CERTAIN INFLATIONARY MODELS OF THE EARLY UNIVERSE-A SURVEY

By

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*I certify that the dissertation entitled "**Certain Inflationary Models of The Early Universe**" submitted by Mrs. Sanchalee Ghosh in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.*

I certify that the sources from which ideas borrowed, have been duly referred to.

The material in this dissertation has not been presenteu for the award of a degree in any University before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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PREFACE

The standard model of hot big-bang cosmology relies on certain initial conditions, which are puzzling in two ways. The first is the assumption of highly homogeneous early universe, in spite of the fact that there exist separated regions which are causally disconnected. The second one is the flatness problem, stressed by Dieke and Peebles.

It is often seen that exciting developments take place when different disciplines come together. This happened with cosmology also, when quantum field theory was drawn into service to explain above problems of initial conditions of the early universe. Incorporating the minimal SU(5) grand unified theory in the model of the early universe, Guth suggested a possible solution to the horizon as well as flatness problems by proposing a cosmological scenario called inflationary model of the universe which leads to exponentially expanding universe. Later on, using Coleman-Weinberg potential a modified scenario, called new inflationary universe, was proposed by Linde, Albrechdt and Steinhardt resolving certain difficulties of Guth's model. This model was further developed by Hawking and Moss, Albrechdt and Steinhardt, Vilnekin, Ford and Vilnekin and others.

In 1983 Ellis, Nanopoulos, Olive and Tamvakis raised one question, "At what stage in the evolution of the universe should inflationary epoch have occurred?" To answer this question they studied super-symmetric grand unified theory and suggested that super-symmetry breaking scale must be considerably smaller than grand unified scale. From the analysis of super-symmetric toy model, they found that inflation could take place before GUT (Grand Unified Theory) phase transition. Inflation before the GUT epoch is called primordial inflation.

In spite of these developments, inflationary scenario can be obtained without using the results of Ground Unified Theory or Supersymmetry theory. In this direction, some attempts were done by Halliwel, Burd and Barrow by using exponential potential. Importance of viscosity in the theory of early universe has also been emphasized by some authors e.g. Hawking, Misner and Weinberg. Paking the effect of bulk viscosity in the dynamics of early universe, Papeni and Weiss have found that inflationary scenario might be possible in certain cases.

This thesis contains a systematic review of the work done in searching possibility of inflationary scenario in different situations.

CHAPTER - I

INTRODUCTION

The purpose of Cosmology is to study the universe from a global point of view and to describe the physical processes throughout its evolution. The models describing the universe and its evolution are based upon an extrapolation of physical theories whose validity has been locally tested out. The extrapolation is based on the theory of Relativity which states that physical Laws are covariant at all times. Gravitation is one of the basic tools to build up a cosmological model. The standard cosmological model is based upon a set of hypotheses which make it the simplest global description of the universe. The first of them is to assume that General Relativity, whose field content is minimum among the metric theories, is the correct gravitation theory. The second assumption is that hyper-surfaces of constant time are homogeneous and isotropic called 'Cosmological Principle'. The other ingredient of a cosmological model is the particle content of the universe. In the standard cosmological model it is assumed that these particles and their interactions correspond to the standard particle physics model.

The standard cosmology[1,2] is very much successful in describing the history of the universe back to times as early as 10^{-45}Sec. and energy as high as 10^{19}GeV. The considered frame work is quantum field theory coupled to classical General Relativity. In this frame-work matter is quantised and geometry is described by General Relativity. The reason for assuming quantum field theory is that it successfully describes particle interaction at high temperatures.

1.1 Grand Unified Theories

By a Grand Unified theory[3], we mean a particle physics in which the standard Low gauge group SU(3)X[SU(2)XU(1)] electroweak is embedded in a simple Gauge group G which is spontaneously broken at a large mass scale. The SU(5) model of Georgi and Glashow[4] is the simplest model of the type G=SU(5) and the symmetry breaking of SU(5) $\longrightarrow$ SU(3)XSU(2)XU(1) which occurs at an energy scale Mx $\approx 10^{15}$GeV.

Kirzhnits and Linde[5] suggested that at high temperature, the Higgs fields of any spontaneously broken gauge theory would lose their expectation values and in this high temperature phase the full gauge symmetry is restored at $T > \sim 10^{14}$GeV which was ultimately formalised by Weinberg and by Dolan andJackiw.[6]

The following are the three consequences of GUTs which are particularly important in Cosmology. The first is the non-conservation of baryon number. This means that proton has a finite life time around 10^{33} years. It also means that the net baryon number of the universe be generated dynamically[7] with no observed quantity which distinguishes the observed universe from the vacuum.

The second consequence is the existence of one or more phase transitions at a critical temperature T_c of masses of order $M_{GUT}=10^{14}$GeV. For $T > T_c$ the full grand unified gauge symmetry, which is spontaneously broken at low temperature, is restored. This phase transition may have had a very significant effect on the evolution of the universe.

The third consequence is the existence of magnetic monopoles in the particle spectrum. t'Hooft and Polyakov[8,9] investigated that magnetic monopoles are produced when the gauge group breaks to SU(3)XU(1). These monopoles are topologically stable knots in the Higgs field expectation value. Their masses are order of $M_{X/\alpha}$ where M_X is the mass of the superheavy vector Bosons and α is the fine-structure constant. In the minimal SU(5) GUT,

$M_X \approx 4 \times 10^{14}$GeV and

$\alpha = \frac{1}{45}$

1.2 Problems of the Standard Model

The standard model of hot big-bang cosmology depends on the assumption of initial conditions.

Here by "Standard model", we mean adiabatically expanding radiation-dominated universe described by a Robertson-Walker metric and the initial conditions are taken as follows. The standard model has a singularity at time t=0. As $t \longrightarrow 0$, the temperature $T \longrightarrow \propto$. So no original problem at the time t=0 can be expressed.

We have Planck mass $M_P = \frac{1}{\sqrt{G}} = 1.22 \times 10^{19}$GeV when T is at the order of Planck mass or even greater, there will be no meaning adhered to standard model.

Near $T \approx 10^{19}$GeV, gravity is expected to be unified with other interactions of nature for example strong interaction, weak interaction and electromagnetism. Hence to study the effects of gravity separately it is wise to start with a temperature below the Planck scale. Due to this reason Guth started his model with temperature $T_o \approx 10^{17}$GeV. So it will be convenient to set initial conditions of the universe to derive the subsequent evolution. On the contrary, the equation of state for matter at these temperatures can not be practically be deduced, however, various hypotheses can be assumed and conclusion may be derived. The initial conditions taken in the standard model are -

(i) The universe is assumed to be homogeneous and isotropic,

(ii) The universe is filled with a gas.

(iii) Particle in the universe effectively massless.

(iv) The temperature is at the state of equillibrium.

(v) The initial temperature is considered as T_o and the Hubble expansion parameter H at initial state is H_0.

So model of the universe can be practically expressed at the initial stage.

The first puzzle is the horizon problem[10,11] which was proposed by Rindler in 1956. The early universe is assumed to be homogeneous. But there are atleast $\sim 10^{83}$ different regions, which are causally disconnected.

The second problem is the flatness problem which was proposed by Dicke and Peebles.[12] The energy density

ρ of the present universe is $3\times10^{-31}h\ gm/cm^3$ and the critical density $\rho_{cr} = \frac{3H^2}{8\pi G}$ is $1.1\times10^{-29}h\ gm/cm^3$. So $\rho \lesssim \rho_{cr}$. Now one can safely assume that[13]

$$0.01 < \Omega_p < 10 \tag{1.1}$$

(subscript p denotes present)

where Ω is defined as

$$\Omega = \rho/\rho_{cr} = \left(\frac{8\pi}{3}\right) \frac{G\rho}{H^2} \tag{1.2}$$

The bounds for Ωp have very powerful implications. The crucial point is $\Omega \approx 1$ which is very much unstable.

To understand the problems of standard model in a precise manner, one can start, like Guth, with Robertson-Walker line-element. The reason for adapting this line element is that it describes the homogeneous and isotropic cosmological model successfully. The Robertson-Walker or Friedmann model[4] is given by the line element

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \text{Sin}^2\theta\, d\phi^2)\right] \tag{1.3}$$

where R(t) is a scale factor and k is the spatial curvature having three possible values +1,0,-1 for closed, flat as well as open models respectively.

The evaluation of R(t) is governed by the Einstein equations

$$\ddot{R} = -\frac{4\pi}{3}G\ (\rho + 3p)\ R \tag{1.4}$$

where p and ρ denotes the pressure and density of the cosmological fluid and dot(.) denotes the derivative w.r.t. time. The above relation (1.4) means that the expansion of the universe is getting deaccelarated due to the presence of matter in the universe. This result is not unexpected since gravitational forces are attractive. Also we have,

$$R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi G\ (\rho - p)R^2 \tag{1.5}$$

Eliminating $\ddot{R}$ between (1.4) and (1.5) we get,

$$H^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{1.6}$$

where $H = \frac{\dot{R}}{R}$ is the Hubble "constant".

Conservation equation is given by

$$\frac{d}{dR}\ (\rho R^3) = -3pR^2 \tag{1.7}$$

In the standard model of cosmology, one can assume that the system is to undergo adiabatic change, in which case

$$\frac{d}{dt}\ (sR^3) = 0 \tag{1.8}$$

where s is the entropy density.

Particles, in the early universe, are in a highly relativistic stage as early universe is supposed to be extremely hot. So matter content, in the early univers, has radiation like behaviour. As a result, it is appropriate to choose the equation of state

$$p = \frac{1}{3} \rho \tag{1.9}$$

Also in the early universe, spin of the particle is very important. On the basis of spin one can classify particles in two groups.

(1) Bosons (Particles with integral spin 0,1,2,.....)

(2) Fermions (Particles with half integral spin $\pm 1/2$, $\pm 3/2$, ...)

Let $N_b(t)$ denotes the number of bosonic spin degrees of freedom which are effectively massless at temperature T (e.g. the photon contributes two units to N_b), and $N_f(t)$ denotes the corresponding number for fermions (e.g. electrons and positrons together contribute four units).

The thermodynamic functions are given by -

$$\rho = 3p = \frac{\pi^2}{30} \Pi(T)\ T^4 \tag{1.10}$$

$$S = \frac{2\pi^2}{45} \Pi(T)\ T^3 \tag{1.11}$$

$$n = \frac{\zeta(3)}{\pi^2} \Pi'(T)\ T^3 \tag{1.12}$$

where $$\Pi(T) = N_b(T) + \frac{7}{8} N_f(T) \tag{1.13}$$

$$\Pi'(T) = N_b(T) + \frac{3}{4} N_f(T) \tag{1.14}$$

Here n denotes the particle number density and $\zeta(3) = 1.20206$----- is the Riemann Zeta function.

Rewriting equation (1.6) in terms of temperature T where T is not near any mass thresholds, the evoluation of the universe is given by

$$\left(\frac{\dot{T}}{T}\right)^2 + \varepsilon(T)\ T^2 = \frac{4\pi^3}{45} G\Pi(T)T^4 \tag{1.15}$$

where $$\varepsilon(T) = \frac{k}{R^2T^2}$$

$$= k\ \left[\frac{2\pi^2}{45} \quad \frac{\Pi(T)}{S} \right]^{2/3} \tag{1.16}$$

and $S=R^3s$ denotes the total entropy in volume specified by the radius of curvature R.

Since the expansion is assumed to be adiabatic, S is conserved, its value in the early universe is determined by the current observations. Therefore,

$$\rho/\rho_{cr} = (8\pi/3)\, G\rho/H^2$$

$$= (H^2 + \frac{k}{R^2})/H^2$$

Taking $\frac{\rho}{\rho_{cr}} < 10$ today, it follows that

$$(H^2 + \frac{k}{R^2})/H^2 < 10$$

i.e. $H^2 + \frac{k}{R^2} < 10H^2$

or $\frac{k}{R^2} < 9H^2$

$$\therefore \left| \frac{k}{R^2} \right| < 9H^2 \qquad (1.17)$$

Henceforth, the value of k will be considered as ±1, as $R \longrightarrow \infty$, k=0 is included.

The present age of the universe is $\sim 9\text{x}10^9$ years. So

$$H \sim t^{-1}$$

It means that

$$R > \frac{1}{3} H^{-1} \sim 3\text{x}10^9 \text{ years}$$

The photon contribution to S is

$$S_\gamma > 3 \times 10^{85} \qquad (1.18)$$

taking the present photon temperature T_γ as $T_\gamma = 2.7°K$

Assuming that there are three species of massless neutrinos ν_e, ν_μ, ν_τ corresponding to electron (e), muon(μ) and τ-lepton (i.e., e, μ, τ) decouple from the massless leptons and photons at a particular time. At the epoch of decoupling $S_\nu = \frac{21}{22} S_\gamma$

where S_ν = Entropy for neutrinos

S_γ = Entropy for photon.

Thus,

$$S = S_\gamma + S_\nu > 10^{86}$$

and $$|\varepsilon| < 10^{-58} \Pi^{2/3} \qquad (1.19)$$

But $$\left| \frac{\rho - \rho_{cr}}{\rho} \right| = \frac{45}{4\pi^3} \frac{M_P^2}{\Pi T^2} |\varepsilon|$$

$$< 3 \times 10^{-59} \Pi^{-1/3} (M_P/T)^2 \qquad (1.20)$$

Taking $T = 10^{17}$GeV and $\Pi \sim 10^2$ (for minimal grand unified theory),

$$\left| \frac{\rho - \rho_{cr}}{\rho} \right| < 10^{-55} \qquad (1.21)$$

This shows that $\rho \approx \rho_{cr}$. From (1.6) implies that if $\rho \approx \rho_{cr}$, $k \approx 0$. But one gets equation (1.20) taking $|k|=1$ and excluding the case $k=0$. Also if $k=0$, $\rho = \rho_{cr}$ is a stable fixed point, but it is difficult to believe that k could be made to vanish with an infinite occuracy unless there exists some symmetry principle. Thus the problem of understanding why our universe was created almost flat, with unbelievable accuracy given above, is called the flatness problem. This is equivalent to understanding why our universe is $\sim 10^{10}$ years old at present. Without ϵT^2 term in eqn.(1.15) can be integrated to

$$T^2 = \frac{M_P}{2\gamma t} \tag{1.22}$$

With the boundary condition $T \longrightarrow \infty$ as $t \longrightarrow 0$. Here M_P is the Planck mass and $\gamma^2 = \left(\frac{4\pi^3}{45}\right)\Pi$. For the minimal SU(5) grand unified model $N_b=82$, $N_f=90$ and $\gamma = 21.05$.

Conservation of entropy gives

$$RT = \text{Const.}$$

So $$R \sim t^{1/2}$$

So the physical distance travelled by a light pulse in time t is

$$l = t^{1/2} \int_0^t \frac{dt'}{t'^{1/2}} = 2t \tag{1.23}$$

which gives the physical horizon distance.

Again using conservation of entropy,

$$L(t) = (s_p/s(t))^{1/3} L_p \qquad (1.24)$$

where s_p is the present entropy density and $L_p \sim 10^{10}$ years is the radius of the currently observed region of the universe.

Now the ratio of the physical volume covered by light pulses in time t to the volume of the observed universe is given by

$$\frac{\ell^3}{L^3} = \frac{11}{43} \left(\frac{45}{4\pi^3} \right)^{3/2} \Pi^{-1/2} \left(\frac{M_p}{L_p T_\gamma T} \right)^3$$

$$= 4 \times 10^{-89} \, \Pi^{-1/2} (M_p/T)^3 \qquad (1.25)$$

For $\Pi \sim 10^2$ and $T_o = 10^{17}$GeV

We have

$$\frac{\ell_o^{\,3}}{L_o^{\,3}} = 10^{-83} \qquad (1.26)$$

(1.26) shows that at $T_o = 10^{17}$GeV, volume covered by light pulses is extremely small compared to the volume of the universe. It means that expansion of the universe is much faster than the speed of light. It leads to the implication that no two regions of the universe can communicate signals among them unless one has tachyonic fields. So every region in the universe was causally disconnected. This is the well known horizon problem.

The third problem is monopole problem. Magnetic monopoles are topologically stable knots which can be tied in the Higgs expectation value. When universe cools, due to its expansion, it becomes thermodynamically favourable for the Higg's field to align uniformly over large distances. However, it takes time for these correlations to be established. In the standard scenario, horizon length is equal to 2t according to eqn.(1.23). Causality demands that the Higg's field correlation length ξ must be less than 2t. Since monopoles are knots in the Higg's expectation value. Roughly their number density is given by

$$n_M \approx \xi^{-3} > \frac{1}{8t^3} \tag{1.27}$$

where t is the time at which universe cools to the critical temperature T_c. So from eqns.(1.11), (1.22) and (1.27)

$$\frac{n_M}{s} > O\,(T_c/M_p)^{1/2} \tag{1.28}$$

(1.28) is independent of time.

The entropy is conserved in the model under consideration, so the ratio n_M/s should be the same today. But we do not observe monopoles at present epoch. Also Preskill [15] has shown that monopole-antimonopole annihilation is ineffective. So to answer the monopole puzzle, we can not think that monopole and anti-monopoles might have annihilated each other in the early universe. Thus we do not have any answer for this problem in the standard model.

1.3 Phase Transitions In Field Theories[1]

Most of cosmological applications of field theories are based upon the theory of phase transitions.[15] In particular, the concept of spontaneous symmetry breaking in gauge theories[16] and symmetry restoration[15] at high temperature plays a fundamental role. In this section the notion of phase transition with few simple examples is illustrated.

