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Classification of homotopy Wall's manifolds

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Abstract

Complete PL and topological classification and partial smooth classification of manifolds homotopy equivalent to a Wall's manifold (defined as a mapping torus of a Dold manifold), introduced by Wall in his 1960 Annals paper on cobordism, have been done by determining: (1) the normal invariants of Wall's manifolds, (2) the surgery obstruction of a normal invariant and (3) the action of the Wall surgery obstruction groups on the smooth, PL and homeomorphism classes of homotopy Wall's manifolds (to be made precise in the body of the paper). Consequently classification results of automorphisms (self homeomorphisms, and self PL-homeomorphisms) of Dold manifolds follow.

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Keywords: Normal invariant; Surgery obstruction; Browder–Livesay invariant; Arf invariant

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1. Introduction

This paper is a sequel to the papers [16,17], giving classification results of manifolds homotopy equivalent to the ones defined by A. Dold (see [4]), J. Milnor (see [14]) in the context of finding concrete generators of (un)oriented cobordism groups (see [20]). In the paper on determination of oriented cobordism ring C.T.C. Wall defined intermediate groups and defined new manifolds as mapping Tori of Dold manifolds, see [23]. In this paper we give classification results for the homotopy types of these manifolds of Wall. Consequently classification results of automorphisms (self homeomorphisms, and self PL-homeomorphisms) of Dold manifolds follow. Let X be a compact, connected, smooth, piecewise linear (PL) or topological manifold with or without boundary ∂X. By a homotopy smoothing (respectively homotopy PL or TOP triangulation) of the manifold X we mean a pair (M, f), where M is a smooth, PL or topological manifold and $f:(M, \partial M) \to (X, \partial X)$ is a simple homotopy equivalence of pairs, for which $f|_{\partial M}: \partial M \to \partial X$ is a diffeomorphism (respectively a PL or TOP homeomorphism). Two homotopy smoothings (respectively homotopy PL or TOP triangulations) (M, f) and (M', f') are said to be equivalent if there is a diffeomorphism (respectively PL or TOP homeomorphism) $h:(M, \partial M) \to (M', \partial M')$ for which the maps $f' \circ h$ and f are homotopic relative to the boundary ∂M. The set of equivalence classes of homotopy smoothings (respectively PL or TOP triangulations) of the manifold X is denoted by $hS(X)$ (respectively $hT_{PL}(X)$, or $hT_{TOP}(X)$) and is a pointed set with base point (X, id_X). If we use the notation $CAT = O$, PL, or TOP, then later in the paper we shall call "homotopy smoothings, homotopy

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PL or TOP triangulations", simply as "homotopy CAT structures". Also $hS(X), hT_{CAT}(X)$ are sometimes referred simply as "structure sets".

The standard method of determining the sets $hS(X)$ (respectively $hT_{PL}(X)$, or $hT_{TOP}(X)$) for various concrete manifolds X is the analysis of the following Sullivan–Wall surgery exact sequences:

$$\to L_{n+1}\big(\pi_1(X), \omega(X)\big) \xrightarrow{\delta_O} hS(X) \xrightarrow{\eta_O} [X/\partial X, G/O] \xrightarrow{\theta_O} L_n\big(\pi_1(X), \omega(X)\big),$$

$$\to L_{n+1}\big(\pi_1(X), \omega(X)\big) \xrightarrow{\delta_{PL}} hT_{PL}(X) \xrightarrow{\eta_{PL}} [X/\partial X, G/PL] \xrightarrow{\theta_{PL}} L_n\big(\pi_1(X), \omega(X)\big),$$

$$\to L_{n+1}\big(\pi_1(X), \omega(X)\big) \xrightarrow{\delta_{TOP}} hT_{TOP}(X) \xrightarrow{\eta_{TOP}} [X/\partial X, G/TOP] \xrightarrow{\theta_{TOP}} L_n\big(\pi_1(X), \omega(X)\big)$$

where $n = \dim X \geqslant 5$, and where the first and the last terms are Wall's surgery obstruction groups, second terms are as defined above and the third terms are sets of normal invariants of X; the first maps δ are the realization maps (or actions), the second maps η are the forgetful type of maps (or Pontrjagin–Thom type maps) and the last maps θ are the surgery obstruction maps. For details one can refer to the book of Wall [24].

In order to determine $hS(X), hT_{PL}(X)$, or $hT_{TOP}(X)$ one must compute the groups $L_n(\pi_1(X), \omega(X)), [X/\partial X, G/CAT]$, where $CAT = O$, PL, TOP, and also the maps θ and the actions δ in the above exact sequences.

The purpose of this paper is to determine completely the sets $hT_{PL}(X)$, and $hT_{TOP}(X)$, and partially the sets $hS(X)$, for $X = D^m \times Q(r,s)$, $r, s > 1$, where D^m is the disk of dimension $m \geqslant 0$ and $Q(r,s)$, is the *Wall's manifold* defined as a mapping torus of a Dold's manifold $P(r,s)$ as introduced by Wall [23] in his determination of the oriented cobordism ring. We recall the definition in more detail in the next section.

The main results of this paper are Theorem 8.5, Theorem 8.6, Theorems 5.4, 5.5, Propositions 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, and 6.8, Propositions 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, and Propositions 4.1, 4.2. In addition to these, many results about Wall's manifolds not very accessible in the literature have been derived e.g. Theorems 3.7, 3.8.

The paper has been arranged in the following fashion: In Section 2 we recall the basic definition and properties of Wall's manifolds (see [23]). In Section 3 we calculate in detail the integral (co)homology of Wall's manifolds by defining a suitable cellular decomposition of Wall's manifolds; these calculations were necessiated because we needed torsion information of Wall's manifolds in studying its normal invariants and complete integral cohomology information of these manifolds to give exact characterization of their structure sets. Moreover this material is not readily available in the literature. In Section 4 we study a map, intimately related to the Browder–Livesay invariants associated with the Wall's manifolds, and prove Propositions 4.1, 4.2. In Section 5 we calculate the normal invariants both in the PL and topological cases for Wall's manifolds and prove Theorems 5.4, 5.5. In Section 6 we study the action map δ_{CAT} ($CAT = PL$, or TOP) of the groups $L_{n+1}(\mathbb{Z}^- \times \mathbb{Z}/2^{\pm})$ on the homotopy CAT structures $hT_{CAT}(D^m \times Q(r,s))$ and prove Propositions 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, and 6.8. In Section 7 we calculate the image of the surgery obstruction maps θ_{CAT}, ($CAT = PL$, or TOP) in various dimensions and orientabilities of Wall's manifolds and prove Propositions 7.2, 7.3, 7.4, 7.5, 7.6, 7.7. In the last Section 8 we first give some remarks about the homotopy smoothings of Wall's manifolds which can be derived from the calculations of Sections 6, and 7; next we summarize the calculations of $hT_{CAT}(X)$ ($CAT = PL$ or TOP), $X = D^m \times Q(r,s)$ in terms of reduced Sullivan–Wall surgery short exact sequences as Theorem 8.5; and finally we determine the structure of $hT_{CAT}(Q(r,s))$ ($CAT = PL$ or TOP) in all cases as Theorem 8.6.

Techniques of the proofs are similar to the ones in Haršiladze [7], Kharshiladze [10], and López de Medrano [13]. We have tried to make the paper as self contained as possible for readability.

2. Wall's manifolds and their orientability

Recall that a *Dold's manifold* is defined as the quotient $P(r,s) \overset{\text{def}}{=} (S^r \times \mathbb{C}P^s)/\sim$, where $(x,z) \sim (x',z')$ if and only if $x' = -x$, and $z' = \bar{z}$. Let us define the involution $\phi : S^r \times \mathbb{C}P^s \to S^r \times \mathbb{C}P^s$ by $\phi(x,z) = (-x, \bar{z})$ and denote by q the quotient map $q : S^r \times \mathbb{C}P^s \to P(r,s)$. Then $q \circ \phi = q$. The projection map $S^r \times \mathbb{C}P^s \to S^r$ is equivariant with respect to the involution ϕ in the domain and the antipodal involution $a : S^r \to S^r$ of the range, so passing to orbit spaces we get that a Dold manifold can also be written as the total space of a fibre bundle over $\mathbb{R}P^r$ with fibre $\mathbb{C}P^s$:

$$\mathbb{C}P^s \xrightarrow{\text{incl}} P(r,s) \xrightarrow{\text{proj}} \mathbb{R}P^r. \tag{$*$}$$

Let $T : S^r \to S^r$ be the reflection of a point of S^r about the hyperspace $x_r = 0$. Define an involution $\widehat{T} : S^r \times \mathbb{C}P^s \to S^r \times \mathbb{C}P^s$ by $\widehat{T}(x, z) = (T(x), z)$. This involution $\widehat{T}$ commutes with the involution ϕ, so passing onto orbit spaces $\widehat{T}$ induce a homeomorphism $A : P(r, s) \to P(r, s)$, and we have a commutative rectangle:

$$\begin{tikzcd}
S^r \times \mathbb{C}P^s \arrow[r, "\widehat{T}"] \arrow[d, "q"] & S^r \times \mathbb{C}P^s \arrow[d, "q"] \\
P(r, s) \arrow[r, "A"] & P(r, s)
\end{tikzcd}$$

In case $P(r, s)$ is orientable (e.g. when r is odd and s is even or when r is even and s is odd) A reverses orientation. Now we are in a position to define the Wall's manifold.

Definition 2.1. A *Wall's manifold* $Q(r, s)$ is defined as the mapping torus of the homeomorphism $A : P(r, s) \to P(r, s)$, that is

$$Q(r, s) \overset{\text{def}}{=} P(r, s) \times [0, 1]/\sim, \quad \text{where } \bigl([x, z], 0\bigr) \sim \bigl(A[x, z], 1\bigr).$$

Define $q_1 : P(r, s) \times [0, 1] \to Q(r, s)$, as this identification map.

We give some more descriptions of $Q(r, s)$. Consider the projection to the third factor $\pi_3 : S^r \times \mathbb{C}P^s \times [0, 1] \to [0, 1]$, given by $\pi_3(x, z, t) = t$. It induces a fibration $P(r, s) \to Q(r, s) \xrightarrow{\beta} S^1$, with fibre $P(r, s)$. The following diagram is commutative:

$$\begin{tikzcd}
S^r \times \mathbb{C}P^s \arrow[r] \arrow[d, "q"] & S^r \times \mathbb{C}P^s \times [0, 1] \arrow[r, "\pi_3"] \arrow[d, "q_1 \circ (q \times 1)"] & {[0, 1]} \arrow[d, "q_2"] \\
P(r, s) \arrow[r] & Q(r, s) \arrow[r, "\beta"] & S^1
\end{tikzcd}$$

q_2 is the identification map which identifies the end points of $[0, 1]$. Consider the projection $\pi_{13} : S^r \times \mathbb{C}P^s \times [0, 1] \to S^r \times [0, 1]$, given by $\pi_{13}(x, z, t) = (x, t)$. It induces another fibration $\mathbb{C}P^s \to Q(r, s) \xrightarrow{\gamma} Q(r, 0)$, with fibre $\mathbb{C}P^s$ and group $\mathbb{Z}/2$. The following diagram is commutative:

$$\begin{tikzcd}
\mathbb{C}P^s \arrow[r] \arrow[d, "\mathrm{id}"] & S^r \times \mathbb{C}P^s \times [0, 1] \arrow[r, "\pi_{13}"] \arrow[d, "q_1 \circ (q \times 1)"] & S^r \times [0, 1] \arrow[d, "q_1 \circ (q \times 1)"] \\
\mathbb{C}P^s \arrow[r] & Q(r, s) \arrow[r, "\gamma"] & Q(r, 0)
\end{tikzcd}$$

For the fibration $\mathbb{C}P^s \to Q(r, s) \xrightarrow{\gamma} Q(r, 0)$ we have a classifying map $Q(r, 0) \xrightarrow{\theta} \mathbb{R}P^{r+1}$ and the following diagram of fibrations:

$$\begin{tikzcd}
\mathbb{C}P^s \arrow[r] \arrow[d] & \mathbb{C}P^s \arrow[d] \\
Q(r, s) \arrow[r, "\Theta"] \arrow[d, "\gamma"] & P(r + 1, s) \arrow[d] \\
Q(r, 0) \arrow[r, "\theta"] & \mathbb{R}P^{r+1}
\end{tikzcd} \tag{$**$}$$

where the map Θ is defined by $\Theta(x, z, t) = (x_0, x_1, \ldots, x_r \cos \pi t, x_r \sin \pi t, z)$. From this it follows that Θ is of degree 1 (in the nonorientable case homology with suitable local coefficients or $\mathbb{Z}/2$ coefficients is to be understood).

Theorem 2.2. *The* mod 2 *cohomology of Wall's manifold* $Q(r, s)$ *is given by*

$$H^*\bigl(Q(r, s); \mathbb{Z}/2\bigr) = \mathbb{Z}/2[x, c, d]/\bigl\langle x^2, c^{r+1} - c^r x, d^{s+1}\bigr\rangle,$$

where $\dim x = 1$, $\dim c = 1$ *and* $\dim d = 2$, *x is a spherical class induced by β, c, d are classes of same name which appear in the* mod 2 *cohomology of $P(r, s)$ and comes from spectral sequence calculation.*

Theorem 2.3. *The total Stiefel–Whitney class of $Q(r,s)$ for $r>0$ is*

$$(1+c+x)(1+c)^{r-1}(1+c+d)^{s+1}.$$

For a proof of the above theorems see [23].

Remark 2.4. From Theorems 2.3 and 2.2 it follows that the first Stiefel–Whitney class of $Q(r,s)$ is given by $w_1(Q(r,s))=(r+s+1)c+x$. So $Q(r,s)$ is *non-orientable* for all $r>0$ and is *orientable* only in the cases $r=0$, s even.

Remark 2.5. From the homotopy exact sequence of the fibration $P(r,s)\to Q(r,s)\xrightarrow{\beta} S^1$, the knowledge of fundamental group of $P(r,s)$ and homotopy groups of S^1, it follows that the fundamental group of $Q(r,s)$ is given by

$$\pi_1\big(Q(r,s)\big)=\begin{cases}\mathbb{Z}/2\times\mathbb{Z} & \text{if } r>1,\\ \mathbb{Z}\times\mathbb{Z} & \text{if } r=1.\end{cases}$$

We define the orientation homomorphism (or the orientation character) of $Q(r,s)$ as follows: It is the composite homomorphism

$$\omega^{Q(r,s)}:\pi_1\big(Q(r,s)\big)\xrightarrow{\cong} H_1\big(Q(r,s);\mathbb{Z}\big)\to H_1\big(Q(r,s);\mathbb{Z}/2\big)\xrightarrow{w_1(Q(r,s))}\mathbb{Z}/2.$$

In the later sections this will be needed to denote the appropriate Wall surgery obstruction groups.

3. Integral (co)homology of Wall's manifolds

We calculate the integral (co)homology groups of $Q(r,s)$ by using a cellular decomposition of this space. We use notations and definitions from Dold [4] and Fujii [5]:

Definition 3.1. *Defining a cell structure for S^r* [4,5]: Define a cellular decomposition for S^r by taking open i-cells:

$$C_i^+=\big\{(x_0,x_1,\dots,x_r)\in S^r \mid x_i>0,\ x_{i+1}=0=x_{i+2}=\cdots=x_r\big\},$$
$$C_i^-=\big\{(x_0,x_1,\dots,x_r)\in S^r \mid x_i<0,\ x_{i+1}=0=x_{i+2}=\cdots=x_r\big\},$$

where $0\leqslant i\leqslant r$. Treating these as the generators of the cellular chain complex of S^r, the boundary operators on these act as:

$$\partial C_i^+=C_{i-1}^++C_{i-1}^-,\quad \partial C_i^-=-\big(C_{i-1}^++C_{i-1}^-\big),\quad 0\leqslant i\leqslant r.$$

The antipodal involution $a:S^r\to S^r$ and the reflection involution $T:S^r\to S^r$ about the hyperspace $\{x_r=0\}$, defined in Section 2, induce chain maps, also denoted by a, and T respectively, of the cellular chain complex of S^r whose values on the generators are given by

$$a\big(C_i^+\big)=(-1)^{i+1}C_i^-,\quad a\big(C_i^-\big)=(-1)^{i+1}C_i^+,\quad 0\leqslant i\leqslant r,$$
$$T\big(C_i^+\big)=C_i^+,\quad T\big(C_i^-\big)=C_i^-,\quad 0\leqslant i<r,\qquad T\big(C_r^+\big)=C_r^-,\qquad T\big(C_r^-\big)=C_r^+.$$

Definition 3.2. *Defining a cell structure for $\mathbb{C}P^s$* [4,5]: Define a cellular decomposition for $\mathbb{C}P^s$ by taking open $2j$-cells:

$$D_j=\big\{[z_0,z_1,\dots,z_s]\in\mathbb{C}P^s \mid z_j>0,\ z_{j+1}=0=z_{j+2}=\cdots=z_s\big\}$$

where $0\leqslant j\leqslant s$. Treating these as the generators of the cellular chain complex of $\mathbb{C}P^s$, the boundary operators on these act as:

$$\partial D_j=0,\ \text{as there are no odd dimensional cells.}$$

The conjugation involution $b:\mathbb{C}P^s\to\mathbb{C}P^s$, $b(z)=\bar{z}$ induces a chain map, also denoted by b, of the cellular chain complex of $\mathbb{C}P^s$ whose values on the generators are given by

$$b(D_j)=(-1)^jD_j,\quad 0\leqslant j\leqslant s.$$

Definition 3.3. *Defining a cell structure for* $S^r \times \mathbb{C}P^s$ [4,5]: Define a cellular decomposition for $S^r \times \mathbb{C}P^s$ by taking open $(i+j)$-cells:

$$\bigl\{C_i^+ \times D_j, C_i^- \times D_j \mid 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s\bigr\}.$$

Treating these as the generators of the cellular chain complex of $S^r \times \mathbb{C}P^s$, the boundary operators on these act as:

$$\begin{aligned}
&\partial\bigl(C_i^+ \times D_j\bigr) = +\bigl(C_{i-1}^+ \times D_j + C_{i-1}^- \times D_j\bigr), \quad 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s,\\
&\partial\bigl(C_i^- \times D_j\bigr) = -\bigl(C_{i-1}^+ \times D_j + C_{i-1}^- \times D_j\bigr), \quad 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s,\\
&\partial\bigl(C_0 \times D_j\bigr) = 0, \quad 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s.
\end{aligned}$$

