FUZZY THEORY AND ITS APPLICATIONS

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CERTIFICATE

I certify that the dissertation entitled " FUZZY THEORY AND ITS APPLICATIONS " submitted by Shri Debojyoti Paul Choudhury in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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INTRODUCTION

The fundamental concept in classical mathematics is that of a set. A classical set is formed by selecting certain objects, called the members or elements of the set and the set is completely determined by its members. Thus, in classical mathematics there are only two acceptable situations for an element : being a member of or not being a member of a set.

The natural world in which we live, is a world of imprecision, inexactitude and fuzziness. Here we come across sets where the membership of an element cannot be determined by two valued Boolean logic. This is because of the fact that much of human cognition and interaction with the outside world involves con' constructs which are not sets in the classical sense, but rather classes with unsharp boundaries in which transition from membership to nonmembership is gradual rather than abrupt.

For clarification let us consider the following example:

Take the collection of all students of class x of a particular school. Suppose, we want to form a collection of good students of this class. As 'goodness' is not boolean in real sense and there is no fixed boundary for defining goodness, it will be an impossible task to select good students in the classical sense. In the present case it will be required to attach to each student a certain degree of goodness possibly varying from 0 to 1 (0 for those students who are undisputedly bad and 1 for those students who are considered to be good without any doubt). Thus we observe that in the process of selecting good students there is some sort of fuzziness / vagueness. Thus the classical sets are not natural, appropriate or useful notions in different phenomenon specially in human behaviour.

In 1965, the theory of fuzzy sets was formally introduced by L.A. Zadeh (32). The motivation for the introduction of such a theory was the need of precision in speaking about vagueness and imprecision.

The theory of fuzzy sets provides an adequate conceptual framework as well as a mathematical tool to handle systems or phenomena which due to intrinsic indefiniteness cannot themselves be characterised precisely. Thus, by providing a basis for a systematic approach to approximate reasoning the theory of fuzzy sets has found applications in nearly every field where human judgement and perception play an important role.

Accprding to Zadeh (32), fuzzy sets are functions $P:X \rightarrow I$ where X is a classical set and I is the closed unit interval [0, 1]. The value P(x) assigns to $x \varepsilon X$ its " grade of membership" in the fuzzy set P, In 1967, Goguen (10) generalised the theory of fuzzy sets introduced by Zadeh, by using a partially ordered set L instead of the unit interval I.

In 1968, Chang (4) defined fuzzy topological spaces and concentrated on basic concepts such as open and closed fuzzy sets, fuzzy continuity etc. In a series of papers published by Warren (29) (1974,77,78) various properties of continuity, neighborhoods, and [illegible] bases in fuzzy structure and boundary of fuzzy set were discussed.

In 1973, Nazaroff (20) developed a generalised theory of optimal control and contributed to the basic ideas of closure, exterior and interior of fuzzy sets. In the same year Goguen(10) presented the fundamental ideas of base, subbase, and product of fuzzy topological spaces. In 1974, Wong (31) proved the product and quotient theorems and two years later studied categories of fuzzy sets. In 1975, Weiss (30) introduced fixed points, separation and induced topologies for fuzzy sets. In the same year Hutton (14) gave a definition of normality in fuzzy topological space. Christoph (5) studied quotient fuzzy topology and local compactness in the year 1977.

In 1976, Lowen (19) presented an alternative definition of fuzzy topological space in order to make the constant functions (between fuzzy topological spaces) continuous. But in the process he lost the concept that "fuzzy topology generalises ordinary topology". In 1979, Faster (9) used the alternative definition to bring together the structure of a fuzzy topological space and that of a fuzzy topological space and that of a fuzzy group (defined earlier by Rosenfield (23) in 1971). In the same year, Anthony and Sherwood (1) observed that some mathematical structures which would intuitively seem to be fuzzy did not satisfy the definition given by Rosenfield (23). They further observed that the notion of Min used by Rosenfield did give a semigroup structure to I. To meet the natural requirement they redefined fuzzy group by replacing Min by some other suitable semigroup structure on I, and established various interesting properties of fuzzy group.

Deficiencies in Chang's definition of compactness started being pointed out in 1970. According to Goguen the deficiency lies in the fact that the Tychonoff theorem is false for infinite products. Consequently new definitions of compactness were given by various mathematicians in different years suiting different requirements. In 1978, Lowen (19) studied the relative merits of as many as seven definitions of compactness in fuzzy topological space.

In 1978, Warren, Gantner and Steinlage (28) presented Hausdorff space in fuzzy structure. In 1980, Ming & Mingg(21) investigated various properties of fuzzy points, T_o, T_1 , quasi-T_o and quasi-T_1 spaces together with their product spaces. They also introduced Q-neighborhood in order to give a new definition of fuzzy Hausdorff space. In 1981, Sarkar (26) studied Hausdorff space, Regular & Normal spaces, P-space in fuzzy structure. In 1982, Rodabaugh (22) established some results in connectedness with respect to the fuzzy unit interval and the fuzzy real line.

Fuzzy vector space, fuzzy uniform space and fuzzy topological vector space were introduced respectively by Lowen (19) in 1974, by Hutton (14) in 1977 and by Katsaras and Liu (17) in 1977. Nazaroff(20) Bellman, Kalaba and Zadeh (2), Capocelle & Luca (3), Ruspini (24), Warren (29), Santos (25) and others applied fuzzy theory to various fields such as pattern recognition, artificial intelligence, optimization, decision theory and aerodynamics etc. *

* * *

The present work has been divided into 7 chapters.

In chapter 1, we define the following:
fuzzy topology, fuzzy open and fuzzy closed sets, neiborhood, closure & interior of fuzzy sets, derived set and boundary of fuzzy sets, base and subbase for fuzzy topology, fuzzy continuity fuzzy subspace. We investigate various results in this connection most of them being analogous to results available in ordinary topology. At the end of the chapter we introduce the functors and 'i' and discuss a few properties.

In chapter 2, we give the notion of fuzzy point and prove various results. In fuzzy theory the notion of ordinary point does not enable us to give, in many situation, natural results analogous to point set topology. It has been observed that fuzzy point in fuzzy theory plays the same role as the point in the classical set theory. Most of the results proved in this chapter in connection with base etc. are analogous to corresponding results in ordinary topology.

In chapter 3, we deal with quotient and product fuzzy topologies. Here we give the definitions and prove certain results. By constructing a counter example we establish that the product of C_1^* spaces is not necessarily C_1^* which is a deviation form correspondin result of ordinary point set topology.

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*The author has taken some informations from the survey article title "Fuzzy Mathematics - In Relation to Topology" by P.Srivastava(appeare in the proceedings, Fifth annual day 1980, The MIR, Allahabad.

In chapter 4 we introduce various fuzzy compactness and prove different results based on them. One of the importent results proved in this chapter is 'Tychonoff Theorem'.

In chapter 5 we define the notion of fizzy Hausdorff space, fuzzy regular space, fuzzy normal space, fuzzy T_0, T_1 spaces, fuzzy quasi-T_0, quasi-T_1 spaces. We establish various results with specific mention of the results which are deviated from the counterparts in ordinary topology. In this chapter we prove Urysohn's Lemma. Moreover we introduce the notion of connectedness and α- connectedness and prove certain results. One of the important results proved in this chapter is the following:-

"A product space (X,T) of a family of fuzzy topological spaces $\{(X_\lambda, T_\lambda):\lambda\in\Lambda\}$ is connected iff each co-ordinate space (X_λ, T_λ) is connected."

In chapter 6, we discuss the algebraic structures in fuzzy space like fuzzy group, fuzzy vector space and mention elementary results in this direction. We also define fuzzy metric space and fuzzy uniformity and prove a few results.

In the last chapter we mention various applications of fuzzy theory in different fields with details of a couple of applications.

CHAPTER I

TOPOLOGICAL STRUCTURES ON FUZZY SET

The fundamental concept of fuzzy sets was introduced by L.A. Zadeh in (32) and provided a natural framework for generalising many concepts of general topology. Chang (4) developed the theory of fuzzy topological spaces. According to Chang's definition of topological space, constant functions from X to I, the unit closed interval [0,1], are not necdssarily continuous. To remedy this, Lowen (19) proposed an alternative. But in the process he lost the concept that fuzzy topology generalises ordinary topology.

This chapter is devoted to the study of various topological structures on fuzzy set like fuzzy topology, fuzzy neighborhood, fuzz closure & fuzzy interior, fuzzy derived set, fuzzy boundary, fuzzy continuity base and subbase for a given fuzzy topology etc.

§ 1. PRELIMINARIES

Let $X = \{x\}$ be a space of points. A function $A : X \to I = [0,1]$ is called a fuzzy set in X and the function value $A(x)$ for $x \in X$ is called the "grade of membership" of x in A.

DEFINITION 1.1 (Chang, 68 (4)) Let A and B be two fuzzy sets in X.

Then $A = B \Leftrightarrow A(x) = B(x)\ \forall\ x$ in X

$A \subset B$ iff $A(x) \leq B(x)\ \forall\ x$ in X

$C = A \cup B$ iff $C(x) = \mathrm{Max}\ \{A(x), B(x)\}\ \forall x$ in X

$D = A \cap B$ iff $D(x) = \mathrm{Min}\ \{A(x), B(x)\}\ \forall x$ in X

$E = A^c$ iff $E(x) = 1 - A(x)\ \forall x$ in X

More generally, for a family of fuzzy sets $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ the union $C = \bigcup_\lambda A_\lambda$ and intersection $D = \bigcap_\lambda A_\lambda$ are defined by

$$C(x) = \sup_{\Lambda} \{ A_{\lambda}(x) \} \; \forall x \text{ in } X$$

$$\text{and } D(x) = \inf_{\Lambda} \{ A_{\lambda}(x) \} \; \forall x \text{ in } X$$

The symbols $\underline{0}$ and $\underline{1}$ are used to denote the empty and the full fuzzy sets defined by $\underline{0}(x) = 0 \;\; \forall x$ in X and $\underline{1}(x) = x \; \forall x$ in X

<u>Note</u>: However in some chapters we shall use the symbol X_{λ} to represent the function $\underline{1}$ in the set X_{λ}.

An ordinary point $x \in X$ is called a crisp point and is identified with its characteristic function $\chi_{\{x\}}$. If S is an ordinary ("crisp") subset of X, then its characteristic function χ_S is a fuzzy set.

§ 2. FUZZY TOPOLOGY

In this section we define fuzzy topological space (Chang). The alternative definition given by Lowen (19) is also included.

DEFINITION 1.2.1 (Chang, 1968 (4)) A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions

(i) $\underline{0}$ and $\underline{1} \in T$

(ii) If $A, B \in T$ then $A \cap B \in T$

(iii) If $A_{\lambda} \in T$ for each $\lambda \in \Lambda$ then $\bigcup_{\lambda \in \Lambda} A_{\lambda} \in T$

T is called a fuzzy topology for X and the pair (X,T) is a fuzzy topological space (or fts).

Every member of T is called T-open or open fuzzy set. A fuzzy set A is called T-closed(or fuzzy closed) if A^c is in T. As in general topology, the indiscrete fuzzy topology contains only $\underline{0}$ & $\underline{1}$ while discrete fuzzy topology contains all fuzzy sets.

THEOREM. 1.2.2 (Warren, 1978 (29)) If $T_\lambda, \lambda\varepsilon\Lambda$ are fuzzy topologies on X, then $\bigcap_{\Lambda} T_\lambda$ is also a fuzzy topology on X.

PROOF: Straight forward.

THEOREM. 1.2.3 (Warren 1978 (29)). Let (X,T) be an fts.

(1) Then $\underline{0}$ and $\underline{1}$ are closed fuzzy sets.

(2) If C_λ ,$\lambda\varepsilon\Lambda$ are closed fuzzy sets then $\bigcap_{\gamma\Lambda} C_\lambda$ is a closed fuzzy set.

(3) If D and E are closed fuzzy sets, then $D\cup E$ is a closed fuzzy set.

PROOF: (1) & (2) follow from definition.

For (3) observe the following result:

If S is a nonempty set of real numbers, then it i

$$\inf\{ x : x\varepsilon S\} = - \sup\{ - x : x\varepsilon S\} \text{ and}$$

$$\inf\{1+x : x\varepsilon S\} = 1+ \inf\{ x : x\varepsilon S\} .$$

DEFINITION. 1.2.4

A fuzzy topology T is said to be coarser than a fuzzy topology S (or S is finer than T) iff $T\subset S$.

In 1976, Lowen(19) introduced an alternative definition of fuzzy topology and in the process the constant fuzzy sets were made continuous.

DEFINITION. 1.2.5 (Lowen 1976 (19)) A fuzzy topology on a set X is a family T of fuzzy sets in X which satisfies the following conditions.

(1) For all $\alpha\varepsilon I$, $k_\alpha \varepsilon T$ where $k_\alpha : X \to I$ such that $k_\alpha(x) = \alpha$ for all $x\varepsilon X$

(2) If

(2) If $A, B \in T$ then $A \cap B \in T$.

(3) If $A_\lambda \in T$ for all $\lambda \in \Lambda$ then $\bigcup_{\lambda \in \Lambda} A_\lambda \in T$.

(X,T) is called a fuzzy topological space.
To avoid confusion, we term this topological space as quasitopological space or qfts.

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§.3. FUZZY NEIGHBORHOOD

In this section we introduce the notion of fuzzy neighborhood of a point and the fuzzy neighborhood of a fuzzy set. We prove a few elementary results in this direction and finally concentrate on the fuzzy neighborhood system of a fuzzy set.

DEFINITION. 1.3.1 (Warren,1978 (29)) A fuzzy set N in a fts(X,T) is said to be a neighborhood of a point $x \in X$ iff there is an element $A \in T$ such that $A \subset N$ and $A(x) = N(x) > 0$

A fuzzy neighborhood of a point $x \in X$ is frequently denoted by N_x. A fuzzy neighborhood N_x of a point $x \in X$ is said to be open iff $N_x \in T$

From the definition,it follows that if $N \in T$ such that $N(x) > 0$ then N is a fuzzy neighborhood of $x \in X$.

THEOREM. 1.3.2 (Warren,1978(29)) If M_x and N_x are two fuzzy neighborhood of x then $M_x \cap N_x$ is also a fuzzy neighborhood of x.

PROOF: Straightforward

DEFINITION. 1.3.3 (Chang, 1968(4)) A fuzzy set N in a fts (X,T) is said to be a fuzzy neighborhood (or f-nbhd) of a fuzzy set A iff there exists $P \in T$ such that $A \subset P \subset N$.

This definition is a generalization of the definition of a neighborhood of a set in a topological space $(X,\Im)$

THEOREM. 1.3.4 (Warren 1978 (29)) Let (X,T) be a fts and N, A two fuzzy sets in X. Then N is a f-nbhd of A iff, given $x \in X$ satisfying $A(x) > 0$, then there exists $M_x \in T$ such that $A(x) \leq M_x(x)$ and $M_x \subset N$.

PROOF: '$\Rightarrow$' Assume that N is a f-nbhd of A

$\therefore \exists G \in T$ such that $A \subset G \subset N$.

Now, given $x \in X$ for which $A(x) > 0$, then choose $G = M_x$.

'$\Leftarrow$' Assume that given $x \in X$ satisfying $A(x) > 0$ then there exists $M_x \in T$ such that $A(x) \leq M_x(x)$ and $M_x \subset N$

Consider $G = \cup\{M_x \in T : 0 < A(x) \leq M_x(x) \text{ and } M_x \subset N\}$

$\therefore G \in T$ and $A \subset G \subset N$ ($\because$ for every $x \in X$ with $A(x) > 0, \exists$ atleast one M_x)

N is a f-nbhd.

THEOREM. 1.3.5 (Lou & Pan 1980 (18)) The intersection of any two f-nbhds of a fuzzy set A in a fts(X,T) is also a f-nbhd of A.

PROOF: Straightforward.

THEOREM. 1.3.6 (Lou & Pan 1980 (18)) Any fuzzy superset of a f-nbhd N of a fuzzy set A in a fts (X,T) is again a f-nbhd of A.

PROOF: Straightforward.

THEOREM. 1.3.7 (Chang, 1968(4)) A fuzzy set A is open in a fts (X,T) iff for each fuzzy set B contained in A, A is a f-nbhd of B.

PROOF: ($\Rightarrow$) obvious

($\Leftarrow$) $\because A \subset A \therefore A$ is a f-nbhd (by hypothesis)

$\therefore \exists C \in T \ni A \subset C \subset A$. $A = C$ and A is fuzzy open.

THEOREM. 1.3.8 (Warren 1978 (29)) A fuzzy set A is open in a fts (X,T) iff for every $x \in X$ satisfying $A(x) > 0 \exists N_x \subset A \ni N_x(x) = A(x)$.

PROOF: ($\Leftarrow$) Let $G = \cup\{N_x \in T: N_x \subset A \text{ and } N_x(x) = A(x) > 0\}$

Then $G \in T$ and $G = A$. For the other part of the result take $N_x = A$.

THEOREM. 1.3.9 (Warren 1978 (29)) A fuzzy set A is open in a fts (X,T) iff it is a f-nbhd of each point at which it assumes a positive value.

PROOF: Follows from Theorem 1.3.8 .

THEOREM. 1.3.10 (Warren 1978 (29)) A fuzzy set A in a fts (X,T) is closed iff, given $x \in X$ for which each N_x satisfies $N_x(x) \neq A^c(x)$ or $N_x(y) > A^c(y)$ for some $y \in X$ then $A(x) = 1$.

PROOF: Apply theorem 1.3.8 to the fuzzy set A^c .

DEFINITION. 1.3.11 (Chang 1968 (4)) The fuzzy neighborhood system of a fuzzy set in a fts is the family of all f-nbhds of the set.

THEOREM: 1.3.12 (Chang 1968(4)) If η be the fuzzy neighborhood system of a fuzzy set, then finite intersections of members of η belong to η and each fuzzy set which contains a member of η belongs to η.

PROOF: Use theorems 1.3.5 and 1.3.6

THEOREM: 1.3.13 (Lou and Pan 1980 (18)) Let η be the fuzzy neighborhood system of a fuzzy set A in a fts (X,T). Then each member $N \in \eta$ is a fuzzy superset of a member $M \in \eta$ and $M \in T$

PROOF: Straightforward

4. CLOSURE AND INTERIOR OF FUZZY SET.

In this section we study the basic notions of closure and interior of a fuzzy set.

DEFINITION. 1.4.1 (Warren 1978(29)) Let A be a fuzzy set in a fts (X,T). The closure of A, denoted by $\bar{A}$, is defined by

$$\bar{A} = \{B : B^c \in T \text{ and } A \subset B\} .$$

THEOREM. 1.4.2 (Warren 1978 (29)) Let A be a fuzzy set in a fts (X,T). Then $\bar{A}$ is closed and is the least closed fuzzy set such that $\bar{A} \supset A$ or $\bar{A} = A$. Also A is closed iff $A = \bar{A}$.

PROOF: If $A = \bar{A}$ then there is a sequence of closed fuzzy sets B_n such that $0 \leqslant B_n(x) - A(x) < \frac{1}{n}$ ∀ x in X.

Hence A = B which is closed. The other results are easily verified.

THEOREM. 1.4.3 (Warren 1978 (29) Let A and B be two fuzzy sets in a fts (X,T). Then (1) $\bar{\underline{0}} = \underline{0}$ (2) $A \subset \bar{A}$ (3) $\bar{\bar{A}} = \bar{A}$

PROOF: (1) and (3): As $\underline{0}$ and $\bar{A}$ are T-closed so, they are equal to their closures.

(2) Follows from definition of closure

Theorem:1,4,4 (Warren,1978 (29)) Let A and B be fuzzy sets in a fts (X,T). If $A \subset B$ then $\bar{A} \subset \bar{B}$

PROOF: $\bar{B}$ is the smallest closed set which contains B so $A \subset B \subset \overset{2}{\bar{B}}$
But $\bar{A}$ is the smallest closed set which contains A . So $\bar{A}$ $\bar{B}$

THEOREM. 1.4.5 (Warren 1978 (29)) Let A and B be two fuzzy sets in a fts (X,T). Then (1) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ (2) $\bar{A} \cap \bar{B} \supset \overline{A \cap B}$

PROOF:(1) $\bar{A} \cup \bar{B}$ is a closed superset of $A \cup B$

By definition it follows that $\bar{A} \cup \bar{B} \supset \overline{A \cup B}$

On the other hand $A \cup B \supseteq A \Rightarrow \overline{A \cup B} \supset \bar{A}$
Similarly $\overline{A \cup B} \supset \bar{B}$. Hence $\bar{A} \cup \bar{B} = \overline{A \cup B}$

(2) By theorem 1.4.4, $\bar{A} \cap \bar{B} \supset \overline{A \cap B}$

DEFINITION. 1.4.6 (Chang 1968 (4)). Let A and B be fuzzy sets in a fts (X,T) and let $A \supset B$. Then B is called an interior fuzzy set of A iff A is a f-nbhd of B.

The union of all interior fuzzy sets A is called the interior of A and is denoted by A^{o}.

THEOREM. 1.4.7 (Chang 1968 (4)). Let A be a fuzzy set in a fts (X,T). Then A^o is open and is the largest open fuzzy set contained in A. The fuzzy set A is open iff $A = A^o$.

PROOF: By definition 1.4.6, clearly, A^o is itself an interior fuzzy set of A. Hence, $\exists$ B ε T such that $A^o \subset B \subset A$

But B is an interior fuzzy set of A. So $B \subset A^o$. $\therefore$ $A^o = B$

$\therefore$ A^o is open and is the largest open fuzzy set contained in A.

If A is open, then A is an interior point of A $\therefore$ $A \subset A^o$ $\therefore$ $A = A^o$.

The converse is obviously true.

From theorem 1.4.7 it follows that $(A^o)^o = A^o$

THEOREM. 1.4.8 (Warren 1978 (29)). Let A and B be two fuzzy sets in a fts (X,T). Then (1) $A \subset B \Rightarrow A^o \subset B^o$

(2) $A^o \cap B^o = (A \cap B)^o$ (3) $A^o \cup B^o \subset (A \cup B)^o$

PROOF: Straightforward.

THEOREM. 1.4.9 (Warren 1978 (29)). Let A be a fuzzy set in a fts (X,T). Then (1) $(A^c)^o = (\bar{A})^c$ (2) $\overline{(A^c)} = (A^o)^c$

PROOF: For any $x \in X$

$$\begin{aligned}(\bar{A})^c(x) &= 1 - \bar{A}(x) = 1 - \inf\{ D(x) : D^c \varepsilon T \ \& \ D \supset A\} \\ &= \sup\{ 1 - D(x) : D^c \varepsilon T \text{ and } D \supset A \} \\ &= \sup\{ D^c(x) : D^c \varepsilon T \text{ and } A^c \supset D^c\} \\ &= \sup\{ B(x) : B \varepsilon T \text{ and } B \subset A^c\} = (A^c)^o \text{ where } B = D^c.\end{aligned}$$

Proof of (2) is similar.

DEFINITION. 1.4.10 (Ming & Ming 1980 (21)).

Let X be a set and $f : I^X \to I^X$.

(1) Then f is called a fuzzy closure operator for X iff for all fuzzy sets A & B in X (C1) $f(\underline{0}) = \underline{0}$, (C2) $f(A) \supset A$

(C3) $f(f(A)) = A$ & (C4) $f(A \cup B) = f(A) \cup f(B)$

(2) f is called a fuzzy interior operator for X iff for all fuzzy sets A & B in X (I1) $f(\underline{1}) = \underline{1}$ (I2) $f(A) \subset A$

(I3) $f(f(A)) = A$ & (I4) $f(A \cap B) = f(A) \cap f(B)$.

The following theorems can be easily verified.

THEOREM. 1.4.10 (Ming & Ming 1980(21)) Let $f: I^X \to I^X$ be a closure operator on X. Then $T = \{ A \in I^X : f(A) = A \}$ is a fuzzy topology and $f(A) = A^c$

THEOREM. 1.4.11 (Wong (33)) Let $f: I^X \to I^X$ be an interior operator on X. Then $T = \{ A \in I^X : f(A) = A \}$ is a fuzzy topology and $f(A) = A^o$.

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§.5. DERIVED FUZZY SET.

