BORDISM, GROUP ACTION AND FIXED POINTS SET : A SURVEY

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CERTIFICATE

I certify that the dissertation entitled ``Bordism, Group action and fixed points set: A survey'' submitted by Miss Sanghita Dutta in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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PREFACE

The classification of differentiable manifolds is a basic problem of differential topology, the most important being bordism classification. Two closed manifolds are said to be bordant if their disjoint union is boundary of some compact manifold. The set of all bordism classes of all closed manifolds having dimension n form an abelian group under the operation of disjoint union of manifolds. This group is termed as n-dimensional bordism group and its theory is termed as bordism theory. It became an important area of differential topology when Pontrjagin [27] in 1947 developed the notion of characteristic numbers of a manifold. His theorem that the characteristic numbers of a closed manifold are zero if the manifold is a boundary was the first to provide some invariants for bordism classification. Then came the monumental discovery of R.Thom [34] in 1954 which revealed that the problem of bordism is equivalent to homotopy problem. This opened new avenues for solving many classification problems. Later Wall [35] in 1960 considered the bordism of oriented manifolds and proved that an oriented manifold bounds if and only if all its Stiefel-Whitney numbers and Pontrjagin numbers are zero. In 1973 Lee and Wasserman [36] developed the notion of equivariant characteristic numbers for G-manifolds. Khare developed these invariants for unoriented singular G-manifolds in 1976 [12], for oriented singular G-manifolds in 1982 [13] and for (F,F′) – free manifolds in 1984 [14].

Bordism theory took a different turn when in 1964 Conner and Floyd introduced bordism method to the study of group action [5] . His technique was applied to equivariant bordism by Stong, Kosniowski, Ossa, Wheeler, Khare and many others in the later period. Analysing the action of $\mathbb{Z}_2^k$-actions on a manifold, Conner and Floyd [5] proved that if $\mathbb{Z}_2^k$ acts smoothly on a closed manifold M without any fixed points then M is a boundary. Stong [32] strengthened this result by proving that under the same condition M is a $\mathbb{Z}_2^k$-boundary. Khare developed this result further first for a finite abelian group [15] and then for any finite group [16]. The crucial role of 2-central component $G_2(C)$ of the centre of a finite group in its action on closed manifolds appeared clear for the first time in [16]. Khare proved that if $G_2(C)$ acts on M without any fixed point then M is a G-boundary. After analysis of torus action in [17], Khare proved a general result for a compact Lie group action in [18]. This is the most general result available in this direction and the results of Conner and Floyd, Stong and Khare's earlier results are direct consequences of this general result.

It is well known that if M is a $\mathbb{Z}_2$ manifold in which the fixed points set has codimension 1, then M is a $\mathbb{Z}_2$-boundary.

The mapping cylinder provides the required $\mathbb{Z}_2$-manifold with boundary. In 1967 J.M.Boardman [3] proved that if M is a $\mathbb{Z}_2$- manifold with M a boundary then dim M > (5/2)x dim F, where dim F is the maximal dimension of the components of fixed points set F of $\mathbb{Z}_2$-action on M. This is known as Boardman's five-halves theorem. In 1978 Kosniowski and Stong [23] proved its equivariant version. They also proved that if M is a $\mathbb{Z}_2$-manifold such that the fixed points set has constant dimension and dim M > 2.dim F, then M is a $\mathbb{Z}_2$-boundary. The basic technique used by them in this work is the caculation of Stiefel-Whitney numbers of M in terms of the fixed points data $\{(F^{n-k}, \gamma^k)\}$ by a specific formula.

In [16], Khare has considered the case when the fixed points set F of $G_2(C)$, the two central component of the centre of the group G, in M is nonempty. Let $F = \bigcup\limits_{0}^{n} F^l$, where F^l denote the ℓ-dimensional component of F. Let $D(\gamma_l)$ be the normal disc bundle of F^l in M^n with the induced action θ_l of G on $D(\gamma_l)$. He has proved that if $G/G_2(C)$ acts trivially on F and $\exists$ some positive dimensional G-representation (W_l, ϕ_l), $0 \leq l \leq n$, such that $[D(\gamma_l), \theta_l] = [F^l][D(W_l), \phi_l]$ in $\mathfrak{N}_*(G: \mathscr{A}, P)$, where P is the family of all subgroups of G which do not contain $G_2(C)$, $\mathscr{A}$ is the family of all subgroups of G and $D(W_l)$ is the unit disc of W_l, then the G-manifold M is a G-boundary. As a particular case, one gets Conner and Floyd result (Theorem 31.3) in [5] that if $(\mathbb{Z}_2)^k$ acts differentiably on a closed n-manifold with n

0, then there can not be precisely one fixed point.

In 1960, M. F. Atiyah and R. Bott in [1], showed that if for a prime p, $\mathbb{Z}_p$ acts on a homology shpere with two isolated fixed points then the representations of $\mathbb{Z}_p$ on the tangent space at each fixed point are same. This shows that the manifold is $\mathbb{Z}_p$-bordant to one with no fixed point, that is, to a free $\mathbb{Z}_p$-manifold. The bordism is obtained by removing a disk about each of the fixed points and attaching a handle equivariantly. In 1976, Ewing [9] has proved the same result for any $\mathbb{Z}_p$-manifold for which equivariant signature vanishes. Very recently Kosniowski and J. Ewing [10] proved that the same result is true for any smooth $\mathbb{Z}_p$-manifold provided that the dimension of the fixed points set is not too small.

They also proved a more general result in the case when number of isolated fixed points is $n \geq 2$. Their result is that if $n < 1 + \frac{2\{n/2\}}{(p-1)\{\log_p n\} - 2}$ then M is $\mathbb{Z}_p$-bordant to a free $\mathbb{Z}_p$-manifold. Here $\{x\}$ denotes the least integer greater than or equal to x.

In 1981, Stong [30] proved that a manifold M^{2n-1} is bordant to a manifold N^{2n-1} with involution T having fixed points set of diemension n ⇔ every Stiefel Whitney number of M^{2n-1} involving a product of two odd dimensional Stiefel Whitney classes is zero. In 1986, Stong further [31] studied the problem: "Which compact Lie-groups G can act smoothly (and effectively) on a closed oriented manifold M^n of positive dimension so that the fixed points set consists of precisely one point?"

In 1989, Demichelis Stefano [8] showed that a finite group G acting effectively, locally linearly and preserving orientation on a $\mathbb{Z}$-homology 4 sphere has a fixed point set a k-sphere , $k \leq 2$. Wu [39] in 1990 extended the study of fixed points set made by Stong in [30]. He considered J_m^k as the group of unoriented bordism classes of m-dimensional smooth manifolds which are represented by manifolds with smooth involutions having m-k dimensional fixed points set and obtained a necessary and sufficient condition for a bordism class to lie in J_n^{2k} and J_n^{2t+1} for $k \leq 2$ and $t \leq 9$. The groups J_{2n-1}^{n-1} and J_{2n-2}^{n-2} were studied by Stong in [30]. Waner [34] in 1990 proved that a unitary $\mathbb{Z}_p$s-manifold cannot have a single fixed point.

In [22], Kosniowski proved that if M is a S^1-manifold with two isolated fixed points then $\exists$ an integer r such that 2^r copies of M bounds as an S^1-manifold. In [21], Kosniowski has shown that if M is a unitary S^1-manifold of dimension not equal to 2 or 6 with the fixed points set a homology sphere then M is an S^1-boundary. He further proves that if M is a unitary S^1-manifold of dimension not equal to 6 with the fixed points set having integral homology of a product of two odd dimensional spheres then M is an S^1-boundary.

Conner, Floyd, Stong, and Khare used the technique of family of subgroups to study the equivariant bordism problems. It was for the first time that Kosniowski [19] realised that considering family of G-slice type is much more stronger (though complicated) tool than considering family of subgroups. He has shown that in equivariant bordism theory , for some families of G-slice types, it is possible for the whole theory to vanish.

This dissertation is aimed at studying bordism, group action and fixed points set. In chapter I, we recall some definitions and we discuss the proof of the theorem due to Thom:

Theorem 1.2.10. For $k > n+1$, the homotopy group $\Pi_{n+k}(MO(k))$ is canonically isomorphic to the unoriented cobordism group $\mathfrak{N}_n$.

In this chapter we have also discussed the oriented version of the above theorem.

In chapter II, we recall some preliminary definitions and the definition of G-characteristic classes of G-manifolds. We study in this section the theorem of Lee and Wasserman [36]:

Theorem 2.2.5. Let $[M] \in \mathfrak{N}_*(G)$. Then $[M] = 0$ if and only if all h^*-characteristic numbers vanish.

Here in this section we also recall the definition of characteristic classes of unoriented and oriented singular G-manifolds defined by Khare and we study his theorem of invariance of charateristic classes of unoriented and oriented singular G-manifolds under bordism:

Theorem 2.4.3. If all the characteristic numbers of the map $f : M^n \longrightarrow X$ for an element $[M^n, f:G] \in \mathfrak{N}_*(X;G)$, corresponding to the theory h^*, are zero, then $[M^n, f;G]$ is zero in $\mathfrak{N}_n(X;G)$.

We also study the oriented version of the same in chapter II.

In chapter III, we study the bordism method of studying the group action which was introduced by Conner and Floyd:

Theorem 3.1.5. If $\mathbb{Z}_2^k$ acts smoothly on a closed manifold M^n without fixed points then $[M^n] = 0$.

Stong has strengthened the above theorem by showing in [32] that:

Proposition 3.2.6. If $\mathbb{Z}_2^k$ acts on a closed differentiable manifold M^n, then $[M^n, \theta] = 0$ in $\mathfrak{N}_n(\mathbb{Z}_2^k)$.

We study the above theorem and also the result that was proved by Khare in the case of a finite group. The crucial role of 2-central component $G_2(C)$ of the centre of a finite group G in its action on closed manifolds appeared clear for the first time in [16]. Khare proved that:

Corollary 3.3.6. If $G_2(C)$ acts on M^n under θ without any stationary point then (M^n, θ) is a G-boundary.

After analysis of torus aciton in [17], Khare proved a general result for a compact Lie group action in [18]. In chapter III, we discuss this result:

Definition 3.4.1. Let H be a compact Lie group. Then H is said to be H-boundary if there exists an H-manifold N such that $\partial N = H$ and the restriction of action θ on ∂N coincides with the given group operation of H.

Let G be a compact Lie Group. By the central elementary H-group in G, we mean the maximal subgroup $H^n = H \times \dots \times H$, n-times, contained in the centre of G.

Let G be a compact Lie group with H^n the central elementary H-group in G, H being H-boundary. Let h_o be a fixed point of H and $p_r : H^n \longrightarrow H$ denote the projection onto the r-th factor, $1 \leq r \leq n$. Let H_r denote the subgroup of H^n with $p_i(H_i) = H$, if $i = r$ and $p_i(H_r) = h_o$, if $i \neq r$. Let a family $\{L_r\}$ of subgroups of G be such that $L_r \cap H_r$ is a nontrivial subgroup of H_r, $1 \leq r \leq n$. By an *$\{L_r\}$-type action* of G, we mean a differential action of G on a differential manifold M such that for every x in M, $p_r(G_x \cap H^n)$ is either trivial or contains L_r for all r, G_x being the isotropy group of x. A point x in M is said to be a *pseudo stationary point* if $p_r(G_x \cap H^n)$ is nontrivial for all r, $1 \leq r \leq n$.

Corollary 3.4.5. Let M^n be a closed G-manifold with $\{L_r\}$-type of action for some family $\{L_r\}$ of subgroups of G such that $L_r \cap H_r$ is nontrivial. If M^n does not have any pseudo stationary point, then M^n is a G-boundary.

We also discuss the work of Wheeler [38] who has shown that:

Theorem 3.3.9. If $F_{\mathbb{Z}_{2^j}}(X,A)$ is 2-even (2-odd) for $0 \leq j \leq k$ then $\Omega_*(\mathbb{Z}_{2^k})(\mathcal{F})(X,A) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even (odd) dimensional generators.

Here the fixed points pair $F_{\mathbb{Z}_2^j}(X,A)$ of the pair (X,A) under $\mathbb{Z}_2^j$ is said to be 2-even (2-odd) if and only if $H_*(X,A;R_2)$ is a free R_2-module on even (odd) dimensional generators.

In chapter IV, we study Boardman's five halves theorem [3] which states that:

Theorem 4.1.17. Let T be smooth involution on a smooth closed n-dimensional manifold V, and let k be the fixed point dimension (that is maximum of the dimensions of different components of the fixed points set). If V is a boundary, then $n \leq 5k/2$. If further the class [V] is indecomposable in the cobordism ring $\mathfrak{N}_*$, then $n \leq 2k+1$.

In this chapter, we also describe the result of Khare [16]:

Definition 4.2.1. Suppose (M^n,θ) is a closed G-manifold and $F = \bigcup_{l=0}^{n} F^l$, where F^l is the l-dimensional component of F, is the fixed points set of M^n under the subgroup $G_2(C)$ of G consisting of all elements of order 2 in the centre C of G. Let $D(\nu_l)$ be the normal disk bundle of F^l in M^n with the induced action θ_l of G on $D(\nu_l)$. Then F is said to have an equivariant trivial normal bundle in M^n, if $G/G_2(C)$ acts trivially on F and there exists some positive dimensional G-representations (W_l,ϕ_l), $0 \leq l \leq n$, such that in $\mathfrak{N}_*(G;\mathscr{A},P)$, where $\mathscr{A}$ is the family of all subgroups of G and P is the family of subgroups of G not containing $G_2(C)$, $[D(\nu_l),\theta_l] = [F^l][D(W_l),\phi_l]$, $D(W_l)$ being the unit disk of W_l.

Theorem 4.2.1. If F has an equivariant trivial normal bundle in M^n, then F is a boundary and (M^n,θ) is a G-boundary.

Further in this chapter, *we* study the *work* of Kosniowski and Stong's work on the calculation of characteristic number of a $\mathbb{Z}_2$-manifold in terms of its fixed points set.

Theorem 4.3.1. Let M^n be a closed n-dimensional manifold with a smooth involution T. Let F^{n-k} be the union of (n-k)-dimensional components of the fixed points set of T and ν^k be the normal bundle of F^{n-k} in M^n. If $f(x_1, \ldots, x_n)$ is a symmetric polynomial over $\mathbb{Z}_2$ in n-variables of degree at most n, then

$$f(x_1, \ldots, x_n)[M^n] = \sum_k \frac{f(1+y_1, \ldots, 1+y_k, z_1, \ldots, z_{n-k})}{\prod_{i=1}^{k}(1+y_i)}[F^{n-k}]$$

where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_i(x)$, $\sigma_i(y)$ and $\sigma_i(z)$ by the Stiefel-Whitney classes $w_i(M^n)$, $w_i(\nu^k)$ and $w_i(F^{n-k})$ respectively and taking the value of the resulting cohomology class on the fundamental homology class of M^n or F^{n-k}.

We described in this chapter the equivalent condition developed by Stong [30] for a manifold to be bordant to another manifold with involution having the fixed points set of dimension n.

Theorem 4.4.8. A class $\alpha \in \mathfrak{N}_{2n-1}$ is represented by a manifold M^{2n-1} with involution T having fixed points set of dimension n if and only if every Stiefel-Whitney number of α involving a product of two odd dimensional Stiefel-Whitney classes is zero.

We also study the compact Lie-groups G that can act smoothly (and effectively) on a closed oriented manifold M^n of positive dimension so that the fixed point set consists of precisely one point. We study various results proved by Stong [31] in this direction which can be summarised as follows:

(a) For every non-abelian group, there exists a closed connected oriented manifold of positive dimension on which the group can act with exactly one fixed point

(b) If G is abelian of positive dimension with G_o the component of identity then

(i) for $|G/G_o|$ even, there exists a closed connected orientable manifold of positive dimension on which G can act with precisely one fixed point

(ii) for $|G/G_o|$ odd, divisible by at least two primes, there exists a closed oriented manifold of positive dimension on which G can act with precisely one fixed point, but no such action on a

connected manifold can be effective

(iii) for $|G/G_o|$ a power of an odd prime, G cannot act on a closed oriented manifold of positive dimension with precisely one fixed point

(C) If G is finite abelian then G cannot act on a closed oriented manifold of positive dimension with precisely one fixed point if either $G = (\mathbb{Z}_2)^k$ or G is an abelian p-group with p odd. For all other finite abelian groups G, there exists a closed connected oriented manifold of positive dimension on which G can act with exactly one fixed point. Finally we give some recent results proved by Demichelis Stefano and Wu regarding fixed points set.

In chapter V, we study the equivariant bordism theory using G-slice types. In this chapter we discuss the development made by Deb and Khare considering any finite group, of the work of Kosnoiwski who has shown by considering finite abelian groups that using G-slice types in equivariant bordism theory it is possible for the theory to vanish.

Let G be an abelian group, M a G-manifold and G_x the isotropy subgroup at x. For every $x \in M$ there is a G_x-module $\tilde{V}_x$ whcih is equivariantly diffeomorphic to a G_x-neighbourhood of x. The G_x-module $\tilde{V}_x$ can be decomposed as $\tilde{V}_x = V_x \oplus V'_x$, where G_x acts trivially on V'_x and no nonzero vector of V_x is fixed by all of G_x, i.e. V_x does not have any trivial G_x-submodule. The pair $[G_x, V_x]$ is called the slice type of $x \in M$. By a G-slice type we mean a pair [H; U], where H is a subgroup of G and U is an H-module having no trivial H-submodule.

A family $\mathcal{F}$ of G-slice types is a collection of G-slice types such that $[H; U] \in \mathcal{F}$ implies that $[G_x, V_x] \in \mathcal{F}$, $\forall x \in G \times_H U$. A G-manifold M is said to be $\mathcal{F}$-free if $\forall x \in M$, the slice type $[G_x, V_x]$ at x belongs to $\mathcal{F}$. In usual way one has the notion of $\mathcal{F}$-bordant. This gives an equivalence relation on the set of all $\mathcal{F}$-Free G-manifolds of dim n. The set of equivalence classes can be given group structure in the usual manner giving rise to the group $N^G_*[\mathcal{F}]$.

In [19], Kosniowski constructed a family $\hat{\mathcal{F}}$ of G-slice types such that $N^G_*[\hat{\mathcal{F}}] = 0$. This gives in particular Khare's result namely if G_2 acts on M without fixed points then M is G-boundary. Considering G to be a finite group or $(S^1)^k$ or a compact abelian Lie group, Khare and Deb have constructed a family $\tilde{\mathcal{F}}(\hat{G})$ of G-slice types for which they have shown that:

<u>Theorem 5.5.5</u>. $\mathfrak{N}^G_n[\tilde{\mathcal{F}}(\hat{G})] = 0$.

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CHAPTER I

Preliminaries

In this chapter we discuss the homotopy interpretation of unoriented bordism, oriented bordism. We also introduce unoriented and oriented singular bordism. Throughout this chapter, unless mentioned otherwise, the manifolds are closed, smooth and compact.

§1. Some definitions and elementary results.

In this section we introduce some elementary and essential notions and give some necessary results that may be used later on.

(A) Unoriented bordism.

For any integer $n \geq 0$ let N_n be the set of all closed unoriented n-dimensional manifolds. A manifold $M^n \in N_n$ is said to bord if M^n is the boundary of some $(n+1)$-dimensional compact unoriented manifold W^{n+1}. A manifold $M_1^n \in N_n$ is said to be bordant to a manifold $M_2^n \in N_n$, denoted by $M_1^n \sim M_2^n$, if their disjoint union $M_1^n \amalg M_2^n$ bords. It is easy to see that $\sim$ is an equivalence relation in N_n. The quotient set $N_n/\sim$, denoted by $\mathfrak{N}_n$, is an abelian group with respect to the operation 'disjoint union', i.e. $[M_1^n] + [M_2^n] = [M_1^n \amalg M_2^n]$. $\mathfrak{N}_n$ is called the n-dimensional unoriented bordism group. More precisely $\mathfrak{N}_n$ is a vector space over $\mathbb{Z}_2$. There is a bilinear symmetric map of $\mathfrak{N}_i \otimes \mathfrak{N}_j$ into $\mathfrak{N}_{i+j}$ induced by the operation 'cartesian product of manifolds' and $\mathfrak{N}_* = \bigoplus_{i=0}^{\infty} \mathfrak{N}_i$ becomes a graded commutative algebra over $\mathbb{Z}_2$. Thom [34] has shown that $\mathfrak{N}_*$ is a free commutative polynomial algebra over $\mathbb{Z}_2$ with one generator in each dimension n except those of the form 2^m-1; for n even, projective space $\mathbb{R}P^n$ is a generator. For n odd and $n \neq 2^m-1$, Dold has shown that the generator can be chosen as the bordism class of the Dold manifolds $P(2^r-1,\ 2^r.s)$ where r and s are given by $n + 1 = 2^r(2s + 1)$ and $P(m,n)$ is obtained from $S^m \times \mathbb{C}P^n$ by identifying $(x,\ [z])$ with $(-x,\ [\bar{z}])$. Dimension of $P(m,n)$ is $m + 2n$.

(B) Universal vector bundle.

Consider $BO(n) = G_n(\mathbb{R}^{\infty})$ = the set of all n − dimensional linear subspaces of $\mathbb{R}^{\infty}$ and $EO(n) = \{ (X,\ x) \mid X \in G_n(\mathbb{R}^{\infty}),\ x \in X \}$. Define $p : EO(n) \longrightarrow BO(n)$ given by $p(X,x) = X$.

Then $\gamma_n : EO(n) \longrightarrow BO(n)$ is the n-dimensional Universal vector bundle dimension n in the sense that if ξ is a vector bundle of dimension n over a paracompact base $B(\xi)$ then there exists a map f, unique upto homotopy from $B(\xi)$ to $BO(n)$ such that $\xi \cong f^*(\gamma_n)$. $BO(n)$ is called the classifying space for the n-dimensional vector bundles and f is called the classifying map for ξ.

Now for each n we have a map

$$i_n : BO(n) \hookrightarrow BO(n+1)$$

such that i_n maps a point $(x_1, x_2, \ldots, x_n)$ of an n-dimensional plane X in $BO(n)$ into the point $(x_1, \ldots, x_n, 0)$ of an $(n+1)$-dimensional plane X' in $BO(n+1)$, and with this

$\{BO(n)\}$ is a directed system.

Consider $BO = \operatorname{dir}\lim\limits_{n\longrightarrow\infty} BO(n)$, $EO = \operatorname{dir}\lim\limits_{n\longrightarrow\infty} EO(n)$ and $\gamma = \operatorname{dir}\lim\limits_{n\longrightarrow\infty} \gamma_n$.

Then $\gamma : EO \longrightarrow BO$ is a universal vector bundle which classifies vector bundles of all dimensions over paracompact base, i.e., if ξ is any vector bundle over a paracompact base $B(\xi)$ then there exists a map $f : B(\xi) \longrightarrow BO$, unique upto homotopy, such that $\xi \cong f^*(\gamma)$.

<u>Definition 1.1.1.</u> Let $\xi : E(\xi) \longrightarrow B(\xi)$ be a real vector bundle and ξ has a Riemannian metric. Then we define the associated <u>disc bundle</u>, denoted by $D(\xi)$, as the subbundle of $E(\xi)$ consisting of $x \in D(\xi)$ with $\| x \| \leq 1$ and the associated <u>sphere bundle</u>, denoted $S(\xi)$, as the subbundle of $E(\xi)$ consisting of $x \in S(\xi)$ with $\| x \| = 1$.

<u>Definition 1.1.2 .</u> Let $\xi : E(\xi) \longrightarrow B(\xi)$ be a real vector bundle with a Riemannian metric. Then the Thom space $T(\xi)$ of the vector bundle ξ is defined to be the space $D(\xi)/S(\xi)$.

<u>Remark 1.1.3.</u> If the base space $B(\xi)$ of a real vector bundle $\xi : E(\xi) \longrightarrow B(\xi)$ is a compact space then $T(\xi)$ can be identified with one-point compactification of $E(\xi)$. The correspondense $x \longmapsto x / \sqrt{1-\|x\|^2}$ maps $D(\xi) \setminus S(\xi)$ diffeomorphically onto $E(\xi)$, inducing the required homeomorphism from $T(\xi) \longrightarrow E(\xi) \cup e_0$, where e_0 is the point at infinity.

<u>Notation 1.1.4.</u> The Thom space $T(\gamma_k)$ of the universal

vector bundle $\gamma_k : EO(k) \longrightarrow BO(k)$ is denoted by $MO(k)$.

Definition 1.1.5. Let M and N be smooth manifolds of dimension m and n respectively and let $f : M \longrightarrow N$ be a smooth map. Then f has a point $y \in N$ as a regular value throughout some subset $X \subseteq M$ if for every $x \in f^{-1}(y) \cap X$ the induced map

$$D_x f : T_x M \longrightarrow T_y N$$

of tangent spaces is surjective.

Lemma 1.1.6. Let $W \subseteq \mathbb{R}^m$ be an open subset and $f : W \longrightarrow \mathbb{R}^k$ be a smooth map. Let f have the origin as a regular value throughout a closed subset $X \subseteq W$. Let K be a compact subset of W. Then there exists a smooth map $g : W \longrightarrow \mathbb{R}^k$ which is homotopic to f and coincides with f outside a compact set and which has the origin a regular value throughout $X \cup K$, (cf. [25]).

Theorem 1.1.7. Every continous map $f : S^{n+k} \longrightarrow MO(k)$ is homotopic to a map g which is smooth throughout $g^{-1}(EO(k))$ and is transverse to the zero-section of $BO(k)$.

Proof. We first approximate f by a map f_0 which is smooth throughout $f_0^{-1}(EO(k))$. Now, we choose an open covering of the compact set $f_0^{-1}(BO(k))$ by open subsets $W_1, \ldots\ldots, W_r$ of $f_0^{-1}(EO(k))$ such that

$$f_0(W_i) \subseteq (\gamma_k)^{-1}(U_i) \cong U_i \times \mathbb{R}^k$$

where U_i is an open subset of B such that $\gamma_k|_{U_i}$ is trivial.

We choose compact sets $K_i \subseteq W_i$ such that $f_0^{-1}(BO(k)) \subseteq$ int $(K_1, \ldots, K_r)$. We now modify f_0 on each subset W_i one after another inductively.

We assume that f_{i-1} has been chosen to satisfy the following conditions :

(i) f_{i-1} is a smooth map on $f_{i-1}^{-1}(EO(k)) = f_0^{-1}(EO(k))$ and coincides with f_{i-1} outside a compact subset of W_{i-1},

(ii) f_{i-1} is transverse to $BO(k)$ on $K_1 \cup \dots \cup K_{i-1}$ and

(iii) the projection $\gamma_k(f_{i-1}(x)) \in BO(k)$ is equal to $\gamma_k(f_0(x))$, $\forall\, x \in f_0^{-1}(EO(k))$.

We define $f_i|_{W_i} : W_i \longrightarrow (\gamma_k)^{-1}(U_i)$. By (iii), f_{i-1} maps W_i into $(\gamma_k)^{-1}(U_i) \cong U_i \times \mathbb{R}^k$. The first co-ordinate $\gamma_k(f_i(x))$ of $f_i(x)$ for $x \in W_i$ is given by (iii). Let $\rho_i : (\gamma_k)^{-1}(U_i) \longrightarrow \mathbb{R}^k$ be the projection to the second factor.

Since $f_{i-1}|_{W_i}$ is transverse to $BO(k)$ throughout $K_1 \cup \dots \cup K_{i-1}$ so $\rho_i \circ f_{i-1}|_{W_i} : W_i \longrightarrow U_i \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ has $0 \in \mathbb{R}^k$ as regular value throughout $(K_1 \cup \dots \cup K_{i-1}) \cap W_i$. By lemma (1.1.6) we can approximate the map $\rho_i \circ f_{i-1}$ by a smooth map from W_i to $\mathbb{R}^k$ such that it agrees with $\rho_i \circ f_{i-1}$ outside a compact subset of W_i and has origin as regular value throughout $(K_1 \cup \dots \cup K_i) \cap W_i$ of W_i.

Let the approximating map be $\rho_i \circ f_i$. Then f_i is transverse to $BO(k)$ throughout $K_1 \cup \dots \cup K_i$. Thus f_i satisfies conditions (i), (ii) and (iii).

Proceeding by induction we define maps $f_1, f_2, \dots, f_r$. Let $g = f_r$. Clearly g is transverse to $BO(k)$ throughout $K_1 \cup \dots \cup K_r$.

Let for $e \in EO(k)$, $0 \le |e| < 1$ denote the Euclidean norm. Thus $|e| = 0$ iff $e \in B$. We set $|e_0| = 1$.

Since $K_1 U \ldots\ldots U K_r$ is a neighborhood of $f_0^{-1}(BO(k))$ in the compact space $\mathbb{S}^{n+k}$, $\exists$ $c > 0$ such that

$$|f_0(x)| \geq c, \quad \forall\ x \notin K_1 U \ldots\ldots U K_r .$$

Let f_i be chosen such that $|f_i(x) - f_{i-1}(x)| < c/r$, $\forall\ x \in S^{n+k}$.

Then clearly $|g(x) - f_0(x)| < c$.

Now for $x \notin K_1 U \ldots U K_r$, $|g(x)| = |f_0(x) + g(x) - f_0(x)|$ $\geq |f_0(x)| - |g(x) - f_0(x)| > c - c = 0$.

Therefore $g^{-1}(BO(k)) \subset K_1 U \ldots\ldots U K_r$. Hence g is transeverse to $BO(k)$, $\forall\ x \in \mathbb{S}^{n+k}$. ∎

§ 2. Homotopy interpretation of unoriented bordism group.

In this section we establish isomorphism between unoriented cobordism group and sufficiently higher homotopy group of MO.

(A) Higher homotopy groups.

Let (X, x_0) be a pointed topological space. We consider the space I with base point 0, and the subspace $\dot{I} \subset I$. Then suspension of $\dot{I}$, $\mathbb{S}(\dot{I})$ is the space

$$(I \times I) \,/\, I \times \{0\} \cup I \times \{1\} \cup \{0\} \times I$$

and we define n-th suspension $\mathbb{S}^n(\dot{I})$ to be $\mathbb{S}^{n-1}(S(\dot{I}))$ inductively.

For any $n \geq 0$, the homotopy group $\Pi_n(X, x_0)$ of the topological space (X, x_0) is a covarient functor defined to be the set of homotopy classes of maps of the pair $(\mathbb{S}^n(\dot{I}), \mathbb{S}^{n-1}(\dot{I}))$ to (X, x_0). But we have $\mathbb{S}^n(\dot{I}) \approx \mathbb{S}^n(\mathbb{S}^0) \approx \mathbb{S}^n$, $\forall$ n. Hence, $\Pi_n(X, x_0) = [\ (\mathbb{S}^n(\dot{I}), \mathbb{S}^{n-1}(\dot{I})), (X, x_0)\ ]$ $\approx [\ (\mathbb{S}^n, \mathbb{S}^{n-1}), (X, x_0)\ ]$.

(B) Thom map.

We first defne Thom map

$$\tau : \Pi_{n+k}(MO(k)) \longrightarrow \mathfrak{N}_n$$

Let $[f] \in \Pi_{n+k}(MO(k))$. Then by theorem (1.1.7) $\exists$ a smooth map g which is homotopic to f and is transverse to $BO(k)$ throughout S^{n+k}. So $g^{-1}(BO(k))$ is a smooth, compact n-dimensional manifold.

We define $\tau([f]) = [g^{-1}(BO(k))]$.

Theorem 1.2.1. Let M^n be a closed differentiable manifold embedded in $\mathbb{R}^{n+k}$. There is a neighborhood of M^n in $\mathbb{R}^{n+k}$ which is diffeomorphic to the top space of the normal bundle of M^n in $\mathbb{R}^{n+k}$ under a diffeomorphism which takes each $x \in M^n$ to the zero normal vector at x.

Proof. Let $\nu^k : E \longrightarrow M^n$ be the normal bundle of M^n in $\mathbb{R}^{n+k}$ and for $\varepsilon > 0$, let $E(\varepsilon) \subseteq E$ be the open subset of E consisting of the points $(x, v) \in E$ with $|v| < \varepsilon$ where $x \in M^n$ and v a normal vector to M^n at x.

We define $e : E(\varepsilon) \longrightarrow \mathbb{R}^{n+k}$ by $e(x, v) = x + v$.

We identify M^n with zero cross-section of E. The tangent space to $E(\varepsilon)$ at any point $(x, 0)$ on the zero cross-section has a natural splitting $T_xM \oplus (T_xM)^{\perp}$ where $(T_xM)^{\perp}$ is the orthogonal complement of T_xM in $T_{(x,0)}(E(\varepsilon))$. Clearly the derivative $D_{(x,0)}e$ is the identity map on T_xM and $(T_xM)^{\perp}$. Therefore, $D_{(x,0)}e$ has rank n+k at all points on the zero croos-section. Applying Inverse Function Theorem at any point $(x, 0)$ on the zero cross-section, we get an *open neighborhood* U_x *of* $(x, 0)$ in $E(\varepsilon)$ *which* is mapped diffeomorphically onto an open subset of $\mathbb{R}^{n+k}$ containing x.

Let $U = \bigcup_x U_x$. Consider $e : U \longrightarrow \mathbb{R}^{n+k}$.

We claim that e is one-one on U. Note that e is already locally one-one.

Suppose e is not one-one. Then for each integer $i > 0$, considering $\varepsilon = 1/i$, there exist two distinct points

$$(x_i, v_i) \neq (x'_i, v'_i)$$

in the neighborhood $E(1/i)$ such that $e(x_i, v_i) = e(x'_i, v'_i)$. Since M is compact, there exists convergent subsequences $\{x_{i_j}\}$ and $\{x'_{i_j}\}$ such that $\lim (x_{i_j}, v_{i_j}) = (x, 0)$

and $\lim (x'_{i_j}, v'_{i_j}) = (x', 0)$.

Therefore, we get

$$x = e(x,0) = \lim e(x_{i_j},v_{i_j}) = \lim e(x'_{i_j},v'_{i_j}) = e(x',0) = x'.$$

But for large j, (x_{i_j}, v_{i_j}) and $(x'_{i_j}, v'_{i_j}) \in U_x$ with $e(x_{i_j}, v_{i_j}) = e(x'_{i_j}, v'_{i_j})$ contradicting the fact that e is locally one-one. So e is one-one.

Hence e is a diffeomorphism which maps U onto an open neighborhood of M^n in $\mathbb{R}^{n+k}$. ■

Definition 1.2.2. Let M^n be a differentiable manifold embedded in $\mathbb{R}^{n+k}$. Let $\xi = (p, E, M)$ be a vector bundle over M and M be identified with the zero cross-section of E. A tubular neighborhood of M^n is a neighborhood of M^n in $\mathbb{R}^{n+k}$ which is diffeomorphic to a neighborhood of M^n in E.

Lemma 1.2.3. Let V,N be manifolds, $A \subseteq N$ be a compact submanifold and $f : V \longrightarrow N$ a map such that f is transverse to A. Let $M = f^{-1}(A)$ and U a tubular neighborhood of M and E a tubular neighborhood of A. Let $D \subseteq U$ be a disk subbundle such that $f(D) \subseteq E$. Then there is a homotopy from f to a map $h : V \longrightarrow N$ such that $h|_D$ is the restriction of a vector bundle map $\Phi : U \longrightarrow E$ over $f : M \longrightarrow A$, (cf. Theorem (4.6.7) of [11]).

Definition 1.2.4. Let $\xi = (E, B, p)$ be a vector bundle. Let Q be a manifold and $g : Q \longrightarrow T(\xi)$ be a map. We say g is in standard form if there is a submanifold $M \subseteq Q$ and a tubular neighborhood $U \subseteq Q$ of M such that $U = g^{-1}(E)$ and $M = g^{-1}(B)$ and the diagram:

$$\begin{tikzcd}
U \arrow[r, "g|U"] \arrow[d] & E \arrow[d] \\
M \arrow[r, "g|M"] & B
\end{tikzcd}$$

is a vector bundle map.

Lemma 1.2.5. Let $f : Q \longrightarrow T(\xi)$ be a map. Then f is homotopic to a map in standard form.