The main point here is that at finite temperatures the equillibrium value of the scalar field ϕ, $\phi(T)$, does not correspond to the minimum of the effective potential $V_{eff}^{T=o}(\bar{\phi})$, but to the minimum of the finite-temperature effective potential $V_{eff}^{\beta}(\bar{\phi})$ where $\bar{\phi}$ is the constant field, and $\beta = \frac{1}{T}$ in units. Thus even if the minimum of $V_{eff}^{T=o}(\bar{\phi})$

occurs at $\bar{\phi} = \sigma \neq o$, very often, for sufficiently large temperatures, the minimum of $V^{\beta}_{eff}(\bar{\phi})$ is at $\bar{\phi}=o$, this phenomenon is known as symmetry restoration at high temperature. This gives rise to the phase transition from $\bar{\phi}=o$ to $\bar{\phi} = \sigma$.

The cosmological scenario can be drawn as follows. According to hot big-bang theory, the universe is initially at very high temperature and depending on the function $V^{\beta}_{eff}(\bar{\phi})$, can be in the symmetric phase $\bar{\phi}=o$ i.e. $\bar{\phi}=o$ can be stable absolute minimum. At some critical temperature T_c the minimum at $\bar{\phi}=o$ becomes metastable and the phase transition may proceed. The phase transition may be of first or second order. When the temperature decreases, the first order phase transitions have supercooled symmetric states $\bar{\phi}=o$ and are of use for inflationary universe. Here we are illustrating these points with a simple example.

1.3(A) A Toy Model

Considering a real scalar field ϕ with the Lagrangian

$$L = \frac{1}{2} (\partial_{\mu}\phi)^2 - V \qquad (1.29)$$

where

$$V(\phi) = -\frac{1}{2} \mu^2\phi^2 + \frac{\lambda}{4} \phi^4 \qquad (1.30)$$

and its solutions to $\frac{dv}{d\phi} = o$, given by $\phi=o$ and $\phi_o = \pm \frac{\mu}{\sqrt{\lambda}}$

At $\phi = o$, the potential is unstable and the energetically favoured state corresponds to the minimum of $V(\phi)$, at $\phi_o = \pm \frac{\mu}{\sqrt{\lambda}}$.

The symmetry $\phi \longleftrightarrow -\phi$ of the original theory is spontaneously broken. The temperature dependent effective potential can be written as

$$V(\phi,T) = -[\frac{\pi^2}{90} T^2 + \frac{\mu^2}{24}]T^2 + \frac{1}{2} [\frac{\lambda}{4} T^2 - \mu^2]\phi^2 + \frac{\lambda}{4} \phi^4 \qquad (1.31)$$

where higher powers in the zero-temperature effective potential are neglected. The curvature of the potential is now T-dependent and

$$m^2(\phi,T) = 3\lambda\phi^2 + \frac{\lambda}{4} T^2 - \mu^2 \qquad (1.32)$$

Its solution to $\frac{dV(\phi,T)}{d\phi} = o$ is given by

$$\phi = o \text{ and } \phi_o(T) = \pm \frac{1}{2} [\frac{4\mu^2}{\lambda} - T^2]^{1/2} \qquad (1.33)$$

Therefore the critical temperature is given by

$$T_c = \frac{2\mu}{\sqrt{\lambda}} = 2\phi_o \qquad (1.34)$$

At $T > T_c$, $m^2(O,T) > o$ and the origin $\phi = o$ is minimum. Only

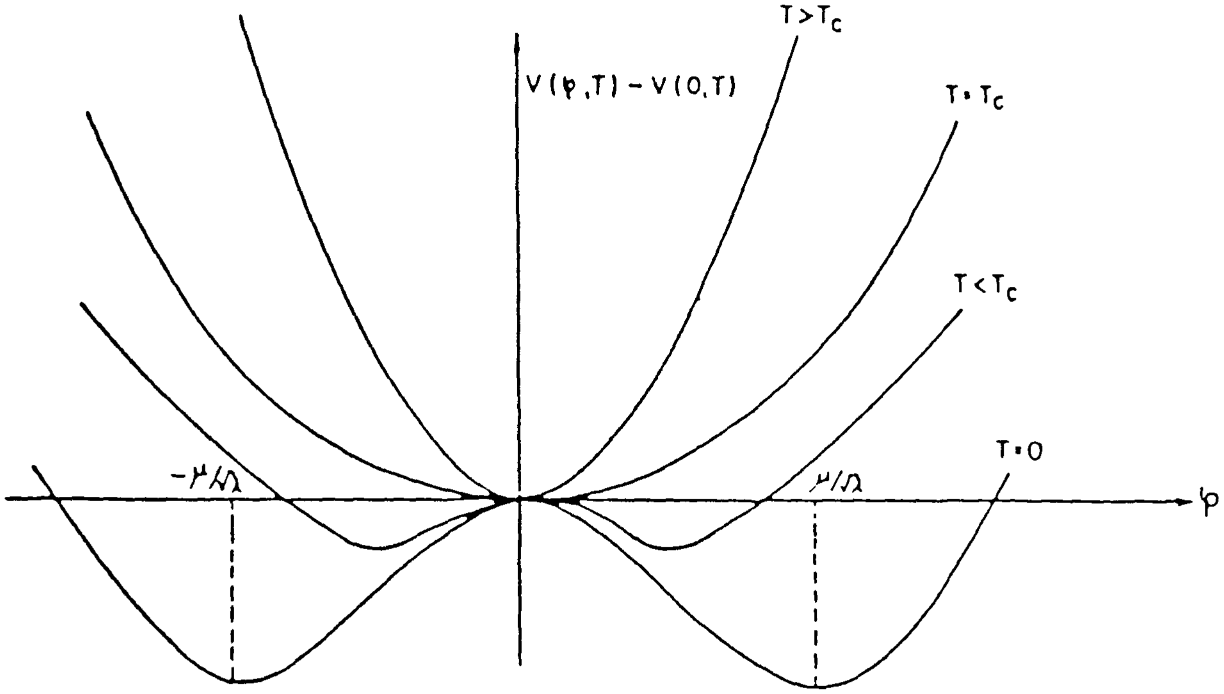

Figure 1·1 Second-order phase transition. The potencial is normalized at φ=0 for all values of T.

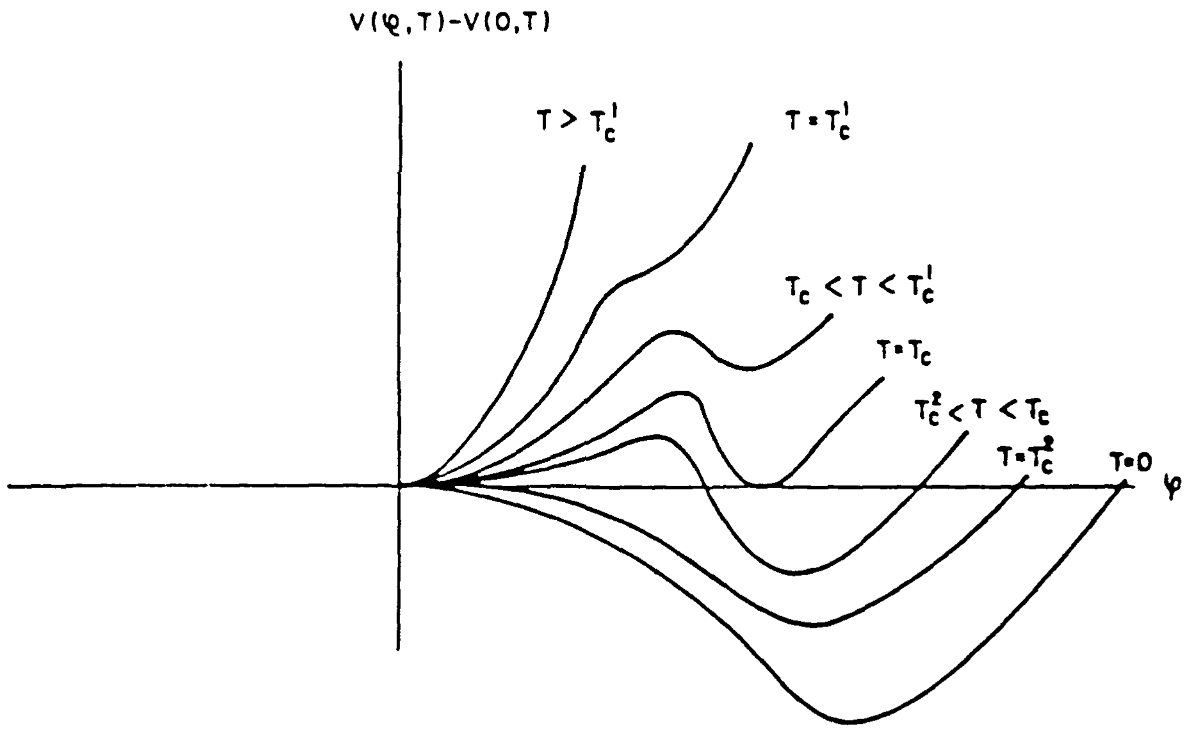

Figure 1·2 A typical first-order phase transition. The normalization of the potential at φ=0 is as in Fig.

the solution ϕ=o exists in (1.33).

At $T=T_c$, $m^2(O,T_c)$=o and at ϕ=o both solutions in (1.33) collapse. The potential (1.31) becomes

$$V(\phi,T_c) = -[\frac{\pi^2}{90} T_c^2 + \frac{\mu^2}{24}]T_c^2 + \frac{\lambda}{4} \phi^4 \quad (1.35)$$

At $T < T_c$, $m^2(O,T_c) < o$ and the origin becomes a maximum. Simultaneously, the solution $\phi_o(T)$ does appear in (1.33). Since there is no barrier between the symmetric and broken phases, this phase transition is called of second order phase transition. When the broken phase is formed, the origin (symmetric phase) becomes a maximum. A typical potential corresponding to a second order phase transition is illustrated in fig.1.1

The phase transition may be achieved by a thermal fluctuation for a field located at the origin.

However in many interesting theories there is a barrier between the symmetric and broken phases. This is characteristic of First-Order phase transitions. A typical example is shown in the fig1.2.

At $T > T_c^1$, the only minimum is at ϕ=o. AT $T=T_c^1$, a local minimum $\phi_o(T)$ appears at $\phi_o \neq$ o with a barrier in between. At $T=T_c$ both minima are degenerate and for $T < T_c$ the minimum at $\phi=o$ becomes metastable and the minimum at $\phi_o \neq o$ becomes the global one. At $T=T_c^*$ the barrier disappears and the origin becomes a maximum. T_c is defined by

$$V(o,T_c) = V_o(\phi_o\,(T_c),\ T_c) \tag{1.36}$$

The phase transition starts at $T=T_c$ by tunnelling. However, if the barrier is high enough the tunnelling effect is very small and the phase transition does effectively starts at a temperature $T_c^2 \lesssim T << T_c$. This corresponds to a supercooled phase transition. In some models T_c^2 can be equal to zero.

1.4 Summary of the work included in other chapters and units

Chapter II contains discussion on Guth's model of inflation. But Guth's model faces two serious problems (1) problem of 'graceful exit' and (2) problem of inhomogeneity in space curved by nucleation of many bubbles. These two problems are evaded in New Inflationary scenario. In this chapter these two models are discussed at length. Later on, a question was raised by Ellis et al. "Is it possible to get inflationary scenario prior to GUT's phase transition as proposed by Guth, Linde and others?" In chpater III possibility of primordial inflation (inflation before GUT's phase transition) has been discussed.

In chapter IV, it is discussed that possibility of getting inflation exists in certain other situations for example if we take exponential potential for scalar fields and connect the sxcalar fields with Einstein's dynamical equation for the early universe. In chapter V, possibility of inflation, driven by bulk viscosity of matter is discussed.

Natural units $8\pi G=\hbar=C=k=1$ are used throughout the entire thesis where G is the Newtonian gravitational constant, C is the speed of light, $\hbar = \frac{h}{2\pi}$ (h is Planck's constand) and k is Boltzmann's constant. $G = \frac{1}{M_p^2}$ where M_p is Planck mass. In these units $1m = 5.068 \times 10^{15} GeV^{-1}$, $1kg = 5.610 \times 10^{26} GeV$, $1Sec = 1.519 \times 10^{24} GeV^{-1}$, and $1°K = 8.617 \times 10^{-14} GeV$.

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CHAPTER II

GUTH's MODEL OF INFLATION AND NEW INFLATIONARY SCENARIO

2.1 Motivation for Inflationary Scenario

In the following paragraphs a scenario is illustrated which can possibly avoid the three cosmological problem of the standard scenario that were discussed in the previous chapter.

In the earlier chapter, it is seen that both the problems could disappear if the assumption of adiabaticity were grossly incorrect. Suppose instead that

$$S_p = Z^3 S_o \tag{2.1}$$

where S_p and S_o denotes the present and initial values of R^3s and Z is some large factor.

Using (2.1), the R.H.S. of (1.20) is multiplied by a factor of Z^2. The 'initial' value (as $T_o = 10^{17}$GeV) $|\rho - \rho_{cr}|/\rho \approx 1$ and flatness problem will be solved

$$\text{if } 10^{-55}\, Z^2 = 1$$

$$\text{i.e. } Z > 3 \times 10^{28} \tag{2.2}$$

Now one can consider the horizon problem. Again the R.H.S. of (1.24) is multiplied by Z^{-1} which means that at any given temperature, length scale of the early universe was smaller by a factor Z. Now if Z is sufficiently larger than the region which evolves to become our observed universe has a size of order 10cm. To see how large Z must be, R.H.S. of (1.25) is multiplied by Z^3. So if T is rescaled to T/Z then

$$10^{-83} \times Z^3 = 1$$

$$\text{i.e. } Z > 5 \times 10^{27} \tag{2.3}$$

Due to the expansion of the early universe, there is a rapid fall of temperature. So as the universe cools through the temperature T_c (critical temperature), the phase transition will begin to occur through the spontaneous nucleation of bubbles of the new phase. It is consider that the nucleation rate for this phase is rather low. As the universe expands, it will continue to cool and then super-cool in the high temperature phase. Let us suppose that T_s be the magnitude of supercooling temperature is below T_c. When phase transition finally takes place at T_s, latent heat is released. The universe is reheated by the latent heat released upto temperature T_γ which is compared to T_c. As a result entropy density is increased by $\left(\frac{T_\gamma}{T_s}\right)^3$ while R remains unchanged. Thus,

$$Z \approx \frac{T_\gamma}{T_s} \tag{2.4}$$

So if the universe supercooled to temperature 28 or more order of magnitude below T_c for some phase transition, these problem will disappear.

However to work in this scenario, the universe must be lagging of any strictly conserved quantities. Let n denote the density of some strictly conserved quantity and $\gamma = n/s$ denote the ratio of this conserved quantity to entropy. So

$$\gamma = \frac{n}{s} \quad \alpha \; \frac{n}{T^3}$$

When T is rescaled to T/Z

$$\text{then} \quad \frac{n}{T} \longrightarrow \frac{Z^3 n}{T^3}$$

Now

$$\frac{\gamma_p}{\gamma_o} = n/S_p \Big/ n/S_o = \frac{S_o}{S_p} = \left(\frac{T_o}{T_p}\right)^3$$

$$\longrightarrow \left(\frac{T_o/Z}{T_p}\right)^3 = Z^{-3}$$

$$\gamma_p = Z^{-3}\gamma_o < 10^{-84}\gamma_o$$

Thus, only an absurdly large value for the initial ratio would lead to a measurable value for the present ratio. Thus, if baryon number were exactly conserved, the inflationary model would be untenable. However, in the context

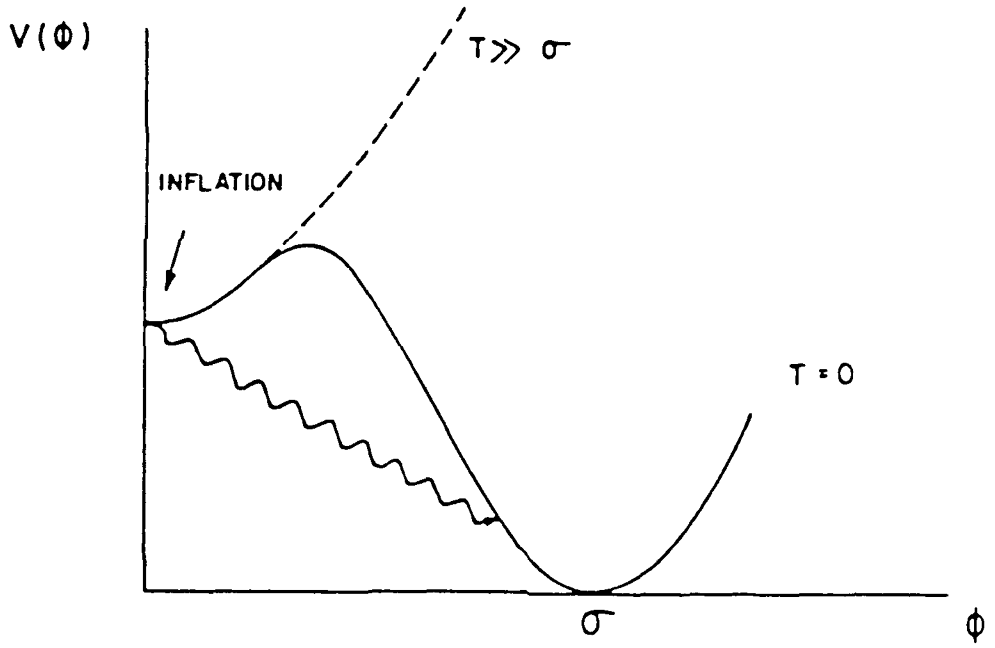

Figure 2.1 Shape of inflationary potential in Guth's original scenario. The wavy line indicates the transition by quantum tunnelling.

of grand unified models, baryon number is not exactly conserved. The net baryon number of the universe is believed to be created by CP-violating interactions at a temperature of 10^{13}-10^{14}GeV.[1] Thus, provided that T_c lies in this range or higher, there is no problem. The baryon production would take place after the reheating.