In this section we deal with fuzzy limit point of a fuzzy set and derived fuzzy set.

DEFINITION. 1.5.1 (Warren 1978 (29)) Let A be a fuzzy set in a fts (X,T). A point $x \in X$ is called a fuzzy limit point of A iff

whenever $A(x) = 1$, then for each fuzzy nbhd N_x, there exists $y \in X - \{ x \}$ such that $(N_x \cap A)(y) \neq 0$

or whenever $A(x) \neq 1$ then $\bar{A}(x) > 0$ and for each open fuzzy nbhd N_x satisfying $N_x^c(x) = A(x)$, there exists $y \in X - \{ x \}$ such that $(N_x \cap A)(y) \neq 0$.

DEFINITION. 1.5.2 (Warren 1978 (29)) Let A be a fuzzy set in a fts (X,T). The derived fuzzy set of A (denoted by A^D) is defined as $A^D(x) = \bar{A}(x)$ if x is a limit point of A

$= 0$ otherwise.

REMARK: 1.5.3 If all fuzzy sets are restricted to the usual concept of general topology then definitions 1.5.1 and 1.5.2 agree with the general topology concept of limit point and derived set.

<u>THEOREM.</u> 1.5.4 (Warren 1978 (29)). Let A be a fuzzy set in a fts (X,T) and let $x \in X$. Then x is a fuzzy limit point of A iff $A^D(x) > 0$.

PROOF: This is a consequence of definitions 1.5.1 & 1.5.2.

It is clear from definition that $A^D \subset \bar{A}$.

Next consider the following theorem.

<u>THEOREM.</u> 1.5.5 (Warren 1978 (29) Let A be a fuzzy set in a fts (X,T) Then (1) A is T-closed iff $A^D \subset A$.

Furthermore (2) $(A^D)^D \subset \bar{A}$, (3) $A \cup A^D = \bar{A}$.

PROOF: (1) If A is closed then $\bar{A} = A$. But $A^D \subset \bar{A}$ $\therefore$ $A^D \subset A$.

Now assume that $A^D \subset A$. We shall show that $A = \bar{A}$. Take $x \in X$

If $A^D(x) = \bar{A}(x)$ then since $A^D \subset A \subset \bar{A}$, it follows that $A(x) = \bar{A}(x)$.

If $A^D(x) \neq \bar{A}(x)$, then x is not a limit point of A and $\bar{A}(x) > 0$

When $A(x) = 1$ then $\bar{A}(x) = A(x)$. So we assume that x is not a fuzzy limit point of A, $\bar{A}(x) > 0$ and $A(x) \neq 1$.

By definition 1.5.1 there is an open f-nbhd N_x such that $N_x^c(x) = A(x)$ and if $y \in X - \{x\}$ then $(N_x \cap A)(y) = 0$

Hence if $y \neq x$ then $N_x^c(y) \geq A(y)$. Since N_x^c is T-closed $\therefore$ $\bar{A} \subset N_x^c$

Therefore, $\bar{A}(x) \leq N_x^c(x) = A(x)$

(2) To see that $(A^D)^D \subset \bar{A}$ use definition 1.5.2 . One gets $(A^D)^D \subset \overline{A^D}$

Therefore $(A^D)^D \subset \bar{A}$ (since $A^D \subset \bar{A}$)

(3) Now we verify that $\bar{A} = A \cup A^D$

We note that if $A^D(x) = \bar{A}(x)$ then $\bar{A}(x) = (A \cup A^D)(x)$ ($\because A(x) \leq \bar{A}(x)$)

On the otherhand, if $A^D(x) \neq \bar{A}(x)$ then $A^D(x) = 0$ and by the argument used in the proof of the converse part of (1) one gets $\bar{A}(x) = A(x)$.

Now we show that the following statement is false:

If $A \subset B$ then $A^D \subset B^D$

Let X be a nonempty set and take an element $x_o \in X$. Define fuzzy sets A and B as follows

$A(x) = 0 = B(x)$ if $x \in X - \{x_o\}$ and $A(x_o) = 1/4$, $B(x_o) = 1/2$.

Let the fuzzy topology on X be $\{\underline{0}, \underline{1}, B\}$. Then $A^D(x_o) = 1/2$ & $B^D(x_o) = 0$. It is noted that x_o is a fuzzy limit point of A but is not a fuzzy limit point of B. $A^D \not\subset B^D$.

In this example note that $(A^D \cup B^D)(x_o) > (A \cup B)^D(x_o)$ and

$$(A^D \cap B^D)(x_o) < (A \cap B)^D(x_o).$$

However the following result is valid.

THEOREM. 1.5.6 (Warren 1978 (29)). Let A and B be fuzzy sets in a fts (X,T) & let $x \in X$.

(1) If x is a fuzzy limit point of $A \cup B$ then $(A^D \cup B^D)(x) \leq (A \cup B)^D(x)$

(2) If x is a fuzzy limit point of both A and B then $(A^D \cap B^D)(x) \geq (A \cap B)^D(x)$.

PROOF: (1): $A^D(x) \leq \bar{A}(x) \leq (\bar{A} \cup \bar{B})(x) = (\overline{A \cup B})(x) = (A \cup B)^D(x)$ and since $B^D(x) \leq (A \cup B)^D(x)$, the result follows.

(2) Since $(A \cap B)^D(x) \leq (A \cap B)(x) \leq (\bar{A} \cap \bar{B})(x) = (A^D \cap B^D)(x)$ the result follows.

6. BOUNDARY OF A FUZZY SET.

In this section we establish a boundary for a fuzzy set in a fts. Based on properties of fuzzy boundary we define a fuzzy boundary operator and observe that the boundary operator is an equivalent way of defining a fuzzy topology. In fact, we shall show that the space defined with the held of boundary operator is identical to the space defined by fuzzy closure operator.

DEFINITION. 1.6.1 (Warren 1977 (29)). Let A be a fuzzy set in a fts (X,T). The fuzzy boundary of A, denoted by A^B, is defined as

$$A^B = \inf\{\, D : D^c \in T \text{ such that } D(x) \geq \bar{A}(x) \text{ for all } x \in X \text{ satisfying } (\bar{A} \cap \overline{A^c})(x) > 0 \,\}\ .$$

Clearly, A^B is a closed fuzzy set and $A^B \subset \bar{A}$. If $\bar{A} \cap \overline{A^c} = \underline{0}$ then

$A^B = \inf\{$ All closed fuzzy sets in X$\} = \underline{0}$

THEOREM. 1.6.2 (Warren 1977 (29)). If $(\bar{A} \cap \overline{A^c})(x) > 0$ then $A^B(x) = \bar{A}(x)$.

PROOF: Easy verification based upon definition 1.6.1

THEOREM. 1.6.3 (Warren 1977 (29)). Let A be a fuzzy set in a fts (X,T). Then (1) $\bar{A} = A^o \cup A^B$ (2) $A^B \supset \bar{A} \cap \overline{A^c} \supset \bar{A} - A^o$

PROOF: (1)If $(\bar{A} \cap \overline{A^c})(x) > 0$ or $\bar{A}(x) = 0$ then $A^B(x) = \bar{A}(x)$.

If $(\bar{A} \cap \overline{A^c})(x) = 0$ and $\bar{A}(x) > 0$, then $\overline{A^c}(x) = 0$.

Hence $A^c(x) = 0$ and so $A(x) = 1$ But $A^o = (\overline{A^c})^c$ $A^o(x) = 1 = \bar{A}(x)$

Hence $\bar{A} = A^o \cup A^B$.

(2) To prove the first inclusion ,it is sufficient to consider those $x \in X$ for which $(\bar{A} \cap \overline{A^c})(x) > 0$.

It follows from theorem 21.6.2 that $A^B(x) = \bar{A}(x) \geq (\bar{A}\ \ \overline{A^c})(x)$.

To prove the second inclusion, it is first noted that $\bar{A}\ \ \bar{A} - A^o$.

But $A^o = (\overline{A^c})^c \therefore \overline{A^c} = (A^o)^c \supset \bar{A} - A^o \therefore \bar{A} \cap \overline{A^c} \supset \bar{A} - A^o$

COROLLARY 1.6.4 (Warren 1977 (29)) $\bar{A} = A \cup A^B$

PROOF: Apply $\bar{A} \supset A \supset A^o$ to the Ist part of theorem 1.6.3.

To see that both inclusions in theorem 1.6.3 (2) may occur, one may consider the following example:

Let X be a nonempty set. Let $A(x) = 3/4$ for all $x \in X$.

Let $T = \{\, \underline{0}, \underline{1}, A, A^c \}$.

__THEOREM.__ 1.6.5 (Warren 1977 (29)). Let A & B be fuzzy sets in a fts (X,T). Then

(1) $(\underline{0})^{B} = \underline{0}$ (2) $(A^{B})^{B} \subset A^{B}$ (3) $(A \cup B)^{B} \subset A^{B} \cup B^{B}$

(4) $(A \cup B) \cup (A \cup B)^{B} = (A \cup B) \cup (A^{B} \cup B^{B})$

(5) If $\bar{A} \cap \overline{A^{c}} = \underline{0}$ then $A^{B} = (A^{c})^{B}$

(6) If $(\bar{A} \cap \overline{A^{c}})(x) > 0$ then $A^{B}(x) \geq A(x)$

(7) If $\bar{A} \cap \overline{A^{c}} \neq \underline{0}$ and given $y \in X$ such that $\overline{A^{c}}(y) = 0$, then $A^{B}(y) = \cap \{ \bar{D}(y) : \bar{D}(x) \geq \bar{A}(x)$ when $(\bar{A} \cap \overline{A^{c}})(x) > 0\}$.

PROOF: (1) $(\underline{0})^{B} \subset (\bar{\underline{0}}) = \underline{0}$, the result is immediate.

(2) A^{B} is closed and $A^{B} \subset \bar{A}$ $\therefore (A^{B})^{B} \subset \overline{A^{B}} = A^{B}$

(3) If $(\overline{A \cup B} \cap \overline{(A \cup B)^{c}})(x) = 0$ then $(A \cup B)^{B}(x) = 0$

So we take $x \in X$ such that $(\overline{A \cup B} \cap \overline{(A \cup B)^{c}})(x) > 0$

We shall show that $(A^{B} \cup B^{B})(x) = (\overline{A \cup B})(x)$

If $\bar{A}(x) = 0$ then $A^{B}(x) = \bar{A}(x)$ $(\because A^{B} \subset \bar{A})$

If $\bar{A}(x) > 0$ then, since $\overline{(A \cup B)^{c}} = \overline{A^{c} \cap B^{c}} \subset \overline{A^{c}}$ it follows that $\overline{A^{c}}(x) \geq \overline{(A \cup B)^{c}}(x) > 0$.

Thus $(\bar{A}\ \overline{A^{c}})(x) > 0$ and $A^{B}(x) = \bar{A}(x)$ (by theorem 1.6.2).

Similarly $B^{B}(x) = \bar{B}(x)$ $\therefore$ $(A^{B} \cup B^{B})(x) = (\bar{A} \cup \bar{B})(x) = (\overline{A \cup B})(x)$

But $A^{B} \cup B^{B}$ is closed From definition 1.6.1, $(A \cup B)^{B} \subset A^{B} \cup B^{B}$

(4) $(A \cup B) \cup (A \cup B)^{B} = (\overline{A \cup B})$ (by Corollary 1.6.4)

$$= \bar{A} \cup \bar{B} = (A \cup A^{B}) \cup (B \cup B^{B}) = A \cup B \cup A^{B} \cup B^{B}$$

(5) Since $\bar{A}\ \overline{A^{c}} = \underline{0}$ $A^{B} = \underline{0} = (A^{c})^{B}$

(6) If $(\bar{A} \cap \overline{A^{c}})(x) > 0$ then by theorem 1.6.2 $A^{B}(x) = \bar{A}(x) \geq A(x)$.

(7) This is a consequence of definition 1.6.1

__THEOREM.__ 1.6.6 (Warren 1977 (29)). Let A & B be two fuzzy sets in a fts (X,T). Then

(1) $A^{B} = \underline{0}$ iff $\bar{A} \cap \overline{A^{c}} = \underline{0}$ (2) A is closed iff $A^{B} \subset A$

(3) $A^{B} \cap A = \underline{0}$ iff A is open & crisp

(4) $A^{B} = \underline{0}$ iff A is open, closed & crisp

(5)

(5) $(A^o)^B \cup (\bar{A})^B \subset A^B$ (6) If $A \subset B$ then $A^B \subset B \cup B^B$

(7) $(A \cap B)^B \subset A^B \cup B^B$ (8) $\bar{A} = \overline{A^o}$ iff $A^B \subset \overline{A^o}$

PROOF: (1) If $A^B = \underline{0}$ then by theorem 1.6.3 $\bar{A}$ $\overline{A^c} = \underline{0}$.

Conversely if $\bar{A} \cap \overline{A^c} = \underline{0}$ then by definition 1.6.4

$A^B = \cap \{$ all closed fuzzy sets in X$\} = \underline{0}$.

(2) Since $A^B \subset \bar{A}$, if A is closed/ then $A^B \subset A$

Since $\bar{A} = A \cup B^B$, if $A^B \subset A$ then $\bar{A} = A$.

(3) Let $A^B \cap A = \underline{0}$. We shall show that $A = (\overline{A^c})^c$ which is open.

When $A(x) = 0$ then $\bar{A}(x) = 1$ Therefore $(\overline{A^c})^c = A$.

But for $Ay(x) > 0$, $A^B(x) = 0 < \bar{A}(x)$ and so $(\bar{A} \cap \overline{A^c})(x) = 0$

$\therefore \overline{A^c}(x) = 0$ $A9(x) = 1$ $\therefore (\overline{A^c})^c = A$ $\therefore$ A is open & crisp.

Next assume A is open & crisp

If $(\bar{A} \cap \overline{A^c})(x) > 0$ then $\overline{A^c}(x) = 1 \geq \bar{A}(x)$ So $A^B \subset A^c$

When $A(y) = 1$, then $A^B(y) = 0$ $A^B \cap A = \underline{0}$

(4) Apply the parts (2) & (3) of this theorem.

(5) Set $D = \overline{A^o} \cap \overline{(A^o)^c}$. Since $(A^o)^c = \overline{A^c}$ $\therefore$ $D \subset \bar{A} \cap \overline{A^c}$

So when $D(x) > 0$ then $\overline{A^o}(x) \leq \bar{A}(x) = A^B(x)$

Since A^B is closed By definition $(A^o)^B \subset A^B$

Since $\bar{A} \cap \overline{(\bar{A})^c} \subset \bar{A} \cap \overline{A^c}$ By similar argument it can be shown that $(\bar{A})^B \subset A^B$.

(6) If $A \subset B$ then $\bar{A} \subset \bar{B}$.

From theorem 1.6.3 $A^B \subset \bar{A}$ and $\bar{B} = B \cup \bar{B}$.

(7) Set $D = \overline{A \cap B} \cap \overline{(A \cap B)^c}$. Since $D \subset (\bar{A} \cap \overline{A^c}) \cup (\bar{B} \cap \overline{B^c})$ it follows that if $D(x) > 0$ then $(\bar{A} \cap \overline{A^c})(x) > 0$ or $(\bar{B} \cap \overline{B^c})(x) > 0$

Hence $A^B(x) = \bar{A}(x)$ or $B^B(x) = \bar{B}(x)$ $\therefore$ $(\overline{A \cap B})(x) < (A^B \cup B^B)(x)$

But $A^B \cup B^B$ is closed. Thus by definition $(A \cap B)^B \subset A^B \cup B^B$.

(8) If $\bar{A} = \overline{A^o}$, then, since $A^B \subset \bar{A}$ it follows that $A^B \subset \overline{A^o}$. On the other hand if $A^B \subset \overline{A^o}$ then $\bar{A} = A^o \cup A^B = \overline{A^o} \cup A^B = \overline{A^o}$

Now we produce an example to show that properties (2) and (5) of the previous theorem can't both hold.

Let X be a nonempty set. Let T be the set consisting of the constant functions $\underline{0}$, $A_{\frac{1}{2}}$, $\underline{1}$ where $A_{\frac{1}{2}}(x) = 1/2$ for all x in X. Let A be the constant function given by $A(x) = 3/4$ for all $x \in X$. Then $\bar{A} = \underline{1}$, $\overline{A^c}(x) = 1/2$ and $A^o(x) = 1/2$ for all x in X. If both (2) and (5) hold then $A^B(x) \leq (\bar{A} \cap \overline{A^c})(x) = 1/2$ and (2) would give $\bar{A}(x) = 1/2$ for all x in X, a contradiction.

Now we define fuzzy boundary operator.

<u>DEFINITION.</u> 1.6.7 (Warren 1977 (29)). Let X be a set and let β be a function from I^X into I^X. Set $\psi(A) = A \cup \beta(A)$ and set $\nu(A) = \psi(A) \cap \psi(A^c)$ where I^X is the collection of all fuzzy sets in X and $A \in I^X$. Then β is called a fuzzy boundary operator for X iff for all fuzzy sets A & B in X the following axioms are satisfied.

(B1) $\beta(\underline{0}) = \underline{0}$ (B2) $\beta(\beta(A)) \subset \beta(A)$. (B3) $\beta(A \cup B) \subset \beta(A) \cup \beta(B)$

(B4) $\psi(A) \cup \psi(B) \subset \psi(A \cup B)$ (B5) If $\nu(A) = 0$ then $\beta(A) = \beta(A^c)$

(B6) If $(\nu(A))(x) > 0$ then $(\beta(A))(x) \geq A(x)$

(B7) If $\nu(A) \not\leq 0$ and given $y \in X$ such that $(\psi(A^c))(y) = 0$ then $(\beta(A))(y) = \inf \{(\psi(E))(y) : (\psi(E))(x) \geq (\psi(A))(x)$ whenever $(\nu(A))(x) > 0\}$.

Clearly in a fts (X,T) if $\beta(A) = A^B$ for each fuzzy set A in X, then β is a fuzzy boundary operator for X by theorem 1.6.5.

__THEOREM.__ 1.6.8 (Warren 1977 (29)). Let β be a fuzzy boundary operator for the set X and define $\theta : I^X \to I^X$ by $\theta(A) = A \cup \beta(A)$ for each fuzzy set A in X. Then θ is a fuzzy closure operator for X and $\beta(A) = A^B$.

PROOF: Clearly $\theta(\underline{0}) = \underline{0} \cup \beta(\underline{0}) = \underline{0}$. It is obvious that $A \subset \theta(A)$ and $\theta(A) \subset \theta(\theta(A))$. Moreover by B2 & B3

$$\theta(\theta(A)) = \theta(A \cup \beta(A)) = A \cup \beta(A) \cup \beta(A \cup \beta(A)$$

$$\subset A \cup \beta(A) \cup \beta(\beta(A)) \subset A \cup \beta(A) = \theta(A)$$

Also $\theta(A \cup B) = A \cup B \cup \beta(A \cup B) \subset A \cup B \cup \beta(A) \cup \beta(B)$

$= \theta(A) \cup \theta(B)$ by B3.

On the otherhand, by B4 $\theta(A \cup B) \supset \theta(A) \cup \theta(B)$. Thus θ is a fuzzy closure operator for X.

To show that $\beta(A) = A^B$ we note that the closure operator θ induces fuzzy topology on X in which $\theta(A) = \bar{A}$. Put $C = \bar{A} \cap \overline{A^c}$.

If $C = \underline{0}$ then $A^B = \underline{0}$. Also by B5, $\beta(A) = \beta(A^c)$ and thus $\beta(A) = 0$.

If $C \neq 0$, then $A^B = \cup \{ E \cap \beta(E) : (E \cup \beta(E))(x) \geq (A \cup \beta(A)(x)$ if $C(x) > 0\}$

When $C(x) > 0$ then by B6, $(\beta(A))(x) > A(x)$.

Hence $A^B(x) = (A \cup \beta(A))(x) = (\beta(A))(x)$

when $C(y) > 0$ then $(A \cup \beta(A))(y) = 0$ or $(A^c \cup \beta(A^c))(y) = 0$

In the first case, $(\beta(A))(y) = 0$ and $A^B(y) = 0$. In the 2nd case, B7 is applied and we conclude that $A^B(y) = (\beta(A))(y)$.

__THEOREM.__ 1.6.9. (Warren 1977 (29)). Let (X,T) be an fts and A^B the fuzzy boundary of a fuzzy set A in X. Define $\theta : I^X \to I^X$ by $\theta(A) = A \cup A^B$. Then θ is a fuzzy closure operator for X and $\theta(A) = \bar{A}$

PROOF: By corollary 1.6.4, $A \cup A^B = \bar{A}$. Hence θ is a fuzzy closure operator.

THEOREM. 1.6.10 (Warren 1977 (29)). There is a natural 1 - 1 correspondence between the set of all fuzzy boundary operators on X and the set of all fuzzy topologies on X.

PROOF: Define Γ to be the set of all fuzzy closure operators on X. Define Ψ tobe the set of all fuzzy topologies on X & define Δ to be the set of all fuzzy boundary operators on X. Based on theorem 1.4.3 and 21.4,5 it is easy to verify that there is a natural map $t : \Psi \to \Gamma$ which is 1 - 1 & onto.
Theorem 1.6.5 establishes a map $f : \Psi \to \Delta$. Therefore from theorem 1.6.8 we deduce that there is a 1-1 map $g : \Delta \to \Gamma$ Such that $g^{-1} \circ t = f$. It follows from theorem 1.6.9 that f is onto.

§.7. CONVERGENCE OF SEQUENCE OF FUZZY SETS.

In this section we define few terms in connection with sequence of fuzzy sets and prove a theorem.

DEFINITION. 1.7.1 (Chang 1968 (4)). A sequence of fuzzy sets $\{A_n : n = 1,2...\}$ is said to be eventually contained in a fuzzy set. A iff there is an integer m such that, if $n \geq m$ then $A_n \subset A$.

The sequence is said to be frequently contained in A iff for each m there is an integer n such that $n \geq m$ and $A_n \subset A$.

DEFINITION. 1.7.2 (Chang 1968 (4)). A sequence of fuzzy sets $\{A_n : n = 1,2,...\}$ in a fts (X,T) is said to converge to a fuzzy set A iff it is eventually contained in each fnbhd of A.

DEFINITION. 1.7.3 (Chang 1968 (4)). Let $\mathcal{J}$ be the set of non-negative integers. The sequence $\{B_i ; i = 1,2,...\}$ is said to be a subsequence of $\{A_n : n = 1,2...\}$ iff there is a map f from $\mathcal{J}$ to $\mathcal{J}$ such that

$B_i = A_{f(i)}$ and for each integer m there is an integer n such that $f(i) \geq m$ whenever $i \geq n$.

DEFINITION. 1.7.4 (Chang 1968 (4)). A fuzzy set A in a fts (X,T) is called a clusterfuzzy set of a sequence of fuzzy sets $\{A_n : n=1,2..\}$ iff the sequence is frequently contained in every f-nbhd of A.

THEOREM. 1.7.5.(Chang 1968 (4)). If the f-nbhd system of each fuzzy set in a fts (X,T) is countable then the following are true.

(a) A fuzzy set A is open iff each sequence of fuzzy sets $\{A_n : n = 1,2,...\}$ which converges to a fuzzy set B contained in A is eventually contained in A.

(b) If A is a cluster fuzzy set of the sequence $\{A_n : n=1,2,..\}$ of fuzzy sets, then there is a subsequence of this sequence converging to A.

PROOF: (a)($\Rightarrow$)Since $A \in T$ and $B \subset A$. Therefore A is a f-nbhd of B. But $\{A_n : n=1,2,\}$ converges to B.

By definition $\{A_n : n=1,2,..\}$ is eventually contained in B.

($\Leftarrow$) For each B A let U_1, U_2,... be the neighborhood system of B. Let $V_n = \bigcap_{n=1}^{n} U_n$. Therefore V_1, V_2,... is a sequence which is eventually contained in each f-nbhd of B.

Therefore $V_1, V_2, ...$ converges to B. Hence there is an m such that for $n \geq m$ $V_n \subset A$. Then V_n's are f-nbds of B.

Therefore by theorem 1.3.7, A is open.

(b) Let $R_1, R_2, ...$ be the nbhd system of A. Let $S_n = \bigcup_{i=1}^{n} R_i$.