Proof. The argument similar to that in theorem (1.1.7) gives that f is homotopic to a map f_0 which is smooth throughout $f_0^{-1}(E)$ and transverse to B. Let $M = f_0^{-1}(B)$. Let $U \subset f_0^{-1}(E)$ be a tubular neighborhood of M in Q and $D \subseteq U$ a disk subbundle. By lemma (1.2.3) we may assume that f_0 is homotopic to a map f_1 such that f_1 agrees in D with a vector bundle map $\Phi : U \longrightarrow E$. We define $g : U \longrightarrow T(\xi)$ given by $g = \begin{cases} \Phi & \text{on } U \\ e_0 & \text{on } Q \setminus U \end{cases}$. Then g and f_1 agrees on D and

g is in the standard form. Since g and f_1 agree on ∂D and both map $Q\setminus \text{int}D$ into the contractible space $T(\xi)\setminus B$, we have $g \simeq f_1$. Hence f is homotopic to a map in standard form. ■

<u>Lemma 1.2.6</u> . τ is a well defined homomorphism.

<u>Proof</u>. Let g, g' be two homotopic maps from S^{n+k} to $MO(k)$, both being differetiable on the inverse image of $EO(k)$ and both being transverse to $BO(k)$.

Let $H : S^{n+k} \times I \longrightarrow MO(k)$ be a homotopy from g to g'. We assume that $H(x, t) = H(x, 0)$ for $t \leq 1/3$

and $H(x, t) = H(x, 1)$ for $t \geq 2/3$.

Then the argument similar to that in theorem (1.1.7) gives a map $G : S^{n+k} \times I \longrightarrow MO(k)$ which coincides with H outside a compact subset of $S^{n+k} \times I$ and which is transverse to $BO(k)$ and G is equal to H on $S^{n+k} \times [0, 1/3] \cup S^{n+k} \times [2/3, 1]$ and is homotopic to H.

Then $G^{-1}(BO(k))$ is a smooth $(n+1)$-dimensional manifold of $S^{n+k} \times I$ whose boundary is $g^{-1}(BO(k)) \amalg g'^{-1}(BO(k))$.

Thus τ is well defined map.

Let $[\alpha]$, $[\beta] \in \Pi_{n+k}(MO(k))$. By lemma (1.2.5) $\exists$ a map $f : S^{n+k} \longrightarrow MO(k)$ which maps lower hemisphere S^{n+k} of e_0 and is homotopic to α and $\exists$ a map $g : S^{n+k} \longrightarrow MO(k)$ which maps upper hemisphere of S^{n+k} to e_0 and is homotopic to β.

We define a map $h : S^{n+k} \longrightarrow MO(k)$ given by

$$h = \begin{cases} f \text{ on } S^{n+k} \setminus E_n^- \\ g \text{ on } S^{n+k} \setminus E_n^+ \end{cases}$$

Then h is transverse to $BO(k)$ throughout S^{n+k} and

$$h^{-1}(BO(k)) = f^{-1}(BO(k)) \amalg g^{-1}(BO(k)).$$

$$\begin{aligned} \text{Therefore } \tau([\alpha] + [\beta]) &= \tau([\alpha + \beta]) \\ &= [h^{-1}(BO(k))] \\ &= [f^{-1}(BO(k)) \amalg g^{-1}(BO(k))] \\ &= [f^{-1}(BO(k))] + [g^{-1}(BO(k))] \\ &= \tau([\alpha]) + \tau([\beta]). \end{aligned}$$

Hense τ is a well defined homomorphism. ■

Lemma 1.2.7 . The Thom homomorphism τ is surjective if $k > n$.

Proof Let $[M^n] \in \mathfrak{N}_n$. We can assume that $M^n \subseteq S^{n+k}$ by Whitney embedding theorem for sufficiently large k. Let $U \subseteq S^{n+k}$ be a tubular neighborhood of M^n.

Now there is a vector bundle map $h : U \longrightarrow EO(k)$. We extend h to a map $g : S^{n+k} \longrightarrow MO(k)$

such that $g = \begin{cases} h & \text{on } U \\ e_0 & \text{on } S^{n+k} \setminus U. \end{cases}$

Then g is transverse to $BO(k)$ and $g^{-1}(BO(k)) = M^n$.

Let $[g]$ denote the homotopy class of g in $\Pi_{n+k}(MO(k), e_0)$. By definition, the cobordism class of M^n is the image of $[g]$.

Therefore, $\tau([g]) = M^n$. ■

We now mention an important result without proof which will be used in the next theorem. For proof see of [4].

Theorem 1.2.8 . Given a vector space bundle ξ^k over a CW complex of dimension $\leq m$, any bundle map of ξ^k, restricted to a subcomplex, into γ^k_m can be extended throughout ξ^k.

Theorem 1.2.9. The Thom homomorphism ξ^k is injective if $k > n+1$.

Proof. Let $f : S^{n+k} \longrightarrow MO(k)$ be a map which is differentiable on $f^{-1}(EO(k))$ and transverse regular to $BO(k)$ such that $\tau([f]) = 0 \in \mathfrak{N}_n$. Let $f^{-1}(BO(k)) = M^n$. Then $[M]^n = 0$. So M^n is the boundary of a compact (n+1)-dimensionalmanifold Q^{n+1}. We shall show that f is homotopic to the constant map. By lemma (1.2.5), we may assume that f is in the standard form.

Let $h : M^n \times [0, 1] \longrightarrow Q^{n+1}$ be a diffeomorphism onto its image which carries $M^n \times 0$ onto ∂Q^{n+1}. We define a map $h_1 : Q^{n+1} \longrightarrow C_{n+k} \times I$, where C_{n+k} is the closed cube $[0,1]^{n+k}$, as follows. Let $x \in Q^{n+1}$, if $x = h(y,t)$ where $y \in M^n$ & $0 \le t \le 1\backslash 2$ then let $h_1(x) = (y,t)$, if $x \notin$ image h, then let $h_1(x) = p$, where p is a fixed point in $\mathrm{int}(C_{n+k} \times I)$, if $x = h(y,t)$ where $y \in M^n$ and $1\backslash 2 \le t \le 1$, let $h_1(x) = (1-\beta(t))h(y, 1\backslash 2) + \beta(t)p$, where $\beta(t)$ is a C^{∞}-function with $\beta'(t) \ge 0$, $\beta(t) = 0$ in a neighborhood of $t = 1\backslash 2$ and $\beta(t) = 1$ in a neighborhood of $t=1$. Then h_1 is a differentiable map of int Q^{n+1} into $\mathrm{int}(C_{n+k} \times I)$. h_1 is a one-one immersion in a neibourhood of ∂Q^{n+1}. Since $\dim (C_{n+k} \times I) > 2(n+1)$, as $k > n+1$, h_1, may be approximated by an one-one immersion $h_2 : Q^{n+1} \longrightarrow C_{n+k} \times I$ which equals h_1 in a neighborhood of ∂Q^{n+1}. Then the one-one immersion h_2 is an embedding of Q^{n+1} into $C_{n+k} \times I$ as Q^{n+1} is compact. Let ψ be a homeomorphism of $C_{n+k} \times I$ into D^{n+k+1}. Let Q^{n+1} now be considered as the subset $\psi \circ h_2(Q^{n+1})$ in D^{n+k+1}.

We have the mao f of S^{n+k} into $MO(k)$ which is a bundle map when restricted to a small tubular neighborhood of M^n in S^{n+k}. By theorem (1.2.8), we can extend this map to a map g of the tubular neighborhood N of Q^{n+1} in D^{n+k+1} into $MO(k)$ which equals f in some neighborhood of ∂Q^{n+1} in D^{n+k+1}. We define $h: D^{n+k+1} \longrightarrow MO(k)$ given by

$$h = \begin{cases} g & \text{on } N \\ e_0 & \text{on } D^{n+k+1} \setminus N. \end{cases}$$

Now $h|S^{n+k} = f$. So f is homotopic to the constant map.■

Theorem 1.2.10. For $k > n+1$, the homotopy group $\Pi_{n+k}(MO(k))$ is canonically isomorphic to the unoriented cobordism group $\mathfrak{N}_n$.

§ 3. Stiefel-Whitney Classes.

In this section, we introduce Stiefel-Whitney classes, Stiefel-Whitney numbers, and establish their invariance with respect to unoriented bordism.

Let ξ be an n-dimensional vector bundle with base space $B(\xi)$. Then there exists a unique sequence of cohomology classes

$$W_i(\xi) \in H^i(B(\xi)), \quad i = 0,\ 1,\ 2, \ldots\ldots$$

satisfying the following four axioms.

Axiom 1 $W_0(\xi) = 1 \in H^0(B(\xi))$ and $W_i(\xi) = 0$ if $i > n$.

Axiom 2 If $f : B(\xi) \longrightarrow B(\eta)$ is covered by a bundle map from ξ to η, then $W_i(\xi) = f^*(W_i(\eta))$.

Axiom 3 If ξ and η are the vector bundle over the same base space then $W_k(\xi \oplus \eta) = \sum_{i=0}^{k} W_i(\xi) W_{k-i}(\eta)$, where $\oplus$ denotes the Whitney sum and the product on the right hand side is the cup product.

Axiom 4 For the canonical line bundle γ_1^1 over the projective space $\mathbb{R}P^1$, the class $W_1(\gamma_1^1) \neq 0$.

Definition 1.3.1. The cohomology classes $W_i(\xi)$, $i = 0$, 1, ... , satisfying the four axioms mentioned above, are known as Stiefel-Whitney classes of ξ.

Also $W(\xi) = 1 + W_1(\xi) + \ldots\ldots\ldots + W_n(\xi)$ is called the total Stiefel-Whitney class of the vector bundle ξ.

Definition 1.3.2. Let M^n be a smooth, closed, oriented, n-dimensional manifold. Using mod 2 co-efficients $\exists$ a unique fundamental homology class $[M^n] \in H_n(M^n, \mathbb{Z}_2)$.

Hence for any cohomology class $v \in H^n(M^n, \mathbb{Z}_2)$, the krönecker index $\langle v, [M^n]\rangle \in \mathbb{Z}_2$ is defined. Here

$$\langle\ ,\ \rangle : H^n(M^n, \mathbb{Z}_2) \otimes H_n(M^n, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

is the bilinear map given by $\langle\ ,\ \rangle(v \otimes z) = x(y) \in \mathbb{Z}_2$, where $[x] = v$ and $[y] = z$.

Let $i_1 + i_2 + \ldots\ldots + i_k = n$ be a partition of n (here $i_j > 0\ \forall\ i \le j \le k$), then we can form the monomial

$$W_{i_1}(M^n)W_{i_2}(M^n) \ldots\ldots\ldots W_{i_k}(M^n) \in H^n(M^n, \mathbb{Z}_2)$$

and the corresponding number $\langle W_{i_1}(M^n) \ldots\ldots W_{i_k}(M^n), [M^n]\rangle$ is called the Stiefel-Whitney number of M^n associated to the partition $i_1 + i_2 + \ldots\ldots + i_k = n$.

Theorem 1.3.3. If N is a smooth compact (n+1)-dimensional manifold with boundary equal to M^n then the Stiefel-Whitney numbers of M^n are zero.

Proof. Given $M^n = \partial N^{n+1}$.

Now $T(N^{n+1})|_{M^n} \cong T(M^n) \oplus \varepsilon^1$, where ε^1 is a trivial line bundle, for, choosing an Euclidean metric on $T(N^{n+1})$, there is a unique outward normal vector field along M^n which spans the trivial line bundle ε^1.

Again $T(N^{n+1})|_{M^n} = i^*(T(N^{n+1}))$ where $i : M^n \hookrightarrow N^{n+1}$.

We have the bundle map

$$\begin{tikzcd}
T(M^n) \oplus \varepsilon^1 \arrow[r] \arrow[d] & T(N^{n+1}) \arrow[d] \\
M^n \arrow[r, hook, "i"] & N^{n+1}
\end{tikzcd}$$

Therefore, $i^*(W_i(N^{n+1})) = W_i(T(M^n) \oplus \varepsilon^1) = W_i(M^n)$ for $i \quad 0$.

Consider any partition p of n given by $n = i_1 + \ldots + i_k$. Let $W_p(M^n)$ be the product of Stiefel-Whitney classes of M^n for the partition p, i.e., $W_p(M^n) = W_{i_1}(M^n) \ldots\ldots W_{i_k}(M^n)$.

The natural homomorphism $\partial : H_{n+1}(N^{n+1}, M^n) \longrightarrow H_n(M^n)$ map $[N^{n+1}]$ to $[M^n]$.

$$\begin{aligned}
\text{Now } \langle W_p(M^n), [M^n]\rangle &= \langle W_p(M^n), \partial[N^{n+1}]\rangle \\
&= \langle \partial W_p(M^n), [N^{n+1}]\rangle \\
&= \langle \partial i^* W_p(N^{n+1}), [N^{n+1}]\rangle \\
&= 0.
\end{aligned}$$

Therefore, the Stiefel-Whitney numbers of M^n are all zero. ■

The converse theorem is due to Thom [1]. The proof is very complicated and is omitted here.

Theorem 1.3.3. If all the Stiefel-Whitney numbers of M^n are zero, then M^n is the boundary of some smooth compact manifold.

§4. Oriented bordism.

In this section, we define oriented Universal vector bundle, discuss oriented bordism, and give homotopy interpretation of oriented bordism group. We also define Pontrjagin numbers and give their invariance with respect to oriented bordism.

(A) Oriented Universal vector bundle.

Let $\tilde{G}_n(\mathbb{R}^{n+k})$ denote the Grassmann manifold consisting of all oriented n-planes in $\mathbb{R}^{n+k}$. Then $\tilde{G}_n(\mathbb{R}^{n+k})$ is a 2-fold covering space of the unoriented Grassmann manifold $G_n(\mathbb{R}^{n+k})$.

Consider $BSO(n) = \tilde{G}_n(\mathbb{R}^{\infty})$ = set of all oriented n-dimensional subspaces of $\mathbb{R}^{\infty}$ and $ESO(n) = \{(X, w, x) \mid (X, w) \in \tilde{G}_n(\mathbb{R}^{\infty}),\ x \in X$ and w is an orientation of $X\}$. Define $\tilde{p}_n : ESO(n) \longrightarrow BSO(n)$ given by $\tilde{p}_n(X, w, x) = (X, w)$.

Then $\tilde{\gamma}_n : ESO(n) \longrightarrow BSO(n)$ is called an *oriented Universal vector bundle*, i.e., every n-dimensional oriented vector bundle ξ over a paracompact base space $B(\xi)$ has a classifying map $f_\xi : B(\xi) \longrightarrow BSO(n)$. $BSO(n)$ is called the classifying space for oriented n-dimensional vector bundles.

Now $\forall$ n we have maps $i_n : BSO(n) \hookrightarrow BSO(n+1)$ induced from the inclusion map of $SO(n)$ into $SO(n+1)$ such that $\{BSO(n)\}$ is a directed system.

Consider $BSO = \operatorname{dir} \lim\limits_{n \to \infty} BSO(n)$, $ESO = \operatorname{dir} \lim\limits_{n \to \infty} ESO(n)$ and $\tilde{\gamma} = \operatorname{dir} \lim\limits_{n \to \infty} \tilde{\gamma}_n$.

Then $\tilde{\gamma} : ESO \longrightarrow BSO$ is an oriented universal vector bundle which classifies oriented vector bundles of all dimensions over paracompact bases.

Notation. The Thom space $T(\tilde{\gamma}_k)$ of the oriented universal vector bundle $\tilde{\gamma}_k : ESO(k) \longrightarrow BSO(k)$ is denoted by $MSO(k)$.

Two closed oriented manifolds (M_1, ω_1) and (M_2, ω_2) where ω_1 and ω_2 are the orientations of M_1 and M_2 respectively, are said to be bordant, denoted by $(M_1, \omega_1) \sim (M_2, \omega_2)$, if there is a compact oriented $(n+1)$-dimensional manifold a (N, θ) such that $\partial N = M_1 \sqcup M_2$ and an orientation preserving diffeomorphism

$$(\partial N, \partial\theta) \approx (M_1, -\omega_1) \cup (M_2, \omega_2).$$

It is easy to see that $\sim$ is an equivalence relation. The set of these equivalence classes is denoted by Ω_n. The operation of disjiont union, i. e.,

$$[(M_1, \omega_1)] + [(M_2, \omega_2)] = [M_1 \sqcup M_2, \omega_1 \sqcup \omega_2]$$

makes Ω_n into an abelian group. Ω_n is called the n-dimensional oriented bordism group with the class of oriented bounding manifolds as the identity element and the inverse of $[M_1, \omega_1]$ as $[M_1, -\omega_1]$. The cartesian product operation of oriented manifolds

$$[M_1, \omega_1] \times [M_2, \omega_2] = [M_1 \times M_2, \omega_1 \times \omega_2]$$

gives rise to an associative, bilinear product operation $\Omega_m \times \Omega_n \xrightarrow{\cdot} \Omega_{n+m}$. $\Omega_* = \bigoplus_{n=0}^{\infty} \Omega_n$ has the structure of a graded ring.

Lemma 1.4.1. Let $0 \longrightarrow E' \xrightarrow{\Phi} E \xrightarrow{\Psi} E''$ be an exact sequence of vector spaces. Then given orientation ω of E, it induces an orientation ω' of E'.

Proof Consider $\Phi(E') \subset E$.

Let $\{s_1, \ldots, s_k\}$ be a basis of $\Phi(E')$. We extend this to a basis $\{s_1, \ldots, s_k, t_1, \ldots t_m\}$ of E. Let $\omega = [u_1, \ldots, u_n]$ be an orientation of E where all u_i's are s_r's and t_k's for

$1 \leq r \leq k$ and $1 \leq x \leq m$. Consider $w' = [w'_1, \ldots., w'_k]$ such that $w'_y = \Phi^{-1}(s_{\iota_y})$ for $1 \leq y \leq k$ where s_{ι_y} is the y-th s_r's in $u_1, \ldots, u_n$. Then w' is an orientation of E' induced by the orientation w of E. ■

Theorem 1.4.2. Every continuous map $f : S^{n+k} \longrightarrow MSO(k)$ is homotopic to a map g which is smooth throughuot $g^{-1}(ESO(k))$ and is transverse to the zero cross-section $BSO(k)$.

The proof is similar to the proof of theorem (1.1.7).

Let $f : S^{n+k} \longrightarrow MSO(k)$ be a map. Then f is homotopic to a map g which is smooth throughout $g^{-1}(ESO(k))$ and transverse to the zero cross-section $BSO(k)$.

Let $M^n = g^{-1}(BSO(k))$. Let M^n be embedded in S^{n+k}. Then the normal bundle $E(\nu^k)$ to M^n is equivalent to the orthogonal complement of the tangent bundle on M^n in the tangent bundle of S^{n+k}. Also the normal bundle $E(\nu^k)$ to M^n is the pull-back of the normal bundle of $BSO(k)$ in $MSO(k)$. Let ν be the orientation of $E(\nu^k)$ which is induced through the pull-back of the canonical orientation of $BSO(k)$.

Let σ be the standard orientation of S^{n+k}. Now

$$g^*(\nu^k) \cong T(S^{n+k})|_{M^n} .$$

We have an exxact sequence of homomorphisms

$$0 \longrightarrow TM^n \longrightarrow T(S^{n+k})|_{M^n} \longrightarrow E(\nu^k) \longrightarrow 0 .$$

Let θ be the orientation of TM^n such that $\theta \oplus \nu = \sigma$ on $T(S^{n+k})|_{M^n}$. Thus M^n is an oriented manifold with orientation θ.

We define the Thom map $\tilde{\tau} : \Pi_{n+k}(MSO(k)) \longrightarrow \Omega_n$

given by $\tilde{\tau}([f]) = [M^n, \theta]$.

Theorem (1.4.3). For $k > n + 1$ the homotopy group $\Pi_{n+k}(MSO(k), e_0)$ of the universal Thom space is isomorphic to the oriented bordism group Ω_n.

Proof of theorem (1.4.3) is similar to the proofs in the case of unoriented cobordism group $\mathfrak{N}_n$.

(A) Euler Class.

Let $\xi = (E, B, \pi)$ be an oriented vector bundle of dimension n. Let B be identified with the zero cross-section of E. Then the projection $\pi : E \longrightarrow B$ induces an isomorphism $\pi^* : H^n(B; \mathbb{Z}) \longrightarrow H^n(E; \mathbb{Z})$.

The inclusion $i : E \hookrightarrow (E, E_0)$ induces a homomorphism $i^* : H^n(E, E_0; \mathbb{Z}) \longrightarrow H^n(E; \mathbb{Z})$ given by $i^*(y) = y|_E$.

Let $u \in H^n(E, E_0; \mathbb{Z})$ be the fundamental cohomology class of $H^n(E, E_0; \mathbb{Z})$ and $i^*(u) \in H^n(E; \mathbb{Z})$ be its image under i^*.

Definition 1.4.4. The Euler class of an oriented vector bundle $\xi = (E, B, \pi)$ of dimension n is the cohomology class $e(\xi) \in H^n(B; \mathbb{Z})$ such that $\pi^*(e(\xi)) = i^*(u)$.

Definitoin 1.4.5. A complex vector bundle ω of dimension n over B consists of a topological space E and a projection map $\pi : E \longrightarrow B$, together with the structure of a complex vector space in each fibre $\pi^{-1}(x)$, satisfying the following condition:

For each point $x \in B$, has a neighborhood U such that the map $h : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^n$ is a homeomorphism which maps each fibre $\pi^{-1}(x)$ complex linearly onto $x \times \mathbb{C}^n$.

It is a well known fact that if $\xi = (E, B, \pi)$ is a complex vector bundle, then the underlying real vector bundle $\xi_{\mathbb{R}}$ has an orientation.

Thus for any complex n-dimensional vector bundle $\xi = (E, B, \pi)$ the Euler class $e(\xi) \in H^{2n}(B, \mathbb{Z})$ is well defined. We consider E_o, the set of non-zero elements of E. We shall construct an (n-1)-dimensional bundle ξ_o over E_o. A point in E_o is specified by a fibre F of ξ and a non-zero vector v in that fibre. Let ξ has a hermitian mectric. A fibre over $v \in E_o$ is defined to be the orthogonal complement of v in vector space F. Then ξ_o is an (n-1)-dimensional complex vector bundle.

Definition 1.4.7 . By induction we define elements

$$c_i(\xi) \in H^{2i}(B; \mathbb{Z}) \text{ for } 0 \leq i \leq n$$

called Chern classes as follows:

Let $c_o(\xi_o) = 1$. If $i \geq 0$ and $c_i(\xi_o)$ is already defined, $c_i(\xi) = \pi_o^{*-1}(c_i(\xi_o))$, where $\pi_o^{*} : H^{2i}(B) \longrightarrow H^{2i}(E_o)$ is an isomorphism for $i < n$, $c_n(\xi) = e(\xi_{\mathbb{R}})$ and $c_i(\xi) = 0$ for $i > n$.

The formal sum $c(\xi) = 1 + c_1(\xi) + \dots\dots\dots + c_n(\xi)$ in the ring $H^{\pi}(B; \mathbb{Z})$ is called the total chern class of ξ. ■

Compactification of real vector bundle.

Let $\xi = (E, B, \pi)$ be a real vector bundle. Let $x \in B$ and F_x be the fibre over x. The tensor product $F_x \otimes \mathbb{C}$ of F_x with complex numbers is a complex vector space and it is called the compactifition of F_x. Complexifying each fibre F_x of ξ we obtain a new vector bundle, denoted by $\xi \otimes \mathbb{C}$, over the base space B. The vector bundle $\xi \otimes \mathbb{C}$ is called the complexifition of the real vector bundle ξ, and it is isomorphic to its own conjugate bundle $\overline{\xi \otimes \mathbb{C}}$.

Definition 1.4.9. Let $\xi = (E, B, \pi)$ be a complex oriented vector bundle. The i-th Pontrjagin class $p_i(\xi) \in H^{4i}(B; \mathbb{Z})$ is defined to be the integral cohomology class $(-1)^i c_{2i}(\xi \otimes \mathbb{C})$. Clearly $p_i(\xi) = 0$ for $i > n/2$.

The total Pontrjagin class is defined to be the unit

$$p(\xi) = 1 + p_1(\xi) + \dots\dots + p_{[n\backslash 2]}(\xi)$$

in the ring $H^{\pi}(B; \mathbb{Z})$.

Definition 1.4.10. Let M^{4n} be a smooth, compact, oriented, 4n-dimensional manifold. Let $i_1 + i_2 + \dots + i_k = 4n$ be a partition of 4n, then the number $\langle p_{i_1}(M^{4n}) p_{i_2}(M^{4n}) \dots p_{i_k}(M^{4n}), [M^{4n}] \rangle \in \mathbb{Z}$ is called the Pontrjagin number of M^{4n} associated to the above partition of 4n.

Thom has shown that

Theorem 1.4.11 . Let M^n be a smooth, compact and oriented manifold. Then some positive multiple $M^n + \dots + M^n$ is an *oriented boundary if and only if every Pontrjagin* number $p_i(M^n)$ is zero.

C.T.C. Wall has proved a much stronger statement, viz.,

Theorem 1.4.12. Let M^n be a smooth, compact, oriented, n-dimensional manifold. Then M^n *is oriented boundary if and* only if all Pontrjagin numbers and all Stiefel-Whitney numbers of M^n are zero.

Oriented singular manifold.

Let (X,A) be a topological pair. An oriented singular manifold in (X,A) is a pair (M^n,f) consisting of a compact, oriented, n-dimensional manifold M^n and a map $f : (M^n,\partial M^n) \longrightarrow (X,A)$.

An oriented singular manifold (M^n,f) in (X,A) is said to bord if and only if there is a compact, oriented, (n+1)-dimensional manifold W^{n+1} and a map $F : W^{n+1} \longrightarrow X$ such that $M^n \subseteq \partial W^{n+1}$, orientation of M^n is induced by that of W^{n+1}, $F|_{M^n} = f$ and $F(W^{n+1}\setminus M^n) \subseteq A$.

A singular manifold (M_1^n, f_1) is said to be bordant to a singular manifold (M_2^n, f_2), denoted by $(M_1^n, f_1) \sim (M_2^n, f_2)$, if their disjiont union $(M_1^n \amalg M_2^n, f_1 \amalg f_2)$ bords. Then clearly $\sim$ is an equivalence relation among the oriented singular manifolds in (X,A). We denote the oriented bordism class of (M^n, f) by $[M^n, f]$ and the collection of all such bordism classes by $MSO_n(X,A)$. An abelian group structure is imposed on $MSO_n(X,A)$ by disjoint union

$$[M_1^n, f_1] + [M_2^n, f_2] = [M_1^n \amalg M_2^n, f_1 \amalg f_2].$$

There is Ω_*-module structure defined on the direct sum $MSO_*(X,A) = \bigoplus_{n=0}^{\infty} MSO_n(X,A)$. From an oriented singular manifold (M_1^n, f_1) in (X,A) and a closed oriented manifold M^m a new singular manifold $(M_1^n \times M^m, g)$ may be defined where $g(x,y) = f(x)$. The module structure is now given by

$$[M_1^n, f][M^m] = [M_1^n \times M^m, g].$$

This also defines on $MSO_*(X,A)$ the structure of a graded right Ω_*-module.

Unoriented singular manifold.

Let (X,A) be a fixed topological pair. Then a singular manifold in (X,A) is a pair (M^n,f) where M^n is a compact n-dimensional manifold and $f : (M^n,\partial M^n) \longrightarrow (X,A)$.

A bordism relation is defined in the set of all unoriented singular manifolds just as in the case of oriented version, except that no orientability requirements are imposed here. The resulting group of bordism classes are denoted by $\mathfrak{N}_n(X,A)$. The unoriented bordism class of (M^n,f) is denoted by $[M^n,f]$, and the direct sum $\mathfrak{N}_*(X,A) = \overset{\infty}{\underset{n=0}{\oplus}} \mathfrak{N}_n(X,A)$ is a graded right $\mathfrak{N}_*$-module.

CHAPTER II

Characteristic numbers for singular G-bordism

The purpose of this chapter is to define G-bordism, singular G-bordism, G-characteristic classes of G-manifolds and singular G-manifolds; to prove their invariance with regard to G-bordism and singular G-bordism.

§1. G-bordism.

(A) G-space, equivariant map, G-manifold.

Let G be a group with discrete topology. A G-space is a topological space X together with a continuous map $\theta : G \times X \longrightarrow X$ such that

(1) For each $x \in X$, and $g_1, g_2 \in G$, $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$ and

(2) For each $x \in X$, $\theta(e, x) = x$ where e is the identity of G.

We denote the G-space by (X, θ) and $\theta(g, x)$ by gx. We call a continuous map $\theta : G \times X \longrightarrow X$ satisfying (1) and (2) an action of G on X. If X is a differentiable manifold, an action $\theta : G \times X \longrightarrow X$ is called differentiable action if the map $\theta_g = \theta|_{\{g\} \times X}$ is differentiable for every $g \in G$. An action θ is said to be a principal action if $gx = g$ implies $g = e$. A topological space together with a principal action of G is called a principal G-space.

A subspace A of a G-space X is called a G-subspace if $ga \in A$ for all $g \in G$ and $a \in A$. By a G-pair we mean a pair

(X,A) where X is a G-space and A is a G-subspace of X. A G-pair (X,A) is principal if the action of G is a principal action. If X is a G-space then two elements x_1 and $x_2 \in X$ are called G-equivalent, denoted by $x_1 \sim x_2$, provided there exists an element $g \in G$ with $x_1 = gx_2$. The relation $\sim$ is an equivalence relation, and the set of all gx, $g \in G$, denoted by G_x, is the equivalence class determined by $x \in X$. The set of disjoint equivalence classes of X under this relation is called the orbit set of X by G, and is denoted by X/G. This set with the quotient topology is called the orbit space of X by G. The equivalence class of $x \in X$ is also denoted by $[x]$.

If X and Y are G-spaces then a continuous map $f : X \longrightarrow Y$ is called a G-map or an eqivariant map if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. A map $f : (X,A) \longrightarrow (Y,B)$ between two G-pairs (X,A) and (Y,B) is called an equivariant map if $f : X \longrightarrow Y$ is an equivariant map. An equivarient map $f : X \longrightarrow Y$ gives rise to map $\bar{f} : X/G \longrightarrow Y/G$, on passing to the quotient, given by $\bar{f}([x]) = [f(x)]$, for all $x \in X$.

An unoriented G-manifold is an unoriented smooth manifold M with an action $\theta : G \times M \longrightarrow M$ of G such that for every $g \in G$ the map $\theta_g : M \longrightarrow M$ given by $\theta_g(x) = gx$, is a diffeomorphism. An oriented G-manifold is an oriented smooth manifold M with an action $\theta : G \times M \longrightarrow M$ of G such that the map θ_g is an orientation preserving diffeomorphism from M to M, for all $g \in G$. We call such an action θ of G on M an orientation preserving action.

Let M^n be an oriented G-manifold. We say M^n bounds if and only if there is a compact oriented (n+1)-dimensional G-manifold W^{n+1} such that ∂W^{n+1} is G-diffeomorphic to M^n under an equivariant orientation preserving map. We say that a G-manifold M_1^n is bordant to a G-manifold M_2^n, denoted by $M_1^n \overset{G}{\sim} M_2^n$, if and only if the G-manifold $M_1^n \amalg M_2^n$ bounds. The relation $\overset{G}{\sim}$ can be shown an equivalence relation. The resulting set of equivalence classes is denoted by $\Omega_n(G)$. By the operation 'disjoint union', this becomes an abelian group. The direct sum $\Omega_*(G) = \bigoplus_{n=0}^{\infty} \Omega_n(G)$ is a graded commutative algebra with identity over Ω_*.

There is an unoriented analogue. The bordism relation is defined as in the case of oriented version, except that no orientability condition is imposed. The unoriented G-bordism algebra is denoted by $\mathfrak{N}_*(G)$.

A G-vector bundle is a vector bundle $p : E \longrightarrow X$ such that E and X are G-spaces, p and the zero-section s are G-maps. The tangent bundle of a smooth G-manifold is a G-vector bundle.

A linear representation of G in V is a homomorphism ϕ from the group G into the group GL(V). If there is a linear representation of G in V, we say that V is a representation space of G.

(B). Classifying spaces $B(O,G)_n$ and $B(SO,G)_n$.

Let G be a finite group and W be a real orthogonal representation of G. Let $BO_n(W)$ be the Grassmannian of n-dimensional subspaces of W with the G-action induced by the linear action on W. Let $EO_n(W) = \{ (X,x) \mid X \in BO_n(W)$

and $x \in X$ }. The bundle $\gamma_n(W) : EO_n(W) \longrightarrow BO_n(W)$ with $\gamma_n(W)(X,x) = X$ is an n-plane bundle. Define an action of G on $EO_n(W)$ by the bundle maps covering the action on $BO_n(W)$ such that the projection is equivariant. Then $\gamma_n(W)$ becomes a G-vector bundle.

Let G have exactly r distinct orthogonal irreducible real finite dimensional representations. Consider $\mathbb{R}^{\infty}(G) = \mathbb{R}^{\infty}_1(G) \oplus \ldots\ldots \oplus \mathbb{R}^{\infty}_r(G)$ where $\mathbb{R}^{\infty}_i(G)$ is the countable copy of i-th represantation of G. It is a well known fact that the G-vector bundle

$$\gamma_n(G) = \gamma_n(\mathbb{R}^{\infty}(G)) : EO_n(\mathbb{R}^{\infty}(G)) \longrightarrow BO_n(\mathbb{R}^{\infty}(G))$$

is a universal n-dimensional G-vector bundle for the category of G-spaces in the sense that if $p : X \longrightarrow B$ is any G-vector bundle of dimension n, then $\exists$ a G-map $f : B \longrightarrow BO_n(\mathbb{R}^{\infty}(G))$ unique upto G-homotopy such that the induced G-vector bundle $f^*(\gamma_n(G))$ is G-isomorphic to the G-vector bundle $p : X \longrightarrow B$. We denote $\gamma_n(\mathbb{R}^{\infty}(G))$ by $\gamma_n(G)$, $EO_n(\mathbb{R}^{\infty}(G))$ by $E(O,G)_n$ and $BO_n(\mathbb{R}^{\infty}(G))$ by $B(O,G)_n$. $B(O,G)_n$ is called the <u>classifying space</u> for real n-dimensional G-vector bundles. Taking the Grassmannian of oriented n-dimensional subspaces of $\mathbb{R}^{\infty}(G)$ we get $B(SO,G)_n$ which we call the classifying space for oriented real n-plane G-bundles. In case $G = \{e\}$, $B(O,G)_n = BO(n)$ the classifying space for real n-plane bundles and $B(SO,G)_n = BSO(n)$ the classifying space for oriented real n-plane bundles.

§2. <u>G-characteristic classes of G-manifolds and its invariance.</u>

Let $p : EG \longrightarrow BG$ denote the universal G-bundle, i. e., EG is a paracompact, contractible, free G-space and BG is the corresponding orbit space. Then, if X is any paracompact G-space, EG×X becomes a free G-space with G acting co-ordinate-wise. Let $EG \times_G X$ be the corresponding orbit space. Then we define a G-cohomology theory as

$$h^*(X;G) = H^*(EG \times_G X; \underset{\sim}{K}),$$

where $\underset{\sim}{K}$ denotes a commutative ring, and H^* denotes singular cohomology theory.

Consider a commutative diagram

$$\begin{tikzcd}
EG \times X \arrow[r, "\pi"] \arrow[d] & EG \arrow[d] \\
EG \times_G X \arrow[r, "q_x"] & BG
\end{tikzcd}$$

where the map q_x, induced by the projection π, is unique upto homotopy and is called the classifying map for the G-bundle $EG \times X \longrightarrow EG \times_G X$.

If h^* is an equivariant cohomology theory then elements of $h^*(B(O,G)_n;G)$ are called <u>universal h^* characteristic classes</u>. If $E \longrightarrow X$ is the G-vector bundle induced by an equivariant map $f : X \longrightarrow B(O,G)_n$, then $f^*(h^*(B(O,G)_n;G)) \leq h^*(X;G)$ is the charecteristic subgroup of the bundle E. This subgroup is well defined since f is unique upto equivariant homotopy.

