \in G$. We denote the module of all such equivariant cochains by $C^q(X;J)$. The coboundary homomorphism

$$\delta \;;\; C^q(X;J) \longrightarrow C^{q+1}(X;J)$$

is defined by $\delta\emptyset(x_o, x_1, \dots, x_n) = \sum_{0 \leqslant i \leqslant q+1} (-1)^i \;\emptyset(x_o, \dots, \hat{x}_i, \dots, x_{n+1})$

clearly $\delta(g\emptyset) = g(\delta\emptyset)$.

We denote the resulting cohomology by $H^n(X/G\,;\,J)$.

Fix W : $o \longleftarrow L \longleftarrow W_o \longleftarrow W_1 \longleftarrow \dots \longleftarrow W_n \longleftarrow \dots\dots,$

a free acyclic resolution of G. (By a free acyclic resolution of G, we mean an exact sequence of L(G) modules, L itself being considered as a trivial L(G) module.) As W is acyclic, there is the following short exact sequence, for each $n \geqslant 0$:

$$0 \longrightarrow Z_n \longrightarrow W_n \longrightarrow Z_{n-1} \longrightarrow 0$$

where $Z_n = \operatorname{Ker}(W_n \longrightarrow W_{n-1})$, L being denoted by Z_{-1}.

We get the corresponding exact cohomology sequence :

4.3.2 $$\dots \longrightarrow H^n(X/G;Z_n) \longrightarrow H^n(X/G;W_n) \longrightarrow H^n(X/G;Z_{n-1}) \xrightarrow{\delta^*} H^{n+1}(X/G;Z_n) \rightarrow \dots$$

Considering such sequences for all $n \geqslant 0$, and joining them together we get the sequence of coboundaries :

4.3.3 $$H^o(X/G;L) \xrightarrow{\delta^*} H^1(X/G;Z_o) \xrightarrow{\delta^*} H^2(X/G;Z_1) \xrightarrow{\delta^*} \dots$$

$$\dots \xrightarrow{\delta^*} H^{n-1}(X/G;Z_{n-2}) \xrightarrow{\delta^*} H^n(X/G;Z_{n-1}) \xrightarrow{\delta^*} \dots$$

Let $1 \in H^0(X/G;L)$ denote its unit class.

We denote by $c(X;L)$, the image $\delta^*(1) \in H^1(X/G;Z_0)$.

By iteration, let $c^n(X;L) = \delta^*(c^{n-1}(X;L)) \in H^n(X/G;Z_{n-1})$.

We have the following results :

4.3.4 If $m : X \longrightarrow Y$ is equivariant, the induced map

$$m^* : H^n(Y/G; Z_{n-1}) \longrightarrow H^n(X/G; Z_{n-1})$$

maps $c^n(Y;L)$ into $c^n(X;L)$.

4.3.5 Clearly $c^n(X;L)$ will also be a function of the particular resolution W. However we have the following : If W' is another free acyclic resolution of G there is an equivariant chain map $\psi : W \longrightarrow W'$. In particular, there is a homomorphism $Z_{n-1} \longrightarrow Z'_{n-1}$ of cycle groups. The induced homomorphism : $H^n(X/G;Z_{n-1}) \longrightarrow H^n(X/G;Z'_{n-1})$ maps the class c^n defined by means of the resolution W, into the class c'^n defined by means of the resolution W'. Now since there is an equivariant chain map from $W' \longrightarrow W$ also, we conclude that $c^n = 0$ if and only if $c'^n = 0$.

4.3.6 For defining the products of these classes, first of all, define the join $W \circ W$, of the free acyclic resolution W of G, with itself, as follows :

$$(W \circ W)_n = \bigoplus_{p+q = n-1} W_p \otimes W_q$$

With the usual boundary ($\bar{\partial}_n(c \otimes c') = \partial_p(c) \otimes c' + (-1)^p c \otimes \partial_q(c')$,

where $c \in W_p$ and $c' \in W_q$, with $p+q = n-1$), $W \circ W$ is also a free acyclic resolution of G. (Note that $L \otimes_{L(G)} L \cong L$). Further, $Z_{m-1}(W) \otimes Z_{n-1}(W) \subset Z_{m+n-1}(W \circ W)$. Now, there exist equivariant chain maps $W \circ W \longrightarrow W$, inducing homomorphisms $Z_{m+n-1}(W \circ W) \longrightarrow Z_{m+n-1}(W)$ on the group of cycles. These in turn induce homomorphisms $Z_{m-1}(W) \otimes Z_{n-1}(W) \longrightarrow Z_{m+n-1}(W)$, and hence homomorphisms $H^m(X/G;Z_{m-1}) \otimes H^n(X/G;Z_{n-1}) \longrightarrow H^{m+n}(X/G;Z_{m+n-1})$. This homomorphism maps $c^m(X;L) \otimes c^n(X;L)$ into $c^{m+n}(X;L)$.

4.3.7 If in the above analysis, we put $G = \underline{Z}_2$, we are back to fixed point free involutions. Consider the free acyclic resolution of $\underline{Z}_2$,

$$W = 0 \longleftarrow L(\underline{Z}_2) \xleftarrow{1+T} L(\underline{Z}_2) \xleftarrow{1-T} L(\underline{Z}_2) \xleftarrow{1+T} \cdots\cdot$$

with each W_i, the group-ring $L(\underline{Z}_2)$ of elements $l_1+T(l_2)$, $(l_1, l_2 \in L$, T is the involution) and $\partial : W_n \longrightarrow W_{n-1}$, multiplication by 1+T if n is even and multiplication by 1-T if n is odd. Then we get only two distinct sequences 4.3.2. Further if we put $L = \underline{Z}_2$, then $Z_{n-1}(W) = \underline{Z}_2$ for all n, and then $c^m \in H^m(X/T;\ Z_{m-1}(W)) = H^m(X/T\ ;\ \underline{Z}_2)$ is the classical fundamental class of the involution (X,T) seen earlier (4.3.1).

Now, we are in a position to define cohomology coindex. Note that by the result 4.3.6 on products, if $c^m = 0$, $c^n = 0$ for all $n \geqslant m$.

4.3.8 <u>Definition</u>: Let a finite group G act freely on a paracompact space X and let L be a P.I.D. Then the cohomology coindex, $\text{coind}_L\ X$ is defined to be the largest n for which the class $c^n(X;L)$ is not zero. We write $\text{coind}_L\ X = -1$, if and only if $X = \emptyset$.

4.4 We now proceed to prove that coind_L defined in section 4.3 is in fact a Coindex function defined in section 4.1. Further we want to establish that it satisfies the requirement of continuity.

4.4.1 4.4.1 (i) follows immediately from 4.3.4. Also 4.1.1 (iii) is clear. We proceed now to establish 4.1.1 (ii), i.e., $\text{coind}_L X \leqslant \text{coind}_L A + \text{coind}_L B + 1$, where $X = A \cup B$ and $A, B \subset X$ are closed and G-invariant.

<u>Proof</u> : Let $\text{coind}_L A = n_1$ and $\text{coind}_L B = n_2$.

Then $i^* : H^{n_1+1}(X/G; Z_{n_1}) \longrightarrow H^{n_1+1}(A/G ; Z_{n_1})$ kills $c^{n_1+1}(X;L)$

and $j^* : H^{n_2+1}(X/G ; Z_{n_2}) \longrightarrow H^{n_2+1}(B/G ; Z_{n_2})$ kills $c^{n_2+1}(X;L)$

Now, consider the following Mayer-Vietoris sequences :

$$\cdots \longrightarrow H^{n_1+n_2+1}(A/G\cap B/G;Z_{n_1+n_2+1}) \xrightarrow{\delta^*} H^{n_1+n_2+2}(A/G\cup B/G;Z_{n_1+n_2+1})$$
$$\xrightarrow{(i^*,j^*)} H^{n_1+n_2+2}(A/G ;Z_{n_1+n_2+1}) \oplus H^{n_1+n_2+2}(B/G ; Z_{n_1+n_2+1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{n_1}(A/G\cap B/G ; Z_{n_1}) \xrightarrow{\delta^*} H^{n_1+1}(A/G\cup B/G ; Z_{n_1}) \xrightarrow{(i^*,j^*)}$$
$$H^{n_1+1}(A/G ; Z_{n_1}) \oplus H^{n_1+1}(B/G ; Z_{n_1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{n_2}(A/G\cap B/G ; Z_{n_2}) \xrightarrow{\delta^*} H^{n_2+1}(A/G\cup B/G ; Z_{n_2}) \xrightarrow{(i^*, j^*)}$$
$$H^{n_2+1}(A/G ; Z_{n_2}) \oplus H^{n_2+1}(B/G ; Z_{n_2}) \longrightarrow \cdots$$

Without loss of generality, assume $n_1 \geqslant n_2$.

Then, $c^{n_1+1} \in H^{n_1+1}(A/G\cup B/G ; Z_{n_1})$ gets killed by (i^*,j^*).

Hence, $c_1^{n_1} \in H^{n_1}(A/G\cap B/G ; Z_{n_1})$ s.t. $\delta^*(c_1^{n_1}) = c^{n_1+1}$.

Consider $c^{n_2+1} \in H^{n_2+1}(A/G\cup B/G ; Z_{n_2})$ and

$$j_1^*(c^{n_2+1}) \in H^{n_2+1}(A/G\cap B/G ; Z_{n_2})$$

where, $j_1 : A/G\cap B/G \hookrightarrow A/G\cup B/G = X/G$

Then, $c_1^{n_1} \cup j_1^*(c^{n_2+1}) \in H^{n_1+n_2+1}(A/G \cap B/G;\ Z_{n_1+n_2+1})$, by 4.3.6

Then we have $\delta^*(c_1^{n_1} \cup j_1^*(c^{n_2+1})) \in H^{n_1+n_2+2}(A/G \cap B/G\ ;\ Z_{n_1+n_2+1})$

But, $\delta^*(c_1^{n_1} \cup j_1^*(c^{n_2+1})) = \delta^*(c_1^{n_1}) \cup c^{n_2+1}$ (cfr. [34], 5.6.12)

$= c^{n_1+1} \cup c^{n_2+1}$

$= 0$ as $j_1^*(c^{n_2+1}) = 0$

Note that j_1: $A/G \cap B/G \longrightarrow X/G = A/G \cup B/G$ is the composition

$A/G \cap B/G \hookrightarrow B/G \xrightarrow{j} X/G$ and hence

$j^*(c^{n_2+1}) = 0$ implies that $j_1^*(c^{n_2+1}) = 0$

Therefore, $c^{n_1+n_2+2} = 0$.

Hence, $\operatorname{coind}_L X \leqslant n_1 + n_2 + 1$.

i.e., $\operatorname{coind}_L X \leqslant \operatorname{coind}_L A + \operatorname{coind}_L B + 1$. @

4.4.2 To establish the continuity of coind_L, we want to prove that if A is a closed invariant subset of X, then there exists an open neighbourhood C of A such that $\overline{C}$ is T-invariant and $\operatorname{coind}_L A = \operatorname{coind}_L \overline{C}$.

To prove this, we shall be using the property of continuity of the cohomology theory we are using. Hence, before we go on to the proof of 4.4.2, we shall have a brief look at continuity and the particular deduction we shall be using. (cfr. [13], X)

4.4.3 Let $\left\{(X_m, A_m),\ \pi^{m_1}_{m_2}\right\}$ be an inverse system of paracompact pairs and maps over some directed set M. Applying the cohomology functor H, we get a direct system $\left\{H^q(X_m, A_m),\ \pi^{m_1}_{m_2}{}^*\right\}$ of cohomology modules and homomorphisms. The cohomology theory H is said to be continuous on the category of paracompact pairs and maps, if

$$\underrightarrow{\lim} \left\{ H^q(X_m, A_m),\ \pi^{m_1}_{m_2}{}^* \right\} \cong H^q\left(\underleftarrow{\lim} \left\{ (X_m, A_m),\ \pi^{m_1}_{m_2} \right\} \right)$$

Now, let H be a continuous cohomology theory.

Let $A \subset X$ be a closed subset of a paracompact space X. Suppose now that $x/A = 0$ for some $x \in H(X)$ (i.e., if $i = A \longrightarrow X$ is the inclusion map, then $i^*(x) = 0$). Then the continuity of the cohomology theory implies that there exists an open neighbourhood V of A such that $x/\overline{V} = 0$.

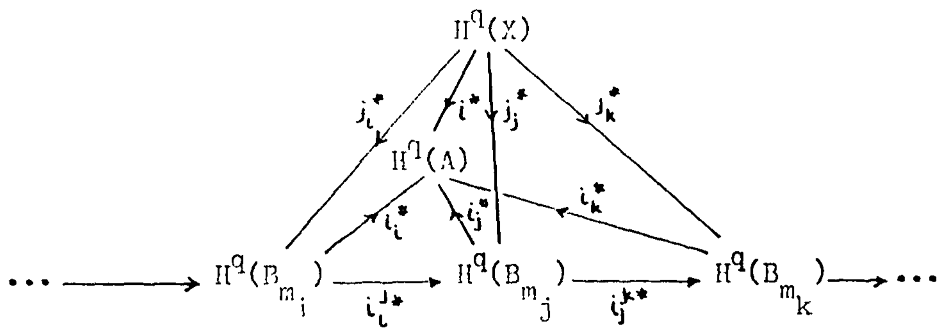
Proof :

Consider $\{B_m\}_{m \in \Lambda}$, the collection of all closed sets of X s.t. $\mathring{B}_m \supset A$, $\forall\ m \in \Lambda$, directed by inclusion. Then $\left\{ B_m, i^{m_1}_{m_2} \right\}$ forms an inverse system and the limit of this system is $\bigcap_{m \in \Lambda} B_m = A$.

The continuity of the cohomology theory H implies that

$$\underrightarrow{\lim}\ H^q(B_m) = H^q(A).$$

Now consider the following diagram :



Now, $i^*(x) = 0$

$\Rightarrow i_i{}^* j_i{}^* (x) = 0.$ If $j_i{}^*(x) = 0$, take $V = \mathring{B}_{m_i}$ and we are through.

If $i_1^*(x) \neq 0$, since $i_1^*(i_1^*(x)) = 0$

$\exists$ some $k > 1$ s.t. $i_k^{1*}\, i_1^*(x) = 0$ (cfr. /L/, VIII.4)

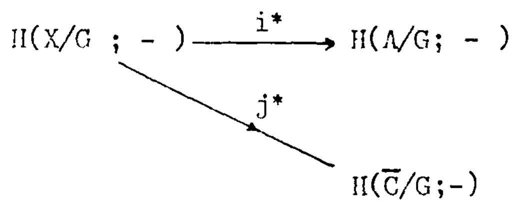
i.e., $j_{n_k}^*(x) = 0$

and we may take $V = \mathring{B}_{n_k}$ and we are through. @

<u>Proof of 4.4.2</u>

First of all, if $A \subset \overline{C}$ and both are invariant, then the inclusion map is itself equivariant and hence $\mathrm{coind}_L\, A \leqslant \mathrm{coind}_L\, \overline{C}$.

To prove the opposite inequality, we first of all assume without loss of generality that $\mathrm{coind}_L\, A$ is finite. By the continuity of the cohomology we consider, we get an open neighbourhood C of A such that $\overline{C}$ may be considered to be invariant (in 4.4.3, take closed invariant subsets B_m of X such that $A \subset \mathring{B}_m$) and in the following diagram :



if $i^*(x) = 0$ for some $x \in H(X/G\ ;\ -\ )$, then $j^*(x) = 0$, where i and j are inclusion maps. As i and j are equivariant, we have that

$\mathrm{coind}_L\, A \geqslant \mathrm{coind}_L\, \overline{C}$ @

Thus coind_L is a continuous Coindex function on the class of finite group actions on paracompact spaces.

5.0 MAPS FROM A SPHERE BUNDLE TO EUCLIDEAN SPACE

Now, we take a step towards generalisation of the Borsuk-Ulam Theorem in another direction, viz., we consider an equivariant map $f:B \longrightarrow \underline{R}^m$ where $p: B \longrightarrow X$ is an n-sphere bundle over a compact space X, $(m \geq n+1)$, and attempt to measure the set $f^{-1}(0)$. In particular, we shall examine how many fibres of $p:B \longrightarrow X$, $f^{-1}(0)$ must intersect. For this purpose, we shall follow P.E. Conner and EE Floyd [6]. We shall be using some of the material we developed in the last chapter and we shall develop a few more notions as we proceed.

5.1 Stiefel Whitney Classes of a Sphere Bundle

We shall consider an n-sphere bundle $p : B \longrightarrow X$ over a paracompact space X, whose structural group is O(n+1). Considering the antipodal map on each fibre B_x $(=S^n)$, we shall obtain a fixed point free involution $T : B \longrightarrow B$. Denote the orbit space B/T by B^*. Thus we get a bundle of projective n-spaces $q : B^* \longrightarrow X$.

Note that the concept of a 0-sphere bundle coincides with the concept of a fixed point free involution. If $p : B \longrightarrow X$ is a zero-sphere bundle, then $q: B^* \longrightarrow X$ is a homeomorphism.

Let $p_i : B_i \longrightarrow X$, $i = 1,2$ be two sphere bundles. Then there is the sphere bundle $p_1 \circ p_2 : B_1 \circ B_2 \longrightarrow X$ whose fibres $(B_1 \circ B_2)_x$ are the ordinary joins $B_{1x} \circ B_{2x}$ of the fibres of B_1 and B_2 defined by

$$B_{1x} \circ B_{2x} = \left\{(t_1 b_1,\ t_2 b_2) \ / \ b_1 \in B_{1x},\ b_2 \in B_{2x},\ \ t_1,\ t_2 \in I,\ t_1+t_2 = 1\right\}.$$

Note that $B_{1x} \circ B_{2x} \subseteq D(B_{1x}) \times D(B_{2x})$ where $D(B_x)$ denotes the disc of which B_x is the boundary. We shall refer to this as the join of the two sphere bundles. For facility of notation, we shall denote a typical element of $(B_1 \circ B_2)_x$ by

$$(1-t)\, b_1 + t b_2 \ , \ 0 \leqslant t \leqslant 1 \ , \ b_i \in B_{ix}$$

Note that the involution $T : B_1 \circ B_2 \longrightarrow B_1 \circ B_2$ has B_1 and B_2 as invariant subspaces and hence $(B_1 \circ B_2)^*$ has $B_1{}^*$ and $B_2{}^*$ as subspaces.

By naturality, the classes $c^i \in H^i(P^n)$ of $S^n \xrightarrow{T} P^n$ are images of the classes $c^i \in H^i(B^*)$ of $T : B \longrightarrow B^*$. Hence the map $i^* : H^i(B^*) \longrightarrow H^i(P^n)$ is epimorphic, i.e., the fibre P^n of $q : B^* \longrightarrow X$ is totally non-homologous to zero. (Throughout this chapter too we shall be considering Alexander-Wallace-Spanier — cohomology with coefficients in $\underline{Z}_2$). Hence the map $H^i(P^n) \longrightarrow H^i(B^*)$ which takes c^i to c^i is a cohomology extension of the fibre. We get the following results :

5.1.1 $q^* : H^k(X) \longrightarrow H^k(B^*)$ is a monomorphism

For,

$$\begin{array}{ccc} H^k(X) & \xrightarrow{\ q^*\ } & H^k(B^*) \\ & \searrow^{id \otimes 1} & \uparrow \phi^*(\cong) \\ & & H^k(X) \otimes H^o(P^n) \end{array}$$

Suppose $q^*(u) = 0$

Then $\phi^*(u \otimes 1) = 0$, where ϕ^* is the Lerray-Hirsch isomorphism.

(cfr.[34], 5.7.9)

$\Rightarrow u \otimes 1 = 0$, as ϕ^* is an isomorphism

$\Rightarrow \quad u = 0$.

5.1.2 Every $\beta_k \in H^k(B^*)$ can be expressed uniquely as $\beta_k = q^* \alpha_k + q^* \alpha_{k-1} \cdot c + \cdots$

$+ q^*\alpha_{k-n}.c^n$

where $\alpha_i \in H^i(X)$, and $c \in H^1(B^*)$ is the fundamental class of the involution $T : B \longrightarrow B^*$.

This follows from the Leray-Hirsch Theorem.

5.1.3 Using 5.1.2 it follows that $c^{n+1} \in H^{n+1}(B^*)$ can be expressed uniquely as $c^{n+1} = q^*w_{n+1} + q^*w_n.c + \dots + q^*w_1.c^n$ for classes $w_i \in H^i(X)$, $i = 1,2, \dots , n+1$.

Now for a given space Y, denote by $H^*(Y)$, the strong direct sum $\sum_0^\infty H^i(Y)$. Consider an element in $H^*(Y)$ as a formal power series $\sum \alpha_i.t^i$, $\alpha_i \in H^i(Y)$. In particular, let $w = w(B)$ denote the element $\sum w_i.t^i \in H^*(X)$, for the n-sphere bundle $p : B \longrightarrow X$. We let $w_0 = 1$.

5.1.4 Proposition : The function assigning to each sphere bundle $p : B \longrightarrow X$, the class $w \in H^*(X)$ is characterised by the following properties:

a) For a 0-sphere bundle, $w = 1 + ct$, where c is the fundamental class of (B,T).

b) If $p_i : B_i \longrightarrow X_i$ are two sphere bundles and $f : B_1 \longrightarrow B_2$, a bundle map, then the induced map $\bar{f} : X_1 \longrightarrow X_2$ has

$\bar{f}^* w(B_2) = w(B_1)$

c) If $p_i : B_i \longrightarrow X$, $i = 1,2$ are sphere bundles, then $w(B_1 \circ B_2) = w(B_1) . w(B_2)$.