Then $S_1, S_2, ...$ is a sequence such that $S_{n+1} \subset S_n$ for each n.

For every non-negetive integer i,choose f(i) such that $f(i) \geq i$ and $A_{f(i)} \subset S_i$. Then, surely$\{A_{f(i)} : i=1,2,.\}$ is a subsequence of the sequence$\{A_n : n=1,2,... .\}$Clearly this subsequence converges to A.

8. BASE AND SUBBASE FOR A GIVEN FUZZY TOPOLOGY.

In this section we establish various results analogous to those in ordinary topology for base and subbase. The task of specifying a fuzzy topology is simplified by taking only enough open fuzzy sets to generate all the topology. We discuss some methods for introducing fuzzy topologies on sets.

DEFINITION. 1.8.1 (Goguen,1973 (10)). Let T be a fuzzy topology on X. A subfamily Ξ of T is called a base for T iff each member of T can be expressed as the union of some members of Ξ .

Also a subfamily Σ of T is called a subbase for T iff the family of all finite intersections of members of Σ form a base for T.

THEOREM. 1.8.2 (Warren, 1978 (29)). Let T be a fuzzy topology on X and Ξ T. Then,the following properties of Ξ are equivalent.

(1) Ξ is a base for T

(2) For each $A \in T$, for each $x \in X$ with $A(x) > 0$ and for each real no $\delta > 0, \exists\ B \in T$ such that $B \subset A$ and $A(x) - B(x) < \delta$

PROOF: $(1) \Rightarrow (2)$: Since $A \in T$, $A(x) > 0$ and Ξ is a base for T

Therefore $A = \bigcup_\lambda B_\lambda$ for some collection $\{B_\lambda\}$ in Ξ .

Therefore $A(z) = \sup_\lambda \{B_\lambda(z)\}$ $\forall$ z in X. Given x & $\delta > 0$ there exists B_λ in Ξ such that $A(x) - B_\lambda(x) < \delta$. Clearly $B_\lambda \subset A$.

$(2) \Rightarrow (1)$ Let $A \in T$ and $A(x) > 0$. $\therefore$ By hypothesis $\exists$ $B_{x,n} \in \Xi$ such that $B_{x,n} \subset A$ and $A(x) - B_{x,n}(x) < 1/n$.

$A = \bigcup_{x,n} \{B_{x,n} : A(x) > 0\ \&\ n = 1,2,\ldots\}$ $\therefore$ Ξ is a base for T.

COROLLARY 1.8.3 (Warren 1978 (29)). Let Ξ be a base for the fuzzy topology T on X and let A be a fuzzy set in X. Then A is open iff $\forall$ $x \in X$ with $A(x) > 0$ and for each $\delta > 0$, there is $B \in \Xi$ such that

$B \subset A$ and $A(x) - B(x) < \delta$.

PROOF: Simple application of theorem 1.8.2

Now we study two general methods for introducing fuzzy topologies on sets.

THEOREM 1.8.4 (Warren 1978 (29)). Given any family $\Sigma = \{ A_\lambda : \lambda \in \Lambda \}$ of fuzzy sets in X, there is always a unique, smallest fuzzy topology T on X such that $\Sigma \subset T$ and Σ is a subbase for T.

PROOF: Take T as the collection consisting of $\underline{0}$, $\underline{1}$, all finite intersections of the A_λ's and all arbitrary unions of these finite intersections. Now using theorem 1.2.2 the results are easily verified.

THEOREM. 1.8.5 (Warren 1978 (29)). Let $\Xi = \{B_\lambda : \lambda \in \Lambda\}$ be any family of fuzzy sets in X such that for each $(\lambda, \mu) \in \Lambda \times \Lambda$ for each $x \in X$ for which $(B_\lambda \cap B_\mu)(x) > 0$ andfor each $\delta > 0$, there is some B_α satisfying $B_\alpha \subset B_\lambda \cap B_\mu$ and $(B_\lambda \cap B_\mu)(x) - B_\alpha(x) < \delta$. Then the collection T consisting of $\underline{0}$, $\underline{1}$ and all unions of members of Ξ is the unique, smallest fuzzy topology on X such that $\Xi \subset T$ and Ξ is a base for T.

PROOF: The verification is straightforward.

The above theorem gives another characterization of a base for a fuzzy topology. We have noted that every base can be associated to unique fuzzy topology. But a fuzzy topology may have several distinct bases.

THEOREM. 1.8.6 (Warren, 1978 (29)).Let Ξ_1 and Ξ_2 be bases for fuzzy topologies T_1 and T_2 on X respectively. Then $T_1 \subset T_2$ iff for each $B_1 \in \Xi_1$ and each $x \in X$ for which $B_1(x) > 0$ and each $\delta > 0$, there is $B_2 \in \mathcal{B}_2$ such that $B_2 \subset B_1$ and $B_1(x) - B_2(x) < \delta$.

PROOF: If $T_1 \subset T_2$, then given $B_1 \varepsilon \Xi_1$ and $x \varepsilon X$ for which $B_1(x) > 0$, it follows that $B_1 \varepsilon T_2$. Since Ξ_2 is a base for T_2, the existence of a suitable B_2 follows from theorem 1.8.2.

For the converse, it follows from corollary 1.8.3 that $\Xi_1 \subset T_2$.

Hence $T_1 \subset T_2$.

COROLLARY 1.8.7 (Warren 1978 (29)). Let $\Xi_1; \Xi_2$ be bases for fuzzy topologies T_1, T_2 on X respectively. Then $T_1 = T_2$ iff both the following conditions hold:

(a) For each $B_1 \varepsilon \Xi_1$, each $x \varepsilon X$ for which $B_1(x) > 0$ and each $\delta > 0$, there is $B_2 \varepsilon \Xi_2$ such that $B_2 \subset B_1$ and $B_1(x) - B_2(x) < \delta$.

(b) For each $B_2 \varepsilon \Xi_2$, each $x \varepsilon X$ for which $B_2(x) > 0$ and each $\delta > 0$, there is $B_1 \varepsilon \Xi_1$ such that $B_1 \subset B_2$ and $B_2(x) - B_1(x) < \delta$.

PROOF: Straight forward.

Finally, we define C_{11} space.

DEFINITION. 1.8.9 (Wong (31)).A fts (X,T) is said to be C_{11} if there exists a countable base for T.

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§. 9. FUZZY CONTINUITY.

In this section we study the generalized notion of continuity introduced by Chang in (4). As a preliminary, we mention various properties of fuzzy sets induced by mappings.

DEFINITION. 1.9.1 (Chang 1968 (4)). Let f be a function from X to Y. Let B be a fuzzy set in Y and A a fuzzy set in X.

The inverse image of B under f is the fuzzy set $f^{-1}(B)$ in X is defined by $(f^{-1}(B))(x) = B(f(x))$ for all x in X.

The image of A under f is the fuzzy set f(A) in Y defined by

$$(f(A))(y) = \sup_{z \varepsilon f^{-1}(y)} \{A(z)\} \text{ if } f^{-1}(y) = \{x : f(x) = y\} \text{ is nonempty}$$
$$= 0, \text{ otherwise, for all } y \varepsilon Y.$$

THEOREM. 1.9.2 (Chang 1968 (4)). Let f be a function from X to Y
Then, (a) $\bar{f}^{1} (B^{c}) = (\bar{f}^{1}(B))^{c}$ for any fuzzy set B in Y

(b) $f(A^{c}) \supset (f(A))^{c}$ for any fuzzy set A in X.

(c) $B_1 \subset B_2 \Rightarrow \bar{f}^{1}(B_1) \subset \bar{f}^{1}(B_2)$ where B_1 and B_2 are fuzzy sets in Y

(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$ where A_1 and A_2 are fuzzy sets in X

(e) $B \supset f(\bar{f}^{1}(B))$ for any fuzzy set B in Y

(f) $A \subset \bar{f}^{1}(f(A))$ for any fuzzy set A in X

THEOREM. 1.9.3 (Chang 1968 (4)). Let f be a function from X to Y and g be a function from Y to Z. Then for any fuzzy set C in Z, $(gof)^{-1}(C) = f^{-1}(g^{-1}(C))$ where gof is the composition of g and f.

THEOREM. 1.9.4 (Warren, 1978 (29)). Let f be a function from X to Y. If A, $A_\lambda, \lambda \in \Lambda_1$ are fuzzy sets in X and if B, $B_\mu, \mu \in \Lambda_2$, are fuzzy sets in Y, then the following relations are valid.

(1) $f(f^{-1}(B)) = B$ when f is onto Y (2) $f(\cap A_\lambda) \subset \cap f(A_\lambda)$
(3) $f^{-1}(\cap B_\mu) = \cap f^{-1}(B_\mu)$ (4) $f(\cup A_\lambda) = \cup f(A_\lambda)$
(5) $f^{-1}(\cup B_\mu) = \cup f^{-1}(B_\mu)$ (6) $f(f^{-1}(B) \cap A) = B \cap f(A)$.

The results given in the above theorems are all consequences of the definitions of f(A) and $f^{-1}(B)$ and can be verified without much difficulty.

Now we are in a position to define the continuous functions in fuzzy structure.

DEFINITION. 1.9.5 (Chang 1968 (4)). A function f from a fts (X,T) to a fts $(Y,\mathcal{U})$ is called fuzzy continuous (or F-continuous) iff the inverse of each $\mathcal{U}$-open fuzzy sets is T-open.

DEFINITION. 1.9.6 (Wong (31)). A function f from a fts (X,T) to a fts $(Y,\mathcal{U})$ is said to be fuzzy open (fuzzy closed) or F-open (F-closed) iff it maps an open (closed) fuzzy set in (X,T) onto an open (closed) fuzzy set in $(Y,\mathcal{U})$.

<u>DEFINITION.</u> 1.9.7 (Chang 1968 (4)). A fuzzy homeomorphism or F-homeomorphism is defined as an F-continuous bijection of a fts (X,T) to a fts $(Y,\mathcal{U})$ such that the inverse of the map is also F-continuous.

<u>DEFINITION.</u> 1.9.8 (Chang 1968 (4)). Two fts's are said to be F-homeomorphic(or topologically F-equivalent) iff there exists a F-homeomorphism of one space onto another.

<u>THEOREM.</u> 1.9.10 (Chang 1968 (4)). Let (X,T) and $(Y,\mathcal{U})$ be fts's. A mapping $f: X \rightarrow Y$ is F continuous iff the inverse image of every $\mathcal{U}$-closed fuzzy set is T-closed.

PROOF: $(\Rightarrow)$ Let B be a $\mathcal{U}$-closed fuzzy set in Y. $\therefore B^c \in \mathcal{U}$

$\therefore f^{-1}(B^c) \in T$ ($\because f$ is F-continuous). But, $f^{-1}(B^c) = (f^{-1}(B))^c$

$(f^{-1}(B))^c \in T$. Hence $f^{-1}(B)$ is T-closed.

$(\Leftarrow)$ Let $B \in \mathcal{U}$ $\therefore B^c$ is closed fuzzy set in Y.

By hypothesis $f^{-1}(B^c)$ is T-closed $\therefore (f^{-1}(B))^c$ is T-closed

$\therefore f^{-1}(B) \in T$. Hence f is F-continuous.

<u>THEOREM.</u> 1.9.11 (Chang 1968(4)). If (X,T) & $(Y,\mathcal{U})$ are fts's and f is a function on X to Y then the conditions below are related as follows: (a) (b) ; (b) (c) and (c) (d)

(a) The function f is F-continuous.

(b) For each fuzzy set A in X, the inverse of every fuzzy nbhd of $f(A)$ is a fuzzy nbhd of A.

(c) For each fuzzy set A in X and each fuzzy nbhd V of $f(A)$, there is a fuzzy nbhd W of A such that $f(W) \subset V$

(d) For each sequence of fuzzy sets$\{A_n : n=1,2,\dots\}$in X which converges to a fuzzy set A in X, the sequence $\{f(A_n), n=1,2,\dots\}$ converges to $f(A)$.

PROOF: (a) $\Rightarrow$ (b) Let A be a fuzzy set in X & V a fuzzy nbhd of f(A). $\therefore$ V contains an open f-nbhd W of f(A)

Since $f(A) \subset W \subset V$ Therefore $f^{-1}(f(A)) \subset f^{-1}(W) \subset f^{-1}(V)$. But $A \subset f^{-1}(f(A))$ and $f^{-1}(W)$ is open. $\therefore$ $f^{-1}(V)$ is a f-nbhd of A.

(b) $\Rightarrow$ (c): Since $f^{-1}(V)$ is a fuzzy nbhd of A $\therefore$ Therefore $f(W) = f(f^{-1}(V)) \subset V$ where $W = f^{-1}(V)$.

(c) $\Rightarrow$ (b): Let V be a f-nbhd of f(A). Then, there is a f-nbhd W of A such that $f(W) \subset V$. So $f^{-1}(f(W)) \subset f^{-1}(V)$. But $W \subset f^{-1}(f(W)$. Hence $f^{-1}(V)$ is a f-nbhd of A.

(c) $\Rightarrow$ (d): If V is a f-nbhd of f(A), then there is a f-nbhd W of A such that $f(W) \subset V$. Since $\{ A_n : n=1,2... \}$ is eventually contained in W $\therefore$ There is an m such that for $n \geq m$, $A_n \subset W$

$f(A_n) \subset f(W)$ for $n \geq m$. $\therefore$ $\{f(A_n) : n = 1,2,...\}$ converges to f(A).

THEOREM. 1.9.12 (Warren 1978 (29). Let (X,T) & (Y,U) be fts and $f: X \to Y$ a mapping. Then the following conditions are equivalent.

(a) The function f is F-continuous.

(b) For every $x \in X$ and every F-nbhd N of f(x), $f^{-1}(N)$ is a f-nbhd of x

(c) For every $x \in X$ and every f-nbhd N of f(x) there is a f-nbhd M of x such that $f(M) \subset N$ and $M(x) = (f^{-1}(N))(x)$.

PROOF: It follows from theorem 1.9.11 that (a) $\Rightarrow$ (b) & (b) $\Rightarrow$ (c). Let us prove that (c) $\Rightarrow$ (a)

Let $B \in U$ & $x \in X$ such that $(f^{-1}(B))(x) > 0$ $\therefore$ $B(f(x)) > 0$ and B is a f-nbhd of f(x) (by definition).

By hypothesis, there is a f-nbhd M of x such that $f(M) \subset B$ and $M(x) = (f^{-1}(B))(x)$. $M \subset f^{-1}(f(M)) \subset f^{-1}(B)$

Therefore by theorem 1.3.8 $f^{-1}(B) \in T$.

THEOREM: 1.9.13 (Warren, 1978(29)) Let (X,T) & $(Y,\mathcal{U})$ be fts and $f: X \to Y$ a function. Then the following conditions are equivalent

(a) The function f is F-continuous

(b) For every fuzzy set A in X, $f(\bar{A}) \subset \overline{f(A)}$

(c) For every fuzzy set B in Y, $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$

Proof: (a)$\Rightarrow$(b) By definition, $\overline{f(A)} = U\{B: f(A) \subset B,\ B^c \in \mathcal{U}\}$

Therefore $f^{-1}(\overline{f(A)}) = U\{f^{-1}(B): f(A) \subset B,\ B^c \in \mathcal{U}\}$

Now by theorem 1.9.10 $f^{-1}(B)$ is closed fuzzy set in X. Also $f^{-1}(B) \supset A$. Therefore $\bar{A} \subset f^{-1}(B)$ $\therefore$ $\bar{A} \subset \mathbb{U}\{f^{-1}(B)\}$

$\therefore$ $\bar{A} \subset f^{-1}(\overline{f(A)})$ $\therefore$ $f(\bar{A}) \subset \overline{f(A)}$.

(b) $\Rightarrow$ (c) Since $f^{-1}(B)$ is a fuzzy set in X, it follows that $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \bar{B}$ $\therefore$ $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$

(c)$\Rightarrow$(a) By theorem 1.9.10 it suffices to prove that the inverse image of every $\mathcal{U}$ closed fuzzy set is T-closed

Let D be a closed fuzzy set in $(Y,\mathcal{U})$ Then $\overline{f^{-1}(D)} \subset f^{-1}(\bar{D}) = f^{-1}(D)$

$\therefore$ $\overline{f^{-1}(D)} = f^{-1}(D)$. $\therefore$ $f^{-1}(D)$ is T-closed $\therefore$ f is F-continuous.

THEOREM 1.9.14 (Ming and Ming 1980 (21)). Let (X,T) and $(Y,\mathcal{U})$ be fts and $f:(X,T) \to (Y,\mathcal{U})$ be a function. Then the following are equivalent.

(1) f is F-continuous

(2) for each member V of a subbase Σ for $\mathcal{U}$, $f^{-1}(V)$ is T-open

Proof: (1)$\Leftrightarrow$(2) can be proved in a similar way as in general topology.

§.10. SUBSPACE

Warren (29) introduced the concept of relative fuzzy topology as well as subspace. In this section we examine certain elementary results.

THEOREM: 1.10.1 (Warren 1978(29)). Let (X,T) be a fts and let $A \quad X$. Then the family $T_A = \{U/_A : U \in T\}$ is a fuzzy topology on A, where $U/_A$ is the restriction of U to A.

Proof: It can be easily verified that T_A satisfies all the axioms of definition 1.2.1.

DEFINITION 1.10.2 (Warren 1978(29)) The fuzzy topology T_A is called the relative fuzzy topology on A or the fuzzy topology on A induced by the fuzzy topology T on X. Also (A,T_A) is called a subspace of (X,T).

THEOREM 1.10.3 (Warren 1978 (29)). Let (A,T_A) be a subspace of the fts (X,T) and let Z be a fuzzy set in A. Further let B be the fuzzy set in X defined by $B(x) = Z(x)$ if $x \in A$ and $= 0$ if $x \in X-A$. Then $\bar{Z} = \bar{B}/_A$ and $B^o/_A = Z^o \cap (\chi_A^o)/_A$ where $\bar{Z}$ and Z^o are with respect to T_A and $\bar{B}$, B^o and χ_A^o are with respect to T.

Proof: $\bar{Z} = \cap \{C/_A : C/_A \supset Z \text{ and } (C/_A)^c \in T_A\}$

$= (\cap \{D : D \supset B \text{ and } D^c \in T\})/_A = \bar{B}/_A.$

The verification of other result is similar.

The following definitions and theorems are due to Foster (9). To avoid confusion we shall use the terms "induced quasi fuzzy topology", "quasi fuzzy subspace", "relatively quasi fuzzy continuous" "relatively quasi fuzzy open" respectively in place

of the terms "induced fuzzy topology", "fuzzy subspace", "relatively fu
fuzzy continuous" & "relatively fuzzy open" used in (9).

DEFINITION 1.10.4 (Foster 1979 (9)). Let A be a fuzzy set in a quasi fts (X,T). The induced quasi fuzzy topology on A is the family of fuzzy subsets of A which are the intersections with A of T-open fuzzy sets in X. The induced quasi fuzzy topology is denoted by $T_{(A)}$, and the pair $(A,T_{(A)})$ is called a quasi fuzzy subspace of (X,T).
Note that the induced quasi fuzzy topology does not in general satisfy the Ist condition of the definition of quasi fuzzy topology. The remaining conditions are satisfied.

DEFINITION 1.10.5 (Foster 1979 (9). Let $(A,T_{(A)})$ and $(B,\mathcal{U}_{(B)})$ be two quasi fuzzy subspaces of qfts's (X,T) & $(Y,\mathcal{U})$ respectively.
(1) A mapping f of (X,T) into $(Y,\mathcal{U})$ is called a mapping of $(A,T_{(A)})$ into $(B,\mathcal{U}_{(B)})$ if $f(A)\subset B$.
(2) A mapping of of $(A,T_{(A)})$ into $(B,\mathcal{U}_{(B)})$ is said to be relatively quasi fuzzy continuous iff for each open fuzzy set V^* in $\mathcal{U}_{(B)}$, the intersection $f^{-1}(V^*)\cap A$ is in $T_{(A)}$

(3) A mapping f of $(A,T_{(A)})$ into $(B,\mathcal{U}_{(B)})$ is said to be relatively quasi fuzzy open iff for each open fuzzy set U^* in $T_{(A)}$ the image $f(U^*)$ is in $\mathcal{U}_{(B)}$.

THEOREM 1.10.6 (Foster 1979 (9) Let $(A,T_{(A)})$ and $(B,\mathcal{U}_{(B)})$ be quasi fuzzy subspaces of quasi fts's (X,T) & $(Y,\mathcal{U})$ respectively, and let f be a fuzzy continuous mapping of (X,T) into $(Y,\mathcal{U})$ such that $f(A)\subset B$. Then f is relatively quasi fuzzy continuous mapping of $(A,T_{(A)})$ into $(B,\mathcal{U}_{(B)})$.
Proof: Let V^* be fuzzy open in $\mathcal{U}_{(B)}$. $\therefore \exists\ V\in\mathcal{U}$ such that $V^* = V\cap B$

The inverse image $f^{-1}(V)$ is open in T. Hence $f^{-1}(V') \cap A =$ $f^{-1}(V) \cap f^{-1}(B) \cap A = f^{-1}(V) \cap A$ is open in $T_{(A)}$.

THEOREM 1.10.7 (Foster 1979 (9)). Let $(A, T_{(A)})$, $(B, \mathcal{U}_{(B)})$ and $(C, \mathcal{V}_{(C)})$ be quasi fuzzy subspaces of qfts's (X, T), $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ respectively. Let f be a relatively quasi fuzzy continuous (resp. relatively quasi fuzzy open) mapping of $(A, T_{(A)})$ into $(B, \mathcal{U}_{(B)})$ and g a relatively quasi fuzzy continuous (resp. relatively quasi fuzzy open) mapping of $(B, \mathcal{U}_{(B)})$ into $(C, \mathcal{V}_{(C)})$. Then the composition is a relatively quasi fuzzy continuous (resp. relatively quasi fuzzy open) mapping of $(A, T_{(A)})$ into $(C, \mathcal{V}_{(C)})$.

Proof. Let W' be open in $\mathcal{V}_{(C)}$ Then $g^{-1}(W') \cap B$ is open in $\mathcal{U}_{(B)}$ and $f^{-1}(g^{-1}(W') \cap B) \cap A$ is open in $T_{(A)}$. But $(g \circ f)^{-1}(W') \cap A =$ and $f^{-1}(g^{-1}(W') \cap B) \cap A$ are equal fuzzy sets& Since $f(A) \subset B$ and so $g \circ f$ is relatively quasi fuzzy continuous. The proof in the case of relatively quasi fuzzy open mappings is trivial.

DEFINITION 1.10.8 (Foster 1979 (9)) Let (X, T) be a qfts and $T_{(A)}$ the induced quasi fuzzy topology on a fuzzy subset A of X. A subfamily Ξ' of $T_{(A)}$ is a base for $T_{(A)}$ iff each member of $T_{(A)}$ can be expressed as the union of members of Ξ'.

It is to be noted that if Ξ is a base for a quasi fuzzy topology T on a set X, then $\Xi_A = \{ U \cap A : U \in \Xi \}$ is a base for the induced quasi fuzzy topology ; $T_{(A)}$ on the fuzzy subset A.

THEOREM 1.10.9 (Foster 1979(9)). Let $(A, T_{(A)})$ and $(B, \mathcal{U}_{(B)})$ be 2 quasi fuzzy subspaces of qfts's (X, T) and $(Y, \mathcal{U})$ respectively. Let Ξ' be a base for $\mathcal{U}_B$. Then a mapping f of $(A, T_{(A)})$ into $(B, \mathcal{U}_B)$ is relatively quasi fuzzy continuous iff for each B' in Ξ' the set $f^{-1}(B') \cap A$ is in $T_{(A)}$.

Proof. Straight forward.

§ 11. THE FUNCTORS 'ω' AND 'i'

One observes that there is z natural way to associate a fuzzy topology with a given topology and vice versa. This type of opposite relationship is established with the help of two functions denoted by 'ω' & 'i'. We shall examine the nature of these functions.