Suppose h^* is given by $H^* \circ A$ where A is a functor from the category of G-spaces and equivariant maps to the category of topological spaces and continuous maps and H^* is singular cohomology theory and let $h_* = H_* \circ A$ denote the associated equivariant homology theory. Let

$$\langle\ ,\ \rangle : h^*(X;G) \underset{H^*(pt.)}{\otimes} h_*(X;G) \longrightarrow H_*(pt.)$$

be the krónecker pairing.

Suppose for each compact G-manifold W, there is a class, $[W,\partial W] \in h_*(W,\partial W;G)$ satisfying

(1) $[W_1 \cup W_2, \partial W_1 \cup \partial W_2] = [W_1,\partial W_1] + [W_2,\partial W_2]$ and

(2) $\partial_*[W,\partial W] = [\partial W]$.

Such an element $[W,\partial W] \in h_*(W,\partial W;G)$ is called a *topological class* of W.

The x-characteristic number of a manifold M is defined by $x(M) = \langle \tau_M^*(x), [M] \rangle \in H_*(pt.)$ where $x \in h^*(B(O,G)_n;G)$.

Theorem 2.2.1. Let $\mathcal{G}$ be the collection of G-manifolds. The x-characteristic number is a $\mathcal{G}$-equivariant bordism invariant if x is in the image of $j^* : h^*(B(O,G)_{n+1};G) \longrightarrow h^*(B(O,G)_n;G)$, where $j : B(O,G)_n \longrightarrow B(O,G)_{n+1}$ is the map classifying the bundle $(\gamma_n \oplus 1)$.

Proof. We have the commutative diagram:

$$\begin{tikzcd}
\partial W \arrow[r, "\tau_{\partial W}"] \arrow[d, "i"] & B(O,G)_n \arrow[d, "j"] \\
W \arrow[r, "\tau_W"] & B(O,G)_{n+1}
\end{tikzcd}$$

and $x = j^*(y)$, $y \in h^*(B(O,G)_{n+1};G)$ as x is in the image of j^*. Then $x(\partial W) = \langle \tau^*_{\partial W}(x), [\partial W] \rangle = \langle \tau^*_{\partial W} j^*(y), [\partial W] \rangle = \langle i^* \tau^*_W(y), \partial_*[W] \rangle = \langle \tau^*_W(y), i_* \partial_*[W] \rangle = 0$.

Hence x-characteristic number is bordism invariant. ■

Theorem 2.2.2. For any finite group G, there is a natural isomorphism of $\mathfrak{N}_*$-modules, $\mathfrak{N}_*(G) \cong \mathfrak{N}_*(BG)$ where $\mathfrak{N}_*(G)$ denotes the set of all bordism classes G-manifolds and $\mathfrak{N}_*(BG)$ denotes the group of all bordism classes of singular manifolds in BG.

Proof. We consider the universal principal G-bundle ν :

$EG \longrightarrow BG$. An element of $\mathfrak{N}_n(BG)$ is represented by $[V^n, f]$ where V^n is an n-dimensional manifold and $f : V^n \longrightarrow BG$ is a map. Then in $V^n \times EG$, let M^n be defined as the set of all (x,y) with $f(x) = \nu(y)$. Now M^n is the principal G-space with action θ of G given by $g(x,y) = (x,gy)$. The quotient map $q : M^n \longrightarrow V^n$ is $q(x,y) = x$, and q is a local homeomorphism. A differential structure is imposed on M^n to *make q a local diffeomorphism so that M^n becomes a closed* manifold together with a principal action of G. The map ψ : $\mathfrak{N}_*(BG) \longrightarrow \mathfrak{N}_*(G)$ given by $\psi([V^n,f]) = [M^n,\theta]$ is a well-defined homomorphism.

Also, let $[M^n,\theta] \in \mathfrak{N}_n(G)$. Then there is an equivariant map $F : M^n \longrightarrow EG$ which is unique upto equivariant homotopy. With $V^n = M^n/G$, this induces a homotopically unique map $f : V^n \longrightarrow BG$. We give a differentiable structure on V^n to make $q : M^n \longrightarrow M^n/G = V^n$ a local diffeomorphism. This defines a map

$$\phi : \mathfrak{N}_*(G) \longrightarrow \mathfrak{N}_*(BG)$$

given by $\phi([M^n,\theta]) = [V^n,f]$.

Thus $\mathfrak{N}_*(G) \cong \mathfrak{N}_*(BG)$. ■

Lemma 2.2.3. If $L \leq G$ then $B(O,G)_n|_L = B(O,L)_n$, i. e., $B(O,G)_n$ thought of as an L-space is just $B(O,L)_n$.

Proof Let $E \longrightarrow X$ be a L-vector bundle. Then $G \times_L E \longrightarrow G \times_L X$ is a G-vector bundle and hence has a G-equivariant classifying map $f : G \times_L X \longrightarrow B(O,G)_n$. Since $X \leq G \times_L X$, $f|L$ is a L-equivarient map to $B(O,G)_n|_L$ which classifies $E \longrightarrow X$. ■

<u>Theorem 2.2.4</u>. If $f : M^n \longrightarrow X$ is an unoriented singular manifold then $[M^n, f] = 0$ iff every characteristic number of the map f, $\langle w_1^{\iota_1} w_2^{\iota_2} \ldots w_n^{\iota_n} \cup f^*(x), [M] \rangle = 0$ for all $x \in H^*(X, \mathbb{Z}_2)$ where w_j is the j-th Steifel-Whitney class of M, (cf. page 56 of [6]).

By theorem (2.2.2), we have $\mathfrak{N}_*(G) \cong \mathfrak{N}_*(BG)$ by the map which sends $[M, \theta]$ to $[M/G, f]$ where $f : M/G \longrightarrow BG$ classifies the principal G-bundle $M \longrightarrow M/G$. Consider the equivariant cohomology $h^*(X;G) = H^*(EG \times_G X; \mathbb{Z}_2)$.

Now $\pi : EG \times_G M \longrightarrow M/G$ is a fibration. The inverse of π, $q : M/G \longrightarrow EG \times_G M$ given by $q([gm]) = [\bar{f}(m), m]$, where $\bar{f} : M \longrightarrow EG$ is an equivariant map, is a homotopy equivalence as EG is contractible. Thus $h_*(M;G) = H_*(EG \times_G M; \mathbb{Z}_2) = H_*(M/G; \mathbb{Z}_2)$ has a topological class if M is compact since M/G is a compact manifold. Hence h^*-characteristic numbers are defined.

<u>Theorem 2.2.5</u>. Let $[M] \in \mathfrak{N}_*(G)$. Then $[M] = 0$ iff all h^*-characteristic numbers vanish.

<u>Proof</u>. We have an inclusion $i : BO_n \longrightarrow B(O,G)_n$ where BO_n is the classifying space for vector bundles with trivial G-action. Since $B(O,e)_n = BO_n$, we have that $i : BO_n \longrightarrow B(O,G)_n$ is a homotopy equivalence. From the diagram of fibrations

$$\begin{tikzcd}
BG \times BO_n \arrow[r, "\approx"] \arrow[d] & EG \times BO_n \arrow[r, "\approx"] \arrow[d, "BO_n"] & EG \times B(O,G)_n \arrow[d, "B(O,G)_n"] \\
BG \arrow[r, phantom, "="] & BG \arrow[r, phantom, "="] & BG
\end{tikzcd}$$

we see that $EG \times_G B(O,G)_n \approx BG \times BO_n$ and hence

$$h^*(B(O,G)_n;G) = H^*(BG;\mathbb{Z}_2) \underset{\mathbb{Z}_2}{\otimes} H^*BO_n;\mathbb{Z}_2).$$

We have the maps

$$M/G \xrightarrow{q} EG \times_G M \xrightarrow{1\times\tau} EG \times_G B(O,G)_n \approx BG \times BO_n$$

and we will denote the composite by $\ell \times k$ where $\ell : M/G \longrightarrow BG$ and $k : M/G \longrightarrow BO_n$.

Using the diagram of principal fibrations below, we can identify ℓ with the classifying map for the bundle $M \to M/G$:

$$\begin{tikzcd}
M \arrow[r, "\bar{f}\times id"] \arrow[d] & EG \times M \arrow[r] \arrow[d] & EG \arrow[d] \\
M/G \arrow[r, "q"] & EG \times_G M \arrow[r] & BG
\end{tikzcd}$$

To identify k we see that there is a bundle map $T(M) \longrightarrow T(M)/G$ and the classifying map $\tau_M : M \longrightarrow B(O,G)_n$ factors as

$$M \xrightarrow{\pi} M/G \xrightarrow{p} BO_n \xrightarrow{i} B(O,G)_n .$$

Taking product with EG we get

$$M/G \xrightarrow{q} EG \times_G M \xrightarrow{\pi\times\pi} BG \times M/G \xrightarrow{id\times p} BG \times BO_n \xrightarrow{\approx} EG \times_G B(O,G)_n$$

and since $\pi \circ q = id$, we have $k = p$ i.e. k classifies $T(M)/G = T(M/G) \oplus T_F/G$ where T_F is the tangent bundle along the fiber $\pi : M \longrightarrow M/G$. T_F is induced by $M/G \xrightarrow{\ell} BG \xrightarrow{ad} BO_s$ where $s = \dim G$ and $ad : G \longrightarrow O_s$ the adjoint representation.

Thus the Stiefel-Whitney classes of M/G may be expressed in terms of the Whitney classes of T(M)/G, $k^*w_1, \dots , k^*w_n$ and the Whitney classes of T_F which are in the image of ℓ^*. Thus all cohomology classes $w_1^{i_1} \dots w_n^{i_n} \cup f^*(x)$ are in the image of $(\ell \times k)^*$ and h^*-numbers vanish if and only if the numbers $\langle w_1^{i_1} \dots w_n^{i_n} \cup f^*(x), [M] \rangle$, for all $x \in H^*(X,\mathbb{Z}_2)$, vanish where w_j is the j-th Steifel-Whitney class of M. Hence the theorem follows from theorem (2.2.2). ■

§3. Singular G-bordism.

(A) Oriented case

Let G be a finite group and (X,A) be a G-pair. A singular oriented n-dimensional G-manifold in the G-pair (X,A) is a triple $(M^n,f;G)$ consisting of a compact oriented G-manifold M^n of dimension n with boundary ∂M^n and and equivariant map $f : (M^n,\partial M^n) \longrightarrow (X,A)$. A singular oriented n-dimensional G-manifold inthe G-pair (X,A) is called a singular oriented n-dimensional principal G-manifold in (X,A) if the action of G on M^n is a principal action.

Two singular oriented n-dimensional G-manifolds $(M^n_1,f_1;G)$ and $(M^n_2,f_2;G)$ in (X,A) are said to be equivalent if there exists an orientation preserving diffeomorphism $\phi : M^n_1 \longrightarrow M^n_2$ such that ϕ is equivariant and $f_2 \circ \phi = f_1$.

A singular oriented n-dimensional G-manifold $(M^n_1,f_1;G)$ in (X,A) is said to be equivalently bordant to $(M^n_2,f_2;G)$ if there exists a compact oriented (n+1)-dimensional G-manifold W^{n+1} with boundary ∂W^{n+1} and an equivariant map $F : W^{n+1} \longrightarrow X$ such that there exists an orientation preserving equivariant map $i : M^n_1 \amalg (-M^n_2) \longrightarrow W^{n+1}$ where topology on $i(M^n_1 \amalg (-M^n_2))$ induced by i is same as that induced by W^{n+1}, $F|(M^n_1 \amalg (-M^n_2)) = f_1 \amalg f_2$ and $F(W^{n+1} \setminus (M^n_1 \amalg (-M^n_2)) \subseteq A$.

The relation 'bordant to' is an equivalence relation in the set of all singular oriented n-dimensional G-manifolds in (X,A). We denote the set of equivalence classes by $\Omega_n(X,A;G)$ and an element of $\Omega_n(X.A;G)$ by $[M^n,f;G]$. The set $\Omega_n(X,A:G)$ is an abelian group under the binary operation

$$[M_1^n, f_1;G] + [M_2^n, f_2;G] = [M_1^n \amalg M_2^n, f_1 \amalg f_2;G]$$

where $[M_1^n, f_1;G]$ and $[M_2^n, f_2;G] \in \Omega_n(X,A;G)$.

(B) Unoriented Case

Let G be a finite group and (X,A) be a G-pair. An unoriented n-dimensional singular G-manifold in the pair (X,A) is a triple $(M^n,f;G)$ where M^n is an unoriented n-dimensional compact G-manifold with boundary ∂M^n and an equivariant map $f : (M^n,\partial M^n) \longrightarrow (X,A)$.

A bordism relation is defined on the set of unoriented n-dimensional singular G-manifolds just as in case of oriented version, except that no orientability requirements are imposed here. The resulting set of bordism classes is denoted by $\mathfrak{N}_n(X,A)$ and an element of $\mathfrak{N}_n(X,A)$ is denoted by $[M^n,f,;G]$. An abelian group operation is defined on $\mathfrak{N}_n(X,A)$ as $[M_1^n,f_1;G] + [M_2^n,f_2;G] = [M_1^n \amalg M_2^n, f_1 \amalg f_2;G]$ where $[M_1^n,f_1;G]$ and $[M_2^n,f_2;G] \in \mathfrak{N}_n(X,A)$.

§4. Characteristic classes of unoriented singular G-manifold and its invariance.

Let X be a finite CW-complex with free action of G, G being a finite group, and let X/G be again a finite CW-complex. Let h^* be an equivariant cohomology theory and h_* be the associated equivariant homology theory. Let $h^* = H^* \circ A$ and $h_* = H_* \circ A$ where A is a functor from the category of G-spaces and equivariant maps to the category of topological spaces and continuous maps, H^* is the singular cohomology theory and H_* is the associated singular homology theory. Let

$$\langle\ ,\ \rangle : h^*(X;G) \underset{H^*(pt)}{\otimes} h_*(X;G) \longrightarrow H_*(pt)$$

be the kronecker pairing.

Suppose $[M^n,f;G]$ is an element of unoriented bordism group $\mathfrak{N}_n(X,A)$ and $x \in h^*(B(O,G)_n;G)$. Then the x-characteristic number of the map $f : M^n \longrightarrow X$ associated with an element $a^m \in h^n(X;G)$ is defined to be $\langle \tau^*_{M^n}(x) \cup f^*(a^m),\ [M^n]\rangle \in H_*(pt)$, where $\tau_{M^n} : M^n \longrightarrow B(O,G)_n$ is the tangent map, i.e., the classifying map for the G-tangent bundle on M^n.

In this section, we shall consider the equivarient cohomology h^* given by

$$h^*(X;G) = H^*(EG \times_G X;\mathbb{Z}_2)$$

and the associated equivariant homology h_* given by

$$h_*(X;G) = H_*(EG \times_G X;\mathbb{Z}_2)$$

where the action of G on EG x X is given by $g(e,x) = (eg^{-1},gx)$, EG being the total space of the universal G-bundle.

The map $q : X/G \longrightarrow EG \times_G X$ given by $q([x]) = [\bar{h}(x),x]$, where $\bar{h}$ is the map covering the classifying map $h : X/G \longrightarrow BG$ of the projection G-bundle $X \longrightarrow X/G$, is homotopy equivalence.

Thus $h^*(X;G) \overset{q^*}{\approx} H^*(X/G;\mathbb{Z}_2)$ and $h_*(X;G) \overset{q_*^{-1}}{\approx} H_*(X/G;\mathbb{Z}_2)$. Therefore, $h_*(M^n;G) \approx H_*(M^n/G;\mathbb{Z}_2)$ has a topological class σ_n in dimension n with $\sigma_n = q_*(\bar{\sigma}_n)$, $\bar{\sigma}_n$ being the fundamental class of the manifold M^n/G in $H_*(M^n/G;\mathbb{Z}_2)$.

Theorem 2.4.1. If $[M^n, f; G] \in \mathfrak{N}_n(X;G)$ is zero, then for all $x \in h^*(B(O.G)_n;G)$, x-characteristic numbers of the map $f : M^n \longrightarrow X$ associated with every $a^m \in h^m(X;G)$ are zero.

Proof. Since $[M^n, f; G] \in \mathfrak{N}_n(X;G)$ is zero, there exists an $(n+1)$-dimensional compact manifold W^{n+1} and an equivariant map $F : W^{n+1} \longrightarrow X$ with $\partial W^{n+1} = M^n$ and $F|M^n = f$. Let $[W, \partial W] \in h_{n+1}(W^{n+1}, \partial W^{n+1}; G)$ be the topological class of W^{n+1}. Then $\partial_*[W, \partial W] = \sigma_n$. We have the following commutative diagram:

$$\begin{tikzcd}
h^*(B(O,G)_n;G) \arrow[r, "\tau^*_{M^n}"] & h^*(M^n;G) \\
h^*(B(O,G)_{n+1};G) \arrow[u, "j^*"] \arrow[r, "\tau^*_{W^{n+1}}"] & h^*(W^{n+1}, \partial W^{n+1}.G) \arrow[u, "\iota^*"']
\end{tikzcd}$$

where $j : B(O,G)_n \longrightarrow B(O,G)_{n+1}$ is the map classifying $(\gamma^n(G) \oplus 1)$, $\gamma^n(G) : E(O,G)_n \longrightarrow B(O,G)_n$ being the universal G-vector bundle. Also

$$\begin{aligned} h^*(B(O,G)_n;G) &= H^*(EG \times_G B(O,G)_n;\mathbb{Z}_2) \\ &\approx H^*(BG \times BO_n;\mathbb{Z}_2) \\ &\approx H^*(BG,\mathbb{Z}_2) \underset{\mathbb{Z}_2}{\otimes} H^*(BO_n;\mathbb{Z}_2) \end{aligned}$$

Similarly $h_*(B(O,G)_n;G) \approx H_*(BG,\mathbb{Z}_2) \underset{\mathbb{Z}_2}{\otimes} H_*(BO_n;\mathbb{Z}_2)$.

We have $j^* = 1 \otimes \alpha^*$, where $\alpha^* : H^*(BO_{n+1},\mathbb{Z}_2) \longrightarrow H^*(BO_n;\mathbb{Z}_2)$ is induced by the inclusion $BO_n \hookrightarrow BO_{n+1}$. Since α^* is surjection, so j^* is a surjection. . Therefore for every $x \in h^*(B(O,G)_n;G)$, $\exists$ $y \in h^*(B(O,G)_{n+1};G)$ such that $j^*(y) = x$.

Therefore $\langle \tau^*_{M^n}(x) \cup f^*(a^m), [M^n] \rangle$

$$= \langle \tau^*_{M^n}(j^*(y)) \cup f^*(a^m), \sigma_n \rangle$$

$$= \langle i^*\tau^*_{W^{n+1}}(y) \cup f^*(a^m), \sigma_n\rangle$$

$$= \langle i^*\tau^*_{W^{n+1}}(y) \cup i^*F^*(a^m), \sigma_n\rangle$$

$$= \langle \tau^*_{W^{n+1}}(y) \cup F^*(a^m), i^*\sigma_n\rangle$$

$$= \langle \tau^*_{W^{n+1}}(y) \cup F^*(a^m), i_*\partial_*[W,\partial W]\rangle = 0.$$

Thus all the x-characteristic numbers are zero.

We now consider the map $\mu : \mathfrak{N}_n(X;G) \longrightarrow h_*(X;G)$ defined by $\mu([M^n,f;G]) = q_*\bar{f}_*(\bar{\sigma}_n)$, where $\bar{\sigma}_n$ is the fundamental class of M^n/G in $H_n(M^n/G;\mathbb{Z}_2)$, and $\bar{f}$ is the map obtained from f by passing onto quotients making the following diagram commutative,

$$\begin{tikzcd} M^n \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ M^n/G \arrow[r, "\bar{f}"] & X/G \end{tikzcd}$$

and the map $q_* : H_*(X/G;\mathbb{Z}_2) \longrightarrow h_*(X;G)$.

Let $\bar{\mu} : \mathfrak{N}_*(X/G) \longrightarrow H_*(X/G;\mathbb{Z}_2)$ be defined as $\bar{\mu}([N^n,g]) = g_*(\bar{\sigma}'_n)$, where $\bar{\sigma}'_n$ is the fundamental class of N^n in $H_n(N^n;\mathbb{Z}_2)$. Suppose $\psi_* : \mathfrak{N}_*(X;G) \longrightarrow \mathfrak{N}_*(X/G)$ is given by $\psi_*([M^n,f;G]) = [M^n/G,\bar{f}]$. Then $\mu = q_*\circ\bar{\mu}\circ\psi_*$, and therefore, μ is an epimorphism, since $\bar{\mu}$ is so. For every $a \in h_*(X;G)$, let $[M^n,f;G] \in \mathfrak{N}_*(X;G)$ such that $\mu([M^n,f;G]) = a$. We define the $\mathfrak{N}_*$-module structure on $h_*(X;G)$ by $[V^m]a = \mu[M^n \times V^m, f';G]$, for every $[V^m] \in \mathfrak{N}_*$, where the action of G on $M^n \times V^m$ is defined as $g(x,y) = (gx,y)$ and the map $f' : M^n \times V^m \longrightarrow X$ is defined as $f'(x,y) = f(x)$. Thus $h_*(X;G) \otimes \mathfrak{N}_*$ is a $\mathfrak{N}_*$-module. Let $\{c_{n,i}\}$ be the additive base of $h_*(X;G)$. Let $[M^n_i,f_i;G] \in \mathfrak{N}_*(X:G)$ with $\mu([M^n_i,f_i;G]) = c_{n,i}$. We define

$$h : h_*(X;G) \otimes \mathfrak{N}_* \longrightarrow \mathfrak{N}_*(X;G)$$

given by $h(c_{n,i} \otimes 1) = [M^n_i, f_i; G]$.

<u>Theorem 2.4.2.</u> The map $h : h_*(X;G) \otimes \mathfrak{N}_* \longrightarrow \mathfrak{N}_*(X;G)$ is an isomorphism.

<u>Proof.</u> We have the following commutative diagram

$$\begin{tikzcd}
h_*(X;G) \otimes \mathfrak{N}_* \arrow[r, "h"] \arrow[d, "q^{-1}_* \otimes 1_{\mathfrak{N}_*}"] & \mathfrak{N}_*(X;G) \arrow[d, "\psi_*"] \\
H_*(X/G;\mathbb{Z}_2) \otimes \mathfrak{N}_* \arrow[r, "\bar{h}"] & \mathfrak{N}_*(X/G),
\end{tikzcd}$$

where $\bar{h} : H_*(X/G;\mathbb{Z}_2) \otimes \mathfrak{N}_* \longrightarrow \mathfrak{N}_*(X/G)$ is defined as

$$\bar{h}(\bar{c}_{n,i} \otimes 1) = [M^n_i/G, \bar{f}_i],$$

$\bar{c}_{n,i} = q^{-1}_*(c_{n,i})$. The vertical maps in the above diagram are isomorphisms and $\bar{h}$ is also an isomorphism by the theorem 17.2 of [5]. The isomorphism of h then follows from the diagram.

<u>Theorem 2.4.3.</u> If all the characteristic numbers of the map $f : M^n \longrightarrow X$ for an element $[M^n, f; G] \in \mathfrak{N}_*(X;G)$, corresponding to the theory h^*, are zero, then $[M^n, f; G]$ is zero in $\mathfrak{N}_n(X;G)$.

<u>Proof.</u> Let $\mu([M^n, f; G]) = c_n \in h_*(X;G)$ and $q^{-1}_*(c_n) = \bar{c}_n \in H_*(X/G;\mathbb{Z}_2)$. Then $\bar{f}_*(\bar{\sigma}_n) = \bar{c}_n$. Suppose $\{c_{n,i}\}_{i \in I}$ is an additive base of $h_n(X;G)$ and $c^{n,i} \in h^n(X;G)$ is the cohomology class dual to $c_{n,i}$ in the sense that $\langle \bar{c}^{n,i}, \bar{c}_{n,j} \rangle = \delta_{ij}$, where $\bar{c}_{n,i} = q^{-1}_*(c_{n,i})$ and $\bar{c}^{n,i} = q^*(c^{n,i})$. Let $c_n = \sum_{i \in S} \pm c_{n,i}$ where S is a finite subset of I. Let $c^n = \sum_{j \in S} \pm c^{n,j}$. By the hypothesis each x-characteristic number of the map $f : M^n \longrightarrow X$ associated with $c^n \in h^*(X;G)$ is zero, i.e., $\langle f^*(c^n), \sigma_n \rangle = 0$, taking x to be the unit class of $h^*(B(0,G)_n;G)$.

So $f^*(c^n), q_*(\bar{\sigma}_n)\rangle = 0 = \langle c^n, f_*q_*(\bar{\sigma}_n)\rangle = 0$

$$\Rightarrow \langle (q^*)^{-1}(\bar{c}^n), f_*q_*(\bar{\sigma}_n)\rangle = 0.$$

$$\Rightarrow \langle (q^*)^{-1}(\bar{c}^n).q_*\bar{f}_*(\bar{\sigma}_n)\rangle = 0,$$

since the diagram

$$\begin{tikzcd}
h_*(M^n;G) \arrow[r, "T_*"] \arrow[d, "q_*^{-1}"] & h_*(X;G) \arrow[d, "q_*^{-1}"] \\
H_*(M^n/G;\mathbb{Z}_2) \arrow[r, "\bar{f}_*"] & H_*(X/G;\mathbb{Z}_2)
\end{tikzcd}$$

is commutative. Therefore $\langle (q_*)^{-1}(\bar{c}^n), q_*(\bar{c}_n)\rangle = 0$, implies that $\langle \bar{c}^n, \bar{c}_n\rangle = 0$. Thus $\bar{c}_n = 0$ and hence $c_n = 0$. Also $h(c_n \otimes 1) = [M^n, f; G]$. Since h is an isomorphism and $c_n = 0$, so $[M^n, f; G] = 0$. This completes the proof of the theorem.

§5. Characteristic classes of oriented singular G-manifold and its invariance.

Let X be a finite CW-complex with an action of a finite group G and let $EG \times_G X$ be again a finite CW-complex. Let $[M^n, \partial M^n]$ be a class assigned to each compact oriented G-manifold M^n in $h_*(M^n, \partial M^n)$ such that

(a) $[M_1 \cup M_2, \partial M_1 \cup \partial M_2] = [M_1, \partial M_1] + [M_2, \partial M_2]$ and

(b) $\partial_*[M, \partial M] = [\partial M]$.

Such a class $[M, \partial M] \in h_*(M, \partial M)$ is called a topological class of the oriented G-manifold M.

Consider $[M^n, f] \in \Omega_n(X, A)$. We define x-characteristic number of the map $f: M^n \longrightarrow X$ associated with an element $a^m \in h^m(X;G)$ to be $\langle \tau^*_{M^n}(x) \cup f^*(a^m), [M^n]\rangle \in h_*(pt.)$ where $\tau_{M^n} : M^n \longrightarrow B(SO,G)_n$ is tangent map and $[M^n] \in h_n(M^n;G)$ is a topological class of M^n.

Theorem 2.5.1. The n-dimensional oriented principal G-bordism group $\Omega_n(X,A;G)$ is isomorphic to the n-dimensional oriented bordism group $\Omega_n(X \times_G EG, A \times_G EG)$.

Proof. Let $[M^n,f;G] \in \Omega_n(X,A;G)$. Then $f : M^n \longrightarrow X$ is an equivariant map. Also $\eta : M^n \longrightarrow M^n/G$ is a principal G-bundle. By classification theorem of G-bundles we have a map $\alpha : M^n \longrightarrow EG$ and $\beta : M^n/G \longrightarrow BG$ making the following diagram commutative:

$$\begin{tikzcd}
M^n \arrow[r, "\alpha"] \arrow[d, "\eta"] & EG \arrow[d, "\xi"] \\
M^n/G \arrow[r, "\beta"] & BG
\end{tikzcd}$$

Let the map $(f,\alpha):M^n \longrightarrow X \times EG$ be defined as $(f,\alpha)(a) = (f(a), \alpha(a))$, for every $a \in M^n$. Let $\tilde{f} = \overline{(f,\alpha)} : M^n/G \longrightarrow X \times_G EG$ be the map obtained from (f,α) on passing to the quotients. This gives a map

$$\mu:\Omega_n(X,A;G) \longrightarrow \Omega_n(X \times_G EG, A \times_G EG)$$

given by $\mu([M^n,f;G]) = [M^n/G,\tilde{f}]$.

We define the inverse map

$$h:\Omega_n(X \times_G EG, A \times_G EG) \longrightarrow \Omega_n(X,A;G)$$

as follows:

Let $[N^n,f] \in \Omega_n(X \times_G EG, A \times_G EG)$. Then $f:(N^n,\partial N^n) \longrightarrow (X \times_G EG, A \times_G EG)$ and we have the following diagram

$$\begin{tikzcd}
 & X \times EG \arrow[d, "\nu"] \\
N^n \arrow[r, "f"] & X \times_G EG
\end{tikzcd}$$

where $\nu:X \times EG \longrightarrow X \times_G EG$ is the projection map. Let $M^n = \{(y,x,t) \in N^n \times (X \times EG) \mid f(y) = \nu(x,t)\}$, and we define an action of G on M^n as $g(y,x,t) = (y,g(x,t))$. Clearly it is a free action and M^n/G is homeomorphic to N^n. Since N^n

is an oriented manifold with $\eta' : M^n \longrightarrow N^n$ being a covering space, a differential structure and an orientation can be given to M^n to make M^n an n-dimensional compact oriented principal G-manifold and the map η' an orientation preserving local diffeomorphism. We define $f' : M^n \longrightarrow X$ as $f'(y,x,t) = x$. Also $f'(\partial M^n) \subseteq A$, since $f(\partial N^n) \subseteq A/G$. Thus $[M^n, f';G] \in \Omega_n(X,A;G)$. This gives a map h, defined by $h([N^n,f]) = [M^n,f';G]$. Then $\mu \circ h$ and $h \circ \mu$ are the identity maps on $\Omega_n(X/G,A/G)$ and $\Omega_n(X,A;G)$ respectively. Hence μ is an isomorphism. ■

In this section, we will be considering the equivariant cohomology and homology $h^*(X;G) = H^*(X \times_G EG; \mathbb{Z}\oplus\mathbb{Z}_2)$ and $h_*(X;G) = H_*(X \times_G EG; \mathbb{Z}\oplus\mathbb{Z}_2)$.

Theorem 2.5.2. If $[M^n,f;G] = 0 \in \Omega_n(X;G)$, then all the x-characteristic numbers of the map $f : M^n \longrightarrow X$ associated with every $a^m \in h^m(X;G)$ are zero.

Proof Let $q : M^n/G \longrightarrow M^n \times_G EG$ be given by $q([x]) = [x, \alpha(x)]$ where α is the map covering the classifying map $\beta : M^n/G \longrightarrow BG$ of the principal G-bundle $M^n \longrightarrow M^n/G$. Then q is homotopy equivalence with an inverse the projection map $\pi : EG \times_G M^n \longrightarrow M^n/G$.

Since G acts freely on M^n,

$h_*(M^n;G) \approx H_*(M^n/G, \mathbb{Z}\oplus\mathbb{Z}_2) \approx H_*(M^n/G,\mathbb{Z}) \oplus H_*(M^n/G,\mathbb{Z}_2)$.

Since M^n/G is an oriented manifold, $h_*(M^n;G)$ has a topological class, say σ_n in dimension n. Since $[M^n,f;G]$ is zero in $\Omega_n(X;G)$, there exists an (n+1)-dimensional principal compact oriented G-manifold W^{n+1} with an equivariant map

$F : W^{n+1} \longrightarrow X$ such that $\partial W^{n+1} = M^n$ and $F|M^n = f$. Let $[W^{n+1}, \partial W^{n+1}] \in h_*(W^{n+1},\partial W^{n+1};G)$ be a topological class of W^{n+1} Then $\partial_*([W^{n+1},\partial W^{n+1}]) = \sigma_n$. We have the following commutative diagram:

$$\begin{tikzcd}
M^n = \partial W^{n+1} \arrow[r, "\tau_{M^n}"] \arrow[d, "i"] & B(SO,G)_n \arrow[d, "j"] \\
W^{n+1} \arrow[r, "\tau_{W^{n+1}}"] & B(SO,G)_{n+1}
\end{tikzcd}$$

where j is the map classifying $\gamma_s^n(G) \oplus 1$; $\gamma_s^n(G) : E(SO,G)_n \longrightarrow B(SO,G)_n$ being the universal oriented G-vector bundle of dimension n.

Also, $h^*(B(SO,G)_n;G) = H^*(B(SO,G)_n \times_G EG ; \mathbb{Z} \oplus \mathbb{Z}_2)$

$\approx H^*(BG \times BSO_n ; \mathbb{Z} \oplus \mathbb{Z}_n)$.

Therefore, $h^*(B(SO,G)_n;G) \overset{\mu^*}{\approx} H^*(BG;\mathbb{Z}\oplus\mathbb{Z}_2) \otimes H^*(BSO_n;\mathbb{Z}\oplus\mathbb{Z}_2)$ and $h_*(B(SO,G)_n;G) \overset{\mu_*}{\approx} H_*(BG;\mathbb{Z}\oplus\mathbb{Z}_2) \otimes H_*(BSO_n;\mathbb{Z}\oplus\mathbb{Z}_2)$. Since j^* is surjective, for every $x \in h^*(B(SO,G)_n;G)$ there exists $y \in h^*(B(SO,G)_{n+1};G)$ such that $j^*(y) = x$.

Therefore, $\langle \tau^*_{M^n}(x) \cup f^*(a^m), \sigma_n \rangle$

$= \langle \tau^*_{M^n} j^*(y) \cup i^* F^*(a^m), \sigma_n \rangle$

$= \langle i^* \tau^*_{W^{n+1}}(y) \cup i^* F^*(a^m), \partial_*[W^{n+1}, \partial W^{n+1}] \rangle$

$= \langle \tau^*_{W^{n+1}}(y) \cup F^*(a^m), i_*\partial_*[W^{n+1}, \partial W^{n+1}] \rangle$

$= 0$.

Hence all x-characteristic numbers are zero. ■

Theorem 2.5.3. If all the x-characteristic numbers of the map $f : M^n \longrightarrow X$ associated with every $a^m \in h^m(X;G)$ is zero then $[M^n,f;G] \in \Omega_n(X;G)$ is zero.

Proof. We have by theorem (2.5.1.) that

$$\Omega_n(X;G) \approx \Omega_n(X \times_G EG).$$

Therefore, it is sufficient to prove that $[M^n/G, \tilde{f}] = 0$, where $\tilde{f}$ is the map as given in the proof of theorem (2.5.1). We have $\langle \tau^*_{M^n}(x) \cup f^*(a^m), [M^n] \rangle = 0$, for all $x \in h^{n-m}(B(SO,G)_n;G)$ and $a^m \in h^m(X;G)$. We have the following commutative diagrams:

$$\begin{tikzcd}
h^*(B(SO,G)_n;G) \arrow[r, "\tau^*_{M^n}"] \arrow[d, "\mu^*"'] & h^*(M^n;G) \arrow[d, "q^*"] \\
H^*(BG;\mathbb{Z}\oplus\mathbb{Z}_2) \otimes H^*(BSO_n;\mathbb{Z}\oplus\mathbb{Z}_2) \arrow[r, "\bar{\tau}^*_{M^n}"] & H^*(M^n/G;\mathbb{Z}\oplus\mathbb{Z}_2)
\end{tikzcd}$$

and

$$\begin{tikzcd}
h^*(X;G) \arrow[r, "f^*"] \arrow[d, equal] & h^*(M^n;G) \arrow[d, "q^*"] \\
H^*(X \times_G EG;\mathbb{Z}\oplus\mathbb{Z}_2) \arrow[r, "\bar{f}^*"] & H^*(M^n/G;\mathbb{Z}\oplus\mathbb{Z}_2)
\end{tikzcd}$$

where $\bar{\tau}^*_{M^n}$ and $\bar{f}^*$ are the maps obtained from $\tau^*_{M^n}$ and f^* on passing to quotients, and q^* is the isomorphism induced from the homotopy equivalence $q : M^n/G \longrightarrow M^n \times_G EG$. Also, $\bar{\tau}^*_{M^n} = (\ell \times k)^*$, where $\ell : M^n/G \longrightarrow BG$ is the classifying map of the principal G-bundle $M^n \longrightarrow M^n/G$ and $k : M^n/G \longrightarrow BSO_n$ is the classifying map for the n-dimensional bundle $T(M^n)/G \longrightarrow M^n/G$.