Proof

a) From 5.1.3, $c = q^*w_1$. Further, for a 0-sphere bundle, X and B^*

are identified. Hence identifying w_1 with its image under the isomorphism q^*, we get the required result.

b) Consider the following diagram of maps induced on the cohomology level:

$$\begin{array}{ccc} H^{n+1}(B_2^*) & \xrightarrow{f^*} & H^{n+1}(B_1^*) \\ \uparrow q_2^* & & \uparrow q_1^* \\ H^{n+1}(X_2) & \xrightarrow{\overline{f}^*} & H^{n+1}(X_1) \end{array}$$

Let $c_2^{n+1} \in H^{n+1}(B_2^*)$ be the $(n+1)^{th}$ power of the fundamental class c_2 of (B_2,T) and let $c_1^{n+1} \in H^{n+1}(B_1^*)$ be the $(n+1)^{th}$ power of the fundamental class c_1 of (B_1,T). Then by the naturality of the fundamental class and product, we shall have $f^*(c_2^{n+1}) = c_1^{n+1}$

Now, by 5.1.3,

$$c_2^{n+1} = q_2^*\, w(B_2)_{n+1} + q_2^*\, w(B_2)_n \cdot c_2 + \cdots + q_2^*\, w(B_2)_1 \cdot c_2^n$$

$$\therefore\ f^*(c_2^{n+1}) = f^*(q_2^*\, w(B_2)_{n+1}) + f^*(q_2^*\, w(B_2)_n \cdot c_2) + \ldots + f^*(q_2^* w(B_2)_1 \cdot c_2^n)$$

$$= f^* q_2^* w(B_2)_{n+1} + f^* q_2^* w(B_2)_n \cdot f^*(c_2) + \ldots + f^*(q_2^* w(B_2)_1 \cdot f^*(c_2^n)$$

$$\text{i.e., } c_1^{n+1} = q_1^* \overline{f}^* w(B_2)_{n+1} + q_1^* \overline{f}^* w(B_2)_b \cdot c_1 + \ldots + q_1^* \overline{f}^* w(B_2)_1 \cdot c_1^n$$

$$= q_1^*\, w(B_1)_{n+1} + q_1^*\, w(B_1)_n \cdot c_1 + \cdots + q_1^*\, w(B_1)_1 \cdot c_1^n$$

By the uniqueness of such an expression and injectivity of q_1^*, the result follows.

c) In $B_1 \circ B_2$, let C_1 denote all points of fibres $(B_1 \circ B_2)_x$ of the form $(1 - t)b_1 + t\, b_2$ with $t \leq \frac{1}{2}$ and let C_2 be all such with $t \geq \frac{1}{2}$.

Then $B_1 \circ B_2 = C_1 \cup C_2$, $C_i \supset B_i$ and B_i is an equivariant deformation retract of C_i, $i = 1, 2$.

Passing to orbit spaces, we get $(B_1 \circ B_2)^* = C_1^* \cup C_2^*$, $C_i^* \supset B_i^*$ and B_i^* is a deformation retract of C_i^*, $i = 1, 2$.

Thus, $H^k(B_i^*) \cong H^k(C_i^*)$

also, if $\beta_i \in H((B_1 \circ B_2)^*)$ and $H((B_1 \circ B_2)^*) \longrightarrow H(B_i^*)$ kills β_i for $i = 1, 2$, then $\beta_1 \cdot \beta_2 = 0$.

(cfr. arguments in 4.4.1, using [34], 5.6.12)

Now, suppose that B_1 is an m-sphere bundle and B_2 an n-sphere bundle. Then $c^{m+1} + q^* w_1(B_1).c^m + \ldots + q^* w_{m+1}(B_1) \in H^{m+1}((B_1 \circ B_2)^*)$

on restriction to B_1^* is $2c^{m+1} = 0$, by 5.1.3.

Also, $c^{n+1} + q^* w_1(B_2).c^n + \ldots + q^* w_{n+1}(B_2) \in H^{n+1}((B_1 \circ B_2)^*)$

on restriction to B_2^* is $2c^{n+1} = 0$.

Hence by what has been established above, their product is zero in $H^{m+n+2}((B_1 \circ B_2)^*)$.

Computing the product and comparing the result with the unique expression

$$c^{m+n+2} = q^* w_{m+n+2}(B_1 \circ B_2) + q^* w_{m+n+1}(B_1 \circ B_2).c + \ldots + q^* w_1(B_1 \circ B_2) . c^{m+n+1}$$

We get the required result. @

Since the Stiefel Whitney classes are also defined by the above properties and by the uniqueness of the Stiefel-Whitney classes, the classes w_i are the Stiefel-Whitney classes. (This way of defining

Stiefel-Whitney classes is due to Grothendieck).

5.2 Further Results

We need the following computations and results before we turn our attention to the specific problem at hand.

5.2.1 Consider the following three sphere bundles over B^*.

a) The 0-sphere bundle $\gamma : B \longrightarrow B^*$

b) Consider B^* as the collection of all antipodal pairs of B. Let $B_1 \dashrightarrow B^*$ be the induced bundle which assigns to each element b^* of B^*, the fibre of $p: B \longrightarrow X$ which contains b^*.

i.e.,

$$\begin{array}{ccc} B_1 = q^*(B) & & B \\ \downarrow & & \downarrow p \\ B^* & \overset{q}{\dashrightarrow} & X \end{array}$$

c) Further, consider the sphere bundle $B_2 \longrightarrow B^*$ which assigns to each b^*, the $(n-1)$-sphere orthogonal to b^* in the n-sphere B_x which contains b^*.

Note that as sphere bundles over B^*, $B_1 = B \circ B_2$. By 5.1.4 (b), the class of $B_1 \dashrightarrow B^*$ is $q^* w$, where w is the class of $B \longrightarrow X$. Also, by 5.1.4 (a), the class of the 0-sphere bundle $B \longrightarrow B^*$ is $1 + ct$.

Hence, 5.1.4 (c) gives :

$$q^*(w) = (1 + ct) \cdot (1 + \alpha_1 t + \ldots + \alpha_n t^n) \text{ where}$$

$\sum \alpha_i \cdot t^i$ is the class of $B_2 \longrightarrow B^*$.

i.e., $1 + q^* w_1 \cdot t + \ldots + q^* w_{n+1} \cdot t^{n+1} = (1+ct)(1 + \alpha_1 t + \ldots + \alpha_n t^n)$

Now, for $w \in H^*(X)$, consider the unique inverse $\bar{w} = \sum \bar{w}_i t^i$ in the ring $H^*(X)$ and for $1 + ct$ in the ring $H^*(B^*)$ consider the unique inverse $1 + ct + c^2t^2 + \ldots\ldots$

We have :-

5.2.2 $$1 + \alpha_1 t + \ldots + \alpha_n t^n = (1 + q^*w_1.t + \ldots + q^*w_{n+1}.t^{n+1}) . (1 + ct + c^2t^2 + \ldots\ldots)$$

and

5.2.3 $$(1 + ct + c^2t^2 + \ldots\ldots) = (1 + \alpha_1 t + \ldots + \alpha_n t^n) . (1 + q^*\bar{w}_1.t + q^*\bar{w}_2.t^2 + \ldots\ldots) .$$

Using notations of 4.3.8, let $\mathrm{coind}_2\, B$ denote $\mathrm{coind}_L\, B$ where $L = \underline{Z}_2$.

5.2.4 <u>Theorem</u> : For every n-sphere bundle, $p : B \longrightarrow X$, $\mathrm{coind}_2\, B = n+k$, where k is the largest integer with the dual whitney class $\bar{w}_k$ non-zero.

<u>Proof</u> : Note that $\mathrm{coind}_2\, B$ is the degree of $1 + ct + c^2t^2 + \ldots$ and that k is the degree of $\bar{w}$.

Hence from the relation 5.2.3, it follows that $\mathrm{coind}_2\, B \leqslant n + k$.

Further, from the relation 5.2.2, comparing the coefficients of t^n, we get $\alpha_n = c^n$ + terms of lower degree in c, while α_{n-1}, α_{n-2}, ... α_1 all consist of terms of degree less than n in c.

Using this in 5.2.3 and comparing the coefficients of t^m, we get :

$$c^m = q^*\bar{w}_{m-n}.c^n + \text{terms of lower degree in } c.$$

Hence, since q^* is a monomorphism, $\bar{w}_{m-n} \neq 0 \Rightarrow c^m \neq 0$.

Therefore, $\mathrm{coind}_2\, B \geqslant m$.

But the greatest value that m - n can take is k.

i.e., $m - n \leq k$

$\therefore \text{coind}_2 B \geqslant n + 1$

Thus we have that $\text{coind}_2 B = n + k$.

5.2.5 <u>Corollary</u> : If a differentiable n-manifold X can be immersed in a differentiable m-manifold Y, then $n + k \leqslant m + l$, where k and l are the largest integers with the dual Stiefel-Whitney classes $\bar{w}_k(X)$ and $\bar{w}_l(Y)$ non-zero.

<u>Proof</u> : If X can be immersed in Y, there is an equivariant map of the bundle of unit tangent vectors to X, $V_1(X)$ into the bundle of unit tangent vectors to Y, $V_1(Y)$.

Then, $\text{coind}_2 V_1(X) \leqslant \text{coind}_2 V_1(Y)$

$\Rightarrow n-1+k \leqslant m-1+l$

$\Rightarrow n+k \leqslant m+l$

5.3 Consider an equivariant map $f : B \longrightarrow \underline{R}^m$, $m \geqslant n + 1$ where $p : B \longrightarrow X$ is an n-sphere bundle over a compact space X. In this section we shall try to measure the set $f^{-1}(0)$. We are particularly interested in seeing how many fibres of $p : B \longrightarrow X$ $f^{-1}(0)$ must intersect.

Note, first of all, that f (together with a homeomorphism) maps $B - f^{-1}(0)$ equivariantly into S^{m-1}

Hence, $\text{coind}_2 (B - f^{-1}(0)) \leqslant m - 1$

5.3.1 <u>Definition</u> : If Λ is a closed subset of the compact space X, and if $\gamma \in H^n(X)$, then we say that Λ is a support of γ if for every neighbourhood U of Λ, the natural map (from the exact sequence for the pair (X, X-U)) $H^n(X, X-U) \longrightarrow H^n(X)$ has γ in its image.

In Čech homology, A is a support of $c \in \check{H}_n(X)$ if c is in the image of $\check{H}_n(A) \xrightarrow{i_*} \check{H}_n(X)$.

From the Poincaré duality theorem we shall have that in a compact n-manifold X, A supports $\gamma \in H^k(X)$ if and only if A supports the dual $c \in \check{H}_{n-k}(X)$ of γ.

5.3.2 Theorem : Let $p : B \longrightarrow X$ be an n-sphere bundle over the compact space X and let A be a closed T-invariant subset of B. If $\mathrm{coind}_2(B-A) \leqslant m - 1$, then $p(A) \subset X$ is a support for every dual Whitney class $\bar{w}_p$, $p \geqslant m - n$.

Proof : Let U be a neighbourhood of $p(A)$.
Then $p^{-1}(U) = V \supset A$.
Hence, $\mathrm{coind}_2(B - V) \leqslant \mathrm{coind}_2(B - A) \leqslant m - 1$.
Thus in $H^m(B^* - V^*)$, $c^m = 0$.
But in B^*, from the proof of 5.2.4,

$$c^m = q^*\bar{w}_{m-n} \cdot c^n + \text{terms of lower degree in } c.$$

Now, consider the diagram :

$$\begin{array}{ccc} (B^* - V^*) & \xrightarrow{\;i\;} & B^* \\ \downarrow q' & & \downarrow q \\ X - U & \xrightarrow[\;j\;]{} & X \end{array}$$

and the induced cohomology diagram

$$\begin{array}{ccc} H^m(B^*) & \xrightarrow{\;i^*\;} & H^m(B^* - V^*) \\ \uparrow q^* & & \uparrow q'^* \\ H^m(X) & \xrightarrow[\;j^*\;]{} & H^m(X - U) \end{array}$$

We have :

$$i^*(c^m) = i^*(q'\bar{w}_{m-n}.c^n + \ldots\ldots) = 0$$

$$\Rightarrow i^*q^*\bar{w}_{m-n}.\ c^n = 0$$

$$\Rightarrow i^*q^*\bar{w}_{m-n} = 0$$

$$\Rightarrow q'^*\ j^*\ \bar{w}_{m-n} = 0$$

$$\Rightarrow \quad j^*\ \bar{w}_{m-n} = 0\ \text{, as } q'^* \text{ is a monomorphism.}$$

Hence from the exact sequence,

$$\ldots \longrightarrow H^{m-n}(X,\ X-U) \longrightarrow H^{m-n}(X) \xrightarrow{j^*} H^{m-n}(X - U) \longrightarrow \ldots$$

we have that $\bar{w}_{m-n}$ is in the image of $H^{m-n}(X\ ,\ X-U) \longrightarrow H^{m-n}(X)$

i.e., $p(A)$ supports $\bar{w}_{m-n}$.

The same arguments carry through for $p > m - n$.

5.3.3 __Corollary__ : Suppose that $p : B \longrightarrow X$ is a bundle of n-spheres over a compact space X, and that $f : B \longrightarrow \underline{R}^m$ $(m \geq n+1)$ is equivariant with respect to the antipodal maps. Then the set $S \subset X$, $S = \{x \in X\ /\ f(b) = 0 \text{ for some } b \in B_x\}$ supports every $\bar{w}_p$, for $p \geq m - n$.

__Proof:__ This is a direct application of 5.3.2 where $A = f^{-1}(0)$. It only suffices to note that $S = p(f^{-1}(0))$ and that $\text{coind}_2\ (B - f^{-1}(0)) \leq m - 1$.

5.3.4 Consider the following diagram where $\cap\mu_S$ and $\cap\mu_X$ are the Poincaré duality maps, and X is a compact, connected manifold.

$$\begin{array}{ccc} H^p(X,\ X - S) & \longrightarrow & H^p(X) \\ \Big\downarrow \cap \mu_S & & \Big\downarrow \cap \mu_X \\ H_{\dim X - p}(S) & \longrightarrow & H_{\dim X - p}(X) \end{array}$$

The corollary then implies that $\dim S \geqslant \dim X - p$ provided that $\overline{w}_p \neq 0$, $p \geqslant m - n$.

5.4 Before we conclude this chapter, we would like to look at a continuous map $f : S^n \longrightarrow X$, where X is a compact Riemannian n-manifold. We want to obtain some result on the set $A(f) = \left\{x \in S^n \ / \ f(x) = f(-x)\right\}$. We shall obtain more detailed results in the next chapter. Let us however have a look at $A(f)$ using some of the tools and procedures obtained in this chapter.

5.4.1 Definition : Let T be a fixed point free involution on a space X. The index of (X,T) is the largest integer n for which there is an equivariant map of S^n into X. We denote the index of (X,T) as $\mathrm{ind}(X,T)$.

Note that this index is not the same as the index of a T-space we defined using a homological construction in 3.0. We shall however call it 'index', as it has been called so in the literature. It is however related to the cohomological coindex of a space X defined in 4.0. We have that $\mathrm{ind}\ X \leqslant \mathrm{coind}_L\ X$, for any coefficient group L which has $\mathrm{coind}_L\ S^n = n$. (cfr [6])

5.4.2 Theorem : Suppose that X is a compact differentiable n-manifold and that $p : B \longrightarrow X$ is the bundle of unit tangent vectors to X. Then $\mathrm{ind}\ (B,T) = n - 1$.

<u>Proof</u> : Since $S^{n-1} \subset B$, ind $(B,T) \geqslant n - 1$.

Suppose now, that ind $(B,T) \geqslant n$; i.e., that there exists an equivariant map $f : S^n \longrightarrow B$.

Then $\overline{f} : \underline{R}P^n \longrightarrow B^*$ induces $\overline{f}^* : H^n(B^*) \longrightarrow H^n(\underline{R}P^n)$ which will map the class $c^n(B) \in H^*(B^*)$ of (B,T) onto the class $c^n(S^n) \in H^n(\underline{R}P^n)$ of (S^n,A).

From 5.1.3, $c^n(B) = q^*w_1.c^{n-1} + \ldots + q^*w_n$, where

$w = 1 + w_1 + \ldots + w_n$ is the Stiefel-Whitney class of X.

Then $c^n(S^n) = \overline{f}^*q^*w_1.c^{n-1} + \ldots + \overline{f}^*q^*w_n$.

Using the Wu formulae for w_i (cfr. [34], 6.10.7), we get that there exist classes $u_i \in H^i(X)$, $1 \leqslant i \leqslant n/2$ such that

$$w_j = \sum_i Sq^{j-i}\, u_i$$

$$\begin{aligned}
\text{Now, } c^n(S^n) &= \overline{f}^*\, (\sum_i q^*w_i .\, c^{n-i}) \\
&= \overline{f}^*\, (\sum_i q^*\, (\sum_j Sq^{i-j}\, u_j) \,.\, c^{n-i}\,) \\
&= \overline{f}^*\, (\sum_{i+j+k\, =\, n} q^*\, Sq^i\, u_j .\, c^k) \\
&= \sum_{i+j+k\, =\, n} Sq^i\, \overline{f}^*\, q^*u_j .\, c^k \\
&= \sum_{i+j+k\, =\, n} Sq^i\, v_j \,.\, c^k \text{ , putting } v_j = \overline{f}^*\, q^*\, u_j
\end{aligned}$$

In the above expression, fix j. Then for that j, all the terms $Sq^i v_j .\, c^k = 0$ if $v_j = 0$.

If $v_j \neq 0$, then $v_j = c^j$, as we are in $H^j(\underline{R}P^n)$, coefficients in $\underline{Z}_2$.

Thus we have $\displaystyle\sum_{\substack{i,j \\ i \leqslant j}} Sq^i v_j \,.\, c^k = (\sum_i \begin{Bmatrix} j \\ i \end{Bmatrix})\, c^n$, where $\begin{Bmatrix} j \\ i \end{Bmatrix}$ is the mod 2 binomial coefficient.

But, $\sum_i \begin{Bmatrix} j \\ i \end{Bmatrix} \equiv 0 \bmod 2$.

Hence $\sum_{i,k} Sq^i v_j \cdot c^k = 0$, for every j.

$\therefore\ c^n(S^n) = 0$.

But this is false as $\text{coind}_2\, S^n = n$.

Therefore, $n - 1 \leqslant \text{ind}\,(B,T) < n$

which immediately implies that $\text{ind}\,(B,T) = n - 1$.

5.4.3 Corollary : Suppose that f is a continuous map of S^n into a compact Riemannian n-manifold X. Then for some x, either $f(x) = f(-x)$ or for some x, $f(x)$ and $f(-x)$ are not joined by a unique geodesic of shortest length.

Proof : Suppose it to be false.

For each $x \in S^n$, let $F(x)$ denote the unit tangent vector to the unique smallest geodesic joining $f(x)$ to $f(-x)$ at the midpoint of the geodesic, and pointing towards $f(-x)$.

Then F maps S^n equivariantly into $V_1(X)$, the bundle of unit tangent vectors to X.

This is not possible, as by 5.4.2, $\text{ind}\,(V_1(X), T) = n - 1$.

6.0 MAPS INTO MANIFOLDS

In the last section of the previous chapter (5.4), we had a quick look at maps from S^n into a Riemannian n-manifold. In this chapter we look at maps $f : S^n \longrightarrow M^k$, with $n \geq k$, where M^k is a smooth k-dimensional manifold, not necessarily closed or even compact. We are, as before, interested in results about $A(f) = \{x \in S^n \,/\, f(x) = f(-x)\}$. In the first section we shall define the twist of an n-plane bundle $p : \eta \longrightarrow X/T$, where (X,T) is a fixed point free involution, and proceed to state (without proof) certain related results on Whitney classes required in the rest of the chapter. The second section will be devoted to the statement of the main theorem and its proof. This chapter has been guided by the work of P.E. Conner [5].

6.1 Preliminaries

6.1.1 Definition : Let (X,T) be a fixed point free involution, and let $p : \eta \longrightarrow X/T$ be an n-plane bundle over the quotient space. Let $\tilde{p} : \tilde{\eta} \longrightarrow X$ be the pull-back of the quotient map $\nu : X \longrightarrow X/T$, thus :

$$\begin{array}{ccc} \tilde{\eta} = \nu^{*}(\eta) & & \eta \\ \tilde{p} \big\downarrow & & \big\downarrow p \\ X & \xrightarrow[\nu]{} & X/T \end{array}$$

Here, $\tilde{\eta} \subseteq X \times \eta$ is given by $\{(x,v) \in X \times \eta \,/\, \nu(x) = p(v)\}$ and $p(x,v) = x$.

Define a fixed point free involution of $\tilde{\eta}$ by $\tilde{T}(x,v) = (Tx, -v)$. Clearly $\tilde{p} : (\tilde{\eta}, \tilde{T}) \longrightarrow (X,T)$ is equivariant and induces a bundle $\hat{p} : \tilde{\eta}/\tilde{T} \; (= \hat{\eta}) \longrightarrow X/T$. This is also an n-plane bundle and is referred to as the twist of $p : \eta \longrightarrow X/T$ by the involution (X,T).

6.1.2 Definition: Consider the orthogonal group $O(n)$. In $O(n)$, consider the diagonal group D of diagonal matrices in which each entry must necessarily be 1 or -1. Hence $D \cong (\underline{Z}_2)^n$, the n-fold product of $\underline{Z}_2$. The inclusion $i : D \longrightarrow O(n)$ induces a homomorphism

$$i^* : H^*(BO(n) ; \underline{Z}_2) \longrightarrow H^*(BD ; \underline{Z}_2)$$

where $BO(n)$ and BD are the classifying spaces of $O(n)$ and D respectively (obtainable by the Milnor construction).