2.2 Guth's Scenario of Inflationary Model

Alan Guth made a brilliant contribution to cosmology by suggesting that the universe went through a phase of exponentially rapid expansion.

The inflationary scenario requires a Higgs field effective potential $V(\phi)$. At zero temperature, the global minimum of the effective potential is called the true vacuum with $\phi = \phi_{true}$ and at zero temperature, the local minimum of the effective potential is called the false vacuum with $\phi = \phi_{false}$; the energy of this will be called ρ_o.

As $T \longrightarrow o$, the system is cooling not towards the true vacuum but rather towards some metastable false vacuum with an energy density ρ_o which is necessarily higher than that of the true vacuum. So the total energy is increased by

$$\rho(T) = \frac{\pi^2}{30} \Pi\ (T)\ T^4 + \rho_o \qquad (2.5)$$

Here we are concerned about the zero point of energy and since classical general relativity couples to an energy-momentum tensor $T_{\mu\nu}$, which obeys a covariant conservation equation, reduces to

$$\frac{d}{dt}\,(R^3\rho) = -p\,\frac{d}{dt}\,(R^3) \tag{2.6}$$

where p is the pressure and ρ is the energy density.

When matter is described by a field theory,

$$T_{\mu\nu} \longrightarrow T_{\mu\nu} + \lambda g_{\mu\nu} \tag{2.7}$$

where λ is a constant, not depending on fields, temperature and phase. Using the above transformation for $T_{\mu\nu}$ is equivalent to introduce the cosmological constants in Einstein's field equations. The cosmological constant Λ is defined by

$$< o\,|T_{\mu\nu}|\,o > \;=\; \Lambda g_{\mu\nu} \tag{2.8}$$

where $|0>$ denotes the true vacuum. Λ is identified as the energy density of the vacuum and, in principle, there is no reason for it to vanish. Empirically Λ is very small ($|\Lambda| < 10^{-46}\,GeV^4$) so Guth has taken its value to be zero. The value of ρ_o is necessarily positive which is determined by the particle theory[2], and it is found that $\rho_o \sim O(T_c^4)$ Using (2.5) in eqn.(1.15), we have

$$\left(\frac{\dot{T}}{T} \right)^2 = \frac{4\pi^3}{45} G \Pi(T)\, T^4 - \varepsilon(T) T^2 + \frac{8\pi}{3} G\rho_o \tag{2.9}$$

When the temperature T is low enough for ρ_o term to dominate over other two terms on R.H.S. of (2.9), solution is

$$T(t) \approx \text{const}\cdot e^{-\chi t} \tag{2.10}$$

$$\text{where} \quad \chi^2 = \frac{8\pi}{3} G\rho_o$$

Since RT = const, one has

$$R(t) = \text{const} \times e^{\chi t}$$

The universe is expanding exponentially in a flase vacuum state of energy density ρ_o. The Hubble constant is given by $H = \dot{R}/R = \chi$

Since false vacuum state is completely Lorentz invariant, the energy momentum tensor must have the form $T_{\mu\nu} = \rho_o\, g_{\mu\nu}$.

Hence $p = -\rho_o$, the false vacuum has a large negative pressure. From (2.6), it can be seen that the negative pressure is also the driving force behind the exponential expansion.

One can consider the process of bubble formation in a Robertson-Walker Universe. The bubbles form randomly, so there is a certain nucleation rate $\lambda(t)$ which is the probability per (physical) volume per time that a bubble will form in any region which is in the high-temperature phase. One assume that bubbles start from a point and expand at the speed of light. When the energy is released by the conservation of false vacuum to true, it is transferred to the accelerating bubble wall. The interior of the bubble wall approaches the true vacuum. Since the bubble wall expand at a speed of light, so the bubble formation does not effect the metric out-side the bubbles which results that the region outside is causally disconnected. Hence the outside region is given by flat Robertson Walker metric

$$ds^2 = -dt^2 + R^2(t)\, d\bar{x}^2 \text{ with } R(t) = e^{\chi t}$$

and $$(d\bar{x})^2 = dx^2 + dy^2 + dz^2.$$

For simplicity, it is assumed that the bubble nucleate with zero size and then grow precisely at the speed of light. Now question arises whether these bubbles will fill up the space? To answer this, one needs to calculate P(t), the probability that any given point remains in the high temperature phase at time t. Guth gives the simplification in the following way.

He ignored the fact that bubbles can not form inside of bubbles and assumed that bubbles nucleate with constant rate λ per physical volume at all points in space. This simplification causes no effect on P(t). The distribution of all bubbles, real and fictitious, is then totally uncorrelated.

P(t) is the probability that there are no bubbles which engulf a given point in space. But the number of bubbles which engulf a given point is $P(t)=\exp[-\bar{N}(t)]$ where $\bar{N}(t)$ is the expectation value of the number of bubbles engulfing the point. This leads to[3]

$$P(t) = \exp\left[-\int_0^{t_1} dt_1 \lambda(t_1) R^3(t_1) V(t,t_1)\right] \tag{2.11}$$

where
$$V(t,t_1) = \frac{4\pi}{3}\left[\int_{t_1}^{t} \frac{dt_2}{R(t_2)}\right]^3 \tag{2.12}$$

is the coordinate volume V at time t of a bubble which formed at time t_1.

Now if the nucleation rate $\lambda(t)$ is sufficiently slow, then no significant nucleation takes place until $T \ll T_c$, when exponential growth has set in. It is assumed further that $\lambda(t)$ becomes λ_o (the zero temperature nucleation rate). Thus

$$P(t) = \exp\left[-\frac{t}{\tau} + O(1)\right]$$

where

$$\tau = \frac{3\chi^3}{4\pi\lambda_o} \qquad (2.14)$$

as $\chi t \longrightarrow \infty$, O(1) refers to terms which approach a constant. So during these time constants, the universe will expand by a factor

$$Z_r = \exp(\chi T) = \exp\left(\frac{3\chi^4}{4\pi\lambda_o}\right) \qquad (2.15)$$

If the phase transition is associated with the expectation value of a Higgs field, then according to Coleman-Callan[4]

$$\lambda_o = A\rho_o \exp(-B) \qquad (2.16)$$

Where B is a barrier penetration term and A is a dimensionless coefficient of order unity. Since Z_r is then an exponential of exponential, one can very easily[5,6] obtain values as large as $\log_{10} Z \approx 28$ or even $\log_{10} Z \approx 10^{10}$.

Thus if the universe reaches a state of exponential growth, it is quite plausible for it to expand and supercool by a huge number of orders of magnitude before a significant fraction of the universe undergoes the phase transition.

2.3 Problems of the Guths model of Inflationary Universe

In the earlier section we have seen that the inflationary scenario model seems to lead some unacceptable consequences. Here we will see what actually happens when a slow first order phase transition occurs in an exponentially expanding model.

In 1977, Coleman[4] and Callan and Coleman analyzed the process of bubble nucleation and its growth in a flase vacuum state. This description has been extended to a curved space time in 1980 by Coleman and De-Luccia[7].

The central problem is the difficulty in finding a smooth ending to the period of exponential expansion. As $t \longrightarrow \infty$, $T \longrightarrow o$. Let us assume that $\lambda(t) \longrightarrow$ constant. To get $Z > 10^{28}$, one needs $\lambda_o/\chi^4 < 10^{-2}$ (as in 2.15) which means that the nucleation rate is small compared to the expansion rate of the universe. So the randomness of the bubble formation process leads to gross inhomogeneities.

The following facts will come in connection with the effects of above randomness.

(i) All of the latent heat released as a bubble expands is transferred to the walls of the bubble[4] initially. So when the bubble walls undergo many collisions, this energy can be thermalized.

(ii) The de-Sitter metic does not single out a comoving frame. The O(4,1) invariance of the de-Sitter metric is maintained even after the formation of one bubble. The memory of the original Robertson-Walker comoving frame is maintained by the probability distribution of bubbles, but the local co-moving frame can be reestablished only after enough bubbles have collided.

(iii) The size of the largest bubbles will exceed that of the smallest bubbles by a factor of Z; so the range of the bubble size is immense. Since the surface energy density grows with the size of the bubble so the energy in the walls of the largest bubbles can be thermalized only by colliding with other large bubbles.

(iv) As times goes on, an arbitrarily large fraction of the space will be in the new phase as in (2.13). Now a natural question is that "Is the region composed of finite separated clusters, or do these clusters join together to form an infinite region?" The possibility of joining these clusters to form an infinite region, is known as 'percolation'. One can observe that the system percolates for large values of λ_o/χ^4, but not for sufficiently small values. The critical values of

λ_o/χ^4 has not been determined, but presumably an inflationary universe would have a value of λ_o/χ^4 below critical. Thus the region of space in the new phase will consist of finite clusters, each totally surrounded by a region in the old phase independent of time.

(v) Each cluster will contain only a few of the largest bubbles. Thus, the collisions discussed in (iii) can not occur.

The above statements do not quite prove that the scenario is impossible, but these consequences are at best very unattractive. Thus, it seems that the scenario will become viable only if some modification can be found which avoids these inhomogeneities.

2.4 The New Inflationary Universe Scenario

In 1982, Linde[8,9] and Albrecht and Steinhardt[10] suggested a modified scenario known as "The new inflationary universe" which was based on the theory of phase transition, in GUTs with the Coleman-Weinberg mechanism of symmetry breaking. This scenario provides a possible solution of the horizon flatness, homogeneity, isotropy and primordial monopole problem and may help us to solve the baryon assymmetry problem.

The key feature of this scenario is that under special circumstances it is possible for the observed universe to have evolved from a single bubble which is not possible for a generic Higgs potential. But[11] if all the mass parameters in the Higgs potential are of the same order, then the total entropy produced within the bubble is always of order one. This problem is evaded in the new inflationary scenario by the use of Coleman-Weinberg (1973)[12] potential for the Higgs field which states that the second derivative of the effective potential is required to vanish at the origin.

For definiteness one can consider the SU(5) grand unified theory.

The Coleman-Weinberg potential for the SU(5) Higgs field, including the one-loop gauge field quantum corrections is given by

$$V(\phi) = \frac{25g^4}{128\pi^2}\left(\phi^4 \ln \frac{\phi}{\phi_o} - \frac{\phi^4}{4} + \frac{\phi_o^4}{4}\right) \qquad (2.17)$$

where $\phi = \phi \sqrt{\frac{2}{15}}$ diag $(1,1,1, -3/2, -3/2)$ (2.18)

and g is the SU(5) gauge coupling constant, $\phi_o \approx 1.4\times10^{15}$GeV.

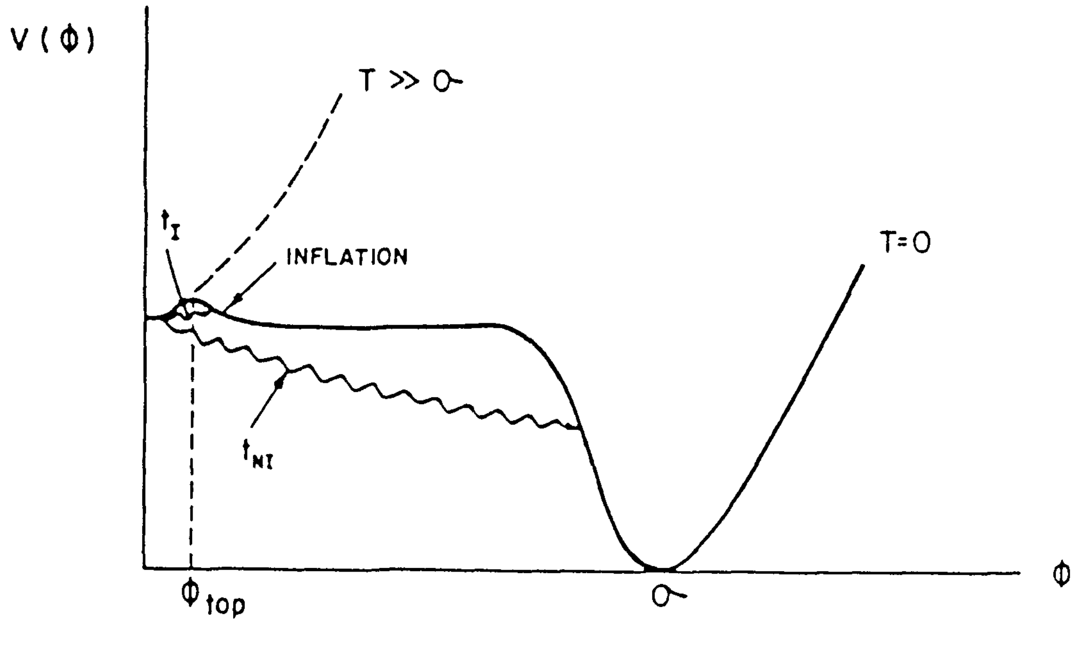

Figure 2·2 Shape of inflationary potential in the new inflationary scenario. Wavy lines indicate possible paths of quantum tunnelling.

The one-loop effective potential for the symmetry breaking SU(5) $\longrightarrow$ SU(3)XSU(2)XU(1) in the Coleman-Weinberg version of this model at finite temperature T is[13,14,15]

$$V(\phi,T)=(18T^4/\pi^2)\int_0^{\infty} dx\, x^2 \ln\{1-\exp[-(x^2+\frac{5g^2\phi^2}{12T})^{\frac{1}{2}}\}$$

$$+\frac{25g^4}{128\pi^2}\phi^4(\ln\frac{\phi}{\phi_o}-\frac{1}{4})+\frac{9}{32\pi^2}M_x^4 \qquad (2.19)$$

where $M_x = \sqrt{\frac{5}{3}}\ \frac{g\phi_o}{2} \approx 5\times10^{14}$GeV,

$g^2 \approx 0.3$, $\phi_o \sim 1.4 \times 10^{15}$GeV. At $T \gg \phi_o$, the only minimum of $V(\varphi)$ is that at $\phi = o$, and the symmetry is restored. With a decrease of temperature the absolute minimum of $V(\phi)$ appears at $\phi \approx \phi_o$. However according to (2.19), at any $T \neq o$ the point $\phi = o$ remains a local minimum of $V(\phi)$.

$$m^2(T) = \frac{d^2V}{d\phi^2}\Big|_{\phi=o} = \frac{5}{4}\ g^2T^2 \qquad (2.20)$$

The scalar field contribution to $V(\phi,T)$, which changes the curvature of $V(\phi,T)$ near ϕ=o is

$$\frac{d^2V}{d\phi^2}\Big|_{\phi=o} = (\frac{5}{4}\ g^2 + \frac{65a + 47b}{30})\ T^2 \qquad (2.21)$$

In 1981, Linde pointed out that at a finite temperature, the bubble of the field ϕ at the moment of its formation has a size $O(T_c^{-1})$ and the field inside the bubble $\phi \ll \phi_o$,

$$\phi \ll \frac{12\pi T_c}{g}\left(5 \ln \frac{M_X}{T_c}\right)^{-\frac{1}{2}} \ll \phi_o \qquad (2.22)$$

This means that the mass squared of the field ϕ inside the bubble is

$$m^2 = \frac{d^2V}{d\phi^2} \lesssim 75\, g^2 T_c^{\,2} \sim 25\, T_c^2 \qquad (2.23)$$

After the bubble formation the field ϕ inside the bubble gradually grows up to its equillibrium value $\phi(T_1) \sim \phi_o$. At the first stages of this process the field ϕ grows approximately as e^{mt}. So it approaches its equillibrium value $\phi(T)$ only after some period of time $\tau > m^{-1} \gtrsim 0.2\, T_c^{-1}$. Though τ is several times greater than m^{-1}; here for simplicity one can take as an estimate

$$\tau \sim T_c^{-1} \qquad (2.24)$$

During most of this period the field ϕ inside the bubble remains much less than ϕ_o. Therefore during some time of the order of $\tau \sim T_c^{-1}$ the vacuum energy density $V(\phi)$ remains almost equal to $V(o)$, and the part of the universe inside the bubble expands exponentially just as it expanded before the bubble creation. This simple observation has very important consequences for the theory of the phase transitions in the Coleman-Weinberg model.

One can assume that the phase transition in the Coleman-Weinberg SU(5) theory occurs at $T_c \sim 2X10^6$GeV. The value of the Hubble constant H in this theory is given by

$$H = \sqrt{(8\pi/3M_p^{\,2})V(o)}$$
$$= \frac{M_x^{\,2}}{2M_p} \sqrt{\frac{3}{\pi}} \approx 10^{10}\text{GeV} \qquad (2.25)$$

Therefore, during the exponential expansion period $\tau \sim T_c^{-1}$, the universe should grow $e^{H\tau}$ times where

$$e^{H\tau} \sim e^{H/T_c} \sim e^{7500} \sim 10^{3260} \qquad (2.26)$$

A typical size of the bubble at the moment of its creation is $O(T_c^{-1}) \sim 10^{-20}$cm. After the period of the exponential expansion this bubble will have a size of

$$10^{-20}\ e^{H\tau}\text{cm} \sim 10^{3240}\text{cm}.$$

which is much greater than the size of the observable part of the universe is contained inside one bubble, so there is no inhomogeneities caused by the wall collisions. After some time the order of τ after the bubble creation all the vacuum energy density V(o) transforms into thermal energy $\sim T_1^{\,4}$ where in this model $T_1 \approx 0.15\,M_X$ $\sim 10^{14}$GeV. However, thermalization occurs not due to the wall collisions, but due to the interactions of particle created by the classical homogeneous field ϕ, convergently oscillating near its equillibrium value $\phi(T_1) \approx \phi_0$ with a frequency of about 10^{14}GeV.