Let $\bar{\sigma}_n$ be the fundamental class of M^n/G. Then $\sigma_n = q_*(\bar{\sigma}_n)$, where $q_* : H_*(M^n/G;\mathbb{Z}\oplus\mathbb{Z}_2) \longrightarrow h_*(M^n;G)$ is an isomorphism. Therefore

$$\langle \tau^*_{M^n}(x) \cup f^*(a^m), \sigma_n \rangle = 0.$$

$$\rightarrow \langle (q^*)^{-1}(\ell \times k)^*\mu^*(x) \cup (q^*)^{-1}\,\bar{f}^*(a^m), q_*(\bar{\sigma}_n) \rangle = 0$$

$$\rightarrow \langle (\ell \times k)^*\mu^*(x) \cup \bar{f}^*(a^m), \bar{\sigma}_n \rangle = 0$$

$$\rightarrow \langle (\ell \times k)^*(y) \cup \bar{f}^*(a^m), \bar{\sigma}_n \rangle = 0$$

for all $y \in H^*(BG; \mathbb{Z}\oplus\mathbb{Z}_2) \otimes H^*(BSO_n; \mathbb{Z}\oplus\mathbb{Z}_2)$ and $a^m \in H^m(X \times_G EG, \mathbb{Z}\oplus\mathbb{Z}_2)$. Therefore, the Stiefel-Whitney classes of M^n/G can be expressed in terms of $k^*(w_1)$, ..., $k^*(w_n)$, where w_i is the i-th Stiefel-Whitney classes of the universal bundle $\gamma_s^n : ESO_n \longrightarrow BSO_n$. It follows that all the Stiefel-Whitney classes of M^n/G are in the image of $(\ell \times k)^*$. Thus

$$\langle w_1^{i_1} \cdots\cdots w_r^{i_r} \cup \bar{f}^*(a^m), \bar{\sigma}_n \rangle = 0$$

for every $a^m \in H^m(X \times_G EG ; \mathbb{Z}\oplus\mathbb{Z}_2)$, $i_1 + 2i_2 + \ldots + ri_r = n-m$, w_j being the j-th Stiefel-Whitney class of M^n/G. Also all the Pontrjagin classes of M^n/G can be expressed in terms of the images of the Pontrjagin classes of the universal bundle $\gamma_s^n : ESO_n \longrightarrow BSO_n$ under k^*. Hence

$$\langle p_1^{i_1} \cdots\cdots p_r^{i_r} \cup \bar{f}^*(a^m), \bar{\sigma}_n \rangle = 0,$$

for all $a^m \in H^m(X \times_G EG; \mathbb{Z}\oplus\mathbb{Z}_2)$, $r \leq n/4$, $n-m = 4(i_1 + 2i_2 + \ldots + ri_r)$ and $p_1, \ldots, p_r$ being the Pontrjagin classes of M^n/G. Thus $[M^n/G, \tilde{f}] \in \Omega_n(X \times_G EG)$ with all the Steifel-Whitney numbers and Pontrjagin numbers of $\tilde{f} : M^n/G \longrightarrow X \times_G EG$ equal to zero. It is given that all the torsion elements of $H^*(X \times_G EG; \mathbb{Z})$ have order two. Hence the theorem follows from theorem (17.5) of [5].

CHAPTER III

Bordism and Group action

In 1964 Conner and Floyd introduced bordism method to the study of group actions. In this chapter we describe the result of Conner and Floyd which states that a closed $\mathbb{Z}_2^k$-manifold bounds if the $\mathbb{Z}_2^k$-action is fixed point free. We shall also develop the generalised version by R.E.Stong, according to which a closed $\mathbb{Z}_2^k$-manifold is $\mathbb{Z}_2^k$-boundary if the $\mathbb{Z}_2^k$-action is stationary point free. Next we discuss the work on finite group action and equivariant bordism, compact Lie group action and bordism by Khare. Finally we study the work of Wheeler on oriented bordism of $\mathbb{Z}_2{}^k$ - action.

§1. Action of $\mathbb{Z}_2^k$ without fixed points.

Defition 3.1.1. Let M^n be a manifold. Then a differentiable map $T : M^n \longrightarrow M^n$ of order two is called an involution on M^n.

Let M^n be a G-manifold. Let F be the set of fixed points of a group action on M^n, i.e., $F = \{ x \in M^n \mid gx = x$ for all $g \in G \}$. For each k, $0 \leq k \leq n$, we denote by $F^k \subseteq M^n$ the union of the k-dimensional components of F. Then $F = \coprod_0^k F^k$ and $\eta^{n-k} \longrightarrow F^k$ will denote the normal bundle to the union of the k-dimensional components of the set of stationary points and $\eta \longrightarrow F$ is the normal bundle to the whole set of fixed points. The group G acts as a group of bundle maps on $\eta \longrightarrow F$ sending each fibre onto itself.

Let T be a differentiable involution on a closed manifold M^n. We assume T is an isometry in a fixed Riemannian metric on the normal bundle $\eta \longrightarrow F$. Then there are also .the bundle involutions T on $S(\eta^{n-k})$, where $\eta \longrightarrow F = \coprod_0^k (\eta^{n-k} \longrightarrow F^k)$ is the normal bundle to F and $S(\eta^{n-k})$ is the sphere bundle. For $k < n$, the total space here has dimension $(n-1)$. Let $(T, S(\eta)) = \coprod_0^n (T, S(\eta^{n-k}))$. then $S(\eta)$ is called the normal sphere bundle to the fixed points set.

Now we have $H^*(\mathbb{R}P(\infty),\mathbb{Z}_2) \cong \mathbb{Z}_2[c]$ is the polynomial ring on $c \in H^1(\mathbb{R}P(\infty).\mathbb{Z}_2)$. A fixed point free involution (T,X) is induced by a homotopically unique map $f : X/T \longrightarrow \mathbb{R}P(\infty)$. We denote $f^*(c) \in H^1(X/T,\mathbb{Z}_2)$ by c once again and call c the characteristic classes of the involution. In general the numbers $\langle w_{i_1} w_{i_2} \ldots w_{i_r} c^m, \sigma(X/T) \rangle \in \mathbb{Z}_2$, are called the involution numbers of (T,X) corresponding to a partion $i_1 + i_2 \ldots + i_r$ of $n-m$ where w_i's are the Stiefel-Whitney classes, and $\sigma(X/T)$ is the fundamental class of X/T.

Theorem 3.1.2. Suppose for each non-negative integer n there is a fixed point free involution T on a closed n-maniflod X^n such that for each n the involution number $\langle c^n,\sigma(X/T)\rangle$ of (T,X^n) is non-zero. Then $\{[T,X^n]\}$ is a homogeneous $\mathfrak{N}_*$-module basis for $\mathfrak{N}_*(\mathbb{Z}_2)$. In particular if (A,S^n) is the antipodal involution on the n-sphere then $\{[A,S^n]\}$ is a basis for $\mathfrak{N}_*(\mathbb{Z}_2)$.

For proof see page 74 of [6].

By theorem (3.1.2), for every fixed point free involution (T,M^n), we have a unique expression

$$[T,M^n] = \sum_{m=0}^{n} [A,S^{n-m}][V^m].$$

Lemma 3.1.3. If (T,M^n) is an involution on a closed manifold with $(T,S(\eta))$ the bundle involution on the normal sphere bundle of fixed point set, then in $\mathfrak{N}_{n-1}(\mathbb{Z}_2)$,

$$[T,S(\eta)] = \sum_{k=0}^{n-1} [T,S(\eta^{n-k})] = 0.$$

Proof. Since F^n does not contribute, we can very well assume that $F^n = \phi$. Let (T,N) be a closed tubular neighborhood of F. Then $W^n = M^n \setminus \mathrm{int}(N)$ is a compact regular T-invarient n-dimensional submanifold of M^n for which (T,W^n) has no fixed points and we have $(T,\partial W^n) = (T,S(\eta)) = (T,\partial N)$. Hence we have $[T,S(\eta)] = \sum_{k=0}^{n-1} [T,S(\eta^{n-k})] = 0$.

Theorem 3.1.4. Let (T,M^n) denote a smooth involution on a closed manifold with $\eta \longrightarrow F$ the normal bundle to the fixed point set. If $\eta \oplus \mathbb{R} \longrightarrow F$ is the Whitney sum of the normal bundle with a trivial line bundle then, $\mathbb{R}P(\eta \oplus \mathbb{R})$ is a closed n-manifold and $[M^n] = [\mathbb{R}P(\eta \oplus \mathbb{R})]$ in $\mathfrak{N}_n$.

Proof. We consider the involutions $(T_1,M^n \times I)$ and $(T_2,M^n \times I)$ where $T_1(x,t) = (x,1-t)$, $T_2(x,t) = (tx,(1-t))$. Then fixed points set of T_1 is $M^n \times (1/2)$ and the normal bundle to this fixed set is a trivial line bundle. The fixed points set of T_2 is $F \times (1/2)$. Identifying F with $F \times (1/2)$, the normal bundle to the fixed points set of T_2 is $\eta \oplus \mathbb{R} \longrightarrow F$.

We define an equivariant diffeomorphism $\phi : (T_1,M^n \times \partial I) \longrightarrow (T_2,M^n \times \partial I)$ by $\phi(x,1) = (Tx,1)$ and $\phi(x,0) = (x,0)$. We adjoin $(T_1,M^n \times I)$ to $(T_2,M^n \times I)$ along thier boundaries by ϕ. This gives an involution (T_3,M^{n+1}) on a closed manifold

M^{n+1}. The fixed points set of M^{n+1} under T_g is $M^n \amalg F$. Therefore by lemma (3.1.3) we have

$$[A,S^0][M^n] + [T,S(\eta \oplus \mathbb{R})] = 0 \in MO_n(\mathbb{Z}_2).$$

Passing to qoutients we get $[M^n] = [\mathbb{R}P(\eta \oplus \mathbb{R})]$. ■

Theorem 3.1.5. If $\mathbb{Z}_2^k$ acts smoothly on a closed manifold M^n without fixed points then $[M^n] = 0$.

Proof. We use the method of induction.

For $k = 1$, (T,M^n) is a fixed point free involution. The mapping cylinder of the quotient map $q : M^n \longrightarrow M^n/T$ is a compact $(n+1)$-manifold whose boundary is M^n.

We assume that the result is true for $\mathbb{Z}_2^{k-1}$. Let $\mathbb{Z}_2^k$ acts smoothly on a closed manifold M^n without fixed points. Now $\mathbb{Z}_2^k = \mathbb{Z}_2 \times \mathbb{Z}_2^{k-1}$. Let F be the fixed points set of $\mathbb{Z}_2$ in M^n. If $F = \phi$, then by the induction hypothesis, the result is true. If $F \neq \phi$, then $\mathbb{Z}_2 \times \mathbb{Z}_2^{k-1}$ acts as a group of orthogonal bundle maps on the normal bundle $\eta \longrightarrow F$. The generator T, of the first $\mathbb{Z}_2$ acts on each fibre as the antipodal map. Since $\mathbb{Z}_2^{k-1}$ has no fixed point in F, the fibres of $\eta \longrightarrow F$ are carried onto distinct fibre by all elements $\mathbb{Z}_2^{k-1}$. We form the Whitney-sum $\eta \oplus \mathbb{R} \longrightarrow F$. We extend the action of $\mathbb{Z}_2 \times \mathbb{Z}_2^{k-1}$ to $\eta \oplus \mathbb{R}$ by $T(v,t) = (-v,-t)$ and for $g \in \mathbb{Z}_2^{k-1}$, $g(v,t) = (gv,t)$. Then we have the sphere bundle $(\mathbb{Z}_2^{k-1}, S(\eta \oplus \mathbb{R}))$ with T acting as the bundle involution and correspondingly an induced bundle $(\mathbb{Z}_2^{k-1}, \mathbb{R}P(\eta \oplus \mathbb{R})) \longrightarrow (\mathbb{Z}_2^{k-1}, F)$. Since $\mathbb{Z}_2^{k-1}$ has no fixed points on F, the actions of $\mathbb{Z}_2^{k-1}$ on $\mathbb{R}P(\eta \oplus \mathbb{R})$ is also fixed point free. Thus by induction hypothesis $[\mathbb{R}P(\eta \oplus \mathbb{R})] = 0$. Therefore, from the theorem (3.1.4), $[M^n] = [\mathbb{R}P(\eta \oplus \mathbb{R})] = 0$.

§2. $(\mathcal{F},\mathcal{F}')$-Free bordism and action of Z_2^k without fixed points.

Definition 3.2.1. A family $\mathcal{F}$ in G is a collection of subgroups of G such that

(a) if $H \in \mathcal{F}$ and K is a subgroup of G with $K \subseteq H$ then $K \in \mathcal{F}$,

(b) if $H \in \mathcal{F}$ then $gHg^{-1} \in \mathcal{F}$, for all $g \in G$.

Definition 3.2.2. Given a family $\mathcal{F}$ in a group G, an action θ of G on a smooth manifold M^n is said to be $\mathcal{F}$-free if for every $x \in M^n$ the isotropy group $G_x \in \mathcal{F}$. We often say that (M^n,θ) is an $\mathcal{F}$-free manifold.

Suppose $\mathcal{F}'$, $\mathcal{F}$ are families in G and $\mathcal{F}' \subseteq \mathcal{F}$. An $(\mathcal{F},\mathcal{F}')$ free n-dimensional singular manifold in a G-pair (X,A) is a 5-tuple (M^n,M_0,M_1,θ,f) where

(a) $\theta : G \times M^n \longrightarrow M^n$ is a group action with (M^n,θ) being $\mathcal{F}$-free.

(b) M_0 and M_1 are compact submanifolds of the boundary ∂M^n with $\partial M^n = M_0 \amalg M_1$, $M_0 \cap M_1 = \partial M_0 = \partial M_1$ which are invariant under the G-action such that $(M_0.\theta|G \times M_0)$ is $\mathcal{F}'$-free.

(c) $f : M^n \longrightarrow X$ is an equivarient map such that $f(M_1) \subseteq A$.

Definition 3.2.3. A 5-tuple $(M^n.M_0,M_1,\theta,f)$ is bordic to another 5-tuple $(M'^n,M'_0,M'_1,\theta',f')$ if $\exists$ a 6-tuple $(V^{n+1},V^+,V_0,V_1,\theta,F)$ where

(a) (V^{n+1},θ) is an $\mathcal{F}$-free manifold and $F : V^{n+1} \longrightarrow X$ is an equivarient map with $F(V_1) \subseteq A$.

(b) $\partial V^{n+1} = M^n \cup M'^n \cup V^+$ where $M^n \cap V^+ = \partial M^n$, $M'^n \cap V^+ = \partial M'^n$, $M^n \cap M'^n = \phi$ and $V^+ \cap (M^n \cup M'^n) = \partial V^+$, V^+ is invariant under θ which restricts to θ on M^n and θ' on M'^n.

(c) $F|M^n = f$ and $F|M'^n = f'$.

(d) V^+ is the union of G-invariant submanifolds V_0 and V_1 with intersection a submanifold V^- such that $\partial V_i = M_i \cup V^- \cup M'_i$, $M_i \cap V^- = \partial M_i$ and $M'_i \cap V^- = \partial M'_i$, $i = 0, 1$.

(e) $(V_0, \theta|G \times V_0)$ is an $\mathcal{F}'$-free manifold.

'To be bordic' is an equivalence relation in the set of all $(\mathcal{F},\mathcal{F}')$-free n-dimensional singular manifolds in (X,A). The equivalence classes of the 5-tuple $(M^n.M_0,M_1,\theta,f)$ under this relation is denoted by $[M^n.M_0,M_1,\theta,f]$ and is called an $(\mathcal{F},\mathcal{F}')$-free n-dimensional bordism element in (X,A). The set of all $(\mathcal{F},\mathcal{F}')$-free n-dimensional bordism elements in (X,A) forms an abelian group with respect to the operation induced by disjoint union and is denoted by $\mathfrak{N}_n(G;\mathcal{F},\mathcal{F}')(X,A)$. We call it $(\mathcal{F},\mathcal{F}')$-free n-dimensional unoriented bordism group of (X,A). Consider

$$\mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')(X,A) = \bigoplus_n \mathfrak{N}_n(G;\mathcal{F},\mathcal{F}')(X,A).$$

It is easy to see that $\mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')(X,A)$ is a module over the unoriented bordism ring $\mathfrak{N}_*$ with multiplication given by

$$[N^m][M^n,M_0,M_1,\theta,f] = [N^m \times M^n, N^m \times M_0, N^m \times M_1, \theta', f']$$

where $\theta' : G \times (N^m \times M^n) \longrightarrow (N^m \times M^n)$ is defined as $g(x,y) = (x,gy)$ for all $(x,y) \in N^m \times M^n$; $g \in G$ and $f' : N^m \times M^n \longrightarrow X$ is $f'(x,y) = f(y)$. The module $\mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')(X.A)$ is called $(\mathcal{F},\mathcal{F}')$-free unoriented bordism module in (X,A) over $\mathfrak{N}_*$.

If $\alpha : (X,A) \longrightarrow (X_1,A_1)$ is an equivarient map between G-pairs, one has the induced homomorphism

$$\alpha^* : \mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')(X,A) \longrightarrow \mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')(X_1,A_1),$$

defined as $\alpha^*[M^n,M_0,M_1,\theta,f] = [M^n,M_0,M_1,\theta,\alpha f]$. The boundary homomorphism

$$\partial_* : \mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')(X,A) \longrightarrow \mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')(A,\phi)$$

of degree -1 is given by

$$\partial_*[M^n, M_0, M_1, \theta, f] = [M_1, \partial M_1, \phi, \theta | G \times M_1, f | M_1].$$

It can be seen that if $\mathcal{F}'' \subseteq \mathcal{F}' \subseteq \mathcal{F}$ are families in G. Then the following sequence

$$\ldots\ldots \longrightarrow \mathfrak{N}_*(G; \mathcal{F}', \mathcal{F}'')(X,A) \xrightarrow{j_*} \mathfrak{N}_*(G; \mathcal{F}, \mathcal{F}'')(X,A) \xrightarrow{j'_*}$$

$$\mathfrak{N}_*(G; \mathcal{F}, \mathcal{F}')(X,A) \xrightarrow{j''_*\,\partial'_*} \mathfrak{N}_*(G; \mathcal{F}', \mathcal{F}'')(X,A) \longrightarrow \ldots\ldots$$

is exact, where $j : (\mathcal{F}, \mathcal{F}') \hookrightarrow (\mathcal{F}, \mathcal{F}'')$, $j' : (\mathcal{F}, \mathcal{F}'') \hookrightarrow (\mathcal{F}, \mathcal{F}')$ and $j'' : (\mathcal{F}, \phi) \hookrightarrow (\mathcal{F}', \mathcal{F}'')$ are inclusion maps.

We now consider a specific group $G = \mathbb{Z}_2^k$, generated by elements $t_1, \ldots, t_k$ with relations $t_i^2 = 1$, and $t_i t_j = t_j t_i$. For $p \leq k$, let $\mathbb{Z}_2^p$ be the subgroup of $\mathbb{Z}_2^k$ generated by $t_1, \ldots, t_p$. Let $\mathcal{F}_p = \{ H \subseteq G \mid H \not\supset \mathbb{Z}_2^p = [t_1, \ldots, t_p] \}$ and $\mathcal{F}_{k+1}$ the family of all subgroups of $\mathbb{Z}_2^k$. Then we have

$$\phi = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots\ldots \subseteq \mathcal{F}_k \subseteq \mathcal{F}_{k+1}.$$

Proposition 3.2.5. For $0 \leq p < k$, the sequence for the triple of families $\mathcal{F}_p \subseteq \mathcal{F}_{p+1} \subseteq \mathcal{F}_{k+1}$ becomes a split exact sequence:

$$0 \longrightarrow \mathfrak{N}_n(\mathbb{Z}_2^k; \mathcal{F}_{k+1}, \mathcal{F}_p) \longrightarrow \mathfrak{N}_n(\mathbb{Z}_2^k; \mathcal{F}_{k+1}, \mathcal{F}_{p+1}) \underset{\rho}{\rightleftarrows}$$

$$\mathfrak{N}_{n-1}(\mathbb{Z}_2^k; \mathcal{F}_{p+1}, \mathcal{F}_p) \longrightarrow 0.$$

Proof. We construct a map

$$\rho : \mathfrak{N}_{n-1}(\mathbb{Z}_2^k; \mathcal{F}_{p+1}, \mathcal{F}_p) \longrightarrow \mathfrak{N}_n(\mathbb{Z}_2^k; \mathcal{F}_{k+1}, \mathcal{F}_{p+1}).$$

Let $[M^n, \theta] \in \mathfrak{N}_{n-1}(\mathbb{Z}_2^k; \mathcal{F}_{p+1}, \mathcal{F}_p)$. Let F be the fixed point set in M^n under $(\mathbb{Z}_2)^p$. F is a submanifold of M^n with $\partial F = \partial M^n \cap F$. Since all isotropy subgroups on ∂M^n belong to $\mathcal{F}_p$, $x \in \partial M^n \rightarrow G_x \not\supset \mathbb{Z}_2^p$. Also $x \in F \rightarrow G_x \supset \mathbb{Z}_2^p$. Hence $F \cap \partial M = \phi$ and F is a closed submanifold embedded in the interior of M.

Let ν be the normal bundle of F in M^n. By $\mathbb{Z}_2^k$-tubular neighborhood theorem, $\exists$ a $\mathbb{Z}_2^k$-tubular neighborhood of F in M^n which is $\mathbb{Z}_2^k$-diffeomorphic to disc bundle $D(\nu)$ on F, the disc bundle $D(\nu)$ having induced action of $\mathbb{Z}_2^k$ through vector bundle maps covering the actions of $\mathbb{Z}_2^k$ on F. Since F is the fixed point set of $\mathbb{Z}_2^p$, $\mathbb{Z}_2^p$ acts as linear transformations of each fibre of $D(\nu)$ into itself with only the zero vector being fixed under the action.

Since all isotropy groups of M^n belong to $\mathcal{F}_{p+1}$, $x \in M^n \rightarrow G_x \not\supset \mathbb{Z}_2^{p+1}$. Also $x \in F \rightarrow G_x \supset \mathbb{Z}_2^p$. So t_{p+1} acts freely on F and hence also acts freely on $D(\nu)$.

Let $q_1 : F \longrightarrow F' = F / \langle t_{p+1} \rangle$ and $q_2 : D(\nu) \longrightarrow D'(\nu) = D(\nu) / \langle t_{p+1} \rangle$ be the quotient maps. Let ν' be the vector bundle obtained from ν on passing to quotients. Then $D(\nu') = D'(\nu)$.

Since t_{p+1} commutes with the action of $\mathbb{Z}_2^k$ on F and $D(\nu)$, we have induced actions of $\mathbb{Z}_2^k$ on F' and $D(\nu')$.

Let C_1 and C_2 be the mapping cylinder of q_1 and q_2, then we have maps $q_1^* : C_1 \longrightarrow F'$ and $q_2^* : C_2 \longrightarrow D'(\nu)$. Also we have the induced maps $\nu' : D'(\nu) \longrightarrow F'$ and $\nu^* : C_2 \longrightarrow C_1$ induced by $\nu : D(\nu) \longrightarrow F$ and the group action ν induces group actions ψ_1^* and ψ_2^* on C_1 and C_2 respectively. Now we have a $\mathbb{Z}_2^k$-diffeomorphism $\phi : (\nu^*)^{-1}(\partial C_1) \longrightarrow D(\nu)$.

We define $W = \dfrac{M^n \times [0,1] \amalg C_2}{(y,1) \sim \phi(y)}$ and a $\mathbb{Z}_2^k$-action X on W which restricts to the action $\theta \times 1$ on $M^n \times I$ and ψ_2^* on C_2.

We note that ∂W is the union of four manifolds $M^n \times \{0\}$,

$\partial M^n \times I$, $(M^n \times \{1\} \setminus D(\nu) \times \{1\})$ and $(\partial C_2 \setminus (\nu^*)^{-1}\partial C_1)$. On $M^n \times \{0\}$, X acts as θ and $\mathbb{Z}_2^{p+1}$ has no fixed point, on $\partial M^n \times [0,1]$, X acts as $\theta \times 1$ and $\mathbb{Z}_2^p$ has no fixed point, on $(M^n \times \{1\} \setminus D(\nu) \times \{1\})$, X acts as $\theta \times 1$ and $\mathbb{Z}_2^p$ has no fixed points, since all fixed points are in $F \subseteq$ int.$D(\nu)$. On $(\partial C_2 \setminus (\nu^*)^{-1}\partial C_1)$, X acts as ψ_2^* and $\mathbb{Z}_2^p$ has no fixed point, for ψ_2^* act on C_2 by bundle maps with $\mathbb{Z}_2^p$ acting as linear transformation on the fibres and having the zero vector as the fixed point set.

Hence $[W,X] \in \mathfrak{N}_n(\mathbb{Z}_2^k;\mathcal{F}_{k+1},\mathcal{F}_{p+1})$. We define $\rho[M^n,\theta] = [W,X]$. Consider $\tilde{W} = \partial W \times [0,1]$. Then $\partial\tilde{W} = \partial(\partial W \times [0,1]) = \partial W \amalg \partial W = \partial W \amalg (M^n \times \{0\} \cup M^n \times I) \cup (M^n \times \{1\} \setminus D(\nu) \times \{1\}) \cup (\partial C_2 \setminus (\nu^*)^{-1}\partial C_1)$, and also $X|_{(\mathbb{Z}_2)^k \times \partial W}$ is equal to θ on M^n. Hence $[\partial W, X|_{(\mathbb{Z}_2)^k \times \partial W}] = [M^n,\theta]$. ■

<u>Proposition 3.2.6</u>. If $\mathbb{Z}_2^k$ acts on a closed differentiable manifold M^n, then $[M^n,\theta] = 0$ in $\mathfrak{N}_n(\mathbb{Z}_2^k)$.

<u>Proof</u>. We have the exact squence of the triple $\mathcal{F}_0 \subseteq \mathcal{F}_k \subseteq \mathcal{F}_{k+1}$,.

$$\mathfrak{N}_*(\mathbb{Z}_2^k;\mathcal{F}_k,\mathcal{F}_0) \xrightarrow{i} \mathfrak{N}_*(\mathbb{Z}_2^k;\mathcal{F}_{k+1},\mathcal{F}_0) \xrightarrow{j} \mathfrak{N}_*(\mathbb{Z}_2^k;\mathcal{F}_{k+1},\mathcal{F}_k)$$

with $\mathfrak{N}_*(\mathbb{Z}_2^k;\mathcal{F}_k,\mathcal{F}_0)$, the bordism group of stationary point free actions and $\mathfrak{N}_*(\mathbb{Z}_2^k;\mathcal{F}_{k+1},\mathcal{F}_0) = \mathfrak{N}_*(\mathbb{Z}_2^k)$. Now we have the map j as the composite $j_{k-1} \circ \dots\dots \circ j_1 \circ j_0$.

$$\mathfrak{N}_*(\mathbb{Z}_2^k;\mathcal{F}_{k+1},\mathcal{F}_0) \xrightarrow{j_0} \mathfrak{N}_*(\mathbb{Z}_2^k;\mathcal{F}_{k+1},\mathcal{F}_1) \xrightarrow{j_1} \dots\dots\dots \xrightarrow{j_{k-1}} \mathfrak{N}_*(\mathbb{Z}_2^k;\mathcal{F}_{k+1},\mathcal{F}_k).$$

Now by the proposition (3.2.5), each j_i is one-one for $0 < i < k$. Hence i is one-one and $i = 0$. So $[M^n] = 0$ in $\mathfrak{N}_*(\mathbb{Z}_2^k)$.

§3. Finite group action and equivariant bordism.

Throughout this section we will take G to be a finite group. In this section we will prove that if G acts without fixed point on a manifold M^n then M^n is a G-boundary.

Definition 3.3.1. Let $(\mathcal{F},\mathcal{F}\ )$ be a pair of families in G and a be a central element in G of order 2. We call $(\mathcal{F}.\mathcal{F}\ )$ an admissible pair of families with respect to $a \in G$ if

(i) $a \notin H$, for all $H \in \mathcal{F} \setminus \mathcal{F}'$.

(ii) $H \in \mathcal{F}' \to [H \cup \{a\}] \in \mathcal{F}'$, and

(iii) the intersection S of all members of $\mathcal{F} \setminus \mathcal{F}'$ is in $\mathcal{F} \setminus \mathcal{F}'$.

Example 3.3.2. Let G be a finite group. The 2-central component $G_2(C)$ can be written as $\mathbb{Z}_2^r = [t_1, \ldots, t_r]$ where $t_1, \ldots, t_r$ are the generators of $\mathbb{Z}_2^r$ with $t_i^2 = 1$ and $t_i t_j = t_j t_i$. Let $\mathcal{F}_k$ be the family of all subgroups of G not containing $\mathbb{Z}_2^k$, $0 \le k \le r$, where $\mathbb{Z}_2^k$ denotes the subgroup of G generated by the first k generators $t_1, \ldots, t_k$. Then $(\mathcal{F}_{k+1},\mathcal{F}_k)$ is an admissible pair with respect to t_{k+1}, $0 \le k \le r$.

Theorem 3.3.3. If $(\mathcal{F},\mathcal{F}')$ is an admissible pair of families in G with respect to $a \in G_2(C)$, then an element $[M^n,\theta]$ in $\mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')$ is zero in $\mathfrak{N}_*(G;\mathcal{F}_a,\mathcal{F}'_a)$, where $\mathcal{F}_a$ is the smallest family in G consisting of all subgroups $[H \cup \{a\}]$; $H \in \mathcal{F}$.

Proof. Let S be the intersection of all members of $\mathcal{F} \setminus \mathcal{F}'$. F denote the fixed point set of M^n under S with $\partial F = \partial M^n \cap F$. But $x \in F \to G_x \supset S$; and $x \in \partial M^n \to G_x \in \mathcal{F}' \to G_x \not\supset S$.

Hence $\partial F = \phi$ and F is a closed submanifold embedded in the interior of M^n. Since $\mathcal{F} \setminus \mathcal{F}'$ is invariant under conjugation, S is normal in G and hence the action θ on M induces an action on F which we denote again by θ.

Claim: a acts freely on F.

Suppose a fixes a point x of F, then $G_x \supset [S \cup \{a\}]$. Therefore $[S \cup \{a\}] \in \mathcal{F}$. Further $S \in \mathcal{F} \setminus \mathcal{F}$ implies that $[S \cup \{a\}] \in \mathcal{F} \setminus \mathcal{F}'$. Hence $a \in [S \cup \{a\}]$ which is absurd. This shows that a acts freely on F.

Let ν be the normal bundle of the embedding of F in the interior of M^n and $D(\nu)$ be the disc bundle with the action θ^* of G on $D(\nu)$ induced by the real vector bundle maps covering the action θ on F. Let $F' = F/[a]$ and $D'(\nu) = D(\nu)/[a]$. The actions θ and θ^* on F and $D(\nu)$ induce actions θ' and $\theta^{*\prime}$ on F' and $D'(\nu)$ respectively. Let C_1 and C_2 be the mapping cylinders of the equivariant double covers $q_1 : F \longrightarrow F'$ and $q_2 : D(\nu) \longrightarrow D'(\nu)$ respectively and ψ_1 and ψ_2 be the induced actions on C_1 and C_2 respectively. We have the following diagram:

$$\begin{tikzcd}
C_2 \arrow[r] \arrow[d, "\alpha"] & D'(\nu) \arrow[d, "\nu'"] \\
C_2 \arrow[r] & F'
\end{tikzcd}$$

where $\alpha : C_2 \longrightarrow C_1$ is the map induced from $\nu' : D'(\nu) \longrightarrow F'$ by going to the mapping cylinders. Then ∂C_1 is homeomorphic to F, $\alpha^{-1}(\partial C_1)$ is homeomorphic to $D(\nu)$ and the action ψ_1 on $\alpha^{-1}(\partial C_1)$ is isomorphic to the action θ^* on $D(\nu)$. We have a G-diffeomorphism $\phi : \alpha^{-1}(\partial C_1) \xrightarrow{\approx} D(\nu)$ and we define $W = ((M^n \times [0,1]) \amalg C_2) / ((y,1) \sim \phi(y))$. The action ψ of G on W is given by $\psi|M^n \times [0,1] = \theta \times 1$ and $\psi|C_2 = \psi_1$. Let V be the set $(\partial M^n \times [0,1]) \cup ((M^n \setminus \operatorname{int}D(\nu)) \times \{1\}) \cup (\partial C_2 \setminus \operatorname{int}\alpha^{-1}(\partial C_1))$. Clearly W is $\mathcal{F}_a$-free and ∂W is the union of V and M^n by identifying ∂V with ∂M^n. Now V is $\mathcal{F}'_a$-free as S is the intersection of all members of $\mathcal{F} \setminus \mathcal{F}'$. Therefore ∂W is $\mathcal{F}'_a$-free. Hence $[M^n,\theta]$ is zero in $\mathfrak{N}_*(G;\mathcal{F}_a,\mathcal{F}'_a)$.

Corollary 3.3.4. For every k, $0 \le k \le r$, the homomorphism $\mathfrak{N}_*(G;\mathcal{F}_{k+1},\mathcal{F}_k) \longrightarrow \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_k)$ induced from the inclusion map $(\mathcal{F}_{k+1},\mathcal{F}_k) \longrightarrow (\mathcal{A},\mathcal{F}_k)$ is zero, where $\mathcal{A}$ denote the family of all subgroups of G.

Proof. $(\mathcal{F}_{k+1},\mathcal{F}_k)$ is an admissible pair of families with respect to t_{k+1} for $0 \le k \le r$. We have $(\mathcal{F}_k)_{t_{k+1}} = \mathcal{F}_k$ and bordism exact sequence

$$\mathfrak{N}_*(G;\mathcal{F}_{k+1},\mathcal{F}_k) \longrightarrow \mathfrak{N}_*(G;(\mathcal{F}_{k+1})_{t_{k+1}},(\mathcal{F}_k)_{t_{k+1}}) \xrightarrow{i_*} \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_k)$$

where $i_* : \mathfrak{N}_*(G;(\mathcal{F}_{k+1})_{t_{k+1}},(\mathcal{F}_k)_{t_{k+1}}) \longrightarrow \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_k)$ is is induced by the inclusion $((\mathcal{F}_{k+1})_{t_{k+1}},(\mathcal{F}_k)_{t_{k+1}}) \longrightarrow (\mathcal{A},\mathcal{F}_k)$.

By theorem (3.3.3) we conclude that the homomorphism

$$\mathfrak{N}_*(G;\mathcal{F}_{k+1},\mathcal{F}_k) \longrightarrow \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_k)$$

is zero.

Corollary 3.3.5. Let the 2-central component $G_2(C)$ of a group G be denoted by $\mathbb{Z}_2^r$. Let P be the family of all subgroups of G which do not contain $G_2(C)$. Then the homomorphism $\mathfrak{N}_*(G;P) \longrightarrow \mathfrak{N}_*(G;\mathcal{A})$ induced from the inclusion map $\mathcal{F}_r \longrightarrow \mathcal{A}$ is the zero homomorphism.

Proof. By corollary (3.3.4) we have

$$\mathfrak{N}_*(G;\mathcal{F}_{k+1},\mathcal{F}_k) \xrightarrow{i_*} \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_k)$$

is the zero homomorphism, $0 \leq k \quad r$. We have the exact sequence for the triple $\mathcal{F}_k \subseteq \mathcal{F}_{k+1} \subseteq \mathcal{A}$,

$$\cdots \longrightarrow \mathfrak{N}_*(G;\mathcal{F}_{k+1},\mathcal{F}_k) \xrightarrow{i_*} \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_k) \xrightarrow{j_*} \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_{k+1}) \longrightarrow \cdots$$

where j_* is the homomorphism induced from the inclusion j : $(\mathcal{A},\mathcal{F}_k) \longrightarrow (\mathcal{A},\mathcal{F}_{k+1})$. Since i_* is the zero homomorphism, j_* is a monomorphism. Therefore the composite

$$\mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_0) \longrightarrow \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_1) \longrightarrow \cdots\cdots\cdots \longrightarrow \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_r)$$

is a monomorphism and hence from the exact sequence of the triple $\mathcal{F}_0 \subseteq P \subseteq \mathcal{A}$, we have $\mathfrak{N}_*(G;P,\mathcal{F}_0) \longrightarrow \mathfrak{N}_*(G;\mathcal{A},\mathcal{F}_0)$ is the zero homomorphism.