Let $(X,\mathcal{J})$ be a topological space. A function $f:X\to I$ will be called upper semi-continuous iff for each real number $\alpha\in I$, $\{x\in X:f(x)<\alpha\}$ is $\mathcal{J}$-open (ie. $\{x:f(x)\geq\alpha\}$ is $\mathcal{J}$-closed) and will be called lower semicontinuous iff for each $\alpha\in I$, $\{x\in X:f(x)>\alpha\}$ is $\mathcal{J}$-open (ie $\{x: f(x)\leq\alpha\}$ is $\mathcal{J}$-closed) and will be called continuous iff it is both upper and lower semicontinuous.

On I consider the usual topology $\{\,]\alpha,1]:\alpha\in I\}\cup\{I\}\cup\{\varphi\}$ and let the resulting topological space be denoted by I_r.

DEFINITION 1.11.1. (Lowen 1976 (19)) for a fuzzy topology T on a sdt X, i(T) is defined to be the topology on X induced by all members $\mu: X\to I_r$ of T.

DEFINITION. 1.11.2 (Weiss 1975 (30)). For a topology $\mathcal{J}$ on a set X, $\omega(\mathcal{J})$ is defined to be the fuzzy topology on X consisting of all lower semicontinuous functions from $(X,\mathcal{J})$ to I_r.

THEOREM 1.11.3 (Weiss 1975 (30)).Let $(X,\mathcal{J})$ be a topological space. $\omega(\mathcal{J})$ is a fuzzy topology on X and so $(X,\omega(\mathcal{J}))$ is a fts.

Proof: Clearly the constant fuzzy sets $\underline{0}$ & $\underline{1}$ are lower semicontinuous. Furthermore, since the supremum of an arbitrary family and infimum of a finite family oflower semicontinuous functions from X into I are each lower semicontinuous. Therefore the unions of arbitrarily many and intersections of finitely many elements of $\omega(\mathcal{J})$ are themselves in $\omega(\mathcal{J})$. $\therefore$ $\omega(\mathcal{J})$ is a fuzzy topology for X.

DEFINITION 1.11.4 (Weiss 1975(30)) Let A be a fuzzy set in an arbitrary set. We define the sets $\sigma_r(A)$ and $w_r(A)$ for $r \in I$ by

$$\sigma_r(A) = \{ x \in X : A(x) > r \}$$

$$\omega_r(A) = \{ x \in X : A(x) \geq r \}$$

These sets will be called the strong and the weak r-cut of A.

THEOREM 1.11.5 (Weiss, 1975 (30)). A fuzzy set A in X is open (resp. closed) iff for each $r > 0$, $\sigma_r(A)$ is open (resp. $\omega_r(A)$ is closed.

Proof: Straight forward.

It is clear that if S is $\mathcal{J}$-open in a topological space $(X, \mathcal{J})$ then the characteristic function χ_S is $\omega(\mathcal{J})$ - open.

THEOREM 1.11.6 (Weiss 1975 (24) A mapping $f: (X, \omega(\mathcal{J})) \to (Y, \omega(\mathcal{S}))$ is F-continuous iff $f: (X, \mathcal{J}) \to (Y, \mathcal{S})$ is continuous with respect to topological spaces.

Proof: ($\Rightarrow$) Let V be a $\mathcal{S}$-open set

$$f^{-1}(V) = \{ x \in X : \chi_V(f(x)) = 1 \} = \{ x \in X : (f^{-1}(\chi_V))(x) > \tfrac{1}{2} \} = \sigma_{\frac{1}{2}}(f^{-1}(\chi_V))$$

But, by theorem 1.11.5 the latter is $\mathcal{J}$-open because χ_V is $\omega(\mathcal{S})$-open and f is F-continuous. Therefore f is continuous.

Conversely, suppose f is continuous and B is an open fuzzy set in Y. For any $r > 0$

$$\sigma_r(f^{-1}(B)) = \{ x \in X : (f^{-1}(B))(x) > r \}$$
$$= \{ x \in X : B(f(x) > r \} = f^{-1}(B^{-1} \,]r,1]\,)$$

But the latter set is open, because B is lower semicontinuous and f is continoous. By theorem 1.11.5 $f^{-1}(B)$ is $\omega(\mathcal{J})$-open Therefore f is F-continuous.

THEOREM. 1.11.7 (Lowen 1976 (19)) If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is F-continuous then $f: (X, i(\mathcal{T})) \to (Y, i(\mathcal{U}))$ is continuous

Proof: Straight forward.

THEOREM 1.11.8 (Srivastava & others (27)) Let $(X,\mathcal{J})$ be a topolotical space and $(Y,\mathcal{U})$ afts. Then $f: (X,\mathcal{J}) \to (Y, i(\mathcal{U}))$ is continuous $\Rightarrow$ $f: (X, \omega(\mathcal{J})) \to (Y,\mathcal{U})$ is F-continuous.

Proof: Straight forward.

Let TOP denote the set of all topologies on a set of X and FTOP the set of all fuzzy topologies on X. Then we have the two mappings

i; FTOP $\to$ TOP given by $T \to i(T)$ and ω : TOP $\to$ FTOP given by $\mathcal{J} \to \omega(\mathcal{J})$.

DEFINITION 1.11.9 (Lowen 1976 (19)) If $T \in$ FTOP be equal to $\omega(\mathcal{J})$ for some $\mathcal{J} \in$ TOP then T is said to be topologically generated.

THEOREM 1.11.10 (Lowen 1976 (19))

(i) $i \circ \omega = id_{TOP}$ (ii) i and ω are respectively an isotone surjection and isotone injection. (iii) $\omega \circ i(T)$ is the smallest topologically generated fuzzy topology which contains T. (We denote it by $\bar{T}$). (iv) T is topologically generated iff $T = \bar{T}$

Proof: Straight forward

DEFINITION 1.11.11 (Lowen 1976 (19)) A function $f: (X,T) \to (Y,\mathcal{U})$ is said to be continuous iff $f: (X, i(T)) \to (Y, i(\mathcal{U}))$ is continuous.

THEOREM 1.11.12 (Lowen, 1976(19)) Let (X,T) and $(Y,\mathcal{U})$ be fts's and g is a function from X to Y. Then the conditions below are related as follows (1)$\Rightarrow$(2), (2)$\Leftrightarrow$(3) and (3)$\Leftrightarrow$(4)

(1) g is F-continuous (2) g is continuous (3) $g: (X,\bar{T}) \to (Y,\bar{\mathcal{U}})$ is F-continuous (4) $g: (X,\bar{T}) \to (Y,\mathcal{U})$ is F-continuous.

Proof: This is straight forward.

Next, we state a few results without giving the proofs. For the proofs we refer the reader to (19(c))

THEOREM 1.11.13 (Lowen 1977 (19)) If $f: X \to (Y,\mathcal{U})$ then

$i(f^{-1}(\mathcal{U})) = f^{-1}(i(\mathcal{U})) = i(f^{-1}(\bar{\mathcal{U}}))$

<u>THEOREM</u> 1.11.14 (Lowen 1977 (19)). If $f: X \to (Y, \omega(\mathcal{G}))$ then $\omega(f^{-1}(\mathcal{G})) = f^{-1}(\omega(\mathcal{G}))$

<u>THEOREM</u> 1.11.15 (Lowen 1977 (19)) If X is a set and $\{\mathcal{J}_\lambda\}_{\lambda\in\Lambda}$ is a family of topologies on X then $\sup\limits_{\lambda\in\Lambda} \omega(\mathcal{J}_\lambda) = \omega(\sup\limits_{\lambda\in\Lambda} \mathcal{J}_\lambda)$

<u>THEOREM</u> 1.11.16 (Lowän 1977 (19)) If X is a set, $\{(Y_\lambda, \mathcal{G}_\lambda)\}_{\lambda\in\Lambda}$ is a family of topological spaces and $f_\lambda : X \to Y_\lambda$ is a family of functions then $\sup\limits_{\lambda\in\Lambda} f_\lambda^{-1}(\omega(\mathcal{G}_\lambda)) = \omega(\sup\limits_{\lambda\in\Lambda} (f_\lambda^{-1}(\mathcal{G}_\lambda)))$

<u>THEOREM</u> 1.11.17 (Lowen 1977(19)) If X is a set and $\{(Y_\lambda, \mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$ is a family of fts's and $f_\lambda : X \to Y_\lambda$ is a family of functions then we have $i(\sup\limits_{\lambda\in\Lambda} f_\lambda^{-1}(\mathcal{U}_\lambda)) = \sup\limits_{\lambda\in\Lambda} f_\lambda^{-1}(i(\mathcal{U}_\lambda))$

<u>THEOREM</u> 1.11.18 (Lowen 1977 (19)) If $f:(X,T)\to Y$ then $i(f(\bar{T})) = f(i(T)) \supset i(f(T))$

<u>THEOREM</u> 1.11.19 (Lowen 1977 (19)) If $f:(X,\omega(\mathcal{J}))\to Y$ then $f(\omega(\mathcal{J}))=\omega(f(\mathcal{J}))$

<u>THEOREM</u> 1.11.20 (Lowen 1977 (19)). If X is a set and $\{\mathcal{J}_\lambda\}_{\lambda\in\Lambda}$ is a family of topologies on X then $\bigcap\limits_{\lambda\in\Lambda} \omega(\mathcal{J}_\lambda) = \omega(\bigcap\limits_{\lambda\in\Lambda} \mathcal{J}_\lambda)$

<u>THEOREM</u> 1.11.21 (Lowen 1977 (19)) If Y is a set, $\{(X_\lambda, \mathcal{J}_\lambda)\}_{\lambda\in\Lambda}$ is a family of topological spaces and $f_\lambda : X_\lambda \to Y$ is a family of functions then $\bigcap\limits_{\lambda\in\Lambda} f_\lambda(\omega(\mathcal{J}_\lambda)) = \omega(\bigcap\limits_{\lambda\in\Lambda} f_\lambda(\mathcal{J}_\lambda))$.

CHAPTER -2

FUZZY POINTS IN FTS

In fuzzy theory the notion of ordinary point does not enable us to give, in many situations, natural results analogous to point set topology. In this we fuzzify the concept of point. It will be observed that fuzzy point in fuzzy theory plays the same role as the point in the classical set theory.

§.1. FUZZY POINT AS DEFINED BY WONG.

In this section we deal with the fuzzy point as introduced by Wong(33) and discuss few properties.

DEFINITION.2.1.1. (Wong (33)). A fuzzy point p in a set X is a fuzzy set p which takes the value 0 for all $y \in X$ except one, say x. If $p(x) = \alpha$ where $0 < \alpha < 1$, then p is said to have support x and value α.

We denote a fuzzy point having support x and value α by x_α. Thus, $x_\alpha(y) = 0$ for all $y \in X$ & $y \neq x$, and $x_\alpha(x) = \alpha$.

DEFINITION2.1.2. (Wong (33)). Let x_α be a fuzzy point and A a fuzzy set in X. Then x_α is said to be in A or A contains x_α, denoted by $x_\alpha \in A$ iff $x_\alpha(y) < A(y)$ for all $y \in X$.

Consider the following theorem.

THEOREM. 2.1.3. (Wong (33)). If $A = \bigcup_{\lambda \in \Lambda} A_\lambda$, where Λ is any index set & $\{A_\lambda\}$ is a family of fuzzy sets in X, then a fuzzy point $x_\alpha \in A$ iff $x_\alpha \in A_\lambda$ for some $\lambda \in \Lambda$.

In ordinary set theory, this theorem is trivial. But this is not trivial in case of fuzzy set theory. If one replaces the inequality in definition 2.1.2 by $x_\alpha(y) \leq A(y)$ then theorem 2.1.3 is no longer true. On the other hand, if we restrict all fuzzy sets to take values

{0 , 1} then definitions 2.1.1 and2.1.2 will reduce to corresponding definitions in ordinary set thoery provided we use $o<\alpha\leq 1$ & $x_\alpha(y)\leq A(y)$ in place of $o<\alpha< 1$ & $x_\alpha(y) < A(y)$. In other words, our current definitions will not reduce to the ordinary case even if we impose the restriction that all fuzzy sets will take values {0 , 1} only. However Wong used this definition and proved the following results. tThe proofs of these results are omitted but references are provided.

<u>THEOREM</u>.2.1.4. (Wong (33)). Let (X,T) be an fts. A subfamily Ξ of T forms a base of T iff for every member A of T & for every fuzzy point $x_\alpha \in A$, there exists a member B of Ξ such that $x_\alpha \in B \subset A$.

<u>DEFINITION</u>.2.1.5. (Wong (33)). Let (X,T) be an fts and x_α is a fuzzy point. A subfamily Ξ of T is called a local base of x_α iff $x_\alpha \in B$ for every member B of Ξ and for every member A of T such that $x_\alpha \in A$, there exists a member D of Ξ such that $x \in D \subset A$.

<u>DEFINITION</u>.2.1.6. (Wong (33)). An fts (X,T) is to be C_1 if every fuzzy point in X has a countable local base.

<u>THEOREM.</u> 2.1.7. (Wong (33)). If an fts (X,T) is C_{11} then it is C_1.

<u>DEFINITION</u>.2.1.8. (Wong (33)). An fts (X,T) is said to be separable iff there exists a countable sequence of fuzzy points $\{(x_\lambda)_{\alpha_\lambda}\}\lambda{=}1,2,\ldots$ such that for every member A of T & $A \neq \underline{0}$ there exists a fuzzy point $(x_\lambda)_{\alpha_\lambda}$ in this sequence such that $(x_\lambda)_{\alpha_\lambda} \in A$.

<u>THEOREM</u>.2.1.9. (Wong (33)). If an fts (X,T) is C_{11},then it is separable

<u>THEOREM</u>.2.1.10 (Wong (33)). Let f be an F-continuous function from a separable fts (X,T) onto an fts $(Y,\mathfrak{U})$ .Then $(Y,\mathfrak{U})$ is also separable.

<u>THEOREM</u>.2.1.11. (Wong (33)). Let f be an F-continuous function from a C_1 fts (X,T) onto an fts $(Y,\mathfrak{U})$. If f is also F-open, then $(Y,\mathfrak{U})$ is C_1

The introduction of fuzzy points enables us to discuss convergence in a meaningful way.

DEFINITION.2.1.12. (Wong (33)). Let $\{(x_n)_{\alpha_n}\}$ $n = 1,2,\ldots$ be a sequence of fuzzy points. Let x_α be a fuzzy point with $x \neq x_n$ $\forall$ $n \geq n_0$ where n_0 is some number. Then $(x_n)_{\alpha_n}$ is said to converge to x_α, written (as $(x_n)_{\alpha_n} \to x_\alpha$, iff for every member A of T such that $x_\alpha \in A$, there exists a number m such that $(x_n)_{\alpha_n} \in A$ for all $n \geq m$.

REMARK.2.1.13. (Wong (33)). If the sequence $\{(x_n)_{\alpha_n}\}, n = 1,2,\ldots$ of fuzzy points in an fts (X,T) converges to the fuzzy point x_α then, in general $\alpha_n \not\to \alpha$. In fact, $(x_n)_{\alpha_n}$ converges to all fuzzy points x_β if $\beta \geq \alpha$. In the theory of general topology we have a similar situation.

DEFINITION.2.1.14. (Wong (33)). Let x_α be a fuzzy point in an fts (X,T) and A a fuzzy set in X. Then x_α is said to be an accumulation point of A iff every member B of T such that $x_\alpha \in B$, $B \cap A_{x_\alpha} \neq \underline{0}$ where A_{x_α} is the fuzzy set defined by

$$A_{x_\alpha}(y) = \begin{cases} 0 & \text{for } y = x \\ A(y) & \text{otherwise.} \end{cases}$$

REMARK.2.1.15 (Wong (33)). If x_α is an accumulation point of A then all fuzzy points x_β with $\beta \geq \alpha$ are accumulation points of A.

THEOREM. 2.1.16. (Wong (33)). Let (X,T) be a C_1 fts. Let A be a fuzzy set and x_α a fuzzy point in X. Then x_α is an accumulation point of A iff there exists a sequence of fuzzy points $\{(x_n)_{\alpha_n}\}$, $n = 1,2,\ldots$ such that $(x_n)_{\alpha_n} \in A$ and $(x_n)_{\alpha_n} \to x$.

THEOREM.2.1.17. (Wong (33)). Let (X,T) be an fts. If there exists a countable sequence of fuzzy points $\{(x_n)_{\alpha_n}\}$ $n = 1,2,\ldots$ in X such that every fuzzy point x_α in X is an accumulation point of the fuzzy set $A = \bigcup_n (x_n)_{\alpha_n}$ then (X,T) is separable.

REMARK.2.1.18. (Wong (33)). The converse of the theorem 2.1.17 is not true in general. This is another departure from general topology. We have the following counter example.

Let X be a point set and $x \in X$. Let A_β, $0 \leq \beta \leq 1$, be fuzzy sets in X defined by $A_\beta(y) = \beta$ for $y = x$, and $A_\beta(y) = 0$ otherwise. Let $T = \{\underline{0}, \underline{1}, A_\beta : 0 \leq \beta \leq 1\}$. Then (X,T) is an fts. Consider the countable sequence of fuzzy points $\{x_{\alpha_n}\}$ where n ranges over the set of rational numbers between 0 and 1. Any member B of T such that $B \neq \underline{0}$ will contain a member of $\{x_{\alpha_n}\}$. Thus (X,T) is separable.

Let x_α be a fuzzy point with $0 < \alpha < 1$. Then x_α is not an accumulation point of the union A of any countable fuzzy points because, $B \cap A_{x_\alpha} = \underline{0}$ for all $B \in T$ containing x_α & $B \neq \underline{1}$.

§.2. 'BELONGINGNESS' OF FUZZY POINT IN FUZZY SET AS INTRODUCED BY SRIVASTAVA & OTHERS.

In an attempt to remove the controversial aspect of the definition 2.1.2 Srivastava and others (27) defined 'belongingness' from a new angle. In this section we study some results related to this definition due to Srivastava & others.

DEFINITION.2.2.1.(Srivastava & others (27)). Let x_α be a fuzzy point and A a fuzzy set in X. Then x_α is said to be in A or A contains x_α, denoted by $x_\alpha \in A$ iff $x_\alpha(x) < A(x)$ and $x_\alpha(y) \leq A(y)$ if $y \neq x$.

So, if x_α is a fuzzy point and A is a fuzzy set in X, then $x_\alpha \notin A \Leftrightarrow x_\alpha(x) \geq A(x)$.

THEOREM.2.2.2. (Srivastava & others (27)). Let Λ be any index set and x_α a fuzzy point in X. Then $x_\alpha \in \bigcup_{\lambda\in\Lambda} A_\lambda$ iff $x_\alpha \in A_\lambda$ for some $\lambda \in \Lambda$ where A_λ's are fuzzy sets in X.

Proof : $(\Rightarrow)$ Let $x_\alpha \in \bigcup_\lambda A_\lambda$. Therefore, $x_\alpha(x) < (\bigcup_\lambda A_\lambda)(x) = \operatorname{Sup}_\lambda A_\lambda(x)$

Thus $x_\alpha(x) < A_\lambda(x)$ for some $\lambda \in \Lambda$ which implies that $x_\alpha \in A_\lambda$ for some $\lambda \in \Lambda$

$(\Leftarrow)$ $x_\alpha \in A_\lambda \Rightarrow x_\alpha(x) < A_\lambda(x) \leq \operatorname{Sup}_\lambda A_\lambda(x) = (\bigcup_\lambda A_\lambda)(x)$

which implies that $x_\alpha \in \bigcup_{\lambda\in\Lambda} A_\lambda$.

THEOREM.2.2.3.(Srivastava & others (27)). If A is a fuzzy set in X and if $A(x) \neq 0$ for some $x \in X$, then there exists a fuzzy point x_α such that $x_\alpha \in A$.

Proof : Simple.

THEOREM.2.2.4.(Srivastava & others 1981 (27)). A fuzzy set A in X is the union of all its fuzzy points.

Proof : If $A = \underline{0}$, then the result is obviously true.

Let $A \neq \underline{0}$. Suppose $\{(x_\lambda)_{\alpha_\lambda}\}_{\lambda \in \Lambda}$ be the collection of all fuzzy points in A . So $(x_\lambda)_{\alpha_\lambda} \in A \Rightarrow (x_\lambda)_{\alpha_\lambda}(x_\lambda) < A(x_\lambda)$ & $(x_\lambda)_{\alpha_\lambda}(y) \leq A(y)$ for $y \neq x_\lambda$

This can also be written as $(x_\lambda)_{\alpha_\lambda} \in A \Rightarrow (x_\lambda)_{\alpha_\lambda}(y) \leq A(y)$ for all $y \in X$ which implies that $\underset{\lambda}{\mathrm{Sup}}\,(x_\lambda)_{\alpha_\lambda}(y) \leq A(y) \Rightarrow \underset{\lambda \in \Lambda}{\cup}(x_\lambda)_{\alpha_\lambda} \subset A$ ....(a).

Now $A \neq \underline{0}$ there exists a point $x \in X$ such that $A(x) \neq 0$.

Define a sequence $\{(x_n)_{\alpha_n}(x)\}$ of non-zero real numbers as follows.

$(x_n)_{\alpha_n}(x) = A(x) - \xi_n$, $n=1,2,\ldots$ where $\{\xi_n\}$ is a sequence of non-negetive real numbers such that $\lim\limits_{n \to \infty} \xi_n = 0$.

Then $\underset{n}{\mathrm{Sup}}\,(x_n)_{\alpha_n}(x) = A(x)$. But, $\underset{n}{\mathrm{Sup}}(x_n)_{\alpha_n}(x) \leq \underset{\lambda}{\mathrm{Sup}}\,(x_\lambda)_{\alpha_\lambda}(x)$

So $A(x) \leq \underset{\lambda}{\mathrm{Sup}}\,(x_\lambda)_{\alpha_\lambda}(x)$ ......(b). Therefore, the inequality (b) holds good at every point $x \in X$ such that $A(x) \neq 0$.

At other points of X the inequality (b) holds trivially.

Hence, $A(y) < \underset{\lambda}{\mathrm{Sup}}\,(x_\lambda)_{\alpha_\lambda}(y)$ for all $y \in X$, i.e. $A \subset \underset{\lambda}{\cup}(x_\lambda)_{\alpha_\lambda}$ ....(c).

Combining (a) and (c) we have $A = \underset{\lambda}{\cup}(x_\lambda)_{\alpha_\lambda}$.

§ .3. FUZZY POINT AS INTRODUCED BY MING AND MING.

In earlier definitionof fuzzy point we have observed that the value of the fuzzy point x of a set X satisfies $0 < \alpha < 1$. However Ming and Ming included the value 1 and defined fuzzy point and 'belongingness in slightly different direction.

DEFINITION.2.3.1.(Ming & Ming 1980 (21)). A fuzzy set in X is called

a fuzzy

a fuzzy point iff it takes the value 0 for all $y \in X$ except one ,say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$, we denote this fuzzy point by x_α where the point $x \in X$ is called the support & α is called the value.

DEFINITION.2.3.2.(M & M 1980 (21)). The fuzzy point x_α is said to be contained in a fuzzy set A or to belong to A, denoted by $x_\alpha \in A$, iff $\alpha \leq A(x)$.

NOW we introduce the notion of neighborhood of a fuzzy point.

DEFINITION.2.3.3.(M & M 1980 (21)). A fuzzy set A in an fts (X,T) is called a neighborhood of a fuzzy point x_α (with support x and value α) iff there exists an element $B \in T$ such that $x_\alpha \in B \subset A$.

A neighborhood is said to be open iff it is open in the fuzzy topology. The family consisting of all the neighborhoods of a fuzzy point x_α is called the system of neighborhoods of x_α.

Corresponding to the above definitions we introduce the following important concepts.

DEFINITION. 2.3.4. (M & M 1980 (21)). A fuzzy point x_α is said to be quasi-coincident (or simply Q-coincident) with a fuzzy set A, denoted by $x_\alpha \, q \, A$, iff $\alpha + A(x) > 1$. A fuzzy set A is said to be quasi-coincident (or simply Q-coincident) with another fuzzy set B , denoted by $A \, q \, B$, iff there $x \in X$ such that $A(x) + B(x) > 1$ & in this case we also say that A and B are Q-coincident.

DEFINITION.2.3.5.(M & M 1980 (21)). A fuzzy set A in an fts (X,T) is called a Q-neighborhood of a fuzzy point x_α iff there exists $B \in T$ such that $\alpha + B(x) > 1$ and $B \subset A$. A Q-neighborhood is denoted by Q-nbhd. The family consisting of all the Q-neighborhoods of a fuzzy point x_α is called the system of Q-neighborhoods of x_α .