The size of the particle horizon at the time of the phase transition was much greater than the bubble $\sim T_c^{-1}$ i.e. all points inside the bubble were causally connected. After the exponential expansion period this causally connected domain covers the whole observable part of the universe, which solves the horizon problem.

From the results it follows that the size of the universe ℓ_1 after the phase transition should exceed 10^{3240}cm and the temperature T_1 is of the order of 10^{14}GeV. Therefore the total entropy of the universe should exceed $(\ell_1 t_1)^3 \sim 10^{10000}$, which explains why the total entropy of the universe exceeds 10^{85} and simultaneously solves the flatness problems.

As it is known that particle creation in the early universe in general can not make the universe completely isotropic but it make quasi-isotropic[16], (i.e. locally isotropic in small domains of space of the size of the same order as or greater than the Planck length $l_p \sim 10^{-33}\text{cm} \sim M_p^{-1}$ at the Planck time $t_p \sim 10^{-43}\text{s}$, when the temperature was $T_p \sim M_p \sim 10^{19}\text{GeV}$.). Since before the phase transition the quantity a(t)T was constant inside each isotropic domain of the universe, at the moment of the phase transition the typical size of the isotropic domain exceeds the bubble size $\sim T_c^{-1}$. So the space time inside the bubble was isotropic and the exponential expansion extends this isotropy to the whole observable part of the universe. The remaining small amisotropy inside the bubble decreases rapidly during the exponential expansion period[17]. This may solve the long-standing problem of the space-time isotropy in our universe.

One of the most important problems of GUTs is the problem of primodial monopoles which are copiously produced during the phase transitions in GUTs. However in this scenario no monopole problem arises. It is known that primordial monopoles in GUTs are created only in the points, in which bubbles with different types of Higgs field ϕ collide[18]. So in this scenario no monopoles are created in the observable part of the universe which solves the primordial monopole problem in GUTs For the same reason there will be no domain walls in the theorites with broken discrete symmetries and in particular in the theories with spontaneously broken CP invariance. This helps to solve the problem of the baryon asymmetry of the universe.

Density fluctuations inside the bubble immediately after its formation are negligibly small as compared with V(O) i.e. the space inside the bubble is almost homogeneous. Then the exponential expansion extends this homogeneity to the whole observable part of the universe, which explains the large scale homogeneity of the universe.

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CHAPTER - III

PRIMORDIAL INFLATION

In the previous chapter, it is seen that when GUT's break around 10^{14}GeV, the early universe undergoes inflation which resolves certain problems of the standard big-bang cosmology. Now it is natural to ask "Does there exist any possibility of getting inflationary scenario prior to GUT's phase transition?" In 1982 Ellis, Nanopoulos, Olive and Tamvakis (ENOT)[1] suggested that this type of possibility also exists if one considers supersymmetric GUT's. Later on, same authors (ENOT) developed this idea which is given below. The use of super-symmetric GUT's is based on two reasons (1) supersymmetry frees from radiative correction[2] calculation and (2) primordial supersymmetry breaking scale M_S must be considerably smaller than the GUT scale, because if M_S is much larger than $O(10^{10})$GeV, then supergravity corrections to scalar boson masses will be larger than 100 GeV which is not acceptable[3].

3.1 Motivations

Assuming that the early universe can be described by a homogeneous Robertson-Walker Friedmann metric of the form

$$ds^2=dt^2-a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right] \tag{3.1}$$

where a(t) is the Robertson-Walker scale factor and k=0, ±1 is a measure of curvature. The expansion of the universe is described by the scale factor a(t) which is governed by the Einstein field equation

$$H^2\equiv\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi\rho}{3m_p^2}-\frac{k}{a^2}+\frac{1}{3}\Lambda \tag{3.2}$$

$$\frac{\dot{\rho}}{\rho}+3(1+p/\rho)H=0 \tag{3.3}$$

Where H is the Hubble parameter, ρ is the mass energy density, p is the isotropic pressure, Λ is the cosmological constant and $m_p=G_N^{-\frac{1}{2}}$ is the Planck mass.

When SU(5) is broken down to SU(3)XSU(2)XU(1) the vacuum energy density of the universe changes by an amount $\delta\alpha\mu^4$ where $\mu=\langle 0|\phi|0\rangle$ is the vacuum expectation value (v.e.v). This constant energy density acts like a cosmological constants and must be cancelled at early time so that today $\Lambda\approx 0$. Thus by taking an initial value for cosmological constant Λ in eqn.(3.2) as

$$\left(\frac{m_p^2}{8\pi}\right)\Lambda=\delta \tag{3.4}$$

Assuming that the universe was radiation dominated at very early times so that eqns.(3.2) and (3.4) gives

$$\rho = CT^4 \qquad (3.5)$$

where $C = \frac{\pi^2}{30} (N_B(T) + \frac{7}{8} N_F(T))$

Here N_B and N_F denotes the number of bosonic and fermionic degrees of freedom and

$$a \propto T^{-1} \propto t^{\frac{1}{2}} \qquad (3.6)$$

where as if $\rho \lesssim \frac{\Lambda m_p^2}{8\pi} = \delta$ (3.6a)

where the temperature of the universe drops to a value such that $\rho \lesssim \Lambda$, the dynamics of the expansion are governed by Λ and one gets from eqn.(3.2) that the universe evolves towards an expansion[4]

$$a \; \alpha \; e^{Ht} \qquad (3.7)$$

where

$$H = (\frac{1}{3} \Lambda)^{\frac{1}{2}} = (\frac{8\pi\delta}{3mp^2})^{\frac{1}{2}} \qquad (3.8)$$

Characterizing the de-Sitter phase and an exponentially expanding universe.

In order to explain the cosmological observations of large-scale isotropy and homogeneity it is required that the scale factor 'a' be inflated by a factor $O(10^{28})$. So one could expect that the expansion factor $e^{H\tau} > O(10^{28})$

i.e.

$$\tau > 65/H \tag{3.9}$$

could neatly solve problems of standard cosmology mentioned in Chapter I. The near flatness of the present universe $[k/a^2 \lesssim O(10) \times (\frac{8}{3}\pi)(\rho/m_P{}^2)]$ despite its great age $[t_o \approx O(10^{61}) x\ t_p]$ is explained by the early exponential growth (3.7) of the scale factor 'a'. However, inflation during the GUT epoch is not a panacea. Since density perturbations $\delta\rho/\rho$ may be O(1) when they cross the horizon[5] so large-scale homogeneity is not assured. It is not[4] clear that the cosmological abundance of grand unified monopoles would be sufficiently suppressed, since the exponential expansion takes place in a GUT-invariant phase, and in the original inflationary scenario there was no way of correlating the directions in which the GUT Higgs fields ϕ would eventually point after spontaneous symmetry breakdown in different parts of the universe. Also, for the vacuum energy to dominate the expansion it is necessary that neither the curvature, nor any possible additional anisotropic shear or rotation terms in eqn.(3.2) be dominate when the mass energy density ρ falls below δ. So considering the dimensionless measure of curvature $\hat{K}$ given by

$$\hat{K} \equiv \frac{K}{a^2 T^2} \tag{3.10}$$

which is of $\sim O(1)$ at the Planck epoch. Taking

$$\frac{8\pi\rho}{3m_p^2} > \frac{K}{a^2} = \hat{K}\, T^2 \tag{3.11}$$

at the onset of the de-Sitter phase and using eqns.(3.5) and (3.6a) it is needed

$$\hat{K} < \left(\frac{1}{30} N \pi^2\right)^{\frac{1}{2}} \frac{8}{3}\pi \left(\frac{\mu^2}{m_p^2}\right) \tag{3.12}$$

which is redolent of some degree of primordial fine-tuning[6] if $\mu = O(10^{15})$ GeV as in conventional GUTs. Let

$$\hat{\omega} \equiv \omega / T \tag{3.13}$$

where $\hat{\omega}$ is a dimensionless variable and which also remains constant during a radiation-dominated epoch[6] and $\sim O(1)$ at the Planck time. To avoid rotation dominance before the onset of de-Sitter expansion only if

$$\hat{w}^2 < \left(\frac{1}{30} N\pi^2 \right)^{\frac{1}{2}} 4\pi \left(\frac{\mu^2}{mp^2} \right) \qquad (3.14)$$

Which poses another fine tunning problem if $\mu = O(10^{15})$GeV. For K= +1 case or if $\omega \neq 0$, the universe may recollapse before it is inflated.

However, as discussed in earlier chapter the original inflationary scenario suffered from the problem of the graceful exit from the de-Sitter phase.[7] Bubbles of the new phase could never fill the entire universe. This difficulty was finessed in the new inflationary scenario of Linde[8,9], which postulated inflation after the Higgs field ϕ has started rolling over from the high temperature value $<0|\phi|0> = 0$ towards its ultimate v.e.v $<0|\phi|0> = \mu$. In this new inflationary scenario the monopole problem disappears through the other fine tuning difficulties[10,11,6,12] mentioned above do persist. In order to break SU(5) down to SU(3)XSU(2)XU(1) one can consider one loop corrections to the Higgs potential. In the region of interest $\phi < T << v$, the potential can be expressed as

$$V(\phi) = A\phi^4 \left(\ln \frac{\phi^2}{v^2} - \frac{1}{2} \right) + D\phi^2 \qquad (3.15)$$

where

$$A = \frac{1}{64\pi^2 v^4} [\sum_B g_B m_B{}^4 - \sum_F g_F m_F{}^4] \tag{3.16}$$

and $g_B(F)$ is the number of helicity states for the bosons (fermions) of mass $m_{B(F)}$ entering into the loop. For standard SU(5)[13] the X and Y bosons dominate eqn.(3.16) so that $g_X = g_Y = 18$ and $m_X{}^2 = m_Y{}^2 = \frac{25}{8} g^2 v^2$. We have

$$A = \frac{5265}{1024\pi^2} g^4 \tag{3.17}$$

where g is the SU(5) gauge coupling:

$$\frac{g^2}{4\pi} = \frac{1}{41} \Longrightarrow g^2 = 0.3 \tag{3.18}$$

The parameter $D\ \phi^2$ in eqn.(3.15) is an effective mass term in the potential which can be expressed as

$$D = \frac{1}{2} (m_o{}^2 + c T^2 + bR - 3\lambda\langle\phi^2\rangle) \tag{3.19}$$

where m_o is the flat-space, zero temperature mass,

$$c = (\frac{75}{8})\ g^2 \tag{3.20}$$

and T is the Hawking temperature $T_H = H/2\pi$ during an inflationary de-Sitter epoch, b is unknown parameter taking the value $\frac{1}{6}$ for conformally coupled scalar fileds ϕ, $R = R^\mu_\mu$

is the scalar curvature, $-\lambda/4$ is an effective scalar self coupling and $\langle\phi^2\rangle$ the quantum expectation value of ϕ^2.[13,9] the $D\phi^2$ term effectively serves as a barrier between true and false vacuua. When the temperature is high, D is large and the field ϕ is confined near the origin. Whereas when the temperature is low, the field ϕ may fluctuate or tunnel across the potential barriers.

The new inflationary scenario only works for a certain range of values of D since otherwise it is not supersymmetric. To get a lower bound on the tunnelling rate, Hawking and Moss[14] proposed that a region of space larger than the de-Sitter horizon volume jumps simultaneously to a local maximum V, of the potential. The rate for such a transition may be approximated by idealizing to a true de-Sitter space and looking for classical solution of the euclidean field equations. According to Hawking and Moss[14], the resulting tunnelling probability to be

$$P \sim D^2 e^{-B} \tag{3.21}$$

where

$$B = \frac{1}{8} m_p^4 \left[\frac{1}{v_o} - \frac{1}{v_1}\right] \approx \frac{1}{8} m_p^4 \left(\frac{v_1 - V_o}{v_o^2}\right) \tag{3.22}$$

in the limit where $v_1 - v_o \ll v_o$. For the Coleman-Weinberg potential

$$v_o = \frac{1}{2} A v^4, \quad v_1 \approx v_o + D^2/8A \tag{3.23}$$

Therefore the Hawking-Moss action is given by

$$B \approx \frac{m_p{}^4 D^2}{16 A^3 v^8} \tag{3.24}$$

If $B > O(1)$, a de-Sitter horizon volume jumps to a local maximum of v_1 since for $B=O(1)$, the universe could just as well jump to another value of ϕ which is much longer than its value at the maximum also if $B=O(1)$, the value of ϕ may be very inhomogeneous after the jump. So $B \gg 1$ is a necessary condition for the revised inflationary scenario to work.

The amount of inflation occurring after the tunnelling event is determined by the equation of motio of the Higgs field

$$\ddot{\phi} + 3H\,\dot{\phi} + 2D\phi = 0 \tag{3.25}$$

which gives a characteristic time scale $\tau = O\,(\frac{3H}{2D})$. To explain the cosmological observations of large-scale isotropy and homogeneity, one must require that the scale factor a is inflated by a factor $O(10^{28})$. Therefore one must have

$$e^{H\tau} = \exp\,(3H^2/2D) > e^{65}$$

$$\text{or} \qquad \frac{3H^2}{2D} > 65 \tag{3.26}$$

combining (3.26) with the $B >> 1$, one gets from equation (3.25) the double inequality

$$4A^{3/2}\, v^4/m_p^{\,2} < D < 3H^2/130 \tag{3.27}$$

Equation (3.27) can not be satisfied by conventional Coleman-Weinberg SU(5). The two inequalities are only consistent if

$$A < 3(\pi/130)^2 = O(10^{-3}) \tag{3.28}$$

which is not the case fro minimal SU(5) for which $A=O(1/10)$ from equation (3.17).

Thus the new inflationary scenario proposed by Linde[8] and developed by Hawking and Moss[14] still needs further improvement which might be found within the context of conventional GUTs. So the problems found in GUTs are solved in super-symmetry by making use of no-renormalization theorems[12]. In an exactly supersymmetric world there would be no Coleman-Weinberg mechanism to govern inflation. This is clear from equation (3.15) since for any boson (fermion) of $m_{B(F)}$ supersymmetry gives corresponding partner with identical degrees of freedom and mass $m_{F(B)}$. So

$$A \equiv 0 \quad \text{and} \quad V(\phi) = D\phi^2$$

Supersymmetry, however, is not a good symmetry of nature. Though there is no degeneracy between bosons and fermions at present energies. Instead, there must exists some scale m_s at which supersymmetry is breaking. In this case, a mass splitting between bosons and fermions are given by

$$m_B^2 - m_F^2 = m_s^2 << m_B^2 \tag{3.29}$$

In a supersymmetric theory, there are no radiative correction when supersymmetry is broken at m_s, the one-loop potential exists and takes the form of eqn.(3.16) with

$$A = \frac{\Sigma g_B \ (= g_F) \ m_s^2 m_B^2}{32\pi^2 v^4} \tag{3.30}$$

where $m_s^2 << m_B^2$, m_F^2. One can consider that the X, Y bosons dominate eqn.(3.29). Thus in analogy with eqn.(3.17) one gets

$$A = \frac{75}{32\pi^2 v^2} \ g^2 \ m_s^2 \tag{3.31}$$

Then the supersymmetric version of eqn.(3.27) becomes

$$0.46g^3 m_s^3 v/m_p^2 < D < 2.3\times10^{-2} g^2 m_s^2 v^2/m_p^2 \qquad (3.32)$$

and eqns.(3.31) and (3.28) yields

$$gm_s < 5\times10^{-2} v$$

which is not a disastrous restriction on the supersymmetry breaking mass splitting m_s.

Supersymmetry can aid inflation in other ways. The effective mass parameter D of eqn(3.19) enables one to tune the bare mass parameter m_o to whatever value we require and fret less about the possible destibilizing effects of radiative corrections[2]. Supersymmetry also enables to make c arbitrary small in a theory where the operational Higgs field ϕ is a gauge singlet, in which case C has the magnitude of an adjustable Yukawa coupling as[15]

$$T = \frac{H}{2\pi} = \left(\frac{Av^4}{3\pi m_p^2}\right)^{\frac{1}{2}} \qquad (3.33)$$

during the de-Sitter phase, making the contribution of the SU(5) gauge multiplet

$$\frac{1}{2}\,\epsilon T^2 = \frac{25g^2Av^4}{16\pi\ m_p^{\ 2}} \tag{3.34}$$

to D somewhat larger than the upper limit (3.27), thus necessitating a cancellation. As for the third equation (3.19) in the effective mass term D, it vanishes in a conventionally Weyl-rescaled model of the chiral supermultiplets couplet to simple N=1 supergravity. This is again good news, since

$$\frac{1}{2}bR = 6bH^2 = \frac{8\pi bAv^4}{m_p^{\ 2}} \tag{3.35}$$

is again larger than the upper bound in eqn.(3.27) for values of b=0(1) such as the conformal coupling $b=\frac{1}{6}$. Finally, Linde[10] has pointed out that the quantum correction term $-3\lambda/2 <\phi^2>$ in eqn.(3.19) must be negligible for the inflationary epoch.(3.26) to be sufficiently long. He reduces[10]

$$\lambda \lesssim \frac{1}{1800}\,\pi^2 \sim 5 \times 10^{-3} \tag{3.36}$$

which is difficult to realize in conventional GUTs but can easily be arranged in supersymmetric GUTs[11].

