Periodically driven underdamped periodic and washboard potential systems: Dynamical states and stochastic resonance

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We have studied the motion of an underdamped Brownian particle in (i) a bistable periodic potential and (ii) washboard potentials subjected to a sinusoidal external field. The particles are shown to be effectively in two dynamical states of their trajectories with distinct amplitudes and phase relationship with the external drive. These dynamical states are stable with fixed energies at low temperatures, but transitions between them take place as the temperature is increased. The average input energy loss to the environment per period of the drive shows a stochastic resonance (SR) peak as a function of temperature for the underdamped system potentials studied. The occurrence of SR in these systems is explained using the statistics of transitions between the two dynamical states.

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I. INTRODUCTION

The phenomenon of stochastic resonance (SR), initially put forth theoretically to explain the occurrence of ice ages at a certain frequency on Earth [1], has been observed experimentally in many physical [2,3] and biological systems [4,5]. SR has been investigated theoretically and experimentally with considerable interest over the past three decades [6,7]. Its attraction lies in the seemingly counterintuitive idea that by tuning noise level (externally or internally) the response of a nonlinear system to a weak external periodic signal can be enhanced considerably; or a nonlinear system itself tunes the noise level in order to enhance a particular chosen signal. Moreover, the noise level at which the response peaks depends on the frequency of the input signal. As a consequence, it may have potential applications in the detection of weak signals as well as in the selection of a signal of a particular frequency out of a host of signals of different frequencies [8–11]. As a corollary, a biological system can tune noise level internally to select and enhance a desired signal [12–14]. Apart from potential practical applications, it offers considerable theoretical challenges.

The occurrence of SR in bistable systems [6,15] has more or less been confirmed in overdamped as well as in underdamped cases [6,7,16]. Interestingly, in the underdamped bistable systems intrawell as well as interwell SR are found to occur simultaneously in a certain small range of drive frequencies [17]. But at small frequencies only interwell SR is observed in the high damping cases. However, in the small damping limits, only the intrawell SR is observed in the high drive frequency range as in the case of monostable systems [18].

SR has also been reported to occur in washboard potentials, albeit only under restricted conditions. SR can occur even in the static washboard potentials in the underdamped cases [19]. But its occurrence is restricted to the vicinity of the critical tilt F_3, where the potential becomes monotonic. SR can also occur in an overdamped static washboard potential but in an inhomogeneous medium [20]. An overdamped washboard potential system driven by a small-amplitude periodic forcing of small frequency (adiabatic limit) shows SR but again only close to the critical tilt F_3 [19,21]. It can also occur in an underdamped washboard potential driven by a small-frequency periodic forcing [19]. However, again its occurrence is restricted to the immediate vicinity of the damping-coefficient (γ) dependent threshold tilt F_2 ($\sim 3.36\gamma$) of the potential. As far as the authors are aware of, there is no investigation of SR in underdamped washboard potentials in the high drive-frequency domain where intrawell motion dominates over the interwell motion [17]. Moreover, the occurrence of SR in underdamped periodic potentials is still debatable [22], although there have been some important investigations [23].

There have been some discussion on SR in underdamped periodic potentials in view of obvious practical importance of these potentials, such as in describing motion of adatoms on crystal surfaces [24], superionic conductivity [25], RCSJ model of Josephson junction [26–28], etc. The conventional SR is considered to occur in a bistable system at a temperature when the signal frequency matches the time rate of passage across the potential barrier of the nonlinear system [15,22]. Analogously, one would expect, for instance, the frequency dependent mobility, as a response, to peak as a function of temperature, at a signal frequency corresponding to the mean passage rate across a potential barrier of the periodic potential. However, instead, the mobility shows monotonic behavior around that frequency. It does show a peak with temperature only at a much higher frequency. This mobility peaking with temperature is termed as a kind of dynamical resonance, unlike the SR satisfying the conventional criterion of frequency matching, and merely confined to intrawell motion [18,22,23]. This aspect has been examined more closely in a recent numerical work [29] on an underdamped sinusoidal potential system considering hysteresis loop area as response to the input signal $F(t) = \Delta F \cos \omega t$.

Hysteresis loop area (HLA), in the average position-forcing $(x - F)$ space, or equivalently, the input energy lost by the system to the environment per period of the external forcing $F(t)$, is considered as an appropriate measure of SR. HLA has been considered as a quantifier of SR earlier too [30,31].

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However, recently HLA was found to show SR behavior close to what was shown by the amplitude of the average particle-position variable $x(t)$ in a bistable system [32–35]. HLA not only includes information of the mean amplitude x_0 of $x(t)$ but also its phase relationship with the external forcing or the input signal $F(t)$. Its significance as an appropriate quantifier of SR has also been pointed out recently [36,37]. Moreover, in a periodic potential, it is very likely that the particle forays into wells far away, in either direction, from the initial well. It makes the position variable unsuitable for any meaningful quantification of SR [17]. It is found that the HLA or input energy loss takes account of motion whether it is in a single well or spread over several wells. Also, HLA shows qualitatively similar behavior as the frequency dependent mobility calculated using the linear response theory [22]. The HLA has recently been justifiably used as a quantifier of SR in an underdamped periodic potential system [29].

In Ref. [29] it was pointed out that the peaking of HLA as a function of temperature at high frequencies ought not to be dismissed as merely a dynamical resonance [22]. The phenomenon can be seen as a result of transitions between two dynamical states of the underdamped driven particle trajectories in the highly nonlinear (sinusoidal) potential. The two and only two dynamical states are shown to exist and are distinguished by their trajectory amplitudes and their phase relationship with the field $F(t)$. The relative stability of the two states changes as the temperature (noise strength) is varied.

It is hard to prove analytically that an underdamped particle moving in a sinusoidal potential driven by a periodic force of frequency close to the natural frequency at the bottom of a well of the potential and subjected to fluctuating forces can have only two dynamical states of its trajectories as it was found numerically. Solution of only the simplest situation of the deterministic motion of a driven pendulum without friction, $m\frac{d^2x}{dt^2} = -\frac{\partial}{\partial x}(-\sin x) + F_0 \cos \omega t$, given the initial condition $(x(0),v(0))$, can be obtained analytically and its phase trajectories drawn [38]. Notice that the periodic potentials $U_0(x) = -\sin x$ and $U_0(x) = \cos x$ are equivalent except for a phase difference of $\pm\frac{\pi}{2}$. Also, the trajectories with initial conditions $(x(0),0)$ and $(2\pi - x(0),0)$ should be identical except that they are in opposite directions in case of closed orbits and in $\pm v$ sectors in case of traveling solutions. In other words, the two trajectories should have the same amplitude but different phase relationship with the external forcing. However, for different initial conditions $(x(0),v(0))$, they can have different trajectories and not necessarily confined to just two particular trajectories. But, when damping is present, it is easy to appreciate that if two trajectories have different phases they will necessarily have different amplitudes and hence different energy contents. This can be seen as follows.

Consider the two extreme cases of particle motion being (i) in-phase, and (ii) completely out-of-phase with the external drive. Figure 1 shows the periodic potential when $F(t) = 0$ (curve O), $F(t) = \Delta F$ (curve A), and $F(t) = -\Delta F$ (curve B), where $F(t) = \Delta F \cos \omega t$, so that the total potential $V(x) = -\sin x - x\Delta F \cos \omega t$. Consider the extreme position P1 of the particle on the curve A when $F(t) = \Delta F$. In the next moment the $F(t)$ will decrease and also the particle position moves to the left so that $F(t)$ and $x(t)$ are in-phase. They

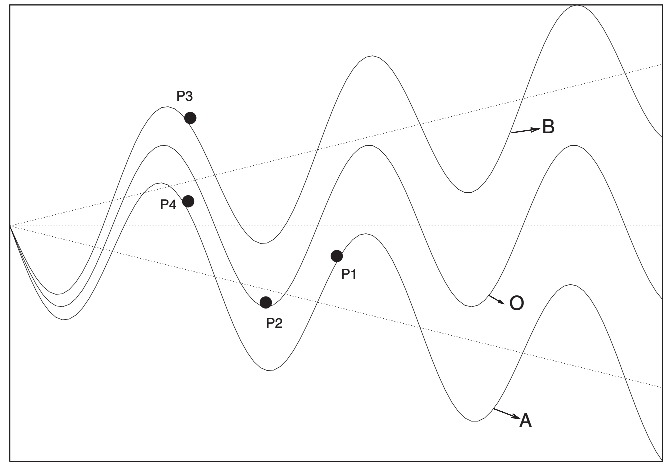

FIG. 1. The figure shows the periodic potential when $F(t) = 0$ (curve O), $F(t) = \Delta F$ (curve A), and $F(t) = -\Delta F$ (curve B).

being in-phase implies that when $F(t) = 0$ the particle is at the potential bottom P2 and when $F(t) = -\Delta F$ the particle is at P3, the extreme position on the left and is about to roll down the potential hill as $F(t)$ begins to rise from $-\Delta F$. From the figure, therefore, one could notice that in this case of in-phase situation the particle always moves on the stiffer slope of the potential $V(x)$. On the other hand, if we were to begin from the particle position P4 on A while $F(t) = \Delta F$, the particle always moves lying on the gentler slope of $V(x)$ in this completely out-of-phase case. Since the force experienced by the particle due to the potential in the two cases have different values one would naturally expect the underdamped particle to have different amplitudes of motion and hence have different energy losses due to frictional forces. The two dynamical states are thus distinct.