Note that $B(\underline{Z}_2)$, the classifying space for $\underline{Z}_2$ can be regarded as $\underline{RP}^\infty = \bigcup_0^\infty \underline{RP}^n$, the infinite dimensional real projective space ([20], 4.11.3). Hence $H^*(B(\underline{Z}_2); \underline{Z}_2) \cong H^*(\underline{RP}^\infty ; \underline{Z}_2) \cong \underline{Z}_2 [t]$, the polynomial ring over $\underline{Z}_2$, generated by a single one-dimensional generator, $t \in H^1(\underline{RP}^\infty ; \underline{Z}_2)$. Now, taking $B(\underline{Z}_2{}^n)$ to be the n-fold cartesian product of $B(\underline{Z}_2)$ with itself, we get

$$H^*(BD ; \underline{Z}_2) \cong \underline{Z}_2 [t_1, t_2, \ldots, t_n] ,$$

The polynomial ring generated by the one-dimensional generators t_i, $\bar{t}_i \in H^1 (B(\underline{Z}_2)_i; \underline{Z}_2)$ where $B(\underline{Z}_2)_i$ is the i^{th} factor of $B(D)$ and t_i is the image of $\bar{t}_i$ under the Künneth map.

Borel has proved that the homomorphism

$$i^* : H^*(BO(n) ; \underline{Z}_2) \longrightarrow H^*(BD ; \underline{Z}_2)$$

is a monomorphism whose image is the subring of $\underline{Z}_2 [t_1, t_2, \ldots, t_n]$ of symmetric polynomials in $t_1, \ldots, t_n$.

Now, let $w_o = 1$

and $w_k = i^{*-1}(v_k)$, where v_k is the k^{th} elementary symmetric function in $t_1, t_2, \ldots, t_n$.

These w_k are the universal Whitney classes. Since every symmetric polynomial can be expressed as a polynomial in the elementary symmetric functions, it follows that $H^*(BO(n);\underline{Z}_2) \cong \underline{Z}_2[w_1, w_2, \ldots, w_n]$.

By classification theorem, an O(n) - bundle $\xi \longrightarrow X$ over a para-compact space X is classified by a homotopically unique map $f: X \longrightarrow BO(n)$. The image of w_k under the induced map

$$f^* : H^k(BO(n) ; \underline{Z}_2) \longrightarrow H^k(X ; \underline{Z}_2),$$

is defined as the k^{th} Whitney class of $\xi \longrightarrow X$ and is also denoted by w_k.

We may thus denote the k^{th} Whitney class w_k as the k^{th} elementary symmetric function in $t_1, t_2, \ldots, t_n$, and the total Stiefel-Whitney class may be expressed as $(1+t_1)(1+t_2) \ldots (1+t_n)$ in the factored form.

6.1.3 Let $p : \eta \rightarrow X/T$ be an n-plane bundle. Let $\xi \longrightarrow X/T$ be the line bundle associated with the involution (X,T). Consider $\tilde{p} : \tilde{\eta} \longrightarrow X$, the pull-back of $\eta \longrightarrow X/T$ by the quotient map $\nu : X \longrightarrow X/T$ given by $\tilde{\eta} = \{(x,v) \in X \times \eta \,/\, \nu(x) = p(v)\}$. Identify X with the O-sphere bundle associated with the line bundle $\xi \longrightarrow X/T$. Then we can think of Tx as -x

Thus consider $X \times \eta \hookrightarrow \xi \times \eta$, to define a map

$$\psi : \tilde{\eta} \longrightarrow \xi \times \eta$$

given by $\psi(x,v) = x \otimes v$.

Note that $\psi(\tilde{T}(x,v)) = \psi(-x, -v) = -x \otimes -v$

$$= x \otimes v$$

Then passing onto quotients, we get a map

$$\varphi : \frac{\tilde{\eta}}{\tilde{T}} \, (= \hat{\eta}) \;\; = \xi \otimes \eta$$

which is an isomorphism.

Let the total Stiefel Whitney class of $\eta \longrightarrow X/T$ be expressed as $(1+t_1)\,(1+t_2)\,\ldots\,(1+t_n)$, and let $c \in H^1(X/T\,;\,\underline{Z}_2)$ be the first Steifel Whitney class of $\xi \longrightarrow X/T$, so that its total class is $1+c$. Then the total Stiefel Whitney class of $\xi \otimes \eta$ is

$$\prod_{j=1}^{n} (1 + c + t_j) = \sum_{k=0}^{n} (1 + c)^k \cdot v_{n-k},$$

where v_{n-k} is the $n-k^{th}$ elementary symmetric function in $t_1, t_2, \ldots, t_n$. (cfr. [29], 7).

6.1.4 Let(M^k,T) be a fixed point free involution on a closed k-manifold M^k, and let (V^{n+m}, T') be an involution on a manifold V^{n+m}. Let F be the set of fixed points of the involution (V^{n+m}, T') and let F^m denote the union of all m-dimensional components of F, which we assume to be non-empty. Let $\eta \longrightarrow F^m$ be the normal n-plane bundle to $F^m \subset V^{n+m}$. Since F^m is evidently T'-invariant, we can have a normal disc bundle $D(\eta) \longrightarrow F^m$ which is equivariantly diffeomorphic to a compact invariant normal tube (N,T') of F^m in V^{n+m}. Further, as the diffeomorphism is equivariant, we may identify the antipodal map A of $D(\eta)$ with $T'/N : N \longrightarrow N$.

Now, $(M^k \times N) \,/\, (T \times T')$ is a tubular neighbourhood of $(M^k \times F^m) \,/\, (T \times T') \cong (M^k/T) \times F^m$ in $(M^k \times V^{n+m}) \,/\, (T \times T')$, and we may identify $(M^k \times N) \,/\, (T \times T')$ with $(M^k \times D(\eta)) \,/\, (T \times A)$. Thus it may be seen that the normal bundle to $(M^k/T) \times F^m$ in $(M^k \times V^{n+m})/(T \times T')$ is obtained from the twist by the involution $(M^k \times F^m, T \times id)$ with the bundle induced by the projection :

$$\begin{array}{ccc} & & \eta \\ & & \downarrow \\ \dfrac{M^k \times F^m}{T \times id} & \longrightarrow & F^m \end{array}$$

From 6.1.3, the total Stiefel Whitney class of the normal bundle to $(M^k/T) \times F^m$ is given by

$$\sum_{j=0}^{n} (1+c)^j \cdot v_{n-j}(\eta)$$

Where $c \in H^1(M^k/T ; \underline{Z}_2)$ is the fundamental class of the involution (M^k, T)

6.2 __Theorem__ : Let M^k be a smooth k-dimensional manifold. For any map $f : S^n \longrightarrow M^k$, let $\Lambda(f) \subset S^n$ denote the set $\{x \in S^n / f(x) = f(-x)\}$. If $n > k$, then $\dim \Lambda(f) \geqslant n - k$. If, for $n = k$, $f^* : H^n(M^n ; \underline{Z}_2) \longrightarrow H^n(S^n ; \underline{Z}_2)$ is trivial, then $\Lambda(f) \neq \emptyset$.

__Proof__ : First of all, we assume that M^k is closed and connected.

Consider the fixed point free involution $(S^n \times M^k \times M^k, T)$ given by $T(x, y, z) = (-x, z, y)$. Then the quotient space $(S^n \times M^k \times M^k) / T$ is a closed n+2k-dimensional manifold. Further, the projection $S^n \times M^k \times M^k \longrightarrow S^n$ induces the bundle projection $p : \dfrac{S^n \times M^k \times M^k}{T} \longrightarrow \underline{RP}^n$ with fibre $M^k \times M^k$ and structure group C_2 (cyclic group of two elements. Let Δ be the diagonal of $M^k \times M^k$. Then $S^n \times \Delta$ is T-invariant and hence $\underline{RP}^n \times \Delta$ is a closed n+k - dimensional subspace of $\dfrac{S^n \times M^k \times M^k}{T}$.

We shall denote by $d \in H^1(\underline{R}P^n \times \Delta ; \underline{Z}_2)$, $d \otimes 1$, the characteristic class of the involution $(S^n \times \Delta, T)$. Expanding the result of 6.1.3 and calculating the highest characteristic class, we get that the k-dimensional characteristic class of the normal bundle to $\underline{R}P^n \times \Delta$ is $\sum_{j=0}^{k} d^j \otimes w_{k-j}$ where w_{k-j} is the (k-j)-dimensional Whitney class of the tangent bundle to M^k.

Now, put $X = S^n \times M^k \times M^k$.

For the inclusion $i : \underline{R}P^n \times \Delta \hookrightarrow X/T$, consider the induced homology map $i_* : H_{n+k}(\underline{R}P^n \times \Delta ; \underline{Z}_2) \longrightarrow H_{n+k}(X/T ; \underline{Z}_2)$.
Then $i_*([\underline{R}P^n \times \Delta]) \in H_{n+k}(X/T ; \underline{Z}_2)$ where $[\underline{R}P^n \times \Delta]$ denotes the fundamental class of $\underline{R}P^n \times \Delta$.

Now consider the duality map

$$\left[\frac{X}{T}\right] \cap - \; : \; H^k(X/T ; \underline{Z}_2) \longrightarrow H_{n+k}(X/T ; \underline{Z}_2)$$

and the element $\varphi_k \in H^k(X/T ; \underline{Z}_2)$ such that $\left[\frac{X}{T}\right] \cap \varphi_k = i_*([\underline{R}P^n \times \Delta])$.

The element φ_k is called the cohomology class dual to the submanifold $\underline{R}P^n \times \Delta \subset X/T$.

It has been shown by Thom ([37]) that $i^*(\varphi_k)$ would be the k^{th} Whitney class of the normal bundle to $\underline{R}P^n \times \Delta$.

$$\text{Thus, } i^*(\varphi_k) = \sum_{j=0}^{k} d^j \otimes w_{k-j} .$$

Let N be a closed tubular neighbourhood of $\underline{R}P^n \times \Delta$ in X/T. Then φ_k

is the image of $[\underline{R}P^n \times \Delta]$ under the composition :

$$\begin{aligned} H_{n+k}(\underline{R}P^n \times \Delta) &\cong H_{n+k}(N) \\ &\cong H^k(N, \dot{N}) \\ &\cong H^k(X/T, X/T - \overset{o}{N}) \quad \text{by excising } \overline{X/T - N} \text{ from } (X/T, X/T - \overset{o}{N}) \\ &\longrightarrow H^k(X/T), \text{ from the exact sequence for the pair } (X/T, X/T - \overset{o}{N}). \end{aligned}$$

Considering the exact sequence

$$\ldots \longrightarrow H^k(X/T, X/T - \overset{o}{N}) \longrightarrow H^k(X/T) \longrightarrow H^k(X/T - \overset{o}{N}) \longrightarrow \ldots$$

and by choosing the tubular neighbourhood properly, we have that for any open neighbourhood $U \supset \underline{R}P^n \times \Delta$, φ_k is in the kernel of the homomorphism $j^* : H^k(X/T) \longrightarrow H^k(X/T - U)$.

Now let us consider maps $f : S^n \longrightarrow M^k$. To each such map we associate a cross-section of $p : X/T \longrightarrow \underline{R}P^n$ by defining

$$s([x]) = [(x, f(x), f(-x))] \text{ where } [x] \in \underline{R}P^n$$

corresponds to $x \in S^n$ and $[(x, f(x), f(-x))]$ corresponds to $(x, f(x), f(-x)) \in X$. Note that by the definition of T, $[(x, f(x), f(-x))] = [(-x, f(-x), f(x))]$, so that $s([x]) = s([-x])$. Hence s is well-defined. If f_t is a homotopy of s, then the induced section of $p : X/T \longrightarrow \underline{R}P^n$ is given by $s_t([x]) = [(x, f_t(x), f_t(-x))]$.

To each $f : S^n \longrightarrow M^k$, we associate the cohomology class $s^*(\varphi_k) \in H^k(\underline{R}P^n)$.

6.2.1 Lemma: If $s^*(\varphi_k) \neq 0$, then $\dim(\Lambda(f)) \geqslant n - k$.

Proof : There is the quotient map $\vartheta : S^n \longrightarrow \underline{R}P^n$.

Put $B(f) = \vartheta(\Lambda(f))$.

Clearly $s : \underline{R}P^n \longrightarrow X/T$ has $s^{-1}(\underline{R}P^n \times \Delta) = B(f)$.

Let U be an open neighbourhood of $\underline{R}P^n \times \Delta$ in X/T.

Consider the diagram

$$\begin{array}{ccc} H^k(X/T) & \xrightarrow{j^*} & H^k(X/T - U) \\ \downarrow s^* & & \downarrow s_1^* \\ H^k(\underline{R}P^n) & \xrightarrow[j_1^*]{} & H^k(\underline{R}P^n - s^{-1}(U)) \end{array}$$

Since $j^*(\varphi_k) = 0$ (by earlier analysis), we shall have that

$$j_1^*(s^*(\varphi_k)) = 0$$

i.e., $j_1^*(d^k) = 0$, as $s^*(\varphi_k) \in H^k(\underline{R}P^n) \neq 0$

$$s^*(\varphi_k) = d^k.$$

Now, given an open neighbourhood V of B(f), we can choose $U = p^{-1}(V)$, so that $s^{-1}(U) \subset V$.

Hence for every open neighbourhood V of B(f), d^k lies in the kernel of $H^k(\underline{R}P^n) \longrightarrow H^k(\underline{R}P^n - V)$. (In terms of Alexander-Wallace-Spanier cohomology, this means that the support of d^k lies in B(f). cfr. 5.3.1).

Suppose now, that the homomorphism $H^{n-k}(\underline{R}P^n) \longrightarrow H^{n-k}(B(f))$ is trivial. Since the cohomology theory we are considering is continuous, we can choose an open neighbourhood V of B(f) such that

d^{n-k} lies in the kernel of $H^{n-k}(\underline{R}P^n) \longrightarrow H^{n-k}(\overline{V})$. (cfr. 4.4.3)

Consider now, the following two sequences :

$$\ldots \longrightarrow H^{n-k}(\underline{R}P^n, \overline{V}) \longrightarrow H^{n-k}(\underline{R}P^n) \xrightarrow{\overline{j}^*} H^{n-k}(\overline{V}) \longrightarrow \ldots$$

and

$$\ldots \longrightarrow H^k(\underline{R}P^n, \underline{R}P^n - V) \longrightarrow H^k(\underline{R}P^n) \xrightarrow{j_1^*} H^k(\underline{R}P^n - V) \longrightarrow \ldots$$

with $\overline{j}^*(d^{n-k}) = 0$ and $j_1^*(d^k) = 0$.

Hence choose $\alpha_{n-k} \in H^{n-k}(\underline{R}P^n, \overline{V})$ with image d^{n-k}

and $\beta_k \in H^k(\underline{R}P^n, \underline{R}P^n - V)$ with image d^k.

Using the relative cup product we find that d^n is the image of $\alpha_{n-k} \cdot \beta_k$ under the homomorphism

$$H^n(\underline{R}P^n, \overline{V} \cup \underline{R}P^n - V) \longrightarrow H^n(\underline{R}P^n).$$

But $\overline{V} \cup \underline{R}P^n - V = \underline{R}P^n$, so that $\alpha_{n-k} \cdot \beta_k = 0$ and hence $d^n = 0$, which is a contradiction.

Hence $H^{n-k}(\underline{R}P^n) \longrightarrow H^{n-k}(B(f))$ is non-trivial.

Therefore, dim (B(f)) and hence dim $(\Lambda(f)) \geqslant n - k$.

Note that in the notation of chapter 4.0, we have just proved that if $s^*(\varphi_k) \neq 0$, $\text{coind}_2(\Lambda(f)) \geqslant n - k$.

Now, for the case $n > k$, to prove that $\dim(\Lambda(f)) \geqslant n - k$, it remains to prove that $s^*(\varphi_k) \neq 0$.

By definition, s^* depends only on the homotopy class of the original map $f : S^n \longrightarrow M^k$. First of all, we shall show that f is homotopic to a map which is constant on the southern hemisphere.

6.2.2 Lemma : Let $f : S^n \longrightarrow M^k$ be a map. Then f is homotopic to a map $\tilde{f} : S^n \longrightarrow M^k$ which takes every point on E^n_- ($= \{x \in S^n$ / the last co-ordinate $\leqslant 0\}$ i.e., the southern hemisphere), to f (0, 0, . . . 0, -1).

Proof :

Every point on S^n (except (o, o, ... o, 1) and (0, 0, ..., 0, -1)) determines a point on the equator $\{x = (x_0, \ldots, x_n) \in S^n / x_n = 0\}$, which lies on the geodesic which joins (0,0, ..., 0, 1) to (0, . . . 0, -1) and passing through that point. Let $\tilde{x} \in S^n$ denote such a point corresponding to the point $x \in S^n$.

Let $w_1 : I \longrightarrow S^n$ be defined by

$$w_1(t) = \frac{(1 - t)(0, \ldots 0, 1) + t(\tilde{x})}{\|(1 - t)(0, \ldots 0, 1) + t(\tilde{x})\|}, \quad t \in I$$

and $w_2(t) : I \longrightarrow S^n$ be defined by

$$w_2(t) = \frac{(1 - t)\tilde{x} + t(0, \ldots, 0, -1)}{\|(1 - t)\tilde{x} + t(0, \ldots, 0, -1)\|}, \quad t \in I.$$

Further define $w_1 * w_2 : I \longrightarrow S^n$ By

$$w_1 * w_2(t) = \begin{cases} w_1(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ w_2(2t - 1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Now, let any $\bar{x} \in S^n$ be expressed as $w_1 * w_2(t)$ for some $t \in I$. Note that (0, 0, . . . 0, 1) and (0, . . ., 0, -1) will be $w_1 * w_2(0)$ and $w_1 * w_2(1)$ respectively, for w_1 and w_2 defined using any point on the equator.

Define $F_s : S^n \times I \longrightarrow S^n$

by $F_s(x, s) = \begin{cases} w_1 * w_2((1+s)t), & 0 \leqslant t \leqslant \frac{1}{2} \\ w_1 * w_2((1-s)t + s), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$

F_s then gives a homotopy between the identity map on S^n, and a map which takes every point on E^n_- to the point (0, . . ., 0, -1).

Hence, $f \circ F_s : S^n \times I \longrightarrow H^k$

gives a homotopy between f and a map $\tilde{f} : S^n \longrightarrow H^k$ which takes every point on E^n_- to $f(0, . . ., -1) = y_0$ (say). @

Thus without loss of generality, we can assume that f is constant on E^n_-, i.e., $f(E^n_-) = y_0 \in H^k$.

Consider $S^{n-1} \subset S^n$ as the equator. Then $f(S^{n-1}) = y_0$.

Hence if we define $s_1 : \underline{RP}^{n-1} \longrightarrow \underline{RP}^n \times$

by $s_1([x]) = [(x, y_0, y_0)]$

for $[x] \in \underline{RP}^{n-1}$, we shall have the following commutative diagram :

$$\begin{array}{ccc} \underline{RP}^{n-1} & \xrightarrow{\ s_1\ } & \underline{RP}^n \times \Delta \\ \big\downarrow i_1 & & \big\downarrow i \\ \underline{RP}^n & \xrightarrow{\ s\ } & X/T \end{array}$$

The corresponding cohomology commutative diagram gives us :-

$$i_1^* s^* (\varphi_k) = s_1^* i^* (\varphi_k)$$

$$= s_1^* \left(\sum_{j=0}^{k} d^j \otimes w_{k-j} \right),$$ by a result established in the proof of 6.2

Note that $s_1 : \underline{R}P^{n-1} \longrightarrow \underline{R}P^n \times \Delta$ can be decomposed as

$$\underline{R}P^{n-1} \xrightarrow{\ \Delta\ } \underline{R}P^{n-1} \times RP^{n-1} \xrightarrow{\ i_1 \times c\ } \underline{R}P^n \times \Delta$$

where Δ is the diagonal map and $c : \underline{R}P^{n-1} \longrightarrow \Delta$ is the constant map $c(x) = (y_o, y_o)$.

$$\begin{aligned}
\text{Then } s_1^* \left(\sum_{j=0}^{k} d^j \otimes w_{k-j} \right) &= \Delta^* \left(i_1^* \times c^* \left(\sum_{j=0}^{k} d^j \otimes w_{k-j} \right) \right) \\
&= \Delta^* \left(\sum_{j=0}^{k} i_1^* d^j \otimes c^* w_{k-j} \right) \\
&= \Delta^* (i_1^* d^k \otimes 1), \quad \text{as } c^*(w_o) = 1 \\
&\qquad \text{and } c^*(w_{k-j}) = 0, \ \forall\, k-j > 0. \\
&= i_1^*(d^k) \in H^k(\underline{R}P^{n-1}). \\
&\neq 0 \text{ as } n > k.
\end{aligned}$$

Hence, $s^*(\varphi_k) \neq 0$ and the proof is complete when $n > k$.

Now, we turn to a map $f : S^n \longrightarrow M^k$ for which $f^* : H^n (M^n) \longrightarrow H^n(S^n)$ is trivial. We continue to assume that M^n is closed and connected and we still require that $f(E^n_-) = y_o \in M^n$.

Consider the equivariant map

$$F : S^n \longrightarrow M^n \times M^n$$

given by $\quad F(x) = (f(x), f(-x))$

Note that, since either x or $-x$ always lies in E^n_-, F actually maps S^n into the wedge $M^n \vee M^n = M^n \times \{y_o\} \cup \{y_o\} \times M^n$.

Also, the involution, $(t, M^n \times M^n)$ given by $t(y, z) = (z, y)$ leaves the wedge invariant.

Further, $(M^n \times M^n)\,/\,t \;\cong\; M^n$ and $F : S^n \longrightarrow M^n \vee M^n$ is equivariant.

6.2.3 <u>Lemma</u> : If $\overline{F} : \underline{R}P^n \longrightarrow M^n$ is the map of quotient spaces associated with F, then $\overline{F}^* : H^n(M^n) \longrightarrow H^n(\underline{R}P^n)$ is trivial.