DEFINITION. 2.3.6.(M & M) 1980 (21)). Two fuzzy sets A and B are said to be intersecting in X iff there exists a point $x \in X$ such that $(A \cap B)(x) \neq 0$ and we say that A and B intersect at x.

NOTE.2.3.7. (M & M 1980 (21)). A Q-nbhd of a fuzzy point generally does not contain the point itself. The neighborhood structure of a point which does not contain the point itself, was studied in ordinary topology by Frechet in 1916 (cf. M. Frechet, '' Les Espaces Abstraits!' Paris, p 172). He formed the foundation upon which the Frechet (V) space theory has been built (cf. W. Sierpinsk, "General Topology", chapter 1. Toronto, 1952). But the fact that A and A^c should not intersect, which is true in the theory of (V)-spaces, is no longer true generally in the theory of fuzzy topological space. Hence, our investigation of the Q-nbhd structure differs from that of Frechet (V)-space theory. The substitute for the fact that A and A^c do not intersect in general topology is the fact that A and A^c are not Q-coincident in fuzzy topology.

THEOREM.2.3.8.(M & M 1980 (21)). Let A & B be two fuzzy sets in an fts (X,T). Then $A \subset B \Leftrightarrow A$ & B^c are not Q-coincident. In particular, $x_\alpha \in A \Leftrightarrow x_\alpha$ is not Q-coincident with A^c .

Proof : This follows from the fact that $A(x) \leqslant B(x) \Leftrightarrow A(x) + B^c(x) \leqslant 1$.

THEOREM.2.3.9. (M & M 1980 (21)). Let υ_p be the family of nbhds(resp. Q-nbhds) of a fuzzy point $p = x_\alpha$ in an fts (X,T).

(1) If $U \in \upsilon_p$ then $p \in U$ (resp. p is Q-coincident with U)

(2) If U , $V \in \upsilon_p$ then $U \cap V \in \upsilon_p$ (3) If $U \in \upsilon_p$ & $U \subset V$ then $V \in \upsilon_p$

(4) If $U \in \upsilon_p$ then there exists $V \in \upsilon_p$ such that $V \subset U_p$ and $V \in \upsilon_q$ for every fuzzy point q such that $q \in V$(resp. q is Q-coincident with V)

Conversely, for each fuzzy point p in X if υ_p is a family of fuzzy sets in X satisfying the above conditions (1),(2),(3), then the family T of all fuzzy sets U such that $U \in \upsilon_p$ whwnever $p \in U$ (resp. p is Q-coincident with U) is a fuzzy topology for X.

If in addition, υ_p satisfy the condition (4), mentioned above, then υ_p is exactly the nbhd (resp. Q-nbhd) system of p relative to the fuzzy topology T.

Proof : Straightforward.

Now we mention some results and leave the proofs of most of them.

THEOREM.2.3.10 (M & M 1980 (21)). Let $\Omega = \{A_\lambda\}$ be a family of fuzzy sets in X. Then a fuzzy point $p = x_\alpha$ is Q-coincident with $\underset{\lambda}{\cup} A_\lambda$ iff there exists some $A_\lambda \in \Omega$ such that p is Q-coincident with A_λ.

THEOREM. 2.3.11. (M & M 1980 (21)). A subfamily Ξ of a fuzzy topology T for X is a base for T iff for each fuzzy point $p = x_\alpha$ in (X,T) and for each open Q-nbhd A of p, there exists a member $B \in \Xi$ such that p is Q-coincident with B and $B \subset A$.

DEFINITION.2.3.12. (M & M 1980 (21)). Let υ_p be a nbhd (resp. Q-nbhd) system of a fuzzy point $p = x_\alpha$ in an fts (X,T). A subfamily Ξ_p of υ_p is called a neighborhood base (resp. Q-neighborhood base) of υ_p iff for each $U \in \upsilon_p$ there exists a member B in Ξ_p such that $B \subset U$.

DEFINITION.2.3.13. (M & M 1980 (21)). An fts (X,T) is said to be C_1 (resp. Q-C_1) iff every fuzzy point in (X,T) has a countable nbhd(resp. Q-nbhd) base.

The C_1 (resp. Q-C_1) defined above will be termed as C_1^* (resp. Q-C_1^*) space throughout thr present work.

NOTE.2.3.14. (M & M 1980 (21)). It has been observed that the definition of C_1 space given by Wong (33) does not take the corresponding definition of C_1 space in ordinary general topology as a special case. But, both the concepts of C_1^*-space and Q-C_1^* space take the C_1 space in ordinary general topology, as a special case.

THEOREM.2.3.15.(M & M 1980(21)). If (X,T) is a C_1^*space then it is also a Q-C_1^* space.

THEOREM . 2.3.16. (M & M 1980 (21)) . If an fts (X,T) is C_{11}, then it is also Q-C_1^* space .

However , the converse of theorem 2.3.15 . is not true in general So we can construct a C_{11}- space which is of course Q-C_1^* . But not C_1^* .

THEOREM. 2.3.17. (M & M 1980 (21)) . Let A^o be the interior of a fuzzy set Ain an fts (X,T) . Then a fuzzy point $p = x_\alpha \in A^o$ iff p has a nbhd contained in A .

The proof is straight forward .

THEOREM . 2.3.18 (M & M 1980 (21)) . A fuzzy point $p = x_\alpha \in \bar{A}$, the closure of a fuzzy set A in an fts (X,T) iff each Q-nbhd of p is Q-coincident with A .

Proof: We know that $x_\alpha \in \bar{A} \Leftrightarrow$ for each closed set $F \supset A$, $x_\alpha \in F$ or $F(x) \geq \alpha$. Therefore , by taking complement , this fact can be stated as : $x_\alpha \in \bar{A} \Leftrightarrow$ for every open set $B \subset A^c$ $B(x) \leq 1 - \alpha$.

In other words , for every open fuzzy set B satisfying $B(x) > 1-\alpha$, B is not contained in A^c.

Now , from theorem 2.3.8. , B is not contained in A^c iff B is Q-coincident with $(A^c)^c = A$.

We have thus proved that $x_\alpha \in \bar{A}$ iff every open Q-nbhd B of x_α is Q-coincident with A, which is evidently equivalent to what we want to prove.

DEFINITION.2.3.19 (M & M 1980 (29)). A fuzzy point $p = x_\alpha$ is called an adherence point of a fuzzy set A iff every Q-nbhd of p is Q-coincident with A

<u>THEOREM</u> 2.3.20 (M & M 1980 (21)) . For a fuzzy set A, $\overline{A}$ is the union of all adherence points of A .

Proof : Straight forward .

<u>DEFINITION</u>.2.3.21 (M & M 1980 (21)) . A fuzzy point $p = x_{\alpha}$ is called a boundary point of a fuzzy set A iff $p \in \overline{A} \cap \overline{A^{c}}$.

The union of all boundary points of a fuzzy set A is called a boundary of the fuzzy set A .

The boundary of A defined by Ming & Ming will be termed as Q-boundary and will be denoted by ^{B}A.

It is clear that $^{B}A = \overline{A} \cap \overline{A^{c}}$.

<u>THEOREM</u> . 2.3.32 . (M & M 1980 (21)). For any fuzzy set A , $\overline{A} \supset A \cup {}^{B}A$ where the inclusion symbol cannot be replaced by an equality .

Proof : The first part follows from the definitiop 2.3.21.

For the last part of the result , we consider the following example :

Let X be any point set . Take $x \in X$.

Let $T = \{\underline{0} , \underline{1} , x_{1/2}\}$, $A = x_{2/3}$ and $p = x_{3/4}$ where x_{α} is a fuzzy point with support x and value α .

Clearly , T is a fuzzy topology .

The Q-nbhd of p in (X,T) are $\underline{1}$ and $x_{1/2}$ which are all Q-coincident with A . Hence , by theorem 2.3.18 , $p \in \overline{A}$.

On the other hand , $p \notin A$ and the Q-nbhd of $\{x_{1/2}\}$ is not Q-coincident with A^{c} i.e. $p \notin {}^{B}A$ and hence $p \notin A \cup {}^{B}A$.

<u>REMARK</u>. 2.3.23. It should be noticed that in general topology we have $\overline{A} = A \cup$ Boundary of A , which is a departure from fuzzy topology .

<u>DEFINITION</u>. 2.3.24. (Weiss 1975 (30)) . Let A be a fuzzy set in an arbitrary set X . The set $\{x \in X : A(x) > 0\}$ is called the support of A and is denoted by supp A or A_{0} .

<u>DEFINITION</u>.2.3.25. (Ming & Ming 1980(21)) . A fuzzy point $p=x_{\alpha}$ is called

an accumulation point of a fuzzy set A iff p is an adherence point of A and every Q-nbhd of p and A are Q-coincident at some point different from supp(p) whenever $p \in A$.

The union of all accumulation points of a fuzzy set A is called the de derived set of A, denoted by ${}^{D}A$.

It is evident that ${}^{D}A \subset \bar{A}$.

The accumulation point and derived set developed by Ming & MIng will be termed as Q-accumulation point and Q-derived set respectively.

THEOREM2.3.26. (M & M 1980 (21)). For any fuzzy set A, $A = \bar{A} \cup {}^{D}A$

Proof : Let $\Omega = \{p : p$ is an adherence point of $A\}$.

Then from theorem 2.3.20, $\bar{A} = \cup \, \Omega$.

On the other hand $p \in \Omega \Rightarrow$ either $p \in A$ or $p \notin A$. For the latter case, by definition of Q-derived set $p \in {}^{D}A$. So, $\bar{A} = \cup \, \Omega \subset A \cup {}^{D}A$.

The other inclusion is obvious. Hence $A = \bar{A} \cup {}^{D}A$.

COROLLARY.2.3.27. (M & M 1980 (21)). A fuzzy set A is closed iff A contains all the Q-accumulation point of A .

Proof : We know that A is closed iff $A = \bar{A}$.

therefore, by theorem 2.3.26 the result follows.

THEOREM.2.3.28. (M & M 1980 (21)). In an fts (X,T) let $A = \{x_\alpha\}$ be a fuzzy set. Then (1) for $y \neq x$, $\bar{A}(y) = {}^{D}A(y)$ (2) if $\bar{A}(x) > \alpha$ then $\bar{A}(x) = {}^{D}A(x)$ and (3) $\bar{A}(x) = \alpha$ iff ${}^{D}A(x) = 0$.

This theorem helps us to establish the following ruselts.

THEOREM2.3.29. (M & M 1980 (21)). In an fts ~~let~~ (X,T) , let $A = \{x_\alpha\}$ be a fezzy set, then (1) when ${}^{D}A(x) > 0$, ${}^{D}A = \bar{A}$ is closed, (2) when ${}^{D}A(x)=0$ ${}^{D}A$ is closed iff there exists a $B \in T$ such $B(x) = 1$ & for $y \neq x$ $B(y) = (\bar{A}^{c})(y) = ({}^{D}A)^{c}(y)$ & (3) ${}^{D}A(x) = 0$ iff there exists a member $B \in T$ such that $B(x) = 1-\alpha$.

THEOREM. 2.3.30. (M & M 1980 (21)). The Q-derived set of each fuzzy point is closed iff the derived set of each fuzzy ~~set~~ point is closed.

DEFINITION.2.3.31.(M & M 1980 (21)). A family $\Omega = \{p_\lambda\}$ $\lambda \in \Lambda$, of fuzzy points $p_\lambda = (x_\lambda)_{\alpha_\lambda}$ in an fts (X,T) is said to be dense in X iff every nonempty T-open set contains some member of Ω . The family Ω is said to be Q-dense iff every nonempty T-open set is Q-coincident with some member of Ω .

DEFINITION.2.3.32.(M & M 1980 (21)). An fts (X,T) is said to be separable (resp. Q-separable) iff there exists a countable family of fuzzy points in X which is dense (resp. Q-dense) in X.

PROPOSITION.2.3.33.(M & M 1980 (21)). In an fts (X,T), a family Ω of fuzzy points in X is Q-dense iff $\overline{\cup\Omega} = X$.

PROOF : Left.

Although the concept of being dense and that of being Q-dense donot imply each other, but we havethe following result.

THEOREM.2.3.34.(M & M 1980 (21)). An fts (X,T) is separable iff it is Q-separable.

PROOF : Omitted.

THEOREM.2.3.35.(M & M 1980 (21)). Let each coordinate space (X_λ,T_λ) $\lambda \in \Lambda$, be separable and $|\Lambda| \leq 2^{\aleph_0}$, then the product space (X,T) is also separable.

THEOREM.2.3.36.(M & M 1980 (21)). Let Λ be an index set. Let each (X_λ,T_λ), $\lambda \in \Lambda$, be an fts such that T_λ has non-intersecting and non-empty members U_λ, V_λ. If the product space (X,T) of these fts's is separable, then each (X_λ,T_λ) is separable and $|\Lambda| \leq 2^{\aleph_0}$.

CHAPTER. 3.

PRODUCT AND QUOTIENT FUZZY TOPOLOGIES.

After Zadeh introduced fuzzy sets in his classical paper (32), Chang (4) developed the theory of fuzzy topological spaces based on Zadeh's concept. A study of the product and the quotient fuzzy topologies was started by Wong in (31).

In this chapter, we generate new fuzzy topologies from given ones and study conditions for some properties to carry over. Product and quotient fuzzy topologies are developed and studied in the same spirit.

§ .1.PRODUCT FUZZY TOPOLOGY.

In this section we deal with product fuzzy topology only. In few results one would notice the difference between general topology and fuzzy topology.

Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be the usual product space and π_λ, the projection from X onto X_λ. Further assume that each X_λ is an fts with fuzzy topology T_λ. Let $\Sigma = \pi_\lambda^{-1}(U_\lambda) : U_\lambda \in T_\lambda \ \& \ \lambda \in \Lambda \}$ Let Ξ be the family of all finite intersections of members of Σ. Let T be the family of all unions of members of Ξ. It is clear that T is indeed a fuzzy topology for X with Ξ as a base & Σ a subbase.

DEFINITION.3.1.1.(Wong, 1974 (31)). Given a family of fts $\{(X_\lambda, T_\lambda)\}$ $\lambda \in \Lambda$, the fuzzy topology T defined as above is called the product fuzzy topology for $X = \prod_{\lambda \in \Lambda} X_\lambda$ and (X, T) is called the product fts.

Some immediate consequences from this definition are given below.

THEOREM.3.1.2.(Wong, 1974(31)). Let (X,T) be the product fts of the family of fts's $\{(X_\lambda, T_\lambda) : \lambda \in \Lambda\}$

(1) For each $\lambda \in \Lambda$, the projection π_λ is F-continuous.

(2) The product fuzzy topology is the smallest fuzzy topology for X such that (1) is true.

(3) Let $(Y,\mathcal{U})$ be an fts and f, a function from Y to X. Then f is F-continuous iff for every $\lambda \in \Lambda$, $\pi_\lambda o\ f$ is F-continuous.

Proof : (1) & (2) follow from the definition of product fuzzy topology.

(3) ( ) Obvious.

( ) Let $U_\lambda \in T_\lambda$. Then $(\pi_\lambda o\ f)^{-1}(U_\lambda) = (f^{-1} o \pi_\lambda)(U_\lambda)$ is $\mathcal{U}$-open. So, $\{ f^{-1}(\pi_\lambda(U_\lambda)) : U_\lambda \in T_\lambda\}$, $\lambda \in \Lambda$ is a family of $\mathcal{U}$-open fuzzysets in Y. As every member of T is the union of finite intersections of the family $\{\pi_\lambda^{-1}(U_\lambda) : U_\lambda \in T_\lambda , \lambda \in \Lambda\}$and f^{-1} preserves union & intersection therefore it follows that f^{-1} maps T-open fuzzy sets onto $\mathcal{U}$-open fuzzy sets. Hence, f is F-continuous.

THEOREM.3.1.3.(Foster, 1979(9)). Let $\{(X_\lambda, T_\lambda)\}$, $\{(Y_\lambda, \mathcal{U}_\lambda)\}$,$\lambda \in \Lambda$be two families of fts's and (X,T), $(Y,\mathcal{U})$ the respective product fts's. For each $\lambda \in \Lambda$, let f_λ be a mapping of (X_λ, T_λ) into $(Y_\lambda, \mathcal{U}_\lambda)$. Then the product mapping $f = \prod_{\lambda \in \Lambda} f_\lambda : (x_\lambda) \mapsto (f_\lambda(x_\lambda))$ of (X,T) onto $(Y,\mathcal{U})$ is F-continuous iff f_λ is F-continuous for each $\lambda \in \Lambda$.

Proof : Note that the mapping f can also be written as $x \to (f_\lambda(\pi_\lambda(x)))$ where $x = (x_\lambda)$. Thus f is F-continuous by theorem 3.1.2.

Let $\{X_\lambda\}$, $\lambda = 1,2,3,\ldots,n$ be a finite family of fuzzy sets & for each $\lambda = 1,2,\ldots,n$, let A_λ be a fuzzy set in X_λ. We define the product $A = \prod_{\lambda=1}^{n} A_\lambda$ of the family $\{A_\lambda\}$, $\lambda = 1,2,\ldots,n$ as a fuzzy set in $X = \prod_{\Lambda=1}^{n} X$ by $A(x_1,x_2,\ldots,x_n) = \text{Min}\ \{A_1(x_1),A_2(x_2),\ldots,A_n(x_n)\}$for all $(x_1,x_2,\ldots,x_n) \in X$

clearly, for each $\lambda = 1,2,\ldots,n$, $\pi_\lambda(A) \subset A_\lambda$, because, for all $x_\lambda \in X_\lambda$

$$(\pi_\lambda(A))(x_\lambda) = \sup_{(z_1,\ldots,z_n)\in\pi^{-1}(x)} A(z_1, z_2,\ldots,z_n)$$
$$= \sup_{(z_1,\ldots,z_n)\in\pi_\lambda^{-1}(x_\lambda)} \mathrm{Min}\{A_1(z_1),A_2(z_2),\ldots,A_n(z_n)\}$$
$$= \mathrm{Min}\,\{\sup_{z_1\in X_1} A_1(z_1),\ldots,\sup_{z_n\in X_n} A_n(z_n)\} \subset A_\lambda(x_\lambda)$$

It follws from the above that if X_λ has fuzzy topology T_λ, $\lambda=1,2,\ldots,n$, the product fuzzy topology T on X has a base the set of product fuzzy sets of the form $\prod_{\lambda=1}^{n} U_\lambda$ where $U_\lambda \in T_\lambda$ $\lambda=1,2,\ldots,n$.

<u>THEOREM</u>. 3.1.4. (Foster 1979(9)). Let $\{(X_\lambda,T_\lambda)\}$ $\lambda=1,2,\ldots,n$ be a finite family of fts's. For each $\lambda=1,2,\ldots,n$, let A_λ be a fuzzy set in X_λ and A the product fuzzy set in X where (X,T) is the product fts of the finite family of fts's. Then the induced topology T_A on A has a base the set of product fuzzy set of the form $\prod_{\lambda=1}^{n} U_\lambda^*$ where $U_\lambda^* \in (T_\lambda)_{A_\lambda}$ $\lambda=1,2,\ldots,n$.

Proof: In accordance with the preceeding remark T has a base $\Xi=\{\prod_{\lambda=1}^{n} U_\lambda : U_\lambda \in T_\lambda,\ \lambda=1,2,\ldots,n.\}$. A base for T_A is therefore given by $\Xi_A=\{(\prod_{\lambda=1}^{n} U_\lambda)\cap A : U_\lambda \in T_\lambda,\ \lambda=1,2,\ldots,n\}$ But $(\prod_{\lambda=1}^{n} U_\lambda)\cap A = \prod_{\lambda=1}^{n}(U_\lambda \cap A)$. The result follows with $U_\lambda^* = U_\lambda \cap A$, $\lambda=1,2,\ldots,n$.

<u>THEOREM</u>.3.1.5. (Foster 1979(9)) Let $\{(X_\lambda,T_\lambda)\}$ $\lambda=1,2,\ldots,n$ be a finite family of quasi-fts's and (X,T) the product fts. For each $\lambda=1,2,\ldots,n$, let A_λ be a fuzzy set in X_λ and A the product fuzzy set on X. Let $(Y,\mathfrak{U})$ be a quasi-fts and B a fuzzy set in Y. Let f be a mapping of the quasi fuzzysubspace $(B,\mathfrak{U}_{(B)})$ into the quasi fuzzy subspace $(A,T_{(A)})$. Then f is relatively quasi fuzzy continuous iff $\pi_\lambda \circ f$ is relatively quasi fuzzy continuous for each $\lambda=1,2,\ldots,n$.

Proof: $(\Rightarrow)$ Since π_λ is F-continuous for each $\lambda=1,2,\ldots,n$, by theorem 1.10.6., each π_λ is relatively quasi fuzzy continuous $\lambda=1,2,\ldots,n$. Therefore, the composition $(\pi_\lambda o f)$ is relatively quasi fuzzy continuous for each $\lambda=1,2,\ldots,n$.

($\Leftarrow$) Let $U^* = U_1^* \times \ldots \times U_n^*$, where $U_\lambda^* \varepsilon (T_\lambda)_{A_\lambda}$ $\lambda=1,2,\ldots,n$. Therefore, by theorem 3.1.4. the set of such U^* forms a base for T_A Since, $f^{-1}(U^*) \cap B = f^{-1}(\pi_1^{-1}(U_1^*) \cap \ldots \cap \pi_n^{-1}(U_n^*)) \cap B =$

$$= \prod_{\lambda=1}^{n} ((\pi_\lambda \circ f)^{-1}(U_\lambda^*) \cap B) \text{ is open in } U_{(B)}\text{, as } \pi_\lambda \text{of}$$

of relatively quasi fuzzy continuous for each $\lambda=1,2,\ldots,n$, it follows from theorem 1.10.9. that f is relatively quasi fuzzy continuous .

THEOREM. 3.1.6. (Foster 1979 (9)). Let $\{(X_\lambda, T_\lambda)\}$ and $\{(Y_\lambda, U_\lambda)\}$ $\lambda=1,2,\ldots,n$ be two finite families of quasi-fts's, and (X,T) and (Y,U) be the respective product fts's. For each $\lambda=1,2,\ldots,n$ let A_λ be a fuzzy set in X_λ, B_λ a fuzzy set in Y_λ and f_λ a mapping of the quasi fuzzy subspace $(A_\lambda,(T_\lambda)_{(A_\lambda)})$ into the quasi fuzzy subspace $(B_\lambda,(U_\lambda)_{(B_\lambda)})$. Let $A = \prod_{\lambda=1}^{n} A_\lambda$ and $B = \prod_{\lambda=1}^{n} B_\lambda$ be the product fuzzy sets in Xand Y respectively. Then the product mapping $f = \prod_{\lambda=1}^{n} f_\lambda$ given by $f(x_1,\ldots,x_n) = (f_1(x_1),\ldots,f_n(x_n))$ of the quasi fuzzy subspace $(A,T_{(A)})$ into the quasi fuzzy subspace $(B,U_{(B)})$ is relatively quasi fuzzy continuous if f_λ is relatively quasi fuzzy continuous for each $\lambda=1,\ldots,n$.

Proof: Analogous to the proof of theorem 3.1.3.

R**THEOREM** .3.1.7. (Foster , 1979 (9)). Let $\{(X_\lambda, T_\lambda)\}$ and $\{(Y_\lambda, U_\lambda)\}$ $\lambda=1,2,\ldots,n$ be two finite families of fts's and (X,T) and (Y,U) the res respective product fts's . for each $\lambda=1,2,\ldots,n$, let f_λ be a mapping of (X_λ, T_λ) into (Y_λ, U_λ). Then the product mapping $f = \prod_{\lambda=1}^{n} f_\lambda$ given by $f(x_1,\ldots,x_n) = (f_1(x_1),\ldots,f_n(x_n))$, of (X,T)into (Y,U) is fuzzy open if f_λ is fuzzy open for each $\lambda=1,2,\ldots,n$.

Proof : Let U^*be open in T. Then , there exists open fuzzy asets $U_{\lambda m}$, $m \varepsilon M$, $\lambda=1,2,\ldots,n$, such that $U^* = \bigcup_{m \varepsilon M} (\prod_{\lambda=1}^{n} U_{\lambda m})$.