At low temperatures, for given $F(t)$, depending only on the initial conditions the system chooses one of the two dynamical states. These two states are quite stable [29] and no transition occurs between them. However, as the temperature is increased transition takes place between the dynamical states and, as a result, the relative population of these states changes. Thus the mean amplitude of the trajectories and the overall phase when averaged over the initial conditions vary with temperature. The hysteresis loop area shows a peak at a temperature indicating stochastic resonance where the transition rate between the dynamical states acquires a particular value. It is also important to notice that well before the HLA peaks, the particle begins to surmount the potential barrier of $V(x)$ and at resonance the motion is no longer confined to a single well of $V(x)$; the interwell transitions become quite numerous. Of course, the interwell transition rate is still quite low compared to the transition rate between the dynamical states [29].

In the present work, we show that a driven underdamped particle exhibits stochastic resonance in (i) a periodic bistable potential, $U(x) = \frac{2}{3}(\cos x + \cos 2x)$, and also in (ii) washboard potentials (tilted sinusoidal as well as tilted periodic bistable potential). The investigation of SR in the underdamped potential $U(x)$ is to show that the occurrence of SR is not specific to sinusoidal potential alone but more general periodic potentials can also exhibit SR and these owe their explanation to the existence of effectively two dynamical states with

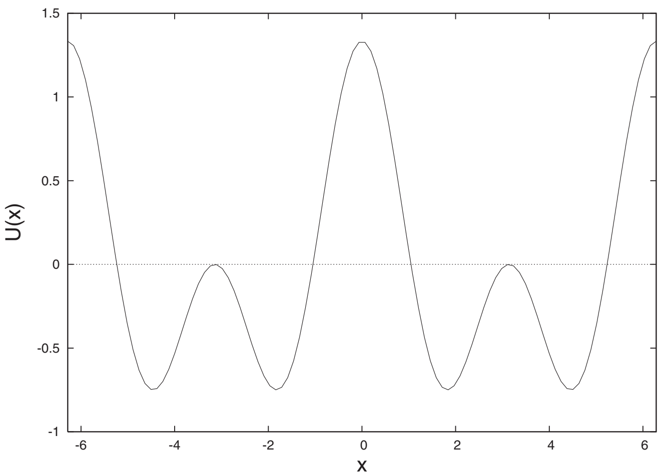

FIG. 2. The periodic bistable potential $U(x) = \frac{2}{3}(\cos x + \cos 2x)$ is shown with two similar subwells in a well of the potential.

different phase lags. Moreover, as mentioned in a previous paragraph, the occurrence of SR in underdamped washboard potentials is shown earlier but only under very restricted conditions of potential slope [19] at small drive frequencies. Here we show that SR occurs in underdamped washboard potentials even for arbitrarily small slopes (including close to zero slopes) of potentials at high drive frequencies. In this work we use HLA as a quantifier of SR.

The importance of periodic bistable potential $U(x)$ in the study of superionic conductors has been discussed in Ref. [25]. In that important work the dynamic structure factor $S^s(q,\omega)$, especially its quasielastic peak ($\omega \sim 0$), has been investigated extensively. The half-width at half-maximum (HWHM) of the quasielastic peak provides important information about the diffusion mechanism of the particle in $U(x)$ (without external drive). HWHM was studied at various values of the ratio Δ of the heights of the two peaks of $U(x)$ (Fig. 2) and also at various values of friction (damping). The results were presented for a fixed temperature. The weight of the side peaks of $S^s(q,\omega)$ at finite ω, which are connected to the oscillatory dynamics around the potential minima and relevant to our present work, is stated without further elaboration to shift toward the lower frequencies ω as T increases. The present work concerns study of particle motion in driven underdamped systems as a function of temperature at a fixed value of $\Delta \approx 0.4$.

As mentioned earlier, in the case of bistable periodic potential too, as in the case of sinusoidal potential, effectively two and only two dynamical states (one in-phase and one out-of-phase) of trajectories of a periodically driven underdamped particle are obtained. Since in a potential well of $U(x)$ there exist two similar subwells (Fig. 2) there are two (one in-phase and one out-of-phase) states in each subwell. The in-phase (out-of-phase) state in one subwell has the same amplitude and phase relationship as the in-phase (out-of-phase) state in the other subwell. Therefore, energetically there exist only one in-phase and one out-of-phase states of trajectories. However, as time progresses, these trajectories can also be in wells and subwells other than the initial ones, the basins of attraction of the dynamical states can be identified in different wells and subwells. Consequently, the basins of attraction of the dynamical states become more complex than in case of sinusoidal potentials where only trajectories in different wells are distinguished.

The washboard potentials, that is a slanted sinusoidal potential and slanted bistable periodic potential, naturally yield finite particle drifts. As a consequence the hysteresis loops do not close. However, upon correcting for the drift factor, as explained in Sec. III, the hysteresis loops close and the closed hysteresis loop areas could be made to agree with the input energy loss. The HLA (or the input energy loss) so obtained again shows peaking behavior as the temperature is varied. This is a clear signature of SR in the washboard potential. The occurrence of SR could again be explained in terms of transition between the two dynamical (one in-phase and one out-of-phase) states of particle trajectories that are realized even in these washboard potentials.

II. THE MODEL

We consider motion of an underdamped particle along (i) a periodic bistable potential $U(x) = \frac{2}{3}V_0(\cos x + \cos 2x)$, (ii) a tilted sinusoidal potential $U_1(x) = -V_0 \sin kx - F_0 x$, and (iii) a tilted bistable potential $U_2(x) = U(x) - F_0 x$, where the tilt F_0 represents a constant force. As mentioned earlier (Fig. 2) $U(x)$ has two subwells of equal depth (bistable) in each periodic well of the potential. $U(x)$ is symmetric about $x = n\pi$, where, $n = \mp 1, \mp 2, \ldots$. The latter two potentials we shall refer to as the washboard potential and the bistable washboard potential, respectively, for $F_0 \neq 0$ but small. In the bistable washboard potential $U_2(x)$ the two subwells exist but are no longer identical as they were in $U(x)$.

A particle of mass m moving in a medium of friction coefficient γ along a potential $V(x)$ and driven by an external periodic forcing $F(t) = \Delta F \cos \omega t$ and subjected to a Gaussian white noise $\xi(t)$ is described here by the Langevin equation

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - \frac{\partial V(x)}{\partial x} + F(t) + \sqrt{\gamma T}\xi(t). \qquad (2.1)$$

The fluctuating forces $\xi(t)$ satisfy $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\delta(t - t')$. The temperature T is in units of the Boltzmann constant k_B. We take $V(x) = U(x), U_1(x)$, or $U_2(x)$ separately to discuss the nature of particle motion.

The equation is written in dimensionless units by setting $m = 1$, $V_0 = 1$, $k = 1$. The Langevin equation, with reduced variables denoted again now by the same symbols, corresponding to Eq. (2.1) is written as

$$\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - \frac{\partial V(x)}{\partial x} + F(t) + \sqrt{\gamma T}\xi(t). \qquad (2.2)$$

The noise variable, in the same symbol ξ, satisfies exactly similar statistics as earlier.

III. NUMERICAL RESULTS

We adopt the same numerical procedures as described in Ref. [29]. The drive (signal) frequency ($\omega = \frac{2\pi}{\tau}$) is chosen to be close to but a little smaller than the natural frequency at the bottom of the wells of the potentials. However, they are not exactly equal to the respective natural frequencies (without damping). For the periodic bistable potential $U(x)$, we take the period τ equal to 4.8. The same optimum τ is taken also for the bistable washboard potential $U_2(x)$. Similarly, we continue with $\tau = 8$ as in Ref. [29] for the washboard potential $U_1(x)$.