Now five reasons have been accumulated for believing the fact that the cosmological inflation would be easier to realize in GUTs with at least an approximate supersymmetry. Supersymmetry is very nearly exact at the GUT scales for which one generally expects

$$m_s \lesssim \sqrt{d} \tag{3.37}$$

where $\sqrt{d}$ is primordial supersymmetry breaking scale and in most scenarios $\sqrt{d} < m_X$. In particular, for the gauge multiplet

$$m_s^2 \lesssim O(1)\ \mathrm{TeV}^2 \tag{3.38}$$

and if $\sqrt{d} < m_X$, it may be much smaller for the X and Y gauge bosons. Furthermore, in some models there are cancellations which enforce

$$\sum_B g_B m_B^4 - \sum_F g_F m_F^4 = O(m_s^4) \tag{3.39}$$

Finally, there are arguments[3,16] based on the coupling to supergravity and the mass of the gravitino which suggest that

$$\sqrt{d} = O(\sqrt{m_G m_p}) \lesssim O(\sqrt{m_W m_p}) = O(10^{10})\mathrm{GeV} \tag{3.40}$$

in which case supersymmetry breaking effects are negligible at or before GUT epoch.

We are therefore led to study a toy model entirely at the tree level and discard all the radiative corrections.

3.2 A Supersymmetric Model

Here it is considered the most general renormalizable tree potential constructed out of a single gauge singlet Higgs field ϕ

$$V(\phi;T) = \alpha\,\phi^4 - \beta\phi^3 + (\gamma + \epsilon T^2)\,\phi^2 + \delta \tag{3.41}$$

Using the v.e.v. of the Higgs field μ to scale dimensional parameter in the potential (3.41)

$$\hat{\beta} \equiv \beta/\mu\ , \qquad \hat{\gamma} \equiv \gamma/\mu^2, \quad \hat{\delta} \equiv \delta/\mu^4 \tag{3.42}$$

The condition that the vacuum energy vanish when $\langle 0|\phi|0\rangle = \mu$ at zero temperature gives

$$V(\mu;0)=0 \qquad \alpha - \hat{\beta} + \hat{\gamma} + \hat{\delta} = 0 \tag{3.43}$$

while the condition that $\langle 0|\phi|0\rangle = \mu$ be a stationary is

$$\frac{\partial V}{\partial \varphi}(\mu;0)=0. \qquad 4\alpha - 3\hat{\beta} + 2\hat{\gamma} = 0 \tag{3.44}$$

and it is minimum if

$$\frac{\partial^2 V}{\partial \varphi^2}(\mu;\ 0) > 0 :\ 12\alpha - 6\hat{\beta} + 2\hat{\gamma} > 0 \tag{3.45}$$

Here the relations between γ and δ to α and β is obtained by using the vacuum condition (3.43, 3.44) is

$$\begin{aligned} \hat{\gamma} &= \frac{3}{2}\hat{\beta} - 2\alpha \\ \hat{\delta} &= \alpha - \frac{1}{2}\hat{\beta} \end{aligned} \tag{3.46}$$

Using these, the minimality condition (3.45) becomes

$$8\alpha - 3\hat{\beta} > 0 \tag{3.47}$$

The potential (3.41) has a local extremum at $\phi = 0$, where it takes the value

$$V_o = V(0;0) = \delta \tag{3.48}$$

at zero temperature. At finite temperature the other two extrema of the potential (3.41) are at

$$\phi = + \frac{3\beta}{8\alpha} \pm \frac{\sqrt{9\beta^2 - 32\alpha(\gamma + cT^2)}}{8\alpha} \tag{3.49}$$

If the vacuum energy δ dominates the expansion rate, then the associated Hawking temperature $T_H = H/2\pi$ which can be calcualted using eqn (3.8) and hence

$$c\,T_H{}^2 = \frac{2c}{3\pi}\,\frac{\mu^4}{m_p{}^2}\,\hat{\delta} \tag{3.50}$$

which can also be written in the scaled form

$$\begin{aligned} c\,T_H{}^2/\mu^2 &\equiv \chi\hat{\delta}, \\ \chi &= \frac{2c}{3\pi}\,\frac{\mu^2}{m_p{}^2} \end{aligned} \tag{3.51}$$

From equations (3.8), (3.9), (3.26) and (3.48), one obtains sufficient inflation if

$$\frac{3H^2}{2D} = \frac{4\pi(\mu^2/m_p{}^2)\hat{\delta}}{\hat{\gamma} + \chi\hat{\delta}} > 65 \tag{3.52}$$

Which can be obtained in the form

$$\hat{\gamma} + \chi\hat{\delta} = y\,\hat{\delta} \tag{3.53a}$$

where $$y = \frac{4\pi}{65}\,\frac{\mu^2}{m_p{}^2}\,\hat{y}; \qquad 0 < \hat{y} \lesssim 1 \tag{3.53b}$$

From eqn (3.49), the local maximum of the potential in the limit $y \ll 1$ is

$$\phi_1 \approx \frac{y}{3\rho}\,\mu \tag{3.54a}$$

where $\rho \equiv 2 + y - \chi$ (3.54b)

The three parameters (3.42) may also be expressed in terms of α and ρ and they are

$$\hat{\beta} = \frac{2\alpha\rho}{\rho+1} \; ; \quad \hat{\gamma} = \frac{\alpha(\rho-2)}{\rho+1} \; , \quad \hat{\delta} = \frac{\alpha}{\rho+1} \tag{3.55}$$

The difference between the value V_1 of the potential at the local maximum ϕ_1 and the value V_O at the local minimum is much smaller than V_O itself, so using eqn (3.22) to the Hawking-Moss[14] action are

$$B \approx \frac{m_p{}^4(\rho+1)y^3}{72\,\mu^4\alpha\rho^2} >> 1 \tag{3.56}$$

If B were only of order unity, there would be no strong reason for large regions of space. The potential (3.41) can be obtained from the superpotential[17]

$$W(\phi,X,Y) = a\,X\,\phi\,(\phi-\mu) + bY\,(\phi^2-\mu^2) \tag{3.57}$$

which yields at zero temperature

$$\begin{aligned} V(\phi,X,Y;O) &= |W_X|^2 + |W_Y|^2 + |W_\phi|^2 \\ &= |\,a\,\phi\,(\phi-\mu)|^2 + |b(\phi^2-\mu^2)|^2 + |aX(2\phi-\mu) \\ &\qquad + 2bY\,\phi|^2 \end{aligned} \tag{3.58}$$

$$\delta V(\phi,X,Y;T) = (2|W_{X\varphi}|^2 + 2|W_{Y\phi}|^2 + |W_{\phi\phi}|^2)\ \frac{1}{8}\ T^2$$

$$= \frac{1}{8}\ T^2[2|a(2\phi-\mu)|^2+2|2b\phi|^2+|2aX+2bY|^2] \qquad (3.59)$$

Eqn.(3.59) yields that finite temperature effects prefer $2aX+2bY=0$ in which case eqn (3.58) tells that the $|W_\phi|^2$ terms prefers $X=0$ and hence $Y=0$ also and the potential does not have a non-zero v.e.v. for the imaginary part of ϕ. So one gets the following finite temperature effective potential are the real part of ϕ

$$V(\phi;T)=(a^2+b^2)\phi^4-2a^2\mu\phi^3+(a^2-2b^2)\mu^2\phi^2+b^4\mu^4+(a^2+b^2)T^2\phi^2$$
$$-a^2\mu T^2\phi+a^2T^2\mu^2 \qquad (3.60)$$

comparing this with (3.41) one can eliminate the errant linear term in eqn (3.60) by a transition

$$\hat{\phi} = \phi-\phi_o$$
$$\phi_o = \frac{a^2T^2}{2(a^2-2b^2)\mu} \qquad (3.61)$$

in the limit of small T. After making the transition (3.61) one can identify the parameters of our toy potential (3.41) as functions of the parameters of our supersymmetric model (3.57) as

$$\alpha = a^2 + b^2 \tag{3.62a}$$

$$\beta = 2a^2\mu \tag{3.62b}$$

$$\gamma = (a^2 - 2b^2)\mu \tag{3.62c}$$

$$c = -\left(\frac{2a^4 + a^2b^2 + 2b^4}{a^2 - 2b^2}\right) \tag{3.62d}$$

$$\delta = b^2\mu^4 \tag{3.62e}$$

$$\rho = a^2/b^2 \tag{3.62f}$$

where the terms of higher order in T/μ is neglected and the quantity χ defined in eqn (3.51) has the value

$$\chi \approx \frac{-2}{3\pi} \quad \frac{\mu^2}{m_p^2} \left[\frac{2a^4 + a^2b^2 + 2b^4}{a^2 - 2b^2} \right]$$

$$\approx \frac{-4\mu^2}{\pi m_p^2} \left[\frac{a^2}{\rho - 2} \right] \quad \text{for} \quad \rho \approx 2 \tag{3.63}$$

Having established the correspondence of our supersymmetric model to the original toy model (3.41), now it is to analyse the inflationary capability of the superpotential (3.57).

Examining the constraints (3.53), (3.56) for a range of values of $\langle 0|\phi|0\rangle = \mu$ between $O(10^{16})$GeV and $O(10^{19})$GeV, allowed the values of a^2 together with the corresponding values of the combination $\rho - 2 = (a^2/b^2 - 2)$ indicating the degree of fine-tuning between a and b necessary in the superpotential (3.57). Both a and b must be rather smaller if $\mu = O\,(10^{16})$GeV.

$$a,\ b \sim O\,(10^{-6}) \tag{3.64}$$

and that their ratio must be finely tuned

$$\rho - 2 = \frac{a^2}{b^2} - 2 \quad = O(10^{-7}) \tag{3.65}$$

However, by pushing μ upto $O(10^{19}$GeV)

$$a,\ b \ \sim O(10^{-3}) \tag{3.66}$$

$$\rho - 2 = \frac{a^2}{b^2} -2 \sim O(10^{-1}) \tag{3.67}$$

Thus while inflation is attainable in a supersymmetric model, it looks rather unnatural (3.64), (3.65) if $\mu = O(10^{16})$GeV but rather less unnatural if $\mu = O(10^{19})$GeV.

This is the observation which leads us to propose primordial inflation i.e. inflation at an epoch prior to the grand unified phase transition and very possibly during the Planck era.

3.3 A different approach for Primordial Inflation

Due to extremely high temperature, the very early universe contains matter in the ionized state. So here one can consider the contribution of fermions along with scalar field, to the energy density of the early universe. Since most of the fermions found in nature are spin - $\frac{1}{2}$ particles so one can consider only spin - $\frac{1}{2}$ fermions. Taking this idea under consideration, an action integral for a spin - $\frac{1}{2}$ field ψ and the scalar field ϕ is written as[18]

$$J = \frac{1}{2} \int (-g)^{\frac{1}{2}} [\bar{\psi}(i\gamma^{\mu}\psi;\mu - m\psi) + g^{\mu\nu}\partial_{\mu}\phi^{*}\partial_{\nu}\phi$$
$$-f\bar{\psi}\,\psi\,(\phi^{*}\phi) - v(\phi)]\, d^{4}x \tag{3.68}$$

where f is the coupling constant, f $\psi\psi(\phi^{*}\phi)$ is an interaction term between ψ and ϕ, γ^{μ} are Dirac matrices, $\bar{\psi}$ is the conjugate of ψ, m is the mass of spinor field ψ, $g^{\mu\nu}$ is the metric tensor, g is the determinant of $g_{\mu\nu}$ and (;) denoters covariant differentiation. The covariant derivative of ψ is defined as

$$\psi;\,\mu = (\delta_{\mu} - \Gamma_{\mu})\,\psi$$

where

$$\Gamma\mu = -\frac{1}{4}\,(\delta_{\mu}\, h^{\rho}_{a} + \{\sigma^{\rho}\mu\ \} h^{\sigma}_{a}\ )g_{\gamma\rho} x\ h^{\gamma}_{b}\ \tilde{\gamma}^{b}\tilde{\gamma}^{a} \tag{3.69}$$

where h^{ρ}_{a} are called tetrad components or viebein defined as

$$g_{\rho\sigma}\quad h^{\rho}_{a}\quad h^{\sigma}_{b} = \eta_{ab} \tag{3.70}$$

where η_{ab} is Minkowskian metric given by

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.71}$$

and $V(\phi,T)$ is the finite temperature Coleman-Weinberg potential in SU(5) gauge symmetry given by[1,11]

$$V(\phi,T) = A\phi^4(\log \frac{\phi^2}{v^2} - \frac{1}{2}) + \frac{1}{2} C T^2\phi^2 \tag{3.72}$$

with

$$A = \frac{1}{64} \pi^2 v^4 \; [\sum_B g_B m_B{}^4 - \sum_F g_F \, m_F{}^4] \tag{3.73}$$

where $g_{B(F)}$ is the number of boson (fermion) helicity states of mass $m_{B(F)}$, $C = (\frac{75}{8}) g^2$, $g^2=0.3$ in minimal SU(5), v stands for the vacuum expectation value of an adjoint multiplet of ϕ and is equal to 1.2×10^{15}GeV, approximately.

So taking the above action as a source of energy density, here one can introduce a cosmological model of the very early universe. It is assumed that the measurement of time(t) is from the Planck epoch as due to our ignorance of quantum gravity it is difficult to investigate before Planck time. In the beginning, the model obeys supersymmetry (SUSY) but after some time SUSY breaks down. So as a result of SUSY breaking, ϕ decreases exponentially and $V(\phi) \longrightarrow V(0) = \frac{1}{2} Av^4$. At high temperature, the expansion of the model is very fast and so the fermion energy density decreases rapidly. As the energy density is dominated by V(O) the breaking of SUSY yields an inflationary scenario even prior to the GUT phase transition. This results supports the idea of supersymmetric inflation or primordial inflation proposed by Ellis, Nanopoulous, Olive and Tamvakis[1,11]

3.4A Classical Solution

The model of the universe is described by the Robertson-Walker (RW) given by the line element (3.1).

The background model given by eqn (3.1) is homogeneous and isotropic hence the spinor field ϕ and the scalar field ψ in the action integral (3.68) will depend only on t. Hence on applying the action principle, the action integral (3.68) yields

$$[\gamma^o \delta_o + \frac{3}{2} \gamma^o \frac{\dot{a}}{a} + i (m + f \ \phi^*\phi)] \ \psi = 0 \tag{3.74}$$

for the spinor field ψ and

$$\ddot{\phi} + (3 \frac{\dot{a}}{a})\dot{\phi} + f \ \bar{\psi} \ \psi \ \phi = - \frac{\partial v}{\partial \phi} \tag{3.75}$$

for the scalar field ϕ.

The equation (3.74) is integrated to yield

$$\psi a^{3/2} = \psi_o a_o^{3/2} \exp\left(-i\gamma^o \int (m+f\ \phi^*\phi)dt\right) \tag{3.76}$$

and its hermitian conjugati $\bar{\psi}$ is

$$\bar{\psi} a^{3/2} = \bar{\psi}_o\ a_o^{3/2} \exp\left(i\gamma^o \int (m+f\ \phi^*\phi)dt\right) \tag{3.77}$$

where $\psi_o(\bar{\psi}_o)$ is the spinor $\psi\ (\bar{\psi})$ at $t = 0$ and $a_o=a(0)$

Now from equations (3.76) and (3.77) one gets

$$\bar{\psi}\ \psi\ a^3 = \bar{\psi}_o \psi_o a_o^3 \tag{3.78}$$

If the model obeys SUSY, so onc has equal number of bosons and fermions. This means that $g_B=g_F$ and $m_B=m_F$. Hence from SUSY (3.73) one gets A=0 in SUSY[1,11]. Now from eqn (3.72)

$$V(\phi) = \frac{1}{2}\ CT^2\ \phi^2 \tag{3.79}$$

Now connecting eqns (3.76) and (3.79) one gets

$$\ddot{\phi} + (3\ \frac{\dot{a}}{a})\dot{\phi} + (f\ a^{-3}\bar{\psi}_o\psi_o a_o{}^3 + CT^2)\phi = 0 \tag{3.80}$$

Under the transformation

$$\tau = \int^t \frac{dt'}{a^3(t')} \qquad (3.81)$$

eqn (3.80) reduces to

$$\frac{d^2\phi}{d\tau^2} + (f\ a^3\bar{\psi}_o\psi_o a_o{}^3 + \mathbb{C}\,T^2 a^6)\phi = 0 \qquad (3.82)$$

The WKB solution of (3.82) is given by

$$\phi = \frac{\alpha\pm}{(fa^3\bar{\psi}_o\psi_o a_o{}^3 + \mathbb{C}T^2a^6)^{1/4}} \exp\left[\pm i \int (f\ a^3\bar{\psi}_o\psi_o a_o{}^3 + \mathbb{C}T^2 a^6)^{1/2} d\tau\right. \qquad (3.83)$$

where $\alpha+$ and $\alpha-$ are normalization constants which can be determined using the normalization prescription that in the flat space-time limit norm approaches unity.

Suppose the temperature is extremely high even grater than 10^{17}GeV; hence from eqn (3.83) one gets

$$\phi \simeq \frac{\alpha\pm}{\mathbb{C}^{1/4}\, a^{3/2}\, T^{1/2}} \exp\left(\pm\, i \int \mathbb{C}^{1/2}\ T\, dt\right) \qquad (3.84)$$

The total energy for ϕ is

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$= (\alpha^2 + \frac{1}{2})\,\frac{\mathbb{C}^{1/2}T}{a^3} + \frac{9\alpha^2}{4c^{1/2}Ta^3}\left(\frac{\dot{a}}{a}\right)^2 \qquad (3.85)$$

and the energy density for ψ is

$$\rho_\psi = \frac{\bar{\psi}_o \psi_o a_o^3}{2a^3} \left(m + \frac{f\alpha^2}{\mathcal{C}^{1/2} a^3 T} \right) \tag{3.86}$$

It is assumed that the early universe is spatially flat. Hence, Einstein equation governing the dynamics during the era is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3M_p^2} (\rho_\phi + \rho_\psi) \tag{3.87}$$

Now from eqn (3.86) we have

$$\rho_\psi = \frac{\bar{\psi}_o \psi_o a_o^3}{2a^3} m \tag{3.88}$$

after neglecting $\frac{f\alpha^2}{\mathcal{C}^{1/2} a^3 T}$ in eqn (3.86) due to the extremely high value of temperature T in the denominator.