Now , for all $y \varepsilon Y$ $(f(U))(y) = \bigcup_{m \varepsilon M} f(\prod_{\lambda=1}^{n} U_{\lambda m})(y)$

$$= \sup_{m \varepsilon M} \sup_{z \varepsilon f^{-1}(y)} (\prod_{\lambda=1}^{n} U_{\lambda m}(z)$$

$$= \sup_{m \varepsilon M} \sup_{z_1 \varepsilon f_1^{-1}(y_1)} \ldots \sup_{z_n \varepsilon f_n^{-1}(y_n)} \operatorname{Min} \{U_{1m}(z_1),\ldots,U_{nm}(z_n)\}.$$

$= \sup_{m\in M} \operatorname{Min}\{ f_1(U_{1m})(y_1),\ldots,f_n(U_{nm})(y_n)\} = \bigcup_{m\in M} \prod_{\lambda=1}^{n}(f_\lambda(U_{\lambda m}))(y) =$

Thus $f(U') = \bigcup_{m\in M} \prod_{\lambda=1}^{n} (f_\lambda(U_{\lambda m}))$.

Since f_λ is fuzzy open for each $\lambda=1,2,\ldots,n$ $f(U')$ is open in U.

THEOREM.3.1.8.(Foster 1979(9)).Let $\{(X_\lambda,T_\lambda)\}$ & $\{(Y_\lambda,U_\lambda)\}$ $\lambda=1,2,\ldots,n$ be two finite families of quasi-fts's and (X,T) & (Y,U) the respective product fts's. For each $\lambda=1,2,\ldots,n$, let A_λ be a fuzzy set in X_λ and B_λ a fuzzy set in Y_λ. Let f_λ be a mapping of the quasi-fuzzy subspace $(A_\lambda,(T_\lambda)_{(A_\lambda)})$ into the quasi-fuzzy subspace $(B_\lambda,(U_\lambda)_{(B_\lambda)})$ for each $\lambda=1,2,\ldots,n$. Let $A = \prod_{\lambda=1}^{n} A_\lambda$ and $B = \prod_{\lambda=1}^{n} B_\lambda$ be the product fuzzy sets in X & Y respectively. Then the product mapping $f = \prod_{\lambda=1}^{n} f_\lambda : (x_1,x_2,\ldots,x_n) \to (f_1(x_1),f_2(x_2),\ldots,f_n(x_n))$ of the quasi-fuzzy subspace $(A,T_{(A)})$ into the quasi-fuzzy subspace $(B,U_{(B)})$ is relatively quasi-fuzzy open if f_λ is relatively quasi-fuzzy open for each $\lambda = 1,2,\ldots,n$.

Proof : Let U' be open in $T_{(A)}$. By theorem 3.1.4 there exists open fuzzy sets $U'_{\lambda m} \in (T_\lambda)_{(A_\lambda)}$, $m \in M$, $\lambda= 1,2,\ldots,n$ such that U' can be written as $U' = \bigcup_{m\in M} \prod_{\lambda=1}^{n} U'_{\lambda m}$

As in the proof of theorem 2.1.7 it follows that

$f(U') = \bigcup_{m\in M} \prod_{\lambda=1}^{n} (f_\lambda(U_{\lambda m}))$. Since f_λ is relatively quasi fuzzy open for each $\lambda=1,2,\ldots,n$, $f(U')$ is open in $U_{(B)}$.

THEOREM.3.1.9 (Foster 1979(9)). Let (X_1,T_1) & (X_2,T_2) be quasi-fts's and (X,T) the product fts. Then for each $a_1 \in X_1$, the mapping $f : x_2 \to (a_1,x_2)$ of (X_2,T_2) into (X,T) is F-continuous.

Proof : The constant mapping $f_1 : x_2 \to a_1$ of (X_2,T_2) into (X_1,T_1) is F-continuous, for, if U_1 is open in T_1, the inverse image $f_1^{-1}(U_1)$ is given by $(f_1^{-1}(U_1))(x_2) = U_1(a_1) = k_c(x_2)$ $\forall x_2 \in X_2$ where k_c is the open fuzzy set in X_2 which is a constant fuction with $c = U_1(a_1)$ and as the identity mapping $i_2 : x_2 \to x_2$ of (X_2,T_2) onto itself is F-continuous the mapping f is FF-continuous by theorem 3.1.2.

THEOREM.3.1.10 (Foster 1979(9)). Let (X_1,T_1) & $(X_2 T_2)$ be two quasi-fuzzy topological spaces and (X,T) the product fts. Let A_1 and A_2 be fuzzy sets in X_1 and X_2 respectively and A the product fuzzy set in X. Then for each $a_1 \varepsilon X_1$ such that $A_1(a_1) \geqslant A_2(x_2)$ for all $x_2 \varepsilon X_2$ the mapping $f : x_2 \to (a_1,x_2)$ of the quasi fuzzy subspace $(A_2,(T_2)_{(A_2)})$ into the quasi-fuzzy subspace $(A,T_{(A)})$ is relatively quasi fuzzy continuous.

Proof : Clearly $f(A_2) \subset A$, because $(f(A_2))(x_1,x_2) = A_2(x_2)$ if $x_1 = \alpha_2$ and $(f(A_2))(x_1,x_2) = 0$ otherwise, and $A(x_1,x_2) = \text{Min}\{A_1(x_1),A_2(x_2)\}$ which is $\geqslant A_2(x_2)$ for all $(x_1,x_2) \varepsilon A$.

The proof of the relatively quasi fuzzy continuity of f is analogous to the proof of the fuzzy continuity of f in theorem 3.1.9.

DEFINITION.3.1.11.(Ming & MIng 1980(21)). Let $X = \prod_{\lambda \varepsilon \Lambda} X_\lambda$ be the product space of a family of fts's $\{(X_\lambda,T_\lambda)\}\lambda\varepsilon\Lambda$ and $x = (x_\lambda)_{\lambda \varepsilon \Lambda} \varepsilon X$ & $\mu \varepsilon \Lambda$ The subset $\tilde{X}_\mu$ of X defined by $\tilde{X}_\mu = \{y = (y_\lambda)$: when $\lambda \neq \mu$, $y_\lambda = x_\lambda\}$ is called the section through $x = (x_\lambda)$ parallel to X_μ.

REMARKE:3.1.12 (M & M 1980 (21)). In general topology, the subspaces $\tilde{X}_\mu$ & X_μ are naturally considered to be homeomorphic. But this is not true in general in fuzzy topology. On account of this we must take care in deducing some property for the coordinate space (X_λ,T_λ) from the product space X which enjoys the same property.

We need the following lemma

LEMMA.3.1.13.(M & M 1980 (21)). Let $U = \bigcap_{\lambda \varepsilon F} \{\pi_\lambda^{-1}(U_\lambda)\}$ be a member of the defining base Ξ for the product topology of the product space of a family of fts's $\{(X_\lambda,T_\lambda)\}$ where $U_\lambda \varepsilon T_\lambda$, and F is a finite subset of the index set Λ, then, when $\mu \notin F$ $\pi_\mu(U)$ is the fuzzy set in X_μ which takes the constant value α on X_μ where

$\alpha = \underset{\lambda \varepsilon F}{\text{Min}}\{ \text{Sup} \{U_\lambda(x_\lambda) ; x_\lambda \varepsilon X_\lambda\}$ &

when $\mu \varepsilon F$, let $F_1 = F - \{\mu\}$, then $\pi_\mu(U) = U_\mu \cap A$ where A is the fuzzy set in X , taking the constant value β such that

$$\beta = \underset{\lambda \in F}{\mathrm{Min}} \{\mathrm{Sup}\{ U_\lambda(x_\lambda) : x_\lambda \in X_\lambda\}\}$$

Proof : This lemma can be directly verified.

THEOREM.3.1.14 (M & M 1980 (21)). Let $\tilde{T}_\mu$ be the relative fuzzy topology of the subspace $\tilde{X}_\mu$. Then there is a F-continuous bijection $f : (\tilde{X}_\mu,\tilde{T}_\mu) \to (X_\mu,T_\mu)$. Moreover when (X_μ,T_μ) is quasi fts , f is a fuzzy homeomorphism.

Proof : Let $y =\{ y_\lambda\}$ be an arbitrary point in $\tilde{X}_\mu$. Let $f(y) = y_\mu$. It is easily seen that f is a bijection. From the definition of product topology it follows that f is F-continuous.

Suppose that (X_μ,T_μ) is quasi fts. We show that f^{-1} is also Fcontinuous. In fact $\Sigma =\{ A : A = \pi_\lambda^{-1}(U_\lambda)\cap \tilde{X}_\mu , U_\lambda \in T_\lambda ,\lambda\in\Delta\}$ is a subbase of $\tilde{T}_\mu$. Therefore, from theorem 1.9.14, in order to prove that f^{-1} is F-continuous, it suffices to show that $f(A)\in \tilde{T}_\mu$.

Obviously $f(A) =\pi_\mu \pi_\lambda^{-1}(U_\lambda)$.

When $\lambda \neq \mu$, from lemma 3.1.13 f(A) is the fuzzy set taking the constant value on X_μ and by definition of quasi fts $f(A)\in T_\mu$.

When $\lambda = \mu$, evidently $f(A) = U_\mu \in T_\mu$.

NOTE.3.1.15. In the light of the correspondence f consteucted above one may directly consider the subspace $(\tilde{X}_\mu,\tilde{T}_\mu)$ of the product space (X,T) to be a fuzzy topological space (X_μ,U_μ), where U_μ is generated by the family of fuzzy sets which consists of all the members of T_μ and in addition, some fuzzy sets taking constant value on X_μ. Ofcourse U_μ is finer than T_μ.

DEFINITION.3.1.16 (M & M 1980(21)). A fts (X,T) is called a purely stratified space iff for each $U\in T$, there is $\alpha\in I$ such that $U(x) =\alpha$ for each $x\in X$. In particular, (X,T) is called simply stratified iff $T = \{\underline{0} , \underline{1}\}$.

NOTE : 3.1.17. In fuzzy topology a purely stratified space plays the role analogous to that played by an indiscrete space in ordinary the role analogous to that played by an indiscrete space in ordinary general topology.

THEOREM.3.1.18.(M & M 1980 (21)). If Λ is an uncountable index set and for each $\lambda \in \Lambda$, (X_λ, T_λ) is purely stratified, so is the product space (X,T) of this collection of fts's.

Proof : Since, T has a subbase whose members are fuzzy sets taking constant value on X ∴ The theorem holds.

THEOREM.3.1.19 (M & M 1980 (21)). A purely stratified space is C_{11}.

Proof : The proof can be obtained from the following simple property of the unit closed interval I of the real line :

For any set $\Omega = \{\alpha_\lambda\} \subset I$ there exists a countable subset Ω_1 of Ω such that for each $\alpha_\lambda \in \Omega$, there is a sequence $\alpha_n \in \Omega_1$ $(n=1,2,\ldots)$ satisfying $\alpha_n \leq \alpha_\lambda$ and $\{\alpha_\lambda\}$ converges to α_λ under the usual topology of the real line.

Now we obtain the product theorems for C_{11} space. Here one would notice the departure from general topology.

THEOREM.3.1.20 (Wong 1974 (31)). Let $\{(X\lambda, T\lambda)\}$, $\lambda=1,2,\ldots$ be a countable family of C_{11} fts's. Then the product fts (X,T) is also C_{11}

Proof : Let Ξ_λ be a countable base for T_λ. Let $\nabla = \{\pi_\lambda^{-1}(B) : B \in \Xi_\lambda\}$ $\lambda=1,2,\ldots$ and Ξ be the family of all finite intersections of members of ∇ . Therefore, Ξ is a countable subfamily of T.

We shall show that Ξ is a base for T.

Let $F \in T$. By definition of product topology, F is the union of open fuzzy sets of the form $\bigcap_{n=1}^{n_0} \pi_{\lambda_n}^{-1}(A_n)$ where $A_n \in T_{\lambda_n}$

Since Ξ_{λ_n} is a base for T_{λ_n} . Therefore, $A_n = \bigcup_{m_n \in M_n} B_{m_n}$, $B_{m_n} \in \Xi_{\lambda_n}$

Consequently, $\pi_{\lambda_n}^{-1}(A_n) = \pi_{\lambda_n}^{-1}\left(\bigcup_{m_n \in M_n} B_{m_n}\right) = \bigcup_{m_n \in M_n} \pi_{\lambda_n}^{-1}(B_{m_n})$

Note that in general,

$\left(\bigcup_{m \in \Lambda_1} C_m\right) \cap \left(\bigcup_{n \in \Lambda_2} D_n\right) = \bigcup_{(m,n) \in \Lambda_1 \times \Lambda_2} (C_m \cap D_n)$ where C_m and D_n are fuzzy sets.

This can be easily generalised to finite intersections.

So $\bigcap_{n=1}^{n_0} \pi_{\lambda_n}^{-1}(A_n) = \bigcap_{n=1}^{n} \bigcup_{m_n \in M_n} \pi_{\lambda_n}^{-1}(B_{m_n})$ is the union of finite intersections of members of ∇. It follows that Ξ is a base.for T.

THEOREM.3.1.20(1) (M & M 1980(21)).Let $\{(X_\lambda, T_\lambda)\}$ $\lambda \in \Lambda$ be a collection of C_{11} fts'Ss and (X,T) their product space. Then (X,T) is C_{11} iff all but a countable number of coordinate spaces are purely stratified.

Proof : Left.

Now we prove that uncountable products of C_{11} spaces may not be C_{11} ; hence , the above result is , in a sense , the best one can get .

THEOREM . 3.1.21. (Wong 1974 (31)) . Let $\{(X_\lambda, T_\lambda)\}$ $\lambda \in \Lambda$be an uncountable family of C_{11} spaces such that (1) none is indiscrete and (2) in each fts (X_λ, T_λ) , for any $F \in T_\lambda$ and $F \neq \underline{0}$, there exists a point $x_\lambda \in X_\lambda$ such that $F(x_\lambda) = 1$. Then the product fts (X,T) is not C_{11}.

Proof. By definition of product fuzzy topology , T has a base B_o with members of the form $B_o = \bigcap_{n=1}^{m} \pi_{\lambda_n}^{-1}(A_{\lambda_n})$, $A_{\lambda_n} \in T_{\lambda_n}$.

Then by (2) $\pi_\lambda(B_o) = X_\lambda$ for $\lambda \neq \lambda_n$, $n=1,2,\ldots,m$ and $B_o \neq \underline{0}$.

Indeed, let $x_n \in X_{\lambda_n}$ such that $A_{\lambda_n}(x_n) = 1$, $n=1,2,\ldots,m$.

Let $x_\lambda \in X_\lambda$. Consider the subset S of X given by

$$S = \{x_1\} \times \ldots \times \{x_m\} \times \{x_\lambda\} \times \prod_{\substack{\mu \neq \lambda_n \\ n=1,2,\ldots,m \\ \mu \neq \lambda}} X_\mu$$ of X . Then $B_o(s) = 1 \;\forall\; s \in S$.

Therefore , $(\pi_\lambda(B_o))(x_\lambda) = 1$. Since x_λ is arbitrarily chosen.

Thus $\pi_\lambda(B_o) = X_\lambda$.

Now , if (X,T) is C_{11}, then T has a countable base $\Xi = \{B_n\}$, $n=1,2,\ldots$

So, for each $B_n \neq \underline{0}$ of ≋, $\exists$ a member $B_o \neq \underline{0}$ of Ξ_o such that $B_n \supseteq B_o$ say $B_o = \bigcap_{n=1}^{m} \pi_{\lambda_n}^{-1}(A_{\lambda_n})$

$A_{\lambda_n} \in T_{\lambda_n}$. Then , $\pi_\lambda(B_n) \supset \pi_\lambda(B_o) = X_\lambda$ for $\lambda \neq \lambda_n$, $n=1,2, \ldots,m$.

$\therefore \pi_\lambda(B_n) = X_\lambda$ for $\lambda \neq \lambda_n$, $n=1,2,\ldots,m$.

Do this for all members of Ξ.

It follows that there exists a countable subset Ω of the index set Λ such that for any $\lambda \in \Lambda - \Omega$, $\pi_\lambda(B_n) = X_\lambda$ for all $B_n \neq \underline{0}$.
Since Λ is uncountable, there exists $\lambda_0 \in \Lambda - \Omega$. Hence $\pi_{\lambda_0}(B_n) = X_{\lambda_0}$ $\forall$ $B_n \neq \underline{0}$
By assumption, X_{λ_0} is not indiscrete. Therefore there exists an open fuzzy set A_0 , $A_0 \neq \underline{0}$, X_{λ_0} such that $\pi_{\lambda_0}^{-1}(A_0)$ is an open fuzzy set in T. Again by definition of base, there exists a member $B_k \neq \underline{0}$ of Ξ such that $\pi_\lambda^{-1}(A_0) \supset B_k$. Thus $\pi_{\lambda_0}(\pi_{\lambda_0}^{-1}(A_0)) \supset \pi_{\lambda_0}(B_n) = X_{\lambda_0}$.
On the other hand, $\pi_{\lambda_0}(\pi_{\lambda_0}^{-1}(A_0)) = A_0$. This is a contradiction.

Next we state theorems on product spaces generated by C_{11} and separable spaces without providing the proofs.

THEOREM.3.1.22.(Wong (33)). Let $\{(X_\lambda, T_\lambda)\}\lambda \in \Lambda$ be a countable family of C_1 fts's. Then product fts (X,T) is also C_1 .

THEOREM.(Wong (33)).Let $\{(X_\lambda, T_\lambda)\}$ $\lambda \in \Lambda$ be an uncountable family of C_1 fts'S sucb that

(1)none is indiscrete i.e, for each $\lambda \in \Lambda$, there exists $U_\lambda \in T_\lambda$ such that $U_\lambda \neq \underline{0}$, X_λ

(2) for each $\lambda \in \Lambda$, there exists a fuzzy point $(x_\lambda)_{\alpha_\lambda} \in U_\lambda$ such that $\bigcap_{\lambda \in \Lambda} \pi_\lambda^{-1}((x_\lambda)_{\alpha_\lambda})$ is a fuzzy point in X and

(3) in each fts (X_λ, T_λ), for any $A \in T_\lambda$ and $A \neq \underline{0}$, there exists a point $x \in X_\lambda$ such that $A(x) \neq 1$. Then the product fts (X,T) is not C_1.

Thus we observe that uncountable product of C_1spaces may not be C_1 .

REMARK :3.1.24(wong (33)). Unlike general topology, given fuzzy points $(x_\lambda)_{\alpha_\lambda}$ in X_λ , $\lambda \in \Lambda$, $\bigcap_{\lambda \in \Lambda} \pi_\lambda^{-1}((x_\lambda)_{\alpha_\lambda})$ is not always a fuzzy point in the product space X. It is either a fuzzy point or the empty fuzzy set $\underline{0}$.
For example let $\Lambda =]0,1[$
For each X_λ , let $(x_\lambda)_\lambda$ be a fuzzy point then $\bigcap_{\lambda \in \Lambda} \pi_\lambda^{-1}((x_\lambda)_\lambda) = \underline{0}$.

THEOREM.3.1.25.(Wong (33)). Let $\{(X_\lambda, T_\lambda)\}$ $\lambda \in \Lambda$ be a countable family of separable spaces. Then the product fts is also separable.

As the definition of C_1 space introduced by Wong does not reduce to the C_1 space of ordinary topology, it is clear that the theorems 3.1.22 and 3.1.23 give partial results concerning the problem on products of so called C_1 space. However the definitions of both C_1^* space and Q-C_1 space given by Ming & Ming take care of the C_1 space in ordinary topology under usual restriction. Now we present a necessary and sufficient condition for the productiveness of Q-C_1 spaces as well and for C_1^* spaces give the necessary condition only. Finally, by constucting a counter example we show that the product space of C_1^* spaces need not be a C_1^* space.

THEOREM.3.1.26. (Ming & Ming 1980(21)). Let (X,T) be the product space of the Q-C_1 (or C_1^*) spaces $\{(X_\lambda, T_\lambda)\}, \lambda \in \Lambda$. If (X,T) is Q-C_1 (or C_1^*) space then all but a countable number of coordinate spaces are purely stratified spaces.

Proof : Omitted.

THEOREM. 3.1.27 (M & M 1980(21)). Let $\{(X_\lambda, T_\lambda)\}\lambda \in \Lambda$ be Q-C_1 spaces such that all but a countable number of them are purely stratified; then their product space (X,T) is a Q-C_1 space.

Proof : Left.

By constructing a counter example we establish the following.

THEOREM.3.1.28. (M & M 1980(21)). There exists C_1^* spaces (X_1, T_1) and (X_2, T_2) whose product space is not C_1^*.

We shall need the following lemma which can be verified directly from the definition of product fuzzy topology.

LEMMA.3.1.29 (M & M 1980(21)). Let $\approx_n$ be a base for the fuzzy topology T_n for X_n $(n=1,2)$. Let $(X,T) = (X_1, T_1) \times (X_2, T_2)$ and B be an open fuzzy neighborhood of a fuzzy point $p = x_\alpha$ in X. Then there is $\tilde{B} \in T$ which is of the form $\tilde{B} = \bigcup_\lambda (\pi_1^{-1}(G_\lambda) \cap \pi_2^{-1}(F_\lambda))$ where π_n are projections $(n=1,2)$, $G_\lambda \in \Xi_1$ & $F_\lambda \in \Xi_2$ such that $\tilde{B} \subset B$ & $\tilde{B}$ is an open fuzzynbhd of p.

<u>COUNTER EXAMPLE.</u> 3.1.30. (M & M 1980 (21)).

(1) Construction of (X_1, T_1).

Let X_1 be the unit interval $I=[0,1]$ if the real line. Denote the zero point by x. For positive integers m & n with $n \geqslant m$ we define the fuzzy set $G_{m,n}$ on X_1 as follows :

$$G_{m,n}(z) = 1 - 1/(1+m) \quad \text{for } z \in [0\ ,\ 1/(n+1)]$$
$$= 0 \quad \text{for } z \in X_1 - [0\ ,\ 1/(n+1)]$$

Denote the fuzzy point $x_{(1-1/(n+1))}$ by h_n .

It is obvious that when $n_1 \geqslant n_2$, $m = \mathrm{Min}(m_1, m_2)$ we have for $l = \mathrm{Min}(n_1, n_2)$

$G_{n_1,m_1} \cap G_{n_2,m_2} = G_{n_2,m}$, $h_{n_1} \cap h_{n_2} = h_{n_2}$ & $G_{n_1,m_1} \cap h_{n_2} = h_l$.

Hence the collection Ξ_1 consisting of allthe $G_{n,m}$, h_n , $\underline{0}$ & $\underline{1}$ is a base for some fuzzy topology T_1 for X_1 . It is easily seen that (X_1, T_1) is both C_1^* and C_{11}. For convenience, let us write $\Xi_1 = \{\underline{0}, g_1, g_2, ..\}$ where each $g_i \neq \underline{0}$.

(2) Construction of (X_2, T_2).

Let $X_2 = X_1$ and denote the zero point of X_2 by y. For each natural number n define the fuzzy set F_n on X_2 as follows :

$F_n(z) = 1$ for $z \in [0\ ,\ 1/n]$ & $F_n(z) = 0$ for $z \in X_2 - [0\ ,\ 1/n]$

It is easily verified that the family T_2 consisting of all the F_n's and $\underline{0}$ is a fuzzy topology foe X_2 . Obviously (X_2, T_2) is both C_1^* & C_{11}.

(3) $(X,T) = (X_1, T_1) \times (X_2, T_2)$ is not C_1^*.

Consider the fuzzy point p in $X = X_1 \times X_2$ whose support is (x,y) and whose value is 1. We shall show that the neighborhood system of p has no countable base.

If this is not the case, there exists an open countable neighborhood base $\{B_n : n=1,2,\ldots\}$ of p.

From lemma 3.1.29 we may assume $B_n = \bigcup_{j=1}^{\infty} (\pi_1^{-1}(G_{\lambda(n,j)}) \wedge \pi_2^{-1}(F_{\mu(n.j)}))$ ..(1)

where $\lambda(n,j)$ and $\mu(n,j)$ are natural numbers, $\underline{0} \neq G_{\lambda(n,j)} \in \Xi_1$ and $\underline{0} \neq F_{\mu(n,j)} \in T_2$.