With the periods τ of the external forcing $F(t) = \Delta F \cos \omega t$, we obtain the trajectories $x(t)$ for given initial conditions $(x(0),v(0))$ numerically [30,39] by solving the Langevin equation (2.2) and calculate the input energy, or work done by the field on the system W in a period τ as [40]

$$W(t_0,t_0+\tau) = \int_{t_0}^{t_0+\tau} \frac{\partial U_e(x(t),t)}{\partial t} dt, \qquad (3.1)$$

where the effective potential $U_e(x(t),t) = V(x) - xF(t)$, and $V(x) = U(x), U_1(x)$, or $U_2(x)$ as applicable. Therefore,

$$W(t_0,t_0+\tau) = -\int_{F(t_0)}^{F(t_0+\tau)} x\, dF = A, \qquad (3.2)$$

where A is the magnitude of the HLA. The average input energy per period $\overline{W}$ averaged over an entire trajectory spanning N_1 periods of $F(t)$ is

$$\overline{W} = \frac{1}{N_1} \sum_{n=0}^{n=N_1} W(n\tau,(n+1)\tau) = \overline{A}. \qquad (3.3)$$

Typically, N_1, ranges between 10^5 to 10^7, as required.

At very low temperatures $\overline{W}$ depends very strongly on the initial conditions $(x(0),v(0))$. This is because whether the trajectories are in the in-phase state [with a small phase difference ϕ_1 between $x(t)$ and $F(t)$] or in the out-of-phase state (with a large phase difference ϕ_2) is determined by the initial condition. And, $\overline{W}$ depends on the kind of trajectory. Moreover, at such temperatures the particle remains trapped in the initial well of the potential and also no transition between in-phase and out-of-phase states of trajectories takes place in a well. However, as the temperature is gradually increased the intrawell transitions between the in-phase and out-of-phase states begin to take place and subsequently, interwell transitions also become frequent. Therefore, as the temperature is increased the dependence of $\overline{W}$ on the initial conditions $(x(0),v(0))$ weakens. However, it is always sensible to ensemble average $\overline{W}$ over all possible initial conditions and obtain the average input energy per period $\langle \overline{W} \rangle$, which is also equal to the mean hysteresis loop area $\langle \overline{A} \rangle$ [Eq. (3.3)].

As mentioned earlier, the response $x(t)$ in the two dynamical phases of trajectories not only have different phase relationship with the forcing $F(t)$ but the mean amplitude x_0 of $x(t)$ in the two cases are very different. Therefore, apart from the inherent stochastic nature of W [Eq. (3.1)], the values of W are different depending on whether during the particular period [of $F(t)$] in question, the system is in the in-phase or in the out-of-phase state of trajectory or makes transition(s) between the states. A clear picture is revealed by the distribution $P(W)$ of input energies at various temperatures. For sinusoidal potentials the evolution of $P(W)$ with temperature furnishes important information on the occurrence of SR [33–35]. Also, since the entire stretch of the trajectories consists of a mixture of in-phase (with phase ϕ_1) and out-of-phase (with phase ϕ_2) states, the average phase lag ϕ must lie between ϕ_1 and ϕ_2. Naturally, ϕ is a function of temperature [41,42]. The phase ϕ is obtained numerically [29] by calculating the mean hysteresis loops $\langle \overline{x}(F(t_i)) \rangle$, where

$$\overline{x}(F(t_i)) = \frac{1}{N_1} \sum_{n=0}^{n=N_1} x(F(n\tau + t_i)) \qquad (3.4)$$

for all $[0 \leqslant t_i < \tau]$.

In the following subsections we present and discuss our numerical results separately for the three cases of underdamped particle motion in potentials $U(x), U_1(x)$, and $U_2(x)$ driven by field $F(t)$ at various temperatures T. We take the dimensionless friction coefficient $\gamma = 0.12$ and initial velocity $v(0) = v(t = 0) = 0$ for all cases. The initial position $x(0) = x(t = 0)$ are chosen at 99 equispaced points $x_i, i = 1,2,\ldots,99$ between the two consecutive peaks, for example, $[0 < x_i < 2\pi]$, for the potential $U(x)$. Unless otherwise explicitly stated the amplitude ΔF of $F(t)$ is taken equal to 0.2 and the tilt $F_0 = 0.1$. Note that the constant slope F_0 is much smaller than either F_2 (in our case $\sim 3.36 \times 0.12 = 0.40$) or F_3, only in the immediate vicinity of which the occurrence of SR was reported earlier [19].

In the case of washboard potential $U_1(x)$ and the bistable washboard potential $U_2(x)$, with constant tilt F_0 the particle does acquire a mean drift [of velocity $\overline{v}(F_0)$] at elevated

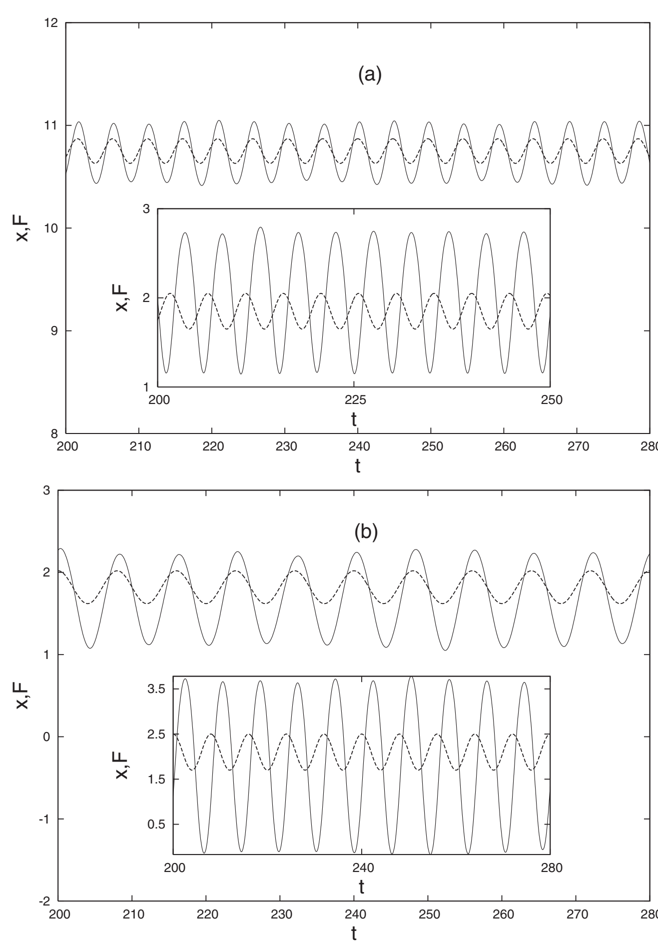

FIG. 3. Plot of particle trajectories $x(t)$ (a) for $U(x)$ at $T = 0.0005$, and (b) for $U_1(x)$ at $T = 0.001$. $F(t)$ (dashed line) is also included for comparison. The phase lags $\phi_1 \sim 0.08\pi$ and $\phi_2 \sim 0.8\pi$ (inset) for $U(x)$ and $\phi_1 \sim 0.08\pi$ and $\phi_2 \sim 0.6\pi$ (inset) for $U_1(x)$.

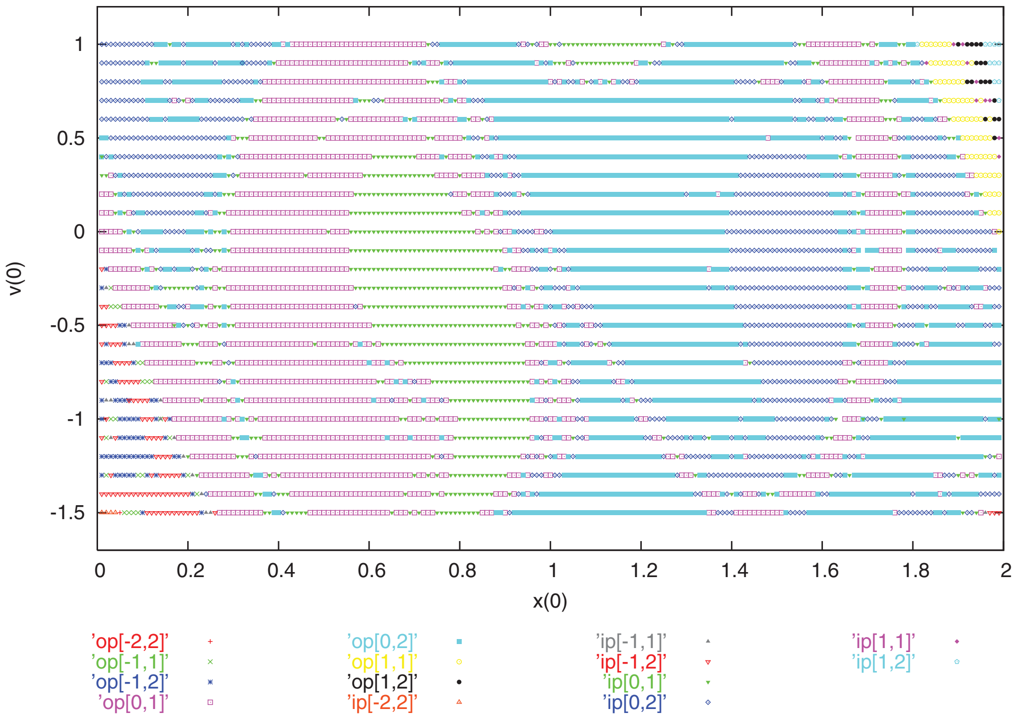

FIG. 4. (Color online) The figure shows the basins of attraction for bistable potential with $\Delta F = 0.2$, $\gamma_0 = 0.12$, $T = 0.0005$, $\tau = 4.8$. The dynamical states of the particle are represented by ip (in-phase) and op (out-of-phase) states with respect to applied periodic force. The wells and subwells in which the dynamical states exist are represented by the indices l and m within braces; e.g., op[0,1] indicates that the out-of-phase trajectory of the particle is in the (left) subwell-1 of the (initial) zeroth well of the potential.