<u>Proof:</u> The validity of the statement is clear from the following commutative diagram :-

$$\begin{array}{ccccccccc}
 & & & & H^n(M^n, y_0) & & & & \\
 \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow \overline{F}^* & & \downarrow \overline{F}^* \\
H^n(S^n) & \cong & H^n(S^n, E^n_-) & \cong & H^n(E^n_+, S^{n-1}) & \cong & H^n(\underline{R}P^n, \underline{R}P^{n-1}) & \cong & H^n(\underline{R}P^n)
\end{array}$$

Where the first isomorphism comes from the cohomology exact sequence for the pair (S^n, E^n_-); the second by excising $\overset{o}{E}{}^n_-$; the third as (E^n_+, S^{n-1}) and $(\underline{R}P^n, \underline{R}P^{n-1})$ are both of the same homotopy type; and the fourth from the cohomology exact sequence for the pair $(\underline{R}P^n, \underline{R}P^{n-1})$.

Note that the first f^* is zero by assumption. This ensures that the last $\overline{F}^*$ is also trivial. @

6.2.4 <u>Lemma</u> : Under the composition $\underline{R}P^n \times (y_0, y_0) \xrightarrow{\;i_1\;} \dfrac{S^n \times (M^n \vee M^n)}{T} \xrightarrow{\;i_2\;} \dfrac{X}{T}$, we have $i_1^*\, i_2^*\,(\varphi_n) \;=\; d^n \otimes 1$.

<u>Proof</u> : Consider the diagram :

$$\begin{array}{ccc}
\underline{R}P^n \times (y_0, y_0) & \xrightarrow{\;i_2 \,o\, i_1\;} & \dfrac{X}{T} \\
\Big\downarrow \tilde{i} & \nearrow i & \\
\underline{R}P^n \times \triangle & &
\end{array}$$

We know that $i^*(\varphi_n) = d^n \otimes 1 + d^{n-1} \otimes w_1 + \ldots + 1 \otimes w_n$

$$= \sum_{j=0}^{n} (d^j \otimes w_{n-j})$$

Note that $\tilde{i} = id \times c$ where $c : \Delta \longrightarrow (y_o, y_o)$ is the constant map

$$c(x) = (y_o, y_o).$$

Then, $$\tilde{i}^*(i^*(\varphi_n)) = id^* \otimes c^* \left(\sum_{j=0}^{k} (d^n \otimes w_{n-j}) \right)$$

$$= \sum_{j=0}^{n} (d^n \otimes c^* w_{n-j})$$

$$= d^n \otimes 1.$$

$$\text{i.e., } i_1^* i_2^* (\varphi_n) = d^n \otimes 1$$

To establish the theorem for $n = k$, all we have to do now is to prove that $s^*(\varphi_n) \neq 0$. We proceed to do that.

Put $Y = S^n \times (M^n \vee M^n)$ and let $s_1 : \underline{R}P^n \longrightarrow Y/T$ be defined by

$$s_1([x]) = ([x, f(x), f(-x)])$$

Then $s : \underline{R}P^n \longrightarrow X/T$ is the composite

$$\underline{R}P^n \xrightarrow{s_1} Y/T \xrightarrow{i_2} X/T.$$

Now, $s_1 : \underline{R}P^n \longrightarrow Y/T$ is a cross-section of the fibre map $p_1 : Y/T \longrightarrow \underline{R}P^n$. Hence $s_1^* p_1^* = id^*$ implies that s_1^* cannot be the zero map, and hence that

$$s_1^* : H^n(Y/T) \longrightarrow H^n(\underline{R}P^n)$$

is an epimorphism.

Set $\Upsilon_n \in H^n(Y/T)$, so that $s_1^*(\Upsilon_n) = d^n$.

For example, we could set $\Upsilon_n = p_1^*(d^n)$.

Under $i_1^* : H^n(Y/T) \longrightarrow H^n(\underline{R}P^n \times (y_0, y_0))$ we shall have

$$i_1^*(\Upsilon_n) = d^n \otimes 1.$$

Then, from 6.2.4, we have that $i_1^*(\Upsilon_n) = i_1^* i_2^*(\varphi_n)$

$$i_1^*(\Upsilon_n + i_2^*(\varphi_n)) = 0$$

Hence, from the cohomology exact sequence for the pair $(Y/T, \underline{R}P^n \times (y_0, y_0))$ we have that $\Upsilon_n + i_2^*(\varphi_n)$ lies in the image of

$$H^n(Y/T, \underline{R}P^n \times (y_0, y_0)) \xrightarrow{j^*} H^n(Y/T).$$

6.2.5 <u>Lemma</u> : The composition $s_1^* j^*$ is trivial.

<u>Proof</u> : Consider the following diagram :

$$\begin{array}{ccc} H^n(Y/T, \underline{R}P^n \times (y_0, y_0)) & \xrightarrow{j^*} & H^n(Y/T) \\ \cong \downarrow \beta^* & & \downarrow s_1^* \\ H^n(M^n, y_0) & \xrightarrow{\bar{F}^*} & H^n(\underline{R}P^n) \end{array}$$

Where the isomorphism β^* is obtained as follows :

consider the map $\beta : Y/T = \dfrac{S^n \times (M^n \vee M^n)}{T} \longrightarrow \dfrac{M^n \vee M^n}{t} = M^n$

which is induced by the projection $S^n \times (M^n \vee M^n) \longrightarrow M^n \vee M^n$.

Note that $S^n \times (M^n \vee M^n) - S^n \times (y_0, y_0) = S^n \times (M^n - y_0) \sqcup S^n \times (M^n - y_0)$

Identification gives : $Y/T - (\underline{R}P^n \times (y_o, y_o)) = S^n \times (M^n - \{y_o\})$

Thus the map β actually gives rise to the projection

$$S^n \times (M^n, y_o) \longrightarrow (M^n, y_o)$$

which induces an isomorphism $H^n(S^n \times M^n, y_o)) \cong H^n (M^n, y_o)$

Hence β^* of the above diagram is an isomorphism.

Further, by 6.2.3 $\overline{F}^*$ is trivial.

Hence, we get $s_1{}^*j^*$ is trivial.

Now, since $\Upsilon_n + i_2{}^*(\varphi_n)$ lies in the image of j^*, we shall have

$$s_1{}^* (\Upsilon_n + i_2{}^* (\varphi_n)) = 0$$

$$\text{i.e., } s_1{}^* i_2{}^* (\varphi_n) = s_1{}^*(\Upsilon_n)$$

$$\text{i.e., } s^*(\varphi_n) = d^n \in H^n(\underline{R}P^n)$$

$$\neq 0$$

The required result now follows by 6.2.1.

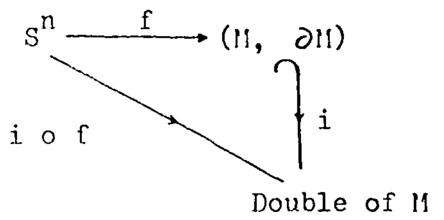
Thus we have established the theorem for closed, connected manifolds.

6.2.6 We can establish the result for compact manifolds with boundary by doubling.

<u>Definition and result</u> : The double of a manifold with boundary $(M, \partial M)$ is the identification space obtained from $(M, \partial M) \times \{0\} \sqcup (M, \partial M) \times \{1\}$ by identifying $(x, 0)$ with $(x, 1)$ if $x \in \partial M$. The double of $(M, \partial M)$ is a manifold without boundary of the same dimension, in which M is

embedded. ([19] , 1.4).

Consider the diagram :



where i is the embedding.

As I is the embedding, note that $A(f) = A(i \circ f)$.

Hence, the result follows from 6.2.

6.2.7 The extension of the result to an open manifold follows as an open manifold can be considered as the increasing union of compact manifolds with boundary, and as S^n is compact.

6.2.8 <u>Corollary</u> : If f is a map of S^n into a connected non-compact n-manifold then there is a point $x \in S^n$ for which $f(x) = f(-x)$.

<u>Proof</u> : The proof follows directly from 6.2 as under the hypothesis of non-compactness, $H^n(M^n, \underline{Z}_2) = 0$. @

6.3 In conclusion, let us consider in brief what generalisation we may make for the involution (S^n, A).

Let (Σ^n, T) be any fixed point free involution on a closed n-manifold which is a homotopy n-sphere. Then by results obtained in [6] there exist equivariant maps $\varphi : (\Sigma^n, T) \longrightarrow (S^n, A)$ and $\varphi' : (S^n, A) \longrightarrow (\Sigma^n, T)$. Further, both φ and φ' are of odd degree. Now, given a map $f : \Sigma^n \longrightarrow M^k$, let $A_T(f)$ be the set of all $x \in \Sigma^n$ such that

$f(x) = f(Tx)$. Then obviously, φ' maps $\Lambda(f \circ \varphi')$ equivariantly to $\Lambda_T(f)$. As we have noted at the end of 6.2.1, 6.2 applied to this situation gives that $\mathrm{coind}_2\, \Lambda(f \circ \varphi') \geq n - k$. Since an equivariant map cannot decrease coind_2, it follows that $\mathrm{coind}_2\, \Lambda_T(f) \geq n - k$. Hence the result 6.2 is valid also for the fixed point free involution (Σ^n, T)

7.0 PARAMETRIZATION OF THE BORSUK-ULAM THEOREM

We attempt now, to parametrize the Borsuk-Ulam Theorem. As we have noted earlier, work in this direction has been done by Nakaoka ([31]), Jaworowski ([22], [23]), Fadell and Husseini ([14]) and Dold ([8]). In this chapter, we shall examine the work of Dold in detail. We consider odd (equivariant) fibre-preserving maps $f : SE \longrightarrow E'$ where $p : E \longrightarrow B \longleftarrow E' : p'$ are vector bundles over the same paracompact space B and $p/SE : SE \longrightarrow B$ is the induced sphere bundle over B. We shall be interested in measuring the set $Z = \left\{ x \in SE \,/\, f(x) = 0 \right\}$. Throughout this section we shall be using Cech cohomology with coefficients in $\underline{Z}_2$.

7.1 Parametrized Borsuk-Ulam theorems for Sphere Bundles with Fixed Point Free Involutions.

Let $p : E \longrightarrow B$ and $p' : E' \longrightarrow B$ be vector bundles of fibre dimensions m and n respectively, over the same paracompact space B. Let $w_j(E)$ and $w_j(E') \in H^j(B)$ be their Stiefel-Whitney classes (henceforth denoted simply as w_j and w'_j as long as there is no confusion) and $w(E;t)$, $w(E';t) \in H^*(B)\,[t]$ defined by

$$w(t) = w(E;t) = \sum_{j=0}^{m} w_j(E)\, t^{m-j}$$

$$w'(t) = w(E';t) = \sum_{j=0}^{m} w_j(E')\, t^{n-j}$$

be their Stiefel-Whitney polynomials. Here $H^*(B)\,[t]$ is the polynomial ring over $H^*(B)$ in one indeterminate t of degree one. Clearly w(t) has degree m, both in the indeterminate and as an element of the graded ring $H^*(B)[t]$.

Let $p/SE : SE \longrightarrow B$ denote the sphere bundle of $p : E \longrightarrow B$. Let $f : SE \longrightarrow E'$ be a map such that $p'f = p/SE$ and $f(-x) = -f(x)$. Let $Z = \{x \in SE \,/\, f(x) = 0\}$. The antipodal action $T : x \longmapsto -x$ is clearly a fixed point free involution on SE and on Z. Thus the projection maps $SE \longrightarrow SE/T \ (\equiv \overline{SE})$ and $Z \longrightarrow Z/T \ (\equiv \overline{Z})$ are two-sheeted covering maps.

For the inclusion $i : \overline{Z} \longrightarrow \overline{SE}$, we have the cohomology homomorphism $i^* : H^1(\overline{SE}) \longrightarrow H^1(\overline{Z})$. Let $u \in H^1(\overline{SE})$ and $i^*(u) \in H^1(\overline{Z})$, denoted by $u/\overline{Z}$, be respectively, their characteristic classes. Thus we have homomorphisms of $H^*(B)$ algebras

$$\sigma : H^*(B)[t] \longrightarrow H^*(\overline{SE}) \longrightarrow H^*(\overline{Z})$$

$$t \longmapsto u \longmapsto u/\overline{Z}$$

We shall, for the sake of clarity, write $p(t)/\overline{SE}$ and $p(t)/\overline{Z}$ for the images $p(u) \in H^*(\overline{SE})$ and $p(u/\overline{Z}) \in H^*(\overline{Z})$ respectively, of $p(t) \in H^*(B)\ t$.

7.1.2 <u>Theorem</u> : If $q(t) \in H^*(B)[t]$ is such that $q(t)/\overline{Z} = 0$, then $q(t).w(E';t) = w(E;t).q'(t)$ for some polynomial $q'(t) \in H^*(B)[t]$. (i.e., either $q(t)/\overline{Z} \neq 0$ or $w(E;t)$ divides $q(t).w(E';t)$).

Before we prove the theorem, let us look at a few important corollaries :

7.1.3 If m, n are the fibre dimensions of E and E' respectively, then $q(t) / \overline{Z} \neq 0$ for all polynomials q(t) whose degree with respect to t is smaller than m - n.

7.1.4 In other words, the H*(B)-homomorphism

$$\bigoplus_{i=0}^{m-n-1} H^*(B).\ t^i \longrightarrow H^*(\overline{Z})$$

$$t^i \longmapsto t^i / \overline{Z}$$

is monomorphic.

7.1.5 In particular if $m > n$, then the cohomological dimensions of $\overline{Z}$ and B satisfy the relation :

$$\text{cohom. dim } (\overline{Z}) \geqslant \text{cohom. dim } (B) + m-n-1.$$

Proof : If the degree of $q(t) < m - n$, then the relation $q(t).w(E';t) = w(E;t).q'(t)$ can never hold for any polynomial q'(t), and hence $q(t) / \overline{Z} \neq 0$.

7.1.4 follows immediately from 7.1.3 and 7.1.5 is an immediate consequence of 7.1.4 . @

Note that if B is a single point, then the result 7.1.5 is a generalisation of the classical Borsuk-Ulam theorem by Bourgin and Yang ([39], discussed earlier in 3.0). In particular, of m = n+1, we get the classical Borsuk-Ulam theorem itself ([3], discussed in 2.0).

7.1.6 If we wish we can work with singular cohomology theory and we would get :

if $q(t) \in H^*(B)[t]$ is such that $q(t) / V = 0$ for some open neighbourhood V of $\bar{Z}$ in $\bar{S}E$, then

$$q(t) \cdot w(E';t) = w(E;t) \cdot q'(t)$$

for some polynomial $q'(t) \in H^*(B)[t]$.

We shall now proceed on to the proof of 7.1.2. We shall however, do it in a much more general setting than stated earlier, as follows :-

7.1.7 Definition : A G-sphere bundle (of dimension m-1) is a fibre map $p: S \longrightarrow B$ together with a free fibre-wise G-action τ on S, such that

(i) (p, τ) is G-locally trivial, i.e., B is covered by open sets U such that $p^{-1}(U) \approx U \times Y$ (where Y is the fibre) as G-spaces over U, $\tau(u,y) = (u, \tau y)$

(ii) The fibre is G-homotopically equivalent to a compact finite dimensional G-space.

and

(iii) $H^*(Y) = H^*(S^{m-1})$, where S^{m-1} is the (m-1) - sphere.

All spaces involved are considered to be paracompact.

For our purposes, G= $\underline{Z}_2$ and we consider cohomology with coefficients in $\underline{Z}_2$. But other subgroups $G \subset S^1$ and other coefficients apart from $\underline{Z}_2$ may also be considered.

Examples for G-sphere bundles are unit sphere-bundles of vector spaces with antipodal action. τ could also be taken to be any fibre-preserving action on a vector bundle, or sphere bundle E and $S = E - \mathrm{Fix}(\tau)$, where Fix (τ) dènotes the fixed points of the action τ . However, one must ensure that the local triviality includes the action as noted in (i) above.

7.1.8 Definition : Y, the fibre of $p : S \longrightarrow B$ is a cohomology sphere and dim (Y) $< \infty$ (by 7.1.7 (iii) above). Hence the orbit space $\overline{Y} = Y/G$ with $G = \underline{Z}_2$, a cohomology projective space,

$$\text{i.e.,} \quad H^*(Y) = H^*(S^{m-1})$$

$$\text{and} \quad H^*(\overline{Y}) = H^*(\underline{RP}^{m-1})$$

$$= \underline{Z}_2 \,[u] /(u^m)$$

where $u \in H^1(\overline{Y})$ is the characteristic class of the $\underline{Z}_2$-action. Since u is also defined on $\overline{S} = S/G$, we can apply the Lerray-Hirsch theorem to the fibre bundle $\overline{p} : \overline{S} \longrightarrow B$ and obtain that $H^*(\overline{S})$ is freely generated over $H^*(B)$ by $1, u, \ldots, u^{m-1}$.

(Note that this result is familiar for singular cohomology. We have worked in a similar vein using A'/S cohomology in 5.0. For Čech cohomology, in this chapter, we assume the result. We shall see in the next chapter how such results can be obtained for generalised cohomology theories).

Now, expressing $u^m \in H^m(\overline{S})$ in terms of these generators we get unique $w_j \in H^j(B)$, $j = 1, 2, \ldots, m$ such that

$$u^m = w_m + w_{m-1}.\, u + \ldots + w_1.\, u^{m-1}$$

$$\text{i.e.,} \quad u^m + w_1.u^{m-1} + \ldots + w_{m-1} \,.\, u + w_m = 0$$

Following Grothendieck, we call these elements, the Stiefel-Whitney classes of (p, τ), putting $w_0 = 1$ and $w_j = 0$, $j > n$.

(cfr. 5.1 where in the AWS cohomology setting, we have proved that these in fact are the Stiefel Whitney classes).

As earlier, we define the Stiefel-Whitney polynomial

$$w(t) = \sum_{j=0}^{m} w_j \cdot t^{m-j}$$

and we immediately have that

$$H^*(B)[t] \,/\, w(t) \cong H^*(\overline{S})$$

$$\text{under} \qquad t \longmapsto u$$

as $H^*(B)$-algebras.

We now reformulate theorem 7.1.2 to include the above generalisations and proceed to prove it.

7.1.9 <u>Theorem</u> : Let $p : S \longrightarrow B$ be a G-sphere bundle and let E' be a space with a G-action τ' and a map $p' : (E' - Z') \longrightarrow B$ (where $Z' = \text{Fix}(\tau')$), such that (p', τ') is a G-sphere bundle, $G = \mathbb{Z}_2$. Let $f : S \longrightarrow E'$ be a G-map which is fibre-preserving (i.e., $p'f = p$) in $S - f^{-1}(Z')$. Put $Z = f^{-1}(Z')$, $\overline{Z} = Z/\tau \subset \overline{S} = S/\tau$. Now, if $q(t) \in H^*(B)[t]$ is a polynomial such that $q(t) / \overline{Z} = 0$,

$$\text{then} \quad q(t) \cdot w'(t) = w(t) \cdot q'(t)$$

for some polynomial $q'(t) \in H^*(B)[t]$, where

$$w(t) = \sum_{i=0}^{m} w_i(E)\, t^{m-i} \quad ; \quad w'(t) = \sum_{j=0}^{n} w_j\,(E' - Z')\, t^{n-j}$$

m and n being the respective cohomological fibre dimensions, and

$q(t) / \overline{Z}$ is obtained from the $H^*(B)$-homomorphism :

$$H^*(B)[t] \longrightarrow H^*(\overline{Z})$$
$$t \longmapsto u / \overline{Z}$$

Proof : Let $q(t) / \overline{Z} = 0$. Then by the continuity of Cech cohomology, there exists an open neighbourhood $V \subset \overline{S}$ of $\overline{Z}$ such that $q(t) / V = 0$. Now by the exactness of

$$\ldots \longrightarrow H^*(\overline{S}, V) \xrightarrow{j^*} H^*(\overline{S}) \longrightarrow H^*(V) \longrightarrow \ldots$$

we get a $v \in H^*(\overline{S}, V)$, such that $j^*(v) = q(t) / \overline{S}$.

On the other hand, the map $f : (S - Z) \longrightarrow (E' - Z')$ induces the cohomology homomorphism

$$\overline{f}^* : H^*(\overline{E}' - \overline{Z}') \longrightarrow H^*(\overline{S} - \overline{Z})$$

on the orbit spaces, and we have

$$\begin{aligned} w'(t) / (\overline{S} - \overline{Z}) &= w'(u), \text{ by definition} \\ &= w'(\overline{f}^* u'), \text{ since as } f \text{ is a map over } B, \\ &\qquad \overline{f}^* \text{ is a } H^*(B)\text{-homomorphism.} \\ &= \overline{f}^* (w' (u')) \\ &= 0, \text{ as } w'(u') = 0. \end{aligned}$$

Consider the exact sequence

$$\ldots \longrightarrow H^*(\overline{S}, \overline{S} - \overline{Z}) \xrightarrow{j^*} H^*(\overline{S}) \longrightarrow H^*(\overline{S} - \overline{Z}) \longrightarrow \ldots$$

there exists an $z \in H^*(\overline{S}, \overline{S} - \overline{Z})$ s.t. $j^*(z) = w'(t) / \overline{S}$.

Now, $v \smile z \in H^*(\overline{S}, V \cup (\overline{S} - \overline{Z})) \cong H^*(\overline{S}, \overline{S}) = 0$

Hence, $q(t).w'(t) / \overline{S} = j^*(v) \smile j^*(z) = j^*(v \smile z)$, by naturality

$$= 0$$

But $H^*(\overline{S}) = H^*(B)[t] / w(t)$

Hence, $q(t).\ w'(t)$ must be a multiple of $w(t)$

i.e., there exists $q'(t) \in H^*(B)[t]$, such that $q(t).w'(t) = w(t).q'(t)$.

@

We may note that in the above p' is not defined on the whole of E' but only on E' - Z'. For an example of a $\underline{Z}_2$ - space E' where p' is not defined on all of E', one may consider the Thom space of a vector bundle.

7.2 A Borsuk-Ulam Theorem in the Presence of Fixed Points

We now want to generalise 7.1.2 by allowing for non-zero fixed points of the action τ, but assuming that f relates the fixed point sets by a cohomology isomorphism. Here we shall restrict ourselves to vector bundles and linear actions, and not concern ourselves with the cohomological version as set forth in 7.1.9.