Corollary 3.3.6. If $G_2(C)$ acts on M^n under θ without any stationary point then (M^n,θ) is a G-boundary.

Proof. The corollary follows from the corollary (3.3.5).

§4. Compact Lie group action and equivariant bordism

Khare has extended the theorem (3.3.3) for a torus action. He finally considers the compact Lie group action and proves that the induced action of the central elementary H-subgroup of G determine G-bordism. This gives us the result of torus action in particular case.

Definition 3.4.1. Let H be a compact Lie group. Then H is said to be H-boundary if there exists an H-manifold N such that $\partial N = H$ and the restriction of action θ on ∂N coincides with the given group operation of H.

Let G be a compact Lie Group. By the central elementary H-group in G, we mean the maximal subgroup $H^n = H \times \cdots\cdots \times H$, n-times, contained in the centre of G.

Let G be a compact Lie group with H^n the central elementary H-group in G, H being H-boundary. Let h_0 be a fixed point of H and $p_r : H^n \longrightarrow H$ denote the projection onto the r-th factor, $1 \leq r \leq n$. Let H_r denote the subgroup of H^n with $p_i(H_r) = H$, if $i = r$ and $p_i(H_r) = h_o$, if $i \neq r$. Let a family $\{L_r\}$ of subgroups of G be such that $L_i \cap H_r$ is a nontrivial subgroup of H_r, $1 \leq r$ n. By an <u>$\{L_i\}$-type action</u> of G, we mean a differential action of G on a differential manifold M such that for every x in M, $p_r(G_x \cap H^n)$ is either trivial or contains L_r for all r, G_x being the isotropy group of x. A point x in M is said to be a <u>pseudo stationary point</u> if $p_r(G_x \cap H^n)$ is nontrivial for all r, $1 \leq r \leq n$.

Let $\mathcal{F}' \subseteq \mathcal{F}$ be families in G such that there exists an H-boundary subgroup H of G satisfying the following conditions:

(a) No nontrivial subgroup of H is contained in K, for all $K \in \mathcal{F} \setminus \mathcal{F}'$.

(b) The intersection I of all the members of $\mathcal{F} \setminus \mathcal{F}'$ is in $\mathcal{F} \setminus \mathcal{F}'$

(c) H is contained in the centre.

We call such a pair $(\mathcal{F}, \mathcal{F}')$ of families an <u>admissible pair</u> in G with respect to $H \leq G$.

<u>Example 3.4.2.</u> Let $\{L_r\}$ be a family of subgroups of G such that $L_r \cap H_r$ be a nontrivial subgroup of H_r. Let $\mathcal{L}_r$ denote the family of all subgroups K of G for which $p_j(K \cap H^n)$ is trivial atleast for one $j = 1, \ldots\ldots, r$, and the nontrivial subgroups $p_j(K \cap H^n)$ of H_j contain L_j. Then

$({}^{L}_{i+1}, {}^{L}_{r})$ is an admissible pair of families in G with respect to H_{i+1}, $0 \leq r \leq n$, ${}^{L}_{0}$ being the empty family.

Theorem 3.4.3. If $(\mathcal{F},\mathcal{F}\ )$ is an admissible pair of families in G with respect to a subgroup H, which is H-boundary. then the homomorphism $\mathfrak{N}_*(G;\mathcal{F},\mathcal{F}') \longrightarrow \mathfrak{N}_*(G;\mathcal{F}_H,\mathcal{F}'_H)$ induced by the inclusion map $(\mathcal{F},\mathcal{F}') \longrightarrow (\mathcal{F}_H,\mathcal{F}'_H)$ is the zero homomorphism, where $\mathcal{F}_H$ denotes the smallest family in G containing all the subgroups $[S \cup Q]$, S $\in$ and Q a subgroup of H.

Proof. Let $[M^n,\theta] \in \mathfrak{N}_*(G;\mathcal{F},\mathcal{F}')$. Let I be the intersection of all members of $\mathcal{F} \setminus \mathcal{F}'$ and F be the fixed points set of M^n under I. Since $\mathcal{F} \setminus \mathcal{F}'$ is invariant under conjugation, I is normal in G, so that the action θ induces an action on F, which we denote by θ again. Let ν be the normal bundle of the embedding of F in the interior of M^n and $D(\nu)$ be the disc bundle with the action θ^* of G on $D(\nu)$ induced by the real vector bundle maps covering the action θ on F. Since F is the fixed points set of I and no non-trivial subgroup of H is contained in K. for all K $\in \mathcal{F} \setminus \mathcal{F}'$, no point of F is fixed by the subgroup $[I \cup Q]$, Q being a nontrivial subgroup of H. Thus H acts freely on F and hence on $D(\nu)$. Let $F' = F/H$ and $D'(\nu) = D(\nu)/H$. The actions θ and θ^* on F and on $D(\nu)$ induce actions θ' and $\theta^{*\prime}$ on F' and $D'(\nu)$ respectively. Since H acts freely on F and $D(\nu)$, the quotient maps $\xi_1 : F \longrightarrow F'$ and $\xi_2 : D(\nu) \longrightarrow D'(\nu)$ are principal H-bundles. Since H is H-boundary, $\exists$ an H-differential closed manifold (N,ϕ) such that the boundary $\partial N = H$ and the restriction of the action ϕ to $\partial N = H$

coincides with the operation in H. We consider the fibre bundles $\tilde{\xi}_1 = \xi_1[N]$ and $\tilde{\xi}_2 = \xi_2[N]$ associated to the principal H-bundles ξ_1 and ξ_2 respectively. The total space E_1 of $\tilde{\xi}_1$ is given by $E_1 = (F \times N)/H$, where the action of H on $F \times N$ is given by $h(x,y) = (xh, h^{-1}y)$, $h \in H$ and $(x,y) \in F \times N$. The boundary ∂E_1 is diffeomorphic to $(F \times H)/H$. We fix a point $\tilde{h}$ of H and define a map $\eta : (F \times H)/H \longrightarrow F$ as $\eta([x,h]) = x\bar{h}$ where $h = \bar{h}\tilde{h}$. Then η is a diffeomorphism. We define an action ψ_1 of G on E_1 as $g[x,t] = [xg,t]$. Then the diffeomorphism η preserves the H-action. Thus E_1 is a G-manifold with ∂E_1 being equivariently diffeomorphic to F. Similarly the total space E_2 of $\tilde{\xi}_2$ is $(D(\nu) \times N)/H$, where the action of H on $D(\nu) \times N$ is given by $h(x,y) = (xh, h^{-1}t)$. Let the action ψ_2 of G on E_2 be given by $g[x,t] = [xg,t]$. Let $\alpha : E_2 \longrightarrow E_1$ be the map induced from $\nu' : D'(\nu) \longrightarrow F'$ by going to the associated fibre bundles. Then we have the commutative diagram:

$$\begin{tikzcd}
\xi_2[N] : E_2 \arrow[r] \arrow[d, "\alpha"'] & D'(\nu) \arrow[d, "\nu'"] \\
\xi_1[N] : E_1 \arrow[r] & F'
\end{tikzcd}$$

Also $\alpha^{-1}(\partial E_1)$ is diffeomorphic to $D(\nu)$ and the action ψ_2 on $\alpha^{-1}(\partial E_1)$ is isomorphic to the action θ^* on $D(\nu)$. We define $W = ((M^n \times [0,1]) \amalg E_2)/(y,1) \sim \phi(y)$. Let the action Φ of G on W be defined by $\Phi|M^n \times [0,1] = \theta \times 1$ and $\Phi|E_2 = \psi_2$. Let V be the set $(\partial M^n \times [0,1]) \cup ((M^n \times \{1\} \setminus \operatorname{int}(D(\nu)) \times \{1\})) \cup (\partial E_2 \setminus \operatorname{int}(\alpha^{-1}(\partial E_1)))$. Since I is the intersection of all members of $\mathcal{I} \setminus \mathcal{I}'$, V is $(\mathcal{I}'_H, \mathcal{I}'_H)$-free. Also W is $(\mathcal{I}_H, \mathcal{I}'_H)$-free with ∂W the union of V and M^n, identifying ∂V

with ∂M^n. Thus $[M^n, \theta]$ is zero in $\mathfrak{N}_*(G; \mathcal{F}_H, \mathcal{F}'_H)$. ■

Corollary 3.4.4. For every r, $0 \leq r \leq n$, the homomorphism $\mathfrak{N}_*(G; \mathcal{F}^L_{r+1}, \mathcal{F}^L_r) \longrightarrow \mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}^L_r)$ induced from the inclusion map $(\mathcal{F}^L_{r+1}, \mathcal{F}^L_r) \longrightarrow (\mathcal{A}, \mathcal{F}^L_r)$ is zero.

Proof. We note that $(\mathcal{F}^L_{r+1}, \mathcal{F}^L_r)$ is an admissible pair of families with respect to the subgroup H_{r+1}, $0 \leq r < n$, and also $(\mathcal{F}^L_r)_{H_{r+1}} = \mathcal{F}^L_r$. We have the composite map

$$\mathfrak{N}_*(G; \mathcal{F}^L_{r+1}, \mathcal{F}^L_r) \longrightarrow \mathfrak{N}_*(G; (\mathcal{F}^L_{r+1})_{H_{r+1}}, (\mathcal{F}^L_r)_{H_{r+1}}) \longrightarrow \mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}^L_r)$$

induced by the inclusion $(\mathcal{F}^L_{r+1}, \mathcal{F}^L_r) \longrightarrow ((\mathcal{F}^L_{r+1})_{H_{r+1}}, (\mathcal{F}^L_r)_{H_{r+1}}) \longrightarrow (\mathcal{A}, {}^L_r)$. By theorem (3.4.3) we obtain that the homomorphism $\mathfrak{N}_*(G; \mathcal{F}^L_{r+1}, \mathcal{F}^L_r) \longrightarrow \mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}^L_r)$ is zero.

Corollary 3.4.5. Let M^n be a closed G-manifold with $\{L_r\}$-type of action for some family $\{L_r\}$ of subgroups of G such that $L_r \cap H_r$ is nontrivial. If M^n does not have any pseudo stationary point, then M^n is a G-boundary.

Proof. It is clear from the hypothesis that, for all $x \in M^n$, $p_r(G_x \cap H^n)$ is either trivial or contains L_r for all r. But M^n does not have any pseudo stationary point, so $p_r(G_x \cap H^n)$ is trivial atleast for one r, $1 \leq r \leq n$. Thus $p_j(G_x \cap H^n)$ is trivial atleast for one $j = 1, \ldots, n$ and the nontrivial subgroup $p_j(G_x \cap H^n)$ of H_j contains L_j. Hence $G_x \in \mathcal{F}^L_n$. Thus $[M^n, \theta] \in \mathfrak{N}_*(G; \mathcal{F}^L_n)$. Now by corollary (3.4.4) and exact bordism sequence for the triple $(\mathcal{A}, \mathcal{F}^L_{r+1}, \mathcal{F}^L_r)$ we obtain that $j_* : \mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}^L_r) \longrightarrow \mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}^L_{r+1})$ is a monomorphism. Therefore the composite

$$\mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}^L_0) \longrightarrow \mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}^L_1) \longrightarrow \cdots\cdot \longrightarrow \mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}^L_n)$$

is a monomorphism and hence by the bordism exact sequence of the triple $(\mathcal{A}, \mathcal{F}_n^L, \mathcal{F}_U^L)$ one gets that the homomorphism $\mathfrak{N}_*(G; \mathcal{F}_n^L, \mathcal{F}_U^L) \longrightarrow \mathfrak{N}_*(G; \mathcal{A}, \mathcal{F}_n^L)$ is the zero homomorphism. Hence $[M^n, \theta]$ is zero in $\mathfrak{N}_*(G; \mathcal{A})$ since $\mathcal{F}_U^L = \phi$.

§5. The oriented bordism of $\mathbb{Z}_2^k$-actions.

In this section we discuss the work of E.R.Wheeler on the oriented bordism of $\mathbb{Z}_2^k$-actions. We shall prove in this section that $\Omega_*(\mathbb{Z}_2^k) \otimes R_2$ is a free $\Omega_* \otimes R_2$ module on even dimensional generators, where $R_2 = \{ n/2^r \mid n, r \in \mathbb{Z} \}$ is a subring of . .

(A) Classifying spaces for bundles with G-actions.

Let G be a finite abelian group and $\mathbb{C}_1, \ldots, \mathbb{C}_r$ be all possible irreducible complex representation of G. We define G action on $\mathbb{C}^\alpha = \mathbb{C}_1^\alpha \oplus \ldots \oplus \mathbb{C}_r^\alpha$ by considering $\mathbb{C}_i^\alpha$ as the countable direct sum of the i-th irreducible reoresentation. Let $B(U,G)_s$ be the Grassmannian of comolex s-planes in $\mathbb{C}^\alpha$ and $E(U,G)_s = \{(W,x) \mid W \in B(U,G)_s$ and $x \in W\}$. Then $\gamma_s : E(U,G)_s \longrightarrow B(U,G)_s$ defined as $\gamma_s(W,x) = W$ is the complex s-plane bundle over $B(U,G)_s$. Since the elements of G acts on $\mathbb{C}^\alpha$ via complex linear transformations, there is induced G-actions on $B(U,G)_s$ and $E(U,G)_s$. This makes $\gamma_s : E(U,G)_s \longrightarrow B(U,G)_s$ the universal complex s-plane equivariat bundle for the family of all complex n-plane G-bundles in the category of G-spaces.

Similarly, we have the universal real s-plane equivarient bundle in the category of G-spaces, $\gamma_s : E(O,G)_s \longrightarrow B(O,G)_s$ over $B(O,G)_s$ where $B(O,G)_s$ is the Grassmannian of real s-planes in $\mathbb{R}^\alpha = \mathbb{R}_1^\alpha \oplus \ldots \oplus \mathbb{R}_t^\alpha$. ${}_1, \ldots, {}_t$ are the irreducible representations of G and $E(O,G)_s = \{(W,x) \mid W \in$

$B(O,G)_s$ and $x \in W\}$.

Now we can define a metric on $E(U,G)_s$ such that G action is orthogonal with respect to this metric. Further, for any complex G-bundle $E \longrightarrow X$ of dimension s over a compact Hausdorff space X, we can define a metric on E such that the G-action on E is orthogonl with respect to this metric and the bundle map covering the classifying map induces $(D(E),S(E)) \longrightarrow (D(E(U,G)_s),S(E(U,G)_s))$ where $D(E)$ and $D(E(U,G)_s)$ denote the unit disc bundles and $S(E(U,G)_s)$ and $S(E)$ denote the unit sphere bundles.

We now consider the G-space $B(U,G)_s$ and the fixed points sets of subgroups of G. Let H be a subgroup of H and X be a compact Hausdorff G-space. The isomorphism classes of G-bundles over X of complex dimension s are in one-one correspondence with the G-homotopy classes of equivariant maps from X into $B(U,G)_s$. Now if H fixes X, any equivariant map takes X into the fixed points set of H acting on $B(U,G)_s$, say $F_H(B(U,G)_s)$. Hence if H fixes X, then the G-homtopy classes of equivariant maps from to X to $F_H(B(U,G)_s)$ are in one-one correspondence with the isomorphism classes of G-bundles over X of complex dimension s. Thus $F_H(B(U,G)_s)$ is the classifying space for G-bundles of complex dimension s over base spaces X such that H fixes X.

Exactly the same analysis is true for equivariant real s-bundles over X and $F_H(B(O,G)_s)$.

<u>Proposition 3.5.1.</u> If H is a subgroup of G, then $F_H(B(U,G)_s)$ is G-homotopy equivalent to $\cup B(U,G)_{t_1} \times \dots \times$

$B(U,G)_{t_i}$, where H has r-irreducible complex representations with $\Sigma t_i = s$.

Proof. Since H has r-irreducible complex representations, a complex G-bundle E over a G-space which is fixed by H decomposes into $E_1 \oplus \dots \oplus E_r$. Therefore $F_H(B(U,G)_s)$ is G-homotopy equivalent to $\cup B(U,G)_{t_1} \times \dots \times B(U,G)_{t_r}$, such that $t_1 + \dots + t_i = s$.

Proposition 3.5.2. $F_{\mathbb{Z}_2}(B(O,\mathbb{Z}_2k)_s)$ is $\mathbb{Z}_2k$ homotopy equivalent to $\cup B(O,\mathbb{Z}_2k)_{t_1} \times B(O,\mathbb{Z}_2k)_{t_2}$, $t_1 + t_2 = s$.

Proof. Since the real irreducible representations of $\mathbb{Z}_2$ are multiplication by +1 and −1 on 1-dimensional vector space, a $\mathbb{Z}_2k$ bundle over a $\mathbb{Z}_2k$ space which is fixed by $\mathbb{Z}_2$, decomposes into $E_{t_1} \oplus E_{t_2}$ where $\mathbb{Z}_2$ acts on the fibres of E_{t_1} and E_{t_2} by multiplication by +1 and −1 respectively. Thus the classifying space for s-dimensional real vector bundles over $\mathbb{Z}_2k$ spaces fixed by $\mathbb{Z}_2$ is $\cup B(O,\mathbb{Z}_2k)_{t_1} \times B(O,\mathbb{Z}_2k)_{t_2}$, $t_1 + t_2 = s$.

Proposition 3.5.3. $F_{\mathbb{Z}_2'}(F^-_{\mathbb{Z}_2}(B(O,\mathbb{Z}_2k)_s))$ is $\mathbb{Z}_2^{k-1}$ homotopy equivalent to $\cup B(U,\mathbb{Z}_2k)_{t_1} \times \dots \times B(U,\mathbb{Z}_2k)_{t_r}$ with $\Sigma t_r = s$.

Proof. Let $F^-_{\mathbb{Z}_2}(B(O,\mathbb{Z}_2k)_s)$ denote the component of $F_{\mathbb{Z}_2}(B(O,\mathbb{Z}_2k)_s)$ above which $\mathbb{Z}_2$ acts as multiplication by −1 in the fibres of the canonical bundle. Now the $\mathbb{Z}_2k$ action restricted to $F_{\mathbb{Z}_2}(B(O,\mathbb{Z}_2k)_s)$ can be considered a $\mathbb{Z}_2^{k-1}$ action. Now $F_{\mathbb{Z}_2'}(F^-_{\mathbb{Z}_2}(B(O,\mathbb{Z}_2k)_s))$ is the classifying space for $\mathbb{Z}_2^k$ bundles $E \longrightarrow X$ which have the properties (a) $\mathbb{Z}_2^{j+1}$ $\mathbb{Z}_2^k$ fixes X and (b) $\mathbb{Z}_2$ $\mathbb{Z}_2^{j+1}$ $\mathbb{Z}_2^k$ acts on the fibres of E as multiplication by -1. Such a bundle E splits into subbundles with respect to the irreducible representations of $\mathbb{Z}_2^{j+1}$. Each irreducible represntations of $\mathbb{Z}_2^{j+1}$, which satisfies the condition (b), is the realification of an irreducible complex representation. Therefore, each of the split subbundles of E has a complex structure. Thus if there are r irreducible representations of $\mathbb{Z}_2^{j+1}$ then $F_{\mathbb{Z}_2^j}(F_{\bar{\mathbb{Z}}_2}(B(0,\mathbb{Z}_2^k)_s))$ is $\mathbb{Z}_2^{k-1}$ homotopy equivalent to $\cup$ $B(U,\mathbb{Z}_2^k)_{t_1} \times \ldots \times B(U,\mathbb{Z}_2^k)_{t_r}$ with $\Sigma\, t_i = s$.

<u>Definition 3.5.4</u>. Let $\xi : E \longrightarrow X$ be a bundle with fibre F. Then the <u>determinant bundle</u> of ξ is a line bundle $\det\xi : \det E \longrightarrow X$ whose fibres are $\Lambda^n F$ where the n-th exterior power $\Lambda^n F = (F \otimes \ldots \otimes F)/S$, where S is the subspace generated by the elements of the type $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ with $x_i = x_j$, $i \neq j$.

<u>Note 3.5.5</u>. Let $\gamma_{2s} : E(0,\mathbb{Z}_2^k)_{2s} \longrightarrow F_{\bar{\mathbb{Z}}_2}(B(0,\mathbb{Z}_2^k)_{2s})$ represent the canonical 2s plane bundle over $F_{\bar{\mathbb{Z}}_2}(B(0,\mathbb{Z}_2^k)_{2s})$. Now $B(0,\mathbb{Z}_2^{k-1})_{2s}$ is $\mathbb{Z}_2^{k-1}$ homotopy equivalent to $F_{\bar{\mathbb{Z}}_2}(B(0,\mathbb{Z}_2^k)_{2s})$. Since $\mathbb{Z}_2$ $\mathbb{Z}_2^k$ acts as multiplication by -1 on the fibres of $E(0,\mathbb{Z}_2^k)_{2s}$ and since the determinant of -1 acting on an even dimensional vector space is $+1$, $\mathbb{Z}_2$ acts trivially on the determinant bundle of γ_{2s}, i.e., $\det\gamma_{2s} : \det E(0,\mathbb{Z}_2^k)_{2s} \longrightarrow F_{\bar{\mathbb{Z}}_2}(B(0,\mathbb{Z}_2^k)_{2s})$ is a $\mathbb{Z}_2^{k-1}$ bundle.

Proposition 3.5.6. If $f : (M, \partial M, \mathbb{Z}_{2^{k-1}}$ actoin$) \longrightarrow (D(\det E(0,\mathbb{Z}_{2^k})_{2s}), S(\det E(0,\mathbb{Z}_{2^k})_{2s}), \det(\mathbb{Z}_{2^k}$ action$))$ is an equivariant map, then f may be equivariantly homotoped to be transeverse regular on the zero section of $\det E(0,\mathbb{Z}_{2^k})_{2s}$. Further, if $A \subseteq M$ is a closed subspace and $f|A$ is already transverse regular, the homotopy can be chosen to fix A.

For proof see page 291 of [29].

Theorem 3.5.7. $\Omega_n(\mathbb{Z}_{2^k})(\mathcal{F},\mathcal{F}_e)(X,A) \approx$
$\bigoplus_{s=0}^{[n/2]} \tilde{\Omega}_{n-2s+1}(\mathbb{Z}_{2^{k-1}})(\mathcal{F})(F_{\overline{\mathbb{Z}}_2}(X)/F_{\overline{\mathbb{Z}}_2}(A) \wedge T(\det\gamma_{2s}))$ where $\gamma_{2s} : E(0,\mathbb{Z}_{2^k})_{2s} \longrightarrow F_{\overline{\mathbb{Z}}_2}(B(0,\mathbb{Z}_{2^k})_{2s})$ is the canonical 2s-plane bundle over $F_{\overline{\mathbb{Z}}_2}(B(0,\mathbb{Z}_{2^k})_{2s})$.

Proof. Let $[M^n,M_0,M_1,T,F] \in \Omega_n(\mathbb{Z}_{2^k})(\mathcal{F},\mathcal{F}_e)(X,A)$ where T generates the $\mathbb{Z}_{2^k}$ action on M^n. Let F_2 be the $(n-2s)$-dimensional component of the fixed set of $\mathbb{Z}_2 < \mathbb{Z}_{2^k}$ acting on M^n. Then F_2 is a submanifold of M^n with induced action of $\mathbb{Z}_{2^{k-1}}$ and $\partial F_2 = F_2 \cap M_1$. Let $\nu : E(\nu) \longrightarrow F_2$ be the normal bundle of F_2 in M^n. Since the disc $D(\nu)$ of the normal bundle can be identified equivariantly with a small tubular neighborhood of F_2, no elements of $D(\nu) \setminus \{$zero-section$\}$ can be fixed by $\mathbb{Z}_{2^j}$ for $1 \leq j \leq k$. Since each fibre of ν is a representation space for $\mathbb{Z}_2$, ν is $\mathbb{Z}_{2^k}$ bundle over F_2 such that $\mathbb{Z}_2$ acts as multiplication by -1 in the fibres. Then $\nu : E(\nu) \longrightarrow F_2$ is classified equivariantly into $F_{\overline{\mathbb{Z}}_2}(B(0,\mathbb{Z}_{2^k})_{2s})$ giving a $\mathbb{Z}_{2^k}$ bundle map

$$\begin{tikzcd}
E(\nu) \arrow[r, "g'"] \arrow[d, "\nu"] & E(0,\mathbb{Z}_2k)_{2s} \arrow[d, "\gamma_{2s}"] \\
F_2 \arrow[r, "g"] & B(0,\mathbb{Z}_2k-1)_{2s}
\end{tikzcd}$$

Now by taking the determinant bundles of ν and γ_{2s} we have a $\mathbb{Z}_2k-1$ bundle map

$$\begin{tikzcd}
\det(E(\nu)) \arrow[r, "\det g'"] \arrow[d, "\det\nu"] & \det(E(0,\mathbb{Z}_2k)_{2s}) \arrow[d, "\det\gamma_{2s}"] \\
F_2 \arrow[r, "g"] & B(0,\mathbb{Z}_2k-1)_{2s}
\end{tikzcd}$$

Then $\det g'$ maps the pair $(D(\det E(\nu)), S(\det E(\nu)))$ into the pair $(D(\det E(0,\mathbb{Z}_2k)_{2s}), S(\det E(0,\mathbb{Z}_2k)_{2s}))$. We have a map $(\det g' \times f)\circ\det\nu : (D(\det E(\nu)), D(\det E(\nu)/\partial F_2) \cup S(\det E(\nu))) \longrightarrow (F_{\mathbb{Z}_2}(X) \times D(\det E(0,\mathbb{Z}_2k)_{2s}), F_{\mathbb{Z}_2}(X) \times S(\det B(0,\mathbb{Z}_2k)_{2s}) \cup F_{\mathbb{Z}_2}(A) \times D(\det E(0,\mathbb{Z}_2k)_{2s}))$ which gives rise to map

$$F : \Omega_*(\mathbb{Z}_2k)(\mathcal{F},\mathcal{F}_e)(X,A) \longrightarrow \bigoplus_{s=0}^{(n/2)} \Omega_{n-2s+1}(\mathbb{Z}_2k-1)(\mathcal{F})(F_{\mathbb{Z}_2}(X,A)$$
$$\times\ (D(\det E(0,\mathbb{Z}_2k)_{2s}), S(\det E(0,\mathbb{Z}_2k)_{2s})).$$

Now we define an inverse map of F. Let $[N,\partial N,S,h] \in \Omega_{n-2s+1}(\mathbb{Z}_2k-1)(\mathcal{F})(F_{\mathbb{Z}_2}(X,A) \times (D(\det E(0,\mathbb{Z}_2k)_{2s}), S(\det E(0,\mathbb{Z}_2k)_{2s}))$. Let $p_2 : F_{\mathbb{Z}_2}(X) \times D(\det(0,\mathbb{Z}_2k)_{2s}) \longrightarrow D(\det E(0,\mathbb{Z}_2k)_{2s})$ be the projection onto second factor. Then we have $p_2\circ h : N \longrightarrow D(\det E(0,\mathbb{Z}_2k)_{2s})$ is an equivariant map. So by the proposition (3.3.5), $p_2\circ h$ may be considered (up to equivariant homotopy) to be transverse regular on the zero-section $B(0,\mathbb{Z}_2k-1)_{2s}$ of $\det E(0,\mathbb{Z}_2k)_{2s}$. Let $N' = (P_2\circ h)^{-1}(B(0,\mathbb{Z}_2k-1)_{2s})$. Since γ_{2s} has a $\mathbb{Z}_2k$ action covering the $\mathbb{Z}_2k-1$ action on $B(0,\mathbb{Z}_2k-1)_{2s}$, $(p_2\circ h)^*(\gamma_{2s}) \xrightarrow{\pi'} N'$ is a bundle with an induced $\mathbb{Z}_2k$ action such that $\mathbb{Z}_2 < \mathbb{Z}_2k$ acts as multiplication by -1 in the fibres. Let S' generates the $\mathbb{Z}_2k$ action on $(p_2 \circ h)^*(\gamma_{2s})$. ALso $D((p_2 \circ h)^*(\gamma_{2s}))$ is oriented and S' is orientation preserving. Thus we can define a map

$K : \Omega_{n-2s+1}(\mathbb{Z}_2k-1)(\mathcal{F})(F_{\mathbb{Z}_2}(X,A) \times (D(\det E(0,\mathbb{Z}_2k)_{2s}),$ $S(\det E(0,\mathbb{Z}_2k)_{2s})) \longrightarrow \Omega_n(\mathbb{Z}_2k)(\mathcal{F},\mathcal{F}_e)(X,A)$ given by

$K[N,\partial N,S,h] = [D((p_2 \circ h)^*(\gamma_{2s})), S((p_2 \circ h)^*(\gamma_{2s})),$ $D((p_2 \circ h)^*(\gamma_{2s})/\partial N''), S', p_2 \circ h \circ \pi']$. Then one can verify that $F \circ K = id$ and $K \circ F = id$. Hence $\Omega_n(\mathbb{Z}_2k)(\mathcal{F},\mathcal{F}_e)(X,A)$

$$\approx \bigoplus_{s=0}^{[n/2]} \tilde{\Omega}_{n-2s+1}(\mathbb{Z}_2k-1)(\mathcal{F})(F_{\mathbb{Z}_2}(X)/F_{\mathbb{Z}_2}(A) \wedge T(\det\gamma_{2s})). \blacksquare$$

Let (X,A) be a CW pair with action of $\mathbb{Z}_2k$ and $q : (X,A) \longrightarrow (X/G,A/G)$ be the quotient map. Using corollary 2.3 of [E.E.floyd : Periodic maps via Smith Theory; Seminar on Transformation Groups, Annals of Mathematics studies, No.46, Princeton university Press, Princeton, N.J., 1960, chapter III] one obtains that if $H_*(X,A;R_2)$ is a free R_2-module on even (odd) dimensional generators, then $H_*(X/G,A/G;R_2)$ is a free R_2-module on even (odd) dimensional generators. A pair (X,A) is defined to be <u>2-even (2-odd)</u> if and only if $H_*(X,A;R_2)$ is a free R_2-module on even (odd) dimensional generators. Using this and the homology exact sequence of the cofibration $S(\det\gamma_{2s}) \longrightarrow D(\det\gamma_{2s}) \longrightarrow T(\det\gamma_{2s})$, Wheeler has shown that if $F_{\mathbb{Z}_2j}(X,A)$ is 2-even (2-odd) for $0 \leq j \leq k$ then $Y = (F_{\mathbb{Z}_2}(X)/F_{\mathbb{Z}_2}(A)) \wedge T(\det\gamma_{2s})$ is a $\mathbb{Z}_2k-1$ space with $F_{\mathbb{Z}_2j}(Y)$ is 2-odd (2-even) for $0 \leq j \leq k-1$.

Lemma 3.3.8. If (X,A) is $\mathbb{Z}_2k$ pair which is 2-even (2-odd), then $\Omega_*(\mathbb{Z}_2k)(\mathcal{F}_e)(X,A) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even (odd) dimensional generators.

Proof. We know that $\Omega_*(G)(\mathcal{F}_e)(X,A)$ is isomorphic to $\Omega_*(X \times_G EG,\ A \times_G EG)$ where EG is the total space of the universal principal G-bundle. Since (X,A) is 2-even (2-odd), one infers that $(X \times_G EG,\ A \times_G EG)$ is 2-even (2-odd). It is easy to see that $\Omega_*(X,A) \otimes R_2 \cong (\Omega_* \otimes R_2) \otimes_{R_2} H_*(X,A;R_2)$, since (X,A) is a CW pair with $H_*(X,A;R_2)$ a torsion free R_2-module. The lemma now follows immediately.

Wheeler finally establishes

Theorem 3.3.9. If $F_{\mathbb{Z}_2{}^j}(X,A)$ is 2-even (2-odd) for $0 \leq j \leq k$ then $\Omega_*(\mathbb{Z}_2k)(\mathcal{F})(X,A) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even (odd) dimensional generators.

Proof. The case $k = 0$ follows from the lemma (3.3.8). Suppose that the theorem is true for $k' < k$. Then by the hypothesis and the lemma (3.3.8) we obtain that $\Omega_*(\mathbb{Z}_2k)(\mathcal{F})(X,A) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even (odd) dimensional generators. Also by theorem (3.3.7) one gets

$$\Omega_n(\mathbb{Z}_2k)(\mathcal{F},\mathcal{F}_e)(X,A) \approx$$

$$\bigoplus_{s=0}^{[n/2]} \tilde{\Omega}_{n-2s+1}(\mathbb{Z}_2k-1)(\mathcal{F})(F_{\mathbb{Z}_2}(X)/F_{\mathbb{Z}_2}(A) \wedge T(\det\gamma_{2s})).$$

Therefore, from the discussion preceeding lemma (3.3.8) and induction hypothesis, we conclude that $\Omega_n(\mathbb{Z}_2k)(\mathcal{F},\mathcal{F}_e)(X,A) \otimes R_2$ is free $\Omega_* \otimes R_2$-module on even (odd) dimensional generators. The result now follows from the following split exact sequence

$$\cdots \longrightarrow \Omega_*(\mathbb{Z}_2k)(\mathcal{F}_e)(X,A) \otimes R_2 \longrightarrow \Omega_*(\mathbb{Z}_2k)(\mathcal{F})(X,A) \otimes R_2$$

$$\longrightarrow \Omega_*(\mathbb{Z}_2k)(\mathcal{F},\mathcal{F}_e)(X,A) \otimes R_2 \longrightarrow \cdots$$

Note that if $(X,A) = (\text{pt.}, \phi)$, the above theorem implies that $\Omega_*(\mathbb{Z}_2k) \otimes R_2$ is free $\Omega_* \otimes R_2$-module on even dimensional generators, since $(\text{pt.},\phi)$ is $\mathcal{E}$-even.

CHAPTER IV

Group action with fixed points set

In this chapter we shall discuss how the normal bundle to the fixed points set of an involution on a closed manifold explicitly determines the bordism class of manifold. We shall discuss Boardman's five-halves theorem which states that if M is a $\mathbb{Z}_2$-manifold **then** $\dim M > (5/2)\dim F$, **if** M is a boundary. We shall discuss the work of Khare about finite group action and fixed points sets. Also we shall develop the work of Kosniowski and Stong on involutions and characteristic numbers. We shall also discuss Stong's works on involutions with n-dimensional fixed points set and group actions having one fixed point.

§1. Fixed points set and Boardman five-halves theorem.

In this section our aim is to study the G-bordism with the help of dimension of the fixed points set. The simplest result proved in this direction is by Conner and Floyd. He proved that if the fixed points set in a $\mathbb{Z}_2$-manifold M^n has codimension 1, then M^n is a $\mathbb{Z}_2$-boundary. The bounding manifold is given by the mapping cylinder C of M^n, i.e., $\frac{M^n \times [0,1]}{(x,1) \sim (\theta(x),1)}$, where θ is the action of G on M^n. The action ϕ on C is given by $\theta \times 1$. With this aciton the boundary of (C,ϕ) is (M^n,θ).

For each positive integer n, and for each real vector bundle ξ consider a well-defined characteristic class $\sigma_n(\xi) \in H^n(X)$, where X is the base space of ξ, which satisfies the folowing axioms:

(a) $\sigma_n(f^*(\xi)) = f^*(\sigma_n(\xi))$, for any bundle map f;

(b) $\sigma_n(\xi \oplus \eta) = \sigma_n(\xi) + \sigma_n(\eta)$, for vector undles ξ and η over the same base space;

(c) $\sigma_n(\xi) = W_1(\xi)^n$, if ξ is a line bundle.

Suppose the symmetric functionn $t_1^n + \dots + t_k^n$ can be expressed as the function $f(s_1, s_2, \dots, s_n)$, where s_i is the i-th elementary function in $t_1, \dots, t_k$. Then $\sigma_n(s) = f(w_1, \dots, w_n)$, ξ being a vector bundle of dimension k.