For any B_n, since $B_n(x,y) = 1$, there is a term $\pi_1^{-1}(G_{\lambda(n,j_n)}) \cap \pi_2^{-1}(F_{\mu(n,j_n)})$ in the expression (1) for B_n, which takes its value at (x,y) greater than $1-1/(1+n)$.

Consequently $G_{\lambda(n,j_n)}(x) > 1 - 1/(n+1)$. Now we take a strictly monotonic increasing sequence of natural numbers $\{s_n\}$ such that $s_n > \mu(n,j)$

Let $A = \bigcup_{m=1}^{\infty} \pi_1^{-1}(h_m) \cap \pi_2^{-1}(F_{s_m})$. Since $h_m(x) = 1 - 1/(n+1)$, A is clearly a neighborhood of p (note that $A(x,y) = 1$).

Take an arbitrary B_n ; there corresponds a point $g_n \in]1/s_n, 1/s_{n-1}[\subset X_2$

Then we have $B_n(x,y_n) \geq (\pi_1^{-1}(G_{\lambda(n,j_n)}) \cap \pi_2^{-1}(F_{\mu(n,j_n)}))(x,y_n)$

$$\geq \mathrm{Min}\{G_{\lambda(n,j_n)}(x), F_{s_{n-1}}(y_n)\} = G_{\lambda(n,j_n)}(x) > 1-1/(n+1).$$

On the other hand, since $m \geq n$, evidently $F_{s_m}(y_n) = 0$, we have

$$A(x,y_n) = \sup_{m=1,2,\ldots}\{\mathrm{Min}(h_m(x), F_{s_m}(y_n))\} \leq \sup_{m=1,2,\ldots,n-1}\{h_m(x)\} = 1-1/n.$$

Consequently, $B_n(x,y_n) > A(x,y_n)$, that is to say, B_n is not contained in A. This contradicts the fact that $\{B_n : n=1,2,\ldots\}$ is an open neighborhood base of p

The space (X,T) in the above example is a C_{11} space according to theorem 3.1.20(1). Hence this example provides a counter example showing that C_{11} space (and hence a Q-C_1 space) need not be a C_1 space.

§ .2. QUOTIENT FUZZY TOPOLOGY.

In this section we discuss another method of constructing new fuzzy topology, which can be regarded as the dual of the product fuzzy topology. At the beginning we shall deal with the quotient fuzzy topology defined by Wong on the set of classes of an equivalence relation in an fts and finally examine the generalised form of Wong's definition.

DEFINITION.3.2.1 (Wong 1974(31)). Let (X,T) be an fts. Let R be an equivalence relation defined on X. Let X/R be the usual quotient set and π the usual projection from X onto X/R. The family $\mathcal{U} = \{B : \pi^{-1}(b) \in T\}$ is a fuzzy topology for X/R and $(X/R,\mathcal{U})$ is called the quotient fts.

We have results similar to theorem 3.1.2.

THEOREM.3.2.2.(Wong 1974 (31)). Let (X,T)be an fts and $(X/R,\mathcal{U})$, the quotient fts. (1) The quotient fuzzy topology $\mathcal{U}$ is the largest fuzzy topology such that π is F-continuous.

(2) If g be a function from the quotient fts $(X/R,\mathcal{U})$to an fts (Y,S) then g is F-continuous iff $g \circ \pi$ is F-continuous.

Proof : (1) Trivial.

(2) ($\Rightarrow$) Let g be F-continuous.

Since $\mathcal{U}$ is quotient topology for X/R, so π is F-continuous and hence $g \circ \pi$ is F-continuous.

($\Leftarrow$) Let $g \circ \pi$ be F-continuous.

Let $V \in S$. So $\pi^{-1}(g^{-1}(V)) = (g \circ \pi)^{-1}(V) \in T$. Therefore by definition of quotient fuzzy topology, $g^{-1}(V) \in \mathcal{U}$. Hence g is F-continuous.

THEOREM3.2.3.(Wong 1974(31)). Let f be an F-continuous function from an fts (X,T) onto an fts (Y,S) such that f is either F-open or Fclosed Then there exists an equivalence relation R on X such that (Y,S) is homeomorphic to the quotient fts $(X/R,\mathcal{U})$.

Proof : Define a relation R on X by agreeing that x is R related to y iff $f(x) = f(y)$. Then R is an equivalence relation.

Let us denote the equivalence class of x by $[x]$.

Now define a function h from (Y,S) to $(X/R,\mathcal{U})$ as follows :

Let $y \varepsilon Y$. Then there exists $x \varepsilon X$ such that $f(x) = y$. Define $h(y)$ $h(y) = [x]$. Clearly, h is 1-1 and onto mapping.

Since, $f = h^{-1} \circ \pi$ is F-continuous, by theorem 3.2.2(2) h^{-1} is F-continuous.

If f is F-open, let Q be an open fuzzy set in X/R. Then $\pi^{-1}(Q)$ is open in (X,T) and $f(\pi^{-1}(Q)) = h^{-1}(\pi(\pi^{-1}(Q))) = h^{-1}(Q)$ is open in (Y,S)

Therefore ,h is F-continuous .

If f is F-closed , then let Q be a closed fuzzy set in X/R . Following the same argument one concludes thar h is F-continuous .

Therefore , h is a homeomorphism .

THEOREM . 3.2.4. (Wong 1974 (31)). Let π be the projection from fts (X,T) onto its quotient fts $(X/R,\mathcal{U})$. If (X,T) is C_{11} and π is F-open then the quotient fts $(X/R,\mathcal{U})$ is also C_{11} .

Proof : Let Ξ be a countable base for T . For every $B \varepsilon \Xi$, $\pi(B) \varepsilon \mathcal{U}$. Since , π is F-open , the family $\{\pi(B) : B \varepsilon \Xi\}$ forms a countable base for $\mathcal{U}$. To see this , take an element $V \varepsilon \mathcal{U}$. So $\pi^{-1}(V) \varepsilon T$.

But , Ξ is a countable base for T . Therefore, $\pi^{-1}(V)$ is the union of members of Ξ , say , $\pi^{-1}(V) = \bigcup_{\lambda \varepsilon \Lambda} B_\lambda$.

Hence, $V = \pi(\pi^{-1}(V)) = \pi(\bigcup_{\lambda \varepsilon \Lambda} B_\lambda) = \bigcup_{\lambda \varepsilon \Lambda} \pi(B_\lambda)$.

Therefore, $(X/R,\mathcal{U})$ is C_{11}.

Christoph (5) generalised Wong's definition of quotient fuzzy topology as follows :

DEFINITION.3.2.5. (Christoph 1977(5)) . Let (X,T) be an fts , Y a set &

$f : X \to Y$ asurjection . The F-quotient topology for Y is the fuzzy topology whose open fuzzy sets are $\{B : f^{-1}(B) \in T\}$. If $f : X \to Y$ is an F-continuous surjection of fts and Y has the F-quotient topology, then f is called an F-quotient map .

THEOREM .3.2.6. (Christoph 1977(5)). Let f-: $X \to Y$ be an F-continuous surjection of fts . Then f is a F-quotient map iff each fts Z and each function $g : Y \to Z$, the F-continuity of the composition $g \circ f$ implies the F-continuity of g .

Proof : ($\Rightarrow$) Let $g : Y \to Z$ be a function of fts such that $g \circ f : X \to Z$ is F-continuous . If U be an open fuzzy set in Z, then
$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.
Thus, $f^{-1}(g^{-1}(U))$ is open in X and hence $g^{-1}(U)$ is open in Y because , f is an F-quotient map. Hence , g is F-continuous .

($\Leftarrow$) Let $f : X \to Y$ be F-quotient map and $g : Y \to Z$ be such that $g \circ f : X \to Z$ is F-continuous .
To show that g is F-continuous let B be F-open in Z .
Then , $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$ is F-open in X .
Since, f is F-quotient map , a fuzzy subset C in Y is F-open iff $f^{-1}(C)$ is F-open in X. This shows that $g^{-1}(B)$ is F-open in Y which implies that g is F-continuous .

THEOREM .3.2.7. (Christoph 1977(5)). If $f : X \to Y$ is an F-continuous, F-open (F-closed) surjection of fts (X,T) to fts (Y,S) , then f is an F-quotient map .

Proof : If $B \in S$, then $f^{-1}(B) \in T$ (Since f is F-continuous).
Conversely , if U is a fuzzy set in Y such that $f^{-1}(U) \in T$.
Then $f(f^{-1}(U)) \in S$. So $U \in S$. Hence f is a F-quotient map .
The proof for F-closed map f is similar .

<u>THEOREM</u>.3.2.8.(Christoph 1977 (5)). Let $f : X \to Y$ be an F-continuous surjection of fts. If there exists an F-continuous function $g : Y \to X$ such that $f \circ g$ is the identity on Y, then f is an F-quotient map.

PROOF : Let B be a fuzzy set in Y such that $f^{-1}(B)$ is open in X. Since g is F-continuous, $g^{-1}(f^{-1}(B))$ is open in Y.

But $g^{-1}(f^{-1}(B)) = (g^{-1} \circ f^{-1})(B) = (f \circ g)^{-1}(B) = B$ as $f \circ g$ is the identity. So B is open in Y.

Hence f is F-quotient map.

<u>THEOREM</u>.3.2.9.(Christoph 1977 (5)). Let $f : X \to Y$ be an F-quotient map. A surjection $g : Y \to Z$ is an F-quotient map iff $g \circ f : X \to Z$ is an F-quotient map.

PROOF : Follows from theorem 3.2.6.

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CHAPTER 4

VARIOUS COMPACTNESS IN FUZZY SPACE

Following the introduction of fuzzy sets by Zadeh(32), Chang(4) developed the theory of fuzzy topological space. In the same paper Chang introduced fuzzy compactness. In 1970, Goguen pointed out the deficiencies in Chang's definition of compactness. Then Weiss introduced a new notion of compact--ness. In 1974, Lowen gave a new definition of compactness and he was able to obtain only a finite Tychonoff theorem. In a second paper Lowen(197 gave another definition and changed the definition of fts and was able to prove Tychonoff theorem. In 1979, Warren aGanter and Steinlage proposed a new definition and introduced α-coppactness. In 1981, Sarkar(26) defined proper compactness and proved certain results of proper compactness in Hausdorff space.

§ 1. FUZZY COMPACTNESS AS DEFINED BY CHANG.

In this section we contribute to the development of fuzzy compactness as introduced by Chang(4).

DEFINITION 4.1.1. (Chang, 1968 (4)). Let (X,T) be a fts. A family ≋ of fuzzy sets is a cover of a fuzzy set B iff $B \subset \cup \{A : A \in ≋\}$.

It is an open cover iff each member of ≋ is T-open.

A subcover of ≋ is a subfamily which is also a cover.

DEFINITION 4.1.2.(Chang 1968 (4)).A fts (X, T)is said to be compact iff each open cover has a finite subcover.

DEFINITION 4.1.3.(Wong, 1973(31)).A fts (X,T)is said to be countably compac iff every countable open cover has a finite subcover.

DEFINITION.4.1.4.(Wong,1973(31)). A fts (X,T) is called Lindelof iff every open cover of X has a countable subcover.

THEOREM 4.1.5.(Wong,1973(31)).If a fts is C_{11},then compactness and countable compactness are equivalent.

Proof;(⟹) Obvious.

($\Leftarrow$) Let $\mathcal{A}=\{A_\lambda ; \lambda\epsilon\Lambda\}$ be any open cover of X. Since, X is C_{11}. So there exists a countable base $\Xi=\{B_n\}$, n=1,2,3,....for T. Therefore each $A_\lambda \epsilon \mathcal{A}$ can be expressed as $A=\bigcup_{k=1}^{\lambda_o} B_{\lambda_k}$ (say) where λ_o may be infinity. Therefore $\Xi_o=\{B_{\lambda_k}\}\lambda\epsilon\Lambda, 1\leq k\leq\lambda_o$, forms a countable open cover of X, because $\Xi_o \subset$ countable family Ξ.

But X is countable compact. There exists a finite subcover $\Xi_1 \subset \Xi_o$

Since each member of Ξ_1 is contained in a member A_λ, these A_λ,s form a finite subfamily of $\mathcal{A}$ and is a cover of X.

THEOREM 4.1.6 (Wong 1973 (31)): If a fts (X,T) is C_{11}, it is also Lindelöf.

Proof; Let $\mathcal{A}=\{A_\lambda\}, \lambda\epsilon\Lambda$ be an open cover iof X. By assumption, T has a countable base $\Xi=\{B_n\}$ n=1,2,... Since $\mathcal{A}\subset T$ $\therefore$ Each A_λ is the union of members of Ξ, say, $A_\lambda=\bigcup_{k=1}^{\lambda_o} B_{\lambda_k}$ where λ_o may be infinity.

$\therefore$ $\Xi_o=\{B_{\lambda_k}\}$ $\lambda\epsilon\Lambda$, $1\leq k\leq\lambda_o$, forms an open cover of X and Ξ_o is countable as it is a subfamily of Ξ .

THEOREM 4.1.7 (Wong 1973 (31)). Let f be an F-continuous function from a compact (countably compact) fts (X,T) onto a fts (Y,U). Then (Y,U) is compact (countably compact).

Proof. Let Ξ be an open cover (countable open cover) of Y

Since $(\bigcup_{B\epsilon\Xi} f^{-1}(B))(x)= \sup_{B\epsilon\Xi}\{[f^{-1}(B)](x)\}= \sup_{B\epsilon\Xi}(B(f(x)))= 1 \ \forall \ x\epsilon X$, the family of all fuzzy sets of the form $f^{-1}(B)$, $B\epsilon\ \Xi$ is an open cover of X which has a finite subcover. However f is onto. Therefore $f(f^{-1}(B))=B$ for any fuzzy set zB in Y . Hence the family of images of members of the subcover is a finite subfamily of Ξ which covers Y. Consequently Y is compact (countably compact).

Using the same argument we have the following

THEOREM 4.1.8.(Wong 1973(31)).Let f be an F-continuous function from a Lindelöf fts (X,T) onto a fts (Y,U). Then Y is Lindelöf.

DEFINITION.4.1.8.(Chang 1968(4)).A family Δ of fuzzy sets in a set X has the finite intersection property iff the intersection of the members of each finite subfamily of Δ is nonempty.

THEOREM.4.1.9.(Chang 1968(4)).A fts is compact iff each family of closed fuzzy sets which has the finite intersection property has a nonempty intersection.

Proof: If Δ is a family of fuzzy sets in a fts (X,T), then Δ is a cover of X iff $\cup\{A: A\in\Delta\} \overset{c}{=} X$ or iff $\{\cup\{A: A\in\Delta\}\}^c = X^c = \underline{0}$

or iff $\cap\{A^c : A\in\Delta\} = \underline{0}$ (by De Morgan's Laws)

Hence the fuzzy space X is compact iff each family of open fuzzy sets in X such that no finite subfamily covers X, fails to be a cover and this is true iff each family of closed fuzzy sets which possesses the finite intersection property has a nonempty intersection.

Next,we present a characterisation of compactness and countable compactness peculiar to fuzzy topological spaces.

Let $\Delta=\{A_\lambda\}$ $\lambda\in\Lambda$ be a cover of X $\therefore$ $\sup\limits_{\lambda\in\Lambda}\{A_\lambda(x)\}=1$ for all $x\in X$

For any $0<\delta<1$ and for any $x\in X$ there exists a fuzzy set A_λ such that $A_\lambda(x)\geq 1-\delta$. Ateach point $x\in X$ select one such A_λ and group together all points x with the same A_λ.

Let $\Gamma_{\lambda,\delta}$ denote the set of all such x's. For fixed δ $\{\Gamma_{\lambda,\delta}\}$ form a partition of X called a δ-partition by Δ

It is to be noted that the partition depends on the initial choice of A_λ's

If in addition, for any $x\in X$ there exists a A_λ such that $A_\lambda(x)=1$, then group all points x with the same A_λ and denote it by $\{\Gamma_{\lambda,0}\}$. Δ is then said to have a 0-partition of X

Now we have the following charecterisation theorem

THEOREM.4.1.10.(Wong 1973(31)). A fts (X,T) is compact (countably compact iff each open cover (countable open cover) has a finite 0-partition of X.

Proof: $(\Rightarrow)$ Let $\Delta=\{A_\lambda\}$ $\lambda\in\Lambda$ be an open cover (countable open cover) of X.

By assumption, it has a finite subcover $\Delta_0=\{A_k\}$ $k=1,2,\ldots,n$. Since $\mathrm{Max}\{A_1(x),A_2(x),\ldots,A_n(x)\} = 1$ for $\forall$ $x\varepsilon X$, one can construct a 0-partition by Δ_0, which is finite because Δ_0 is a finite family. As Δ_0 is a subfamily of Δ, thus a 0-partition by Δ_0 is also a 0-partition by Δ.

($\Leftarrow$) Suppose that Δ has a finite 0 partition $\{\Gamma_{k,0}\}$ $k=1,2,\ldots,n$. Let A_k be the fuzzy set defining $\Gamma_{k,0}$. Clearly $\{A_k\}$ $k=1,2,\ldots,n$ is a finite subcover of Δ.

As a consequence of this we have

COROLLARY.4.1.11.(Wong 1973 (31)).If there exists an open cover (countable open cover)Δ of X and a point $x\varepsilon X$ such that $A_\lambda(x)<1$ for all $A_\lambda\varepsilon\Delta$ then (X,T) is not compact (countably compact)

Let us turn our attention to Lindelöf space.

THEOREM.4.1.12.(Wong 1973 (31)). A fts (X,T) is Lindelöf iff each open cover has a countable δ-partition of X for all δ such that $0<\delta<1$

Proof: ($\Rightarrow$) Let Δ be an open cover. Then Δ has a countable subcover $\Delta_0=\{A_k\}$, $k=1,2,\ldots$. For each $0<\delta<1$ one can construct a δ-partition of X by Δ_0. Such a partition is clearly countable as Δ_0 is countable. A δ-partition of X by Δ_0 is also a δ-partition by Δ since Δ_0 is a subfamily of Δ.

($\Leftarrow$) Let Δ be an open cover. For each $0<\delta<1$ let $\{\Gamma_{\lambda,\delta}\}$ $\lambda\varepsilon\Lambda(\delta)$ be a countable δ-partition of X by Δ. Let $\Gamma_{\lambda,\delta}$ be defined by the member $A_{\lambda,\delta}$ of Δ. Let $\delta=\frac{1}{n}$, $n=2,3,4,\ldots$. Then the family of fuzzy sets $\{A_{\lambda,\delta}\}$ $\lambda\in\Lambda(\delta)$; $\delta=1/n$, $n=2,3,4,\ldots$ forms a countable subcover of Δ.

Next we consider the following result on qutient space

THEOREM.4.1.13.(Wong 1974(31)). If a fts (X,T) is compact (countably compact) then the quotient fts (X/R,U) is also compact (countably compact).

Proof : Only point to be noted is the following : -

" The projection π : X $\to$ X/R is onto & F - continuous .".

The result now follows from theorem 4.1.7.

Next, we extend the concept of sequential compactness and semicompactness in fuzzy set theory and in this connection follow the definitions of convergence and clustering as introduced in 1.7.0 (4).

DEFINITION. 4.1.14. (Wong 1973 (31)). Afts (X,T) is sequentially compact iff every sequence of fuzzy sets has a convergent subsequence .

DEFINITION. 4.1.15. (Wong 1973 (31)) Afts (X,T) is semi-compact iff every sequence of fuzzy sets hassa cluster fuzzy set .

THEOREM. 4.1.16. (Wong 1973 (31)). In a fts (X,T) every sequence of fuzzy sets $\{A_n\}$, n=1,2,.......has a limit , which may not be unique . Consequently , every fts is both sequentially compact & semicompact .

Proof : Let A be the fuzzy set defined by $A = \bigcup_n A_n$. Then A_n converges to A because any open set containing A will contain all A_n , n=1,2,... Furthermore , any fuzzy set containing A is also a limit of $\{A_n\}$.

At the end of this section we present the finite form of Tychonoff theorem on the product of compact spaces . It is well known that the classical theorem of Tychonoff is unquestionably the most important single theorem of general topology . But the deficiency in the present definition of compactness forces us to extend the Tychonoff theorem in finite form only .

THEOREM . 4.1.17. (Wong 1974)(31)) . Let $\{(X_n, T_n)\}$ n=1,2,.....,m be a finite family of compact (countably compact) fts's. Then the product fts (X,T) is also compact (countably compact).

Proof : It suffices to show that the theorem is true for the case m=2. In this case the product topology can be charracterised as follows

$T=\{A_1\times A_2 : A_n\in T_n,\ n=1,2\}$ where $A_1\times A_2$ is a fuzzy set in X defined
T=spa
by $(A_1\times A_2)(x_1,x_2)=\min\{A_1(x_1),A_2(x_2)\}$.

We shall prove this theorem for the case of compactness only since the case of countable compactness is similar .

Let $\Delta=\{B_\lambda\}\ \lambda\in\Lambda$ be an open cover of the product fts (X,T).

Let $B_\lambda=A_1^{(\lambda)}\times A_2^{(\lambda)}$ $A_1^{(\lambda)}\in T_1$ & $A_2^{(\lambda)}\in T_2$ for all $\lambda\in\Lambda$.

Let ybe any point in X_2. Consider the subset $S_y=X_1\times\{y\}$ of X.

For any one $1\geq\delta>0$, let $V_{y,\delta}$ be the subfamily of Δ such that $A_1^{(\lambda)}\times A_2^{(\lambda)}\in V_{y,\delta}$ iff $A_1^{(\lambda)}(x_1)>1-\delta$ for atleast one point $x_1\in X_1$
$A_2^{(\lambda)}(y)>1-\delta$.

This is actually an open set of T. Clearly, $V_{y,\delta}$ forms an open cover of the subset S_y. To see this , only note that if $(x,y)\in S_y$, by definition of Δ, there exists a countable subfamily $\{A_1^{(k)}\times A_2^{(k)}\}$ $k=1,2,\ldots$ of Δ such that $\lim_{k\to\infty}(A_1^{(k)}\times A_2^{(k)})(x,y)=1$.Hence for k large $A_1^{(k)}\times A_2^{(k)}\in V_{y,\delta}$.

Without loss of generality , one can assume that $\{A_1^{(k)}\times A_2^{(k)}\}$ $k=1,2,\ldots$ is a subfamily of $V_{y,\delta}$. Do this for all $(x,y)\in S_y$, one concludes that $V_{y,\delta}$ contains a cover of S_y. Let $W_{y,\delta}=\{A_1^{(\lambda)}: A_1^{(\lambda)}\times A_2^{(\lambda)}\in V_{y,\delta}\}$ Then $W_{y,\delta}$ is an open cover of (X_1,T_1). Since for any $x\in X_1$, there exists a countable subfamily $\{A_1^{(k)}\times A_2^{(k)}\},k=1,2,\ldots,$of $V_{y,\delta}$ such that $\lim_{k\to\infty}(A_1^{(k)}\times A_2^{(k)})(x,y)=1$; Consequently, $\lim_{k\to\infty}(A_1^{(k)}(x))=1$.