Connecting eqns. (3.85) and (3.88) one gets

$$\left(a^3 - \frac{6\pi\alpha^2}{\mathcal{C}^{1/2} M_P^2 T}\right) \left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi}{3M_P^2} \left(m \bar{\psi}_o \psi_o a_o^3 + \alpha^2 \mathcal{C}^{1/2} T \right) \tag{3.89}$$

Since M_P^2 and T is in the denominator so neglecting $\frac{6\pi\alpha^2}{\mathcal{C}^{1/2} M_P^2 T}$ from eqn. (3.89)

$$a^3 . (\frac{\dot{a}}{a})^2 = \frac{4\pi}{3M_P^2} (m \bar{\psi}_o \psi_o a_o^3 + \alpha^2 c^{1/2} T)$$

i.e. $a . \dot{a}^2 = \beta^2$ (3.90)

where $\beta^2 = \frac{8\pi}{3M_P^2} (m \bar{\psi}_o \bar{\psi}_o a_o^3 + \alpha^2 c^{1/2} T)$ (3.91)

The eqn. (2.90) yields the solution

$$a = a_o (1 + \beta t)^{2/3} \tag{3.92}$$

3.4B SUSY Breaking and Sclar Field

As SUSY is not a natural symmetry, it is assumed that it breaks down at a certain epoch after the Planck time. so the mass splitting between bosons and fermions in this case[1,11]

$$m_S^2 \sim m_B^2 = m_F^2$$

This implies

$$A \neq O$$

Hence

$$V(\phi) = A\phi^4 [\log (\phi^2/v^2) - \frac{1}{2}] + \frac{1}{2} CT^2\phi^2 \tag{3.93}$$

Since at high temperature ϕ is confined near the origin, so $V(\phi)$ can be approximated as

$$V(\phi) \approx V(O) + \frac{1}{2} CT^2\phi^2 - \frac{1}{2} A \phi^4 \tag{3.94}$$

Using eqn.(3.75) in eqns.(3.93) and (3.94), the differential equation for ϕ , when SUSY breaks, as

$$(1+\beta t)^2 \phi + 2\beta (1+\beta t)\phi + f \bar{\psi}_o \psi_o \phi$$
$$= 2A (1+\beta t)^2 \phi^3 - CT^2 \phi \qquad (3.95)$$

with the approximation $CT^2(1+\beta t)^2 \simeq CT^2$ since from eqn.(3.91) we have β is $O(T^{1/2}/M_P) << 1$ and also t is small. The differential equation (3.95) is integrated to

$$\phi^{-2} = C_1(1+\beta t)^{-1+\beta^{-1}(2c)^{-1\,2}T} + C_2(1+\beta t)^{-1-B^{-1}(2c)^{1\,2}T}$$
$$- \frac{4A(1+\beta t)^2}{8\beta^2 - 2f \bar{\psi}_o \psi_o - 2CT^2} \qquad (3.96)$$

where C_1 and C_2 are integration constants.

Due to the very high temperature and smalla t (3.96) may be approximated as

$$\phi^{-2} \simeq C_1 \exp\{-(2c)^{1/2} Tt\} + (2A/CT^2) \times \exp(2\beta t) \qquad (3.97)$$

Since $C_1CT^2 \exp\{(2C)^{1/2} Tt\}$ will dominate the term 2A exp $(2\beta t)$ ϕ_o is ϕ at t=0. So eqn.(3.97) yields

$$\phi^2 \simeq \phi_o^2 \exp[-(2c)^{1/2} Tt] \qquad (3.98)$$

When SUSY breaks, the total energy is given by

$$\rho = \frac{\bar{\psi}_o \psi_o}{2(1+\beta t)^2} \left[m + \frac{f \alpha^2}{c^{1/2} a_o^3 (1+\beta t)^2 T} \right]$$

$$+ \frac{\phi_o^2}{2T} \exp \left[-(2c)^{1/2} Tt \right] + V(0)$$

$$- \frac{1}{2} A\phi^4 \exp \left[-(2c)^{1/2} Tt \right]$$

$$+ \frac{1}{2} C T^2 \phi_o^2 \exp \left[-(2c)^{1/2} Tt \right] \qquad [3.99)$$

Due to high temperature, terms containing exponential function in eqn.(3.99) will damp rapidly. The second term within the square bracket also will loose its effect after some time for the same reason. So, one can find that after some time the energy density is given by

$$\rho \simeq \frac{m \bar{\psi}_o \psi_o}{2(1+\beta t)^2} + V(0) \qquad (3.100)$$

As time increases, the first term on the right hand side of eqn (3.100) also become less effective in comparison to V(O). Thus $\rho \longrightarrow V(0)$. Substituting this value of ρ in Einsteins equation one finds that the universe expands as the de-Sitter model like

$$a(t) \sim \exp \left(\frac{8\pi V(0)}{3M_P^2} t \right)$$

Thus one can see that the inflationary situation may start earlier than the GUT phase transition provided that SUSY breaks.

This inference is another support for the hypothesis of primordial inflation by Ellis et al.

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CHAPTER - IV

INFLATIONARY MODELS WITH EXPONENTIAL POTENTIALS

4.1 Introduction

In last chapters the possibility of inflationary expansion in the early universe as suggested by Guth, Linde and Albrecht, Ellis, Olive, Nanopoulas and Tomvakis has been discussed. In this chapter it has been discussed that inflationary scenario can arise in certain cases in cosmological models containing a self-interacting scalar field possessing exponential potential. These kinds of potentials are very much interesting because they arise generically in a number of circumstances. Although they do not possess minima, they can model the gradient of a potential which possesses a minimum in an orthogonal degree of freedom, as for example in the Salam-Sezgin model of N=2 supergravity coupled to matter[1,2]. Also, as Halliwell has pointed out[3], most theories undergoing dimensional reduction to an effective four-dimensional theory result in a linear combination of exponential potentials and one of these will always dominate asymptotically. It is also known that an analogous structure exists in quadratic lagrangian theories of gravity which are conformally equivalent to general relativity plus a scalar field which asymptotically tends to an exponential form[4,5,6].

Whereas there exists a considerable detailed understanding of the asymptotic evolution of anisotropic cosmological models undergoing exponential expansion[7,8], comparatively little is known about the general behaviour of universes undergoing power-law inflation[9,10]. Here analysis given by Burd and Barrow has been discussed regarding asymptotic behaviour of cosmological models dominated by the exponential potential. Extension of the results obtained by Burd, Burrow and Lancaster[11] has been discussed using the method of Halliwell[3] to obtain an autonomous system of equations to analyse the fate of Friedmann-models with N space dimensions. Also this method has been adapted to study the evolution of open, closed and flat anisotropic universes. Some new exact solutions given by Burd and Barrow are given which exhibit the transition from free-scalar field expansion to power-law inflation[12]. These solutions and further general aspects of exponential expansion are given in section 4.2. The general behaviour of N-dimensional Friedmann models is classified in section 4.3 and that of an anisotropic models in section 4.4. Some discussion of the results is given in section 4.5.

4.2 Exact Solutions

Here the case of homogeneous isotropic space-time with N space like dimensions containing a scalar field which possess an exponential self-interaction potential is considered. The space-time metric is given by

$$ds^2 = dt^2 - a^2(t)\ {}^{(n-1)}g_{ij}\ dx^i\ dx^j \tag{4.1}$$

where g_{ij} is the metric of an N-dimensional space of constant spatial curvature:

$$^{(n-1)}g_{ij}\ dx^i\ dx^j = \frac{dr^2}{1-Kr^2} + r^2(\ (d\theta_1^2 + \mathrm{Sin}^2\theta_1)$$

$$(d\theta_2^2 + \mathrm{Sin}^2\theta_2) - - - - - (\qquad)\ ) \tag{4.2}$$

with k is constant. Here the coordinates are generalized speherical polars and for convention Latin indices run from 1 to N and Greek indices run from 0 to N. If the matter content of the space-time is a self-interacting scalar field ϕ (t) with a potential V(ϕ), then it is equivalent to a perfect fluid with pressure p(t) and energy density ρ(t) given by

$$p = \frac{1}{2}\dot{\phi}^2 - V(\varphi) \tag{4.3}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{4.4}$$

The dynamics of the N-dimensional Friedmann universe are the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{16\pi G}{N(N-1)} \left[\frac{1}{2} \dot{\phi}^2 + V(\varphi) \right] - \frac{k}{a^2} \qquad (4.5)$$

and the equation of motion of the scalar field is given by

$$\ddot{\phi} + N \frac{\dot{a}}{a} \dot{\phi} + V'(\varphi) = 0 \qquad (4.6)$$

where

$$V' \equiv \frac{dv}{d\phi}$$

When k=0, several exact solutions can be found to eqns.(4.5) and (4.6) and the potential has an exponential form

$$V(\phi) = \Lambda e^{-\lambda\phi} \qquad (4.7)$$

With Λ and λ are non-negative constants. Firstly when k=0, there exists a power law solution to (4.5) and (4.7). If it is assume

$$a(t) = t^A \qquad (4.8)$$

where A is a constant and

$$\phi(t) = \frac{2}{\lambda} \ln t \qquad (4.9)$$

then (4.5) and (4.7) are satisfied if

$$\Lambda\lambda^2 = 2(NA-1) = \frac{2(2+\Lambda\lambda^2)\Lambda}{A^2N(N-1)} \tag{4.10}$$

Hence, assuming $A \neq 0$ one can find,

$$A = \frac{4}{(N-1)\lambda^2} \tag{4.11}$$

and from the first equation in (4.10), Λ is given in terms of N and λ. Also from equation (4.11), the power-law inflation will occur when $\lambda < \frac{2}{\sqrt{N-1}}$. In the limit $\lambda \longrightarrow 0$ one approaches in de-Sitter solution with

$$a(t) = \exp\left(t\sqrt{\frac{2\Lambda}{N(N-1)}}\right)$$

when k=0 further, more general solution to equations (4.5) and (4.6) can be found:

$$a(t) = (t^4 + Ct)^{1/N} \tag{4.12}$$

$$\phi(t) = \sqrt{\frac{N-1}{N}}\ \ln\left[t^2\left(1+\frac{C}{t^3}\right)\right] \tag{4.13}$$

Where C is a constant, $\Lambda = 6(N-1)/N$ and $\lambda = \sqrt{\frac{N}{N-1}}$. Now from (4.12) one gets, as $t \longrightarrow 0$ when $C > 0$, the solution approaches the behaviour of the free scalar field in N-dimensions, $a(t) \;\alpha\; t^{1/N}$. When $t \longrightarrow \infty$, the solution given by (4.12) asymptotes to

$$a(t) \;\alpha\; t^{4/N} \tag{4.14}$$

which is a power-law asymptote of the class (4.11) when $\lambda = \sqrt{\frac{N}{N-1}}$. Power-law inflation occurs only when $N<4$, because for $N=4$, the situation is marginal. Also as $t \longrightarrow 0$, the solution (4.12) is dominated by the kinetic energy of the scalar field, so $a(t) \sim t^{1/N}$.

For an N-dimensional universe, the necessary condition for inflation to occur is that

$$\Delta \equiv (N-2)\;\rho + N_p < 0 \tag{4.15}$$

using (4.12) & (4.13)

$$\Delta = \frac{2\;(N-1)}{Nt^2(t^3+C)^2}\;[2(N-5)t^6-2Ct^3(1+2N)+C^2(N-1)] \tag{4.16}$$

Hence for $t \longrightarrow \infty$, $N<5$, $\Delta < 0$ and inflation occurs, but for $t \longrightarrow 0$ one always gets $\Delta > 0$. Also at $t=0$, these solutions possess singularities.

The solutions (4.8)-(4.11) and (4.12), (4.13) are the only known exact cosmological solutions to the Friedmann equations (4.5), (4.6) for a non-constant $V(\phi)$. When $k=+1$, from equation (4.6) one gets for static solution ($\ddot{a}=\dot{a}=0$)

$$\ddot{\phi} = \Lambda \lambda e^{-\lambda\phi} \tag{4.17}$$

which has the solution

$$\phi = -\frac{1}{\lambda} \ln \left[\frac{\lambda\beta}{2\alpha}\right] + \frac{2}{\lambda} \ln \left[\mathrm{Cosh} \left(\frac{1}{2} \lambda\sqrt{\beta}\, t\right)\right] \tag{4.18}$$

where $\alpha = \lambda\Lambda$ and β is a constant. Also equation (4.5) yields the constant scale factor,

$$a = \sqrt{\frac{N(N-1)}{\beta}} \tag{4.19}$$

As $t \longrightarrow \infty$, $\phi \sim t$, $\dot{\phi} \sim$ constant and hence the potential becomes negligible.

This static example shows a close connection to the n-body Toda lattice problem[13]. The hamiltonian of the Toda lattice is given by

$$H = \frac{1}{2}\sum_{j=1}^{n} P_j^{\,2} + \sum_{j=1}^{n-1} \exp\left[2(q_j - q_j - 1)\right] \tag{4.20}$$

Where (P_j, q_j) are the generalized coordinates of the jth particle. For n=2 and putting $q=q_2-q_1$, (4.20) reduces to

$$H = \frac{1}{2} (p^2 + e^{-2q}) \qquad (4.21)$$

Now Hamilton's equations have the solution

$$q(t) = \ln (\cosh t) \qquad (4.22)$$

which gives free geodesic motion with energy $\frac{1}{2}$ on a space of constant negative curvature.

4.3 The N-dimensional isotropic models

For dependence on dimensionality in (4.12) one can investigate the generic behaviour of Friedman universes in N space dimensions containing a massless, self-interacting scalar field ϕ with an exponential potential given by (4.7) which is also the generalization of the Halliwell's[3] analysis. Let the new time variable τ is defined by

$$\frac{d}{dt} = V^{\frac{1}{2}} \frac{d}{d\tau} \qquad (4.23)$$

and a new scale factor $\alpha(\tau)$ is

$$a(\tau) = e^{\alpha(\tau)} \quad .$$

Hence using these variables, the equations (4.5) and (4.6) become respectively

$$\alpha' = \kappa \left(\frac{1}{2} \phi'^2 + 1\right) - \frac{k}{Va^2} \qquad (4.24)$$

$$\phi'' + \frac{1}{2}\frac{1}{v}\frac{dv}{d\phi}\phi'^2 + N\alpha'\phi' + \frac{1}{V}\frac{dv}{d\phi} = 0 \tag{4.25}$$

where a prime denotes differentiation with respect to τ and $\kappa=2/N(N-1)$. Here analysis is concerned only with the case when $\lambda \geqslant 0$ since $\lambda < 0$ case can be obtained from the $\lambda > 0$ case by making the transformation $\phi \longrightarrow -\phi$

By defining the following new variables (7)

$$\begin{aligned} \alpha' &= x \\ \alpha \qquad \phi' &= y \end{aligned} \tag{4.26}$$

the system of equations in a two dimensional autonomous system are given by

$$\begin{aligned} x' &= \frac{1}{2}\lambda xy - x^2 - \frac{1}{N}\left(y^2 - \frac{2}{N-1}\right) \\ y' &= \frac{1}{2}\lambda y^2 - Nxy + \lambda \end{aligned} \tag{4.27}$$

Hence the critical points of the system are given by

$$\text{I} \qquad (x_o, y_o) = \left(\pm \frac{\lambda}{\sqrt{2(N-1)}}, \pm\sqrt{\frac{2}{N-1}}\right)$$

$$\text{II} \qquad (x_o, y_o) = \left(\pm \frac{2\sqrt{2}}{\sqrt{(N-1)(4N-\lambda^2(N-1))}} \pm \frac{\lambda\sqrt{2(N-1)}}{\sqrt{4N-\lambda^2(N-1))}}\right)$$

The points I always lie on lines y = constant and for the special case of $\lambda = 0$, they lie on the y-axis, whereas the points II always lie somewhere on the trajectory representing the flat (k=0) model given by,

$$\alpha'^2 = \kappa \left(\frac{1}{2}\phi'^2+1\right)$$

Putting $x-x_o = X$ and $y-y_o = Y$ in (4.27)

$$X' = \left(\frac{1}{2}\lambda y_o-2x_o\right)X + \left(\frac{1}{2}\lambda x_o-\frac{2}{N}y_o\right)Y + P(X,Y)$$

$$Y' = (-Ny_o)X + (Xy_o-Nx_o)Y + Q(X,Y)$$

where $P(X,Y) = \frac{1}{2}\lambda XY - X^2 - \frac{Y^2}{N}$ and

$$Q(X,Y) = \frac{1}{2}\lambda Y^2 - NXY$$

such that P and Q $\longrightarrow$ 0 as X and Y $\longrightarrow$ 0. Now the characteristic equation for the linearized autonomous system

$$X' = \left(\frac{1}{2}\lambda y_o-2x_o\right) X + \left(\frac{\lambda}{2}x_o - \frac{2}{N}y_o\right) Y$$

$$Y' = (-Ny_o) X + (\lambda y_o - Nx_o)$$

is

$$\xi^2 - \left\{\frac{3}{2}\lambda y_o - (2+N)x_o\right\}\xi + \left(\frac{1}{2}\lambda^2 y_o^2 - 2\lambda x_o y_o + 2Nx_o^2 - 2y_o^2\right) = 0$$

which yields the solution

$$\xi = \frac{1}{2}\left[\frac{3}{2}\lambda y_o - (2+N)x_o \pm \left\{\frac{\lambda^2}{4} y_o^{\,2} + (2-N)^2 x_o^{\,2} + (2-3N)\lambda x_o y_o + 3y_o^{\,2}\right\}^{\frac{1}{2}}\right]$$

The critical points I will always exists for values of $N \geqslant 1$. But the critical points II shoot off to infinity when $\lambda = 2\sqrt{N}/(\sqrt{N-1})$ and for values of $\lambda > 2\sqrt{N}/(\sqrt{N-1})$, they are purely imaginary. If one equate the corresponding x_o, and y_o values for the critical point I and II then one can find they coalesce to form a single critical point when $\lambda = 2/\sqrt{N-1}$ which lies on the trajectory of k=0 models.