The involution $\phi : S^r \times \mathbb{C}P^s \to S^r \times \mathbb{C}P^s$, and the involution $\widehat{T} : S^r \times \mathbb{C}P^s \to S^r \times \mathbb{C}P^s$, defined in Section 2, induce chain maps, also denoted by ϕ, and $\widehat{T}$ respectively, of the cellular chain complex of $S^r \times \mathbb{C}P^s$ whose values on the generators are given by

$$\phi\bigl(C_i^+ \times D_j\bigr) = (-1)^{i+j+1}\bigl(C_i^- \times D_j\bigr), \qquad \phi\bigl(C_i^- \times D_j\bigr) = (-1)^{i+j+1}\bigl(C_i^+ \times D_j\bigr),$$

where $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant s$.

$$\begin{aligned}
&\widehat{T}\bigl(C_i^+ \times D_j\bigr) = \bigl(C_i^+ \times D_j\bigr), \quad \widehat{T}\bigl(C_i^- \times D_j\bigr) = \bigl(C_i^- \times D_j\bigr), \quad 0 \leqslant i < r,\ 0 \leqslant j \leqslant s,\\
&\widehat{T}\bigl(C_r^+ \times D_j\bigr) = \bigl(C_r^- \times D_j\bigr), \quad \widehat{T}\bigl(C_r^- \times D_j\bigr) = \bigl(C_r^+ \times D_j\bigr), \quad 0 \leqslant j \leqslant s.
\end{aligned}$$

Definition 3.4. *Defining a cell structure for* $P(r,s)$ [4,5]: Define a cellular decomposition for $P(r,s)$ by taking open $(i+j)$-cells:

$$\bigl\{(C_i, D_j) \overset{\text{def}}{=} q\bigl(C_i^+ \times D_j\bigr) \mid 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s\bigr\},$$

where $q : S^r \times \mathbb{C}P^s \to P(r,s)$ is the quotient map defined in Section 2. Treating these as the generators of the cellular chain complex of $P(r,s)$, and treating q as a cellular chain map, the boundary operators on these generators can easily be seen to act as:

$$\begin{aligned}
&\partial(C_i, D_j) = \bigl(1 + (-1)^{i+j}\bigr)(C_{i-1}, D_j), \quad 1 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s,\\
&\partial(C_0, D_j) = 0, \quad 0 \leqslant j \leqslant s.
\end{aligned}$$

The homeomorphism $A : P(r,s) \to P(r,s)$ defined in Section 2 induces a chain map, also denoted by A, of the cellular chain complex of $P(r,s)$ whose values on the generators can be calculated using the property $A \circ q = q \circ \widehat{T}$, between the relevant cellular chain complexes. These are given by

$$A(C_i, D_j) = \begin{cases} (-1)^{r+j+1}(C_r, D_j) & \text{if } i = r,\ 0 \leqslant j \leqslant s,\\ (C_i, D_j) & \text{if } 0 \leqslant i < r,\ 0 \leqslant j \leqslant s. \end{cases}$$

Definition 3.5. *Defining a cell structure for* $Q(r,s)$: Define a cellular decomposition for $P(r,s) \times [0,1]$ by taking open cells:

$$\bigl\{(C_i, D_j) \times (0,1), (C_i, D_j) \times \{0\}, (C_i, D_j) \times \{1\} \mid 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s\bigr\},$$

where $[0,1]$ is considered as a cell complex with open cells $\{\{0\}, \{1\}, (0,1)\}$. Using the quotient map $q_1 : P(r,s) \times [0,1] \to Q(r,s)$, defined in Section 2, we define a cellular decomposition for $Q(r,s)$ by taking open cells

$$\begin{aligned}
&\bigl\{\bigl[(C_i, D_j) \times (0,1)\bigr] \overset{\text{def}}{=} q_1\bigl((C_i, D_j) \times (0,1)\bigr) \mid 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s\bigr\},\\
&\bigl\{\bigl[(C_i, D_j)\bigr] \overset{\text{def}}{=} q_1\bigl((C_i, D_j) \times \{0\}\bigr) \mid 0 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant s\bigr\}.
\end{aligned}$$

Treating these as the generators of the cellular chain complex of $Q(r,s)$, treating q_1 as a cellular chain map, denoted again by q_1 (images of the other cells of $P(r,s) \times [0,1]$ under q_1 can be written in terms of these generators using the property of the cellular chain map A of the cellular chain complex of $P(r,s)$), the boundary operators on these generators can be calculated. These are given as:

$$\partial\big[(C_i,D_j)\times(0,1)\big]=\begin{cases}(1+(-1)^{r+j})[(C_{r-1},D_j)\times(0,1)]+(-1)^{r+2j}((-1)^{r+j+1}+1)[(C_r,D_j)]\\ \quad\text{if } i=r,\ 0\leqslant j\leqslant s,\\ (1+(-1)^{i+j})[(C_{i-1},D_j)\times(0,1)]+(-1)^{i+2j}.2[(C_i,D_j)]\\ \quad\text{if } 0<i<r,\ 0\leqslant j\leqslant s,\\ 0\ \text{ if } i=0,\ 0\leqslant j\leqslant s,\end{cases}$$

$$\partial\big[(C_i,D_j)\big]=\begin{cases}(1+(-1)^{i+j})[(C_{i-1},D_j)] & \text{if } 0<i\leqslant r,\ 0\leqslant j\leqslant s,\\ 0 & \text{if } i=0,\ 0\leqslant j\leqslant s.\end{cases}$$

Definition 3.6. *Cochain complex dual to the cellular chain complex of $Q(r,s)$*: Let $[(c^i,d^j)\times(0,1)]$ and $[(c^i,d^j)]$ be the cochain duals of the open cells $[(C_i,D_j)\times(0,1)]$ and $[(C_i,D_j)]$ of $Q(r,s)$ respectively, then these generate the cellular cochain complex of $Q(r,s)$. The coboundary operators on these generators act as follows:

$$\delta\big[(c^i,d^j)\times(0,1)\big]=\big(1+(-1)^{i+j+1}\big)\big[(c^{i+1},d^j)\times(0,1)\big],\quad 0\leqslant i\leqslant r-1,\ 0\leqslant j\leqslant s,$$

$$\delta\big[(c^i,d^j)\big]=\begin{cases}(-1)^{r+2j}((-1)^{r+j+1}+1)[(c^r,d^j)\times(0,1)] & \text{if } i=r,\ 0\leqslant j\leqslant s,\\ (-1)^{i+2j}.2[(c^i,d^j)\times(0,1)]+(1+(-1)^{i+j+1})[(c^{i+1},d^j)] & \text{if } 0\leqslant i<r,\ 0\leqslant j\leqslant s.\end{cases}$$

Now we are in a position to write down the (co)homology groups of $Q(r,s)$ with integer coefficients.

Theorem 3.7 *(Homology of $Q(r,s)$ with integer coefficients). The total homology group $H_*(Q(r,s);\mathbb{Z})$ is a direct sum of the following free* (*abelian*) *and torsion parts*:

(i) *If r* is even, *its free abelian part is generated by*

$$\big\{\big[(C_0,D_{2j})\times(0,1)\big],\big[(C_0,D_{2j})\big]\mid 0\leqslant j\leqslant[s/2]\big\},$$

and its torsion part is generated by

$$\big\{\big[(C_{2i},D_{2j+1})\big],\big[(C_{2i-1},D_{2j})\big]\mid 0\leqslant 2i,\ 2i-1\leqslant r,\ 0\leqslant j\leqslant[s/2]\big\},$$

whose order are 2.

(ii) *If r* is odd, *its free abelian part is generated by*

$$\big\{\big[(C_0,D_{2j})\times(0,1)\big],\big[(C_0,D_{2j})\big]\mid 0\leqslant j\leqslant[s/2]\big\},$$

and its torsion part is generated by

$$\big\{\big[(C_{2i},D_{2j+1})\big],\big[(C_{2i-1},D_{2j})\big]\mid 0\leqslant 2i,\ 2i-1\leqslant r,\ 0\leqslant j\leqslant[s/2]\big\},$$

whose order are 2.

Theorem 3.8 *(Cohomology of $Q(r,s)$ with integer coefficients). The total cohomology group $H^*(Q(r,s);\mathbb{Z})$ is a direct sum of the following free* (*abelian*) *and torsion parts*:

(i) *If r* is even, *its free abelian part is generated by*

$$\big\{\big[(c^r,d^{2j})\big]\mid 0\leqslant j\leqslant[s/2]\big\},$$

and its torsion part is generated by

$$\big\{\big[(c^{2i},d^{2j})\times(0,1)\big],\big[(c^{2i-1},d^{2j+1})\times(0,1)\big],\big[(c^r,d^{2j+1})\times(0,1)\big]\mid\\ 0\leqslant 2i,\ 2i-1\leqslant r,\ 0\leqslant j\leqslant[s/2]\big\},$$

whose order are 2.

(ii) *If r* is odd, *its free abelian part is generated by*

$$\big\{\big[(c^r,d^{2j+1})\big]\mid 0\leqslant j\leqslant[s/2]\big\},$$

and its torsion part is generated by

$$\big\{\big[(c^{2i},d^{2j})\times(0,1)\big],\big[(c^{2i-1},d^{2j+1})\times(0,1)\big],\big[(c^r,d^{2j})\times(0,1)\big]\mid 0\leqslant 2i,\ 2i-1\leqslant r,\ 0\leqslant j\leqslant[s/2]\big\},$$

whose order are 2.

In particular, we have

Theorem 3.9.

$$H^4\big(Q(r,s);\mathbb{Z}\big)=\begin{cases}\mathbb{Z}/2 & \textit{if } r>4,\ s\geqslant 2,\\ \mathbb{Z}\times\mathbb{Z}/2 & \textit{if } r=4,\ s\geqslant 2,\\ \mathbb{Z}/2\times\mathbb{Z}/2 & \textit{if } r=3,\ s\geqslant 2,\\ \mathbb{Z}/2 & \textit{if } r=2,\ s\geqslant 2,\end{cases}$$

and in particular,

$$H^4\big(Q(r,0);\mathbb{Z}\big)=\begin{cases}0 & \textit{if } r>4,\ s=0,\\ \mathbb{Z} & \textit{if } r=4,\ s=0,\\ \mathbb{Z}/2 & \textit{if } r=3,\ s=0,\\ 0 & \textit{if } r=2,\ s=0.\end{cases}$$

Remark 3.10. There is another way in which one can determine the homology groups of $Q(r,s)$, which is the mapping torus of the homeomorphism $A: P(r,s)\to P(r,s)$ defined in Section 2, from the homology groups of $P(r,s)$, (see e.g. [16]) and the information of the homomorphism $A_*: H_*(P(r,s);\mathbb{Z})\to H_*(P(r,s);\mathbb{Z})$ induced by A as determined above, from the following exact homology sequence of the mapping torus (see [8, p. 151, Example 2.48]): Coefficients of all the following homology groups are $\mathbb{Z}$:

$$\to H_k\big(P(r,s)\big)\xrightarrow{1-A_*}H_k\big(P(r,s)\big)\xrightarrow{i_*}H_k\big(Q(r,s)\big)\to H_{k-1}\big(P(r,s)\big)\xrightarrow{1-A_*}H_{k-1}\big(P(r,s)\big)\to .$$

4. The Browder–Livesay invariants associated to Wall's manifold

Let $X=Q(r,s)$, and $Y=Q(r-1,s)$, $r\geqslant 3$, $s>1$ then the inclusion $Y\subset X$ induces isomorphism of fundamental groups $\pi_1(Y)\cong\pi_1(X)=\mathbb{Z}\times\mathbb{Z}/2$. It easily follows from the alternative descriptions of Wall's manifolds given in Section 2 that the pair (X,Y) is a Browder–Livesay pair according to the definition of Kharshiladze [10]. Let $n=\dim X=r+2s+1$, and let t denote the generator of the factor $\mathbb{Z}/2$ of the group $\pi_1(X)\cong\pi_1(Y)\cong\mathbb{Z}\times\mathbb{Z}/2$. Let $\omega^X:\pi_1(X)\to\mathbb{Z}/2=\{+1,-1\}$ denote the orientation homomorphism (or orientation character) of X as defined in Section 2 and ω^Y the same for Y. Further, let $\varepsilon=\pm1$ denote the number $\omega^X(t)$, $0\neq t\in\mathbb{Z}/2$. It then follows from the definition of a Browder–Livesay pair that $\omega^Y(t)=-\varepsilon$. Let $BL(X,Y)$ *denote the Wall surgery obstruction group* $L_{n+\varepsilon}(\mathbb{Z},\omega^X\mid\mathbb{Z})$. The geometric meaning of this group can be seen as follows:

Suppose we have a simple homotopy equivalence $f:(M,\partial M)\to(X,\partial X)$, where M is some manifold for which $f|_{f^{-1}(\partial Y)}:f^{-1}(\partial Y)\to\partial Y$ is also a simple homotopy equivalence. For every such simple homotopy equivalence there is defined *the Browder–Livesay invariant* η (see [3]) with values $\eta(f)$ in the group $BL(X,Y)=L_{n+\varepsilon}(\mathbb{Z},\omega^X\mid\mathbb{Z})$, such that $\eta(f)=0$ if and only if the map f is homotopic rel ∂M to a map f_1 for which the map $f_1|_{f_1^{-1}(Y)}:f_1^{-1}(Y)\to Y$ is a simple homotopy equivalence. We will say in short that f_1 *is a splitting along* Y, *or* f *splits along* Y.

Let, for a manifold with boundary $(X,\partial X)$, and a proper submanifold with boundary $(Y,\partial Y)$, we define $L(X)\overset{\text{def}}{=}L_n(\pi_1(X),\omega^X)$. Further, let $V=X\backslash U$, where U is a tubular neighbourhood of Y in X. In this case $\partial V=\partial X_{\partial Y}\cup\widehat{Y}$, $\widehat{Y}$ is a double covering of Y and $\partial X_{\partial Y}$ is the complement of a tubular neighbourhood of ∂Y in ∂X. One then has the following diagram of chain complexes (see [10, diagram (8)], [11, Theorem 11 and its proof]):

$$\begin{tikzcd}
{} \arrow[r] & L(Y\times I^2) \arrow[r] \arrow[d, phantom, "\wr"] & L(V\times I) \arrow[r] \arrow[d, phantom, "\wr"] & L(X\times I) \arrow[r, "\partial"] \arrow[d, phantom, "\wr"] & BL(X,Y) \arrow[r] \arrow[d, phantom, "\wr"] & L(Y) \\
{} \arrow[r] & L(X\times I^2) \arrow[r] & BL(X\times I,Y\times I) \arrow[r] & L(Y\times I) \arrow[r] & L(V) \arrow[r] & L(X)
\end{tikzcd} \tag{D$_1$}$$

which can be extended indefinitely on the left and where the vertical wavy lines denote isomorphism of the homology groups of the chain complexes (see [9]).

In algebraic notation the diagram looks like the following. Before we proceed further we should give some equivalent notations for typographical convenience: For denoting Wall's surgery obstruction groups we will use either of the notations: $L_*(\pi, \omega)$ and $L_*(\pi^{\omega})$.

$$\begin{tikzcd}
{} \arrow[r, "r"] & L_{n+1}(\mathbb{Z}^{\omega^V}) \arrow[r, "c"] & L_{n+1}((\mathbb{Z} \times \mathbb{Z}/2)^{\omega^X}) \arrow[r, "\partial=r\circ t"] & L_{n+\varepsilon}(\mathbb{Z}^{\omega^X}) \arrow[r, "t\circ c"] & {} \\
& \wr & \wr & \wr & \\
{} \arrow[r, "\partial=r\circ t"] & L_{n+\varepsilon+1}(\mathbb{Z}^{\omega^V}) \arrow[r, "t\circ c"] & L_n((\mathbb{Z} \times \mathbb{Z}/2)^{\omega^Y}) \arrow[r, "r"] & L_n(\mathbb{Z}^{\omega^V}) \arrow[r, "c"] & {}
\end{tikzcd} \tag{D$_2$}$$

$\omega^V, \omega^X \mid \mathbb{Z}$, and $\omega^Y \mid \mathbb{Z}$ can all be denoted by ω as they have the same value -1 on the generator. So we can rewrite the above diagram as

$$\begin{tikzcd}
{} \arrow[r, "r"] & L_{n+1}(\mathbb{Z}^{\omega}) \arrow[r, "c"] & L_{n+1}(\mathbb{Z}^{\omega} \times \mathbb{Z}/2^{\varepsilon}) \arrow[r, "\partial=r\circ t"] & L_{n+\varepsilon}(\mathbb{Z}^{\omega}) \arrow[r, "t\circ c"] & {} \\
& \wr & \wr & \wr & \\
{} \arrow[r, "\partial=r\circ t"] & L_{n+\varepsilon+1}(\mathbb{Z}^{\omega}) \arrow[r, "t\circ c"] & L_n(\mathbb{Z}^{\omega} \times \mathbb{Z}/2^{-\varepsilon}) \arrow[r, "r"] & L_n(\mathbb{Z}^{\omega}) \arrow[r, "c"] & {}
\end{tikzcd} \tag{D$_3$}$$

All the horizontal maps in the diagram are expressible in terms of algebraically defined maps

$$c : L_n\big(\mathbb{Z}^{\omega}\big) \to L_n\big(\mathbb{Z}^{\omega} \times \mathbb{Z}/2^{\varepsilon}\big),$$

defined by functoriality,

$$r : L_n\big(\mathbb{Z}^{\omega} \times \mathbb{Z}/2^{-\varepsilon}\big) \to L_n\big(\mathbb{Z}^{\omega}\big),$$

the transfer, and

$$t : L_n\big(\mathbb{Z}^{\omega} \times \mathbb{Z}/2^{\varepsilon}\big) \to L_{n-1+\varepsilon}\big(\mathbb{Z}^{\omega} \times \mathbb{Z}/2^{\varepsilon}\big),$$

multiplication of a quadratic form by the generator $t \in \mathbb{Z}/2$. The map

$$\partial : L(X \times I) \to BL(X, Y)$$

which factors through the Browder–Livesay invariant η *as follows*

$$\partial = \eta \circ \delta : L(X \times I) \xrightarrow{\delta} hT_{CAT}(X) \xrightarrow{\eta} BL(X, Y),$$

($CAT = PL$ or TOP), coincides with

$$r \circ t : L_{n+1}\big(\mathbb{Z}^{\omega} \times \mathbb{Z}/2^{\varepsilon}\big) \to L_{n+\varepsilon}\big(\mathbb{Z}^{\omega}\big).$$

We need to compute ∂ in various cases. We have seen that $\omega = -$, that is ω maps the fundamental group (in particular its $\mathbb{Z}$ factor) onto $\{+1, -1\}$. Consider the diagram (D$_3$), which now looks like

$$\begin{tikzcd}
{} \arrow[r, "r"] & L_{n+1}(\mathbb{Z}^{-}) \arrow[r, "c"] & L_{n+1}(\mathbb{Z}^{-} \times \mathbb{Z}/2^{\varepsilon}) \arrow[r, "\partial=r\circ t"] & L_{n+\varepsilon}(\mathbb{Z}^{-}) \arrow[r, "t\circ c"] & {} \\
& \wr & \wr & \wr & \\
{} \arrow[r, "\partial=r\circ t"] & L_{n+\varepsilon+1}(\mathbb{Z}^{-}) \arrow[r, "t\circ c"] & L_n(\mathbb{Z}^{-} \times \mathbb{Z}/2^{-\varepsilon}) \arrow[r, "r"] & L_n(\mathbb{Z}^{-}) \arrow[r, "c"] & {}
\end{tikzcd} \tag{D$_4$}$$

There are two cases to be considered, namely: $\varepsilon = +1$, and $\varepsilon = -1$.