If the base space X is a n-dimensional manifold with fundamental class $z_X \in H_n(X)$, we define $e_n(\xi) = \sigma_n(\xi), z_X\rangle \in \mathbb{Z}_2$. We also define $\sigma_n(X) = \sigma_n(\tau)$ and $e_n(X) = e_n(\tau)$, where τ is the tangent bundle of X. In terms of e_n, the result of Thom described under unoriented bordism in section 1 of chapter I can be reformulated in the following form.

<u>Theorem 4.1.1.</u> The ring $\mathfrak{N}_*$ is a graded polynomial algebra over $\mathbb{Z}_2$ with one generator in each degree not of the form $2^q - 1$. The class $[V] \in \mathfrak{N}_n$ serves as a polynomial generator in degree n if and only if $e_n(V) = 1$.

The following lemma is significant for calculation of Stiefel-Whitney numbers of the Milnor manifolds.

<u>Lemma 4.1.2.</u> Let $H_{m,n}$ be the surface in $\mathbb{R}P_m \times \mathbb{R}P_n$ defined by the equation $t_0u_0 + t_1u_1 + \dots + t_mu_m = 0$, where $(t_0, \dots, t_m)$ are the homogeneous co-ordinates in $\mathbb{R}P_m$ and $(u_0, \dots, u_n)$ are homogeneous coordinates in $\mathbb{R}P_n$, $m \leq n$. Then for any class $\phi \in H^*(\mathbb{R}P_m \times \mathbb{R}P_n)$ we have $\langle j^*\phi, z_H\rangle =$

$\phi.(\alpha + \beta), z_{p\times p}\rangle$, where z_H and $z_{p\times p}$ are the fundamental classes of the Milnor manifold $H_{m,n}$ and $\mathbb{RP}_m \times \mathbb{RP}_n$ respectively and j is the inclusion map of $H_{m,n}$ into $\mathbb{RP}_m \times \mathbb{RP}_n$ and α, β are the generators of $H^1(\mathbb{RP}_m \times \mathbb{RP}_n)$ inherited from $\mathbb{RP}_m$ and $\mathbb{RP}_n$ respectively.

Proof. The one dimensional cohomology class in $\mathbb{RP}_m \times \mathbb{RP}_n$ dual to the submanifold $H_{m,n}$ $\alpha + \beta$, i.e., $j_* z_H = z_{p\times p} \cap (\alpha + \beta)$. Then $\langle j^*\phi, z_H\rangle = \langle \phi, j_* z_H\rangle = \langle \phi, z_{p\times p} \cap (\alpha+\beta)\rangle = \langle \phi.(\alpha+\beta), z_{p\times p}\rangle$.

The module structure of $\mathfrak{N}_*(X)$ over $\mathfrak{N}_*$ is given by the following:

Theorem 4.1.3. For a CW space X, the $\mathfrak{N}_*$-module $\mathfrak{N}_*(X)$ is free, and $\mu : \mathfrak{N}_*(X) \longrightarrow H_*(X)$ defined by $\mu[V,f] = f_* z_V$, where z_V is the fundamental class of V induces an isomorphism $\mathfrak{N}_*(X) \cong H_*(X) \otimes \mathfrak{N}_*$. The elements $z_\alpha \in \mathfrak{N}_*(X)$ form a $\mathfrak{N}_*$ base of $\mathfrak{N}_*(X)$ if and only if $\mu(z_\alpha)$ form a $\mathbb{Z}_2$- base of $H_*(X)$.

Proof. Let $\{c_{n,i}\}$ be a homogeneous base for the vector space $H_*(X;\mathbb{Z}_2)$ over $\mathbb{Z}_2$. For each $c_{n,i}$ choose an element $[V^n_i, f_i] \in \mathfrak{N}_*(X)$ with $\mu[V^n_i, f_i] = c_{n,i}$. $\{[V^n_i, f_i]\}$ forms a homogeneous base of $\mathfrak{N}_*X)$ over $\mathfrak{N}_*$. ■

The following lemma describes polynomial algebra structure of $\mathfrak{N}_*(BO)$ over $\mathfrak{N}_*$.

Lemma 4.1.4. The ring $\mathfrak{N}_*(BO)$ is a polynomial algebra over N with one generator in each positive degree. The class $[\xi]$ of a vector bundle ξ over a n-dimensional manifold serves as a polynomial generator if and only if $e_n(\xi) = 1$.

Proof. The homology $H_*(BO)$ is a polynomial algebra on the generators $a_i \in H_i(BO(1)) \cong H_i(BO)$, for i 0. Also if ξ is a vector bundle with classifying map $f : X \longrightarrow BO(1) \longrightarrow BO$ and $u \in H_n(X)$, the class $f_*(u)$ serves as one of the polynomial generators of $H_*(BO)$ if and only if $\langle \sigma_n(\xi), u \rangle = 1$. Combining this with theorem (4.1.1.) we get the lemma. ▪

For convenience we write $b_i(\xi) = e_i(V)$ whenever ξ is a vector bundle over a manifold V. Thus $b_i(\xi) = e_i(p_*[V,f])$, where $f : V \longrightarrow BO$ is the map classifying the bundle ξ and p_* is the homomorphism induced from the constant map $p : BO \longrightarrow pt.$

Definition 4.1.5. A bordism class of a manifold is decomposable if and only if it can be expressed as a sum of products of lower dimensional bordism classes.

Lemma 4.1.6. Suppose given for each positive integer n, elements x'_n and x''_n of $\mathfrak{N}_n(BO)$, with x'_n absent if n has the form $2^q - 1$. Then these elements form a system polynomial generators of the ring $\mathfrak{N}_*(BO)$ over Z_2 if and only if for each n the pairs of numbers $(b_n(x'_n), e_n(x'_n))$, $(b_n(x''_n), e_n(x''_n))$ and $(0,0)$ are distinct.

Proof. Let $p_* : \mathfrak{N}_*(BO) \longrightarrow \mathfrak{N}_*$ be the homomorphism induced by the map $p : BO \longrightarrow pt.$, and $i_* : \mathfrak{N}_* \longrightarrow \mathfrak{N}_*(BO)$ be the inclusion of rings induced by the inclusion of a point into BO. For each n, we choose an element $y_n \in \mathfrak{N}_n(BO)$ such that $e_n(y_n) = 1$, so that $\mathfrak{N}_*(BO)$ is a polynomial algebra over $\mathfrak{N}_*$ on the generators y_n. We may assume that $b_n(y_n) = 0$, by replacing y_n by $y_n + i_* p_* y_n$, if necessary, since $b_n(y_n) = e_n(p_* y_n)$. We now choose polynomial generators z_n;

n not of the form $2^q - 1$. for $\mathfrak{N}_*$, included in $\mathfrak{N}_*(BO)$ by i_*; then $e_n(z_n) = 0$ and $b_n(z_n) = 1$. So $\mathfrak{N}_*(BO)$ is a polynomial algebra over $\mathbb{Z}_2$ with y_n and z_n as generators.

Now any element x in $\mathfrak{N}_n(BO)$ can be expressed as a polynomial in y_r and z_r. Since b_n and e_n vanish on lower dimensional bordism classes. We have $x = e_n(x)y_n + b_n(x)z_n$ + decomoosable elements. Hence the lemma follows.

Let M_n denote the set of bordism classes of all n-dimensional manifolds with an involution that is free on the boundary. Let $[V^n,T] \in M_n$. Then the set F of fixed points is the disjoint union of various submanifolds F^i of dimension i and each F^i admits a tubular neighborhood N_i that is isomorphic to the disk bundle $D(\nu_i)$ of the normal bundle ν_i to F^i in V^n, $D(\nu_i)$ having the fibrewise antipodal involution on each fibre. Assuming N_i to be disjiont, T acts freely on the complement of the union N of N_i. acts freely. So V^n is bordant to N in M_n. But N is determined by the normal bundles to the fixed points sets F^i. So M_n may be regarded as the bordism group of smooth vector bundles over manifolds, where n is the total dimension.

Definition 4.1.7. Let F_n denote the set of bordism classes of all n-dimensional manifolds with free involution. Then the homomorphism $\partial : M_n \longrightarrow F_{n-1}$ defined by $\partial[V,T] = [\partial V, T/\partial V]$ is called the bordism J-homomrphism.

Definition 4.1.8. We call the element $s = [D^1,T]$ of M_1 the suspension element where T is the antipodal involution on the unit disc D^1.

Let $f : V \longrightarrow W$ be a smooth map between smooth manifolds. Let $[Q,g] \in \mathfrak{N}_*(W)$. By transverse regularity theorem, $g : Q \longrightarrow W$ can be considered to be transverse to f (that is $f \times g : V \times Q \longrightarrow W \times W$ is transverse to the diagonal submanifold ΔW). Let P be the pull back of f and g, i.e., $P = \{(v,q) \in V \times Q \mid f(v) = g(v)\} \subset V \times Q$. Define $h : P \longrightarrow V$ as $h(v,q) = v$.

Definition 4.1.9. The $\mathfrak{N}_*$-module homomorphism $\tilde{f} : \mathfrak{N}_*(W) \longrightarrow \mathfrak{N}_*(V)$ defined as $\tilde{f}[Q,g] = [P,h]$ is called the transfer homomorphism induced by f.

Definition 4.1.10. For any integer n, A sufficiently large (at least n), we define $L_n = \text{inv lim}_k\ \mathfrak{N}_{n+k}(\mathbb{R}P_{n+k})$, where the inverse limit is formed using the homomorphisms $\tilde{h} : \mathfrak{N}_{n+(k+1)}(\mathbb{R}P_{A+(k+1)}) \longrightarrow \mathfrak{N}_{n+k}(\mathbb{R}P_{A+k})$, induced by the inclusion $h : \mathbb{R}P_{A+k} \longrightarrow \mathbb{R}P_{A+k+1}$ so that the graded group L is a $\mathfrak{N}_*$-module. We define the stable bordism J-homomorphism $J : M_{i+n} \longrightarrow L_{i+n}$ on the class $[\xi]$ from the sequence of elements $\partial s^{k+1}[\xi] \in \mathfrak{N}_{i+n+k}(\mathbb{R}P_{A+k})$, where $\partial s^{k+1}[\xi]$ is the class of the manifold $S(\xi \oplus \varepsilon^{k+1})$ with involution.

Theorem 4.1.11. Let I denote the graded group of the bordism classes of manifolds with involution. Let $x \in M$. Then $x \in I$ if and only if Jx is an element of the subalgebra K of power series in θ, $\theta = \theta_1 \in L_{-1}$ is the element represented by the inclusion $\mathbb{R}P_{-1+k} \hookrightarrow \mathbb{R}P_{A+k}$.

Proof. We have the split short exact sequence

$$0 \longrightarrow I_n \xrightarrow{\rho} M_n \xrightarrow{\partial} F_{n-1} \longrightarrow 0.$$

Then we have $I_n \cong \sum_{i \neq 1} \mathfrak{N}_{n-i}(BO(1))$ and also $I_n \cong \operatorname{im} \rho = \ker \partial$. Consider the map $P : L_n \longrightarrow \mathfrak{N}_{n+k}(\mathbb{R}P_{A+k})$. Let $\theta_n \in L_n$ be the element represented by the inclusion $\mathbb{R}P_{-n+k} \subseteq \mathbb{R}P_{A+k}$, defined for $n \geq -A$. Then we have $\theta_m \theta_n = \theta_{m+n}$. We have for any integer n, $\theta_n = \theta^n$. By theorem (4.1.3.), for $-A \leq n \leq k$, θ^n is mapped to a $\mathfrak{N}_*$-base of $\mathfrak{N}_*(\mathbb{R}P_{A+k})$ and for $n > k$ θ^n is mapped to 0. As we let k vary, we find that L is the graded algebra of Laurent series in θ with coefficient in $\mathfrak{N}_*$ and as θ has degree -1, there are only finitely many terms with negative powers of θ. Then it is easy to see that P is a surjective map with kernel $\theta^{k+1}K_{n+k+1}$.

Now taking $k = -1$, we find that kernel is K_n and the homomorphism ∂ is the composite $M_n \xrightarrow{J} L_n \xrightarrow{P} \mathfrak{N}_{n-1}(\mathbb{R}P_\alpha)$. Now, $x \in I \Leftrightarrow x \in \ker \partial \Leftrightarrow \partial x = 0 \Leftrightarrow P \circ Jx = 0 \Leftrightarrow Jx \in K$. Hence the theorem follows. ■

Theorem 4.1.12. Let V be a closed manifold with involution T. Then $J[V,T] = [V]$ + terms with positive powers of θ.

Proof. Let $[V,T] \in I$. To determine the constant term in the power series $J[V,T]$, consider $k = 0$. So that we have surjective map $L_n \longrightarrow \mathfrak{N}_n(\mathbb{R}P_A)$ with kernel θK_{n+1} and only the constant term survives . For the image in $\mathfrak{N}_n(\mathbb{R}P_A)$, we have $\partial s[V,T] = [\partial(V \times D^1), T \times T] = [V \times S^0, T \times T] = [V \times S^0, 1 \times T] = [V].1$, because the manifolds with involution $(V \times S^0, T \times T)$ and $(V \times S^0, 1 \times T)$ are isomorphic. Hence $J[V,T] = [V]$ + terms with positive powers of θ.

We adjoint to M a formal inverse s^{-1} of s, to get $M[s^{-1}]$. Then $M[s^{-1}] = \mathfrak{N}_*(BO) \otimes \mathbb{Z}_2[s, s^{-1}]$ and the stable J-homomorphism extends uniquely to an algebra homomorphism $J : M[s^{-1}] \longrightarrow L$.

Definition 4.1.13. Let $x \in M$. Then the filtration of an element x, denoted by a fil(x), is the highest dimension of the various components of a base manifold of a representative vector bundle.

The following lemma gives generators x_n of $M[s^{-1}]$ and z_n of $\mathfrak{N}_*$ on which the effect of J is known. The lemma also gives filtration of x_n.

Lemma 4.1.14. For each n not of the form $2^q - 1$, we can find elements $x_n \in \mathfrak{N}_*(Z_2) \subseteq M_n$ and $z_n \in \mathfrak{N}_n$ with the properties

(a) the x_n form a system of polynomial generators of $M[s^{-1}]$ over the subalgebra $Z_2[s,s^{-1}]$;

(b) the z_n form a system of polynomial generators of $\mathfrak{N}_*$;

(c) $Jx_n = z_n$ + terms involving θ;

(d) $\mathrm{fil}(x_n) = n/2$ if n is even, or $(n-1)/2$ if n is odd;

(e) the filtration of a polynomial in the x_n is the maximum of the filtrations of its terms.

Proof. We define an involution T_i on the projective space $\mathbb{R}P_{2i}$ by

$T_i[t_0, \dots, t_i, t'_1, \dots, t'_i] = [t_0, \dots, t_i, -t'_i, \dots, -t'_i]$

in terms of homogeneous co-ordinates $(t_0, \dots, t_i, t'_1, \dots, t'_i)$ on $\mathbb{R}P_{2i}$. Its fixed points sets are the linear subspaces $\mathbb{R}P_i$ defined by $t'_1 = t'_2 = \dots = t'_i = 0$, and $\mathbb{R}P_{i-1}$ defined by $t_0 = t_1 = \dots = t_i = 0$. For the normal bundle ν of $\mathbb{P}P_i$ we have $\sigma_i(\nu) = i\alpha^i$, where α generates the cohomology $H^1(\mathbb{R}P_i)$. For the normal bundle ν of $\mathbb{P}P_{i-1}$ we have $\sigma_i(\nu) = (i+1)\alpha^{i-1}$. The product involution $T_i \times T_j$ on $\mathbb{R}P_{2i} \times \mathbb{P}P_{2j}$ maps the Milnor manifold $H_{2i,2j}$ into itself, $i \leq j$. We consider the fixed points sets of $T_i \times T_j|H_{2i,2j}$, which are just the intersections with $H_{2i,2j}$ of the four fixed points sets of $T_i \times T_j$ on $\mathbb{RP}_{2i} \times \mathbb{RP}_{2i}$, namely $\mathbb{RP}_i \times \mathbb{RP}_j$, $\mathbb{RP}_i \times \mathbb{RP}_{j-1}$, $\mathbb{RP}_{i-1} \times \mathbb{RP}_j$ and $\mathbb{RP}_{i-1} \times \mathbb{RP}_{j-1}$. Now $\mathbb{RP}_i \times \mathbb{RP}_j$ intersects $H_{2i,2j}$ in a copy of $H_{i,j}$. The normal bundle ν_1 of $H_{i,j}$ in $H_{2i,2j}$ is the restriction to $H_{i,j}$ of the normal bundle of $\mathbb{RP}_i \times \mathbb{RP}_j$ in $\mathbb{RP}_{2i} \times \mathbb{RP}_{2j}$ and we have $\sigma_{i+j-1}(\nu_1) = i.\alpha^{i+j-1} + j.\beta^{i+j-1}$ where α and β generates the cohomologies of $\mathbb{RP}_{2i}$ and $\mathbb{RP}_{2j}$. The sets $\mathbb{RP}_i \times \mathbb{RP}_{j-1}$ and $\mathbb{RP}_{i-1} \times \mathbb{RP}_j$ are entirely contained in $H_{2i,2j}$. The normal bundle ν_2 of $\mathbb{RP}_i \times \mathbb{RP}_{j-1}$ in $H_{2i,2j}$ becomes the normal bundle of $\mathbb{RP}_i \times \mathbb{RP}_{j-1}$ in $\mathbb{RP}_{2i} \times \mathbb{RP}_{2j}$ if we add the normal line bundle of $H_{2i,2j}$ in $\mathbb{RP}_{2i} \times \mathbb{RP}_{2j}$; therefore

$$\sigma_{i+j-1}(\nu_2) = i.\alpha^{i+j-1} + (j+1).\beta^{i+j-1} + (\alpha+\beta)^{i+j-1}.$$

Similarly, for the normal bundle ν_3 of $\mathbb{RP}_{i-1} \times \mathbb{RP}_j$ in $H_{2i,2j}$ we find that

$$\sigma_{i+j-1}(\nu_3) = (i+1).\alpha^{i+j-1} + j.\beta^{i+j-1} + (\alpha+\beta)^{i+j-1}.$$

Finally, $\mathbb{RP}_{i-1} \times \mathbb{RP}_{j-1}$ intersects $H_{2i,2j}$ in a copy of $H_{i-1,j-1}$, which has dimension $i+j-3$ and we can ignore them.

We shall list the generators x_n according to the form of n.

Case I. $n = 4k-2$.

We take $x_{4k-2} = [\mathbb{RP}_{4k-2}, T_{2k-1}]$. The fixed points sets F is the disjoint union $\mathbb{RP}_{2k-1} \amalg \mathbb{RP}_{2k}$. But the normal bundle of $\mathbb{RP}_{2k}$ is the Whitney sum $2k\xi \longrightarrow \mathbb{RP}_{2k}$, where ξ is the canonical line bundle on $\mathbb{RP}_{2k}$ and hence $e_{2k}(\xi) = 0$. The normal bundle to $\mathbb{RP}_{2k-1}$ is the Whitney sum $(2k-1)\xi \longrightarrow \mathbb{RP}_{2k-1}$. Thus, $\sigma_{2k-1}(\nu) = (2k-1).\alpha^{2k-1}$. Hence, $e_{2k-1}(x_{4k-2})$

$= 1$ and $b_{2k-1}(x_{4k-2}) = e_{2k-1}(\mathbb{RP}_{2k-1}) = 0$.

Case II. $n = 4k-1$, n not of the form $2^q - 1$.

We choose positive integers i and j such that $i+j = k$ and the binomial coefficient $\begin{pmatrix} k \\ i \end{pmatrix} = 1$. We take $x_{4k-1} = [H_{4i,4j}, T_{2i} \times T_{2j}|H_{4i,4j}]$. The fixed point dimension is $2k-1$. To find $e_{2k-1}(x_{4k-1})$, we evaluate $\sigma_{2k-1}(\nu)$ on the fundamental classes of the various components and add. For $H_{2i,2j}$ we have $\sigma_{2k-1}(\nu_1) = 2i.\alpha^{2k-1} + 2j.\beta^{2j-1} = 0$. To find σ_{2k-1} of the normal bundle on $\mathbb{RP}_{2i} \times \mathbb{RP}_{2j-1}$, we note that the coefficient of $\alpha^{2i}\beta^{2j-1}$ in $(\alpha+\beta)^{2k-1}$ is $\begin{pmatrix} 2k-1 \\ 2i \end{pmatrix}$. Also, for $\mathbb{RP}_{2i-1} \times \mathbb{RP}_{2j}$, the coefficient of $\alpha^{2i-1}\beta^{2j}$ in $(\alpha + \beta)^{2k-1}$ is $\begin{pmatrix} 2k-1 \\ 2j \end{pmatrix} = \begin{pmatrix} 2k-1 \\ 2i-1 \end{pmatrix}$. Therefore,

$$e_{2k-1}(x_{4k-1}) = \begin{pmatrix} 2k-1 \\ 2i \end{pmatrix} + \begin{pmatrix} 2k-1 \\ 2i-1 \end{pmatrix} = \begin{pmatrix} 2k \\ 2i \end{pmatrix} = 1.$$

Also, $b_{2k-1}(x_{4k-1}) = e_{2k-1}(H_{2i,2j}) = \begin{pmatrix} 2i+2j \\ 2i \end{pmatrix} = 1$.

Case III. $n = 4k$.

We take $x_{4k} = [\mathbb{RP}_{4k}, T_{2k}]$. For the normal bundle of fixed points set $\mathbb{RP}_{2k}$ we have $\sigma_{2k}(\nu) = 2k.\alpha^{2k} = 0$, so that $e_{2k}(x_{4k}) = 0$. Also, $b_{2k}(x_{4k}) = e_{2k}(\mathbb{RP}_{2k}) = 1$.

Case IV $n = 4k+1$.

We take $x_{4k+1} = [H_{2,4k}, T_1 \times T_{2k}|H_{1,4k}]$. For the fixed points set $H_{1,2k}$ we have $\sigma_{2k}(\nu_1) = \alpha^{2k} + 2k.\beta^{2k} = \alpha^{2k}$. To evaluate this on $H_{1,2k}$, we use lemma (4.1.2), which yields zero. For the set $\mathbb{RP}_1 \times \mathbb{RP}_{2k-1}$, we find $\sigma_{2k}(\nu_2) = \alpha^{2k} + (2k+1).\beta^{2k} + (\alpha+\beta)^{2k}$, in which the coefficient of $\alpha\beta^{2k-1}$ is $\begin{pmatrix} 2k \\ 1 \end{pmatrix} = 0$. For the set $\mathbb{RP}_0 \times \mathbb{RP}_{2k}$ we find $\sigma_{2k}(\nu_3) = 2.\alpha^{2k}$

$+ 2k.\beta^{2k} + (\alpha+\beta)^{2k}$ which does not contain the term β^{2k}. Hence $e_{2k}(x_{4k+1}) = 1$ and $b_{2k}(x_{4k+1}) = e_{2k}(H_{1,2k}) = 0$.

These x_n's form polynomial generators of $M[s^{-1}]$ over $\mathbb{Z}[s,s^{-1}]$. The elements $z_n \in \mathfrak{N}_n$ are given by the equation $Jx_n = z_n$ + terms involving θ. Assertion (b) follows from theorem (4.1.1) and assertion (c) follows from theorem (4.1.12). Assertion (d) is obvious from the choice of x_n and above discussion. Since $M[s^{-1}] = \mathfrak{N}_*(BO) \otimes \mathbb{Z}_2[s,s^{-1}]$, by lemma (4.1.6), for each n, we choose an element $y_n \in M[s^{-1}]$ satisfying (a), (d) and (e) of the lemma. We can adjust y_n, by replacing y_n by $y_n + i_*p_*y_n$ if necessary, where i_* and p_* are the maps defined in lemma (4.1.6), so that we have $x_n = y_n$ + decomposable terms; from which assertion (e) for x_n is clear. ■

<u>Corollary 4.1.15.</u> $J : M \longrightarrow L$ and $J : M[s^{-1}] \longrightarrow L$ are monomorphism.

Proof of this follows from lemma (4.1.15).

<u>Corollary 4.1.16.</u> Given $x \in M$, Jx is a finite Laurent series if and only if x is a polynomail in s with coefficient in $\mathfrak{N}_*$.

<u>Proof.</u> Since J is a homomorphism of $\mathfrak{N}_*$-algebra and $Js = \theta^{-1}$, corollary (4.1.15) gives that Jx is a finite Laurent series in $\theta \Leftrightarrow x$ is a finite Laurent series in s with coefficients in $\mathfrak{N}_*$. Since $x \in M$, negative powers of do not occur. Hence x is a polynomial in s with coefficients in $\mathfrak{N}_*$.

Finally we give the Boardman five halves theorem.

<u>Theorem 4.1.17.</u> Let T be smooth involution on a smooth

closed n-dimensional manifold V, and let k be the fixed point dimension (that is maximum of the dimensions of different components of the fixed points set). If V is a boundary, then $n \leq 5k/2$. If further the class $[V]$ is indecomposable in the cobordism ring $\mathfrak{N}_*$, then $n \leq 2k+1$.

Proof. Let V be an n-dimensional manifold with involution, with fixed point dimension k, so that $[V,T]$ has filtration at most k. We write $[V,T]$ by lemma (4.1.14) as a polynomial in x_i, s and s^{-1}; by the lemma (4.1.14) and the theorem (4.1.11) we find that this polynomial must take the form

$$[V,T] = \lambda_0 + \lambda_1 s^{-1} + \lambda_2 s^{-2} + \ldots.$$

where each λ_i is a polynomial over $\mathbb{Z}_2$ in x_r. By theorem (4.1.12) and lemma (4.1.14), $[V]$ is a polynomial in z_r corresponding to λ_0.

Case (a), $\lambda_0 \neq 0$. Now for all generators x_r, $\mathrm{fil}(x_r) \geq 2r/5$ and so $k \geq \mathrm{fil}([V,T]) \geq \mathrm{fil}\, x_n \geq 2n/5$.

Case (b), $\lambda_0 = 0$. Here the term x_n actually appears in λ_0, and so $k \geq \mathrm{fil}\lambda_0 \geq \mathrm{fil}\, x_n \geq (n-1)/2$. ∎

§2. The stationary points set $F_{G_2(C)}$ and the normal bundle.

Definition 4.2.1. Let (M^n,θ) be a closed G-manifold and $F = \bigcup_{i=0}^{n} F^i$, where F^i is the i-dimensional component of F, be the fixed points set of M^n under the subgroup $G_2(C)$ of G consisting of all elements of order 2 in the centre C of G. Let $D(\nu_i)$ be the normal disk bundle of F^i in M^n with the induced action θ_i of G on $D(\nu_i)$. Then F is said to have an equivariant trivial normal bundle in M^n, if $G/G_2(C)$ acts trivially on F and there exists some positive dimensional G-representations (W_i,ϕ_i), $0 \le i \le n$, such that in $\mathfrak{N}_*(G;\mathcal{A},P)$, where $\mathcal{A}$ is the family of all subgroups of G and P is the family of subgroups of G not containing $G_2(C)$, $[D(\nu_i),\theta_i] = [F^i][D(W_i),\phi_i]$, $D(W_i)$ being the unit disk of W_i.

Theorem 4.2.2. If F has an equivariant trivial normal bundle in M^n, then F is a boundary and (M^n,θ) is a G-boundary.

Proof. Since F has an equivariant trivial normal bundle in M^n, we have $[D(\nu_i),\theta_i] = [F^i][D(W_i),\phi_i]$ for some positive dimensional G-representations (W_i,ϕ_i), $0 \le i \le n$. We have the exact sequence of the triple $\phi \subseteq P \subseteq \mathcal{A}$,

$$\cdots \longrightarrow \mathfrak{N}_*(G;\mathcal{A}) \xrightarrow{j_*} \mathfrak{N}_*(G;\mathcal{A},P) \xrightarrow{\partial_*} \mathfrak{N}_{*-1}(G;P) \longrightarrow \cdots$$

Now, $j_*\ [M^n,\theta] = \sum_{i=0}^{n} [D(\nu_i),\theta_i] = \sum_{i=0}^{n} [F^i][D(W_i),\phi_i]$.

Therefore, $\partial_* j_*\ [M^n,\theta] = \sum_{i=0}^{n} [F^i][S(W_i),\phi_i] = 0$ in $\mathfrak{N}_*(G;P)$.

Hence, there exists a P-free manifold (W,η) of dimension n such that $[\partial W^n,\eta] = \sum_{i=0}^{n} [F^i][S(W_i),\phi_i]$ in $\mathfrak{N}_*(G;P)$, i.e.,

$$(\partial W^n,\eta) \sim \Big(\bigcup_{i=0}^{n} F^i \times (S(W_i),\phi_i)\Big) \qquad \ldots . (1)$$

Let $(V_1,\psi_1), \ldots ,(V_m,\psi_m)$ be the set of all irreducible representations of G. Now any G-representation can be written as $\bigoplus_{k=1}^{m} (V_k,\psi_k)^{f(k)}$ for some map $f : \{1, \ldots ,m\} \longrightarrow \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of non-negative integers. Therefore, $W_i = \bigoplus_{k=1}^{m} (V_k,\psi_k)^{f_i(k)}$ for some $f_i : \{1, \ldots ,m\} \longrightarrow \mathbb{Z}_+$. Since W_i is positive dimensional, there exist at least one member $k(i) \in \{1, \ldots, m\}$ such that $f_i(k(i)) \neq 0$. Now we fix some $\beta \in \{0, \ldots, n\}$. Then there exist $k(\beta) \in \{1, \ldots, m\}$ such that $f_\beta(k(\beta)) \neq 0$. We consider the irreducible representation $(V_{k(\beta)}, \psi_{k(\beta)})$ of G. Let $\rho = (\tilde{V}_{k(\beta)}, \tilde{\psi}_{k(\beta)})$ be an irreducible factor of the $G_2(C)$-representation induced by $(V_{k(\beta)}, \psi_{k(\beta)})$. As ρ is 1-dimensional, $\ker\rho$ is isomorphic to the subgroup $H_{k(\beta)}$ of $\mathbb{Z}_2^k$ isomorphic to $\mathbb{Z}_2^{k-1}$. Now $g \in H_{k(\beta)} \to \lambda_g : \mathbb{F} \longrightarrow \mathbb{F}$ is identity, i.e., $H_{k(\beta)}$ fixes $\tilde{V}_{k(\beta)}$. Let $A_{k(\beta)}$ be the largest subset of $\{1, \ldots, m\}$ such that $H_{k(\beta)}$ fixes $\tilde{V}_j$, $j \in A_{k(\beta)}$. Clearly $k(\beta) \in A_{k(\beta)} \neq \phi$. Let $\Delta(i,\beta) = \sum_{j \in A_{k(\beta)}} f_i(j)$.

Now $\Delta(\beta,\beta) \neq 0$. From (1) we get,

$$F_{H_{k(\beta)}}(\partial W^n, \eta) \sim \bigcup_{l=0}^{n} (F^l \times F_{H_{k(\beta)}}(S(W_l), \phi_l))$$

i.e., $$\partial F_{H_{k(\beta)}}(W^n, \eta) \sim \bigcup_{l=0}^{n} (F^l \times F_{H_{k(\beta)}}(S(W_l), \phi_l)).$$

Let $\mathbb{Z}_{2,\beta}$ be the subgroup complement to $H_{k(\beta)}$ in $G_2(C)$. As W^n is P-free, $\mathbb{Z}_{2,\beta}$ acts freely on $F_{H_{k(\beta)}} W'' = F^*$ so that we have $(\partial F^*, \eta|\mathbb{Z}_{2,\beta}) \sim \bigcup_{l=0}^{n} (F^l \times (S^{\Delta(l,\beta)-1}, a))$ where a is the antipodal involution. Also $[\partial F^*, \eta|\mathbb{Z}_{2\beta}]$ is zero in $\mathfrak{N}_*(\mathbb{Z}_{2,\beta};\mathcal{F}_1)$, $\mathcal{F}_1$ being the family consisting of only trivial subgroup of $\mathbb{Z}_{2,\beta}$.

Therefore, $0 = [\partial F^*, \eta|\mathbb{Z}_{2,\beta}] = \sum_{l=0}^{n} [F^l \times (S^{\Delta(l,\beta)-1}, a)]$.

But $\mathfrak{N}_*(\mathbb{Z}_{2,\beta}:\mathcal{F}_1)$ is free $\mathfrak{N}_*$-module with generators $[S^n, a]$, $n \in \mathbb{Z}_+$. Since $\Delta(\beta,\beta) \neq 0$, $[F^\beta] = 0$, for all $\beta \in \{0, \ldots, n\}$. Hence $[F] = 0$ in $\mathfrak{N}_*$. Therefore,

$j_* [M^n,\theta] = \sum_{i=0}^{n} [F^i].[D(W_i),\phi_i] = 0$ in $\mathfrak{N}_*(G;\mathscr{A},P)$. But we have $j_* : \mathfrak{N}_*(G;\mathscr{A}) \longrightarrow \mathfrak{N}_*(G;\mathscr{A},P)$ is an injection. Therefore $[M^n,\theta]$ is zero in $\mathfrak{N}_*(G;\mathscr{A})$. ∎

§3. Characteristic number of a manifold with involution.

In this section we shall discuss the calculation of characteristic classes of a manifold with involution which was worked out by Kosniowski and Stong.

Theorem 4.3.1. Let M^n be a closed n-dimensional manifold with a smooth involution. Let F^{n-k} be the union of $(n-k)$-dimensional components of the fixed points set of T and ν^k be the normal bundle of F^{n-k} in M^n. If $f(x_1, \dots, x_n)$ is a symmetric polynomial over Z_2 in n-variables of degree at most n, then

$$f(x_1, \dots ,x_n)[M^n] = \sum_k \frac{f(1+y_1, \dots, 1+y_k, z_1, \dots, z_{n-k})}{\prod_{i=1}^{k} (1 + y_i)} [F^{n-k}]$$

where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_i(x)$, $\sigma_i(y)$ and $\sigma_i(z)$ by the Stiefel-Whitney classes $w_i(M^n)$, $w_i(\nu^k)$ and $w_i(F^{n-k})$ respectively and taking the value of the resulting cohomology class on the fundamental homology class of M^n or F^{n-k}.

Proof. Let N be a closed manifold of dimension n-k and ξ be a k-plane bundle over N. Let M be the projective space bundle $\mathbb{R}P(\xi \oplus 1)$ of the lines in the fibres of $\xi \oplus 1$. Let T be the involution induced by multiplication by -1 in the fibres of ξ and by $+1$ in the fibres of the trivial line bundle 1. The fixed points set of T is then the union of N

$= \mathbb{R}P(1)$ with normal bundle ξ and $\mathbb{R}P(\xi)$ with normal bundle the standard line bundle. We consider the equation

$$f(x)[M] = \Sigma\, \frac{f(1+y,\ z)}{\prod\,(1 + y)}\,[F].$$

Then both sides of the above equation are invariants of the equivariant bordism class of the involution (M,T). Also it is additive on the disjoint union of the involutions. Hence it is enough to check the formula for the involutions which generate the equivariant bordism additively.

As both sides of the equation are additive functions of f, we shall check the formula for the smallest symmetric polynomial $s_\omega(x) = \Sigma x_1^{i_1}\ldots x_r^{i_r}$ containing the given monomial, where $\omega = (i_1, \ldots, i_r)$ is a partition of n.

Now for any k-plane bundle η over N the cohomology $H^*(\mathbb{R}P(\eta);\mathbb{Z}_2)$ is the free $H^*(N;\mathbb{Z}_2)$ module on $1, c, \ldots, c^{k-1}$ where c is the Stiefel-Whitney class of the standard line bundle.

Now Stiefel-Whitney class of $\mathbb{R}P(\eta)$ is given by

$$w(\mathbb{R}P(\eta)) = w(N).\{(1+c)^k + (1+c)^{k-1}w_1(\eta) + \ldots. + w_k(\eta)\}$$

and if $\alpha \in H^*(N;\mathbb{Z}_2)$, then

$$c^i\alpha[\mathbb{R}P(\eta)] = \begin{cases} 0 & \text{if } i < k-1 \\ \bar{w}_{i-k+1}(\eta)\alpha[N] & \text{if } i \geq k-1 \end{cases}$$

where $\bar{w}$ denotes the dual Stiefel-Whitney class defined by $w(\eta)\bar{w}(\eta) = 1$.