Now by studying the eigen-values (ξ_i) and the eigenvectors of the characteristic equation, one can determine the nature of the critical points (x_o, y_o). For different ranges of the parameter λ, the nature of the critical points I are as follows:-

(i) For $\lambda=0$, the eigenvalues are $\pm \frac{2}{\sqrt{N-1}}$, both real and have different signs. So the point is a saddle point and hence unstable; since $x_o=0$ for $\lambda=0$ so it represents a static model and hence the scale factor will not grow. Putting (x_o, y_o) for I in (4.24) one finds that $\frac{k}{va^2} > 0$ which means that these critical points lie in the region occupied by $k > 0$ models.

(ii) For $0 < \lambda < 2/\sqrt{N-1}$, eigenvalues are real and opposite in sign, so the critical point is an unstable saddle point and no longer lies on the y-axis but has shifted along the line $y = \sqrt{2/(N-1)}$. The critical point only occurs for those models with positive curvature, which can be seen from (4.24) putting (x_0, y_0) for I.

(iii) For $\lambda = \frac{2}{\sqrt{N-1}}$, the two critical points merge to form a single non-linear equillibrium point and one of the eigenvalues of the characteristic equation is zero. When one of the eigen values of the linearized system is zero, the stability can not be unambiguously decided using just a linear analysis. In such cases one must take the higher order terms in the equations. Critical variable in this case is found to be

$$Z_0 = x - \frac{y}{\sqrt{2N}} \tag{4.28}$$

and choosing the remaining variable to be

$$Z_1 = \frac{x\sqrt{2N}}{2-N} + y \tag{4.29}$$

then the linearized equations become

$$Z_0' = 0$$

$$Z_1' = \frac{\sqrt{N}\,(1-N)\,Z_1}{\sqrt{N-1}}$$

these equations are already in Jordan canonical form and one gets as $Z_o \longrightarrow 0$

$$Z_o' = -\frac{4Z_o^2}{(N-3)^2}$$

and hence the critical point is unstable. It lies only on the k=0 separatrix and there are no other critical point in this case.

(iv) For $2/\sqrt{N-1} < \lambda < 2\sqrt{N}/(N-1)$, the eigenvalues are again of opposite signs and the critical point is in general unstable saddle point.

(v) For $\lambda = 2\sqrt{N}/(N-1)$, the critical point on the k=0 trajectory has gone off to infinity. The critical point is a stable spiral.

Now for the other set of critical points II i.e.,

$$(x_o, y_o) = \left(\pm\frac{2\sqrt{2}}{\sqrt{(N-1)(4N-\lambda^2(N-1))}}, \ \pm\frac{\lambda\sqrt{2(N-1)}}{\sqrt{4N-\lambda^2(N-1)}} \right)$$

these points lie only on the k=0 Friedmann trajectory for all values of λ and N.

(1) For λ=0, the eigen values are $-N\sqrt{\frac{2}{N(N-1)}}$ and

$-2\sqrt{\frac{2}{N(N-1)}}$. Hence the critical point is a stable node. The critical point is an attractor for all K= -1 model and for some k= +1 model. The closed models will not recollapse for some values of the initial conditions but expand forever. But it is quite difficult to determine for a simple model[14] whether a particular model will expand forever or recollapse. But whenever a model evolves into a state such that

$$(N-2)\rho + Np = 2(\dot{\phi}^2 - V) > 0$$

then the model will expand forever rather than recollapse. The critical point is located at $(x_0, y_0) = (\kappa, 0)$ for all values of N.

(ii) For $0 < \lambda < 2/\sqrt{N-1}$, both the eigenvalues are real and negative for all integral values of N and so the critical point is a stable node and is an attractor for all $k < 0$ models and for some $k > 0$ models.

(iii) For $\lambda = \frac{2}{\sqrt{N-1}}$, the critical points merge and the linear analysis can not be used to determine unambiguously the stability of the point. Since this point is unstable, the critical point is no longer an attractor for $k > 0$ models.

(iv) For $2/\sqrt{N-1} < \lambda < 2\sqrt{N}/(N-1)$, the eigenvalues are of different sign for $N > 2$ and hence the critical point can no longer be a stable node and must be a saddle point. The equillibrium point now represents a model with scale factor $a \sim t^P$ where $P < 1$ and hence can not solve the horizon and flatness problems.

(v) For $\lambda = 2\sqrt{N}/\sqrt{(N-1)}$, the critical point has moved to infinity and for all values of $\lambda > 2\sqrt{N}/\sqrt{(N-1)}$, the coordinates of the critical point are imaginary. The results in this section are summarised in table I below.

Table I

Summary of behaviour of isotropic model, showing the asymptotic behaviour of a(t) and ϕ (t) as $t \longrightarrow \infty$ together with the resultant asymptotic stability of the critical points listed

Critical point	λ	a(t)	$\phi(t)$	Stability
	O	constant.	$\sqrt{\frac{2\Lambda}{N-1}}\ t$	no
	$O < \lambda < \frac{2}{\sqrt{N-1}}$	t	$\sim \ln t$	no
I	$\frac{2}{\sqrt{N-1}}$	t	$\sqrt{N-1}\ \ln\left[\frac{t\sqrt{2\Lambda}}{N-1}\right]$	no
	$\frac{2}{\sqrt{N-1}} < \lambda < \frac{2\sqrt{N}}{N-1}$	t	$\sim \ln t$	no
	$\frac{2\sqrt{N}}{N-1}$	t	$\frac{N-1}{\sqrt{N}} \ln\left[\frac{t}{N-1}\sqrt{\frac{2\Lambda N}{N-1}}\right]$	no

table contd..

Table - 1 contd..

	O	$e^{t\sqrt{\frac{2\Lambda}{N(N-1)}}}$	O	yes
II	$O < \lambda < \frac{2}{\sqrt{N-1}}$	t^P $(P > 1)$	$\sim \ln t$	yes
	$\frac{2}{\sqrt{N-1}}$	t	$\sqrt{N-1}\ \ln\left[\frac{t\sqrt{2\Lambda}}{N-1}\right]$	yes
	$\frac{2}{\sqrt{N-1}} < \lambda < \frac{2\sqrt{N}}{N-1}$	$t^P (P < 1)$	$\sim \ln t$	yes

4.4 Anisotropic cosmological models

In this section the previous analysis of cosmological isotropic models is extended to anisotropic models with three space dimensions. Here estimation is done on a class of models which include the Bianchi type I and III and the Kantowski-Sachs models[15,16,17] which generalize k=O and $K = \pm 1$ isotropic cosmological models respectively.

The line-element for different models can be written in the following form

$$ds^2 = dt^2 - X^2(t)\, dr^2 - Y^2(t)\, (d\theta^2 + f(\theta) d\phi^2) \tag{4.30}$$

where $f(\theta) = 1$, $\sinh^2\theta$, $\sin^2\theta$ in the Bianchi types I, III and Kantowski-Sachs models respectively. The Einstein equations for a perfect fluid source with energy density ρ and pressure p are given by

$$\left(\frac{\dot{Y}}{Y}\right)^2 + \frac{k}{Y^2} + 2\,\frac{\dot{X}\dot{Y}}{XY} = \rho$$

$$2\,\frac{\ddot{Y}}{Y} + \frac{k}{Y^2} + \left(\frac{\dot{Y}}{Y}\right)^2 = -p$$

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} = -p \qquad (4.31)$$

where a dot denotes the differentiation with respect to time t and K=0, -1, +1 are the spatial curvatures for hyper-surface in the type Bianchi I, III and Kantowski-Sachs models respectively. The conservation equation is given by

$$\dot{\mu} + (\mu+p)\left(\frac{\dot{X}}{X} + \frac{2\dot{Y}}{Y}\right) = 0 \qquad (4.32)$$

These equations can be written in the form

$$\dot{\theta} + \frac{1}{3}\,\theta^2 + 2\sigma^2 + \dot{\phi}^2 - V(\phi) = 0$$

$$\dot{\sigma} + \sigma\,\theta = \frac{\mathcal{R}}{2\sqrt{3}},$$

$$= \sqrt{\frac{1}{3}}\left(\frac{1}{2}\,\dot{\phi}^2 + V(\phi) + \sigma^2 - \frac{1}{3}\,\theta^2\right)$$

$$\ddot{\phi} + \theta\,\dot{\phi} + \frac{dV}{d\phi} = 0 \qquad (4.33)$$

where

$$\theta^2 = 3\sigma^2 + 3\left(\frac{1}{2}\,\dot{\phi}^2 + V(\varphi)\right) - \frac{3\mathcal{R}}{2}$$

and the spatial 3-curvature is defined by $\mathcal{R} = \frac{2k}{Y^2}$. In these equations, θ is the rate of volume expansion given by

$$\theta = \frac{\dot{X}}{X} + 2\,\frac{\dot{Y}}{Y}$$

and shear scalar σ is given by $\sigma = \frac{1}{2}\,\sigma_{\mu\nu}\,\sigma^{\mu\nu}$ where $\sigma_{\mu\nu}$ is the shear tensor. Thus

$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right)$$

and ϕ is a scalar field with a self-interaction potential $V(\varphi)$ The following are the 3 special cases viz. (i) for $\mathcal{R}=0$, the equations corresponds to a Bianchi I and (ii) for $\mathcal{R}<0$, the equations corresponds to a Bianchi III model and finally (iii) for $\mathcal{R}>0$, the equations describe a Kantowski-Sachs model.

Let a(t) be an average length scale defined by $a(t)=(XY^2)^{\frac{1}{3}}$ and $a(t) = e^{\alpha(t)}$. Also defining a new time variable τ as before and a new set of variables defined by (4.26) and $Z = \sigma / \sqrt{V}$ so the system of equations (4.33) can be written as a three-dimensional autonomous system

$$X' = \frac{1}{2}\lambda xy - x^2 - \frac{2}{3}Z^2 - \frac{1}{3}(y^2-1)$$

$$Y' = \frac{1}{2}\lambda y^2 - 3xy + \lambda$$

$$Z' = -3xz + \frac{1}{2}\lambda yz + \sqrt{\frac{1}{3}}\left(\frac{1}{2}y^2+1+z^2-3x^2\right) \qquad (4.35)$$

One can analyse above equations in a similar way to the isotropic model. The critical points of the system are

$$\text{III} \quad (x_0, y_0, z_0) = \left(\pm\frac{\sqrt{2(\lambda^2+1)}}{3\sqrt{\lambda^2+2}}, \ \pm\frac{\lambda\sqrt{2}}{\sqrt{\lambda^2+2}}, \ \pm\frac{(2-\lambda^2)\sqrt{\lambda^2+2)}}{\sqrt{6}(\lambda^2+2)}\right)$$

$$\text{IV} \quad (x_0, y_0, z_0) = \left(\pm\frac{\sqrt{2}}{\sqrt{6-\lambda^2}}, \ \pm\frac{\lambda\sqrt{2}}{\sqrt{6-\lambda^2}}, \ 0\right)$$

The critical points IV lie on the curve corresponding to the k=0, Z=0 isotropic model and when $\lambda = \sqrt{6}$ these points go off to infinity when $\lambda < \sqrt{2}$, The other critical points all have positive spatial curvature and for $\lambda > \sqrt{2}$, the spatial curvature becomes negative just as in the isotropic case. When $\lambda = \sqrt{2}$, the critical points III and IV coalesce.

Putting $x-x_0 = X$, $y-y_0=Y$ and $z-z_0=Z$ in (4.35)

$$X' = \left(\frac{1}{2}\lambda y_0 - 2x_0\right)X + \left(\frac{1}{2}\lambda x_0 - \frac{2}{3}y_0\right)Y - \frac{4}{3}Z_0 z + P(x,Y,Z)$$

$$Y' = -3y_0 X + (\lambda y_0 - 3x_0)Y + Q(X,Y,Z)$$

$$Z' = (-3Z_0 - 2\sqrt{3}\,x_0)X + \left(\frac{1}{2}\lambda z_0 + \frac{1}{\sqrt{3}}Y_0\right)Y$$
$$+ \left(-3x_0 + \frac{1}{2}\lambda y_0 + \frac{2}{\sqrt{3}}Z_0\right)Z + R(X,Y,Z)$$

where

where

$$P(X,Y,Z) = \frac{1}{2}\lambda XY - X^2 - \frac{2}{3}Z^2 - \frac{1}{3}Y^2$$

$$Q(X,Y,Z) = \frac{1}{2}\lambda Y^2 - 3XY$$

$$R(X,Y,Z) = -3XZ + \frac{1}{2}\lambda YZ + \frac{1}{\sqrt{3}}\left(\frac{1}{2}Y^2+Z^2-3X^2\right)$$

such that $P,Q,R \longrightarrow 0$ as $X,Y,Z \longrightarrow 0$. Now the characteristic equation for linearized autonomous system

$$X' = \left(\frac{1}{2}\lambda y_o - 2x_o\right)X + \left(\frac{1}{2}\lambda x_o - \frac{2}{3}y_o\right)Y - \frac{4}{3}z_o Z$$

$$Y' = -3y_o X + (\lambda y_o - 3x_o)Y$$

$$Z' = (-3z_o - 2\sqrt{3}x_o)\,X + \left(\frac{1}{2}\lambda z_o + \frac{1}{\sqrt{3}}y_o\right)Y + \left(-3x_o + \frac{1}{2}\lambda y_o + \frac{2}{3}z_o\right)Z$$

is

$$\xi^3 - \left(2\lambda y_o - 8x_o + \frac{2}{\sqrt{3}}z_o\right)\xi^2 + \Big\{21x_o^2 - 2y_o^2 + \frac{5}{4}\lambda^2 y_o^2 - 4z_o^2 - 9\lambda x_o y_o + \left(\frac{1}{\sqrt{3}} + \frac{2}{3}\right)\lambda y_o z_o - \frac{1}{\sqrt{3}}(12+2\sqrt{3})\,x_o z_o\Big\}\xi -$$

$$\Big\{\left(\frac{1}{2}\lambda y_o - 2x_o\right)(\lambda y_o - 3x_o)\left(-3x_o + \frac{1}{2}\lambda y_o + \frac{2}{\sqrt{3}}z_o\right) - \left(\frac{1}{2}\lambda x_o - \frac{2}{3}y_o\right)(-3y_o)\left(-3x_o + \frac{1}{2}\lambda y_o + \frac{2}{\sqrt{3}}z_o\right) + \frac{4}{3}z_o(+3y_o)\left(\frac{1}{2}\lambda z_o + \frac{1}{\sqrt{3}}y_o\right) - \frac{4}{3}z_o(3z_o + 2\sqrt{3}\,x_o)(\lambda y_o - 3x_o)\Big\} = 0$$

For different values of λ the nature of the critical points are given below:

(i) For $\lambda=0$, the critical points are

$$(x_o, y_o, z_o) = (\frac{1}{3}, 0, \frac{1}{\sqrt{3}}) \tag{4.36}$$

$$(x_o, y_o, z_o) = (\frac{1}{\sqrt{3}}, 0, 0) \tag{4.37}$$

For the critical point (4.36), the eigenvalues have different signs, and the critical point is a saddle and hence is unstable to small perturbations in certain directions. This point represents a solution with cosmological constant, which remains anisotropic.

For the critical point (4.37) the eigen values of the linearized system are all real and negative so the critical point is a stable node and all trajectories crossing a sphere centered on the critical point tend towards the point as $t \longrightarrow \infty$.

When $\lambda=0$, the coordinates t and τ are simply proportional to each other and so the scale factor $a(t) \sim e^{t\sqrt{\Lambda/3}}$. This solution is an example of conventional inflation with the scale factor increasing exponentially. As $\lambda=0$, the potential reduces to a constant and as the model expands the kinetic terms ϕ^2 will damp away leaving an effective cosmological constant. The solution has $Z=\sigma/\sqrt{\Lambda}=0$ and so $\sigma \longrightarrow 0$ as the trajectories to the critical point, hence the solution asymptotically tends to isotropy in the sense that $\sigma/\theta \longrightarrow 0$.

(ii) For $0 < \lambda < \sqrt{2}$, by choosing a fixed value of λ in this range, one can study the behaviour of the critical point for that particular value of λ. Suppose $\lambda=1$, so for this critical point are located at

$$(x_o, y_o, z_o) = \left(\frac{2\sqrt{2}}{3\sqrt{3}} , \sqrt{\frac{2}{3}} , \frac{1}{3\sqrt{2}} \right) \tag{4.38}$$

$$(x_o, y_o, z_o) = \left(\sqrt{\frac{2}{5}} , \sqrt{\frac{2}{5}} , 0 \right) \tag{4.39}$$

For the critical points (4.38), the eigenvalues $\{\xi_i\}$ satisfy $\xi_1 < \xi_2 < o < \xi_3$, and the critical point is unstable in the direction corresponding to ξ_3. The shear evolves as $\sigma \sim 1/t$ and since $\theta \sim 1/t$ as well, the ratio of shear to the expansion factor remains constant; the model does not isotropize.