temperatures and consequently the hysteresis loops do not close. Therefore, a correction is required to make the loops close. In a period, $F(t)$ changes from $F_0 - \Delta F$ to $F_0 + \Delta F$ and then back to $F_0 - \Delta F$ in a cosinusoidal manner. During a period the particle moves on the average a distance of $\tau \overline{v}(F_0)$. Approximating the variation of $F(t)$ to be linear, the mean work done during a period τ as a rel Therefore, the required correction to the hysteresis loop area $\langle A \rangle$ due to the mean drift equals $\Delta F \tau \overline{v}(F_0)$. This simple approximate correction closes the hysteresis loops, thereby enabling us to calculate the HLA and equivalently the $\langle \overline{W} \rangle$.

A. Dynamical states of trajectories

When the underdamped particle moves along the potentials $U(x)$ $[=\frac{2}{3}(\cos x + \cos 2x)]$ and $U_1(x)$ $(=-\sin x - F_0 x)$ and driven by the external periodic field $F(t)$ the trajectories are essentially of just two kinds (states). The in-phase states correspond to trajectories $x(t)$ which lag behind $F(t)$ by a small phase ϕ_1, whereas the out-of-phase states lag behind $F(t)$ by a large phase ϕ_2. Of course ϕ_1 and ϕ_2 are approximate average values which weakly depend on the potential and the temperature. At low temperatures ϕ_1 and ϕ_2 essentially maintain the same values at each period of $F(t)$ (Fig. 3). The potential $U(x)$ has two similar subwells (Fig. 2) in a well, the trajectories in either subwell are just the same two states. Here, these in-phase and out-of-phase states are identified by the symbols ip[l,m] and op[l,m], respectively. l indicates well number, for example, $l = 0$ for the initial well, $l = -1$ $(+1)$ for the first well to the left (right) of the initial well. m indicates the subwell number: 1 (2) for the left (right) subwell. These states are truly dynamical states. The basins of attraction of these states for the potential $U(x)$ is given in Fig. 4 as an illustration at temperature $T = 0.0005$.

At temperature $T = 0.0005$ the two states are quite stable and no transition between them could be observed. Though the potential barrier between any two consecutive wells have almost the same value for $U(x)$ and $U_1(x)$, their well bottoms are quite dissimilar and hence the input energy per period $\overline{W}$ are very different (Fig. 5). For $U_1(x)$, $\overline{W}$ are about 0.1 (in-phase) and 1.15 (out-of-phase), whereas for $U(x)$ they are about 0.04 (in-phase) and 0.32 (out-of-phase). Naturally, transitions between the two states for $U(x)$ occur at lower temperature than in case of $U_1(x)$ (Fig. 6).

In the case of $U(x)$ by the temperature $T = 0.001$ the indications of out-of-phase state going to the in-phase state, in a subwell, could be found and by $T = 0.0015$ a substantial fraction of the out-of-phase states have jumped to the in-phase state. At $T = 0.002$ all the out-of-phase states have made way to the in-phase state due to thermal fluctuations. Therefore, close to $T = 0.002$ the system have the lowest average input energy $\langle \overline{W} \rangle$ (or $\langle \overline{A} \rangle$) per period of $F(t)$. And, from upward of $T = 0.003$, the in-phase states begin to jump to the out-of-phase state. At $T = 0.005$ even intersubwell transitions could also be observed. The corresponding temperatures for the case of $U_1(x)$ are much higher.

For the system with potential $U_1(x)$ the out-of-phase states begin going over to the in-phase state at around $T = 0.004$ and the process completes at about $T = 0.006$. The transition from the all-in-phase state to the out-of-phase state begins at about $T = 0.009$. Interestingly, however, the interwell transitions

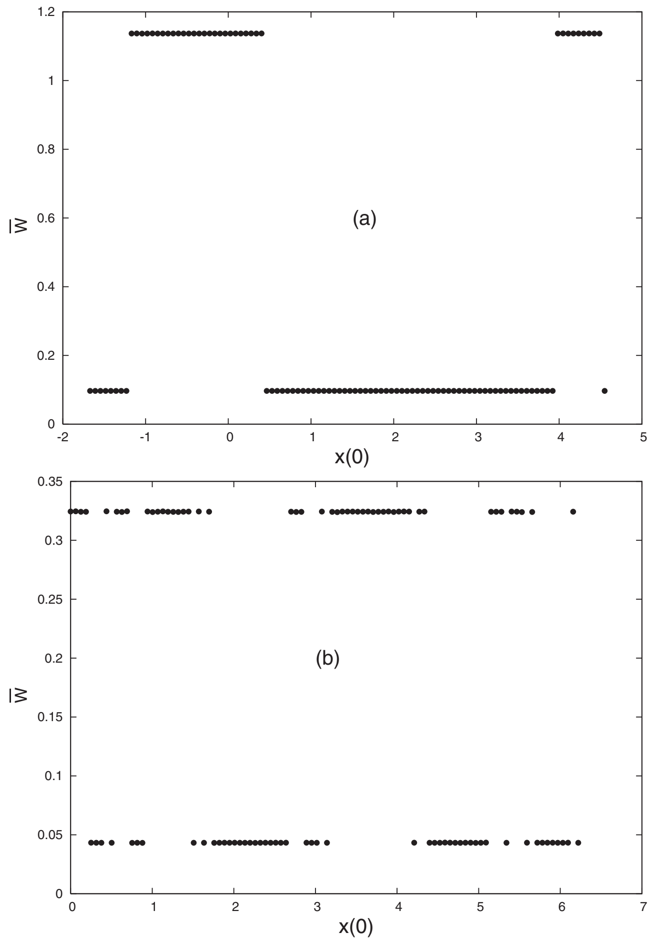

FIG. 5. Plot of $\overline{W}$ with $x(0)$ for $U_1(x)$ at $T = 0.001$ (a) and $U(x)$ at $T = 0.0005$ (b).

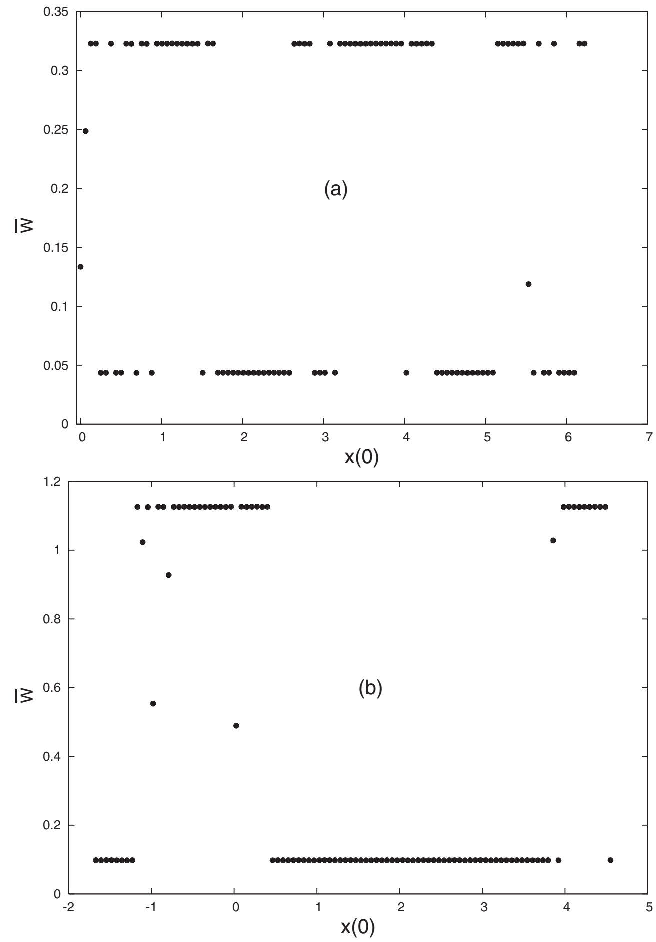

FIG. 6. Plot of $\overline{W}$ with $x(0)$ for $U(x)$ at $T = 0.0015$ (a) and $U_1(x)$ at $T = 0.004$ (b).

for $U_1(x)$ takes place at a much lower temperature (about $T = 0.01$) than in the case of the system with potential $U(x)$ which is only at about $T = 0.23$ (Fig. 7).