Case I ($\varepsilon = +1$): In this case the diagram (D$_4$) tell us that:

$$\frac{\ker[L_{n+1}(\mathbb{Z}^{-} \times \mathbb{Z}/2^{+}) \overset{\partial=r\circ t}{\longrightarrow} L_{n+1}(\mathbb{Z}^{-})]}{\operatorname{im}[L_{n+1}(\mathbb{Z}^{-}) \overset{c}{\longrightarrow} L_{n+1}(\mathbb{Z}^{-} \times \mathbb{Z}/2^{+})]} \cong \frac{\ker[L_n(\mathbb{Z}^{-} \times \mathbb{Z}/2^{-}) \overset{r}{\longrightarrow} L_n(\mathbb{Z}^{-})]}{\operatorname{im}[L_{n+2}(\mathbb{Z}^{-}) \overset{t\circ c}{\longrightarrow} L_n(\mathbb{Z}^{-} \times \mathbb{Z}/2^{-})]}.$$

From the computations of Wall's surgery groups in [24]:

$\pi^{(\pm)}$	L_0	L_1	L_2	L_3
$(\mathbb{Z})^+$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$(\mathbb{Z})^-$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}^+ \times \mathbb{Z}/2^+$	$\mathbb{Z} \times \mathbb{Z}\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
$\mathbb{Z}^+ \times \mathbb{Z}/2^-$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}^- \times \mathbb{Z}/2^-$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}^- \times \mathbb{Z}/2^+$	$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
$\mathbb{Z}^+ \times \mathbb{Z}^-$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$

one derives readily the following:

Proposition 4.1. *If* $\varepsilon = +1$*, taking* $n \pmod 4$*, we have*

$$\partial : L_{n+1}\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big) \xrightarrow{rot} L_{n+1}(\mathbb{Z}^-) = \begin{cases} \textit{zero map} & \textit{if } n = 0, 1, 2, \\ \textit{epi. } (\mathbb{Z}/2)^3 \to \mathbb{Z}/2 & \textit{if } n = 3. \end{cases}$$

Proof. In cases $n \equiv 0, 1 \pmod 4$ the stated maps are the only choice for ∂, as c is injective (see [9] proof of Lemma 2). For $n \equiv 2, 3 \pmod 4$ the choice of ∂ is a consequence of the fact that the map c is injective, r, and $t \circ c$ are zero maps (see [9] proof of Lemma 2). □

Case II ($\varepsilon = -1$): In this case from diagram (D_4), and the fact that $t \circ c$ is injective (see [9] proof of Lemma 2) one readily derives the following:

Proposition 4.2. *If* $\varepsilon = -1$*, we have*

$$\partial = r \circ t : L_{n+1}\big(\mathbb{Z}^- \times \mathbb{Z}/2^-\big) \to L_{n-1}\big(\mathbb{Z}^-\big)$$

is the zero map for all $n, n \equiv 0, 1, 2, 3 \pmod 4$.

Proof. The horizontal sequences of (D_4) are chain complexes. □

5. Normal invariant of Wall's manifolds

Let us first recall the following well-known theorem due to Sullivan and Kirby–Siebenmann (see [12,21]):

Let $K(A, n)$ denote an Eilenberg–Mac Lane space, Y denote the space with two nontrivial homotopy groups $\pi_2(Y) = \mathbb{Z}/2$; $\pi_4(Y) = \mathbb{Z}$, and k-invariant $\delta Sq^2 \in H^5(K(\mathbb{Z}/2, 2); \mathbb{Z})$, where Sq^2 is the Steenrod square and δ is the Bockstein homomorphism in cohomology, corresponding to the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. For a topological space X, let $X_{(2)}$ denotes its localization at 2, $X_{(0)}$ denotes its rationalization and $X[1/2]$ denotes its localization away from 2 (see [22]), then

Theorem 5.1 *(Sullivan, Kirby–Siebenmann). We have the following homotopy equivalences*:

$$G/TOP_{(2)} \cong \prod_{i>1} K(\mathbb{Z}/2, 4i - 2) \times \prod_{i>1} K(\mathbb{Z}_{(2)}, 4i),$$

$$G/PL_{(2)} \cong Y_{(2)} \times \prod_{i>1} K(\mathbb{Z}/2, 4i - 2) \times \prod_{i>1} K(\mathbb{Z}_{(2)}, 4i),$$

and the following homotopy equivalences:

$$G/TOP[1/2] \overset{h_{TOP}}{\cong} BO^{\otimes}[1/2] \overset{h_{PL}}{\cong} G/PL[1/2].$$

As a consequence of this the normal invariants for a manifold X can be calculated using the following fibre squares, where $CAT = PL$ or TOP:

$$\begin{tikzcd}
G/CAT \arrow[r, "P_{(2)}^{G/CAT}"] \arrow[d, "P^{G/CAT}[1/2]"'] & G/CAT_{(2)} \arrow[d, "u"] \\
BO^{\otimes}[1/2] \cong G/CAT[1/2] \arrow[r, "ph\circ h_{CAT}"] & G/CAT_{(0)} = \prod_{i>0} K(\mathbb{Q}, 4i)
\end{tikzcd}$$

where ph stands for the Pontrjagin character, and u_* in homotopy coincides with the inclusion ϕ: $\mathbb{Z}_{(2)} \subset \mathbb{Q}$ for $k > 1$, and with 2ϕ for $k = 1$. These give by definition, the following exact sequence for any CW-complex X:

$$0 \to [X, G/TOP] \overset{\Phi^{G/TOP}}{\longrightarrow} KO^0(X)[1/2] \times \sum_{i>1} H^{4i-2}(X; \mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X; \mathbb{Z}_{(2)}) \overset{\Psi^{G/TOP}}{\longrightarrow} \sum_{i>0} H^{4i}(X; \mathbb{Q}) \to 0$$

and

$$\begin{aligned} 0 \ \to \ & [X, G/PL] \overset{\Phi^{G/PL}}{\longrightarrow} KO^0(X)[1/2] \times [X, Y_{(2)}] \oplus \sum_{i>1} H^{4i-2}(X; \mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X; \mathbb{Z}_{(2)}) \\ & \overset{\Psi^{G/PL}}{\longrightarrow} \sum_{i>0} H^{4i}(X; \mathbb{Q}) \to 0, \end{aligned}$$

where $\Phi^{G/CAT} = ((P^{G/CAT}[1/2])_* \oplus (P_{(2)}^{G/CAT})_*)\Delta$, $\Psi^{G/CAT} = \nabla(-ph \circ h_{cat} \oplus u)$, $\Delta(x) = (x, x)$ and $\nabla(x, y) = x + y$.

Let

$$\begin{aligned} \Pi &= Y \times \prod_{i>1} K(\mathbb{Z}/2, 4i-2) \times \prod_{i>1} K(\mathbb{Z}, 4i), \\ \Pi_{(2)} &= Y_{(2)} \times \prod_{i>1} K(\mathbb{Z}/2, 4i-2) \times \prod_{i>1} K(\mathbb{Z}_{(2)}, 4i), \\ \Pi[1/2] &= Y[1/2] \times \prod_{i>1} K\big(\mathbb{Z}[1/2], 4i\big), \end{aligned}$$

then from the fibre square:

$$\begin{tikzcd}
\Pi \arrow[r, "P_{(2)}^{\Pi}"] \arrow[d, "P^{\Pi}[1/2]"] & \Pi_{(2)} \arrow[d, "u"] \\
\Pi[1/2] \arrow[r, "j"] & \Pi_{(0)} = \prod_{i>0} K(\mathbb{Q}, 4i)
\end{tikzcd}$$

we also get an exact sequence:

$$\begin{aligned} 0 \ \to \ & [X, \Pi] \overset{\Phi^{\Pi}}{\longrightarrow} \big[X, Y[1/2]\big] \oplus \sum_{i>1} H^{4i}\big(X; \mathbb{Z}[1/2]\big) \times [X, Y_{(2)}] \oplus \sum_{i>1} H^{4i-2}(X; \mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X; \mathbb{Z}_{(2)}) \\ & \overset{\Psi^{\Pi}}{\longrightarrow} \sum_{i>0} H^{4i}(X; \mathbb{Q}) \to 0, \end{aligned}$$

Φ^{Π} and Ψ^{Π} have similar definitions as above. Now, we have the following result of Rudyak [19, Theorems 1]:

Theorem 5.2 *(Rudyak). Let X be a finite* CW*-complex with no odd torsion in homology, then*

$$\big(P_{(2)}^{\Pi}\big)_* : [X, \Pi] \to [X, \Pi_{(2)}]; \qquad \big(P_{(2)}^{G/CAT}\big)_* : [X, G/CAT] \to \big[X, (G/CAT)_{(2)}\big],$$

are monomorphisms.

We shall prove:

Theorem 5.3. *For* $X = Q(r,s)$, $r,s > 1$, *if we identify* $G/PL_{(2)}$ *and* $\Pi_{(2)}$ *under the homotopy equivalence given in Theorem* 5.1, *then the groups* $\mathrm{Im}(P^{\Pi}_{(2)})_*$ *and* $\mathrm{Im}(P^{G/PL}_{(2)})_*$ *are isomorphic.*

Proof. Since $X = Q(r,s)$, $r,s > 1$ is a finite complex with no odd torsion in homology, by the last Theorem 5.2 it follows that $[X, G/PL]$, and $[X, \Pi]$ are a finitely generated abelian groups which do not have any odd torsions and whose $\mathbb{Z}$-ranks and 2-torsions are same as the $\mathbb{Z}_{(2)}$-ranks and 2-torsions of

$$[X, G/PL] \otimes \mathbb{Z}_{(2)} \cong [X, G/PL_{(2)}] \cong [X, \Pi_{(2)}] \cong [X, \Pi] \otimes \mathbb{Z}_{(2)},$$

so, if we identify $G/PL_{(2)}$ and $\Pi_{(2)}$ under the homotopy equivalence given in Theorem 5.1 $\mathrm{Im}(P^{\Pi}_{(2)})_*$ and $\mathrm{Im}(P^{G/PL}_{(2)})_*$ are isomorphic. This proves the theorem. □

The above discussion yield that:

Theorem 5.4. *For* $X = Q(r,s)$, $r,s > 1$,

$$[X, G/TOP] \cong \sum_{i>1} H^{4i-2}(X;\mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X;\mathbb{Z}), \quad \textit{and}$$
$$[X, G/PL] \cong [X, \Pi] \cong [X, Y] \oplus \sum_{i>1} H^{4i-2}(X;\mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X;\mathbb{Z}).$$

Hence the normal invariant of $X = Q(r,s)$, $r,s > 1$ in the topological case is completely determined and the normal invariant in the PL case is determined once we determine $[X, Y]$. We recall for ready reference the following calculations in [13,19,16]: $[\mathbb{C}P^n, Y] = \mathbb{Z}$ for $n \geqslant 2$,

$$\bigl[\mathbb{R}P^n, Y\bigr] = \begin{cases} \mathbb{Z}/2 & \text{if } n = 2,3, \\ \mathbb{Z}/4 & \text{if } n \geqslant 4, \end{cases}$$
$$\bigl[P(r,s), Y\bigr] = \begin{cases} \mathbb{Z} \times \mathbb{Z}/4 & \text{if } r \geqslant 4,\ s \geqslant 2, \\ \mathbb{Z} \times \mathbb{Z}/2 & \text{if } r = 3,\ s \geqslant 2, \\ \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{if } r = 2,\ s \geqslant 2. \end{cases}$$

Towards determining $[X, Y]$ we may have to use the various alternative descriptions of $X = Q(r,s)$ given in Section 2. To start with let us use the descriptions:

$$\mathbb{C}P^s \xrightarrow{\text{incl}} X = Q(r,s) \xrightarrow{\text{proj}} Q(r,0), \quad \text{and} \quad \mathbb{R}P^r = P(r,0) \xrightarrow{\text{incl}_1} Q(r,0) \xrightarrow{\text{proj}_1} S^1.$$

First recall from the calculations of [16] and Section 3

$$H^4\bigl(P(r,s);\mathbb{Z}\bigr) = \begin{cases} \mathbb{Z} \times \mathbb{Z}/2 & \text{if } r \geqslant 4,\ s \geqslant 2, \\ \mathbb{Z} & \text{if } r = 3,\ s \geqslant 2, \\ \mathbb{Z} \times \mathbb{Z} & \text{if } r = 2,\ s \geqslant 2, \end{cases}$$
$$H^4\bigl(Q(r,s);\mathbb{Z}\bigr) = \begin{cases} \mathbb{Z}/2 & \text{if } r > 4,\ s \geqslant 2, \\ \mathbb{Z} \times \mathbb{Z}/2 & \text{if } r = 4,\ s \geqslant 2, \\ \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{if } r = 3,\ s \geqslant 2, \\ \mathbb{Z}/2 & \text{if } r = 2,\ s \geqslant 2, \end{cases}$$
$$H^4\bigl(Q(r,0);\mathbb{Z}\bigr) = \begin{cases} 0 & \text{if } r > 4,\ s = 0, \\ \mathbb{Z} & \text{if } r = 4,\ s = 0, \\ \mathbb{Z}/2 & \text{if } r = 3,\ s = 0, \\ 0 & \text{if } r = 2,\ s = 0. \end{cases}$$

Case I: $r > 4$, $s \geqslant 2$. Let $H^4(\mathbb{R}P^r;\mathbb{Z}) = \mathbb{Z}/2$ be generated by a; $H^2(\mathbb{R}P^r;\mathbb{Z}/2) = \mathbb{Z}/2$ be generated by c; $H^4(\mathbb{C}P^s;\mathbb{Z}) = \mathbb{Z}$ be generated by α; $H^2(\mathbb{C}P^s;\mathbb{Z}/2) = \mathbb{Z}/2$ be generated by d; the summands of $H^4(P(r+1,s);\mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}/2$ be generated respectively by α', β; $H^4(Q(r,s);\mathbb{Z}) = \mathbb{Z}/2$ be generated by β'; $H^4(Q(r,0);\mathbb{Z}) =$

0; the summands of $H^2(P(r+1,s);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by c'^2, d'; the summands of $H^2(Q(r,s);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by $c''^2, c''.x, d''$; the summands of $H^2(Q(r,0);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by $c'''^2, c'''.x'$.

Now from the definition of Y we have a fibration $K(\mathbb{Z},4)\xrightarrow{j} Y\xrightarrow{p} K(\mathbb{Z}/2,2)$ for which ΩY has zero k-invariant $k\in H^4(K(\mathbb{Z}/2,1);\mathbb{Z})$, and also for $Q(r,s)$, the operation $\delta Sq^2: H^2(Q(r,s);\mathbb{Z}/2)\to H^5(Q(r,s);\mathbb{Z})$ is zero. So we have the following commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(r,s);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(r,s),Y]} \arrow[r, "p_*"] & H^2(Q(r,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(Q(r,0);\mathbb{Z}) \arrow[r, "j_*"] \arrow[u, "\mathrm{proj}^*"] & {[Q(r,0),Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & H^2(Q(r,0);\mathbb{Z}/2) \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0
\end{tikzcd} \qquad I\,(\mathrm{i})$$

which reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(r,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r, "j_*"] \arrow[u, "\mathrm{proj}^*"] & {[Q(r,0),Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0
\end{tikzcd} \qquad I\,(\mathrm{ii})$$

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(r,s);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(r,s),Y]} \arrow[r, "p_*"] & H^2(Q(r,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(P(r+1,s);\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\Theta^*"] & {[P(r+1,s),Y]} \arrow[r, "p_*"] \arrow[u, "\Theta^Y"] & H^2(P(r+1,s);\mathbb{Z}/2) \arrow[r] \arrow[u, "\Theta^*"] & 0
\end{tikzcd} \qquad I\,(\mathrm{iii})$$

where Θ is the map of degree one defined in the diagram $(**)$ in the paragraph following Definition 2.1. This reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(r,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}\times\mathbb{Z}/2 \arrow[r, "\chi"] \arrow[u, "\Theta^*"] & \mathbb{Z}\times\mathbb{Z}/4 \arrow[r, "p_*"] \arrow[u, "\Theta^Y"] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] \arrow[u, "\Theta^*"] & 0
\end{tikzcd} \qquad I\,(\mathrm{iv})$$

Consider the diagram $I\,(\mathrm{iv})$, and note that the first vertical map is surjective (obtained by examining the commutative diagram induced by Θ on the Bockstein exact cohomology sequences corresponding to $0\to\mathbb{Z}\xrightarrow{\times 2}\mathbb{Z}\to\mathbb{Z}/2\to 0$) sending $\alpha'\mapsto 0$, and $\beta\mapsto\beta'$ and the third vertical map sending $c'^2\mapsto c''^2$ and $d\mapsto d'$ is injective, and hence by a diagram chase one can see that the middle vertical map sends the $\mathbb{Z}/4$ summand injectively, and the map χ is given by $(\alpha',0)\mapsto(2\alpha',0)$ and $(0,\beta)\mapsto(0,2\beta)$. Therefore, j^* maps the generator β' of $H^4(Q(r,s);\mathbb{Z})$ to 2 times an element.