Now using splitting principle we have $w(N) = \prod_1^{n-k} (1+\alpha_i)$, $w(\xi) = \prod_1^{k} (1+\beta_i)$. Then

$$w(\mathbb{R}P(\xi \oplus 1)) = \prod_1^{n-k} (1+\alpha_i).(1+c).\prod_1^{k} (1+c+\beta_i)$$

and $$w(\mathbb{R}P(\xi)) = \prod_{1}^{n-k} (1+\alpha_\iota) . \prod_{1}^{k} (1+c+\beta_\iota).$$

For $f(x_1, \dots, x_n) = s_\omega(x_1, \dots, x_n)$ we have $f(x_1, \dots, x_n)[\mathbb{R}P(\xi \oplus 1)] = s_\omega(x_1, \dots, x_n)[\mathbb{R}P(\xi \oplus 1)] = s_\omega(\tau(\mathbb{R}P(\xi \oplus 1)))[\mathbb{R}P(\xi \oplus 1)]$, where $s_\omega(\tau(\mathbb{R}P(\xi \oplus 1)))$ denotes the characteristic class. The characteristic class is stable, so the right hand side becomes

$s_\omega(c, c+\beta_1, \dots, c+\beta_k, \alpha_1, \dots, \alpha_{n-k})[\mathbb{R}P(\xi \oplus 1)]$.

Also $\dfrac{f(1+y, z_1, \dots, z_{n-1})}{(1+y)}[\mathbb{R}P(\xi)]$

$$= \frac{s_\omega(1+y, z_1, \dots, z_{n-1})}{(1+y)}[\mathbb{R}P(\xi)]$$

$$= \left(\frac{s_\omega(z_1, \dots, z_{n-1})}{(1+y)} + \sum_{\iota>0} \frac{(1+y)^\iota s_{\omega-(\iota)}(z_1, \dots, z_{n-1})}{(1+y)}\right)$$

$[\mathbb{R}P(\xi)]$, where $s_{\omega-(\iota)} = 0$ if $\iota \notin \omega$ and is the obvious partition if $\iota \in \omega$, and the right hand side becomes

$$\left(\frac{s_\omega(\tau(\mathbb{R}P(\xi)))}{(1+c)} + \sum_{\iota>0} (1+c)^{\iota-1} s_{\omega-(\iota)}(\tau(\mathbb{R}P(\xi)))\right)[\mathbb{R}P(\xi)]$$

$$= \left(\frac{s_\omega(c+\beta_1, \dots, c+\beta_k, \alpha_1, \dots, \alpha_{n-k})}{(1+c)} + \sum_{\iota>0} (1+c)^{\iota-1} s_{\omega-(\iota)}(c+\beta_1, \dots, c+\beta_k, \alpha_1, \dots, \alpha_{n-k})\right)[\mathbb{R}P(\xi)].$$

Finally applying the splitting principle to $\tau(N)$ and ξ,

$$\frac{f(1+y_1, \dots, 1+y_k, x_1, \dots, x_{n-k})}{\prod(1+y_\iota)}[N]$$

$$= \frac{s_\omega(1+\beta_1, \dots, 1+\beta_k, \alpha_1, \dots, \alpha_{n-k})}{\prod(1+\beta_\iota)}[N].$$

On expanding the expression $s_\omega(1+\beta_1, \dots, 1+\beta_k, \alpha_1, \dots, \alpha_{n-k}) = \sum_{\iota=0}^{n-k} \gamma_\iota \in H^*(N, \mathbb{Z}_2)$ with $\gamma_\iota \in H^\iota(N, \mathbb{Z}_2)$, we get

$s_\omega(c+\beta_1, \dots, c+\beta_k, \alpha_1, \dots, \alpha_{n-k}) = \sum_{\iota=0}^{n-k} c^{|\omega|-\iota} \gamma_\iota$, where $|\omega| = \iota_1 + \dots + \iota_r$.

Then $\frac{s_\omega(1+\beta,\alpha)}{\Pi(1+\beta_\iota)}[N] = \left(\sum_{\iota=0}^{n-k} \gamma_\iota\right)(1 + \bar{w}_1(\xi) + .. + \bar{w}_{n-k}(\xi))[N]$

$$= \sum_{\iota=0}^{n-k} \gamma_\iota . \bar{w}_{n-k-\iota}[N]$$

and $\frac{s_\omega(c+\beta,\alpha)}{(1+c)}[\mathbb{R}P(\xi)] = \left(\sum_{\iota=0}^{n-k} c^{|\omega|-\iota}\gamma_\iota\right).(1+c+c^2+ \ldots)[\mathbb{R}P(\xi)]$

with $c^{|\omega|-\iota}\gamma_\iota(1 + c + c^2 + \ldots)[\mathbb{R}P(\xi)]$

$= \begin{cases} 0 & \text{if } |\omega|-\iota > n-1-\iota \\ c^{n-1-\iota}\gamma_\iota[\mathbb{R}P(\xi)] & \text{if } |\omega|-\iota \leq n-1-\iota \end{cases}$ by selecting $(n-1)$ degree terms.

Thus $\frac{s_\omega(c+\beta,\alpha)}{(1+c)}[\mathbb{R}P(\xi)]$

$$= \begin{cases} \sum_{\iota=0}^{n-k} c^{k-1} \bar{w}_{n-k-\iota}(\xi)\gamma_\iota[\mathbb{R}P(\xi)], & \text{if } |\omega| \leq n-1 \\ 0 & , \text{if } |\omega| > n \end{cases}$$

$$= \begin{cases} \frac{s_\omega(1+\beta,\alpha)}{\Pi(1+\beta)}[N], & \text{if } |\omega| \leq n-1 \\ 0 & , \text{if } |\omega| > n-1 \end{cases}.$$

Similarly $\frac{s_\omega(c+\beta,\alpha)}{(1+c)}[\mathbb{R}P(\xi \oplus 1)] = \begin{cases} \frac{s_\omega(1+\beta,\alpha)}{\Pi(1+\beta)}[N], & \text{if } |\omega| \leq n \\ 0 & , \text{if } |\omega| > n \end{cases}.$

Now we have $f(x)[\mathbb{R}P(\xi \oplus 1)] = s_\omega(c,c+\beta,\alpha)[\mathbb{R}P(\xi \oplus 1)]$

$$= s_\omega(c+\beta,\alpha)[\mathbb{R}P(\xi \oplus 1)] + \sum_{\iota>0} c^\iota s_{\omega-(\iota)}(c+\beta,\alpha)[\mathbb{R}P(\xi \oplus 1)]$$

$$= s_\omega(c+\beta,\alpha)[\mathbb{R}P(\xi \oplus 1)] + \sum_{\iota>0} c(1+c)^{\iota-1}s_{\omega-(\iota)}(c+\beta,\alpha)[\mathbb{R}P(\xi\oplus 1)]$$

where using $|\omega| \leq n$ the terms added from $c(1+c)^{\iota-1}-c^\iota$ evaluate to zero. Thus,

$f(x)[\mathbb{R}P(\xi \oplus 1)]$

$$= s_\omega(c+\beta,\alpha)[\mathbb{R}P(\xi \oplus 1)] + \sum_{\iota>0} (1+c)^{\iota-1}s_{\omega-(\iota)}(c+\beta,\alpha)[\mathbb{R}P(\xi)]$$

as $c.\gamma[\mathbb{R}P(\xi\oplus 1)] = i^*.\gamma[\mathbb{R}P(\xi)]$ where i is the inclusion from $\mathbb{R}P(\xi)$ to $\mathbb{R}P(\xi\oplus 1)$. Then the above equation reduces to

$$f(x)[\mathbb{R}P(\xi \oplus 1)]$$

$$= s_{\omega}(c+\beta,\alpha)[\mathbb{R}P(\xi \oplus 1)] + \frac{f(1+y,z)}{(1+y)}[\mathbb{R}P(\xi)] + \frac{s_{\omega}(c+\beta,\alpha)}{(1+c)}[\mathbb{R}P(\xi)].$$

Finally, if $|\omega| \leq n - 1$, $s_{\omega}(c+\beta,\alpha)[\mathbb{R}P(\xi \oplus 1)] = 0$ and $\frac{s_{\omega}(c+\beta,\alpha)}{(1+c)}[\mathbb{R}P(\xi)] = \frac{s_{\omega}(1+\beta,\alpha)}{\Pi(1+\beta)}[N] = \frac{f(1+y,z)}{\Pi(1+y)}[N]$, while for $|\omega| = n$, $\frac{s_{\omega}(c+\beta,\alpha)}{(1+c)}[\mathbb{R}P(\xi)] = 0$, and $s_{\omega}(c+\beta,\alpha)[\mathbb{R}P(\xi\oplus 1)] =$

$$s_{\omega}(c+\beta,\alpha)(1 + c + c^2 + \dots.)[\mathbb{R}P(\xi\oplus 1)] = \frac{s_{\omega}(c+\beta,\alpha)}{(1+c)}[\mathbb{R}P(\xi\oplus 1)]$$

$$= \frac{s_{\omega}(1+\beta,\alpha)}{\Pi(1+\beta)}[N] = \frac{f(1+y,z)}{\Pi(1+y)}[N].$$

Thus, if $|\omega| \leq n$, we have

$$f(x)[\mathbb{R}P(\xi\oplus 1)] = \frac{f(1+y,z)}{(1+y)}[\mathbb{R}P(\xi)] + \frac{f(1+y,z)}{\Pi(1+y)}[N].$$

Hence the theorem follows. ■

As a consequence of the above theorem Stong and Kosniowski have proved the stronger version of Boardman theorem proved in §1.

Theorem 4.3.2. If (M^{n+r},T) is an involution with fixed point set F and $\dim M > (5/2)\dim F$, then (M^n,T) is a boundary.

Proof. Suppose F^n denotes the n-dimensional part of the fixed points set, with $r > (3/2).n$ and for $i > n$,

(F^i, ν^{n+r-i}) is zero in $\mathfrak{N}_i(BO(n+r-i))$. Consider a pair of partitions $\omega = (i_1, \ldots, i_s)$ and $\omega' = (j_1, \ldots, j_t)$ for which $|\omega| + |\omega'| = n$, ω being a nondyadic partition. Let $f_{\omega,\omega'}(x)$ be the symmetric function defined as

$$f_{\omega,\omega'}(x) = \{ \Sigma \, (1+x_1)^{i_1+1} . x_1^{i_1} .... (1+x_s)^{i_s+1} . x_s^{i_s} \} .$$
$$\{ \Sigma \, (1+x_1)^{j_1+1} . x_1^{j_1} .... (1+x_t)^{j_t+1} . x_t^{j_t} \} .$$

This symmetric function satisfies the following properties

(a) In $f_{\omega,\omega'}(1+y, z)/\Pi(1+y)$, the term of least degree is of degree atleast n and $f_{\omega,\omega'}(1+y_1, \ldots, 1+y_r, z_1, \ldots, z_n)/\Pi(1+y) = s_\omega(z).s_{\omega'}(y_1, \ldots, y_r, z_1, \ldots, z_n)$ + terms of higher degree.

(b) If $i_\alpha \in \omega$, $i_\alpha > 1$, so $2i_\alpha + 1 \leq (5/2)i_\alpha$. Thus the highest degree term in $f_{\omega,\omega'}(x)$ is of degree $2|\omega| + s + 2|\omega'| < (5/2)(|\omega| + |\omega'|) = (5/2)n$.

Therefore, we have

(i) $f_{\omega,\omega'}(x)[M^{n+r}] = 0$, since $f_{\omega,\omega'}(x)$ has degree $<$ n+r.

(ii) If $k > n$, (F^k, ν^{n+r-k}) bounds and so all characteristic numbers vanish. For $k < n$, all the terms of $f_{\omega,\omega'}(1+y, z)/\Pi(1+y)$ have degree $> k$. Thus,

$(f_{\omega,\omega'}(1+y, z)/\Pi(1+y))[F^k] = 0$, $k \neq n$.

(iii) $(f_{\omega,\omega'}(1+y, z)/\Pi(1+y))[F^n] = s_\omega(\tau).s_{\omega'}(\tau \oplus \nu)[F^n]$, from the calculation of the degree n term.

From theorem (4.3.1), we have $s_\omega(\tau).s_{\omega'}(\tau\oplus\nu)[F^n] = 0$ for all partitions ω, ω' with $|\omega| + |\omega'| = n$ and ω nondyadic. One knows that $f : N^n \longrightarrow X$ bounds if all characteristic numbers $s_\omega(\tau_N).f^*(x_i)[N] = 0$, where $|\omega| + i = n$, x_i are base elements of $H^i(X; \mathbb{Z}_2)$ and ω is a nondyadic partition. Thus the map $\tau \oplus \nu : F^n \longrightarrow BO_{n+r}$ classifying the bundle $\tau \oplus \nu$

bounds. Let $f : V^{n+1} \longrightarrow BO_{n+r}$ be a map with boundary $\tau \oplus \nu : F^n \longrightarrow BO_{n+1}$. Let $g : V^{n+1} \longrightarrow BO_m$ classify the normal bundle of V in some $\mathbb{R}^{m+n+1}$. Then $\oplus \circ (f \times g) : V^{n+1} \longrightarrow BO_{n+r+m}$ has boundary map $\nu \oplus (n \oplus m) : F^n \longrightarrow BO_{n+1+m}$ classifying the stabilization of ν. Since the stabilization $BO_r \longrightarrow BO_{n+m+r}$ is monic at bordism level, $[F^n, \nu^r]$ is zero in $\mathfrak{N}_n(BO_r)$. Thus, inductively, all the fixed data of (M,T) are zero in bordism. Hence (M,T) bounds. ■

Kosniowski and Stong have also provided a formula for the evaluation of the characteristic numbers of manifolds with $\mathbb{Z}_2^k$ action.

Let M be a closed n-dimensional manifold with a smooth action of the group $\mathbb{Z}_2^k$. Let $f(x) = f(x_1, \ldots, x_n : x_1^1, \ldots, x_{n_1}^1; \ldots; x_1^s, \ldots, x_{n_s}^s)$ be a polynomial over $\mathbb{Z}_2$ which is symmetric in each of the sets of variables $\{x_1, \ldots, x_n\}$ and $\{x_1^i, \ldots, x_{n_i}^i\}$, $i = 1, \ldots, s$. We write $\mathbb{Z}_2^k = \{T_1, \ldots, T_k \mid T_i^2 = 1 : T_iT_j = T_jT_i\}$. Assuming $\alpha_1, \ldots, \alpha_k$ to be formal variable, let $\alpha_\rho = \Sigma \{\alpha_i \mid \rho(T_i) = -1\}$ for ρ an irreducible representation of $\mathbb{Z}_2^k$. Let F_0 be a component of F and K be the quotient field of $\mathbb{Z}_2[\alpha_1, \ldots, \alpha_k]$. The restriction of the tangent bundle of M to F_0 decomposes into subbundles under the action of $\mathbb{Z}_2^k$ as $\tau_M|F_0 \cong \tau_{F_0} \oplus (\oplus_{\rho \neq 0} \nu_\rho)$ where ν_ρ is the subbundle on one space which $\mathbb{Z}_2^k$ acts via ρ; and the subbundle on which $\mathbb{Z}_2^k$ acts trivially is identified with the tangent bundle of F_0. Let $p = \dim F_0$, $q_\rho = \dim \nu_\rho$. Then $n = p + \Sigma q_\rho$. Also each subbundle ξ^{n_i} decomposes such that $\xi^{n_i}|F_0 \cong \oplus \xi_\rho^{n_i}$, $\mathbb{Z}_2^k$ acting via ρ. If $q_\rho^i = \dim \xi_\rho^{n_i}$ then $n_i =$

$\sum_{\rho} o^{i}_{\rho}$. Kosniowski and Stong have proved the following theorem generalising theorem (4.3.1):

Theorem 4.3.5. The element $\sum \frac{f(z;\ \alpha+y;\ \alpha+w)}{\prod(\alpha+y)}[F_{\upsilon}] \in r$ is a polynomial in $\alpha_1, \ldots, \alpha_k$ and has constant term equal to $f(x)[M]$, where the expression is evaluated by replacing the j-th elmentary symmetric function in $\{z_1, \ldots, z_{\rho}\}$ by $w_j(F_{\upsilon}) = w_j(\tau_{F_{\upsilon}})$, $\{y^1_{\rho}, \ldots, y^{q_\rho}_{\rho}\}$ by $w_j(\nu_{\rho})$ and $\{w^{i,1}_{\rho}, \ldots, w^{i,q^i_\rho}_{\rho}\}$ by $w_j(\xi^{n_i}_{\rho})$; (for proof cf. [32]).

§4. Involutions with fixed points set of constant dimension.

In this section, we shall study the works done by Kosniowski and Stong who investigated involutions (M,T) having fixed points set of constant dimension. This study was made in 1978 by Kosniowski and Stong [23]. Later Stong [30] almost completed this study in 1981. One needs some facts about symmetric functions to make this investigation. We mention below some of them without proof.

Lemma 4.4.1. If $\sigma_i(x)$ denotes the i-th elementary symmetric function $\Sigma\, x_1.x_2 \ldots x_i$, then

(a) $\sigma_i(1+y_1, \ldots, 1+y_k) = \sum_{j=0}^{i} \binom{k-j}{i-j} \sigma_i(y_1, \ldots, y_k)$

(b) $\sigma_i(1+y_1, \ldots, 1+y_k, z_1, \ldots, z_n) = \sum_{p+q\leq i} \binom{k-p}{i-p-q} \sigma_p(y_1, \ldots, y_k).\sigma_q(z_1, \ldots, z_n)$.

Lemma 4.4.2. For $1 \leq i \leq \inf(k,n)$, $\exists$ symmetric polynomials $f_i(x_1, \ldots, x_{n+k})$ and $g_i(x_1, \ldots, x_{n+k})$ of degree $\leq 2i$ such that

$f_i(1+y_1, \dots, 1+y_k, z_1, \dots, z_n) = \sigma_i(z)$ + higher degree terms,

$g_i(1+y_1, \dots, 1+y_k, z_1, \dots, z_n) = \sigma_i(z)$ + higher degree terms.

Lemma 4.4.3. There exists symmetric polynomials $h_i(x_1, \dots, x_{2n})$, $1 \le i \le 2n$, of degree $\le i$ for which the degree i term in $h_i(x_1, \dots, x_{2n})$ is $\sigma_i(x_1, \dots, x_{2n})$.and such that $h_i(1+y_1, \dots, 1+y_k, z_1, \dots, z_n)$ has no non-zero terms of degree $(i/2)$.

In [23] Kosniowski and proved the following results regarding fixed points set of constant dimension.

Proposition 4.4.4. Let (M^{n+k}, T) be an involution with fixed points set F^n of constant dimension n. If either $k >$ n or $k = n$ with M^{n+k} being boundary, then the involution (M^{n+k}, T) bounds.

Proof. If $i_1 + \dots + i_r + j_1 + \dots + j_s = n$, then

$$w_{i_1}(F)\dots w_{i_r}(F).w_{j_1}(\nu^k)\dots w_{j_s}(\nu^k)[F^n]$$

$$= \frac{(f_{i_1}\dots f_{i_r}.g_{j_1}\dots g_{j_s})(1+y,z)}{\Pi(1+y)}[F^n]$$

$$= (f_{i_1}\dots f_{i_r}.g_{j_1}\dots g_{j_s})(x)[M^{n+k}] = 0, \text{ since } M^{n+k} \text{ bounds.}$$

Thus, all Stiefel-hitney numbers of the mao\p $\nu : F^n \longrightarrow BO_k$ classifying the normal bundle of F^n in M^{n+k} are zero. Hence (M^{n+k}, T) bounds.

Proposition 4.4.5. If (M^{2n}, T) is an involution with fixed points set F^n of constant dimension n, then (M, T) is bordant to $(F \times F$, twist).

Proof. For $i_1 + \dots + i_r = 2n$, the Stiefel-Whitney number $w_{i_1}\dots w_{i_r}[M]$ is given by

$$\frac{(h_{i_1}\dots h_{i_r})(1+y_1, \dots, 1+y_n, z_1, \dots, z_n)}{\Pi(1+y_i)}[F^n]$$

and $h_i(1+y,z)$ has no non-zero term of degree $<$ $(i/2)$ by lemma (4.4.3). Therefore, $h_{2j}(1+y,z) = \alpha_j$ + higher terms and $h_{2j+1}(1+y,z) = \alpha_{j+1}$ + higher term. Also $(h_{i_1}\dots h_{i_r})(1+y,z)$ has all non-zero terms of degree $>$ n, if any i_α is odd. Thus, all Steifel-Whitney numbers of M divisible by an odd w_i are zero (i.e. allmonomials of degree 2n in Stiefel-Whitney classes involving odd w_i evaluated at $[M^{2n}]$ is zero). Therefore by Milnor M^{2n} is bordant to $N^n \times N^n$ for some n-dimensional manifold. Consider the involution $(M^{2n} \cup N^n \times N^n$, T $\cup$ twist) bounds as an involution. Thus (M^{2n},T) is bordant to $(N^n \times N^n$, twist). Hence $[F^n,\nu] = [N^n,\tau]$ in $\mathfrak{N}_n(BO_n)$, so that N^n is bordant to F. This implies that (M^{2n},T) is bordant to (F $\times$ F, twist).

<u>Proposition 4.4.5</u>. If (M^{n+k}, T) is an involution with fixed points set F^n of constant dimension $n > k$, then every Stiefel-Whitney number of M^{n+k} divisible by a monomial $w_{2i_1+1}\dots w_{2i_s+1}$ with $s > n-k$ is zero

<u>Proof</u>. or $2.(\Sigma i_\alpha) + s + 2.(\Sigma j_\beta) = n+k$, the Stiefel-Whitney number $w_{2i_1+1}\dots w_{2i_s+1}.w_{2j_1}\dots w_{2j_t}[M^{n+k}]$ is given by

$$\frac{(h_{2i_1+1}\dots.h_{2i_s+1}.h_{2j_1}\dots.h_{2j_t})(1+y,z)}{\Pi(1+y)}[F^n]$$

The degree of the least non-zero term in $(h_{2i_1+1}\dots.h_{2j_t})(1+y,z)$ is $\Sigma i_\alpha + s + \Sigma j_\beta = q$ or higher. One has $2q = n+k + s$, so if $s > n-k$ then $q > n$ and therefore the Stiefel-Whitney number of F^n given above is zero.

The following proposotion enables us to determine an involution (M,T) in terms of F and M up to bordism.

Proposition 4.4.6. Suppose $[M^{2n-1}] = 0$ and $M^{2n-1} = \partial V^{2n}$. Consider the involution $(\Gamma(M^{2n-1}),s)$, where

$$\Gamma(M^{2n-1}) = \frac{S^1 \times M^{2n-1}}{(z,m) \sim (-z,Tm)}$$

and $s[z,m] = [\bar{z},m]$. The fixed data of this involution is $(M^{2n-1},1) \cup (F^n, \nu^{n-1}\oplus 1)$. Consider a manifold W obtained by taking disjiont union of involutions $(\Gamma(M) \times [0,1], s \times 1)$ and $(V \times [-1,1], 1 \times -1)$ and then identify $M \times [-1,1]$ with an equivarient tubular neighborhood of $M \times \{1\}$ in $\Gamma(M) \times \{1\}$. This gives an equivarient bordism of $(\Gamma(M),s)$ to an involution (N^{2n},s^1) with fixed data $(F^n,\nu^{2n-1}\oplus 1)$. By proposition (4.4.5), (N^{2n},s^1) is equivariently bordant to the involution $(F^n \times F^n$, twist which has fixed data $(F^n,\tau$, where τ is the tangent bundle on F. Now $[F^n,\tau] \in \mathfrak{N}_n(BO_n)$ is completely determined by $[F^n]$. So if $[F^n] = 0$ in $\mathfrak{N}_*$, then $[F^n,\nu^{n-1}] \in \mathfrak{N}_n(BO_{n-1})$ is mapped to zero in $\mathfrak{N}_n(BO_n)$. Hence $[F^n,\nu^{n-1}] = 0$ in $\mathfrak{N}_n'(BO_{n-1})$. This shows that (M^{2n-1},T) bounds. This completes the proof of the proposition.

This study started by Kosniowski and Stong in [23] was almost completed by Stong in 1981 by proving that a class $\alpha \in \mathfrak{N}_{2n-1}$ is represented by a manifold M^{2n-1} with involution T having fixed points set of constant dimension $n \Leftrightarrow$ every Stiefel-Whitney number of α involving a product of two odd dimensional Stiefel-Whitney classes is zero. To establish tthis first Stong proved the following lemma:

Lemma 4.4.7. Let $\Lambda_* \subseteq \mathfrak{N}_*$ be defined as $\Lambda_m = \{ \alpha \in \mathfrak{N}_m$ for which all Stiefel-Whitney numbers divisible by a product

$w_{2i+1}w_{2j+1}$ are zero). For every non-dyadic odd integer 2p+1, there is an indecomposable $\lambda_{2p+1} \in \Lambda_{2p+1}$. For any such choice of indecomposables $\{\lambda_{2p+1}\}$, every $\alpha \in \Lambda_{2n-1}$ can be expressed $\sum_{p \le n-1} \lambda_{2p+1} \cdot (\beta^{n-1-p})^2$, for some $\beta^{n-1-p} \in \mathfrak{N}_{n-1-p}$.

Proof. For each non-dyadic odd i, let us choose polynomial generators $x_i = \lambda_i$ for $\mathfrak{N}_* \cong Z_2[x_i \mid i \ne 2^t-1]$. Let N^i be the manifold representing the classes $x_i = \lambda_i$. For given $\alpha \in \Lambda_{2n-1}$, let M^{2n-1} be its representative. Then upto bordism , we have

$$M^{2n-1} \sim \cup N^{i_1} \times \dots \times N^{i_r}. \qquad \dots(1)$$

If M^{2n-1} is indecomposable, this union contains the single term N^{2n-1} with all other terms having length r 1. Therefore, by adding λ_{2n-1} to α, we may assume that all terms have length $r > 1$. We iterate this process. So assume that all products in the expression have length $\ge s$. suppose that the product

$$N^{j_1} \times \dots \times N^{j_1} \times N^{j_2} \times \dots \times N^{j_2} \times \dots\dots \times N^{j_k} \times \dots \times N^{j_k},$$

where $N^{j_i} \times .. \times N^{j_i}$ has p_i factors with $p_1 + p_2 + .. + p_k = s$, occurs as a term in M^{2n-1}.

For $\omega = (i_1 \dots i_r)$, let $s_\omega = s_{i_1 \dots i_r}$ be the the symmetric function $\Sigma x^{i_1} \dots x^{i_r}$ of the classes x_i for which the Stiefel-Whitney class is $w = \Pi(1+x_i)$. Then,

$$s_{j_1 \dots j_1, j_2 \dots j_2, \dots j_k \dots j_k}[M^{2n-1}] \ne 0,$$

since it has the value one for the term

$$s_{j_1 \dots j_1, j_2 \dots j_2, \dots, j_k \dots j_k}[N^{j_1} \times \dots \times N^{j_k}] = \prod_{i=1}^{k} (s_{j_i}[N^{j_i}])^{p_i}$$

and it is zero on all other products in the expression (1). Note that $s_j[N^j] = 1$, since N^j is decomposable.

Now consider the product

$$\phi = s_{j_1 \ldots j_1} s_{j_2 \ldots j_2} \ldots s_{j_k \ldots j_k},$$

By rules for products

$$\phi = s_{j_1 \ldots j_1, j_2 \ldots j_2, \ldots, j_k \ldots j_k} + \Sigma\, s_\omega,$$

where each ω in the sum has indices $\leq s$ so that $s_\omega[M] = 0$, and hence $\phi[M] \neq 0$. Let us consider the classes $s_{j \ldots j}$ with p odd. If $j = 2^u(2v+1)$, $u > 0$, then $s_{j \ldots j} = (s_{2v+1 \ldots 2v+1})^{2^u}$ is an even power of an odd dimensional class and hence any Stiefel-Whitney number of M^{2n-1} divisible by $s_{j \ldots j}$ is zero. If j is odd, then $s_{j \ldots j}$ is an odd dimensional class.

Since $\phi[M^{2n-1}] \neq 0$ and $\alpha = [M^{2n-1}] \in \Lambda_{2n-1}$, at themost one of the numbers p_i can be odd, and for that odd p_i the corresponding j_i must be odd. Since $2n-1 = \Sigma\, p_i j_i$ is odd, at least one odd p and j must occur. Let p_i and j_i is odd. Then

$$N^{j_1} \times \ldots \times N^{j_1} \times N^{j_2} \times \ldots \times N^{j_2} \times \ldots\ldots \times N^{j_k} \times \ldots \times N^{j_k},$$

is equal to $N^{j_\rho} \times V \times V$ and hence represents a class $\lambda_{j_\rho}(\beta^{n-1-p})^2$. Therefore by subtracting from classes of the form $\lambda_{2p+1}(\beta^{n-1-p})^2$, one can remove all products of length s. Now induction completes the proof of the proposition.

<u>Theorem 4.4.8.</u> A class $\alpha \in \mathfrak{N}_{2n-1}$ is represented by a manifold M^{2n-1} with involution T having fixed points set of

constant dimension n $\Leftrightarrow$ every Stiefel-Whitney number of α involving a product of two odd dimensional Stiefel-Whitney classes is zero.

Proof. Direct implication follows from proposition (4.4.5). Converse follows from proposition (4.4.7) and induction on n.

§5. Group actions having one fixed point.

In this section our effort is to study the compact Lie-groups G that can act smoothly and effectively on closed and oriented manifold M^n of positive dimension such that the fixed points set of M^n consist of orecisely one point. One notes that we may consider manifold to be connected as the component of M containing the fixed point is invarient under the action. From the works of Conner and Floyd (mentioned in chapter III), one easily notes that G can not be $\mathbb{Z}_2^k$. Also $\mathbb{Z}_4$ acts on $\mathbb{R}P^2$ via the action $t[x_0,x_1,x_2] = [-x_1,x_0,x_2]$ with exactly one fixed point namely $[0,0,1]$. Further Conner and Floyd conjectured in 1964 that for p an odd prime, $\mathbb{Z}_{p^k}$ can not act on an oriented closed manifold with exactly one fixed point. This conjecture was established by Atiya and Bott [1]. We begin with an algebraic result.

Theorem 4.5.1. If G is a compact non abelian Lie group, there is a faithful complex representation of G having no one dimensional irreducible subrepresentations.

Proof. We know that G has a faithful finite dimensional complex representation W. Let $W = W_1 \oplus W'$, where W_1 is the sum of all one dimensional irreducible subrepresentations, W' being the sum of all irreducible representations of dimension greater than one. Let $N \leq G$ be the kernel of the representation W', so that N is a closed normal subgroup of G. let $[G,G]$ denote the closure of the subgroup generated by the commutators.

Now the commutators act trivially on any one dimensional representations, so $[G,G]$acts trivially on W_1. Thus any element in $N \cap [G,G]$ acts trivially on both W_1 and W' and so on W. Since W is faithful, this gives $N \cap [G,G] = \{1\}$.

If $n \in N$ and $g \in G$, then $ngn^{-1}g^{-1} = n(gn^{-1}g^{-1})$ belongs to N by normality of N. But $ngn^{-1}g^{-1}$ is a commutator, so $ngn^{-1}g^{-1} = 1$, i.e., $ng = gn$. Thus N is central in G.

Now $N \times [G,G]$ is a closed subgroup of G, and if ℓ is a one dimensional irreducible representation of N and ρ a nontrivial irreducible representation of $[G,G]$, then $\ell \otimes \rho$ is an irreducible representationof $N \times [G,G]$, and is a constituent in some irreducible representation U_ℓ of G. Then U_ℓ must have dimension greater than one, since its restriction to $[G,G]$ is non-trivial.

We.have some finite sum of representations ℓ is faithful for N, so there is a representation V which is the sum of W' and a finite sum of representations U_ℓ which will be a faithful representation of G with no one dimensional irreducible subrepresentaions. ■

Let V be a representation of G over F ($= \mathbb{R}$ or $\mathbb{C}$) having no one dimensional irreducible reoresentations. Let $FP(V \oplus F)$ be the projective space of F-lines in the representation space $V \oplus F$, where F is the trivial one dimensional representation of G. The action of G on $V \oplus F$ induces an action on the lines and hence on $FP(V \oplus F)$. Now, if G fixes $x \in FP(V \oplus F)$, then x is a one dimensional invarient subspace of $V \oplus F$, but the only such subspace is given by the trivial summand F. Thus the action of G on $FP(V \oplus F)$ has exactly one fixed point. Therefore theorem (4.5.1) gives the following result

Corollary 4.5.2. If G is a compact non-abelian Lie group the G has an effective action on an oriented manifold with exactly one fixed point.

Proposition 4.5.3. If G is a compact abelian Lie group $\neq \mathbb{Z}_2^k$, $\exists$ an effective G-action on a closed manifold of positive dimension with exactly one fixed point.

Proof. If dim G 0, then the component of the identity is $G_0 = (\mathbb{S}^1)^r$ for some $r > 0$. The representation of G_0 on $\mathbb{C}^r$ given by $(z_1, .. ,z_r)(w_1, ... ,w_r) = (z_1 w_1, ... ,z_r w_r)$ has no one dimensional invarient subspace. Therefore $(\mathbb{S}^1)^r$ acts on $N = \mathbb{C}P(\mathbb{C}^r \oplus \mathbb{C})$ with exactly one fixed point. Let M be the space of $(\mathbb{S}^1)^r$ equivarient map from G to N and let G act on M by $(gf)(g') = f(g'g)$, $f \in M$. Since G/H is finite , number of coset representatives are finite. Also $f : G \longrightarrow N$ is determined by the values $f(g_i)$, where g_i are coset representatives. Therefore M is a closed connected manifold diffeomorphic to $|G/H|$ copies of N. Also $gf = f$, for all g implies that $f(g = (gf)(1) = f(1)$, i. e., f is a constant map. Further $hf(1) = f(h.1) = f(h) = f(1)$, for $h \in (\mathbb{S}^1)^r$, so that $f(1)$ is the fixed point of $N = \mathbb{C}P(\mathbb{C}^r \oplus \mathbb{C})$ under $(\mathbb{S}^1)^r$. Hence the action of G on M has exactly one fixed point.

Proposition 4.5.4. If G is finite abelian group $\neq \mathbb{Z}_2^k$, then $\exists$ an effective G- action on a closed manifold of positive dimension with one fixed point.

Proof. By the hypothesis, G has at least one element of order 4 or p for some odd prime p. Consider H = $\mathbb{Z}_4$ or $\mathbb{Z}_p$. The action of $\mathbb{Z}_4$ on $\mathbb{RP}^2$ given by $t[x_0,x_1,x_2] = [x_1,-x_0,x_2]$ has exactly one fixed point. For H = $\mathbb{Z}_p$, the representation on $\mathbb{C}$ given by $(e^{2\pi i/p},z) \longrightarrow e^{2\pi i/p}.z$ has no invarient one dimensional space. Therefore $\mathbb{Z}_p$ acts on $\mathbb{CP}(\mathbb{C} \oplus \mathbb{C})$ with exactly one fixed point. Now, since G/H is finite, using the same construction as in proposition (4.5.3), we get a closed manifold of positive dimension on which G acts with exactly one fixed point.

Proposition 4.5.5. If G is a finite abelian p group with p an odd prime, then G cannot act on a closed oriented manifold of positive dimension with precisely one fixed one fixed point.

Proof. We prove the result by induction on $|G|$. If $|G|$ = p, G is cyclic. Then G cannot act on a closed oriented manifold with a single fixed point (see the proposition (4.5.4))for proof cf. [2]). Now let G be finite abelian p group acting on some oriented manifold M of positive dimension with one fixed point $x \in M$. Let J be a subgroup with $\{e\} \subsetneq J \subsetneq G$ and let F be the component of the fixed points set of J which contains x. G/J acts on F, which is orientable, fixing precisely the point x. Therefore by induction F must be zero dimensional. G then acts on an invariant neighborhood of x so that no proper subgroup of G fixes any point other than x in the neighborhood, so G acts freely on neighborhood $\setminus\{x\}$. Thus $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot be subgroup of G, and G must be cyclic. This is a contradiction, as cyclic p groups cannot act on a closed oriented manifold with precisely one fixed point.