The set up is as follows :-

We have vector bundles E, F, E', F' over B and we let $G = \underline{Z}_2$ act on $E \oplus F$ and $E' \oplus F'$ by $\tau(x, y) = (-x, y)$, i.e., antipodal action on E, E' and trivial action on F, F'. We now consider a G-map

$$f : S(E \oplus F) \longrightarrow E' \oplus F'$$

over B and aim to obtain the cohomology dimension of $Z = f^{-1}(0)$. (Note that $S(E \oplus F)$ denotes the unit sphere bundle of $E \oplus F$ over B.) We have that $S(E \oplus F)$ is homeomorphic to the fibre-wise join

$SE \underset{B}{*} SF$ (which we shall denote by $SE * SF$).

As f is a G map, $f(\tau(0, y)) = \tau(f(0, y))$

$$\Rightarrow f(0, y) = \tau(f(0, y))$$

so that $f(0, y) \in F'$.

Hence, f maps $SF = \{z \in SE * SF / \tau z = z\}$ into

$$F' = \{z \in E' \oplus F' / \tau z = z\}.$$

We assume in addition that $f(SF) \subset F' - \{0\}$ and that $f / SF : SF \longrightarrow F' - \{0\}$ has odd degree in the fibres. In particular, F and F' must have the same fibre dimension, say k.

As earlier, we denote by $\overline{S}(E \oplus F)$, $\overline{Z}$, $\overline{E}$ etc., the corresponding orbit spaces of the G-action. Further, we use the notations $p(t) / \overline{Z}$, $w(E;t)$ etc., as earlier. We shall also use the following obbreviations:

$$S = S(E \oplus F) - SF$$

$$\text{and} \quad \overline{S} = \overline{S}(E \oplus F) - SF$$

7.2.1 Theorem : If $q(t) \in H^*(B)[t]$ is such that $q(t) / \overline{Z} = 0$, then

$$q(t).w(E';t) = w(E;t).q'(t)$$

for some polynomial $q'(t) \in H^*(B)[t]$.

Proof : Note that τ and τ' operate freely on $S = S(E \oplus F) - SF$ and $(E' - \{0\}) \oplus F' = (E' \oplus F') - F'$ respectively.

Thus we get characteristic classes of the action $u \in H^1(\overline{S}(E \oplus F) - SF)$ and $u' \in H^1((E' \oplus F')/\tau - F')$.

As earlier we may substitute these classes for the indeterminate in the polynomials $p(t) \in H^*(B)[t]$. (For example, we may have $q(t) / \overline{Z}$ and $q(t) / \overline{S}$. Note that $f(SF) \subset F' - \{0\}$ so that $f^{-1}(0)$ (=Z) $\not\subset$ SF. Hence $Z \subset S$ making the former polynomial possible.)

If $q(t) / \overline{Z} = 0$, then by the continuity of the Cech cohomology, there exists an open neighbourhood $V \subset^{\circ} \overline{S}$ of $\overline{Z}$ such that $q(t) / V = 0$.

By the exactnes of the sequence

$$\cdots \longrightarrow H^*(\overline{S}, V) \xrightarrow{j^*} H^*(\overline{S}) \longrightarrow H^*(V) \longrightarrow \cdots$$

we get an element $v \in H^*(\overline{S}, V)$ such that $j^*(v) = q(t) / \overline{S}$.

Further, the map $f : S - f^{-1}(F') \longrightarrow (E' \oplus F') - F'$
$$\approx (E' - \{0\}) \oplus F'$$
induces $\overline{f}^* : H^*(E' - \{0\}) \longrightarrow H^*(\overline{S} - \overline{f}^{-1}(F'))$

on the cohomology of the orbit spaces. (Note that as F' is contractible $(E' \oplus F') - F' \approx (E' - \{0\}) \oplus F'$ is homotopic to $E' - 0$

$$\begin{aligned} \text{Now consider } w'(t) / \overline{S} - \overline{f}^{-1}(F') &= w'(u) \\ &= w'(\overline{f}^*(u')) \\ &= \overline{f}^*(w'(u')) \\ &= 0. \end{aligned}$$

Then the exact sequence

$$\cdots \longrightarrow H^*(\overline{S}, \overline{S} - \overline{f}^{-1}(F')) \xrightarrow{j^*} H^*(\overline{S}) \longrightarrow H^*(\overline{S} - \overline{f}^{-1}(F')) \longrightarrow \cdots$$

gives an element $z \in H^*(\overline{S}, \overline{S} - \overline{f}^{-1}(F'))$ such that $j^*(z) = w'(t)/ \overline{S}$.

We have already seen that $S(E \oplus F) = SE * SF$

and $\overline{S}(E \oplus F) = \overline{SE} * \overline{SF}$

where $*$ denotes fibre-joins over B.

Now, consider the two vector bundles

$p_1 : S(E \oplus F) - SF \longrightarrow SE$, where p_1 is the natural retraction of $S(E \oplus F) - SF$ ($\approx SE * SF - SF$) onto SE

and $\tilde{p} : p^*F \longrightarrow SE$ where $p = p/SE : SE \longrightarrow B$ is the fibre-projection. Since the fibres of the two vector bundles are isomorphic, we get a vector bundle isomorphism $p^*F = S$

Similarly $\overline{p}^*F \cong \overline{S}$ ($\cong \overline{S}(E \oplus F) - \overline{S}F$), where $\overline{p} : \overline{S}E \longrightarrow B$

Note that $(S, S - SE)$ is the Thom space of p^*F and that $(\overline{S}, \overline{S} - \overline{S}E)$ is the Thom space of $\overline{p}^*F$.

Let $s \in H^k(\overline{S}, \overline{S} - \overline{S}E)$, $s \in H^k(F, F - \{0\})$ and $s' \in H^k(F', F' - \{0\})$ be respectively the Thom classes of $\overline{p}^*F$, F and F' respectively.

Let TSF be an open tubular neighbourhood of SF in S which is small enough so that $f(TSF) \cap E' = \emptyset$. If B is compact we will be able to construct it with uniform radius for the fibres of TSF. But if B is not compact, we may have to resort to a variable radius r_b for the fibres T_b of TSF.

Consider the following diagram

$$\begin{array}{ccccc}
(\overline{S}, \overline{S} - \overline{f}^{-1}(E')) & \xleftarrow{\tilde{i}} & (\overline{S}, \overline{T}SF) & \xrightarrow{i} & (\overline{S}, \overline{S} - \overline{S}E) \\
& \overline{f} \searrow & \downarrow \overline{f} & & \\
& & (\overline{E}' \oplus F', \overline{E}' \oplus F' - \overline{E}') & & \\
& & \downarrow \text{proj.} & & \\
& & (F', F' - \{0\}) & &
\end{array}$$

and the corresponding cohomology diagram :

$$\begin{array}{ccccc} H^*(\bar S,\ \bar S - \bar f^{\,-1}(E')) & \xrightarrow{\ \tilde j^*\ } & H^*(\bar S,\ \overline{T}SF) & \xleftarrow[\ \cong\ ]{\ i^*\ } & H^*(\bar S,\ \bar S - \overline{S}E) \\ & {\scriptstyle \bar f^*}\nwarrow & \uparrow {\scriptstyle \bar f^*} & & \\ & & H^*(\bar E' \oplus F',\ \bar E' \oplus F' - \bar E') & & \\ & & \cong \uparrow {\scriptstyle \text{proj}\,^*} & & \\ & & H^*(F',\ F' - \{0\}) & & \end{array}$$

Note that $\tilde i^*$ is an isomorphism as $\overline{T}SF$ is of the same homotopy type as $\bar S - \bar S E$. Also proj* is an isomorphism for similar reasons. Further f (or $\bar f$) maps SF into $F' - \{0\}$, by assumption, with odd-degree in the fibres. Similarly for $\overline{T}SF$. Therefore $\bar f^*$ takes Thom classes to Thom classes.

$$\text{i.e., } \bar f^*(s') = i^*(s) \in H^*(\bar S,\ \overline{T}SF)$$

Now consider $\bar f^*(s') \in H^*(\bar S,\ \bar S - \bar f^{\,-1}(E'))$, and the product

$$v \cup z \cup \bar f^*(s') \in H^*(\bar S,\ V \cup (\bar S - \bar f^{-1}(F')) \cup (\bar S - \bar f^{-1}(E')))$$

$$= H^*(\bar S, \bar S)$$

$$= 0$$

We now apply j^* to bring $v \cup z$ into $H^*(\bar S)$, and $\tilde i^*$ to bring $j^*(v \cup z) \cup \tilde i^*(\bar f^*(s'))$ into $H^*(\bar S,\ \overline{T}SF)$.

Thus we have

$$0 = j^*(v) \cup j^*(z) \cup \tilde i^*(\bar f^*(s'))$$

$$= (q(t)/\bar S) \cup (w'(t)/\bar S) \cup \bar f^*(s')$$

$$= q(t)\ .\ w'(t)\ /\ \bar S \cup i^*(s)$$

$$= i^*(q(t)\ .\ w'(t)/\ \bar S \cup s\ )$$

But it is an isomorphism, so that $q(t).w'(t)/\overline{S} \cup s = 0$.

Further $\cup s$ is the Thom isomorphism, which gives

$$q(t) \,.\, w'(t) \,/\, \overline{S} = 0$$

As we have noted earlier, $\overline{S} = \overline{S}(E \oplus F) - SF$ has a natural retraction to $\overline{S}E$, which implies that

$$H^*(\overline{S}(E \oplus F) - SF) \cong H^*(\overline{S}E) \cong H^*(B)[t]/ w(t).$$

Therefore, $q(t) \,.\, w'(t)$ is a multiple of $w(t)$.

i.e., there exists $q'(t) \in H^*(B)[t]$, s.t. $q(t).w'(t) = w(t).q'(t)$

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7.3 Examples and Remarks

To illustrate the formula of 7.1.2, we consider a linear map $\emptyset : E \longrightarrow E'$ of constant rank. Then we know that Kernel, Image and Cokernel of $\emptyset$ are vector bundles, which we shall denote by K, I and K' (cfr. [20], 3.8).

Then $E = K \oplus \eta$ and $E' = I \oplus K'$, where $\eta \cong I$.

Hence $w(E;t) = w(t) = w(K;t).\, w(I;t)$

and

$w'(t) = w(E';t) = w(I;t) \,.\, w(K';t)$

Thus we get

$$w(K;t).w'(t) = w(K;t).w(I;t).w(K';t)$$

$$= w(t).w(K';t)$$

Also, by definition, $Z = SE \cap \emptyset^{-1}(0) = SK$

and $w(K;t)/SK = 0$

Conversely, note that $q(t) / \overline{S}K = 0 \iff q(t)$ is a multiple of $w(K;t)$, say $q(t) = w(K;t) \,.\, \lambda(t)$.

Then
$$\begin{aligned} q(t) \,.\, w'(t) &= \lambda(t) \,.\, w(K;t) \,.\, w'(t) \\ &= \lambda(t) \,.\, w(K;t) \,.\, w(I;t) \,.\, w(K';t) \\ &= \lambda(t) \,.\, w(t) \,.\, w(K';t) \end{aligned}$$

so that $q(t) \,.\, w'(t) = w(t) \,.\, q'(t)$ *holds with*
$$q'(t) = \lambda(t) \,.\, w(K';t).$$

7.3.1 The set of polynomials $q(t) \in H^*(B)[t]$ such that $q(t) / \overline{Z} = 0$ is clearly an ideal in $H^*(B)[t]$. Fadell-Husseini called it the index of the G-space Z [14]. Further, the set of polynomials $q(t)$ which satisfy $q(t).w'(t) = w(t).q'(t)$ for some $q'(t) \in H^*(B)[t]$ is also an ideal in $H^*(B)[t]$, denoted by $[w(t) : w'(t)]$.

The theorems we have proved in this chapter state that
$$\text{index}\ (Z) \subset [w(t) : w'(t)].$$

Now, consider the ring $H^*(B)\,[t, t^{-1}]$ of finite Laurent series, obtained from $H^*(B)[t]$, by inverting t. If $\dim(B) < \infty$ or E' is a bundle of finite type, then $(w'(t))^{-1} \in H^*(B)\,[t, t^{-1}]$.

Then we have in $H^*(B)[t, t^{-1}]$,
$$[w(t) : w'(t)] = (w'(t)^{-1} \,.\, w(t)\, H^*(B)[t]\,) \cap H^*(B)[t]$$
which is an intersection of two $H^*(B)$-free modules.

Clearly $[w(t):w'(t)]$ is independant of the map f. Hence the ideal $J = [w(t) : w'(t)]$ contains $Z(f)$ for all odd maps $f : SE \longrightarrow E'$

If we replace E and E' by $E \oplus F$ and $E' \oplus F$ respectively where F is any vector bundle over B, the relations $w(t) = w(E \oplus F;\ t) = w(E;t).w(F;t)$ and $w'(t) = w(E' \oplus F;\ t) = w(E';t).w(F;t)$ imply that J is unchanged. Hence we have that J contains $Z(\emptyset)$ for all odd maps $\emptyset : S(E \oplus F) \longrightarrow E' \oplus F$, for all F.

7.3.2 Note that in all cases the theorem 7.1.2 is a richer result than the corollaries that follow.

Let us first of all consider the case when $m \leqslant n$, where m = dim E and n = dim E'. Clearly in this case the corollaries do not give any information. However, we can use the result of the theorem to study some standard obstructions to immersions. Consider the following set up : Let M_1 and M_2 be manifolds of dimensions m_1 and m_2 respectively, $m_1 \leqslant m_2$. We would require that M_1 be paracompact. Let $g : M_1 \longrightarrow M_2$ be a differentiable map. We want to find a necessary condition for it to be an immersion. Let $dg : TM_1 \longrightarrow TM_2$ be the resultant map of tangent bundles. If g is an immersion,

$$dg_m = dg/TM_{1,m} : TM_{1,m} \longrightarrow TM_{2,g(m)} \quad , m \in M_1$$

is a monomorphism.

Define $\quad f' : TM_1 \longrightarrow g^*(TM_2)$, g^*TM_2 is the bundle induced by g by $f'(m,v) = (m,\ dg_m(v))$.

Then, f' is a monomorphism as dg_m is a monomorphism for every m. Now, consider STM_1, the sphere bundle of TM_1 and $f = f'/STM_1$: $STM_1 \longrightarrow g^*TM_2$. Clearly $Z = f^{-1}(0) = \emptyset$, i.e., $f: STM_1 \longrightarrow g^*TM_2 - (0)$.
As $Z = \emptyset$, $1/\bar{Z} = 0$
and hence, $w'(t) = w(t) \cdot q'(t)$

$$q'(t) = w'(t) \cdot w(t)^{-1} \in H^*(M_1)[t]$$

This means that the class $[w'(t).w(t)^{-1}] \in H^*(M_1)[t, t^{-1}]/ H^*(M_1)[t]$ is an obstruction for $g : M_1 \longrightarrow M_2$ to be an immersion. (The symbols $w(t)$, $w'(t)$, $q(t)$ etc., have obvious meanings analogous to those developped in this section). Apart from this, every $q(t)$ for which $q(t) \cdot w'(t) \cdot w(t)^{-1} \notin H^*(M_1)[t]$ gives a measure of how much every effort to immerse M_1 in M_2 will fail. —

In the case when m=n, if there is a G-map $SE \longrightarrow E' - (0)$, we shall have that $w(t) = w'(t)$.

Finally, if $m > n$ also, the theorem provides more information than the corollary. For example, consider E = the complex Hopf bundle over $B = \underline{C}P^1 \approx S^2$, and E' = the trivial line bundle. Then $m = 2 > n = 1$. Clearly it is impossible to have a relation of the form $t.w'(t) = w(t) \cdot q'(t)$ for any $q'(t) \in H^*(B)[t]$. Hence the theorem necessitates that $t/\overline{Z} \neq 0$, which is something the corollaries do not tell us. However, the theorem does not give any better estimate for the dimension of Z than the corollary.

8.0 BORSUK-ULAM THEOREM AND GENERALISED HOMOLOGY THEORIES

The purpose of this chapter is to give an introductory survey on how one could develop the theorems of the last chapter using generalised homology and cohomology theories like K-theory and cobordism theory. The motivation for doing this is, as suggested by Dold himself, widening the applicability of the results proved in these more generalised set ups.

It is clear from the work of the last chapter, that to obtain analogues of parametrized Borsuk-Ulam theorems in the context of generalised cohomology theories, we would have to develop the notions of characteristic classes and generalise the Lerray-Hirsch theorem. One could obtain details of this work in several works, notably Schwitzer [52]. However, we shall give a very brief account here for completeness and for making the reading smoother. After developping these preliminary notions, we shall give brief descriptions of K-theory and cobordism theory and using the notion of a Ring Spectrum, we shall explicitate the multiplicative cohomology structures inherited by them. We then indicate the construction of characteristic classes in the context of these theories. We conclude the section by proposing analogues of the results of the last chapter and indicating what problems and what new avenues this work opens.

8.1 Homology and Cohomology theories and characteristic classes.

8.1.1 Definition : Let $\mathcal{T}^{2'}$ be the homotopy category of topological pairs. On $\mathcal{T}^{2'}$ we have the restriction functor $R : \mathcal{T}^{2'} \longrightarrow \mathcal{T}^{2'}$

$R(X, A) = (A, \emptyset)$ and $R(f) = f/A$.

An unreduced homology theory h_* on $\mathcal{T}^{2\prime}$ is a sequence of functors $h_n : \mathcal{T}^{2\prime} \longrightarrow \mathcal{A}$ (the category of abelian groups) for each $n \in \underline{Z}$ and natural transformations $\partial_n : h_n \longrightarrow h_{n-1} \circ R$, $n \in \underline{Z}$, satisfying the following two axioms :

(i) Exactness : For every pair $(X, A) \in \mathcal{T}^{2\prime}$, the sequence

$$\ldots \longrightarrow h_{n+1}(X, A) \xrightarrow{\partial_{n+1}(X,A)} h_n(A, \emptyset) \xrightarrow{h_n[i]} h_n(X, \emptyset) \xrightarrow{h_n[j]} h_n(X,A) \longrightarrow \ldots$$

is exact, where $i : (A, \emptyset) \longrightarrow (X,\emptyset)$ and $j : (X,\emptyset) \longrightarrow (X,A)$ are inclusions.

(ii) Excision : For every pair (X,A) $\mathcal{T}^{2\prime}$ and subset $U \subset A$ with $\bar{U} \subset \overset{o}{A}$, the inclusion $j : (X-U, A-U) \longrightarrow (X, A)$ induces an isomorphism

$$h_n[j] : h_n(X-U, A-U) \longrightarrow h_n(X, A)$$

for all $n \in \underline{Z}$.

Now consider $\mathcal{PT}'$, the homotopy category of pointed topological spaces. On $\mathcal{PT}'$ we have the suspension functor $S : \mathcal{PT}' \longrightarrow \mathcal{PT}'$ defined by $S(X, x_o) = (SX , *)$ and $s([f]) = [Sf] = [1_{S'} \wedge f]$.

A reduced homology theory k_* on $\mathcal{PT}'$ is a collection of functors $k_n : \mathcal{PT}' \longrightarrow \mathcal{A}$ and natural equivalences $\sigma_n : h_n \longrightarrow h_{n+1} \circ S$, $n \in \underline{Z}$, satisfying the exactness axiom :

For every pointed pair (X, A, x_0) with inclusions $i: (A, x_0) \longrightarrow (X, x_0)$ and $j: (X, x_0) \longrightarrow (X \cup CA, *)$ where $X \cup CA$ is the mapping cone of $i : (A, x_0) \longrightarrow (X, x_0)$), the sequence

$$k_n(A, x_0) \xrightarrow{k_n[i]} k_n(X, x_0) \xrightarrow{k_n[j]} k_n(X \cup CA, *)$$

is exact.

We shall often impose on a reduced homology theory

(i) <u>the wedge axiom</u>

For every collection $\{(X_\alpha, x_\alpha) : \alpha \in \Lambda\}$ of pointed spaces, the inclusions $i_\alpha : X_\alpha \longrightarrow \bigvee_\beta X_\beta$ induce an isomorphism

$$\{i_{\alpha *}\} : \bigoplus_{\alpha \in \Lambda} k_n(X_\alpha) \longrightarrow k_n(\bigvee_{\alpha \in \Lambda} X_\alpha),\ n \in \underline{Z}.$$

(ii) <u>Weak Homotopy Equivalence Axiom (WHE axiom)</u>

If $f : X \longrightarrow Y$ is a weak homotopy equivalence, then

$$f_* : k_n(X, x_0) \longrightarrow k_n(Y, f(x_0))$$

is an isomorphism for all $n \in \underline{Z}$, $x_0 \in X$.

When we are dealing with CW-spaces (or with topological spaces with the reduced homology theory satisfying WHE axiom) it may be shown that there is a one-one correspondence between reduced and unreduced homology theories (cfr. [32], 7.35).

We also have the dual notion of cohomology theories. All that we have said about homology theories is true about cohomology theories. To avoid needless repetition, here we shall only define a reduced cohomology theory.

A <u>reduced cohomology theory</u> k^* on $\mathcal{PT}'$ is a collection of cofunctors $k^n : \mathcal{PT}' \longrightarrow \mathcal{A}$ and natural equivalences $\sigma^n : k^{n+1} \circ S \longrightarrow k^n$, $n \in \underline{Z}$ satisfying the exactness axiom :

For every pointed pair (X, A, x_o) with inclusions $i : (A, x_o) \longrightarrow (X, x_o)$, $j : (X, x_o) \longrightarrow (X \cup CA, *)$, the sequence

$$k^n(A, x_o) \xleftarrow{k^n[i]} k^n(X, x_o) \xleftarrow{k^n[j]} k^n(X \cup CA, *)$$

is exact.