As stated earlier the periodic bistable potential $U(x)$ have essentially two states because the two subwells are energetically identical. In the case of the bistable washboard potential $U_2(x)$ [$=U(x) - F_0x$] the two subwells become dissimilar and the right subwell is energetically lower than the left one and we get two states corresponding to each subwell. However, the in-phase state in the left subwell becomes unstable for amplitude $\Delta F > 0.19$ of the drive $F(t)$. Therefore, in our case, we have only one in-phase state to consider in the left subwell.

In case of $U_2(x)$, $\overline{W}$ is about 0.27 for the out-of-phase state in the left subwell (henceforth called subwell-1) and $\overline{W}$ for the two states in the other subwell (subwell-2) are about 0.03 (in-phase) and 0.37 (out-of-phase) at $T = 0.0005$ [inset of Fig.10(c)]. By the temperature $T = 0.001$ most of the (out-of-phase) states in subwell-1 go over to the states in subwell-2 and by $T = 0.0015$ no states in subwell-1 survives. As the temperature is gradually increased the out-of-phase states in subwell-2 begin to jump over to the in-phase state in the same subwell and by $T = 0.003$ we only have in-phase states in subwell-2 and $\langle \overline{W} \rangle$ acquires a minimum value. By $T = 0.009$ the particles begin to leave the in-phase state for the out-of-phase state in subwell-2 and we get a mixture of the two states in the same subwell. As the temperature is increased further, the transitions back and forth to subwell-1 begin at about $T = 0.015$. Interwell transitions begin, at a very slow rate, at about $T = 0.08$, which is much higher than the corresponding temperature for the washboard potential $U_1(x)$ but lower than that for $U(x)$. The presence of two subwells in $U(x)$ and $U_2(x)$ makes the effective bottom of the wells flatter (smaller curvature) than in case of $U_1(x)$ and hence the interwell transition rates smaller [see Eq. (3.5)].

At $T > 0.003$ ($T > 0.01$) transitions from in-phase states to the out-of-phase states increase rapidly for $U(x)$ [$U_1(x)$], and consequently so does the $\langle \overline{W} \rangle$, leading ultimately to the SR condition. Similar is the situation for $U_2(x)$.

B. Hysteresis loss and stochastic resonance

Figure 8 shows the variation of average input energy $\langle \overline{W} \rangle$ or the average hysteresis loop area $\langle \overline{A} \rangle$ as a function of temperature for the underdamped washboard potential $U_1(x)$. The same functions are plotted for the potentials $U(x)$ and $U_2(x)$ in Fig. 9. $\langle \overline{W} \rangle$ peaks at the temperatures $T = 0.2$, 0.04, and 0.08 for the potentials $U_1(x)$, $U(x)$, and $U_2(x)$, with peak values of about 0.36, 0.16, and 0.146, respectively. These maximum values of $\langle \overline{W} \rangle$ lie between the $\overline{W}$ values of the in-phase and out-of-phase states for the corresponding

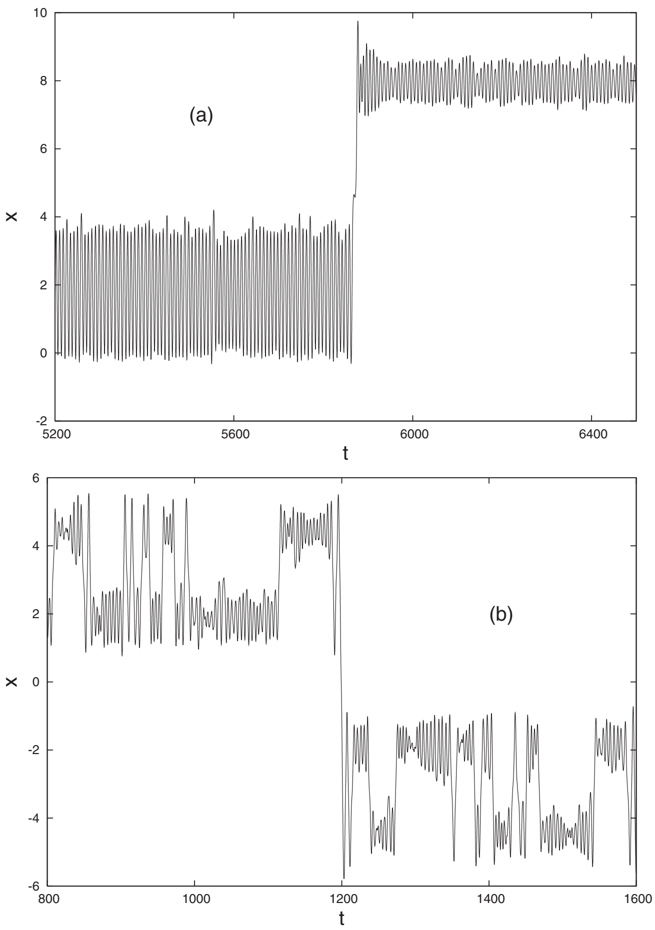

FIG. 7. Plot of $x(t)$ shows an interwell transition for $U_1(x)$ at $T = 0.01$ (a). A similar transition is seen for $U(x)$ at $T = 0.23$ (b).

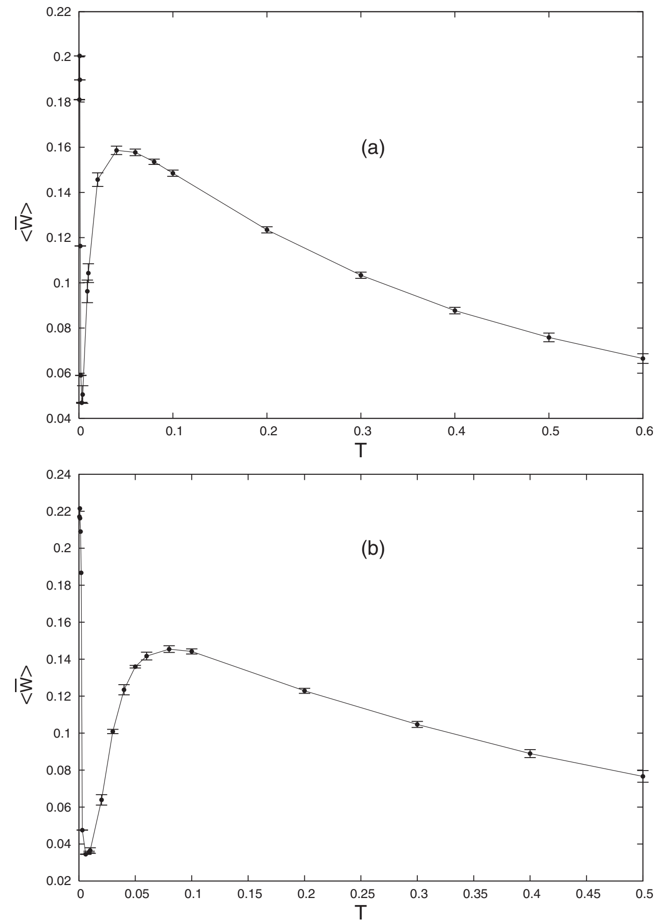

FIG. 9. Plot of $\langle\overline{W}\rangle$ as a function of T for $U(x)$ (a) and $U_2(x)$ (b).

potentials. There is, however, a remarkable difference between the nature of motion of particles at the temperature (T_{SR}) of maximum $\langle\overline{W}\rangle$ for $U_1(x)$ on one hand and $U(x)$ and $U_2(x)$ on the other.