In the diagram $I\,(\mathrm{ii})$ note that the first vertical map and the third vertical map sending $(c'''^2,0)\mapsto(c''^2,0,0)$ and $(0,c'''.x')\mapsto(0,c''.x,0)$ are injections, and hence by five lemma the middle vertical map is also an injection.

Putting together the information obtained in the last two paragraphs we get

$$\bigl[Q(r,s),Y\bigr]\cong\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2,\quad \text{for } r>4,\ s\geqslant 2.$$

Case II: $r=4$, $s\geqslant 2$. Let $H^4(\mathbb{R}P^4;\mathbb{Z})=\mathbb{Z}/2$ be generated by a; $H^2(\mathbb{R}P^4;\mathbb{Z}/2)=\mathbb{Z}/2$ be generated by c; $H^4(\mathbb{C}P^s;\mathbb{Z})=\mathbb{Z}$ be generated by α; $H^2(\mathbb{C}P^s;\mathbb{Z}/2)=\mathbb{Z}/2$ be generated by d; the summands of $H^4(P(5,s);\mathbb{Z})=\mathbb{Z}\times\mathbb{Z}/2$ be generated respectively by α',β; the summands of $H^4(Q(4,s);\mathbb{Z})=\mathbb{Z}\times\mathbb{Z}/2$ be generated respectively by α'',β'; $H^4(Q(4,0);\mathbb{Z})=\mathbb{Z}$ be generated by α'''; the summands of $H^2(P(5,s);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by c'^2,d'; the summands of $H^2(Q(4,s);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by $c''^2,c''.x,d''$; the summands of $H^2(Q(4,0);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by $c'''^2,c'''.x'$.

Now from the definition of Y we have a fibration $K(\mathbb{Z},4) \xrightarrow{j} Y \xrightarrow{p} K(\mathbb{Z}/2,2)$ for which ΩY has zero k-invariant $k \in H^4(K(\mathbb{Z}/2,1);\mathbb{Z})$, and also for $Q(4,s)$, the operation $\delta Sq^2 : H^2(Q(4,s);\mathbb{Z}/2) \to H^5(Q(4,s);\mathbb{Z})$ is zero. So we have the following commutative diagrams:

II(i)

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(4,s);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(4,s),Y]} \arrow[r, "p_*"] & H^2(Q(4,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(Q(4,0);\mathbb{Z}) \arrow[r, "j_*"] \arrow[u, "\mathrm{proj}^*"] & {[Q(4,0),Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^*"] & H^2(Q(4,0);\mathbb{Z}/2) \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0
\end{tikzcd}$$

which reduces to

II(ii)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\times\mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(4,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "j_*"] \arrow[u, "\mathrm{proj}^*"] & {[Q(4,0),Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0
\end{tikzcd}$$

II(iii)

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(4,0);\mathbb{Z}) \arrow[r, "j_*"] \arrow[d, "\mathrm{incl}_1^*"'] & {[Q(4,0),Y]} \arrow[r, "p_*"] \arrow[d, "\mathrm{incl}_1^Y"'] & H^2(Q(4,0);\mathbb{Z}/2) \arrow[r] \arrow[d, "\mathrm{incl}_1^*"'] & 0 \\
0 \arrow[r] & H^4(\mathbb{R}P^4;\mathbb{Z}) \arrow[r, "\chi"] & {[\mathbb{R}P^4,Y]} \arrow[r, "p_*"] & H^2(\mathbb{R}P^4;\mathbb{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

which reduces to

II(iv)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "j_*"] \arrow[d, "\mathrm{incl}_1^*"'] & {[Q(4,0),Y]} \arrow[r, "p_*"] \arrow[d, "\mathrm{incl}_1^Y"'] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] \arrow[d, "\mathrm{incl}_1^*"'] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\chi"] & \mathbb{Z}/4 \arrow[r, "p_*"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

II(v)

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(4,s);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(4,s),Y]} \arrow[r, "p_*"] & H^2(Q(4,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(P(5,s);\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\Theta^*"] & {[P(5,s),Y]} \arrow[r, "p_*"] \arrow[u, "\Theta^Y"] & H^2(P(5,s);\mathbb{Z}/2) \arrow[r] \arrow[u, "\Theta^*"] & 0
\end{tikzcd}$$

where Θ is the map of degree one defined in the diagram $(**)$ in the paragraph following Definition 2.1.This reduces to

II(vi)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\times\mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(4,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}\times\mathbb{Z}/2 \arrow[r, "\chi"] \arrow[u, "\Theta^*"] & \mathbb{Z}\times\mathbb{Z}/4 \arrow[r, "p_*"] \arrow[u, "\Theta^Y"] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] \arrow[u, "\Theta^*"] & 0
\end{tikzcd}$$

Consider the diagram *II*(iv), and note that the first vertical map sending $\alpha''' \mapsto a$, is onto (obtained by examining the commutative diagram induced by incl_1 on the Bockstein exact cohomology sequences corresponding to $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$) and the third vertical map sending $c''^2 \mapsto c^2$ is onto; hence by five lemma the middle vertical map is also onto, and the map χ is given by $a \mapsto 2a$. Therefore the map j_* sends the generator α'' of $\mathbb{Z}$ to two times of an element. From these facts it follows that $[Q(4,0),Y] \cong \mathbb{Z}\times\mathbb{Z}/2$.

Plug in the information of the last paragraph in the diagram *II*(ii), and note that the first vertical map sending $\alpha''' \mapsto (\alpha'',0)$ and third vertical map sending $(c'''^2,0) \mapsto (c''^2,0,0)$ and $(0,c'''.x') \mapsto (0,c''.x,0)$ are injections, and hence by five lemma the middle vertical map is also an injection. Also the map j_* in the lower and hence the upper horizontal sequence maps the generator α' of $\mathbb{Z}$ to two times of an element.

Plug in the informations obtained in the last two paragraph to the diagram *II*(vi), and note that the first vertical map is injective (obtained by examining the commutative diagram induced by Θ on the Bockstein exact cohomology sequences corresponding to $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$) sending $(\alpha', 0) \mapsto \alpha''$ and $\beta \mapsto \beta'$ and the third vertical map sending $(c'^2, 0) \mapsto (c''^2, 0, 0)$ and $(0, d') \mapsto (0, d'', 0)$ is injective, therefore by five lemma the middle vertical map is also injective, and the map χ is given by multiplication by 2 on each generator; hence the map j_* sends both the generators to 2 times some elements.

Putting together the information obtained in the last three paragraphs we get

$$\bigl[Q(4, s), Y\bigr] \cong \mathbb{Z} \times \mathbb{Z}/4 \times \mathbb{Z}/2, \quad \text{for } s \geqslant 2.$$

Case III: $r = 3$, $s \geqslant 2$. Let $H^4(\mathbb{R}P^4; \mathbb{Z}) = \mathbb{Z}/2$ be generated by a; $H^2(\mathbb{R}P^4; \mathbb{Z}/2) = \mathbb{Z}/2$ be generated by c; $H^4(\mathbb{C}P^s; \mathbb{Z}) = \mathbb{Z}$ be generated by α; $H^2(\mathbb{C}P^s; \mathbb{Z}/2) = \mathbb{Z}/2$ be generated by d; the summands of $H^4(P(4, s); \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}/2$ be generated respectively by α', β; the summands of $H^4(Q(3, s); \mathbb{Z}) = \mathbb{Z}/2 \times \mathbb{Z}/2$ be generated respectively by β', γ; $H^4(Q(3, 0); \mathbb{Z}) = \mathbb{Z}/2$ be generated by γ'; the summands of $H^2(P(4, s); \mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2$ be generated respectively by c'^2, d'; the summands of $H^2(Q(3, s); \mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ be generated respectively by $c''^2, c''.x, d''$; the summands of $H^2(Q(3, 0); \mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2$ be generated respectively by $c'''^2, c'''.x'$.

Now, as in the previous cases the operation $\delta Sq^2 : H^2(Q(3, s); \mathbb{Z}/2) \to H^5(Q(3, s); \mathbb{Z})$ is zero. So we have the following commutative diagrams:

III(i)

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(3,s);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(3,s),Y]} \arrow[r, "p_*"] & H^2(Q(3,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(Q(3,0);\mathbb{Z}) \arrow[r, "j_*"] \arrow[u, "\mathrm{proj}^*"] & {[Q(3,0),Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & H^2(Q(3,0);\mathbb{Z}/2) \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0
\end{tikzcd}$$

which reduces to

III(ii)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(3,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "j_*"] \arrow[u, "\mathrm{proj}^*"] & {[Q(3,0),Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0
\end{tikzcd}$$

III(iii)

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(3,0);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(3,0),Y]} \arrow[r, "p_*"] & H^2(Q(3,0);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(\mathbb{R}P^4;\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\theta^*"] & {[\mathbb{R}P^4,Y]} \arrow[r, "p_*"] \arrow[u, "\theta^Y"] & H^2(\mathbb{R}P^4;\mathbb{Z}/2) \arrow[r] \arrow[u, "\theta^*"] & 0
\end{tikzcd}$$

where θ is the classifying map (see diagram $(**)$) defined in the paragraph following Definition 2.1. This reduces to

III(iv)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(3,0),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "\chi"] \arrow[u, "\theta^*"] & \mathbb{Z}/4 \arrow[r, "p_*"] \arrow[u, "\theta^Y"] & \mathbb{Z}/2 \arrow[r] \arrow[u, "\theta^*"] & 0
\end{tikzcd}$$

III(v)

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(3,s);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(3,s),Y]} \arrow[r, "p_*"] & H^2(Q(3,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(P(4,s);\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\Theta^*"] & {[P(4,s),Y]} \arrow[r, "p_*"] \arrow[u, "\Theta^Y"] & H^2(P(4,s);\mathbb{Z}/2) \arrow[r] \arrow[u, "\Theta^*"] & 0
\end{tikzcd}$$

where Θ is the map of degree one defined in the diagram $(**)$ in the paragraph following Definition 2.1. This reduces to

III(vi)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(3,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \times \mathbb{Z}/2 \arrow[r, "\chi"] \arrow[u, "\Theta^*"] & \mathbb{Z} \times \mathbb{Z}/4 \arrow[r, "p_*"] \arrow[u, "\Theta^Y"] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] \arrow[u, "\Theta^*"] & 0
\end{tikzcd}$$

Consider the diagram *III*(iv), and note that the first vertical map is injective (obtained by examining the commutative diagram induced by θ on the Bockstein exact cohomology sequences corresponding to $0\to\mathbb{Z}\xrightarrow{\times 2}\mathbb{Z}\to\mathbb{Z}/2\to 0$) and the third vertical map sending $c^2\mapsto(c''^2,0)$ is also injective, and hence by five lemma the middle vertical map is also injective. From this one can readily derive that $[Q(3,0),Y]\cong\mathbb{Z}/4\times\mathbb{Z}/2$, and the map j_* is sends $\gamma'\mapsto(2\gamma',0)$.

Plug in this information of the last paragraph in the diagram *III*(ii), and note that the first vertical map sending $\gamma'\mapsto(0,\gamma)$ and the third vertical map sending $(c''^2,0)\mapsto(c'^2,0,0)$ and $(0,c''.x')\mapsto(0,c'.x,0)$ are injections, and hence by five lemma the middle vertical map is also an injection. From this we get that $[Q(3,s),Y]\cong G\times\mathbb{Z}/4\times\mathbb{Z}/2$, and the map j_* in the upper horizontal sequence sends $(0,\gamma)$ to 2 times an element. G, remains to be determined.

Plug in the information obtained in the last two paragraphs to the diagram *III*(vi), and note that the first vertical map sending $(\alpha',0)\mapsto(0,\gamma)$ and $(0,\beta)\mapsto(\beta',0)$ is onto (obtained by examining the commutative diagram induced by Θ on the Bockstein exact cohomology sequences corresponding to $0\to\mathbb{Z}\xrightarrow{\times 2}\mathbb{Z}\to\mathbb{Z}/2\to 0$), and the third vertical map sending $(c'^2,0)\mapsto(c''^2,0,0)$ and $(0,d')\mapsto(0,d'',0)$ is an injection. By a diagram chase one gets that the $\mathbb{Z}/4$ summand in the middle group below maps injectively by the middle vertical map; the map χ sends each generator by multiplication by 2; therefore the map j_* maps the generators to 2 times some elements; and the summand $\mathbb{Z}$ in the middle group below maps by the middle vertical map as a reduction mod 4.

Putting together the information obtained in the last three paragraphs we get

$$\big[Q(3,s),Y\big]\cong\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/2,\quad \text{for } s\geqslant 2.$$

Case IV: $r=2$, $s\geqslant 2$. Let $H^2(\mathbb{R}P^2;\mathbb{Z}/2)=\mathbb{Z}/2$ be generated by c; $H^4(\mathbb{C}P^s;\mathbb{Z})=\mathbb{Z}$ be generated by α; $H^2(\mathbb{C}P^s;\mathbb{Z}/2)=\mathbb{Z}/2$ be generated by d; $H^4(P(3,s);\mathbb{Z})=\mathbb{Z}$ be generated by α'; $H^4(Q(2,s);\mathbb{Z})=\mathbb{Z}/2$ be generated by β; $H^4(Q(2,0);\mathbb{Z})=0$; the summands of $H^2(P(3,s);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by c'^2,d'; the summands of $H^2(Q(2,s);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by $c'^2,c'.x,d'$; the summands of $H^2(Q(2,0);\mathbb{Z}/2)=\mathbb{Z}/2\times\mathbb{Z}/2$ be generated respectively by $c''^2,c''.x'$.

Now, as in the last case the operation $\delta Sq^2: H^2(Q(2,s);\mathbb{Z}/2)\to H^5(Q(2,s);\mathbb{Z})$ is zero. So we have the following commutative diagrams:

IV(i)

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(2,s);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(2,s),Y]} \arrow[r, "p_*"] & H^2(Q(2,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(Q(2,0);\mathbb{Z}) \arrow[r, "j_*"] \arrow[u, "\mathrm{proj}^*"] & {[Q(2,0),Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & H^2(Q(2,0);\mathbb{Z}/2) \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0
\end{tikzcd}$$

which reduces to

IV(ii)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(2,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r, "j_*"] \arrow[u, "\mathrm{proj}^*"] & {[Q(2,0),Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0
\end{tikzcd}$$

IV(iii)

$$\begin{tikzcd}
0 \arrow[r] & H^4(Q(2,s);\mathbb{Z}) \arrow[r, "j_*"] & {[Q(2,s),Y]} \arrow[r, "p_*"] & H^2(Q(2,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(P(3,s);\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\Theta^*"] & {[P(3,s),Y]} \arrow[r, "p_*"] \arrow[u, "\Theta^Y"] & H^2(P(3,s);\mathbb{Z}/2) \arrow[r] \arrow[u, "\Theta^*"] & 0
\end{tikzcd}$$

where Θ is the map of degree one defined in the diagram $(**)$ in the paragraph following Definition 2.1. This reduces to

IV(iv)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "j_*"] & {[Q(2,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\chi"] \arrow[u, "\Theta^*"] & \mathbb{Z}\times\mathbb{Z}/2 \arrow[r, "p_*"] \arrow[u, "\Theta^Y"] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r] \arrow[u, "\Theta^*"] & 0
\end{tikzcd}$$

Consider the diagram IV(ii), and note that the first vertical map injects 0 and the third vertical map sending $(c''^2, 0) \mapsto (c'^2, 0)$ and $(0, c''.x') \mapsto (c'.x, 0)$ is injection, and hence by five lemma the middle vertical map is also an injection.