8.1.2 In the context of limits, we have the following result for a cohomology theory, which we shall need later.

Let (X, x_o) be a CW-complex and let $X^o \subset X^1 \subset \ldots \subset X^n \subset \ldots \subset X$ be subcomplexes with $\bigcup_{n \geq 0} X^n = X$, $j_n^m : X^n \longrightarrow X^m$, $i_n : X^n \longrightarrow X$ the inclusions $n \geq m$. Then clearly $\{k^q(X^n), j_n^{m*}, \underline{N}\}$ is an inverse system for every $q \in \underline{Z}$.

For every $q \in \underline{Z}$, and for $x = (x_o, x_1, \ldots) \in \prod_n k^q(X^n)$

define $\delta : \prod_n k^q(X^n) \longrightarrow \prod_n k^q(X^n)$ component-wise

$$\text{as } \delta(x_n) = (-1)^n x_n + (-1)^{n+1} j_n^{n+1*}(x_{n+1}).$$

Note that $\ker\delta \subset \prod_n k^q(X^n)$ is the set of all x such that

$j_n^{n+1*}(x_{n+1}) = x_n$, $n \geq 0$. Hence by definition,

$$\ker\delta = \varprojlim k^q(X^n), \text{ the inverse limit of the system.}$$

Define $\lim^1 k^q(X^n) = \operatorname{coker}\delta$.

If now, k^* satisfies the wedge axiom on $\mathscr{PW}$ (the homotopy category of pointed CW-complexes), then there is an exact sequence —

$$0 \longrightarrow \lim^1 k^{q-1}(X^n) \longrightarrow k^q(X) \xrightarrow{\{i_n^*\}} \varprojlim k^q(X^n) \longrightarrow 0.$$

Thus, in cohomology, $\{i_n^*\} : k^q(X) \longrightarrow \varprojlim k^q(X^n)$ is an isomorphism if and only if $\lim^1 k^{q-1}(X^n) = 0$.

8.1.3 We shall also include here, the statement of the Lerray-Hirsch Theorem.

Let $p : (E,\dot{E}) \longrightarrow B$ be a fibration pair with B 0-connected and fibres $(F, \dot{F})$. Let h^* be a cohomology theory with products. Suppose $e_1, e_2, \ldots, e_r \in h^*(E,\dot{E})$ are elements such that $i^*e_1, i^*e_2, \ldots, i^*e_r \in h^*(F,\dot{F})$ form a free basis for $h^*(F,\dot{F})$ as a module over $h^*(pt.)$. Then $h^*(E,\dot{E})$ is a free $h^*(B)$ - module with generators $\{e_1, e_2, \ldots, e_r\}$ and the module action given by $be = p^*(b) \cup e$, for $b \in h^*(B)$.

8.1.4 Now we are equipped with all that we require for the discussion of characteristic classes. We start with the following theorem :

Theorem : Suppose h^* is a cohomology theory with products such that for each n there are elements $x_n \in h^2(\underline{C}P^n)$ satisfying

(i) $h^*(\underline{C}P^n) \cong h^*(pt)[x_n]/(x_n^{n+1})$

(ii) the inclusion $i : \underline{C}P^n \longrightarrow CP^{n+1}$ gives

$i^*(x_{n+1}) = x_n, \; n \geqslant 1.$

Then for each U(n)-bundle ξ over a CW-complex X, there are uniquely defined elements $c_o(\xi)$, $c_1(\xi)$, . . ., $c_n(\xi)$ with $c_i(\xi) \in h^{2i}(X)$ depending only on the isomorphism class of ξ and satisfying

a) if $\xi \longrightarrow X$ is a bundle and $f: Y \longrightarrow X$, a map, then

$c_i(f^*\xi) = f^*(c_i(\xi))$, $0 \leqslant i \leqslant n$

b) if $\gamma \longrightarrow \underline{C}P^n$ is the Hopf U(1)-bundle over $\underline{C}P^n$, then $c_1(\gamma) = x_n$

c) $c_o(\xi) = 1$ for all ξ

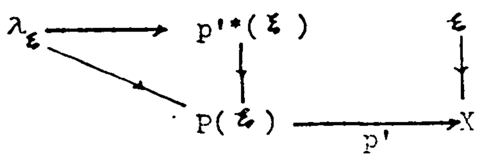
d) If ξ is a U(m)-bundle and η is a U(n)-bundle both over X, then

$$c_i(\xi \oplus \eta) = \sum_{j+k=i} c_j(\xi)\, c_k(\eta), \quad 0 \leqslant i \leqslant n+m$$

where $c_j(\xi) = 0$ if $j \geqslant m$, and $c_k(\eta) = 0$ if $k \geqslant n$.

Here we shall just describe how $c_i(\xi)$ are obtained, and omit the detailed proofs.

Consider $\underline{C}P^n$ as the space of all complex lines in $\underline{C}^{n+1}$, passing through O. Then $E(\gamma) = \{(1,y) \in \underline{C}P^n \times \underline{C}^{n+1} / y \in 1\}$.
Now consider the U(n)-bundle $\xi \longrightarrow X$, and let $P(\xi)$ be the space of all lines through O in all the fibres of ξ. Then we get a fibre bundle $p': P(\xi) \longrightarrow X$ with fibre $\underline{C}P^{n-1}$. Let λ_ξ be a line bundle over $P(\xi)$ where $E(\lambda_\xi) = \{(1,y) \in P(\xi) \times E(\xi) / y \in 1\}$. Then there is a monomorphism $\lambda_\xi \longrightarrow p'^*(\xi)$ of the bundles where $p'^*(\xi)$ is the induced bundle over $P(\xi)$. The following is a diagram of the situation :



Note that if $j : \underline{C}P^{n-1} \longrightarrow P(\xi)$ is the inclusion of a fibre, then $j^*(\lambda_\xi) = \gamma$. Let μ_ξ be the orthogonal complement of λ_ξ in $p'^*(\xi)$. Then $p'^*(\xi) \simeq \lambda_\xi \oplus \mu_\xi$. μ_ξ is a U(n-1)-bundle.

From the assumptions (i) and (ii) and 8.1.2, we obtain that $h^*(\underline{C}P^\infty) = h^*(pt)\,[[x_\infty]]$ where $x_\infty \in h^2(\underline{C}P^\infty)$ is the unique element such that $i_n^*(x_\infty) = x_n$, $i_n : \underline{C}P^n \longrightarrow \underline{C}P^\infty$ is the inclusion.

Considering BU(1) as $\underline{C}P^\infty$, let $f : P(\xi) \longrightarrow \underline{C}P^\infty$ be the classifying map for the line bundle λ_ξ. Thus we have that $f \circ j = i_{n-1}$ and so $j^*f^*(x_\infty) = x_{n-1}$. Let $y = f^*(x_\infty) \in h^2(P(\xi))$. Then $1, y, y^2, \ldots, y^{n-1}$ are elements in $h^*(P(\xi))$ s.t. $\{j^*1, j^*y, \ldots, j^*y^{n-1}\} = \{1, x_{n-1}, \ldots, x_{n-1}^{n-1}\}$ form a basis for $h^*(\underline{C}P^{n-1})$ over h^*(pt.).

The Lerray-Hirsch theorem (8.1.3) now applied to the fibration $\underline{C}P^{n-1} \longrightarrow P(\xi) \longrightarrow X$ says that $h^*(P(\xi))$ is a free $h^*(X)$-module generated by $\{1, y, \ldots, y^{n-1}\}$. Hence we can express y^n as a linear combination

$$y^n = (-1)^{n+1} c_n(\xi).1 + (-1)^n c_{n-1}(\xi).\, y + \ldots + c_1(\xi).\, y^{n-1},$$

for appropriate coefficients $c_1(\xi), c_2(\xi), \ldots, c_n(\xi) \in h^*(X)$.

We define $c_0(\xi) = 1$.

That these coefficients are uniquely determined, depend only on the isomorphism class of ξ and satisfy all the conditions (a), (b), (c) and (d) can be proved. (cfr. [32] 16.2)

For real bundles we have the following similar result :

Suppose h^* is a cohomology theory with products, such that for each $n \geq 1$, there are elements $x_n \in h^1(\underline{R}P^n)$ satisfying

(i) $h^*(\underline{R}P^n) = h^*(pt.)[x_n] / (x_n^{n+1})$

(ii) the inclusion $i : \underline{R}P^n \longrightarrow \underline{R}P^{n+1}$ gives $i^*(x_{n+1}) = x_n$.

Then for each O(n)-bundle $\xi \longrightarrow X$, over a CW-complex X, there are uniquely defined elements $w_0(\xi), w_1(\xi), \ldots, w_n(\xi)$ with $w_i(\xi) \in h^i(X)$, depending only on the isomorphism class of ξ and satisfying

a) $w_i(f^* \xi) = f^*(w_i(\xi))$ for all $f : Y \longrightarrow X$;

b) If $\gamma \longrightarrow \underline{R}P^n$ is the Hopf O(1)-bundle over $\underline{R}P^n$, then

$w_1(\gamma) = x_n$

c) $w_0(\xi) = 1$ for all ξ

d) $w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi).w_j(\eta)$ where ξ is a O(m)-bundle and η is a O(n)-bundle, both over X and $w_j(\xi) = 0$ if $j > m$ and $w_i(\eta) = 0$ if $i > n$.

We may have similar results for symplectic bundles.

Brown's representation Theorem showed that many of the important functors in Algebraic Topology were essentially homotopy functors. Before we conclude this section, we shall state this theorem which will prove to be essential in the later parts of the chapter.

8.1.5 Brown's Representation Theorem.

Let $h : \mathcal{PW}' \longrightarrow \mathcal{PS}$ be a contravariant functor from the homotopy category of pointed CW-complexes to the category of pointed sets, satisfying the following axioms :

(i) wedge axiom : If $\{X_\alpha\}_{\alpha \in \Lambda}$ be a family of objects of $\mathcal{PW}$ and $i_\alpha : X_\alpha \longrightarrow \bigvee_\alpha X_\alpha$ be the natural injection of all α, then $\{i^*_\alpha\} : h(\bigvee_\alpha X_\alpha) \longrightarrow \prod_\alpha (h(X_\alpha))$ is a bijection;

(ii) Mayer-Vietoris axiom : If (X, A_1, A_2) be a CW-triad of objects of $\mathcal{PW}$ and $x_1 \in h(A_1)$ and $x_2 \in h(A_2)$ are such that $x_1/ A_1 \cap A_2 = x_2/A_1 \cap A_2$, then there exists $x \in h(X)$ such that $x/A_1 = x_1$ and $x/A_2 = x_2$. (If $i_j: A_1 \cap A_2 \longrightarrow A_j$, $j=1,2$ are inclusions, then by $x_j/A_1 \cap A_2$ we mean $i_j^*(x_j) \in h(A_1 \cap A_2)$ etc.).

Then there exists a pointed CW-complex Y and a universal element $u \in h(Y)$ such that $T_u : [-, Y] \longrightarrow h(-)$ is a natural equivalence, where if $X \in \mathcal{PW}$, then $T_u : [X, Y] \longrightarrow h(X)$ is defined by $T_u([f]) = f^*(u)$.

(u is called a universal element if $T_u : [S^n, Y] \longrightarrow h(S^n)$ as defined above, is a bijection for every $n \geqslant 0$.)

In the above situation, we call Y, a classifying space for the functor h.

8.2 K-Theory and Cobordism Theory

In this section we intend to present the elements of K-theory and cobordism theory in the form of mainly definitions and results. We start with K-theory.

8.2.1 K - Theory

We say that two vector bundles ξ, η over the same base X (but not necessarily of the same dimension) are stably equivalent (notation $\xi \simeq_s \eta$) if and only if there are trivial bundles ϵ', ϵ'' over X with $\xi \oplus \epsilon' \simeq \eta + \epsilon''$. We denote by $\tilde{K}O$ (X) the set of all stable equivalence classes of real vector bundles over X. The stable equivalence class of ξ will be denoted by $\{\xi\}_s$.

By taking $\{\xi\}_s + \{\eta\}_s = \{\xi \oplus \eta\}_s$ we may introduce an 'addition' on $\tilde{K}O(X)$. The addition so defined is both associative and commutative. The class $\{\epsilon\}_s$ of all trivial bundles over X is the identity element. Further, if ξ is a vector bundle over a finite-dimensional CW-complex X, then there is a vector bundle η over X such that $\xi \oplus \eta \simeq \epsilon^N$ for some N. Hence for every $\{\xi\}_s \in \tilde{K}O$ (X), there exists $\{\eta\}_s \in \tilde{K}O(X)$ such that $\{\xi\}_s + \{\eta\}_s = \{\xi \oplus \eta\}_s = \{\epsilon\}_s$, so that $\tilde{K}O(X)$ is an abelian group when $X \in \mathcal{P}\mathcal{C}\mathcal{W}'_F$, the homotopy category of connected, pointed, finite CW-complexes. $\tilde{K}O(X)$ is called the reduced real K-group of X.

The natural inclusions $G_{k,\ k+n} \subset G_{k+1,k+n+1}$ of Grassmanian manifolds induce inclusions $BO(k) \subset BO(k+1)$ in the classifying spaces for vector bundles, for all $k \geqslant 1$. We let $BO = \bigcup_{k \geqslant 1} BO(k)$ with the weak topoloty. Let γ_k be the universal vector bundle over BO(k).

If X is a finite CW-complex, then X is compact and hence any map $f : X \longrightarrow BO$ factors through $BO(k)$ for some k.

If we define $T : [- ; BO,*] \longrightarrow \widetilde{KO}(-)$

by $T([f]) = \{f^*(\gamma_k)\}_s$

then T is a natural equivalence on $\mathcal{P}\ell\mathcal{W}_F'$.

In a similar fasion we can define $\widetilde{K}(X)$, the stable equivalence classes of complex vector bundles over X and $\widetilde{KSp}(X)$, the stable equivalence classes of quaternionic vector bundles over X, called the reduced K-group of X and the reduced symplectic K-group of X, respectively. We have also the natural equivalences :

$$[- ; BU,*] \longrightarrow \widetilde{K}(-)$$

$$[- ; BSp,*] \longrightarrow \widetilde{KSp}(-)$$

on $\mathcal{P}\ell\mathcal{W}_F'$, defined similarly.

8.2.2 Now, let $V_{\underline{R}}(X)$ denote the set of all isomorphism classes of real vector bundles over X, $X \in \mathcal{W}_F'$ (the homotopy category of finite CW-complexes). Define an addition on $V_{\underline{R}}(X)$ by taking $\{\xi\}+\{\eta\} = \{\xi\oplus\eta\}$ Since this addition is both commutative and associative, $V_{\underline{R}}(X)$ becomes an Abelian semi-group. Let $K(V_{\underline{R}}(X))$ be its group completion and denote it by $KO(X)$. The elements of $KO(X)$ may be denoted by $\{\xi\} - \{\eta\}$ where ξ and η are real vector bundles over X. Clearly KO is a contravariant functor on $\mathcal{W}_F'$.

Define a map $\psi : V_{\underline{R}}(X) \longrightarrow \tilde{K}O(X)$

$$\text{by } \psi(\{\xi\}) = \{\xi\}_s$$

Then by the universal property of group completion, we get a homomorphism $\theta : KO(X) \longrightarrow \tilde{K}O(X)$.

Further, we define $\varkappa : \tilde{K}O(X) \longrightarrow KO(X)$

$$\text{by } \varkappa(\{\xi\}_s) = \{\xi\} - \{\epsilon^n\} \text{ where } n = \dim \xi .$$

We see that $\theta \circ \varkappa = 1_{\tilde{K}O(X)}$

Let $(X, x_o) \in \mathcal{PLW}_F$. Consider the obvious maps

$$i : \{x_o\} \longrightarrow X$$

$$\text{and} \quad c : X \longrightarrow \{x_o\}$$

and the induced maps $i^* : KO(X) \longrightarrow KO(\{x_o\})$

$c^* : KO(\{x_o\}) \longrightarrow KO(X)$

which give a splitting for the exact sequence

$$0 \longrightarrow \tilde{K}O(X) \xrightarrow{\varkappa} KO(X) \underset{c^*}{\overset{i^*}{\rightleftarrows}} KO(\{x_o\}) \longrightarrow 0$$

Hence $KO(X) = \tilde{K}O(X) \oplus KO(\{x_o\})$

Since the elements of $V_{\underline{R}}(X)$ are determined by $\dim \xi$, hence $V_{\underline{R}}(\{x_o\}) = \underline{N}$.

so that $KO(\{x_o\}) = K(V_{\underline{R}}(\{x_o\})) = K(\underline{N}) = \underline{Z}$.

Thus, $KO(X) = \tilde{K}O(X) \oplus \underline{Z}$.

Therefore, we have a natural equivalence

$$[- ; \underline{Z} \times BO] \simeq KO(-)$$

Similar discussions can be carried out for $V_{\underline{C}}(X)$, the set of all equivalence classes of complex vector bundles over X, and $V_{\underline{H}}(X)$, the set of all equivalence classes of quaternionic vector bundles over X, and define

$$K(X) = K(V_{\underline{C}}(X)) \quad \text{and } KSp(X) = K(V_{\underline{H}}(X)).$$

We get natural equivalences

$$[- ; \underline{Z} \times BU] \simeq K(-)$$

$$[- ; \underline{Z} \times BSp] \simeq KSp (-).$$

In particular, we may compute the groups $\tilde{K}O(S^q)$, $\tilde{K}(S^q)$ and $\tilde{K}Sp(S^q)$ by using the above results, the Bott Periodicity theorem which states :

$$\underline{Z} \times BU \simeq \Omega^2 BU$$

$$\underline{Z} \times BO \simeq \Omega^4 BSp$$

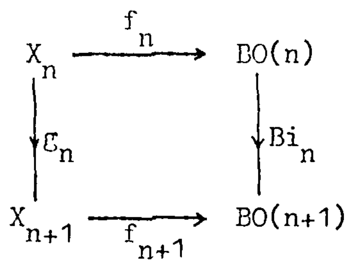
$$\underline{Z} \times BSp \simeq \Omega^4 BO$$

and some computations of the homotopy groups of BU, BO and BSp. The results may be tabulated as follows :

q(mod 8)	1	2	3	4	5	6	7	8
$KO(S^q)$	$\underline{Z}$	$\underline{Z}_2$	$\underline{Z}_2$	0	$\underline{Z}$	0	0	0
$K(S^q)$	$\underline{Z}$	0	$\underline{Z}$	0	$\underline{Z}$	0	$\underline{Z}$	0
$KSp(S^q)$	$\underline{Z}$	0	0	0	$\underline{Z}$	$\underline{Z}_2$	$\underline{Z}_2$	0

8.2.3 Cobordism Theory

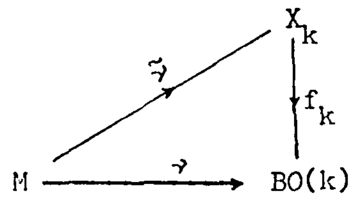
Consider a collection of spaces X_n and strictly commutative diagrams



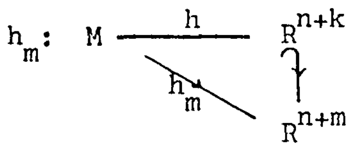
where f_n are fibrations. We shall denote such a system by X.

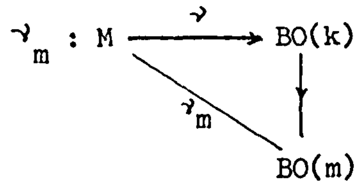
An X-structure on a smooth manifold M is a pair $(h, \tilde{\nu})$ where $h : M \longrightarrow \underline{R}^{n+k}$ is an embedding with $\nu : M \longrightarrow BO(k)$ the classification map of the normal bundle of the embedding h, and $\tilde{\nu} : M \longrightarrow X_k$ is a lifting of

i.e., $f_k \circ \tilde{\nu} = \nu$

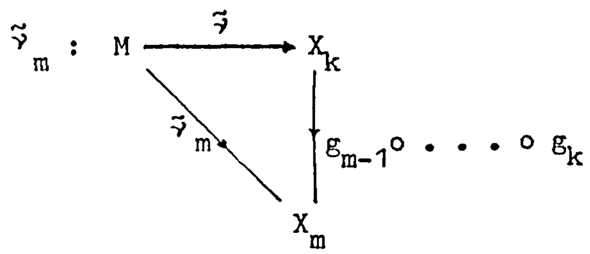


Given an X-structure $(h, \tilde{\nu})$ on M, we can define a sequence $(h_m, \tilde{\nu}_m)$ of X-structures as follows (note $m > k$) :





and



Two X-structures $(h, \tilde{\nu})$ and $(h', \tilde{\nu}')$ on M are equivalent (where

$h : M \longrightarrow \underline{R}^{n+k}$ and $h' : M \longrightarrow R^{n+k'}$ are embeddings) if there exists $k'' \geqslant \max (k, k')$ such that

$$h_{k''} = T \circ h'_{k''} \quad \text{where } T : \underline{R}^{n+k''} \longrightarrow \underline{R}^{n+k''} \text{ is some translation.}$$

and $\tilde{\nu}_{k''} \simeq \tilde{\nu}'_{k''}$ by a homotopy equivalence which covers

$$\nu_{k''} = \nu'_{k''} : M \longrightarrow BO(k'').$$

By an n-dimensional X-manifold we shall mean a manifold M of dimension n together with an equivalence class $\{(h, \tilde{\nu})\}$ of X-structures on M.

If $(W, \partial W)$ is an n-dimensional manifold with boundary and if $(h, \tilde{\nu})$ is an X-structure on W where

$$\begin{array}{ccc} \partial W & \xrightarrow{\ h / \partial W\ } & R^{n+k-1} \\ \downarrow i & & \downarrow \\ W & \xrightarrow{\ h\ } & R^{n+k} \end{array}$$

then $(h/\partial W, \tilde{\nu} \circ i)$ gives an X-structure on ∂W called the X-structure induced by $(h, \tilde{\nu})$.