As the temperature T goes through $T_{SR} \sim 0.2$ corresponding to maximum $\langle\overline{W}\rangle$, the particle feels the periodic nature

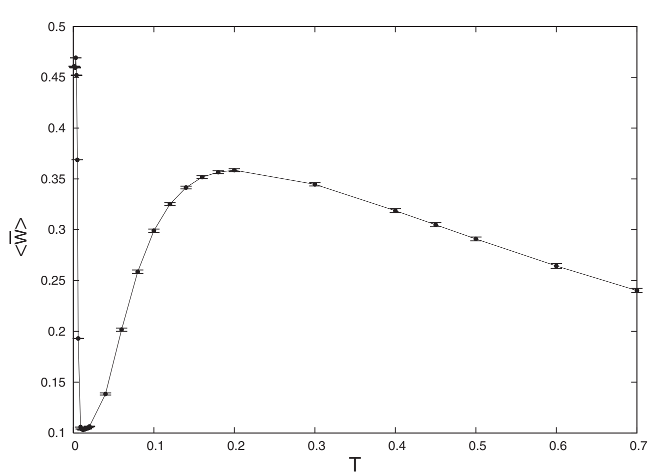

FIG. 8. Plot of $\langle\overline{W}\rangle$ as a function of T for the washboard potential $U_1(x)$.

of the potential $U_1(x)$ as the interwell transitions are quite frequent, though not as frequent as the intrawell transitions between the in-phase to out-of-phase states. The particle motion is truly in a periodic potential implying the presence of stochastic resonance in the periodic potential $U_1(x)$. On the other hand, the particle has not yet begun the interwell transitions [in $U(x)$] or have just started [in $U_2(x)$] as the temperature goes across the $\langle\overline{W}\rangle$ maximum. Hence the particle does not feel the periodicity of the potentials but the presence of the two subwells. Therefore, the resonance effectively occurs in a single well with two subwells. However, the bistability in a well of the potentials is required for that to happen. Moreover, the existence of dynamical states of trajectories is necessary for the occurrence of $\langle\overline{W}\rangle$ maximum at that low temperature, $T_{SR} = 0.04$ for $U(x)$ and $T_{SR} = 0.08$ for $U_2(x)$.

Consider the simple Smoluchowski limit of Kramers rate [43]:

$$k = \frac{\omega_0 \omega_b}{2\pi\gamma} \exp\left(\frac{-E_b}{T}\right), \tag{3.5}$$

where ω_0^2 and ω_b^2, respectively, are the curvatures at the bottom of the wells (subwells) and at the top of the barrier across the wells (subwells) of the potential $U_1(x)$ [$U(x)$ and $U_2(x)$]. This rate calculation shows that $k^{-1} \sim 3450$ across the potential barrier between two consecutive wells of $U_1(x)$ ($T = 0.2$, $\gamma = 0.12$, $E_b = 1.686$) and $k^{-1} \sim 47 \times 10^6$ and 760 across the potential barrier between the two subwells in a well of the potentials $U(x)$ ($T = 0.04$, $\gamma = 0.12$, $E_b = 0.75$) and $U_2(x)$ ($T = 0.08$, $\gamma = 0.12$, $E_b = 0.618$), respectively.

Note that the periods $\tau = 8.0$, 4.8, and 4.8 of the drive field $F(t)$ taken, respectively, for the potentials $U_1(x)$, $U(x)$, and $U_2(x)$ were too small compared to the calculated k^{-1}. Therefore, the observed SR cannot be considered as the conventional stochastic resonance following Ref. [22]. However, the resonance is brought about by the existence of and transition between the two dynamical (in-phase and out-of-phase) states of trajectories. The rates of these transitions are of the order of the period τ of the drive field. Viewed in this perspective of transition between the two dynamical states they indeed indicate SR. Moreover, the potential $U_1(x)$ at $T = T_{SR}$ a substantial number of interwell transitions are observed. But in case of $U(x)$ and $U_2(x)$, interwell transitions are seldom observed at $T = T_{SR}$. However, intersubwell transitions are comparable in number to the transitions between the dynamical states at T_{SR}. SR nature of these transitions is further supported by the behavior of input energy distributions $P(W)$ across the temperature (T_{SR}) of maximum $\langle \overline{W} \rangle$.

C. Input energy distribution and SR

The input energy distributions $P(W)$ have earlier been used in the discussion of SR in underdamped [33] and in overdamped [29,34,35] systems. $P(W)$ in the present case are shown in Fig. 10. At low temperatures, say $T = 0.0005$, $P(W)$ shows the usual bimodal distribution for the potentials $U(x)$ and $U_1(x)$. The peaks occur exactly at $W = \overline{W}$ values shown in Fig. 5, corresponding to the $\overline{W}$ in the in-phase and out-of-phase states. Figure 10 also shows $P(W)$ for the potential $U_2(x)$ exhibiting three peaks for the amplitude $\Delta F = 0.2$ of $F(t)$. This is because, as mentioned earlier, the two subwells are not equivalent because of the finite average tilt $F_0 = 0.1$ in the potential $U_2(x)$. On general considerations one would expect four peaks. The four peak $P(W)$ appears only with $\Delta F < 0.19$. The three peak $P(W)$ instead occurs because of the instablity of the in-phase state in the subwell-1 for $\Delta F \geqslant 0.19$. Here again the peaks are centered at $W = \overline{W}$ values corresponding to the three dynamical states. The correspondence of the peaks of $P(W)$ and the dynamical states is thus unambiguous at low temperatures.

In Fig. 11 are shown $P(W)$ at various elevated temperatures. As the temperature is gradually increased two important common features could be readily observed: (i) the variation of the strength of the $P(W)$ peaks, and (ii) the intrusion of the in-phase peak into the negative W domain and the appearance of a long negative W tail of $P(W)$. The analysis of these two features provides a better understanding of SR in the three potential systems.

For the potentials $U_1(x)$ and $U(x)$ the higher energy peak (corresponding to the out-of-phase state) first diminishes, disappears, and then reappears, as the temperature is gradually increased. For $U_2(x)$, the peak of $P(W)$ corresponding to the out-of-phase state in the left subwell, that is, op[0,1], first disappears, and then the peak corresponding to op[0,2] too disappears. As the temperature is increased further a broad peak (almost a plateau) appears roughly spanning the earlier two out-of-phase peaks. As the temperature is increased further the newly formed peaks begin merging with the sole

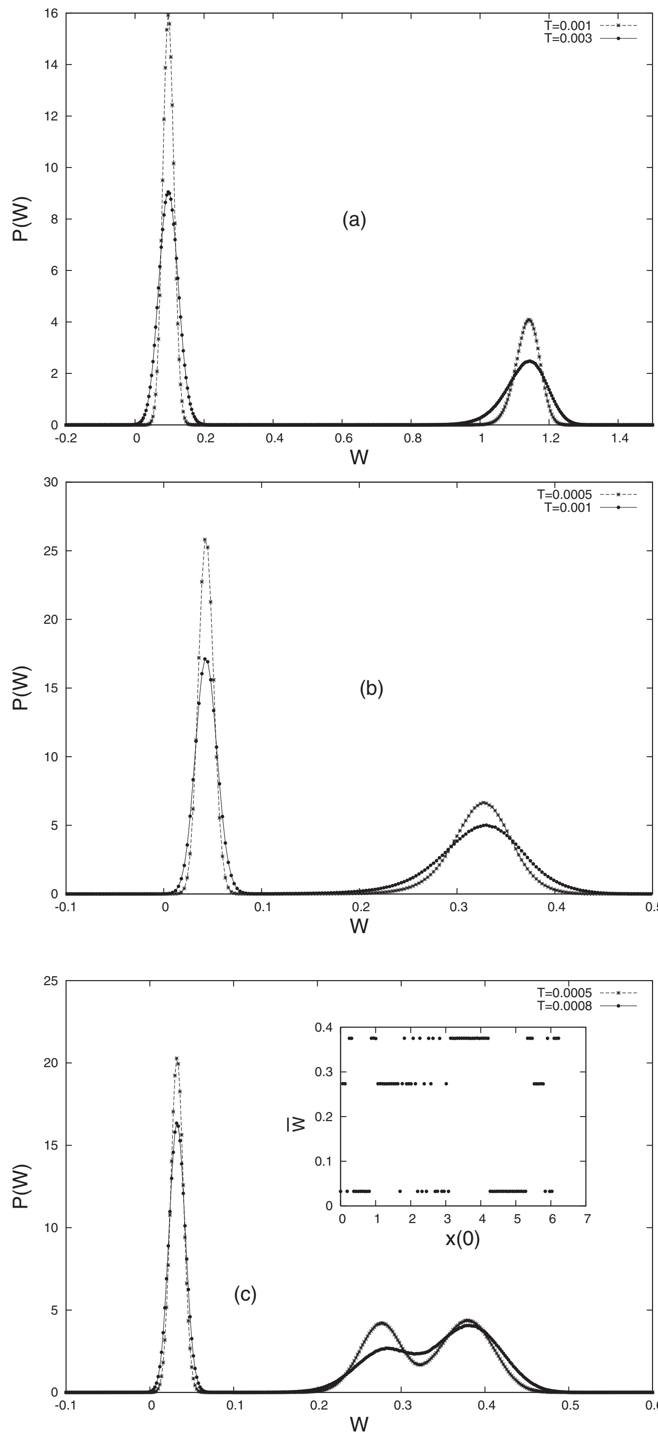

FIG. 10. Plot of $P(W)$ at different values of T for $U_1(x)$ (a), $U(x)$ (b), and $U_2(x)$ (c). For small T, the distribution is bimodal for $U_1(x)$ and $U(x)$. However, $P(W)$ shows three peaks for $U_2(x)$.