Plug in the information obtained in the last paragraph to the diagram IV(iv), and note that the first vertical map sending $\alpha' \mapsto \beta$ is onto (obtained by examining the commutative diagram induced by Θ on the Bockstein exact cohomology sequences corresponding to $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$) and third vertical map sending $(c'^2, 0) \mapsto (c''^2, 0, 0)$ and $(0, d') \mapsto (0, 0, d'')$ is onto; therefore by a diagram chase the middle vertical map injects the $\mathbb{Z}/2$ summand in the below middle group onto a $\mathbb{Z}/2$ summand of the top middle group. The map χ is multiplication by 2; so j_* maps the generator to 2 times an element. So the middle vertical map maps the $\mathbb{Z}$ summand of the below middle group as reduction modulo 4. Putting together the information obtained in the last two paragraphs we get

$$\big[Q(2, s), Y\big] \cong \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2, \quad \text{for } s \geqslant 2.$$

We can summarize the above calculations in the form of the following

Theorem 5.5. *For Wall's manifolds $Q(r, s)$, $r, s \geqslant 2$,*

$$\big[Q(r, s), Y\big] = \begin{cases} \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{if } r > 4, \\ \mathbb{Z} \times \mathbb{Z}/4 \times \mathbb{Z}/2 & \text{if } r = 4, \\ \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/2 & \text{if } r = 3, \\ \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{if } r = 2. \end{cases}$$

6. The action δ_{CAT} of $L_{n+1}(\mathbb{Z}^- \times \mathbb{Z}/2^\pm)$ on the homotopy CAT structures of $Q(r, s)$

We begin by stating a lemma which follows from the commutative diagram of page 560 of [18], in the standard notations of Ranicki, involving (algebraic) surgery exact sequences of a pair (X, Y) of Poincaré complexes equipped with orientation character w. Let X be a Poincaré complex with orientation character $w : \pi_1(X) \to \{\pm 1\}$. Let X^w be the orientation double cover of X and $p : X^w \to X$ be the projection map. Considering p as an inclusion into its mapping cylinder which is homotopy equivalent to X, we treat the double covering as a pair (X, X^w), and can write the following commutative diagram:

$$\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & H_n(X^w; \mathbb{L}_0) \arrow[r, "\sigma_*"] \arrow[d] & L_n(\mathbb{Z}[\pi_1(X^w)]) \arrow[r] \arrow[d] & \mathcal{S}_n(X^w) \arrow[r] \arrow[d] & H_{n-1}(X^w; \mathbb{L}_0) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_n(X; \mathbb{L}_0) \arrow[r, "\sigma_*"] \arrow[d] & L_n(\mathbb{Z}[\pi_1(X)]) \arrow[r] \arrow[d] & \mathcal{S}_n(X) \arrow[r] \arrow[d] & H_{n-1}(X; \mathbb{L}_0) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_n(X, X^w; \mathbb{L}_0) \arrow[r, "\sigma_*"] \arrow[d] & L_n(\mathbb{Z}[\pi_1(X)] \longrightarrow \mathbb{Z}[\pi_1(X^w)]) \arrow[r] \arrow[d] & \mathcal{S}_n(X, X^w) \arrow[r] \arrow[d] & H_{n-1}(X, X^w; \mathbb{L}_0) \arrow[r] \arrow[d] & {} \\
& {} & {} & {} & {} &
\end{tikzcd}$$

If we take (X, w) with $\pi_1(X) = \mathbb{Z}$ and nontrivial w then the above diagram reduces to

$$\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & H_n(X^w; \mathbb{L}_0) \arrow[r, "\sigma_*"] \arrow[d] & L_n(\mathbb{Z}^+) \arrow[r] \arrow[d, "\text{vert}"'] & \mathcal{S}_n(X^w) \arrow[r] \arrow[d] & H_{n-1}(X^w; \mathbb{L}_0) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_n(X; \mathbb{L}_0) \arrow[r, "\sigma_*"] \arrow[d] & L_n(\mathbb{Z}^-) \arrow[r] \arrow[d] & \mathcal{S}_n(X) \arrow[r] \arrow[d] & H_{n-1}(X; \mathbb{L}_0) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_n(X, X^w; \mathbb{L}_0) \arrow[r, "\sigma_*"] \arrow[d] & L_n(\mathbb{Z}^+ \to \mathbb{Z}^-) \arrow[r] \arrow[d] & \mathcal{S}_n(X, X^w) \arrow[r] \arrow[d] & H_{n-1}(X, X^w; \mathbb{L}_0) \arrow[r] \arrow[d] & {} \\
& {} & {} & {} & {} &
\end{tikzcd}$$

By applying Theorem 12.6 of Wall [24] and doing some calculations one can see that the vertical map denoted by *vert* in the last diagram is onto. So, using this fact and Browder ([1, Theorem 1.3 and 1.4 (PL and TOP version)] and [2, Theorem II.2.13]) we get the

Proposition 6.1. *If X is a manifold with orientation character $-:\pi_1(X)\cong\mathbb{Z}\twoheadrightarrow\{\pm 1\}$, the action δ_{CAT} of $L_{n+1}(\mathbb{Z}^-)$ on $hT_{CAT}(X)$ is trivial.*

Let $X=D^m\times Q(r,s)$, $r,s>1$, where D^m is the standard m-dimensional disk, $m\geqslant 0$.

Proposition 6.2. *Let $n=\dim X=m+r+2s+1\equiv 0 \pmod 4$, $r+s+1$ even. So $\varepsilon=+$. Then the action δ_{CAT} of the group $L_1(\mathbb{Z}^-\times\mathbb{Z}/2^+)=0$ on $hT_{CAT}(X)$ is obviously trivial.*

Proposition 6.3. *Let $n=\dim X=m+r+2s+1\equiv 1 \pmod 4$, $r+s+1$ even, or odd. So $\varepsilon=\pm$. Then the action δ_{CAT} of the group $L_2(\mathbb{Z}^-\times\mathbb{Z}/2^\pm)=\mathbb{Z}/2$ on $hT_{CAT}(X)$ is trivial.*

Proof. Since $L_2(\mathbb{Z}^-\times\mathbb{Z}/2^\pm)=L_2(0)$, this group acts on the homotopy CAT structures by taking connected sum with a homotopy sphere, which in the *CAT* ($=$ *PL* or *TOP*) cases is trivial. □

Proposition 6.4. *Let $n=\dim X=m+r+2s+1\equiv 2 \pmod 4$, $r+s+1$ even. So $\varepsilon=+$. Then the action δ_{CAT} of the group $L_3(\mathbb{Z}^-\times\mathbb{Z}/2^+)=\mathbb{Z}/2\times\mathbb{Z}/2$ on $hT_{CAT}(X)$ is trivial.*

Proof. Refer to the diagram (D$_4$) of Section 4 with $n=2$, and $\varepsilon=+$, this looks like:

$$\begin{tikzcd}
{} \arrow[r, "r"] & L_3(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "c=\mathrm{inj}"] \arrow[d, phantom, "\wr"] & L_3(\mathbb{Z}^-\times\mathbb{Z}/2^+)=\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r, "\partial=r\circ t=0"] \arrow[d, phantom, "\wr"] & L_3(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "t\circ c"] \arrow[d, phantom, "\wr"] & {} \\
{} \arrow[r, "\partial=r\circ t"] & L_4(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "t\circ c=0"] & L_2(\mathbb{Z}^-\times\mathbb{Z}/2^-)=\mathbb{Z}/2 \arrow[r, "r=0"] & L_2(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "c"] & {}
\end{tikzcd} \tag{D$_5$}$$

First we analyze the action of the image of the nonzero element of $L_3(\mathbb{Z}^-)=\mathbb{Z}/2$ under $1{-}1$ map c. Let $\widehat{U}$ be a tubular neighbourhood of $Q(r-1,s)$ in $Q(r,s)$, then $Q(r,s)\backslash\widehat{U}$ is homeomorphic to say $V(r,s)$, where $V(r,s)$ is the total space of a $D^r\times\mathbb{C}P^s$ fibration over S^1, with nontrivial first Stiefel–Whitney class. Realize the nonzero element of the group $L_3(\mathbb{Z}^-)=\mathbb{Z}/2$ by a normal map $f:M\to D^m\times V(r,s)\times I$, such that

$$f|_{\partial_-M}:\partial_-M\to D^m\times V(r,s)\times 0\cup\partial\big(D^m\times V(r,s)\big)\times I$$

is a CAT homeomorphism (*CAT* $=$ *PL* or *TOP*), and $f|_{\partial_+M}:\partial_+M\to D^m\times V(r,s)\times 1$ is a simple homotopy equivalence which is a CAT-homeomorphism on the boundary, so $f|_{\partial_+M}$ is a homotopy CAT-structure on X. From Lemma 6.1 it follows that $f|_{\partial_+M}$ is a trivial CAT-structure on X.

Now realize the nonzero element

$$x\in\operatorname{im}\big[\mathbb{Z}/2=L_3(\mathbb{Z}^-)\overset{c=\mathrm{inj}}{\longrightarrow}L_3\big(\mathbb{Z}^-\times\mathbb{Z}/2^+\big)=\mathbb{Z}/2\times\mathbb{Z}/2\big]$$

by a normal map $F:N\to D^m\times Q(r,s)\times I$ as follows: Let U be a tubular neighbourhood of $D^m\times Q(r-1,s)\times I$ in $D^m\times Q(r,s)\times I$, then $D^m\times Q(r,s)\times I\backslash U$ is homeomorphic to $D^m\times V(r,s)\times I$, or $D^m\times Q(r,s)\times I=U\cup D^m\times V(r,s)\times I$. Attach U to the manifold M along the boundary $f^{-1}(\partial(D^m\times V(r,s))\times I)\subseteq\partial_-M$. Extending the map f over U by identity, we obtain the map

$$F:N\overset{\mathrm{def}}{=}M\cup U\to\big(D^m\times V(r,s)\times I\big)\cup U=D^m\times Q(r,s)\times I.$$

As $F|_{\partial_+M}=f|_{\partial_+M}$ it follows that restriction of F on each piece of N gives trivial CAT structure and hence F represent the trivial CAT-structure on X. So the action δ_{CAT} of x on $hT_{CAT}(X)$ is trivial.

Next let us consider the nontrivial element

$$x'\notin\operatorname{im}\big[\mathbb{Z}/2=L_3(\mathbb{Z}^-)\overset{c=\mathrm{inj}}{\longrightarrow}L_3\big(\mathbb{Z}^-\times\mathbb{Z}/2^+\big)=\mathbb{Z}/2\times\mathbb{Z}/2\big].$$

Realize it by a normal map $f':M'\to D^m\times Q(r,s)\times I$ such that

$$f'|_{\partial_-M'}:\partial_-M'\to D^m\times Q(r,s)\times 0\cup\partial\big(D^m\times Q(r,s)\big)\times I$$

is a CAT homeomorphism (*CAT* $=$ *PL* or *TOP*). The map $\partial:L_3(\mathbb{Z}^-\times\mathbb{Z}/2^+)\to L_3(\mathbb{Z}^-)$ is the zero map. Therefore, by the relation between ∂ and the Browder–Livesay invariant mentioned in Section 4, the homotopy CAT structure $f'|_{\partial_+M'}:\partial_+M'\to D^m\times Q(r,s)\times 1$ is split along $D^m\times Q(r-1,s)\times 1$. We denote the map $f'|_{\partial_+M'}$ by f_1'.

Thus the map

$$f_1'|_{f_1'^{-1}(D^m\times Q(r-1,s))} : f_1'^{-1}\bigl(D^m\times Q(r-1,s)\bigr)\to D^m\times Q(r-1,s)$$

is a simple homotopy equivalence, and is a CAT-homeomorphism on the boundary, that is, it is a homotopy CAT structure. Now the map

$$f'|_{f'^{-1}(D^m\times Q(r-1,s)\times I)} : f'^{-1}\bigl(D^m\times Q(r-1,s)\times I\bigr)\to D^m\times Q(r-1,s)\times I$$

is normal and realizes the element $y'\in L_2(\mathbb{Z}^-\times\mathbb{Z}/2^-)$ which corresponds to the element $x'\in L_3(\mathbb{Z}^-\times\mathbb{Z}/2^+)$ (refer to the diagram (D_5) above where $\wr$ is an isomorphism of the homologies of the chain complexes given by the horizontal sequences.) According to Proposition 6.3 the action of y' on $hT_{CAT}(D^m\times Q(r-1,s))$ is trivial, so the homotopy CAT structure

$$f'|_{f'^{-1}(D^m\times Q(r-1,s)\times 1)} : f'^{-1}\bigl(D^m\times Q(r-1,s)\times 1\bigr)\to D^m\times Q(r-1,s)\times 1$$

is trivial. Now if U' is the tubular neighbourhood of $D^m\times Q(r-1,s)$ in $D^m\times Q(r,s)$, then $D^m\times Q(r,s)\backslash U' = D^m\times V(r,s)$, $V(r,s)$ as defined in the last paragraph, so $D^m\times Q(r,s)=U'\cup D^m\times V(r,s)$. Similarly $\partial_+M' = f_1^{-1}(U')\cup f_1^{-1}(D^m\times V(r,s))$, and f_1' gives simple homotopy equivalences on each piece. From the above observation $f_1'|_{f_1'^{-1}(U')} : f_1^{-1}(U')\to U'$ is trivial. Also

$$f_1'|_{f_1'^{-1}(D^m\times V(r,s))} : f_1^{-1}\bigl(D^m\times V(r,s)\bigr)\to D^m\times V(r,s)$$

is a CAT homeomorphism, because it is the outcome of the action of an element of the image of c such as

$$x\in \operatorname{im}\bigl[\mathbb{Z}/2 = L_3(\mathbb{Z}^-)\overset{c=\mathrm{inj}}{\longrightarrow} L_3\bigl(\mathbb{Z}^-\times\mathbb{Z}/2^+\bigr)=\mathbb{Z}/2\times\mathbb{Z}/2\bigr]$$

on $hT_{CAT}(X)$, considered in the first para of the proof, which is trivial. Therefore $f_1' : \partial_+M'\to D^m\times Q(r,s)\times 1$ is a CAT-homeomorphism. Thus the action δ_{CAT} of x' on $hT_{CAT}(X)$ is trivial. This completes the proof of the proposition. □

Proposition 6.5. *Let* $n=\dim X = m+r+2s+1\equiv 3 \pmod 4$, $r+s+1$ *odd. So* $\varepsilon=-$. *Then the action* δ_{CAT} *of the group* $L_0(\mathbb{Z}^-\times\mathbb{Z}/2^-)=\mathbb{Z}/2$ *on* $hT_{CAT}(X)$ *is trivial.*

Proof. Refer to the diagram (D_4) of Section 4 with $n=3$, and $\varepsilon=-$, this looks like:

$$\begin{tikzcd}
{} \arrow[r, "r"] & L_4(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "c=0"] \arrow[d, phantom, "\wr"] & L_4(\mathbb{Z}^-\times\mathbb{Z}/2^-)=\mathbb{Z}/2 \arrow[r, "\partial=r\circ t=0"] \arrow[d, phantom, "\wr"] & L_2(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "t\circ c"] \arrow[d, phantom, "\wr"] & {} \\
{} \arrow[r, "\partial=r\circ t"] & L_3(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "t\circ c=\mathrm{inj}"] & L_3(\mathbb{Z}^-\times\mathbb{Z}/2^+)=\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r, "r=0"] & L_3(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "c"] & {}
\end{tikzcd} \tag{D_6}$$

Consider the nonzero element $x\in L_0(\mathbb{Z}^-\times\mathbb{Z}/2^-)$. Realize it by a normal map $f : M\to D^m\times Q(r,s)\times I$ such that

$$f|_{\partial_-M} : \partial_-M\to D^m\times Q(r,s)\times 0\cup\partial\bigl(D^m\times Q(r,s)\bigr)\times I$$

is a CAT homeomorphism (*CAT* = *PL* or *TOP*). The map $\partial : L_0(\mathbb{Z}^-\times\mathbb{Z}/2^-)\to L_2(\mathbb{Z}^-)$ is the zero map. Therefore, by the relation between ∂ and the Browder–Livesay invariant mentioned in Section 4, the homotopy CAT structure $f|_{\partial_+M} : \partial_+M\to D^m\times Q(r,s)\times 1$ is split along $D^m\times Q(r-1,s)\times 1$. We denote the map $f|_{\partial_+M}$ by f_1.

Thus the map

$$f_1|_{f_1^{-1}(D^m\times Q(r-1,s))} : f_1^{-1}\bigl(D^m\times Q(r-1,s)\bigr)\to D^m\times Q(r-1,s)$$

is a simple homotopy equivalence, and is a CAT-homeomorphism on the boundary, that is, it is a homotopy CAT structure. Now the map

$$f|_{f^{-1}(D^m\times Q(r-1,s)\times I)} : f^{-1}\bigl(D^m\times Q(r-1,s)\times I\bigr)\to D^m\times Q(r-1,s)\times I$$

is normal and realizes the nonzero element

$$y\notin \operatorname{im}\bigl[L_3(\mathbb{Z}^-)=\mathbb{Z}/2\overset{t\circ c=\mathrm{inj}}{\longrightarrow} L_3\bigl(\mathbb{Z}^-\times\mathbb{Z}/2^+\bigr)=\mathbb{Z}/2\times\mathbb{Z}/2\bigr]$$

which corresponds to the element $x \in L_3(\mathbb{Z}^- \times \mathbb{Z}/2^+)$ (refer to the diagram (D_6) above where $\wr$ is an isomorphism of the homologies of the chain complexes given by the horizontal sequences.) According to Proposition 6.4 the action of y on $hT_{CAT}(D^m \times Q(r-1,s))$ is trivial, so the homotopy CAT structure

$$f|_{f^{-1}(D^m \times Q(r-1,s)\times 1)} : f^{-1}\bigl(D^m \times Q(r-1,s) \times 1\bigr) \to D^m \times Q(r-1,s) \times 1$$

is trivial. Now if U is the tubular neighbourhood of $D^m \times Q(r-1,s)$ in $D^m \times Q(r,s)$, then $D^m \times Q(r,s)\backslash U = D^m \times V(r,s)$, $V(r,s)$ as defined in the proof of Proposition 6.4, so $D^m \times Q(r,s) = U \cup D^m \times V(r,s)$. Similarly $\partial_+ M = f_1^{-1}(U) \cup f_1^{-1}(D^m \times V(r,s))$, and f_1 gives simple homotopy equivalences on each piece. From the above observation $f_1|_{f_1^{-1}(U)} : f_1^{-1}(U) \to U$ is trivial. Also

$$f_1|_{f_1^{-1}(D^m \times V(r,s))} : f_1^{-1}\bigl(D^m \times V(r,s)\bigr) \to D^m \times V(r,s)$$

is a CAT homeomorphism, because it is the outcome of the action of an element of the image of $c = 0$ on $hT_{CAT}(X)$, which is trivial. Therefore $f_1 : \partial_+ M \to D^m \times Q(r,s) \times 1$ is a CAT-homeomorphism. Thus the action δ_{CAT} of x on $hT_{CAT}(X)$ is trivial. This completes the proof of the proposition. □

Proposition 6.6. *Let* $n = \dim X = m + r + 2s + 1 \equiv 2 \pmod 4$, $r + s + 1$ *odd. So* $\varepsilon = -$. *Then the action* δ_{CAT} *of the group* $L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-) = \mathbb{Z}/2$ *on* $hT_{CAT}(X)$ *is trivial.*

Proof. Refer to the diagram (D_4) of Section 4 with $n = 2$, and $\varepsilon = -$, this looks like:

$$\begin{tikzcd}
{} \arrow[r, "r"] & L_3(\mathbb{Z}^-) = \mathbb{Z}/2 \arrow[r, "c=\mathrm{inj}"] \arrow[d, phantom, "\wr"] & L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-) = \mathbb{Z}/2 \arrow[r, "\partial=r\circ t=0"] \arrow[d, phantom, "\wr"] & L_1(\mathbb{Z}^-) = 0 \arrow[r, "t\circ c"] \arrow[d, phantom, "\wr"] & {} \\
{} \arrow[r, "\partial=r\circ t"] & L_2(\mathbb{Z}^-) = \mathbb{Z}/2 \arrow[r, "t\circ c=\mathrm{inj}"] & L_2(\mathbb{Z}^- \times \mathbb{Z}/2^+) = \mathbb{Z}/2 \arrow[r, "r=0"] & L_2(\mathbb{Z}^-) = \mathbb{Z}/2 \arrow[r, "c"] & {}
\end{tikzcd} \qquad (D_7)$$

As c in this case is an isomorphism, the proof of the proposition follows from the first part of the proof of Proposition 6.4. □

Proposition 6.7. *Let* $n = \dim X = m + r + 2s + 1 \equiv 3 \pmod 4$, $r + s + 1$ *even. So* $\varepsilon = +$. *Then the action* δ_{CAT} *of the group* $L_0(\mathbb{Z}^- \times \mathbb{Z}/2^+) = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ *on* $hT_{CAT}(X)$ *is trivial when restricted to the subgroup* $\operatorname{Ker} \partial = \mathbb{Z}/2 \times \mathbb{Z}/2$, *and is free on the remaining summand.*

Proof. Refer to the diagram (D_4) of Section 4 with $n = 3$, and $\varepsilon = +$, this looks like:

$$\begin{tikzcd}
{} \arrow[r, "r"] & L_4(\mathbb{Z}^-) = \mathbb{Z}/2 \arrow[r, "c=\mathrm{inj}"] \arrow[d, phantom, "\wr"] & L_4(\mathbb{Z}^- \times \mathbb{Z}/2^+) = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r, "\partial=r\circ t=\mathrm{onto}"] \arrow[d, phantom, "\wr"] & L_4(\mathbb{Z}^-) = \mathbb{Z}/2 \arrow[r, "t\circ c"] \arrow[d, phantom, "\wr"] & {} \\
{} \arrow[r, "\partial=r\circ t"] & L_1(\mathbb{Z}^-) = 0 \arrow[r, "t\circ c=\mathrm{inj}"] & L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-) = \mathbb{Z}/2 \arrow[r, "r=0"] & L_3(\mathbb{Z}^-) = \mathbb{Z}/2 \arrow[r, "c"] & {}
\end{tikzcd} \qquad (D_8)$$

Action δ_{CAT} of the element $x \in \operatorname{im}[Z/2 = L_0(Z^-) \overset{c=\mathrm{inj}}{\longrightarrow} \ker \partial = \mathbb{Z}/2 \times \mathbb{Z}/2]$ on $hT_{CAT}(X)$ is trivial by the first part of the proof of Proposition 6.4.