A map of X-manifolds is a smooth map $f : M \longrightarrow M'$ between manifolds M and M' with X-structures $(h, \tilde{\nu})$ and $(h', \tilde{\nu}')$ respectively such that $h' \circ f = T \circ h$ for some translation T and $\tilde{\nu}' \circ f \simeq \tilde{\nu}$ (homotopy over $\nu = \nu' \circ f$).

The empty set $\emptyset$ will be regarded as an n-dimensional smooth manifold for all $n \geqslant 0$, having a unique vacuous X-structure.

Two n- dimensional X-manifolds $(M_1, \{(h_1, \tilde{\nu}_1)\})$ and $(M_2, \{(h_2, \tilde{\nu}_2)\})$

are said to be X-cobordant, (written $M_1 \sim_X M_2$) if there exists an X-manifold $((W, \partial W), (\hat{h}, \hat{\vec{\nu}}))$ with boundary, of dimension n+1, such that $(\partial W, \{(h/\partial W, \hat{\vec{\nu}} \circ i)\})$ is X-diffeomorphic to $(M_1 \sqcup M_2, \{(h_1, \vec{\nu}_1)\} \sqcup \{(h_2, \vec{\nu}_2)\})$.

Note that $\sim_X$ is an equivalence relation. We shall write Ω_n^X for the set of all X-cobordism classes of closed n-dimensional X-manifolds, $n \geqslant 0$. The operation of taking disjoint unions $\sqcup$ defines a binary operation on Ω_n^X and it constitutes Ω_n^X into an abelian group $n \geqslant 0$, with the class $\{\emptyset\}$ as the zero-element.

<u>The Unoriented cobordism group Ω_*^O</u>

If we take $X = \{BO(0), BO(1), \ldots, BO(k), \ldots\}$

and $g_k : Bi_k : BO(k) \longrightarrow BO(k+1)$

$f_k : BO(k) \longrightarrow BO(k)$, the identity map,

then Ω_*^X is denoted by Ω_*^O, and is called the unoriented cobordism group.

<u>The Oriented cobordism Group Ω_*^{SO}</u>

Take $X = \{BSO(0), BSO(1), \ldots, BSO(k), \ldots\}$

$g_k = BSi_k : BSO(k) \longrightarrow BSO(k+1)$, $Si_k : SO(k) \longrightarrow SO(k+1)$

and $f_k : BSO(k) \xrightarrow[Bj_k]{} BO(k)$, $j_k : SO(k) \longrightarrow O(k)$.

Then Ω_*^X is denoted by Ω_*^{SO} or simply Ω_* and is called the oriented cobordism group.

The Complex or Unitary Cobordism Group Ω_*^U

$X = \{X_o, X_1, X_2, \ldots .\}$

is given by $X_{2k} = BU(k) = X_{2k+1}$

$\varepsilon_{2k} : X_{2k} \longrightarrow X_{2k+1}$ is the identity map

$\varepsilon_{2k+1} : X_{2k+1} \longrightarrow X_{2k+2}$ is Bj_k where $j_k : U(k) \longrightarrow U(k+1)$

$f_{2k} = f_{2k+1} : BU(k) \longrightarrow BO(2k)$ are the classifying maps for the universal bundle $\gamma_k \longrightarrow BU(k)$, considered as a real vector bundle. Ω_*^X denoted by Ω_*^U is called the complex or the Unitary cobordism group.

In an analogous fashion, if we take $X_{2k} = BSU(k) = X_{2k+1}$ we get the special Unitary cobordism group Ω_*^{SU}. On the other hand if we take $X_{4k} = X_{4k+1} = X_{4k+2} = X_{4k+3} = BSp(k)$, we get the simplectic cobordism group Ω_*^{Sp}.

8.3 We turn our attention now to see how generalised homology and cohomology theories may be defined using the K-theory and cobordism theory. To do this, we need notions of spectra, and products of spectra. We proceed now to develop these ideas.

8.3.1 Spectra

A spectrum E is a system $\{X_n, \varepsilon_n\}$ where each $X_n \in \mathcal{PW}'$ (the homotopy category of pointed CW-complexes) and $\varepsilon_n : SX_n \longrightarrow X_{n+1}$ is a homotopy equivalence into a subcomplex of X_{n+1}.

For example, given a space X, we have the suspension spectrum $E(X)$ given by $E(X)_n = \begin{cases} *, & n < 0 \\ S^n(X), & n \geqslant 0 \end{cases}$

where the identity maps $S^n(X) \longrightarrow S^n(X)$ serve as the map ε_n, $n \geqslant 0$.

The sphere spectrum $S^o = \{S^o, S(S^o), S^2(S^o), \ldots\}$ is a particular example of the suspension spectrum.

An Ω- spectrum is a spectrum $\{X_n, \varepsilon_n\}$ such that the adjoint $\varepsilon'_n : X_n \longrightarrow \Omega X_{n+1}$ of the inclusion $\varepsilon_n : SX_n \longrightarrow X_{n+1}$ (where ΩX_{n+1} is the loop space of X_{n+1}) is a weak homotopy equivalence for every n.

8.3.2 Spectra and associated homology and cohomology theories

We can now define the reduced homology and cohomology theories associated with any spectrum $E = \{E_n, \varepsilon_n\}$.

Define $E_n(X) = \Pi_n(E \wedge X) = [\Sigma^n(S^o), E \wedge X] \overset{def}{=} \lim\limits_{k \to \infty} [S^{n+k}, (E \wedge X)_k]$

and $E^n(X) = [E(X), \Sigma^n E] = [\Sigma^{-n}(S^o) \wedge X, E] \overset{def}{=}$

$\lim\limits_{k \to \infty} [S^k \wedge X, E_{n+k}]$

For $f : (X, x_o) \longrightarrow (Y, y_o)$, we take

$E_n(f) = (1 \wedge f)_*$

and $E^n(f) = E(f)^*$

(Note that ΣE is the spectrum defined by $\Sigma E_n = E_{n+1}$).

We define $\sigma_n : E_n(X) \longrightarrow E_{n+1}(SX)$ to be the composite :

$$E_n(X) = [\Sigma^n(S^o),\ E \wedge X] \xrightarrow[\cong]{\Sigma} [\Sigma^{n+1}(S^o),\ \Sigma E \wedge X] \xrightarrow[\cong]{} [\Sigma^{n+1}(S^o),\ E \wedge S^1 \wedge X] = E_{n+1}(SX).$$

σ_n is a natural equivalence.

We define $\sigma^n : E^{n+1}(SX) \longrightarrow E^n(X)$ to be the composite :

$$E^{n+1}(SX) = [E(SX),\ \Sigma^{n+1}E] \xleftarrow[\cong]{i^*} [\Sigma E(X),\ \Sigma^{n+1}E] \xrightarrow[\cong]{\Sigma^{-1}} [E(X),\ \Sigma^n E] = E^n(X),$$

where i^* is induced by $i : E(SX) \longrightarrow \Sigma E(X)$,

and σ^n is a natural equivalence. ([32], 8.33)

Let (X, A, x_o) be a CW-pair. Then the sequence

$$[\Sigma^n(S^o),\ E \wedge A] \xrightarrow{(1 \wedge i)_*} [\Sigma^n(S^o),\ E \wedge X] \xrightarrow{(1 \wedge j)_*} [\Sigma^n(S^o), E \wedge (X \cup CA)]$$

is exact. I.e., the sequence

$$E_n(A) \xrightarrow{i_*} E_n(X) \xrightarrow{j^*} E_n(X \cup CA) \text{ is exact.}$$

Also the sequence

$$[E(A),\ \Sigma^n E] \xleftarrow{E(i)^*} [E(X),\ \Sigma^n E] \xleftarrow{E(j)^*} [E(X \cup CA),\ \Sigma^n E]$$

is exact. I.e., the sequence

$$E^n(A) \xleftarrow{i^*} E^n(X) \xleftarrow{j^*} E^n(X \cup CA) \text{ is exact.}$$

Thus, by the definitions 8.1.1, E_* is a homology theory on $\mathcal{PW}'$ and E^* is a cohomology theory on $\mathcal{PW}'$.

The coefficient groups of the homology theory E_* are

$$E_n(S^o) = [\Sigma^n(S^o),\ E \wedge S^o] = \pi_n(E \wedge S^o) = \pi_n(E)$$

and the coefficient groups of the cohomology theory E^* are

$$E^n(S^o) = [\Sigma^{-n}(S^o) \wedge S^o,\ E] = [\Sigma^{-n}(S^o),\ E] = \pi_{-n}(E).$$

For any spectrum E, since E^* satisfies the wedge axiom, we have that for any filtration $\{X^n\}$ of a CW complex X, we have the following exact sequence :

$$0 \longrightarrow \lim{}^1\ E^{q-1}(X^n) \longrightarrow E^q(X) \xrightarrow{\{i_n{}^*\}} \varprojlim\ E^q(X^n) \longrightarrow 0$$

Further, if E is an Ω- spectrum, then for every pointed CW-complex $(X,\ x_o)$, we have a natural isomorphism $E^n(X) = [X, x_o;\ E_n, *]$.

8.3.3 Products on Spectra

To obtain a product structure on homology and cohomology defined using spectra, we would like to define products of spectra.

The Smash product of two spectra E and F is denoted by $E \wedge F$. The complexity of its construction precludes it's inclusion here. However, we shall look at what is called the naive smash product, because the smash product, in most of the important cases that we consider, is equivalent to the naive smash product.

For any $B \subset \underline{N}$ (or any ordered set, order isomorphic to $\underline{N}$), we have a monotonic function $\beta : \underline{N} \longrightarrow \underline{N}$, defined by $\beta(n)$ = the number of elements $b \in B$ with $b < n$. In particular we have $\alpha : \underline{N} \longrightarrow \underline{N}$ associated with the inclusion $\underline{N} \subset \underline{N}$. Let $\underline{N}$ be now partitioned by

B and C, i.e., $B \cup C = \underline{N}$ and $B \cap C = \emptyset$. We then have maps β and $\gamma : \underline{N} \longrightarrow \underline{N}$, associated with the inclusions $B \subset \underline{N}$ and $C \subset \underline{N}$. We define the naive smash product

$$E \wedge_{BC} F \quad \text{by} \quad (E \wedge_{BC} F)_{\alpha(n)} = E_{\beta(n)} \wedge F_{\gamma(n)}, \quad n \in \underline{N}.$$

We state below the conditions under which $E \wedge_{BC} F \cong E \wedge F$.

(i) If B and C are infinite.

(ii) If B has d elements and $SE_r = E_{r+1}$, $\forall\, r \geqslant d$.

(iii) If C has d elements and $SF_r = F_{r+1}$, $\forall\, r \geqslant d$.

In particular, let F be any spectrum and let X be a CW-complex. Then $F \wedge E(X) = F \wedge_{\underline{N}\,\emptyset} E(X) = F \wedge X$ where E(X) is the suspension spectrum of X.

8.3.4 Ring Spectra and Homology and Cohomology Products

Now that we have defined the smash product of spectra, we are equipped to define ring spectra.

A ring spectrum E is a spectrum with a product $\mu : E \wedge E \longrightarrow E$ (a map of spectra) and an identity $i : S^o \longrightarrow E$ such that the following diagrams are homotopy commutative.

(i)

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{\mu \wedge 1} & E \wedge E \\ {\scriptstyle 1 \wedge \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ E \wedge E & \xrightarrow{\mu} & E \end{array}$$

(ii)

$$\begin{array}{ccccc} E \wedge S^o & \xrightarrow{1 \wedge i} & E \wedge E & \xleftarrow{i \wedge 1} & S^o \wedge E \\ & {\scriptstyle r}\,\cong \searrow & \downarrow{\scriptstyle \mu} & \swarrow \cong\,{\scriptstyle l} & \\ & & E & & \end{array}$$

Note that l is the homotopy equivalence

$$S^o \wedge E \cong S^o \wedge_{\emptyset} \underline{N} E \cong E$$

and r is the homotopy equivalence

$$E \wedge S^o \cong E \wedge_{\underline{N}} \emptyset S^o \cong E$$

is said to be commutative if

(iii)
$$\begin{array}{ccc} E \wedge E & \xrightarrow{\mu} & E \\ \tau \downarrow & \nearrow_{\mu} & \\ E \wedge E & & \end{array}$$

is commutative up to homotopy.

Given a ring spectrum (E, μ, i) We can introduce external products in the homology $E_*(X)$ and the cohomology $E^*(X)$ as follows :

(i) $\wedge \ : \ E_n(X) \otimes E_m(Y) \longrightarrow E_{m+n}(X \wedge Y)$

(ii) $\wedge \ : \ E^n(X) \otimes E^m(Y) \longrightarrow E^{m+n}(X \wedge Y)$

(iii) $/ \ : \ E^p(X \wedge Y) \otimes E_q(Y) \longrightarrow E^{p-q}(X)$

(iv) $\backslash \ : \ E^p(X) \otimes E_q(X \wedge Y) \longrightarrow E_{p-q}(Y)$

The definitions of these products are quite natural. We shall indicate here the construction of

$$\wedge \ : \ E_n(X) \otimes E_m(Y) \longrightarrow E_{n+m}(X \wedge Y)$$

Let $z \in E_n(X)$ and $y \in E_m(Y)$ be represented by $f: S^n \longrightarrow E \wedge X$ and $g : S^m \longrightarrow E \wedge Y$ respectively. We take $x \wedge y \in E_{n+m}(X \wedge Y)$ to be represented by the composite

$$S^{n+m} \simeq S^n \wedge S^m \xrightarrow{f \wedge g} (E \wedge X) \wedge (E \wedge Y) \xrightarrow{1 \wedge \tau \wedge 1} E \wedge E \wedge X \wedge Y \xrightarrow{\mu \wedge 1} E \wedge X \wedge Y.$$

(For the definitions of the other products, cfr. [32], 13.50).

For CW-pairs (X,A), (Y,B) we have the relative groups $E_*(X,A)$ etc., and corresponding external products which result from the fact that $(X/A) \wedge (Y/B) \cong (X \times Y)/(X \times B \cup A \times Y)$.

Thus we get products

(i) $\times$: $E_n(X,A) \otimes E_m(Y,B) \longrightarrow E_{n+m}(X \times Y, X \times B \cup A \times Y)$

(ii) $\times$: $E^n(X,A) \otimes E^m(Y,B) \longrightarrow E^{n+m}(X \times Y, X \times B \cup A \times Y)$

(iii) $/$: $E^p(X \times Y, X \times B \cup A \times Y) \otimes E_q(Y,B) \longrightarrow E^{p-q}(X,A)$

(iv) $\backslash$: $E^p(X,A) \otimes E_q(X \times Y, X \times B \cup A \times Y) \longrightarrow E_{q-p}(Y,B)$

Now that we have external products, we can use the diagonal map to define internal products $\cup$ and $\cap$.

Regard Δ as the map $\Delta : (X, A \cup B) \longrightarrow (X \times X, X \times B \cup A \times X) = (X,A) \times (X,B)$

for pairs (X,A, (X,B).

$\cup$ - product is defined as the composite

$$E^n(X,A) \otimes E^m(X,B) \xrightarrow{\times} E^{n+m}((X,A) \times (X,B)) \xrightarrow{\Delta^*} E^{n+m}(X, A \cup B)$$

and $\cap$-product is defined as the composite

$$E^n(X,A) \otimes E_m(X, A \cup B) \xrightarrow{1 \times \Delta_*} E^n(X,A) \otimes E_m((X,A) \times (X,B)) \xrightarrow{\backslash} E_{m-n}(X,B)$$

8.3.5 Sufficient Condition for the Existence of Ring Spectrum Structure on a Spectrum.

In the following theorem we state a sufficient condition for the existence of a ring spectrum structure on a spectrum. This will enable us later, to equip certain spectra we are concerned with, with a ring spectrum structure.

Theorem : Let E be a spectrum and $\bar{i}_n : S^n \longrightarrow E_n$, and $\bar{\mu}_{nm} : E_n \wedge E_m \longrightarrow E_{n+m}$ be systems of maps such that the following diagrams are homotopy commutative for every m, n, p :

$$\text{(i)} \quad \begin{array}{ccc} S^1 \wedge S^n & \xrightarrow{1 \wedge \bar{i}_n} & S^1 \wedge E_n \\ \cong & & \downarrow \epsilon_{1,n} \\ S^{n+1} & \xrightarrow{\bar{i}_{n+1}} & E_{n+1} \end{array}$$

$$\text{(ii)} \quad \begin{array}{ccccc} S^n \; E_m & \xrightarrow{\bar{i}_n \wedge 1} & E_n \wedge E_m & \xleftarrow{1 \wedge \bar{i}_m} & E_n \wedge S^m \\ & \searrow^{\epsilon_{n,m}} & \downarrow \bar{\mu}_{m,n} & & \downarrow \tau \\ & & E_{n+m} & \xleftarrow{\epsilon_{m,n}} & S^m \wedge E_n \end{array}$$

$$\text{(iii)} \quad \begin{array}{ccc} E_n \wedge E_m \wedge E_p & \xrightarrow{1 \wedge \bar{\mu}_{mp}} & E_n \wedge E_{m+p} \\ \downarrow \bar{\mu}_{nm} \wedge 1 & & \downarrow \bar{\mu}_{n,\, m+p} \\ E_{n+m} \wedge E_p & \xrightarrow{\bar{\mu}_{n+m,p}} & E_{n+m+p} \end{array}$$

where $\epsilon_{nm} : S^n \wedge E_m \longrightarrow E_{n+m}$ is the inclusion.

Suppose further that $\lim^1 \tilde{E}^{n-1}(E_n) = \lim^1 \tilde{E}^{n-1}(E_{\beta(n)} \wedge E_{\gamma(n)})$

$$= \lim^1 \tilde{E}^{n-1}(E_{\beta(n)} \wedge E_{\gamma(n)} \wedge E_{j(n)})$$

$$= 0$$

for any partition $\underline{N} = B \cup C$ or $\underline{N} = B \cup C \cup D$.

Then there is a map $i : S^o \longrightarrow E$ and for every partition $\underline{N} = B \cup C$ with B and C infinite, a map

$$\mu_{BC} : \begin{array}{c} E \wedge_{BC} E \\ \wr\| \\ E \wedge E \end{array} \longrightarrow E$$

such that the homotopy class $[\mu] = [\mu_{BC}]$ is indépendant of B, C and such that the diagrams

$$\text{(i)} \quad \begin{array}{ccc} S^n & \longrightarrow & \Sigma^n S^o \\ \downarrow \bar{i}_n & & \downarrow \Sigma^n i \\ E_n & \hookrightarrow & \Sigma^n E \end{array}$$

$$\text{(ii)} \quad \begin{array}{ccccc} E_{\beta(n)} \wedge E_{\gamma(n)} & \hookrightarrow & \Sigma^n(E \wedge_{BC} E) & \xrightarrow{\cong} & \Sigma^n(E \wedge E) \\ \downarrow \bar{\mu}_{\beta(n)\ \gamma(n)} & & & & \downarrow \Sigma^n \mu \\ E_n & & \hookrightarrow & & \Sigma^n(E) \end{array}$$

commute upto homotopy, making (E, μ, i) a ring spectrum.

If moreover,

$$\begin{array}{ccc} E_{4n} \wedge E_{4m} & \xrightarrow{\tau} & E_{4m} \wedge E_{4n} \\ & \bar{\mu}_{4n,4m} \searrow \quad \swarrow \bar{\mu}_{4m,4n} & \\ & E_{4n+4m} & \end{array}$$

is homotopy commutative, then (E, μ, i) is a commutative ring spectrum.

8.3.6 We turn our attention now to the construction of spectra for K-theory and cobordism theory, and equip them with a ring spectrum structure using 8.3.5.

The following two Ω-spectra are the K-theory and KO-theory spectra respectively.

(i) $K : \ldots K_0 \quad K_1 \quad K_2 \quad K_3 \quad K_4 \ldots$

$$\begin{array}{ccccc} K_0 & K_1 & K_2 & K_3 & K_4 \\ \| & \| & \| & \| & \| \\ \underline{Z} \times BU & \Omega BU & \underline{Z} \times BU & \Omega BU & \underline{Z} \times BU \end{array} \ldots$$

with $\epsilon'_{2i} : K_{2i} \longrightarrow \Omega K_{2i+1}$

$\underline{Z} \times BU \longrightarrow \Omega^2 BU$, the homotopy equivalence of the Bott Periodicity Theorem (8.2.2)

and $\epsilon'_{2i+1}: K_{2i+1} \longrightarrow \Omega K_{2i}$

$\Omega BU \longrightarrow \Omega(\underline{Z} \times BU) \cong \Omega BU$, the identity map.

(ii) $KO : \ldots KO_0 \quad KO_1 \quad KO_2 \quad KO_3 \quad KO_4 \quad KO_5$

$$\begin{array}{cccccc} KO_0 & KO_1 & KO_2 & KO_3 & KO_4 & KO_5 \\ \| & \| & \| & \| & \| & \| \\ \underline{Z} \times BO & \Omega^3 BSp & \Omega^2 BSp & \Omega BSp & \underline{Z} \times BSp & \Omega^3 BO \end{array}$$

$$\begin{array}{ccc} KO_6 & KO_7 & KO_8 \\ \| & \| & \| \\ \Omega^2 BO & \Omega BO & \underline{Z} \times BO \end{array} \ldots$$

with $\epsilon'_i : K_i \longrightarrow \Omega K_{i+1}$ given by the Bott Periodicity Theorem if $i \equiv 0 \pmod 8$ or $4 \pmod 8$.

and $\epsilon'_i : K_i \longrightarrow K_{i+1}$, the identity map otherwise.

We then have homology and cohomology theories K_*, KO_* and K^*, KO^* defined by these spectra. Since these are -spectra, we shall have :

$$K^q(X) = [X; K_q] \quad \text{and} \quad KO^q(X) = [X; KO_q]$$

$$\tilde{K}^q(X) = [X,x_o; K_q,*] \quad \text{and} \quad \tilde{KO}^q(X) = [X,x_o; KO_q,*]$$

In particular $\tilde{KO}^o(X) = [X,x_o; \underline{Z} \times BO, *] = \tilde{KO}(X)$

and $\tilde{K}^o(X) = [X,x_o; \underline{Z} \times BU, *] = \tilde{K}(X)$.