in-phase peak (for $\Delta F = 0.2$) and at the temperature T_{SR} the out-of-phase peak is left only as a receding shoulder leaving no distinguishable trace of either a hump or a plateau. However, in the process the strength of the in-phase $P(W)$ peak also diminishes. This can be seen in either of two ways: (i) by fitting the in-phase $P(W)$ peak by a Gaussian and calculating its area and (ii) by finding the ratio of the number of points in

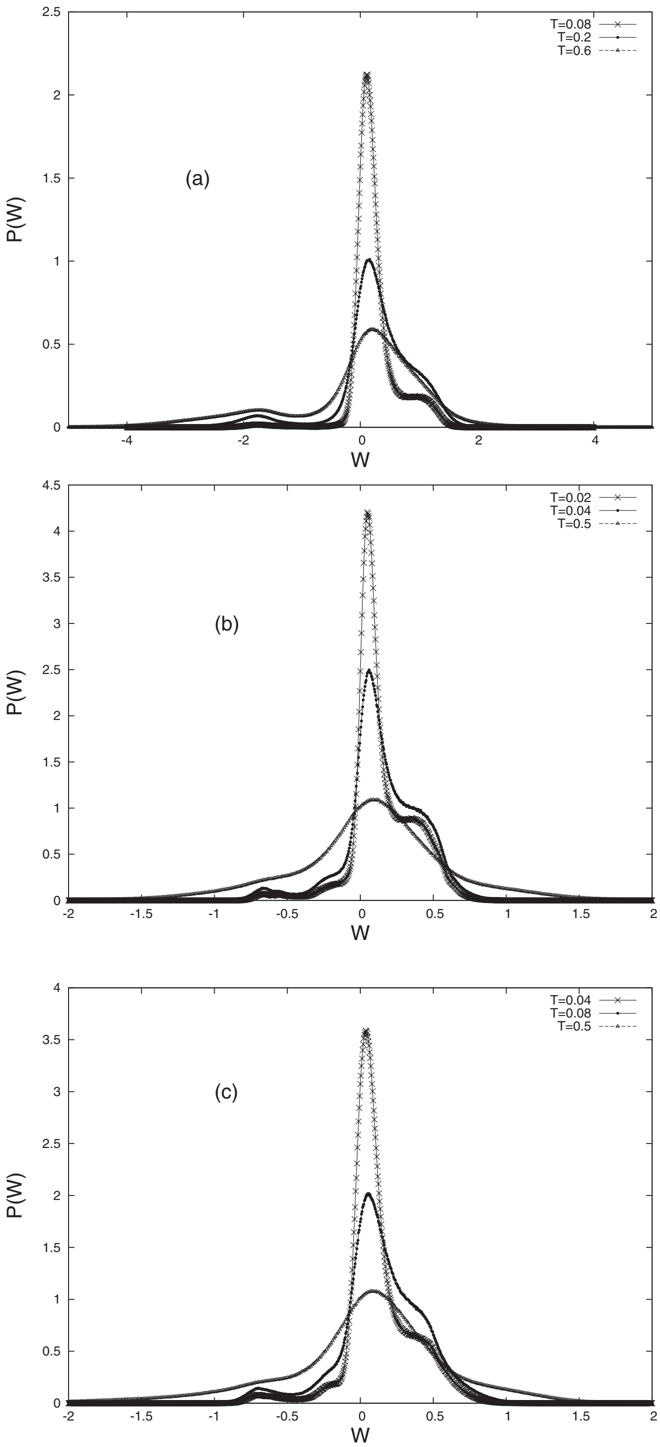

FIG. 11. Plot of $P(W)$ at different values of T for $U_1(x)$ (a), $U(x)$ (b), and $U_2(x)$ (c). At $T_{\rm SR}$, $P(W)$ is on the verge of losing its multimodal character; at higher T the distribution has a single peak structure.

the stroboscopic (Poincaré) plots falling in the in-phase region to the total number of points.

The fraction (contribution) of in-phase states in the trajectory reduces from 1 [at $T = T_{\rm min} \sim 0.02$, 0.003, and 0.003 for $U_1(x)$, $U(x)$, and $U_2(x)$, respectively, corresponding to the respective minimum of $\langle\overline{W}\rangle$] gradually and becomes almost equal to 0.5 at the temperature $T_{\rm SR}$ of maximum

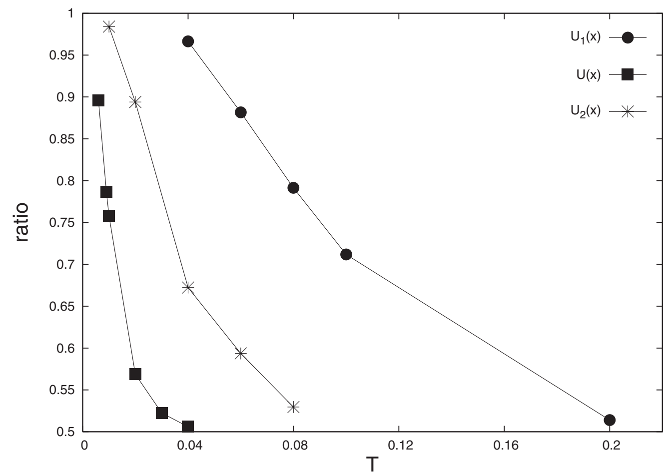

FIG. 12. The figure shows the plot of the fraction (ratio) of time spent by the particle in its in-phase state of trajectories as a function of temperature T for the potentials $U(x)$ ($T_{\rm SR} = 0.04$), $U_2(x)$ ($T_{\rm SR} = 0.08$), and $U_1(x)$ ($T_{\rm SR} = 0.2$).

$\langle\overline{W}\rangle$ [$T_{\rm SR} \sim 0.2$, 0.04, and 0.08 for $U_1(x)$, $U(x)$, and $U_2(x)$, respectively] (Fig. 12). At this temperature, it becomes hard to clearly distinguish the regions of in-phase and out-of-phase states of trajectories, just as in the liquid-gas system at the critical temperature. The temperature of maximum $\langle\overline{W}\rangle$ is said to fall in the region of kinetic phase transitions [44].

The second important feature of $P(W)$ is its intrusion into the negative W region. As mentioned earlier it has two components: (i) a systematic broadening of the in-phase peak of $P(W)$ and spilling over to $W < 0$ region and (ii) the emergence of a long $W < 0$ tail. The former happens at a temperature much lower than the temperature at which the $W < 0$ tail begins to emerge. The phase lag ϕ_1 in case of in-phase is small [$-\phi_1 \sim 0.1\pi$, 0.08π, and 0.08π, respectively, for the $U_1(x)$, $U(x)$, and $U_2(x)$ potentials]. Also the in-phase $P(W)$ peak is centered close to $W \geqslant 0$. As the temperature is increased the fluctuations in the value of ϕ_1 occur leading sometime to make $\phi_1 > 0$. In other words, the thermal effect makes $x(t)$ occasionally lead the forcing $F(t)$. Thus, a few individual HLAs acquire a sign opposite to what is observed in the usual case when causality is respected. Therefore, occasionally, W becomes negative and the in-phase peak of $P(W)$ broadens into the $W < 0$ region. This process continues and becomes more evident as the temperature is increased.

Analyses of the trajectories $x(t)$ and the accompanying hysteresis loops $x(F(t))$ reveal that the violation of causality (leading to $W < 0$) occurs during the transitions between the dynamical (in-phase and out-of-phase) states. In these situations the magnitude of negative W (hysteresis loop area with opposite sign) often becomes very large. The origin of negative tail of $P(W)$ lies in these transitions between the dynamical phases. Since before the temperature at which $\langle\overline{W}\rangle$ becomes minimum at most one (out-of-phase to in-phase) transition occurs in the entire history of any trajectory spanning about 10^5 periods of $F(t)$ the negative tail of $P(W)$ does not show up till the temperature $T_{\rm min}$ of minimum $\langle\overline{W}\rangle$ [$T_{\rm min} \sim 0.02$, 0.003, and 0.003 for $U_1(x)$, $U(x)$, and $U_2(x)$, respectively].