Next let us consider the nontrivial element x' of $\ker \partial$

$$x' \notin \operatorname{im}\bigl[Z/2 = L_0(Z^-) \overset{c=\mathrm{inj}}{\longrightarrow} \ker \partial = \mathbb{Z}/2 \times Z/2\bigr].$$

Realize it by a normal map $f' : M' \to D^m \times Q(r,s) \times I$ such that

$$f'|_{\partial_- M'} : \partial_- M' \to D^m \times Q(r,s) \times 0 \cup \partial\bigl(D^m \times Q(r,s)\bigr) \times I$$

is a CAT homeomorphism ($CAT = PL$ or TOP). As $\partial(x') = 0$, by the relation between ∂ and the Browder–Livesay invariant mentioned in Section 4, the homotopy CAT structure $f'|_{\partial_+ M'} : \partial_+ M' \to D^m \times Q(r,s) \times 1$ is split along $D^m \times Q(r-1,s) \times 1$. We denote the map $f'|_{\partial_+ M'}$ by f_1'.

Thus the map

$$f_1'|_{f_1'^{-1}(D^m\times Q(r-1,s))}: f_1'^{-1}\big(D^m \times Q(r-1,s)\big) \to D^m \times Q(r-1,s)$$

is a simple homotopy equivalence, and is a CAT-homeomorphism on the boundary, that is, it is a homotopy CAT structure. Now the map

$$f'|_{f'^{-1}(D^m\times Q(r-1,s)\times I)}: f'^{-1}\big(D^m \times Q(r-1,s)\times I\big) \to D^m \times Q(r-1,s)\times I$$

is normal and realizes the element $y' \in L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-)$ which corresponds to the element $x' \in L_0(\mathbb{Z}^- \times \mathbb{Z}/2^+)$ (refer to the diagram (D_8) above where $\wr$ is an isomorphism of the homologies of the chain complexes given by the horizontal sequences.) According to Proposition 6.6 the action of y' on $hT_{CAT}(D^m \times Q(r-1,s))$ is trivial, so the homotopy CAT structure

$$f'|_{f'^{-1}(D^m\times Q(r-1,s)\times 1)}: f'^{-1}\big(D^m \times Q(r-1,s)\times 1\big) \to D^m \times Q(r-1,s)\times 1$$

is trivial. Now if U' is the tubular neighbourhood of $D^m \times Q(r-1,s)$ in $D^m \times Q(r,s)$, then $D^m \times Q(r,s)\backslash U' = D^m \times V(r,s)$, $V(r,s)$ as defined earlier, so $D^m \times Q(r,s) = U' \cup D^m \times V(r,s)$. Similarly $\partial_+ M' = f_1^{-1}(U') \cup f_1^{-1}(D^m \times V(r,s))$, and f_1' gives simple homotopy equivalences on each piece. From the above observation $f_1'|_{f_1'^{-1}(U')}: f_1^{-1}(U') \to U'$ is trivial. Also

$$f_1'|_{f_1'^{-1}(D^m\times V(r,s))}: f_1^{-1}\big(D^m \times V(r,s)\big) \to D^m \times V(r,s)$$

is a CAT homeomorphism, because it is the outcome of the action of an element of the image of c such as

$$x \in \operatorname{im}\big[\mathbb{Z}/2 = L_0(\mathbb{Z}^-) \xrightarrow{c=\mathrm{inj}} \ker\partial = \mathbb{Z}/2\times\mathbb{Z}/2\big]$$

on $hT_{CAT}(X)$, considered in the first part of the proof, which is trivial. Therefore $f_1': \partial_+ M' \to D^m \times Q(r,s)\times 1$ is a CAT-homeomorphism. Thus the action δ_{CAT} of x' on $hT_{CAT}(X)$ is trivial. The nonzero element of the remaining summand of $L_0(\mathbb{Z}^- \times \mathbb{Z}/2^+) = \mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$ is not in $\ker\partial$ hence not in $\ker\delta_{CAT} \subset \ker\partial$. This completes the proof of the proposition. □

Proposition 6.8. *Let* $n = \dim X = m + r + 2s + 1 \equiv 0 \pmod 4$, $r + s + 1$ *odd. So* $\varepsilon = -$. *Then the action* δ_{CAT} *of the group* $L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-) = \mathbb{Z}/2$ *on* $hT_{CAT}(X)$ *is trivial.*

Proof. Refer to the diagram (D_4) of Section 4 with $n = 0$, and $\varepsilon = -$, this looks like:

$$\begin{tikzcd}
{} \arrow[r, "r"] & L_1(\mathbb{Z}^-)=0 \arrow[r, "c"] \arrow[d, phantom, "\wr"] & L_1(\mathbb{Z}^-\times\mathbb{Z}/2^-)=\mathbb{Z}/2 \arrow[r, "\partial=r\circ t=0"] \arrow[d, phantom, "\wr"] & L_3(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "t\circ c"] \arrow[d, phantom, "\wr"] & {} \\
{} \arrow[r, "\partial=r\circ t"] & L_0(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "t\circ c=\mathrm{inj}"] & L_0(\mathbb{Z}^-\times\mathbb{Z}/2^+)=\mathbb{Z}/2\times\mathbb{Z}/2\times Z/2 \arrow[r, "r=\mathrm{onto}"] & L_0(\mathbb{Z}^-)=\mathbb{Z}/2 \arrow[r, "c"] & {}
\end{tikzcd} \qquad (D_9)$$

Consider the nonzero element $x \in L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-) = \mathbb{Z}/2$. Realize it by a normal map $f: M \to D^m \times Q(r,s)\times I$ such that

$$f|_{\partial_- M}: \partial_- M \to D^m \times Q(r,s)\times 0 \cup \partial\big(D^m \times Q(r,s)\big)\times I$$

is a CAT homeomorphism (*CAT* = *PL* or *TOP*). As $\partial(x) = 0$, by the relation between ∂ and the Browder–Livesay invariant mentioned in Section 4, the homotopy CAT structure $f|_{\partial_+ M}: \partial_+ M \to D^m \times Q(r,s)\times 1$ is split along $D^m \times Q(r-1,s)\times 1$. We denote the map $f|_{\partial_+ M}$ by f_1.

Thus the map

$$f_1|_{f_1^{-1}(D^m\times Q(r-1,s))}: f_1^{-1}\big(D^m \times Q(r-1,s)\big) \to D^m \times Q(r-1,s)$$

is a simple homotopy equivalence, and is a CAT-homeomorphism on the boundary, that is, it is a homotopy CAT structure. Now the map

$$f|_{f^{-1}(D^m\times Q(r-1,s)\times I)}: f^{-1}\big(D^m \times Q(r-1,s)\times I\big) \to D^m \times Q(r-1,s)\times I$$

is normal and realizes the element $y \in L_0(\mathbb{Z}^- \times \mathbb{Z}/2^+)$, $y \in \ker r = \ker r \circ t$, $y \notin \operatorname{im} t \circ c = \operatorname{im} c$, as t is an isomorphism, which corresponds to the element $x \in L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-)$ (refer to the diagram (D_9) above where $\wr$ is an isomorphism of the homologies of the chain complexes given by the horizontal sequences). According to Proposition 6.7 the action of y on $hT_{CAT}(D^m \times Q(r-1,s))$ is trivial, so the homotopy CAT structure

$$f|_{f^{-1}(D^m \times Q(r-1,s)\times 1)} : f^{-1}\big(D^m \times Q(r-1,s) \times 1\big) \to D^m \times Q(r-1,s) \times 1$$

is trivial. Now if U is the tubular neighbourhood of $D^m \times Q(r-1,s)$ in $D^m \times Q(r,s)$, then $D^m \times Q(r,s)\backslash U = D^m \times V(r,s)$, $V(r,s)$ as defined earlier, so $D^m \times Q(r,s) = U \cup D^m \times V(r,s)$. Similarly $\partial_+ M = f_1^{-1}(U) \cup f_1^{-1}(D^m \times V(r,s))$, and f_1 gives simple homotopy equivalences on each piece. From the above observation $f_1|_{f_1^{-1}(U)} : f_1^{-1}(U) \to U$ is trivial. Also

$$f_1|_{f_1^{-1}(D^m \times V(r,s))} : f_1^{-1}\big(D^m \times V(r,s)\big) \to D^m \times V(r,s)$$

is a CAT homeomorphism, because it is the outcome of the action of an element of the image of c, which is zero, on $hT_{CAT}(X)$, so it is trivial. Therefore $f_1 : \partial_+ M \to D^m \times Q(r,s) \times 1$ is a CAT-homeomorphism. Thus the action δ_{CAT} of x on $hT_{CAT}(X)$ is trivial. This completes the proof of the proposition. □

Thus in Propositions 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, and 6.8 the Kernel of the action map δ_{CAT} in the Sullivan–Wall exact sequence for manifolds of the form $D^m \times Q(r,s)$, $m \geqslant 0$ has been calculated in all possible cases.

7. The surgery obstruction map $\theta_{CAT} : [D^m \times Q(r,s)/\partial(D^m \times Q(r,s)), G/CAT] \to L_n(\mathbb{Z}^- \times \mathbb{Z}/2^\pm)$

We first recall that if $X = D^m \times Q(r,s)$, the map $\partial : L_{n+1}(\mathbb{Z}^- \times \mathbb{Z}/2^\varepsilon) \to L_{n+\varepsilon}(\mathbb{Z}^-)$ is defined as the composite:

$$L_{n+1}\big(\mathbb{Z}^- \times \mathbb{Z}/2^\varepsilon\big) \xrightarrow{\delta_{CAT}} hT_{CAT}\big(D^m \times Q(r,s)\big) \xrightarrow{\eta} BL\big(D^m \times Q(r,s), D^m \times Q(r-1,s)\big).$$

Lemma 7.1. *Let* $X = D^m \times Q(r,s)$, $r,s > 1$, $m \geqslant 0$, $\dim X = n+1$. *If* $x \in L_{n+1}(\mathbb{Z}^- \times \mathbb{Z}/2^\varepsilon)$ *is realized by a normal map* $F : M \to X$, *which is CAT homeomorphism on the boundary* (*CAT* = *PL or TOP*), *then* $\partial(x) = 0$.

Proof. Let $X_1 = D^m \times Q(r-1,s)$, then $\pi_1(X_1) = \pi_1(X)$, and

$$\pi_1(X)^{\omega^X} = \mathbb{Z}^- \times \mathbb{Z}/2^\varepsilon \implies \pi_1(X_1)^{\omega^{X_1}} = \mathbb{Z}^- \times \mathbb{Z}/2^{-\varepsilon}.$$

Realize the element $-x$ by a normal map $f : N \to X_1 \times I$, such that $f|_{\partial_- N} : \partial_- N \to X_1 \times 0 \cup \partial X_1 \times I$ is a CAT homeomorphism (*CAT* = *PL* or *TOP*). By definition of ∂, the obstruction to splitting the homotopy CAT structure $f|_{\partial_+ N} : \partial_+ N \to X_1 \times 1$ along the submanifold $Y_1 = D^m \times Q(r-2,s)$ is equal to $-\partial x$. Consider the connected sum of the manifolds M and N, and also the sum of X and $X_1 \times I$. The normal maps F and f define a normal map $F_1 : M \# N \to X \# X_1 \times I$. According to the construction of surgery obstructions the map F_1 is a simple homotopy equivalence, and considered as an element of the group $L_{n+1}(\mathbb{Z}^- \times \mathbb{Z}/2^\varepsilon)$, is equal to zero; but $\pi_1(X \# X_1 \times I) \neq \mathbb{Z} \times \mathbb{Z}/2$. However, by Wall [24, Theorem 9.4], one can change F_1 using simultaneous surgeries along 1-cycles in the manifolds $M \# N$ and $X \# X_1 \times I$, without changing the boundaries, which make the fundamental groups equal to $\mathbb{Z} \times \mathbb{Z}/2$. We obtain as a result of these surgeries a normal map $F_2 : M_2 \to X_2$. Since on one component of the boundary the map F_2 splits, it follows from a generalization of [13, Lemma 1, Section 1.2.2] that F_2 splits on the other component of the boundary too. Therefore $\partial(x) = 0$. □

Proposition 7.2. *The groups* $L_2(\mathbb{Z}^- \times \mathbb{Z}/2^\pm) \cong \mathbb{Z}/2$ *are completely realized by normal maps into the manifolds* $X = D^m \times Q(r,s), m \geqslant 0$ *which are CAT-homeomorphism* (*CAT* = *PL or TOP*) *on the boundary* (*for* $m = 0$ *the normal maps can be taken from closed manifolds*).

Proof. In both cases the nontrivial element of the group $L_2(\mathbb{Z}^- \times \mathbb{Z}/2^\pm) \cong \mathbb{Z}/2$ belongs to the image of the natural isomorphism $c : \mathbb{Z}/2 \cong L_2(\mathbb{Z}^-) \to L_2(\mathbb{Z}^- \times \mathbb{Z}/2^\pm) \cong \mathbb{Z}/2$.

Case I ($m > 0$): Let y be the nontrivial element of the group $L_2(\mathbb{Z}^- \times \mathbb{Z}/2^\pm)$ which is realized by a normal map

$$f : M \to D^m \times Q(r,s) = D^{m-1} \times Q(r,s) \times I,$$

such that

$$f|_{\partial_- M} : \partial_- M \to D^{m-1} \times Q(r,s) \times 0 \cup \partial\big(D^{m-1} \times Q(r,s)\big) \times I$$

is a CAT-homeomorphism ($CAT = PL$ or TOP). As $m + r + 2s + 1 \equiv 2 \pmod 4 \Longrightarrow m - 1 + r + 2s + 1 \equiv 1 \pmod 4$ by Proposition 6.3 action of y on $hT_{CAT}(D^{m-1} \times Q(r,s))$ is trivial. So

$$f|_{\partial_+ M} : \partial_+ M \to D^{m-1} \times Q(r,s) \times 1$$

is a CAT-homeomorphism ($CAT = PL$ or TOP). Hence $f|_{\partial M}$ is a CAT-homeomorphism ($CAT = PL$ or TOP), and the proof of the proposition is complete in this case.

Case II ($m = 0$): Recall the definition of $V(r,s)$ given in the proof of Proposition 6.4. It can be verified that $V(r,s) \cong V(r-1,s) \times I$, where I can be considered as a I-bundle over a point and $V(r-1,s) \times I$ is a product of two bundles (see [15]). Now let $x \in L_2(\mathbb{Z}^-) \cong \mathbb{Z}/2$ be the nontrivial element, so that $c(x) = y$ is as in case I with $m = 0$. Let x be realized by a normal map $f_1 : M_1 \to V(r,s) \cong V(r-1,s) \times I$, such that

$$f_1|_{\partial_- M_1} : \partial_- M_1 \to V(r-1,s) \times 0 \cup \partial V(r-1,s) \times I$$

is a CAT homeomorphism ($CAT = PL$ or TOP). The map

$$f_1|_{\partial M_1} : \partial M_1 \to V(r-1,s) \times \{0,1\} \cup \partial V(r-1,s) \times I$$

can be assumed to be a CAT homeomorphism because $\dim V(r,s) \equiv 2 \pmod 4$, and so for M_1 we can take the manifold $V(r,s) \# K$, where K is a Kervaire manifold, and for f_1 we can take the map which is identity on $V(r,s)$ and takes K to a point.

This makes it possible to 'glue' $V(r,s)$ to $Q(r,s)$ and in exactly the same way to 'glue' M_1 to some closed manifold M, giving us a normal map $f : M \to Q(r,s)$ of closed manifolds realizing the element $c(x) = y$. (Here the words 'glue' have the following meaning: Let $Q(r-1,s) \subset Q(r,s)$ has a tubular neighbourhood $\widehat{U}$, then $Q(r,s) \backslash \widehat{U} = V(r,s)$. So $Q(r,s) = \widehat{U} \cup V(r,s)$. with $\widehat{U} \cap V(r,s) = V(r-1,s) \times \{0,1\} \cup \partial V(r-1,s) \times I$ which is a nontrivial $S^{r-1} \times \mathbb{C}P^s$-bundle over S^1 and $f_1|_{\partial M_1} : \partial M_1 \to V(r-1,s) \times \{0,1\} \cup \partial V(r-1,s) \times I$ is a CAT homeomorphism. Thus $f_1 : M_1 \to V(r,s)$ extends to $f : M = M_1 \cup_{f_1|_{\partial M_1}} \widehat{U} \to V(r,s) \cup \widehat{U} = Q(r,s)$, where $f|_{\widehat{U}} : \widehat{U} \to \widehat{U}$ is identity.) □

Proposition 7.3. *The group* $L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-) \cong \mathbb{Z}/2$ *is completely realized by normal maps into the manifolds* $X = D^m \times Q(r,s), m \geqslant 0$ *which are CAT-homeomorphism* (*CAT* = *PL or TOP*) *on the boundary* (*for* $m = 0$ *the normal maps can be taken from closed manifolds*).