Since $\Sigma^2 K = K$ and $\Sigma^8 KO = KO$, the coefficient groups $K^q(S^o)$ have period 2 and the coefficient groups $KO^q(S^o)$ have period 8. These appear in the table at the end of 8.2.2.

These spectra are equipped with a ring-spectrum structure as follows : Let F be $\underline{R}$ or $\underline{C}$. Given two F-vector bundles, $\xi \longrightarrow X$ $\eta \longrightarrow Y$, of dimensions m and n respectively, we have the tensor product bundle $\xi \otimes_F \eta \longrightarrow X \times Y$ of dimension mn with fibre $\xi_x \otimes_F \eta_y$ over $(x,y) \in X \times Y$.

If X is a finite CW-complex, the map

$$V_{\underline{C}}(X) \times V_{\underline{C}}(Y) \longrightarrow V_{\underline{C}}(X \times Y)$$

defined by $(\{\xi\}, \{\eta\}) \longmapsto \{\xi \otimes_{\underline{C}} \eta\}$ induces a unique map

$\times : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$, which is natural, associative and commutative.

From x, we also get a product

$$\wedge : \tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \wedge Y)$$

for pointed, finite CW-complexes X,Y. $\wedge$ is natural, associative and commutative.

To extend x and $\wedge$ to infinite complexes and spectra, we note first of all, that as $BU = \bigcup_{k \geqslant 1} BU(k)$ and $BU(k) = \bigcup_{n \geqslant k} G_{k,n}(\underline{C})$, we can see that $BU \cong \bigcup_{n \geqslant 1} G_{n,2n}(\underline{C})$. Let us denote $G_{n,2n}$ by G_n. Let ξ_n be the universal bundle and ϵ^n, the trivial bundle over G_n. Let $u_n = \{\xi_n\}_s - \{\epsilon^n\}_s \in \tilde{K}(G_n)$. If $i_n : G_n \longrightarrow G_{n+1}$ is the inclusion, we shall have that $i^*_n\,(u_{n+1}) = u_n$. Hence $\{u_n\}$ is an element of $\varprojlim \tilde{K}(G_n)$. This is in fact, the class $u = [1_{BU}] \in [BU; BU] = \tilde{K}\,(BU)$. The exactness of the sequence

$$0 \longrightarrow \lim{}^1 \tilde{K}^{-1}(G_n \wedge G_n) \longrightarrow \tilde{K}(BU \wedge BU) \longrightarrow \varprojlim \tilde{K}(G_n \wedge G_n) \longrightarrow 0$$

implies that there is an element $u \wedge u \in \tilde{K}(BU \wedge BU)$ such that $u \wedge u / G_n \wedge G_n = u_n \wedge u_n$ for all $n \geqslant 1$. This element is unique as $\lim^1 \tilde{K}^{-1}(G_n \wedge G_n) = 0$. Call this element μ. Thus $\mu : BU \wedge BU \longrightarrow BU$.

Now, if we define

$$\mu_{2n,2m} : K_{2n} \wedge K_{2m} \longrightarrow K_{2n+2m}$$

to be $\quad \mu : BU \wedge BU \longrightarrow BU$

and $\quad i_{2n} : S^{2n} \longrightarrow K_{2n}$

to be $S^{2n} \cong S^2 \wedge S^{2n-2} \xrightarrow[i_2 \wedge i_{2n-2}]{} BU \wedge BU \xrightarrow{\mu} BU$

(where $i_2 : S^2 \longrightarrow BU$ is the classifying map of $\{\eta\}_s - \{1\}_s \in K(S^2)$, $\eta \longrightarrow S^2 \cong \underline{C}P^1$ is the Hopf line bundle and $1 \longrightarrow S^2$, the trivial line bundle), for $n \geqslant 2$, then the conditions of 8.3.5 may be seen to be satisfied. We thus get a commutative ring spectrum

(K, μ, i) such that the product $K(X) \otimes K(Y) \longrightarrow K(X \times Y)$ is induced by the tensor product of bundles.

In an analogous fashion, the spectrum KO may also be equipped with a ring spectrum structure. (cfr. [32], 8. 91)

8.3.7 Generalised homology and cohomology theories using cobordism theory are constructed using what are known as Thom Spectra.

Let $X \in \mathcal{W}_F$ (the category of finite CW-complexes). Let $\xi = p : E(\xi) \longrightarrow X$ be a vector bundle over X of dimension n. We define $M(\xi) = \dfrac{E(D(\xi))}{E(S(\xi))}$ where $D(\xi)$ and $S(\xi)$ are respectively the disc bundle and the sphere bundle associated with ξ, and call it the Thom space of ξ.

Let $\pi : E(D(\xi)) \longrightarrow M(\xi)$ be the quotient map. If $f : \xi \longrightarrow \eta$ be a morphism of O(n) bundles, then this induces a map of $D(\xi)$ into $D(\eta)$ which takes $S(\xi)$ into $S(\eta)$. Thus we get a map $M(f) : M(\xi) \longrightarrow M(\eta)$. We shall have $M(g \circ f) = M(g) \circ M(f)$ and $M(1_\xi) = 1_{M(\xi)}$.

If ξ_1 and ξ_2 are bundles over X_1 and X_2 respectively, then it may be proved that $M(\xi_1 \times \xi_2) \simeq M(\xi_1) \wedge M(\xi_2)$. In particular if ϵ^n is the trivial n-bundle over X, we shall have that

$M(\xi \oplus \epsilon^n) \simeq S^n(M(\xi))$.

Now, consider a system $X = \{X_k, f_k, g_k\}$ as considered in 8.2.3. Let γ_k be the universal k-dimensional vector bundle $\gamma_k \longrightarrow BO(k)$ and let $\gamma_k(X) = f_k^*(\gamma_k)$ be the induced bundle over X_k, $\forall$ k. So we get Thom Spaces $M(\gamma_k(X))$ and maps of Thom spaces $M(f_k)$: $M(\gamma_k(X)) \longrightarrow M(\gamma_k)$. (We shall agree to denote $M(\gamma_k(X))$ and $M(\gamma_k)$ by MX_k and MO_k respectively).

Since $g_k^*(\gamma_{k+1}(X)) \cong \gamma_k(X) \oplus \epsilon^1$, g_k will induce a bundle map from $\gamma_k(X) \oplus \epsilon^1 \longrightarrow \gamma_{k+1}(X)$. Hence we get a map of Thom spaces

$$\epsilon_k : M(g_k) : SMX_k \longrightarrow MX_{k+1}.$$

Thus we get a spectrum $MX = \{MX_k, \epsilon_k\}$ called the Thom spectrum defined by the system $X = \{X_k, f_k, g_k\}$. In particular, if we consider in order the five systems we considered in 8.2.3, we get respectively— the Thom spectra MO, MSO, MU, MSU and MSp.

As soon as one has a spectrum, one gets homology and cohomology theories. Thus we get bordism theories $MX_*(-)$ and the cobordism theories $MX^*(-)$. A theorem of Thom states that $\Omega_*^X = \pi_*(MX)$. This result enables us to compute the coefficient groups of these homology and cohomology theories for the various Thom spectra.

To obtain ring spectrum structure on the above spectra, we proceed as follows :

For G = O, SO, U, SU and Sp, we have maps

$$\nu_{nm} : BG(n) \times BG(m) \longrightarrow BG(n+m)$$

such that if ξ_n is the universal G(n)-bundle over BG(n), then $\nu_{nm}{}^* \xi_{n+m} \simeq \xi_n \times \xi_m$. The induced maps of Thom complexes then have the form

$$\bar{\mu}_{nm} = M(\nu_{nm}) : MG(n) \wedge MG(m) \cong M(\xi_n \times \xi_m) \longrightarrow$$
$$M(\xi_{n+m}) = MG(n+m).$$

Further, the inclusion i : * $\longrightarrow$ BG(n), induces

$$\bar{i}_n : S^n \cong M(\varepsilon^n) \longrightarrow MG(n).$$

These maps satisfy the requirements of theorem 8.3.5. Thus these spectra are ring spectra, and products can be defined in the homology and cohomology theories defined by these spectra.

8.4 In this section, we shall define a complex orientation, and see that a ring spectrum equipped with a complex orientation satisfies the conditions of theorem 8.1.4. This will immediately give us characteristic classes in the setting of the generalised cohomology theories that we have considered in this chapter.

Let E be a ring spectrum. An element $y_E \in \tilde{E}^*(\underline{C}P^\infty)$ is said to be a complex orientation of E if the inclusion $i : \underline{C}P^1 \longrightarrow \underline{C}P^\infty$ sends y_E to a generator $i^*(y_E) \in \tilde{E}^*(\underline{C}P^1) = E^*(S^2)$ over $E^*(S^0)$.

Examples

The K- Spectrum K : Consider the K-spectrum K. Let $\{y_k\}_S$ =

$\{\gamma^1\}_s - \{1\}_s \in \tilde{K}^0(\underline{C}P^\infty)$ where γ^1 is the Hopf line bundle over $\underline{C}P^\infty$ and 1 is the trivial bundle. Then y_K is a complex orientation of K.

<u>The Thom Spectrum MU</u> : We have a homotopy equivalence $BU(1) \simeq MU_1$. Consider the map $f : \underline{C}P^\infty \cong BU(1) \simeq MU_1 \xrightarrow{M(i)} \Sigma^2 MU$ where M(i) is the map of spectra induced by the inclusion of MU_1 as the second term of the spectrum MU. The element $[f] = y_{MU} \in \widetilde{MU}^2(\underline{C}P^\infty)$ is a complex orientation of MU.

8.4.1 <u>Theorem</u> : Let E be a ring spectrum with a complex orientation $y_E \in \tilde{E}^q(\underline{C}P^\infty)$. Then

$$\tilde{E}^*(\underline{C}P^n) \cong E^*(pt.)[y_E] / (y_E^{n+1})$$

and $$\tilde{E}^*(\underline{C}P^\infty) \cong E^*(pt.)[[y_E]].$$

The proof consists in showing that the Atiyah - Hirzebruch spectral sequence for the cohomology theory E*

$$\tilde{H}^*(\underline{C}P^n, E^*(S^0)) \Rightarrow \tilde{E}^*(\underline{C}P^n)$$

is trivial. (cfr. [32], 16.29). It immediately follows that $E^*(CP^n)$ has an $E^*(S^0)$-basis $\{y_E, y_E^2, y_E^3, \ldots, y_E^n\}$. This proves that $\tilde{E}^*(\underline{C}P^n) = E^*(pt.)[y_E] / (y_E^{n+1})$. We then have that $\lim^1 \tilde{E}^*(\underline{C}P^n) = 0$ and hence

$$\tilde{E}^*(\underline{C}P^\infty) \cong \varprojlim \tilde{E}^*(\underline{C}P^n) = E^*(pt)[[y_E]].$$

8.4.2 Parametrized Borsuk-Ulam Theorem for Generalised Cohomology Theories

Let E be a ring spectrum equipped with a complex orientation (e.g., K or MU) which defines homology and cohomology theories E_* and E^* respectively. Let $p : X \longrightarrow B$ and $p' : X' \longrightarrow B$ be complex vector bundles of complex fibre dimensions m and n respectively over the same paracompact space B. Let $c_i(X) \in E^{2i}(B)$ and $c_i(X') \in E^{2i}(B)$ be the E^*-theoretic chern classes.

Let
$$c\,(X,t) = \sum_{i=0}^{m} (-1)^i c_i(X)\, t^{m-i} \quad \text{and}$$
$$c\,(X',t) = \sum_{i=0}^{n} (-1)^i c_i(X') t^{n-i}$$

belonging to $E^*(B)\,[t]$ be their respective chern polynomials.

Let T and T' be free, fibre-preserving, norm-preserving $S^1 = U(1)$-actions on the given vector bundles respectively. Let $f : SX \longrightarrow X'$ be a fibre-preserving equivariant map, where $SX \longrightarrow B$ is the (2m-1)-sphere bundle associated with $p : X \longrightarrow B$, and let T itself denote the induced S^1 action. Then we shall have the complex projective bundle : $\tilde{p} : SX/T \longrightarrow B$. Let $Z = \{x \in SX \,/\, f(x)=0\}$ and let $\bar{Z} = Z/T$.

Then the theorem 7.1.2 and its proof are valid in the context of these generalised cohomology theories and chern classes and we may state :

8.4.3 Theorem :

If $q(t) \in E^*(B)[t]$ is such that $q(t) / V = 0$ for some open neighbourhood V of $\overline{Z}$, then

$$q(t) \cdot c(X',t) = c(X,t) \cdot q'(t)$$

for some polynomial $q'(t) \in E^*(B)[t]$.

(cfr. 7.1.6 for its counterpart)

If we use Čech cohomology $\check{E}^*$ instead of E^*, we get :

8.4.4 Theorem :

If $q(t) \in \check{E}^*(B)[t]$ is such that $q(t) / \overline{Z} = 0$, then

$$q(t) \cdot c(X',t) = c(X,t) \cdot q'(t)$$

for some polynomial $q'(t) \in \check{E}^*(B)[t]$.

(Cfr. 7.1.2 for the counter part of the result).

For $\check{E}^*$-theory, the more generalised version of Dold's result, 7.1.9 is also valid.

Atiyah has shown in /1/ that the K-theoretic characteristic classes give sharper results about immersions and embeddings. Hence the above results which contain K-theoretic or cobordism chern polynomials are clearly stronger results about immersions mentioned earlier in 7.2.4.

8.5 New Avenues

In this section we would like to indicate a few directions that new research in this field can take.

One may attempt to further generalise the domain, the range or the actions considered in Borsuk-Ulam theorem. In particular one may attempt to obtain results akin to that of Dold (cfr. 7.1.2) in the context of cyclic subgroups G S^1 of order k 2 and study G-sphere bundles p: S —— B and their "Chern-Grothendieck" classes g_i $H^i(B; \underline{Z}/k)$. Then projective spaces and bundles would be replaced by different kinds of lens-spaces and bundles.

We have studied above, how the results of Dold can be extended to Generalised Cohomology Theories, but we have had to assume a Cech construction. The reason is that we have required "continuity" of the cohomology theory for the proof. One may conjecture that it is possible to obtain the results in the absence of the continuity hypothesis; i.e., that the results 8.4.4 and its more general version (a result in the spirit of 7.1.9) are valid for E*-cohomology theory (cfr. 8.4).

In this work, we have at different stages, studied various indices (e.g., the index defined in 3.0, the coind_L defined in 4.0). One could compute some of these indices for specific spaces. We have mentioned in the introduction, the work of Stoltz [35] who has computed a certain coindex (cfr. 1.1.) for the real projective space. Certain computations can prove to be fruitful.

We have had occasion to point out various applications to which the results that have been obtained so far, can be put (cfr. for example, 5.2.5 and 7.3.2). We could explore further applications for the results of the Borsuk-Ulam Theorem. We could also study in depth some of the applications we have mentioned, notably that of the result 7.1.2 for study of immersions of manifolds in manifolds.

One may even want to study in more detail the structure of the set $\Lambda(f)$ or the set Z that we have studied in this work.

Several avenues are open and questions and problems abound. It is our wish that the above work and the problems we have posed will stimulate further research and investigation in this field.

BIBLIOGRAPHY

(The following Bibliography inclues not only books or research articles directly referred to in the above work, but also those to which the reader may refer for further information on the subject).

1. ATIYAH, M.F., Immersions and Embeddings of Manifolds, Topology, 1961, vol 1, pp. 125-132.
2. BACON, PHILIP, Equivariant Formulations of the Borsuk-Ulam Theorem, Canad. J. Math. 18(1966), 492-502.
3. BORSUK, K., Drei Satze uber die n-dimensionale Euklidische Spare, Fund. Math. vol 20(1933), pp. 177-190.
4. BOURGIN, D.G., On Some Separation and Mapping Theorems, Comment. Math. Helv. vol 29(1955), pp. 199-214.
5. CONNER, P.E., Differentiable Periodic Maps, Springer-Verlag, 1970.
6. CONNER, P.E., and FLOYD, E.E., Fixed Point Free Involutions and Equivariant Maps, Bull. of Am. Math. Soc., 66(1960), pp. 416-441.
7. DOLD, ALBRECHT, Lectures on Algebraic Topology, Springer-Verlag, 1972.
8. DOLD, ALBRECHT, Parametrized Borsuk-Ulam Theorems, Comment. Math. Helv. 63(1988), pp. 275-285.
9. DUAN, HAI BAO, Some Borsuk-Ulam Type Theorems for Maps from Riemannian Manifolds into Manifolds, Proc. Roy. Soc. Edinberg, Sect. A III(1989), no. 1-2, pp. 61-67.
10. DUGUNDJI. JAMES, Topology, Prentice Hall of India Pvt. Ltd., New Delhi, 1975.
11. DYSON, F.J., Continuous Functions Defined on Spheres, Ann. of Math(2), vol 54(1951), pp. 534-536.

12. EILENBERG S., Homology of Spaces with Operators I, Tras. Am. Math. Soc. vol 61(1947), pp. 378-417.

13. EILENBERG, S. and STEENROD, NORMAN, Foundations of Algebraic Topology, Princeton University Press, Princeton, 1952.

14. FADELL, E., and HUSSEINI, S., An Ideal-Valued Cohomological Index Theory with Applications to Borsuk-Ulam and Bourgin-Yang Theorems, Ergodic Theory and Dynamical Systems, 1987.

15. FADELL, E., HUSSEINI, S., and RABINOWITZ, P.H., Borsuk-Ulam Theorems for Arbitrary S^1 actions and Applications, Trans. Am. Math. Soc., 274, no. 1, nov. 1982, pp. 345-359.

16. GREENBERG, MARVIN J., Algebraic Topology, W.A. Benjamin Inc., Massachussetts, 1973.

17. GUILLEMIN, V., and POLLACK A., Differential Topology, Prentice-Hall Inc., New Jersey, 1974.

18. HIRSCH, GUY, Sur des theorems de Borsuk-Ulam et de Knaster, Bull. Soc., Roy. Sci. Liege, 13(1944), pp. 137-145.

19. HIRSCH, MORRIS W., Differential Topology, Springer-Verlag, 1976.

20. HUSEMOLLER, DALE, Fibre Bundles, Springer-Verlag, 1966.

21. JAWOROWSKI, J.W., and MOSZYNSKI, K., A Theorem on Mappings of the Sphere into the Projective Space, Bull, Acad., Polon. Sci. Cl. III, 4(1956), pp. 75-77.

22. JAWOROWSKI, JAN, A Continuous Version of Borsuk-Ulam Theorem, Forschungsinstitut fur Mathematik, ETH, Zurich, June 1980.

23. ________ Fibre-Preserving Maps of Sphere Bundles into Vector Space Bundles, Fixed Point Theory, LNM 886, Springer-Verlag, 1981, pp. 154-162.

24. ________ A Borsuk-Ulam Theorem for O(m), Topics in Equivariant Topology, (Montreal, PQ, 1986), pp. 105-118, Sem. Math. Sup. 108, Presses Univ. Montreal, Montreal, PQ, 1989.

25. ________ Maps of Stieffel Manifolds and a Borsuk-Ulam Theorem, Proc. Edinberg Math. Soc. (2) 32(1989) no.2, pp. 271-279.

26. KAKUTANI, S., A Proof that there exists a Circumscribing Cube Around any Bounded Closed Convex Set in R^3, Ann. of Math. (2), vol 43,(1942), pp.739-741.

27. MARZANTOWICZ, WACLAW, A Borsuk-Ulam Theorem for Orthogonal T(k) and Z(p,r) Actions and Applications, 137(1989), no.1, pp. 99-121.

28. MILNOR, JOHN W., Topology From the Differentiable Viewpoint, University Press of Virginia, Charlottesville, 1978.

29. MILNOR, JOHN W., and STASHEFF, JAMES D., Characteristic Classes, Ann. of Math. Studies, Princeton University Press, 1974.

30. MUNKHOLM, H.J., A Borsuk-Ulam Theorem for Maps from a Sphere to a Compact Topological Manifold, Illinois J. Math. vol 13 (1969), pp.167-185.

31. NAKAOKA, M., Equivariant Point Theorems for Fibre-Preserving Maps, Osaka J. Math., 21(1984) pp. 809-815.

32. SCHWITZER, ROBERT M., Algebraic Topology - Homotopy and Homology, Springer-Verlag, 1975.

33. SMITH P.A., Fixed points of Periodic Transformations, Appendix B of LEFSCHETZ, Algebraic Topology, Colloq. Pub. Amer. Math. Soc., vol. XXVII, 1942.

34. SPANIER, EDWIN H., Algebraic Topology, Tata McGraw-Hill Publishing Co., Ltd., Bombay-New Delhi, 1966.

35. STOLZ, STEPHEN, Level of Real Projective Spaces, Comment. Math. Helv. 64(1989), pp. 661-674.

36. STONG, ROBERT E., Notes on Cobordism, Princeton University Press, 1968.

37. THOM, R., Espaces Fibres en Spheres et Carres de Steenrod, Ann. Sci. Ecole Norm. Sup. vol 69(1952), 109-182.

38. YAMABE, H., and YUJOBO, Z., On the Continuous Functions Defined on a Sphere, Osaka Math. J., vol 2(1950), pp. 19-22.

39. YANG, CHUNG-TAO, On Theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobo and Dyson I, Ann. of Math. (2), vol 60(1954), pp. 262-282.

40. ________, On Theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobo

and Dyson II, Ann. Math. vol 62(1955), pp. 271-283.

41. __________, Continuous Functions from Spheres to Euclidean Spaces, Ann. of Math. vol 62(1955) pp. 284-292.

42. LIULEVICIUS, A., Borsuk-Ulam Theorems and K-theory Degrees of Maps, Algebraic Topology, Arhaus 1982, LNM 1051, Springer-Verlag 1983.