As the temperature is gradually increased further, the transitions, initially from the (all) in-phase to the out-of-phase states take place and the high energy peak (plateau) reappears. Thereafter, transitions in both directions become more and more frequent ultimately causing the newly emerged high energy (out-of-phase) peak (plateau) to merge with the in-phase peak at the resonance temperature T_{SR}. The increase of negative tail of $P(W)$ goes hand in hand with increasing T. At temperatures $T \geqslant T_{SR}$ the transition regions dominate over the regions of in-phase and out-of-phase dynamical states. At these temperatures the increasing long negative tail and the ever broadening in-phase peak of $P(W)$ bring down $\langle\overline{W}\rangle$ from its maximum value at (T_{SR}). Note that on the average causality is always respected and $\langle\overline{W}\rangle$ never becomes negative.

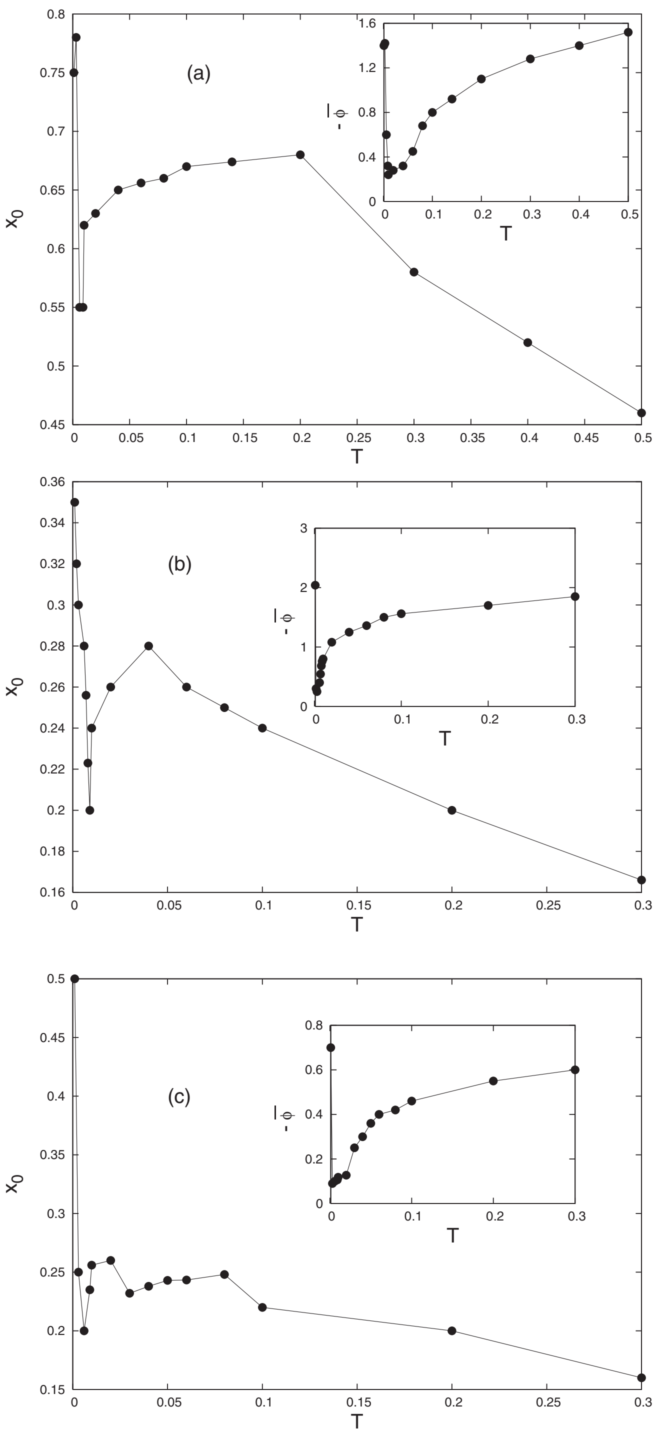

FIG. 13. Plot of x_0 and $-\overline{\phi}$ (insets) with T for $U_1(x)$ (a), for $U(x)$ (b), and for $U_2(x)$ (c).

D. Average amplitude and phase of hysteresis loops

The average hysteresis loops $\langle\overline{x}(F(t_i))\rangle$ are calculated using Eq. (3.4). Since the amplitude ΔF of the external drive $F(t) = \Delta F \cos(\frac{2\pi}{\tau}t)$ (=0.2) is much smaller than the periodic barrier heights (∼2.0) between two consecutive wells, the amplitude x_0 of the average response $\langle\overline{x}(F(t_i))\rangle$ is small and the relation $\langle\overline{x}(F(t_i))\rangle = x_0 \cos(\frac{2\pi}{\tau}t + \overline{\phi})$, with $t = n\tau + t_i$, is found to follow quite well for all three potentials at low temperatures. However, at higher temperatures the relation serves reasonably well for the potential $U_1(x)$, and only approximately for $U(x)$ and $U_2(x)$.

The amplitude x_0 and phase $|\overline{\phi}|$ acquire their respective minima at the temperature T_{min} of minimum $\langle\overline{W}\rangle$ corresponding to the sole dynamical (in-phase) state (Fig. 13). At this temperature $\overline{\phi} \sim \phi_1$. As the temperature is gradually increased from $T = T_{min}$ x_0 peaks at $T \sim T_{SR}$. However, $\overline{\phi}$ shows a monotonic behavior. At these temperatures $|\phi_1| < |\overline{\phi}| < |\phi_2|$ because the trajectories consist of a mixture of in-phase and out-of-phase states. The variation of $\overline{\phi}$ is similar to what is reported earlier [41] and does not show a peak [42].

IV. DISCUSSION AND CONCLUSION

The existence of two dynamical states in a sinusoidally driven underdamped system is quite special to periodic potentials. The (quadratic) harmonic potentials or the (quartic) Landau potentials show only one (in-phase) state. The periodic nature of the potentials allows the particle to explore (in space) regions of high nonlinearity. The high nonlinearity of periodic potentials appears to be responsible for the occurrence of these two states (especially the additional high amplitude out-of-phase state) in a well in a sinusoidal potential or in a subwell in the bistable periodic or washboard potential.

The investigation of dependence of the existence of two dynamical states on the amplitude of the drive field could also be of interest in periodic potentials. As observed earlier, the in-phase state in the left subwell of the potential $U_2(x)$ disappears for $\Delta F > 0.19$. However, if ΔF is chosen to be too small the particle may not have the opportunity to explore the highly nonlinear regions of the potential and the out-of-phase trajectories may disappear. Therefore, the choice of ΔF too is crucial for the study of SR in these potentials.

In the conventional SR it is the bistability of the potential that plays a crucial role. In the present periodic (or washboard) potential case SR is brought about by the bistability (or multistability) of the states of trajectories in a well (subwell) together with the adjacent wells (subwells) of the potential. Moreover, if the system is initially prepared in a given well of a double-well potential the conventional SR occurs when the probability distribution of particles becomes equal in both the wells in a finite (small) time (typically, of the order of the period of the drive field). In other words, the rate of passages across the potential barrier between the two wells becomes equal. Exactly similar is the condition in the periodic potential cases considered in the present work. The ratio of time spent by the particle in either of the two states of trajectories to the total time reaches 0.5 as the temperature rises to T_{SR} (Fig. 12).

In order to observe SR in the underdamped "periodic" potentials discussed above it is necessary to choose the period τ of the drive field judiciously. As has been pointed out in Ref. [17] the natural period of small (harmonic) oscillations about the bottom of the potential wells increases with temperature (and also with damping). In the mechanism of occurrence of SR in terms of the transitions between the dynamical states the period τ of drive field is required to be close to the natural period of oscillation. It is therefore understandable that T_{SR} increases monotonocally with τ. The lower limit of τ is, however, determined by the natural period τ_0 [$\sim 2\pi$ for the potential U_1 and $2\pi\sqrt{\frac{6}{13}}$ for $U(x)$], corresponding to temperature zero (or the undamped oscillation condition). In our case, the lower bound of τ is seen to be $\tau \geqslant 1.1\tau_0$ for all three potentials U, U_1, and U_2. However, no such upper cutoff for τ could be ascertained. One can, therefore, arbitrarily set the upper limit at $\tau \sim 2\tau_0$. At this large τ, $T_{SR} > 0.7$, which is comparable to the potential barrier (~ 2) between any two adjacent wells of the potentials. Also, at these large values of τ and larger, the $\langle\overline{W}\rangle$ peak becomes too broad and hence loses its relevance as a phenomenon of practical significance. On the other hand, if τ is taken a little larger than the lower limiting value (for example, $\tau = 5.0$ for U) the interwell transitions become quite frequent at T_{SR} without losing the fine quality of SR and hence the motion is truly in the periodic potential and not limited to a single well. This observation provides further support to the thesis of the presence of SR in these underdamped "periodic" potentials.

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