The roots of the characteristic equations for the critical point (4.39) are all real and negative and so the critical point is stable. The point is a stable node and lies on the k=0 trajectory. Again the ratio $\sigma/\theta \longrightarrow 0$ as the equillibrium point is approached and the model can be said to approach isotropy as $t \longrightarrow \alpha$.

The critical point represents an isotropic model with a scale factor $a(t) \sim t^2$ which is an example of power-law inflation[18] and since the scale factor grows faster than the horizon, so this model also solves the horizon and flatness problems.

(iii) For $\lambda = \sqrt{2}$, the two critical points merge as in the isotropie case. The single critical point is on the k=0 trajectory and has coordinates.

$$(x_o, y_o, z_o) = \left(\frac{1}{\sqrt{2}} , 1, 0 \right)$$

and the eigenvalues of the linearized system are

$$\xi_1 = -\sqrt{2}$$

$$\xi_2 = -\sqrt{2}$$

$$\xi_3 = 0$$

Once again a critical case come into existence where one of the eigen values is zero and the stability of the point can not be determined from the linear terms alone.

(iv) For $\sqrt{2} < \lambda < \sqrt{6}$, by choosing a particular value in this range one can study the behaviour of the trajectories in this range of the parameter. For this let $\lambda = 2$(say), the critical points are

$$(x_o, y_o, z_o) = \left(\frac{5}{3\sqrt{3}} , \frac{2}{\sqrt{3}} , \frac{-1}{3} \right)$$

$$(x_o, y_o, z_o) = (1,\ 2,\ 0)$$

In the case of second critical point, the eigenvalues are no longer of the same sign and hence it is not a stable node. The point still lies on the k=0 trajectory but it is not an attractor in all directions.

For the remaining critical point two of the eigenvalues of the characteristic equation are complex and such that one is the conjugate of the other. The remaining eigenvalue (ξ_1) is real. The real parts of the complex eigen values satisfy the inequality

$$\xi_1 > \mathrm{Re}(\xi_2) = \mathrm{Re}(\xi_3) < 0$$

Hence the critical point is a stable spiral point. Trajectories tend to the critical point faster in the (ζ_2, ζ_3) plane than in the ζ_1 direction and there is a rotation in the (ζ_2, ζ_3) plane where $\{\zeta_i\}$ are the eigenvectors corresponding to the eigenvalues $\{\xi_i\}$. The solution at the attractor represents an expanding universe with a scale factor $a \sim t^P$ with $P < 1$. The scale factor expands at a slower rate than the horizon size and hence the horizon and flatness problem are not solved. The ratio $\sigma/\theta \longrightarrow$ constant and so the model remains anisotropic.

(v) For $\lambda = \sqrt{6}$, the critical point which lies on the k=0 trajectory goes off to infinity as in isotropic model and the remaining critical points are

$$(x_o, y_o, z_o) = \left(\frac{7}{6} , \frac{\sqrt{6}}{2} , -1/\sqrt{3} \right)$$

Two eigenvalues are complex and one gets

$$Re(\xi_1) > Re(\xi_2) = Re(\xi_3) < 0$$

Hence the critical point is again an attracting spiral. There is a contraction in the ζ_1 direction with a faster contraction and rotation in the (ζ_2, ζ_3) plane. The model has $\sigma/\theta \longrightarrow$ constant and so will not isotropize. The attractor represents a model expanding with a scale factor $a \sim t^P$ with $P < 1$ and so will not solve the horizon and flatness problems. The results of this section are summarised in table 2. In the table certain values of λ is chosen to indicate the asymptotic behaviour of the trajectories.

Table 2

Summary of solutions anisotropic models

Critical Point	λ	a(t)	σ(t)	ϕ(t)	Stability
$\left(\frac{1}{3}, 0, \frac{1}{\sqrt{3}}\right)$	0	$e^{\frac{t\sqrt{\Lambda}}{3}}$	$\sqrt{\frac{\Lambda}{3}}$	constant	No
$\left(\frac{1}{\sqrt{3}}, 0, 0\right)$	0	$e^{t\sqrt{\Lambda/3}}$	0	constant	Yes
$\left(\frac{2\sqrt{2}}{3\sqrt{3}}, \sqrt{\frac{2}{3}}, \frac{1}{3\sqrt{2}}\right)$	1	$t^{4/3}$	$\frac{1}{t\sqrt{3}}$	$2 \ln \left(t \sqrt{\frac{\Lambda}{6}}\right)$	No
$\left(\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, 0\right)$	1	t^2	0	$2 \ln \left(t \sqrt{\frac{\Lambda}{10}}\right)$	Yes
$\left(\frac{1}{\sqrt{2}}, 1, 0\right)$	$\sqrt{2}$	t	0	$\sqrt{2} \ln \left(t \sqrt{\frac{\Lambda}{2}}\right)$	No
$\left(\frac{5}{3\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{-1}{3}\right)$	2	$t^{5/6}$	$-\frac{1}{2\sqrt{3t}}$	$\ln \left[2 \sqrt{\frac{\Lambda}{3}}\, t\right]$	Yes
(1, 2, 0)	2	$\sqrt{t}$	0	$\ln [2 t\sqrt{\Lambda}]$	No
$\left(\frac{7}{6}, \frac{\sqrt{6}}{2}, -\frac{1}{\sqrt{3}}\right)$	$\sqrt{6}$	$t^{7/9}$	$-\frac{2}{3\sqrt{3t}}$	$\frac{2}{\sqrt{6}} \ln \left[\frac{3}{2} \sqrt{\Lambda t} \right]$	Yes

The asymptotic behaviour for a(t), σ (t) and ϕ (t) are shown as $t \longrightarrow \infty$ together with the resultant asymptotic stability of the critical points III and IV listed in the text.

4.5 Discussion of the results

In this chapter both isotropic and anisotropic cosmological models with a scalar field having an exponential $V(\varphi) = \Lambda \exp(-\lambda\phi)$ with Λ and λ non-negative constants, which lead to power-law inflation when $\lambda > 0$, have been studied.

The highest constraints on power-law inflation occur from the Universe which reheats to a sufficiently high temperature to allow the baryogenesis to occur. It has been shown[18] that a model with $a(t) \sim t^P$ where $P \geqslant 10$ satisfies these constraints and those imposed by the isotropy of the microwave background radiation. However, the lower value of P would be admissible if a low-temperature scenario for baryogenesis is possible which occurs in supersymmetric models where the supersymmetric fermion are allowed to decay[19]. This model easily copes with re-heating temperatures of about 10^{14}GeV[20] with a value of $P \sim 2$, consistent with the constraints from the microwave background.

The classification of cosmological models possessing a self-interacting scalar field with an exponential poletial which are expressed earlier can evaluate the possibility of inflation in various gravity theories which generalize general realtivity. For example, theories in which the gravitational lagrangian is an arbitrary analytic function of the scalar curvature[21] can all be shown to be conformally equivalent to general relativity plus a self-interacting scalar field which possess a potential which is typically of the exponential form in the limit of large ϕ[22]. So the results obtained earlier can be used in determining whether such theories ingeneral possess no-hair theorems in the presence of inflation.

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CHAPTER - V

BULK VISCOSITY AND THE INFLATIONARY UNIVERSE

Though most cosmological models are based on an inviscid dust or perfect-fluid description of matter, the importance of viscosity in the theory of the early universe has at times been emphasized[1]. This gives the justification in work of Israel and Vardalas[2] who have shown that the bulk viscosity ζ of simple gases does not vanish at temperatures intermediate between the non-realivistic and extreme-relativistic limits. Weinberg's also supported in his calculations regarding fluid mixtures of material particles with short mean free time, and radiation quanta such as photons, neutrinos and gravitons[3].

The exact solutions of Einsteins equations have been found for classes of cosmological models that incorporate bulk[4] and even shear viscosity[5]. According to Neugebauer and Strobel[6], Treciokas and Ellis[7] and Nightingale[8], for homogeneous, isotropic models equation of state are of the type

$$p = \alpha\,\rho \quad , \; 0 \lesssim \alpha \lesssim \frac{1}{3} \tag{5.1}$$

$$p = (\gamma - 1)\,\rho - \zeta\,\theta \;, \quad 1 \lesssim \nu \lesssim 4/3 \tag{5.2}$$

the cosmic scale factor R incrases with viscosity according to

$$R(t) = \{ \frac{2}{3\zeta} \;(\; \exp\,[\; \frac{3\zeta t}{2}\;]-1 \;)\}^{2/3\gamma} \tag{5.3}$$

where ζ is constant. The above results will not have any place in classical cosmology where one expects the expansion to slow down because of viscosity. The question is readdressed here in an inflationary context[9].

In order to determine the effect of viscosity on the onset of inflation, (5.1) or (5.2) are replaced by the appropriate equation of state[10]

$$p = \; -\rho \tag{5.4}$$

The viscosity in the source term includes in Einstein's equations by writing the energy momentum tensor for a viscous fluid in the form.

$$T_{\mu\nu} = (\rho+p)\; U_\mu\; U_\nu \;-\; p g_{\mu\nu} + \Lambda_{\mu\nu}$$

where U_μ is the velocity vector such that

$$U_\alpha U^\alpha = 1$$

and

$$\Lambda_{\mu\nu} = \zeta \; P_{\mu\nu} \; U^\lambda ; \; \lambda + \eta \, P_{\mu\alpha} \; P_{\nu\beta} \; (U^{\alpha;\beta} + U^{\beta;\alpha} - \frac{2}{3} \, g^{\alpha\beta} \; U^\lambda ; \lambda) \quad (5.5)$$

where ζ and η are the coffecients of bulk and shear viscosity and

$$P_{\mu\alpha} = g_{\mu\alpha} - U_\mu \, U_\alpha \quad (5.6)$$

It is convenient to consider the spherically symmetric metric

$$ds^2 = \exp \, [2\nu] dt^2 - \exp[2\lambda] \; dr^2 - Y^2 d\theta^2 - Y^2 \mathrm{Sin}^2\theta \; d\phi^2 \quad (5.7)$$

Here the unknown functions ν, λ and Y depends only on r and t. In the comoving frame of the fluid, U is defined as

$$U = \exp \; [\nu] \delta^o{}_\mu$$

Einstein's equations become,

$$-\frac{1}{Y^2}+\frac{2\exp[-2\lambda]}{Y}\left\{Y''-Y'\lambda'+\frac{Y'^2}{2Y}\right\}-\frac{2\exp[-2\nu]}{Y}\left\{\dot{Y}\dot{\lambda}+\frac{\dot{Y}^2}{2Y}\right\}=-8\pi\rho \tag{5.8}$$

$$\frac{2\exp[-2\lambda]}{Y}\left\{-\dot{Y}'+Y'\dot{\lambda}+\dot{Y}\nu'\right\}=0 \tag{5.9}$$

$$-\frac{1}{Y^2}+\frac{2}{Y}\exp[-2\lambda]\left\{Y'\nu'+\frac{Y'^2}{2Y}\right\}-\frac{2\exp[-2\nu]}{Y}\left\{\ddot{Y}+\frac{\dot{Y}^2}{2Y}-\dot{Y}\dot{\nu}\right\}=$$
$$=-8\pi\left\{-p-\exp[-\nu]\left[\left(-\frac{4\eta}{3}-\zeta\right)\dot{\lambda}+\left(\frac{4\eta}{3}-2\zeta\right)\frac{\dot{Y}}{Y}\right]\right\} \tag{5.10}$$

$$\exp[-2\lambda]\left\{\frac{Y''}{Y}+\nu''+\nu'^2-\frac{Y'\lambda'}{Y}-\lambda'\nu'+\frac{Y'\nu'}{Y}\right\}+$$
$$+\exp[-2\nu]\left\{-\ddot{\lambda}-\dot{\lambda}^2-\frac{\ddot{Y}}{Y}+\dot{\lambda}\dot{\nu}\ \frac{\dot{Y}\dot{\nu}}{Y}-\frac{\dot{Y}\dot{\lambda}}{Y}\right\}=$$
$$=-8\pi\left\{-p-\exp[-\nu]\left[\left(\frac{2\eta}{3}-\zeta\right)\dot{\lambda}+\left(\frac{-2\eta}{3}-2\zeta\right)\frac{\dot{Y}}{Y}\right]\right\} \tag{5.11}$$

where $\dot{} = \frac{\partial}{\partial t}$ and $' = \partial/\partial r$

The canonical form of the Robertson-Walker metric is obtained from (5.7) by setting

$$\nu=0,\ \exp[2\lambda]=R^2(t)(1-kr^2)^{-1},\ Y^2=R^2(t)r^2 \tag{5.12}$$

Then equations (5.8)-(5.11) reduce to[11]

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi}{3} \rho \tag{5.13}$$

$$\frac{\ddot{R}}{R} - 12 \pi\zeta \frac{\dot{R}}{R} = - \frac{4\pi}{3} \left(\rho + 3p\right) \tag{5.14}$$

combining (5.13), (5.14) and (5.4), one obtains

$$\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 - 12 \pi\zeta \frac{\dot{R}}{R} = \frac{k}{R^2} \tag{5.15}$$

From eqns.(5.15) it follows that universes for which $\zeta \alpha \rho$ or $\zeta = \text{const} > 0$ or $\zeta t = \zeta_1$ with $\zeta_1 = \text{const} > 0$ are not compatible with an exponential expansion. In fact the inflationary expansion is only possible if at the same time ζ decreases exponentially which can be found out from (5.15) by substitution .

If in this process viscosity virtually disappears then only the phase transitions from a universe with non-negligible bulk viscosity to an inflationary one can occur. This and the fact that the Higgs fields in metastable false vacua of inflationary cosmologies correspond to superfluids[12] satisfying the special equation of state(5.4), strongly suggest the phase-change involved coresponds to a normal fluid to superfluid transition.

The argument advanced, makes the viscous fluid model a strong candidate for the pre-inflationary universe. Here some new exact solution are given which correspond to different functional forms of ζ and the equation of state(5.1).

If $\zeta t = \zeta_1 > 0$, a solution of (5.13) and (5.14) for a dust-filled universe is

$$R(t) = \left(\frac{k}{24\pi\zeta_1 - 1}\right)^{\frac{1}{2}} t \,,$$

$$\rho = \frac{9\zeta_1}{t^2} \qquad (5.16)$$

With $\zeta_1 > 1/24\pi$ if $k=1$ and $0 < \zeta_1 < 1/24\pi$ $k = -1$. The expansion is $\Theta = 3/t$

when $\gamma = 1$, more general dust model can be obtained from (5.2) and (5.3).

Other conformally flat universes for which the conformal factor depends exclusively on t can be obtained by setting

$$\exp\,[2\nu] = \exp\,[2\lambda] = Y^2/r^2 = f^2(t)$$

in equations (5.8) and (5.1). The following solutions have been found.

For viscous fluid with $p = \alpha \rho$ and ζ = const one obtains from (5.11) and (5.13)

$$\dot{f} - \frac{24\pi\zeta}{3\alpha+3} f^2 - c_1 f^{1-3\alpha/2} = 0 \tag{5.17}$$

where c_1 is a constant. For the particular case $c_1=0$, eqn. (5.17) integrates to

$$f = \frac{1}{c_2 - (8\pi\zeta/(\alpha+1))t} \quad , \qquad \alpha \neq -1$$

where C_2 is constant. For a radiation - filled universe ($\alpha = \frac{1}{3}$), one has in particular

$$\rho = \frac{27}{7} \pi\zeta^2 \quad , \qquad \Theta = 18\ \pi\zeta$$

and $\alpha_\mu = 0, \quad \omega_{\mu\nu} = \sigma_{\mu\nu} = 0$

In this model, the bulk viscosity affects the line element and H.

For $P = \alpha\rho$, $\zeta = \zeta_1/f$, ζ_1 = const > 0

From equations (5.8) and (5.11), one obtains

$$f = \left\{ c_2 \exp\left[-4\pi\zeta_1 \frac{1+3\alpha}{1+\alpha} t\right] - \frac{C_1(\alpha+1)}{8\pi\zeta_1} \right\}^{2/(1-3\alpha)}$$

$$\rho = \frac{24\pi\zeta_1^2}{(1+\alpha)^2} \exp\left\{-8\pi\zeta_1 \frac{(1+3\alpha)t}{1+\alpha}\right\} *$$

$$* \left\{ C_2 \exp\left[-4\pi\zeta_1 \frac{(1+3\alpha)t}{1+\alpha}\right] - \frac{C_1(\alpha+1)}{8\pi\zeta_1} \right\}^{\frac{2(1-3\alpha)}{1+3\alpha}} \qquad (5.18)$$

where C_2 is a constant.

The expansion is given by

$$\Theta = \frac{24\pi\zeta_1}{1+\alpha} \exp\left[-4\pi\zeta_1 \frac{(1+3\alpha)t}{1+\alpha}\right] * \left[C_2 \exp\left[-4\pi\zeta_1 \frac{(1+3\alpha)t}{1+\alpha}\right] - C_1 \frac{\alpha+1}{8\pi\zeta_1}\right]^{\frac{1-3\alpha}{1+3\alpha}}$$

and $a_\mu = 0$, $\omega_{\mu\nu} = \sigma_{\mu\nu} = 0$. For $t \longrightarrow \infty$ the metric becomes flat, whereas ρ vanishes in the same limit, for any positive value of α. For $\alpha < -1/3$, $f \longrightarrow 0$ exponentially with t, while for $-1/3 < \alpha < 0$, f increases exponentially.

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