Proof. In this case again the nontrivial element of the group $L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-) \cong \mathbb{Z}/2$ belongs to the image of the natural isomorphism $c : \mathbb{Z}/2 \cong L_3(\mathbb{Z}^-) \to L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-) \cong \mathbb{Z}/2$.

Case I ($m > 0$): Let y be the nontrivial element of the group $L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-)$ which is realized by a normal map

$$f : M \to D^m \times Q(r,s) = D^{m-1} \times Q(r,s) \times I,$$

such that

$$f|_{\partial_- M} : \partial_- M \to D^{m-1} \times Q(r,s) \times 0 \cup \partial\big(D^{m-1} \times Q(r,s)\big) \times I$$

is a CAT-homeomorphism ($CAT = PL$ or TOP). As $m + r + 2s + 1 \equiv 3 \pmod 4 \Longrightarrow m - 1 + r + 2s + 1 \equiv 2 \pmod 4$ by Proposition 6.6 action of y on $hT_{CAT}(D^{m-1} \times Q(r,s))$ is trivial. So

$$f|_{\partial_+ M} : \partial_+ M \to D^{m-1} \times Q(r,s) \times 1$$

is a CAT-homeomorphism ($CAT = PL$ or TOP). Hence $f|_{\partial M}$ is a CAT-homeomorphism ($CAT = PL$ or TOP), and the proof of the proposition is complete in this case.

Case II ($m = 0$): With the same notations as in the proof of the last Proposition 7.2 let $x \in L_3(\mathbb{Z}^-) \cong \mathbb{Z}/2$ be the nontrivial element, so that $c(x) = y$ is as in case I with $m = 0$. Let x be realized by a normal map $f_1 : M_1 \to V(r,s) \cong V(r-1,s) \times I$, such that

$$f_1|_{\partial_- M_1} : \partial_- M_1 \to V(r-1,s) \times 0 \cup \partial V(r-1,s) \times I$$

is a CAT homeomorphism (*CAT* = *PL* or *TOP*), and $f_1|_{\partial_+ M_1}: \partial_+ M_1 \to V(r-1,s) \times 1$ is a simple homotopy equivalence which is a CAT-homeomorphism on the boundary, so $f_1|_{\partial_+ M_1}$ is a homotopy CAT-structure on $V(r-1,s)$. From Lemma 6.1 it follows that $f_1|_{\partial_+ M_1}$ is a trivial CAT-structure on $V(r-1,s)$. Therefore, the map

$$f_1|_{\partial M_1}: \partial M_1 \to V(r-1,s) \times \{0,1\} \cup \partial V(r-1,s) \times I$$

can be assumed to be a CAT homeomorphism.

Rest of the proof is same as the proof of the last Proposition 7.2. □

Proposition 7.4. *In the group* $L_0(\mathbb{Z}^- \times \mathbb{Z}/2^+) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ *elements belonging to* $\ker\partial \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ *are realized by normal maps into the manifolds* $X = D^m \times Q(r,s), m \geqslant 0$ *which are CAT-homeomorphism* (*CAT* = *PL or TOP*) *on the boundary* (*refer to diagram* (D$_8$) *for notations*). *Remaining elements are not realized by any normal maps which are CAT-homeomorphism on the boundary.*

Proof. *Case* I ($m > 0$): Let y be a nontrivial element of subgroup $\ker\partial \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ of the group $L_0(\mathbb{Z}^- \times \mathbb{Z}/2^+)$ which is realized by a normal map

$$f: M \to D^m \times Q(r,s) = D^{m-1} \times Q(r,s) \times I,$$

such that

$$f|_{\partial_- M}: \partial_- M \to D^{m-1} \times Q(r,s) \times 0 \cup \partial\big(D^{m-1} \times Q(r,s)\big) \times I$$

is a CAT-homeomorphism (*CAT* = *PL* or *TOP*). As $m + r + 2s + 1 \equiv 0 \pmod 4 \Longrightarrow m - 1 + r + 2s + 1 \equiv 3 \pmod 4$ by Proposition 6.7 action of y on $hT_{CAT}(D^{m-1} \times Q(r,s))$ is trivial. So

$$f|_{\partial_+ M}: \partial_+ M \to D^{m-1} \times Q(r,s) \times 1$$

is a CAT-homeomorphism (*CAT* = *PL* or *TOP*). Hence $f|_{\partial M}$ is a CAT-homeomorphism (*CAT* = *PL* or *TOP*), and the proof of the proposition is complete in this case.

Case II ($m = 0$): With the same notations as in the proof of the last Proposition 7.3 we consider two cases: (i) Let $x \in L_0(\mathbb{Z}^-) \cong \mathbb{Z}/2$ be the nontrivial element, so that $c(x) = y$ is as in case I with $m = 0$. Let x be realized by a normal map $f_1: M_1 \to V(r,s) \cong V(r-1,s) \times I$, such that

$$f_1|_{\partial_- M_1}: \partial_- M_1 \to V(r-1,s) \times 0 \cup \partial V(r-1,s) \times I$$

is a CAT homeomorphism (*CAT* = *PL* or *TOP*), and $f_1|_{\partial_+ M_1}: \partial_+ M_1 \to V(r-1,s) \times 1$ is a simple homotopy equivalence which is a CAT-homeomorphism on the boundary, so $f_1|_{\partial_+ M_1}$ is a homotopy CAT-structure on $V(r-1,s)$. From Proposition 6.1 it follows that $f_1|_{\partial_+ M_1}$ is a trivial CAT-structure on $V(r-1,s)$. Therefore, the map

$$f_1|_{\partial M_1}: \partial M_1 \to V(r-1,s) \times \{0,1\} \cup \partial V(r-1,s) \times I$$

can be assumed to be a CAT homeomorphism.

(ii) Let y as in case I with $m = 0$ with $y \notin \operatorname{im} c$, then the first two Browder–Livesay invariants of y are zero, so its image under the natural forgetful map $L_4(\mathbb{Z}^- \times \mathbb{Z}/2^+) \to L_4^p((\mathbb{Z}^- \times \mathbb{Z}/2^+)$ (isomorphism in this case as $\widetilde{K}_0(\mathbb{Z}(\mathbb{Z} \times \mathbb{Z}/2)) = 0$) is either in the image of $L_4^p(0)$ or is an Arf invariant in codimension 1 or 2. Hence y is realized by a normal map of a closed manifold into $Q(r,s)$ (see [11], and [6]).

Rest of the proof of this case is same as the proof of the last Proposition 7.3.

Lastly if an element is not in $\ker\partial$ then by Lemma 7.1 this element is not realized by any normal map which are CAT-homeomorphism on the boundary. □

Proposition 7.5. *In the group* $L_3(\mathbb{Z}^- \times \mathbb{Z}/2^+) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 = \ker\partial$ *all elements are realized by normal maps into the manifolds* $X = D^m \times Q(r,s), m \geqslant 0$ *which are CAT-homeomorphism* (*CAT* = *PL or TOP*) *on the boundary.*

Proof. *Case* I ($m > 0$): Let y be a nontrivial element of the group $L_3(\mathbb{Z}^- \times \mathbb{Z}/2^+)$ which is realized by a normal map

$$f: M \to D^m \times Q(r,s) = D^{m-1} \times Q(r,s) \times I,$$

such that

$$f|_{\partial_- M}: \partial_- M \to D^{m-1} \times Q(r,s) \times 0 \cup \partial\big(D^{m-1} \times Q(r,s)\big) \times I$$

is a CAT-homeomorphism (*CAT* = *PL* or *TOP*). As $m + r + 2s + 1 \equiv 3 \pmod 4 \Longrightarrow m - 1 + r + 2s + 1 \equiv 2 \pmod 4$ by Proposition 6.4 action of y on $hT_{CAT}(D^{m-1} \times Q(r,s))$ is trivial. So

$$f|_{\partial_+ M} : \partial_+ M \to D^{m-1} \times Q(r,s) \times 1$$

is a CAT-homeomorphism (*CAT* = *PL* or *TOP*). Hence $f|_{\partial M}$ is a CAT-homeomorphism (*CAT* = *PL* or *TOP*), and the proof of the proposition is complete in this case.

Case II ($m = 0$): With the same notations as in the proof of the last Proposition 7.4 let $x \in L_0(\mathbb{Z}^-) \cong \mathbb{Z}/2$ be the nontrivial element, so that $c(x) = y$ is as in case I with $m = 0$. Let x be realized by a normal map $f_1 : M_1 \to V(r,s) \cong V(r-1,s) \times I$, such that

$$f_1|_{\partial_- M_1} : \partial_- M_1 \to V(r-1,s) \times 0 \cup \partial V(r-1,s) \times I$$

is a CAT homeomorphism (*CAT* = *PL* or *TOP*), and $f_1|_{\partial_+ M_1} : \partial_+ M_1 \to V(r-1,s) \times 1$ is a simple homotopy equivalence which is a CAT-homeomorphism on the boundary, so $f_1|_{\partial_+ M_1}$ is a homotopy CAT-structure on $V(r-1,s)$. From Proposition 6.1 it follows that $f_1|_{\partial_+ M_1}$ is a trivial CAT-structure on $V(r-1,s)$. Therefore, the map

$$f_1|_{\partial M_1} : \partial M_1 \to V(r-1,s) \times \{0,1\} \cup \partial V(r-1,s) \times I$$

can be assumed to be a CAT homeomorphism.

Rest of the proof of this case is same as the proof of the last Proposition 7.4. For the other element $z \in L_3(\mathbb{Z}^- \times \mathbb{Z}/2^+)$ which is not in $\operatorname{im} c$, the first two Browder–Livesay invariants are zero so its image under the natural forgetful map $L_3(\mathbb{Z}^- \times \mathbb{Z}/2^+) \to L_3^p((\mathbb{Z}^- \times \mathbb{Z}/2^+)$ (isomorphism in this case as $\widetilde{K}_0(\mathbb{Z}(\mathbb{Z} \times \mathbb{Z}/2)) = 0$) is either in the image of $L_3^p(0)$ or is an Arf invariant in codimension 1 or 2. Hence z is realized by a normal map of a closed manifold into $Q(r,s)$ (see [11], and [6]). □

Proposition 7.6. *The group $L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-) \cong \mathbb{Z}/2$ is completely realized by normal maps into the manifolds $X = D^m \times Q(r,s)$, $m > 0$ which are CAT-homeomorphism (CAT = PL or TOP) on the boundary. If $m = 0$ then the nonzero element in $L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-) \cong \mathbb{Z}/2$ is not realized by any normal map of a closed manifold into $Q(r,s)$.*

Proof. *Case* I ($m > 0$): Let y be the nontrivial element of the group $L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-)$ which is realized by a normal map

$$f : M \to D^m \times Q(r,s) = D^{m-1} \times Q(r,s) \times I, \quad m > 0$$

such that

$$f|_{\partial_- M} : \partial_- M \to D^{m-1} \times Q(r,s) \times 0 \cup \partial\bigl(D^{m-1} \times Q(r,s)\bigr) \times I$$

is a CAT-homeomorphism (*CAT* = *PL* or *TOP*). As $m + r + 2s + 1 \equiv 1 \pmod 4 \Longrightarrow m - 1 + r + 2s + 1 \equiv 0 \pmod 4$ by Proposition 6.8 action of y on $hT_{CAT}(D^{m-1} \times Q(r,s))$ is trivial. So

$$f|_{\partial_+ M} : \partial_+ M \to D^{m-1} \times Q(r,s) \times 1$$

is a CAT-homeomorphism (*CAT* = *PL* or *TOP*). Hence $f|_{\partial M}$ is a CAT-homeomorphism (*CAT* = *PL* or *TOP*), and the proof of the proposition in this case is complete.

Case II ($m = 0$): In this case the only element which can be realized by any normal maps lie in the image of $L_1(\mathbb{Z}^-)$, or $L_1(\mathbb{Z}/2^{\pm})$ or $L_1(0)$, but all these groups are 0 groups and the proof of the proposition in this case is also complete (see [11], and [6]). □

Proposition 7.7. *The group $L_0(\mathbb{Z}^- \times \mathbb{Z}/2^-) \cong \mathbb{Z}/2$ is completely realized by normal maps into the manifolds $X = D^m \times Q(r,s)$, $m \geqslant 0$ which are CAT-homeomorphism (CAT = PL or TOP) on the boundary.*

Proof. *Case* I ($m > 0$): Let y be the nontrivial element of the group $L_0(\mathbb{Z} \times \mathbb{Z}/2^-)$ which is realized by a normal map

$$f : M \to D^m \times Q(r,s) = D^{m-1} \times Q(r,s) \times I, \quad m > 0$$

such that

$$f|_{\partial_- M} : \partial_- M \to D^{m-1} \times Q(r,s) \times 0 \cup \partial\big(D^{m-1} \times Q(r,s)\big) \times I$$

is a CAT-homeomorphism (*CAT* = *PL* or *TOP*). As $m + r + 2s + 1 \equiv 0 \pmod 4 \Longrightarrow m - 1 + r + 2s + 1 \equiv 3 \pmod 4$ by Proposition 6.5 action of y on $hT_{CAT}(D^{m-1} \times Q(r,s))$ is trivial. So

$$f|_{\partial_+ M} : \partial_+ M \to D^{m-1} \times Q(r,s) \times 1$$

is a CAT-homeomorphism (*CAT* = *PL* or *TOP*). Hence $f|_{\partial M}$ is a CAT-homeomorphism (*CAT* = *PL* or *TOP*), and the proof of the proposition is complete in this case.

Case II ($m = 0$): Let y as in case I with $m = 0$ then the first two Browder–Livesay invariants of y are zero, so its image under the natural forgetful map $L_4(\mathbb{Z}^- \times \mathbb{Z}/2^-) \to L_4^p((\mathbb{Z}^- \times \mathbb{Z}/2^-)$ (isomorphism in this case as $\widetilde{K}_0(\mathbb{Z}(\mathbb{Z} \times \mathbb{Z}/2)) = 0$) is either in the image of $L_4^p(0)$ or is an Arf invariant in codimension 1 or 2. Hence y is realized by a normal map of a closed manifold into $Q(r,s)$ (see [11], and [6]). □

Thus in Propositions 7.2, 7.3, 7.4, 7.5, 7.6, 7.7 we have determined the elements of Wall surgery obstruction groups which can be realized by normal maps into $D^m \times Q(r,s)$ which are CAT-homeomorphisms on the boundary (for $m > 0$), and in some cases by normal maps of closed manifolds into $Q(r,s)$.

8. PL and TOP classification theorems and remarks on homotopy smoothings

There exists natural maps of the smooth version of Sullivan Wall surgery exact sequence to the CAT versions of Sullivan Wall surgery exact sequences (*CAT* = *PL* or *TOP*):

$$\begin{tikzcd}
{} \arrow[r] & L_{n+1}(\pi_1(X), \omega(X)) \arrow[r, "\delta_O"] \arrow[d, "="] & hS(X) \arrow[r, "\eta_O"] \arrow[d] & {[X/\partial X, G/O]} \arrow[r, "\theta_O"] \arrow[d] & L_n(\pi_1(X), \omega(X)) \arrow[d, "="] \\
{} \arrow[r] & L_{n+1}(\pi_1(X), \omega(X)) \arrow[r, "\delta_{CAT}"] & hT_{CAT}(X) \arrow[r, "\eta_{CAT}"] & {[X/\partial X, G/CAT]} \arrow[r, "\theta_{CAT}"] & L_n(\pi_1(X), \omega(X))
\end{tikzcd} \tag{D$_{10}$}$$

so, using these one can draw many conclusions from the results of Section 6 (Propositions 6.2, 6.3, 6.4, 6.5, 6.6, 6.7,6.8) and Section 7 (Propositions 7.2, 7.3, 7.4, 7.5, 7.6, 7.7).

Proposition 8.1. *Let* $n = \dim X = m + r + 2s + 1 \equiv 0 \pmod 4$, $r + s + 1$ *even. So* $\varepsilon = +$. *Then the action* δ_O *of the group* $L_1(\mathbb{Z}^- \times \mathbb{Z}/2^+) = 0$ *on* $hS(X)$ *is obviously trivial.*

Proposition 8.2. *Let* $n = \dim X = m + r + 2s + 1 \equiv 3 \pmod 4$, $r + s + 1$ *even. So* $\varepsilon = +$. *Then the action* δ_O *of the group* $L_0(\mathbb{Z}^- \times \mathbb{Z}/2^+) = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ *on* $hS(X)$ *is trivial when restricted to a subgroup of* $\operatorname{Ker} \partial = \mathbb{Z}/2 \times \mathbb{Z}/2$, *and is free on the remaining summand.*

Proposition 8.3. *The groups* $L_2(\mathbb{Z}^- \times \mathbb{Z}/2^\pm) \cong \mathbb{Z}/2$ *are completely realized by normal maps of closed smooth manifolds into the manifolds* $X = Q(r,s)$.