<u>Proposition 4.5.7.</u> *If $G = (S^1)^r \times H$, $r > 0$ and H abelian of odd order, then G cannot act effectively on a closed connected oriented manifold of positive dimension with exactly one fixed point.*

<u>Proof.</u> Suppose G acts effectively on a closed connected oriented manifold M of positive dimension with precisely one fixed point x. This action can have a finite number of distinct isotropy groups G, H_1, H_2, ... , H_m. Now each H_i can contain only a finite number of the groups $G_s = (\mathbb{Z}_{p^s})^r \times P$, for if H_i contain an infinite number of these subgroups, then H_i being closed it must be equal to G. being closed. Hence there is a G_s, for some s, which is not contained in any of H_i, and that G_s is a finite abelian p group acting on M with precisely one fixed point x. This contradicts proposition (4.5.5). Hence the proposition follows.

<u>Proposition 4.5.7.</u> If $G = (S^1)^r \times$, $r > 0$ and H abelian of odd order, then G cannot act effectively on a closed connected oriented manifold of positive dimension with exactly one fixed point.

<u>Proof.</u> Suppose G acts effectively on M^n which is closed connected oriented with positive dimension having exactly one fixed point x. Now the tangent space T_yM of M at x splits irreducible repersentation of G, which are two

dimensional, and at least one of which is nontrivial on $(\mathbb{S}^1)^r$. Let $V \cong \mathbb{R}^2$ be one such irreducible representation of G. The action of G on V factors through a quotient group $G/H \cong \mathbb{S}^1$. Let F be tne component of the fixed points set of H containing x. F is a closed oriented manifolo with an $\mathbb{S}^1$ action having exactly one fixed point . Since V occurs as a subrepresentation of T_xM, F is positive dimensional which is a contradictionto the fact that $\mathbb{S}^1$can not have any such action. cannot act effectively on a closed oriented manifold of positive dimension with precisely one fixed point.

§6. Bordism of $\mathbb{Z}_p$-manifold and $\mathbb{S}^1$-manifolds and some recent results.

In this section we shall briefly mention $\mathbb{Z}_p$ and $\mathbb{S}^1$-bordism and fixed points set. In 1968, Atiyah and Bott [1] proved that if for a prime p, $\mathbb{Z}_p$ acts on a homology sphere with two isolated fixed points then the representations of $\mathbb{Z}_p$ on the tangent space at each fixed point are same. This implies that the given such $\mathbb{Z}_p$-manifold to one with no fixed point which in turn is $\mathbb{Z}_p$-bordant to p copies of some manifold with $\mathbb{Z}_p$ action being the permutation.

Definition 4.6.1. If a free $\mathbb{Z}_p$-manifold M is $\mathbb{Z}_p$-bordant to p copies of some manifold with $\mathbb{Z}_p$ action being the permutation, then we say that the $\mathbb{Z}_p$-manifold is $\mathbb{Z}_p$-boundary mod p.

In view of this definition Atiya and Bott result can be reformulated as below:

If $\mathbb{Z}_p$ acts on a homology sphere with two isolated fixed points then M is a $\mathbb{Z}_p$-boundary mod p.

In 1983, Ewing and KOsniowski has proved the similar result for any $\mathbb{Z}_p$-manifold. They showed that if M is an 2n-dimensional $\mathbb{Z}_p$-manifold with two isolated points and if $n > p-3$ then M is a $\mathbb{Z}_p$-boundary mod p. In the same paper they further generalised this result for more number of isolated fixed points given as follows.

Theorem 4.6.2. Let M be a 2n-dimensional $\mathbb{Z}_p$-manifold with l number of isolated fixed points satisfying $l < 1 + \frac{2\{n/2\}}{(p-1)\{\log_p n\}-2}$ then M is $\mathbb{Z}_p$-bordant to a free $\mathbb{Z}_p$-manifold.

Here $\{x\}$ denotes the least integer greater than or equal to x.

One notes here that if M has a pair of fixed points where the representation of $\mathbb{Z}_p$ on the tangent space at each fixed points is isomorphic by an orientation reversing isomorphism then M is $\mathbb{Z}_p$-bordant to a $\mathbb{Z}_p$-manifold with two less fixed points. The bordism can be achieved by removing a disk about each of the concerned fixed points and attaching a handle equivariently. Therefore we may ignore such pairs of fixed points.

Coming to S^1-action, we have a similar result for S^1-manifold proved by Kosniowski [22] in 1983.

Theorem 4.6.3. Let M be an S^1-manifold with two fixed points, then there is an integer r such that 2^r copies of M bound as an S^1-manifold.

Further he showed in [21] that if M is a unitary S^1-manifold of dimension ≠ 2 or 6 and if the fixed points set is a homology sphere then M is an S^1-boundary. In the same paper Kosniowski also proved that if M is a unitary S^1-manifold of dimension $\neq 6$ and if the fixed points set has the integral homology of a product of two odd dimensional spehres then M is an S^1-boundary.

In 1989, Demichelis Stefano [8] showed that a finite group G acting effectively, locally linearly and preserving orientation on a $\mathbb{Z}$-homology 4 sphere has a fixed point set a k-sphere , $k \leq 2$. Wu [39] in 1990 extended the study of fixed points set made by Stong in [30]. He considered J^k_m as the group of unoriented bordism classes of m-dimensional smooth manifolds whcih are represented by manifolds with smooth involutions having m-k dimensional fixed points set and obtained a necessary and sufficient condition for a bordism class to lie in J^{2k}_n and J^{2t+1}_n for $k \leq 2$ and $t \leq 9$. The groups J^{n-1}_{2n-1} and J^{n-2}_{2n-2} were studied by Stong in [29]. Waner [34] in 1990 proved that a unitary $\mathbb{Z}_p$s-manifold cannot have a single fixed point.

CHAPTER V

Vanishing of equivariant bordism groups

In this chapter, we shall discuss, that in equivariant bordism theory, using families of slice types, it is possible for the theory to vanish. This work was developed by Kosniowski by considering finite abelian groups. Here in this section, we shall discuss the development made by Khare and Dev, considering any finite group, not necessarily an abelian group. We also mention the results obtained by Deb and Khare in the case of k-torus action and actions of a compact abelian Lie group.

§1. G-slice type and bordism of families of G-slice types.

Let M be a G-manifold and $x \in M$ then G_x is the subgroup of G that fixes x. For each $x \in M$ there is a G_x module $\bar{V}_x$ which is equivariently diffeomorphic to a G_x tubular neighborhood of x. This module $\bar{V}_x$ decomposes as $\bar{V}_y$ = $V_x \oplus V'_x$, where G_x acts trivially on V'_x and no non-zero vector of V_x is fixed by all of G_x.

Definition 5.1.1. The pair $[G_x;V_x]$ is called the G-slice type of the point $x \in M$.

Definition 5.1.2. If H is a subgroup of G then a pair [H;U], where U is an H-module with no trivial H-submodule, is called a G-slice type.

Definition 5.1.3. A family $\mathcal{F}$ of G-slice type is a collection such that if $[H:U] \in \mathcal{F}$ then for every $x \in G \times_H U$, the G-slice type $[G_x;V_x] \in \mathcal{F}$.

Corresponding to a family $\mathcal{F}$ of G-slice types we say that a <u>G-manifold is of type $\mathcal{F}$</u> if for every $x \in M$, $[G_x:V_x] \in \mathcal{F}$. Similarly we call a G-manifold M with boundary M to be of type $(\mathcal{F},\mathcal{F})$ if for all $x \in M$ the G-slice type $[G_x;V_x] \in \mathcal{F}$.

<u>Definition 5.1.4.</u> Let M_1 and M_2 be two n-dimensional G-manifolds of type $\mathcal{F}$. Then M_1 is said to be $\mathcal{F}$-bordant to M_2 if there is an (n+1)-dimensional G-manifold N of type $(\mathcal{F},\mathcal{F})$ such that the disjoint union $M_1 \amalg M_2$ is equivariantly diffeomorphic to ∂N.

This equivalence relation on the set of G-manifolds of type $\mathcal{F}$ gives rise to a bordism theory $\mathfrak{N}^G_*[\mathcal{F}]$. The relative bordism group $\mathfrak{N}^G_*[\mathcal{F},\mathcal{F}]$ is defined in a way similar to the definition (3.2.3)..

We also have a result similar to (3.2.4):

<u>Theorem 5.1.5.</u> There is a long exact sequence

$$\cdots \longrightarrow \mathfrak{N}^G_n[\mathcal{F}',\mathcal{F}''] \xrightarrow{i_*} \mathfrak{N}^G_n[\mathcal{F},\mathcal{F}''] \xrightarrow{j_*} \mathfrak{N}^G_n[\mathcal{F},\mathcal{F}'] \xrightarrow{\partial_*}$$

$\mathfrak{N}^G_{n-1}[\mathcal{F}',\mathcal{F}''] \longrightarrow \cdots$ for any triple $\mathcal{F}'' \subseteq \mathcal{F}' \subseteq \mathcal{F}$ of families of G-slice types, where $i : (\mathcal{F}',\mathcal{F}'') \hookrightarrow (\mathcal{F},\mathcal{F}'')$ and $j : (\mathcal{F},\mathcal{F}'') \hookrightarrow (\mathcal{F},\mathcal{F}')$ are the inclusions and ∂ is the boundary map.

§2. <u>Conjugate class of G-slice types, bundle bordism and the map $\nu_{\bar{\rho}}$.</u>

Let $[G_x;V_x]$ be the G-slice type of a point x of a G-manifold M. The orbit G(x) of x is a closed and compact submanifold of M. The normal bundle $\nu(i)$ to G(x) in M is a smooth G-vector bundle and its disk bundle is a closed G-invariant tubular neighborhood of $G(x)$. Also G acts as a group of bundle maps on $\nu(i)$ and the fibre over x is G_x-invariant and contains no trivial G_x-subspace.

Let g_* be the map on the total space $E(\nu(i))$ induced by the action of g on the base space $G(x)$. The G-slice type of $gx \in G(x)$ is $[gG_xg^{-1};g_*V_x]$. The underlying vector space of V_x and g_*V_x are same and the action of ghg^{-1}, $h \in G_x$ on $v \in g_*V_x$ is same as the action of h on $v \in V_x$.

Definition 5.2.1. If $\rho = [H;V]$ be a G-slice type then the collection $\{[gHg^{-1};g_*V] \mid g \in G\}$ is called a conjugate class of G-slice types and is denoted by $\bar{\rho}$ or $[H;V]^g$.

Let us consider a family $\mathcal{F}$ of G-slice types and $\rho = [H;V] \in \mathcal{F}$. Then for every point $x \in G \times_H V$, the G-slice type $[G_x;V_x] \in \mathcal{F}$. Also the G-slice type of any other point $y \in G \times_H V$, is of the form $[K;U]$ where $K \leq H$ and U is a subspace of V with K- action as the restriction of H-acton on V. The conjugate class $[K;U]^g$ gives the G-slice type of all points in $G(y)$. So we see that $\rho \in \mathcal{F}$ implies that $\bar{\rho} \subseteq \mathcal{F}$.

Definition 5.2.2. Let $\bar{\rho}$ be a conjugate class of G-slice types. A *G-vector bundle $\xi : E(\xi) \xrightarrow{p} B(\xi)$ is said to be of type $\bar{\rho}$* if the set of points in $E(\xi)$ having the slice type in $\bar{\rho}$ is precisely the zero-section.

Definition 5.2.3. Let $\xi : E(\xi) \longrightarrow B(\xi)$ be a G-vector bundle of type $\bar{\rho}$. Then ξ is said to be a boundary if there exist a G-vector bundle $\xi' : E(\xi') \longrightarrow B(\xi')$ of type $\bar{\rho}$ such that $\partial B(\xi') = B(\xi)$.

Bordism of bundles of type $\bar{\rho}$ leads to the bundle bordism group $\mathfrak{N}^G_n[\bar{\rho}]$, where n denotes the total dimension of the vector bundles in question.

Let $\mathcal{F}' \subseteq \mathcal{F}$ be the families of G-slice types with $\mathcal{F} = \mathcal{F}' \cup \{\bar{\rho}\}$ and let M be a G-manifold of type $(\mathcal{F},\mathcal{F}')$. The set $N_{\bar{\rho}}$ of all points $x \in M$ having G-slice type in $\bar{\rho}$ is a closed G-submanifold of M and a small equivariant tubular neighborhood of $N_{\bar{\rho}}$ is equivalent to a disk bundle of the normal bundle to N in M. This normal bundle is s G-vector bundle type $\bar{\rho}$ and the total dimension of it is n, if M is n-dimensional. The assignment of a bordism class of a G-manifold M of type $(\mathcal{F},\mathcal{F}')$ to the bordism class of the normal bundle of $N_{\bar{\rho}}$ in M defines a natural $\mathfrak{N}_*$-module homomorphism

$$\nu_{\bar{\rho}} : \mathfrak{N}^G_n[\mathcal{F},\mathcal{F}'] \longrightarrow \mathfrak{N}^G_n[\bar{\rho}].$$

Theorem 5.2.4. Let $\mathcal{F}' \subseteq \mathcal{F}$ be families of G-slice types with $\mathcal{F} = \mathcal{F}' \cup \{\bar{\rho}\}$ then, $\nu_{\bar{\rho}} : \mathfrak{N}^G_n[\mathcal{F},\mathcal{F}'] \longrightarrow \mathfrak{N}^G_n[\bar{\rho}]$ is an $\mathfrak{N}_*$-isomorphism.

Proof. We define an inverse map of the map $\nu_{\bar{\rho}}$, $\mu : \mathfrak{N}^G_n[\bar{\rho}] \longrightarrow \mathfrak{N}^G_n[\mathcal{F},\mathcal{F}']$. Let $[\xi] \in \mathfrak{N}^G_n[\bar{\rho}]$. Then the disc bundle $D(\xi)$ is a G-manifold of type $(\mathcal{F},\mathcal{F}')$. We define $\mu([\xi]) = [D(\xi)]$.

Let $[M] \in \mathfrak{N}^G_n[\mathcal{F},\mathcal{F}']$. If B be the set of all points of M having G-slice type of $\bar{\rho}$, then $\nu_{\bar{\rho}}[M]$ is the bordism class of the normal bundle $\nu(i) : E(\nu) \longrightarrow B$ of B in M. Then $D(\xi)$ is a submanifold of dimension n and it is $(\mathcal{F},\mathcal{F}')$ bordic to M. So we get $\mu \circ \nu_{\bar{\rho}} = \mathrm{id}$. Similarly $\nu_{\bar{\rho}} \circ \mu = \mathrm{id}$. ■

Now we have the following result:

Theorem 5.2.5. If $\mathcal{F}' \subseteq \mathcal{F}$ are families of G-slice types with $\mathcal{F} = \mathcal{F}' \cup \{\bar{\rho}\}$, then there exists a long exact sequence

$$\cdots \longrightarrow \mathfrak{N}^a_n[\mathcal{F}'] \longrightarrow \mathfrak{N}^a_n[\mathcal{F}] \xrightarrow{\nu_{\bar\rho}} \mathfrak{N}^a_n[\bar\rho] \xrightarrow{\partial} \mathfrak{N}^a_{n-1}[\mathcal{F}'] \longrightarrow \cdots$$

where ∂ is the homomorphism defined by $\partial[\xi] = S(\xi)$; $S(\xi)$ being the sphere bundle of the bundle ξ. ■

§3. Extension map and families of G-slice types, isomorphic bundle bordism groups.

Let G_2 be the subgroup of G generated by the elements of order 2 in the centre of G. Then G_2 is isomorphic to $\mathbb{Z}_2^k$ for some integer $k \geq 0$. We choose once and for all a basis $g_1, g_2, \ldots\ldots, g_k$ of G_2 and order the elements of G_2 by $g_1 < g_2 < \ldots . < g_k < g_1 + g_2 < \ldots\ldots < g_1 + g_k < \ldots .$.

Suppose that K is a subgroup of H such that $H = \langle x \rangle \oplus K$, where $0 \neq x \in G_2$, then we write $K \underset{2}{\subseteq} H$. Let $x \in G_2$ be the minimal element in G_2 which satisfies $H = \langle x \rangle \oplus K$. We define a homomorphism $p = p_{H,K} : H \longrightarrow K$, given by $p(ax + k) = k$ and call it the distinguished projection from H to K. now if U is a K-module and $K \underset{2}{\subseteq} H$ then we obtain an H-module $p^*(U)$, where $p^*(U)$ is the H-module with the underlying space is the same as U together with the action of H given by $h(u) = p(h).u$, $u \in U$.

Corresponding to a G-slice type $[K;U]$ with $K \underset{2}{\subseteq} H$ we have an extension function $e = e_{K,H}$ given by

$$e_{K,H}[K;U] = [H;\ V(K) \oplus p^*(U)]$$

where $V(K)$ denotes the set of real numbers with $h \in H$ acting on it by multiplication by $+1$ if $h \in K$ and by -1 if $h \notin K$.

When $H = \langle x \rangle \oplus K$ we have $gHg^{-1} = \langle x \rangle \oplus gKg^{-1}$ and hence $e[gKg^{-1}\ ;\ g_*U] = [gHg^{-1}\ ;\ V(gKg^{-1}) \oplus p^*(g_*U)]$

$$= [gHg^{-1} ; g_*(V(K) \oplus p^* U]$$

Thus $e_{K,H}$ induces a map $e^g = e^g_{K,H}$ on the collection of conjugate classes of G-slice types $[K;U]^g$ and

$$e^g_{K,H}[K;U]^g = [H; V(K) \oplus p^*(U)]^g$$

Let $\hat{G}$ be a subgroup of G containing G_2. We define three families of G-slice types:

$\mathcal{F}(\hat{G}) = \{[gHg^{-1};g_*V] \mid [H;V]$ is a G-slice type with $H \subseteq \hat{G}$, $g \in G\}$, $\mathcal{F}'(\hat{G}) = \{[K;U] \in \mathcal{F}(\hat{G}) \mid K \cap G_2 \neq G_2\}$ and $\tilde{\mathcal{F}}(\hat{G}) = \mathcal{F}'(\hat{G}) \cup \{e_{K,H}[K;U] \mid [K;U] \in \mathcal{F}'(\hat{G})$ and $K \underset{2}{\subseteq} H$ with $H \cap G_2 = G_2\}$.

Suppose that $\bar{\rho} = [H;V]^g$ be a conjugate class of G-slice types and $\xi : E(\xi) \xrightarrow{p} B(\xi)$ be a G-vector bundle of type $\bar{\rho}$. Since the zero section of ξ is precisely the set of all points of $E(\xi)$ having G-slice type in $\bar{\rho}$, the fibres of ξ are isomorphic to V as vector spaces. If $\bar{\rho}$ is now a conjugate class of G-slice types of an orbit of the G-manifold $E(\xi)$ then we can take the set of all points having G-slice types in $\bar{\rho}$ and form the normal bundle to it in $S(\xi)$, which is a G-vector bundle of type $\bar{\rho}'$, $\bar{\rho}'$ being a G-slice type. This correspondence gives rise to an $\mathfrak{N}_*$-homomorphism $\psi : \mathfrak{N}^G_n[\bar{\rho}] \longrightarrow \mathfrak{N}^G_n[\bar{\rho}']$ given by $\psi[\xi] = [\nu_{\bar{\rho}'}(S(\xi))]$. For $\bar{\rho} = e^g[\bar{\rho}']$, ψ becomes an isomorphism. Precisely we have

Lemma 5.3.1. Let $K \underset{2}{\subseteq} H$ and $\bar{\rho} = [H;V]^g$, $\bar{\rho}' = [K;U]^g$ be two classes of conjugte G-slice types such that $e^g(\bar{\rho}') = \bar{\rho}$. Then $\psi : \mathfrak{N}^G_n[\bar{\rho}] \longrightarrow \mathfrak{N}^G_n[\bar{\rho}']$ is an $\mathfrak{N}_*$-isomrphism.

For proof see [6].

Theorem 5.3.2. Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$, be a sequence of subfamilies of $\tilde{\mathcal{F}}(\hat{G})$ with (i) $\mathcal{F}_0 = \bar{\rho}_0 = \{[e,\mathbb{R}^0]\}$, (ii) $\mathcal{F}_i$

$= \mathcal{F}_{i-1} \cup \{\bar{\rho}_i\}$ for all $i \geq 1$, (iii) $\bigcup_{i \geq 0} \mathcal{F}_i = \mathcal{F}$, and (iv) $e^g(\bar{\rho}_{2i}) = \bar{\rho}_{2i+1}$ for all $i \geq 0$. Then $\mathfrak{N}_*^G[\mathcal{F}] = 0$.

Proof. We have the exact sequence

$$\cdots \longrightarrow \mathfrak{N}_n^G[\mathcal{F}_{i-1}] \longrightarrow \mathfrak{N}_n^G[\mathcal{F}_i] \longrightarrow \mathfrak{N}_n^G[\bar{\rho}_i] \xrightarrow{\partial} \mathfrak{N}_{n-1}^G[\mathcal{F}_{i-1}] \longrightarrow \cdots,$$

for $i \geq 0$, $\mathcal{F}_{-1}$ is the empty family. Then $\mathfrak{N}_n^G[\mathcal{F}_0] \cong \mathfrak{N}_n^G[\bar{\rho}_0]$. By lemma (5.3.1), $\mathfrak{N}_n^G[\bar{\rho}_1] \cong \mathfrak{N}_{n-1}^G[\bar{\rho}_0]$. From the long exact sequence and the following diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathfrak{N}_n^G[\mathcal{F}_0] \arrow[r] & \mathfrak{N}_n^G[\mathcal{F}_1] \arrow[r] & \mathfrak{N}_n^G[\bar{\rho}_1] \arrow[r, "\partial"] & \mathfrak{N}_{n-1}^G[\mathcal{F}_0] \arrow[r] & \cdots, \\ & & & \mathfrak{N}_{n-1}^G[\rho_0] & & \end{tikzcd}$$

it follows that ∂ is an isomorphism so that $\mathfrak{N}_n^G[\mathcal{F}_1] = 0$. Proceeding this way, we get $\mathfrak{N}_*^G[\mathcal{F}_{2i}] = \mathfrak{N}_*^G[\bar{\rho}_{2i}]$ and $\mathfrak{N}_*^G[\mathcal{F}_{2i+1}] = 0$. Thus $\mathfrak{N}_*^G[\mathcal{F}] = \varinjlim_{i} \mathfrak{N}_*^G[\mathcal{F}_i] = 0$. ■

§4. Ordering the conjugate classes of G-slice types.

Three distinct relations $\leq$ are defined on the collection $\tilde{A}$ of all subgroups conjugate to subgroups of $\hat{G}$, on the collection of all H-modules, $H \in \tilde{A}$ and finally on the collection of all conjugate classes of G-slice types of the family $\tilde{\mathcal{F}}(\hat{G})$ and extend each of these relations into a total ordering on the respective collection.

Every subgroup H_2 of G_2 has a distinguished base $h_1 \quad h_2 < \dots < h_m$ such that h_1 is the least element in H and for $i > 1$, h_i is the least element among those elements of H which are not in $\langle h_1, h_2, \dots, h_{i-1} \rangle$ the subgroup generated by $h_1, h_2, \dots, h_{i-1}$. Using the distinguished base, the subgroups of G_2 are first totally ordered as follows:

$\langle e \rangle < \langle g_1 \rangle < \langle g_2 \rangle < \dots\dots < \langle g_1 . g_2 \dots g_k \rangle < \langle g_1, g_2 \rangle < \dots\dots,$

i.e., first by the order of the subgroup and then lexicographically on the distinguished base.

A relation $\leq$ is defined in the set of subgroups of G by the following rules:

Rule A. Let H and K belong to $\tilde{A}$. We define $\leq$ as below

(i) if $|H| \leq |K|$ then $H \leq K$

(ii) if $|H| = |K|$ and $|K_2| \leq |H_2|$ then $H \leq K$, where $K_2 = K \cap G_2$ and $H_2 = H \cap G_2$,

(iii) if $|H| = |K|$, $|H_2| = |K_2|$ but $H_2 \leq K_2$ then $H \leq K$ and

(iv) if $|H| = |K|$, $H_2 = K_2$ then we order them arbitrarily so as to make the relation $\leq$ a total ordering on $\tilde{A}$.

Next, a relation $\leq$ is introduced in the set of all non-trivial irreducible H-modules, $H \in \tilde{A}$ as follows: $U \leq V$ if $U = V$ or else there exists a subgroup K such that $K \underset{2}{\subseteq} H$ and $U = p^* i^*(V)$ where $i : K \hookrightarrow H$ is the natural inclusion and $p\ H \longrightarrow K$ is the distinguished projection.

Lemma 5.4.1. The relation $\leq$ is a partial ordering on the collection of all nontrivial irreducible H-modules (for proof cf. [6]).

A total ordering on the set of all irreducible H-modules having same dimension is chosen compatible with the partial ordering introduced. The total ordering is extended to all non-trivial irreducible H-modules by writing $U \leq V$ if and only if $\dim U \leq \dim V$. Since any H-module can be expressed uniquely as the sum of all irreducible H-modules, the total ordering on all H-modules is extended lexicographically.

Rule B. Let U and V be two H-modules.

(i) If $\dim U \leq \dim V$, then $U \leq V$.

(ii) If $\dim U = \dim V$ and V follows U lexicographically, then $U \leq V$.

Lastly, the order $\leq$ is defined by Rule C on the collection of all classes of conjugate G-slice types of the family $\tilde{\mathcal{F}}(\hat{G})$ as follows:

Rule C. Let $\bar{\rho} = [H;U]^g$ and $\bar{o}' = [K;V]^g$ be two conjugate classes of G-slice types of $\tilde{\mathcal{F}}(\hat{G})$.

(i) If $\dim U \leq \dim V$, then $\bar{\rho} \leq \bar{\rho}'$.

(ii) If $\dim U = \dim V$ and $H \leq K$, then $\bar{\rho} \leq \bar{\rho}'$.

(iii) If $\dim U = \dim V$, $H = K$ and $U \leq V$ then $\bar{\rho} \leq \bar{\rho}'$.

§5. Decomposition of a family.

If the dimension of a conjugate class of G-slice types is defined as the dimension of the module present there in, then there are only a finite number of conjugate classes of G-slice types of a given dimension. The classes of the family $\tilde{\mathcal{F}}(\hat{G})$ are totally ordered by the Rule C and can be indexed by non-negative integers as $\bar{\rho}_0 < \bar{\rho}_1 < \bar{\rho}_2 < \ldots.$ where $\bar{\rho}_0 = \{[\langle e\rangle;\mathbb{R}^0]\}$. Let $\mathcal{F}_j = \bigcup\limits_{i \leq j} \bar{\rho}_i$. Then $\mathcal{F}_i$ is a family of G-slice types. Corresponding to the family $\mathcal{F}_j$ a collection $\bar{\mathcal{F}}_j = \{\bar{\rho_0}, \bar{\rho_1}, \ldots, \bar{\rho_j}\}$ is formed.

Let A_j, B_j and C_j be three mutually disjoint subcollections of $\bar{\mathcal{F}}_j$ such that $\bar{\mathcal{F}}_j = A_j \cup B_j \cup C_j$. For $j = 0$, $\bar{\mathcal{F}}_j = \{\bar{\rho}_0\}$. We set $A_0 = \{\bar{\rho}_0\}$, $B_0 = \phi$ and $C_0 = \phi$. Let A_{j-1}, B_{j-1} and C_{j-1} be defined for some $j \geq 1$. Then we have $\bar{\mathcal{F}}_{j-1} = A_{j-1} \cup B_{j-1} \cup C_{j-1}$ and $\bar{\mathcal{F}}_j = \bar{\mathcal{F}}_{j-1} \cup \{\bar{\rho}_i\}$. There are two possibilities: (i) either $\bar{\rho}_j = e^g(\bar{\rho})$ for some $\bar{\rho} \in A_{j-1}$ or else (ii) $\bar{\rho}_j \neq e^g(\bar{\rho})$ for any $\bar{\rho} \in A_{j-1}$.

In case (i), A_j, B_j and C_j are defined as $A_j = A_{j-1} \cup$

$\{\bar{\rho}\}$, $B_j = B_{j-1} \cup \{\bar{\rho}_j\}$ and $C_j = C_{j-1} \cup \{\bar{\rho}_j\}$ and in case (ii) they are defined as $A_j = A_{j-1} \cup \{\bar{\rho}_j\}$, $B_j = B_{j-1}$ and $C_j = C_{j-1}$

We note some lemmas which will be used later (for proof of these, cf. [6]).

Lemma 5.5.1. There is at most one conjugate class of G-slice types $\bar{\rho} \in A_{j-1}$ such that $e^g(\bar{\rho}) = \bar{\rho}_j$.

Lemma 5.5.2. If N is sufficiently large compared to n then A_N consists of conjugate classes of G-slice types of dimension greater than n.

Lemma 5.5.3. If $[H;U]^g$ is a conjugate class of G-slice types and $\bar{\rho} \in A_j$ be a conjugate class of G-slice types of an orbit of a point of $G \times_H U$ then either $\bar{\rho} = [H;U]^g$ or else $\overline{[H;U]}^g \notin \mathcal{F}_j$.

Theorem 5.5.4. There is an isomorphism

$$\oplus \nu_i : \mathfrak{N}^G_*[\mathcal{F}_j] \longrightarrow \bigoplus_{\bar{\rho}_i \in A_j} \mathfrak{N}^G_*[\bar{\rho}_i].$$

Proof. Induction on j is used to prove the result. The result is clearly true for $j = 0$.

We now suppose it is true for $(j-1)$, i.e.,

$$\oplus \nu_i : \mathfrak{N}^G_*[\mathcal{F}_{j-1}] \longrightarrow \bigoplus_{\bar{\rho}_i \in A_{j-1}} \mathfrak{N}^G_*[\bar{\rho}_i]$$

is an isomorphism.

We consider the long exact sequence

$$\ldots \longrightarrow \mathfrak{N}^G_n[\mathcal{F}_{j-1}] \longrightarrow \mathfrak{N}^G_n[\mathcal{F}_j] \longrightarrow \mathfrak{N}^G_n[\bar{\rho}_j] \xrightarrow{\partial_j} \mathfrak{N}^G_{n-1}[\mathcal{F}_{j-1}] \longrightarrow \ldots$$

and $\nu_i : \mathfrak{N}^G_{n-1}[\mathcal{F}_{j-1}] \longrightarrow \mathfrak{N}^G_{n-1}[\bar{\rho}_i]$. We have the composite $\nu_i \circ \partial_j : \mathfrak{N}^G_n[\bar{\rho}_j] \longrightarrow \mathfrak{N}^G_{n-1}[\bar{\rho}_i]$. If $\nu_i \circ \partial_j \neq 0$, then $\bar{\rho}_i$ is a conjugate class of G-slice types of $G \times_H V$, where $\overline{[H;V]}^g = \bar{\rho}_j \in \mathcal{F}_j$. Then by lemma (5.5.3) $\bar{\rho}_i \notin A_j$.

Now for the class $\bar{\rho}_j$ there exists at most one conjugate class of G-slice types $\bar{\rho}_i$ such that $e^g(\bar{\rho}_i) = \bar{\rho}_j$. If there does not exist any such $\bar{\rho}_i \in A_{j-1}$, then for any $\bar{\rho}_i \in A_{j-1}$ both $\bar{\rho}_i$ and $\bar{\rho}_j$ belong to A_j and so $\nu_i \circ \partial_j = 0$ for every $\bar{\rho}_i \in A_{j-1}$. Thus $(\bigoplus_{\bar{\rho}_i \in A_{j-1}} \nu_i) \circ \partial_j = 0$ and consequently $\partial_j = 0$. We have a short exact sequence

$$0 \longrightarrow \mathfrak{N}_n^G[\mathcal{F}_{j-1}] \xrightarrow{\nu_i} \mathfrak{N}_n^G[\mathcal{F}_j] \xrightarrow{\nu_j} \mathfrak{N}_n^G[\bar{\rho}_j] \longrightarrow 0.$$

If for $\bar{\rho}_i$, we have $\bar{\rho}_i \in A_{j-1}$ such that $\bar{\rho}_j = e^g(\bar{\rho}_i)$ then neither $\bar{\rho}_j$ nor $\bar{\rho}_i$ belong to A_j and by lemma (5.3.1)

$$\nu_i \circ \partial_j : \mathfrak{N}_n^G[\bar{\rho}_j] \longrightarrow \mathfrak{N}_{n-1}^G[\bar{\rho}_i]$$

is an isomorphism and once again we have a short exact sequence $0 \longrightarrow \mathfrak{N}_n^G[\bar{\rho}_i] \longrightarrow \mathfrak{N}_n^G[\mathcal{F}_{j-1}] \longrightarrow \mathfrak{N}_n^G[\mathcal{F}_j] \longrightarrow 0$ and the monomorphism of ∂_j.

Both the short exact sequences split as the modules involved are vector spaces over $\mathbb{Z}_2$. Thus one concludes the theorem for the first case.

For the second case we note that $A_j = A_{j-1} \setminus \{\bar{\rho}_i\}$ and this gives $\mathfrak{N}_n^G[\mathcal{F}_j] \cong \bigoplus_{\bar{\rho}_i \in A_j} \mathfrak{N}_n^G[\bar{\rho}_i]$.

Theorem 5.5.5. $\mathfrak{N}_n^G[\tilde{\mathcal{F}}(\hat{G})] = 0$.

Proof. Corresponding to the positive integer n we take all conjugate classes of G-slice types of dimension $\leq$ n+1. If $\mathcal{F}_N$ is the union of all these classes then

$$\mathfrak{N}_n^G[\tilde{\mathcal{F}}(\hat{G})] = \mathfrak{N}_n^G[\mathcal{F}_n] \cong \bigoplus_{\bar{\rho}_i \in A_N} \mathfrak{N}_n^G[\bar{\rho}_i].$$

If now N is made sufficiently large compared to n then by lemma (5.5.2), A_N consists of all conjugate classes of G-slice types of dimension > n and the isomorphism $\oplus\, \nu_i$ is zero.

Corollary 5.5.6. Suppose that G is a finite group. If M is a G-manifold in which G_2 acts without fixed points then M is a boundary.

Proof. The corollary now follows from theorem (5.5.5) because if G_2 acts without fixed points, then an isotropy subgroup H of a point in M satisfies the condition that $H \cap G_2 \neq G_2$. Thus M is then a maniold of type $\mathcal{F}''(G)$ amd cosequently of type $\tilde{\mathcal{F}}(G)$. ■

Finally we mention the result obtained by Deb and Khare in the case when the group is a compact abelian Lie group.

By the structure theorem compact abelian Lie group $G = T^k \times \Gamma$, where T^k is the k-torus and Γ a is afinite abelian group. If the elements of G are denoted by $(y_1, y_2, \ldots, y_k, g)$ where $y_i \in S^1$ for $1 \leq i \leq k$ and $g \in \Gamma$, then there are homomorphisms $\rho_i : G \longrightarrow G$ given by $p_i(y_1, y_2, \ldots, y_k, g) = (0, 0, \ldots, y_i, 0, \ldots, e)$, $1 \leq i \leq k$.

Consider the elementary abelian 2-group $\mathbb{Z}_2^k$ contained in $T^k \subseteq G$. If x_i is the generator of $p_i(\mathbb{Z}_2^k)$ then these elements also form a base of $\mathbb{Z}_2^k$. consider the following collection of G-slice types for $0 \leq j < k$:

$\mathcal{F}_j = \{ [H;V] \mid p_i(H) = \text{finite or } S^1 \text{ for } 1 \leq i \leq k \text{ and } x_i \notin p_i(H) \text{ for atleast } (k-j) \text{ values of } i\}$. An extensio map is defined on each of the families $\mathcal{F}_j$ by $e[K;U] = [H;V(K)\oplus p^*U]$ for $[K;U] \in \mathcal{F}_j$, $0 \leq j < k$. Let $\tilde{\mathcal{F}}_j = \mathcal{F}_j \cup e(\mathcal{F}_j) = 0$. Using similar techniques to those used for the proof of theorem (5.5.5) Dev and Khare proved that $\mathfrak{N}_n^G[\tilde{\mathcal{F}}_j] = 0$.

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