Proposition 8.4. *In the group* $L_0(\mathbb{Z}^- \times \mathbb{Z}/2^+) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ *elements belonging to a subgroup of* $\ker \partial \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ *are realized by normal maps of smooth manifolds into the manifolds* $X = D^m \times Q(r,s)$, $m > 0$ *which are diffeomorphism* (*refer to diagram* (D$_8$) *for notations*). *Reamaining elements are not realized by any normal maps of smooth manifolds which are diffeomorphism on the boundary.*

Finally we summarize the calculations made in the previous sections in the form of the following:

Theorem 8.5 *(Classification theorem 1). Let* $X = D^m \times Q(r,s)$, $r, s > 1$, *where* D^m *is an* m*-dimensional disk,* $m \geqslant 0$. *Then there are following exact sequences* (*CAT* = *PL or TOP*):

(1) *If* $\dim X = m + r + 2s + 1 \equiv 3 \pmod 4$, $r + s + 1$ *even, that is* $\varepsilon = +$, *and* $m \geqslant 0$, *then*

$$\to L_0\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_3\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big),$$

reduces to

$$0 \to \mathbb{Z}/2 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} \mathbb{Z}/2 \times \mathbb{Z}/2 \to 0;$$

(2) *If* $\dim X = m + r + 2s + 1 \equiv 3 \pmod 4$, $r + s + 1$ *odd, that is* $\varepsilon = -$, *and* $m \geqslant 0$, *then*

$$\to L_0(\mathbb{Z}^- \times \mathbb{Z}/2^-) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-),$$

reduces to

$$0 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} \mathbb{Z}/2 \to 0;$$

(3) *If* $\dim X = m + r + 2s + 1 \equiv 0 \pmod 4$, $r + s + 1$ *even, that is* $\varepsilon = +$, *and* $m \geqslant 0$, *then*

$$\to L_1\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_0\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big),$$

reduces to

$$0 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} \mathbb{Z}/2 \times \mathbb{Z}/2 \to 0;$$

(4) *If* $\dim X = m + r + 2s + 1 \equiv 0 \pmod 4$, $r + s + 1$ *odd, that is* $\varepsilon = -$, *and* $m \geqslant 0$, *then*

$$\to L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_0(\mathbb{Z}^- \times \mathbb{Z}/2^-),$$

reduces to

$$0 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} \mathbb{Z}/2 \to 0;$$

(5) *If* $\dim X = m + r + 2s + 1 \equiv 1 \pmod 4$, $r + s + 1$ *even, that is* $\varepsilon = +$, *and* $m \geqslant 0$, *then*

$$\to L_2\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_1\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big),$$

reduces to

$$0 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} 0;$$

(6) *If* $\dim X = m + r + 2s + 1 \equiv 1 \pmod 4$, $r + s + 1$ *odd, that is* $\varepsilon = -$, *and* $m > 0$, *then*

$$\to L_2(\mathbb{Z}^- \times \mathbb{Z}/2^-) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-),$$

reduces to

$$0 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} \mathbb{Z}/2 \to 0;$$

(7) *If* $\dim X = m + r + 2s + 1 \equiv 1 \pmod 4$, $r + s + 1$ *odd, that is* $\varepsilon = -$, *and* $m = 0$, *then*

$$\to L_2(\mathbb{Z}^- \times \mathbb{Z}/2^-) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_1(\mathbb{Z}^- \times \mathbb{Z}/2^-),$$

reduces to

$$0 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} 0;$$

(8) *If* $\dim X = m + r + 2s + 1 \equiv 2 \pmod 4$, $r + s + 1$ *even, that is* $\varepsilon = +$, *and* $m \geqslant 0$, *then*

$$\to L_3\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_2\big(\mathbb{Z}^- \times \mathbb{Z}/2^+\big),$$

reduces to

$$0 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} \mathbb{Z}/2 \to 0;$$

(9) *If* $\dim X = m + r + 2s + 1 \equiv 2 \pmod 4$, $r + s + 1$ *odd, that is* $\varepsilon = -$, *and* $m \geqslant 0$, *then*

$$\to L_3(\mathbb{Z}^- \times \mathbb{Z}/2^-) \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} L_2(\mathbb{Z}^- \times \mathbb{Z}/2^-),$$

reduces to

$$0 \xrightarrow{\delta_{CAT}} hT_{CAT}(X) \xrightarrow{\eta_{CAT}} [X/\partial X, G/CAT] \xrightarrow{\theta_{CAT}} \mathbb{Z}/2 \to 0.$$

Proof. Part (1) of the theorem follows from Propositions 6.7 and 7.5. Part (2) follows from Propositions 6.5 and 7.3. Part (3) follows from Propositions 6.2 and 7.4. Part (4) follows from Propositions 6.8 and 7.7. Part (5) is trivial. Part (6) and (7) follows from Propositions 6.3 and 7.6. Part (8) follows from Propositions 6.4 and 7.2. Part (9) follows from Propositions 6.6 and 7.2. □

This theorem and Theorems 5.4 and 5.5 together determines $hT_{CAT}(Q(r,s))$ completely, where $CAT = PL$ or TOP, once we analyze the maps θ_{CAT}, η_{CAT}, and δ_{CAT} bit more closely:

Theorem 8.6 *(Classification theorem 2). Consider Wall's manifolds $Q(r,s)$, $r,s>1$, $r+2s+1=4k+j$, $j=1$, or 2, or 3, or 4. Then for $k\geqslant 1$: (Coefficients of integral cohomologies are dropped)*

$$hT_{TOP}\big(Q(r,s)^{4k+1}\big) \cong \sum_{i=2}^{k} H^{4i-2}\big(Q(r,s);\mathbb{Z}/2\big) \oplus \sum_{i=2}^{k} H^{4i}\big(Q(r,s)\big); \tag{1(i)}$$

$$\begin{aligned} &hT_{PL}\big(Q(r,s)^{4k+1}\big)\\ &\quad\cong \begin{cases} (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r>4,\\ (\mathbb{Z}\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=4,\\ (\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=3,\\ (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=2; \end{cases} \end{aligned} \tag{1(ii)}$$

$$hT_{TOP}\big(Q(r,s)^{4k+2}\big) \cong \sum_{i=2}^{k} H^{4i-2}\big(Q(r,s);\mathbb{Z}/2\big) \oplus \sum_{i=2}^{k} H^{4i}\big(Q(r,s)\big); \tag{2(i)}$$

$$\begin{aligned} &hT_{PL}\big(Q(r,s)^{4k+2}\big)\\ &\quad\cong \begin{cases} (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r>4,\\ (\mathbb{Z}\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=4,\\ (\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=3,\\ (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=2; \end{cases} \end{aligned} \tag{2(ii)}$$

$$hT_{TOP}\big(Q(r,s)^{4k+3}_{-+}\big) \cong \sum_{i=2}^{k} H^{4i-2}\big(Q(r,s);\mathbb{Z}/2\big) \oplus \sum_{i=2}^{k} H^{4i}\big(Q(r,s)\big); \tag{3(i)}$$

$$\begin{aligned} &hT_{PL}\big(Q(r,s)^{4k+3}_{-+}\big)\\ &\quad\cong \begin{cases} (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r>4,\\ (\mathbb{Z}\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=4,\\ (\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=3,\\ (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=2; \end{cases} \end{aligned} \tag{3(ii)}$$

$$hT_{TOP}\big(Q(r,s)^{4k+3}_{--}\big) \cong \sum_{i=2}^{k} H^{4i-2}\big(Q(r,s);\mathbb{Z}/2\big) \oplus \mathbb{Z}/2 \oplus \sum_{i=2}^{k} H^{4i}\big(Q(r,s)\big); \tag{4(i)}$$

$$\begin{aligned} &hT_{PL}\big(Q(r,s)^{4k+3}_{--}\big)\\ &\quad\cong \begin{cases} (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r>4,\\ (\mathbb{Z}\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=4,\\ (\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=3,\\ (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=2; \end{cases} \end{aligned} \tag{4(ii)}$$

$$hT_{TOP}\big(Q(r,s)^{4k+4}_{-+}\big) \cong \sum_{i=2}^{k} H^{4i-2}\big(Q(r,s);\mathbb{Z}/2\big) \oplus \mathbb{Z}/2\times\mathbb{Z}/2 \oplus \sum_{i=2}^{k} H^{4i}\big(Q(r,s)\big); \tag{5(i)}$$

$$\begin{aligned} &hT_{PL}\big(Q(r,s)^{4k+4}_{-+}\big)\\ &\quad\cong \begin{cases} (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\times\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r>4,\\ (\mathbb{Z}\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\times\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=4,\\ (\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\times\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=3,\\ (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\times\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=2; \end{cases} \end{aligned} \tag{5(ii)}$$

$$hT_{TOP}\big(Q(r,s)^{4k+4}_{--}\big) \cong \sum_{i=2}^{k+1} H^{4i-2}\big(Q(r,s);\mathbb{Z}/2\big) \oplus \sum_{i=2}^{k} H^{4i}\big(Q(r,s)\big); \tag{6(i)}$$

$$\begin{aligned} &hT_{PL}\big(Q(r,s)^{4k+4}_{--}\big)\\ &\quad\cong \begin{cases} (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k+1}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r>4,\\ (\mathbb{Z}\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k+1}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=4,\\ (\mathbb{Z}/4\times\mathbb{Z}/4\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k+1}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=3,\\ (\mathbb{Z}/4\times\mathbb{Z}/2\times\mathbb{Z}/2)\oplus\sum_{i=2}^{k+1}H^{4i-2}(Q(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(Q(r,s)) & \textit{if } r=2. \end{cases} \end{aligned} \tag{6(ii)}$$

Proof. *Case* (1(i), (ii)). This case follows directly from (Theorem 8.5 parts (5) and (7)), and Theorems 5.4, 5.5.

Case (2(i), (ii)). $\dim Q(r,s) \equiv 2 \pmod 4$, $\geqslant 6$, $P(r,s)$ orientable or not. Let $\dim Q(r,s) = 4k+2$, $k \geqslant 1$.

In this case $\theta_{CAT} : [(Q(r,s))^{4k+2}, G/CAT] \twoheadrightarrow \mathbb{Z}/2$ coincides with the projection map $\phi_{4k+2} : [Q(r,s), G/CAT] \to H^{4k+2}(Q(r,s);\mathbb{Z}/2) = \mathbb{Z}/2$. Hence the result follows from (Theorem 8.5 parts (8) and (9)), and Theorems 5.4, 5.5.

Case (3(i), (ii)). $\dim Q(r,s) \equiv 3 \pmod 4$, $\geqslant 6$, $P(r,s)$ orientable, so r has to be necessarily even, and s necessarily odd. Let $\dim Q(r,s) = 4k+3$, $k \geqslant 1$.

In this case $i_1 : Q(r-1,s) \hookrightarrow Q(r,s)$ induces

$$\big[\big(Q(r,s)\big)^{4k+3}_{-+}, G/CAT\big] \xrightarrow{i_1^*} \big[\big(Q(r-1,s)\big)^{4k+2}_{--}, G/CAT\big],$$

and $i_2 : P(r,s) \hookrightarrow Q(r,s)$ induces

$$\big[\big(Q(r,s)\big)^{4k+3}_{-+}, G/CAT\big] \xrightarrow{i_2^*} \big[\big(P(r,s)\big)^{4k+2}_{+}, G/CAT\big],$$

and $\theta_{CAT} : [(Q(r,s))^{4k+3}, G/CAT] \twoheadrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$ coincides with the composite

$$\begin{aligned} &\big[\big(Q(r,s)\big)^{4k+3}, G/CAT\big] \xrightarrow{\text{diag}} \big[\big(Q(r,s)\big)^{4k+3}, G/CAT\big] \times \big[\big(Q(r,s)\big)^{4k+3}, G/CAT\big]\\ &\quad\xrightarrow{\phi'_{4k+2}\circ i_1^*\times\phi''_{4k+2}\circ i_2^*} H^{4k+2}\big(\big(Q(r-1,s)\big)^{4k+2};\mathbb{Z}/2\big) \times H^{4k+2}\big(\big(P(r,s)\big)^{4k+2};\mathbb{Z}/2\big) = \mathbb{Z}/2\times\mathbb{Z}/2. \end{aligned}$$

So the result follows from (Theorem 8.5 part (1)), and Theorems 5.4, 5.5.

Case (4(i), (ii)). $\dim Q(r,s) \equiv 3 \pmod 4$, $\geqslant 6$, $P(r,s)$ nonorientable, so r, and s have to be necessarily even. Let $\dim Q(r,s) = 4k+3$, $k \geqslant 1$.

In this case $i : P(r,s) \hookrightarrow Q(r,s)$ induces

$$\big[\big(Q(r,s)\big)^{4k+3}_{--}, G/CAT\big] \xrightarrow{i^*} \big[\big(P(r,s)\big)^{4k+2}_{-}, G/CAT\big],$$

and $\theta_{CAT} : [(Q(r,s))^{4k+3}, G/CAT] \twoheadrightarrow \mathbb{Z}/2$ coincides with the composite

$$\big[\big(Q(r,s)\big)^{4k+3}, G/CAT\big] \xrightarrow{\phi_{4k+2}\circ i^*} H^{4k+2}\big(\big(P(r,s)\big)^{4k+2};\mathbb{Z}/2\big) = \mathbb{Z}/2.$$

So the result follows from (Theorem 8.5 part (2)), and Theorems 5.4, 5.5.

Case (5(i), (ii)). $\dim Q(r,s) \equiv 0 \pmod 4$, $\geqslant 6$, $P(r,s)$ orientable, so r has to be necessarily odd, and s necessarily even. Let $\dim Q(r,s) = 4k+4$, $k \geqslant 1$.

In this case $i : Q(r, s-1) \hookrightarrow Q(r,s)$ induces

$$\big[\big(Q(r,s)\big)^{4k+4}_{-+}, G/CAT\big] \xrightarrow{i^*} \big[\big(Q(r,s-1)\big)^{4k+2}_{--}, G/CAT\big],$$

and $\theta_{CAT} : [(Q(r,s))^{4k+4}, G/CAT] \twoheadrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$ coincides with the composite

$$\big[\big(Q(r,s)\big)^{4k+4}, G/CAT\big] \xrightarrow{\text{diag}} \big[\big(Q(r,s)\big)^{4k+4}, G/CAT\big] \times \big[\big(Q(r,s)\big)^{4k+4}, G/CAT\big]$$
$$\xrightarrow{\phi_{4k+4} \times \phi'_{4k+2} \circ i^*} H^{4k+4}\big(\big(Q(r,s)\big)^{4k+4}; \mathbb{Z}\big) \times H^{4k+2}\big(\big(Q(r,s-1)\big)^{4k+2}; \mathbb{Z}/2\big) = \mathbb{Z}/2 \times \mathbb{Z}/2.$$

So the result follows from (Theorem 8.5 part (3)), and Theorems 5.4, 5.5.

Case (6(i), (ii)). $\dim Q(r,s) \equiv 0 \pmod 4$, and $\geqslant 6$, $P(r,s)$ nonorientable, so r and s have to be necessarily odd. Let $\dim Q(r,s) = 4k+4$, $k \geqslant 1$.

In this case $\theta_{CAT} : [(Q(r,s))^{4k+4}, G/CAT] \twoheadrightarrow \mathbb{Z}/2$ coincides with

$$\big[\big(Q(r,s)\big)^{4k+4}, G/CAT\big] \xrightarrow{\phi_{4k+4}} H^{4k+4}\big(\big(Q(r,s)\big)^{4k+4}; \mathbb{Z}\big) = \mathbb{Z}/2.$$

So the result follows from (Theorem 8.5 part (4)), and Theorems 5.4, 5.5. □

References

[1] W. Browder, Manifolds with $\pi_1 = \mathbb{Z}$, Bull. Amer. Math. Soc. 72 (2) (1966) 238–244.

[2] W. Browder, Surgery on Simply-Connected Manifolds, Ergeb. Series, Band 65, Springer, Berlin, 1972.

[3] W. Browder, G.R. Livesay, Fixed point free involutions on homotopy spheres, Bull. Amer. Math. Soc. 73 (1967) 242–245.

[4] A. Dold, Erzeugende der Thomschen Algebra $\mathfrak{N}$, Math. Z. 65 (1956) 25–35.

[5] M. Fujii, K_U-groups of Dold manifolds, Osaka J. Math. 3 (1966) 49–64.

[6] I. Hambleton, Projective surgery obstructions on closed manifolds, in: Algebraic K-Theory, Part II, Oberwolfach, 1980, in: Lecture Notes in Math., vol. 967, Springer, Berlin, 1982, pp. 101–131.

[7] A.F. Haršiladze, Manifolds of the homotopy type of the product of two projective spaces, Math. USSR Sbornik 25 (1975) 471–486, Uspekhi Mat. Nauk 42 (4) (1987) 55–85.

[8] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002.

[9] A.F. Kharshiladze, Obstruction to surgery for the group $(\pi \times \mathbb{Z}_2)$, Math. Notes 16 (1974) 1085–1090.

[10] A.F. Kharshiladze, Smooth and piecewise-linear structures on products of projective spaces, Math. USSR Izvestiya 22 (2) (1984) 339–355.

[11] A.F. Kharshiladze, Surgery on manifolds with finite fundamental group, Uspekhi Mat. Nauk 42 (4) (1987) 55–85.

[12] R.C. Kirby, L.C. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings, and Triangulations, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1977.

[13] S. López de Medrano, Involutions on Manifolds, Ergeb. Math. Grenz., Band 59, Springer, Berlin, 1971.

[14] J.W. Milnor, On the Stiefel–Whitney numbers of complex manifolds and spin manifolds, Topology 3 (1965) 223–230.

[15] J.W. Milnor, J.D. Stasheff, Characteristic Classes, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1974.

[16] H.K. Mukerjee, Classification of homotopy Dold manifolds, New York J. Math. 9 (2003) 1–23.

[17] H.K. Mukerjee, Classification of homotopy real Milnor manifolds, Topology Appl. 139 (2004) 151–184.

[18] A.A. Ranicki, Exact Sequences in the Algebraic Theory of Surgery, Mathematical Notes, vol. 26, Princeton University Press, Princeton, NJ, 1981.

[19] Y.B. Rudyak, On normal invariants of certain manifolds, Math. Notes 16 (1974) 1050–1053.

[20] R.E. Stong, Notes on Cobordism Theory, Mathematical Notes, Princeton University Press, Princeton, NJ, 1968.

[21] D. Sullivan, Geometric Topology, Seminar Notes, Princeton, NJ, 1967, in: The Hauptvermutung Book, in: K-Theory Monographs, vol. 1, Kluwer, 1996, pp. 69–103.

[22] D. Sullivan, Localization, Periodicity and Galois Symmetry, Seminar Notes MIT, 1970.

[23] C.T.C. Wall, Determination of the Cobordism Ring, Annals of Math. 72 (2) (1960) 292–311.

[24] C.T.C. Wall, Surgery on Compact Manifolds, Academic Press, New York, 1970, and second ed. edited by A. Ranicki, 1999.