By compactness of (X_1,T_1), there exists a finite subcoverof $W_{y,\delta}$ say $Z_{y,\delta}$ .To each $A_1^{(\lambda)}\in Z_{y,\delta}$ select one $A_2^{(\lambda)}$ such that $A_1^{(\lambda)}\times A_2^{(\lambda)}\in V_{y,\delta}$. The finite family $\{A_1^{(\lambda)}\times A_2^{(\lambda)}\}$ thus constructed willbe called $H_{y,\delta}$ and the finite family of the corresponding $A_2^{(\lambda)}$'s will be called $G_{y,\delta}$. Members of $G_{y,\delta}$ are T_2-open fuzzy sets . let their intersection be $D_{y,\delta}$. Then $D_{y,\delta}$ is open in (X_2,T_2). Do this for all $y\in X_2$ and

for all $\delta\varepsilon]0,1]$. Clearly the family $\{D_{y,\delta}\}$ $y\varepsilon X_2$, $\delta\varepsilon]0,1]$ forms an open cover of (X_2,T_2). By compactness of (X_2,T_2), there exists a finite subcover , say $\{D_{y_\lambda,\delta_\lambda}\}$ $\lambda=1,2,\ldots,p$.
Finally $\{H_{y_\lambda,\delta_\lambda}\}$ $\lambda=1,2,\ldots,p$,forms a finite subcover of Δ . It is finite , because it is a finite collection of finite families .
To see it is a cover note that for any $(x,y)\varepsilon X$, there exists a $D_{y_\lambda,\delta_\lambda}$ such that $D_{y_\lambda,\delta_\lambda}(y)=1$. therefore $A(y)=1$ for all $A\varepsilon G_{y_\lambda,\delta_\lambda}$
On the other hand , there existd $B\varepsilon Z_{y_\lambda,\delta_\lambda}$ such that $B(x)=1$.
Select a member of $H_{y_\lambda,\delta_\lambda}$ With B as the first co-ordinate , say $B\times B_0$. Then $(B\times B_0)(x,y)=1$,since $B_0\varepsilon G_{y_\lambda,\delta_\lambda}$.

Finally we construct a counter example to show that the product of countable family of compact (countably compact) fts's is not compact (countably compact). We need the following definition.

DEFINITION. 4.1.18. (Wong 1974(31)). Let X be a space of points . Let Q be a sub set of X . A family Δ of fuzzy sets is a cover Q iff S $\sup\limits_{A\varepsilon\Delta} A(x)=1$ for all $x\varepsilon Q$.

COUNTEREXAMPLE. 4.1.19.(Wong 1974(31)). Let Y be any space of points. Let n be any positive integer . Let A_n be the fuzzy set in Y defined by $A_n(y)=1-1/n$ for all $y\varepsilon Y$.
Let $X_n=Y$ and $T_n=\{\underline{0}, A_n, Y\}$. Then $(X_n,T_n)$is compact (countably compact) fts since X_n is the only open cover of X_n. However , the product fts of the countable family $\{(X_n,T_n)\}$ $n=1,2,\ldots$ is not compact (countably compact) .
To see this , note that for the fuzzy set $\pi_n^{-1}(A_n)$ we have $(\pi_n^{-1}(A_n)(x)=1-1/n$ for all $x\varepsilon X=\prod\limits_{n=1}^{\infty} X_n$.
By definition of product fts , the family $\Xi=\{\underline{0}$, X, $\pi_n^{-1}(A_n)$, $n=1,2,..\}$ is used to generate the product fuzzy topology T by first taking the finite intersections and then the unions of these intersections

Clearly the product fuzzy topology thus generated is exactlyΞ itselfe. The family $\{\pi_n^{-1}(A_n)\}$ n=1,2,... is an open cover (countable open cover) of (X,T) which has no finite subcover.

§.2. LOCAL COMPACTNESS AS DEVELOPED BY WONG

Let us turn our attention to local compactness with respect to fuzzy point of a fts. Here we follow the definition of fuzzy point and belongingness as defined by wong.

DEFINITION4.2.1.(Wong 1974(31)). A fts (X,T) is said to be local compact iff for every fuzzy point x in X there exists a member A in T such that (1) $x \varepsilon A$ and (2) A is compact.

The next result shows the ramification of fuzzy topology from general topology.

THEOREM.4.2.2.(Wong 1974(31)). A discrete fts (X,T)is not locally compact.

Proof: Note that the discrete fuzzy topology comtains all fuzzy sets.

The proofs of the following results are omitted. For reference one may see (31).

THEOREM.4.2.3.(Wong 1974(31)). Let f be a F-continuous function from a locally compact fts (X,T) onto a fts (Y,U). If f is F-open then (Y,U) is locally compact.

THEOREM.4.2.4.(Wong 1974(31)).Let $\{(X_\lambda,T_\lambda)\}$ $\lambda=1,2,.....,m$ be a finite family of locally compact fts's. Then the product fts (X,T) is also locally compact.

THROREM.4.2.5.(Wong 1974(31)). If (X,T) is a locally compact fts and the projection $\pi: X/R \to X$ is F-open, then the quotient fts (X/R,U) is locally compact.

§.3. C* ?LOCAL COMPACTNESS AND QUOTIENTS.

According to definition 4.2.1 local compactness at a fuzzy point x means that there exists a compact open fuzzy set containing x . This definition is more restricted than the usual definition of local compactness in ordinary topology. SO we define local compactness in more natural way. We shall follow the definitions of fuzzy point and belongingness as given by Wong(31).

DEFINITION.4.3.1.(Christoph 1977(5)). A fts (X,T) is locally compact if for each fuzzy point x_λ in X there is a compact fuzzy set A and a fuzzy set B in T such that $x_\lambda \in B \subset A$.

The local compact space due to Christoph will be termed as C^* local compact space.

If we consider X=]0,1[with the usual topology, then the associated fuzzy topology which has the characteristic functions of open sets as its open fuzzy sets is not locally compact according to definition 4.2.1., but it does ~~not~~ satisfy 4.3.1.

Since Wong's definition 4.2.1 agrees with 4.3.1.for discrete space so we conclude that no discrete fts is C^* locally compact.

DEFINITION.4.3.2.(Christoph 1977(5)). Let $f:X\to Y$ be a F-closed, F-continuous surjection of fts. Then f is said to be F-perfect if $f^{-1}(y_\mu)$ is compact for each fuzzy point y_μ in Y.

NOTE4.3.3. One deviation of fuzzy topology from ordinary topology is the fact that a fuzzy point is not necessarily compact. However, if f:X $f:X\to Y$ is a F-perfect map of fts, then each fuzzy point in Y is compact because $f(f^{-1}(y_\mu)) = y_\mu$.

THEOREM4.3.4.(Christoph 1977(5)). Let $f:X\to Y$ be a F-perfect map of fts. If X is C^* locally compact then so is Y.

Proof: Let $q=y_\mu$ be a fuzzy point in Y. For each fuzzy point $p=x_\lambda$ in

$f^{-1}(q)$ there exists a compact fuzzy set A_p and an open fuzzy set B_p such that $p \in B_p \subset A_p$ ($\because$ X is C^* locally compact).

Now $\Delta = \{B_p : p \in f^{-1}(q)\}$ is an open cover of the fuzzy set $f^{-1}(q)$.

Since f is F-perfact. Therefore $f^{-1}(q)$ is compact.

So $\triangle$ has a finite subcover $B_p^1, B_p^2, \ldots\ldots, B_p^m$. Now $U = \bigcup_{n=1}^{m} A_p^n$ is compact & $f^{-1}(q)$ is a subset of U. Therefore, f(U) is compact (theorem 4.1.8) and $q \in f(U)$.

Let $V = \bigcup_{n=1}^{m} B_p^n$. Then $(f(V^c))^c = W$ is open in Y and $q \in W$

Also $f^{-1}(W) = f^{-1}((f(V^c))^c) \subset (V^c)^c = V \subset U$.

Thus $q \in W \subset f(U)$. Hence Yis C^* locally compact.

<u>DEFINITION</u>.4.3.5.(Christoph 1977(5)). A fts (X,T) is called a F-k space iff a fuzzy set A is F-closed whenever $A \cap C$ is closed for each compact fuzzy set C in X.

<u>THEOREM</u>.4.3.6.(Christoph 1977(5)).If $f: X \to Y$ is an F-quotient map of fts and X is C^* locally compact then Y is an F-k space.

Proof: let A be a fuzzy set in Y such that $A \cap C$ is closed for each compact fuzzy set C in Y. Assume that A is not closed.

Then $f^{-1}(A)$ is not closed in X. $\therefore (f^{-1}(A))^c$ is not F-open.

Therefore, there exists a fuzzy point p in $(f^{-1}(A))^c$ such that for each open fuzzy set V with $p \in V$ it is true that $V \cap f^{-1}(A) \neq \underline{0}$

Since X is C^* locally compact and p is a fuzzy point in X.

So there exists a compact fuzzy set U and an open fuzzy set B such that $p \in B \subset U$. Again f(U) is compact $\therefore A \cap f(U)$ is closed.

If W is an open fuzzy set containing f(p) then $B \cap f^{-1}(W)$ is an open fuzzyset containing p. Consequently $B \cap f^{-1}(W) \cap f^{-1}(A) \neq \underline{0}$ and thus $U \cap f^{-1}(W) \cap f^{-1}(A) \neq \underline{0}$. Therefore, $f(U) \cap W \cap A \neq \underline{0}$ & f(p) is not in $A \cap f(U)$

This contradicts the fact that $A \cap f(U)$ is closed.

Hence A is closed in Y and thus Y is an F-k space.

<u>COROLLARY</u>.4.3.7.(Christoph 1977(5)). A C^*locally compact fts is an F-k space.

Proof: Only observe that the identity fuction is an F-quotient map.

<u>NOTE</u>.4.3.8. In case of ordinary topology, we know that if a sequence $\{x_n\}$ converges to x then ($\bigcup_n \{x_n\}$) $\cup$ $\{x\}$ is a compact set. But this is not necessarily true in fuzzy topological space.

Consider the following example:

Let $X=\{x\}$ be a space of points.with atleast two ordinary points.

Let $0 < \alpha < 1$ be given. In the family of all fuzzy sets defined on X consider the subfamily $T_\alpha=\{A_\beta\}$ where $\beta\epsilon[0,\alpha\,]$or $\beta=1$ and A_β is the fuzzy set defined by $A_\beta(x) = \beta$ for all $x \epsilon X$. Clearly (X,T_α) is an fts. This is called a semidiscrete fts. Choose a fuzzy point p with support x_o and value α.Take $x_1 \neq x_o$ in X. If q be the fuzzy point with support x_1 end value $\alpha/2$, then the constant sequence $\{q\}$ converges to p.

But $\{q\} \cup \{p\}$ is not compact.

Thus we make the following definition:

<u>DEFINITION</u>.4.3.9.(Christoph 1977(5)).Affts (X,T) is s-compact iff for each sequence of fuzzy points p_n converging to p, the fuzzy set $(\bigcup_n p_n) \cup p$ is compact.

This property is critical in considering whether C_1spaces are F-k spaces.

<u>THEOREM</u>.4.3.10.(Christoph 1977(5)). A C_1 s-compact fts (X,T) is an F-k space.

Proof: Let A be a fuzzy set in X which is not closed but $A\cap C$ is closed for each compact fuzzy set C.

Then there is a fuzzy point p in A^c such that for each popen fuzzy set U with p in U, it is true that $U\cap A \neq \underline{0}$

Therefore, there is a sequence of fuzzy points p_n converging to p with each $p_n \epsilon$ A.

Now $V = (\bigcup_{n} p_n) \cup p$ is compact because (X,T) is s-compact. ∴ $V \cap A$ is closed. But p is not in $V \cap A$ and every open fuzzy set U with p in U is such that $U \cap V \cap A \neq \underline{0}$. This contradicts the closedness of $V \cap A$. Therefore X is an F-k space.

Now we introduce the following concept.

DEFINITION.4.3.11.(Christoph 1977 (5)). An fts (X,T) is called an F-k^* space if a fuzzy set A in X is closed whenever $f^{-1}(A)$ is closed for each F-continuous function $f : C \to X$ with C as a compact fts.

THEOREM.4.3.12.(Christoph 1977 (5)). If $f : X \to Y$ is an F-quotient map and X is an F-k^* space, then Y is an F-k^* space.

THEOREM.4.3.13.(Christoph 1977 (5)). If X is an F-k^* space, then X is an F-quotient image of a C^* locally compact fts.

THEOREM.4.3.14.(Christoph 1977 (5)). An F-k^* space is an F-k space.

The proofs of these theorems are omitted.

§.4. N-COMPACTNESS.

In this section we introduce a new fuzzy compactness defined with the help of fuzzy nets. A fuzzy net $S = \{S(n) : n \in D\}$ is a function $S : D \to T_X$ where D is a directed set with order relation $\geqslant$ and T_X is the collection of all the fuzzy points in X.

In this section we follow the definitions of fuzzy point and 'belongingness' given in (21) by Ming and Ming.

DEFINITION.4.4.1.(Wang Guojun 1983 (34)). Let (X,T) be an fts, p a fuzzy point and C a closed fuzzy set in X. Then C is called a Remoted-neighborhood, or briefly, R-nbhd of p , if $p \notin C$

DEFINITION.4.4.2.(Wang Guojun 1983 (34)). Let (X,T) be an fts and A a fuzzy set in X. A closed fuzzy set C is called an R-neighborhood (or R-nbhd) of A if for each crisp point $x \in X$ satisfying $A(x)=\alpha>0$

C is an R-nbhd of x_α.

It is easy to verify that the intersection of arbitrarily many R-nbhds of a fuzzy point p and the union of finite number of R-nbhds of p are R-nbhds of p .

DEFINITION.4.4.3.(Wang Goujun 1983 (34)). A fuzzy point p is called an adherence point of a fuzzy set A if for each R-nbhd P of p we have $A \not\subset P$.

It is not difficult to verify that this definition is equivalent to definition 2.3.19 .

DEFINITION.4.4.4.(Wang Goujun 1983 (34)). A fuzzy point p is called a limit point of a fuzzy net $S = \{S(n) : n \in D\}$ (or S converges to p, in symbols $S \to p$) if for each R-nbhd P of P we have eventually $S(n) \not\leq P$

DEFINITION.4.4.5.(Wang Goujun 1983 (34)). A fuzzy point p is called a cluster point of a fuzzy net $S = \{S(n) : n \in D\}$, in symbols $S \infty p$, if for each R-nbhd P of p we have frequently $S(n) \not\leq P$.

DEFINITION.4.4.6.(Wang Goujun 1983 (34)). Let (X,T) be an fts. For a fuzzy point x_α, we call α the value of x_α, in symbols $V(x_\alpha) = \alpha$. For a fuzzy net $S = \{S(n) : n \in D\}$, let α_n be the value of S(n); then we obtain a crisp net $\{\alpha_n : n \in D\}$ in the half-open interval]0,1] . It will be called the value net of S and denoted by V(S). If V(S) converges to a real number $\alpha \in]0,1]$, then we say that S is an α-net. Specifically, if $\alpha_n = \alpha$ holds for all $n \in D$, then we say that S is a constant α-net.

EXAMPLE.(Wang Goujun (34)). Put $D = \mathbb{N}$. For each $n \in D$, let $S(n)(x) = 1/2 + \sin((-1)^n/2n)$, if $x = 1/n$, & $S(n)(x) = 0$ otherwise. Then $S = \{S(n) : n \in D\}$ is a $\frac{1}{2}$-net.

From now on, for an fts (X,T) we write T' for the set $\{A^c : A \in T\}$.

DEFINITION.4.4.7.(Wang Goujun 1983 (34)). Let (X,T) be an fts and $\Xi \subset T'$, $\Sigma \subset T'$. Then Ξ is called a closed base for T' if for each closed fuzzy set is the intersection of sets of Ξ ; Σ is called a closed subbase for T'if the finite union of sets of Σ constitute a closed base Ξ for T'(we call it the closed base generated by Σ).

The proofs of the following lemmas are simple and hence are left.

LEMMA.4.4.8.(Wang Guojun 1983 (34)). Let (X,T) be an fts and $\Xi \subset T'$; then Ξ is a closed base (subbase) for T' iff Ξ' is an open base(subbase) for T.

LEMMA.4.4.9.(Wang Guojun 1983 (34)). Let (X,T) be an fts and Σ a closed subbase for T'; then a fuzzy net $S = \{S(n) : n \in D\}$ converges to a fuzzy point p iff for each $P \in n(p) \cap \Sigma$ we have eventually $S(n) \not\leq P$. where n(p) is the collection of all R-nbhds of p.

Now we introduce the notion of N-compactness.

DEFINITION.4.4.10.(Wang Guojun 1983 (34)). Let (X,T) be an fts and A a fuzzy set in X. The set A is called N-compact if each α-net , ($\alpha \in]0,1]$)contained in A has atleast in A a cluster point with value α . Specifically, when A = $\underline{1}$ is N-compact, we call (X,T) an N-compact fts.

The property of N-compactness is hereditary with respect to closed fuzzy subsets.

THEOREM.4.4.11.(Wang Guojun 1983 (34)). Let (X,T) be an fts. If A is N-compact, then each closed fuzzy subset contained in A is N-compact.

PROOF : Assume that $B \in T'$ and $B \subset A$;$S = \{S(n) : n \in D\}$ is an α-net in B ($\alpha \in]0,1]$).

Since S is also in the N-compact set A, there exists in A a cluster point x_α^α of S.We shall show that $x_\alpha^\alpha \in B$.

If $x_\alpha^\alpha \not\leq B$, then it follows from $B \in T'$ that B is an R-nbhd of x_α^α.

But $S(n) \in B$ holds for all $n \in D$; this contradicts the fact that x_α^α

is a cluster point of S and B is an R-nbhd of x_α^α .

Now we state a few theorems and some corollaries and omit the proofs.

THEOREMS4.4.12.(Wang Guojun 1983 (34)). A fuzzy set A in an fts (X,T) is N-compact iff for each α-net ($\alpha \in]0,1]$) contained in A has a fuzzy subnet converging to some fuzzy point with value α in A.

THEOREM.4.4.13.(W.G 1983(34)). A fuzzy set A in an fts (X,T) is N-compact iff each fuzzy net S contained in A has a cluster point $x_\alpha \in A$ with value α whenever its value net V(S) has the crisp cluster point $\alpha \in]0,1]$.

THEOREM.4.4.14.(W.G 1983(34)). Let (X,T) be an fts and A an N-compact set in X. Then there exists a crisp point $x \in X$ so that $A(x) = \mathrm{Sup}\{A(t) : t \in X\}$.

COROLLARY.4.4.15.(W.G 1983(34)). Each closed fuzzy set, in an N-compact fts (X,T), as a function has a maximum and each open fuzzy set as a function has a minimum.

COROLLARY.4.4.16.(W.G 1983(34)). Let (X,T) be a compact crisp topological space. Then the lower semicontinuous(upper semicontinuous) function from (X,T) into I has a minimum(maximum).

THEOREM.4.4.17.(W.G 1983(34)). Let (X_λ, T_λ) ,$\lambda \in \Lambda$ be a collection of fts and (X,T) their product fts. Then (X,T) is N-compact iff (X_λ, T_λ) is N-compact for each $\lambda \in \Lambda$.

§.5. α-COMPACTNESS AND α^*-COMPACTNESS.

In this section we introduce the concept of α-compactness and α^*-compactness and study various properties.

DEFINITION.4.5.1.(Gantner,Steinlage,Warren 1978(28)).Let (X,T)be an fts, $\alpha \in [0,1]$. A collection $\Gamma \subset T$ is called an α-shading(resp.α^*-shading) of X if for each $x \in X$, there exists $U \in \Gamma$ with $U(x) > \alpha$ (resp.$U(x) \geq \alpha$).

DEFINITION.4.5.2.(G , S & W 1978(28)). A subcollection Ω of an α-shading (resp.α^*-shading) Γ of X which is also an α-shading (resp.α^*-shading)is called an α-subshading (resp. α^*-subshading).

DEFINITION.4.5.3.(G, S, & W 1978(28)).A fts (X,T) is called α-compact (resp. α^*-compact) if each α-shading (resp. α^*-shading) of X has a finite α-subshading (resp. α^*-subshading).

It is clear from these definitions that any finite fts is α-compact and α^*-compact for any $\alpha \in [0,1]$
Also every fts is 1-compact and 0^*-compact.

Example : Let X be any infinite set. Let $\alpha \in]0,1[$. For each $a \in X$ define U_a^α , $V_a^\alpha : X \to I = [0,1]$ by $U_a^\alpha(x) = \alpha$ if $x = a$ and $U_a^\alpha(x) = 0$ if $x \neq a$
and $V_a^\alpha(x) = 1$ if $x = a$ and $V_a^\alpha(x) = 0$ if $x \neq a$

Let T_0^α denote the fuzzy topology on X generated by $\{U_a^\alpha : a \in X\}$
Then (X,T_0^α) is β-compact $\Leftrightarrow \alpha \leq \beta \leq 1$ & β^*-compact $\Leftrightarrow \beta = 0$ or $\alpha < \beta \leq 1$
Let T_1^α denote the fuzzy topology on X generated by $\{V_a^\alpha : a \in X\}$
Then (X,T_1^α) is β-compact $\Leftrightarrow \beta = 1$ or $0 \leq \beta < \alpha$ & β^*-compact $\Leftrightarrow 0 \leq \beta \leq \alpha$
In particular, (X,T_0^α) is α-compact but not α^*-compact & (X,T_1^α) is α^*-compact but not α-compact.
Moreover if $0 < \alpha < \beta < 1$ and if $\gamma \in [0,1]$ satisfies $\alpha < \gamma < \beta$ then the following hold:
(X,T_0^γ) is β-compact & β^*-compact but is neither α-compact nor α^*-compact. (X,T_1^γ) is α-compact & α^*-compact, but is neither β-compact nor β^*-compact .
Similarly if $0 < \alpha < 1$ then (X,T_0^α) is 1^*-compact but not α^* -compact and is not β-compact when $0 \leq \beta < \alpha$ and (X,T_1^α) is 0-compact but is neither α-compact nor β^*-compact when $\alpha < \beta \leq 1$.

[illegible]

DEFINITION.4.5.4. (G,S& W 1978 (28)) . Let $\alpha \in I$. A collection ξ of fuzzy sets on X is called α-centered if for all $C_1, C_2, \ldots, C_m \in \xi$ there exists $x \in X$ with $C_n(x) \geq 1-\alpha$ for all $n = 1,2,\ldots,m$.
Also ξ is α^*-centered if for all $C_1, C_2, \ldots, C_m \in \xi$ there exists $x \in X$ with $C_n(x) > 1-\alpha$ for all $n = 1,2,\ldots,m$.

Now we state some results.

THEOREM.4.

Now we state some results .

THEOREM. 4.5.5. (G,S&W 1978(28)).Let (X,T) be a fts

(1) (X,T) is α-compact $\Leftrightarrow$ for every α-centered system Γ of T-closed fuzzy setsin X, there exists $x \in X$ such that $F(x) \geq 1-\alpha$ for all $F \in \Gamma$

(2) (X,T) is α^*-compact $\Leftrightarrow$ for every α^*-centered system Γ of T-closed fuzzy sets in X, there exists $x \in X$ such that $F(x) > 1-\alpha$ $\forall$ $F \in \Gamma$

COROLLARY.4.5.6. G,S&W 1978 (28)) Let (X,T) be a fts .

(1) (X,T)isα-compact $\Leftrightarrow$ for every α-centered system Γ of fuzzy sets in X, there exists $x \in X$ such that $\overline{C}(x) \geq 1-\alpha$ for all $C \in \Gamma$

(2) (X,T) is α^*-compact $\Leftrightarrow$ for every α^*-centered system Γ of fuzzy sets in X there exists $x \in X$ such that $\overline{C}(x) > 1-\alpha$ for all $C \in \Gamma$

THEOREM.4.5.7.(G,S&W1978 (28)).Let F be a closed crisp subset of the fts (X,T) . (1) If X is α-compact then F is α-compact as a subspace of X. (2) If X is α^*-compact then F is α^*-compact as a subspace of X.

THEOREM.4.5.8.(G,S&W 1978 (28)). Let (X,T) & (Y,U) be two fts's . Let f: $X \rightarrow Y$ be an F-continuous map . (1) If X is α-compact then f(X) is α-compact as a subspace of Y. (2) If X is α^*-compact then f(X) is α^*-compact as a subspace of Y .

COROLLARY. 4.5.9. (G,S&W 1978(28)). every quotient space of an α-compact (resp. α^*-compact) fts is again α-compact (resp. α^*-compact).

THEOREM.4.5.10. (G,S&W 1978(28)). (Alexander subbase theorem) Let Σ be a subbase for the fuzzy topology T on a set X and let $\alpha \in I$. If every α -shading of X consisting of members of Σ has a finite α-subshading , then (X,T) is α-compact .

THEOREM.4.5.11.(G ,S & W 1978(28)).The product fts (X,T) of the family $\{(X_\lambda,T_\lambda)\}\lambda\in\Lambda$ of nonempty fts is α-compact iff for each $\lambda\in\Lambda$ (X_λ,T_λ) is α-compact where $\alpha\in I$.

By means of examples, now we shall show, how the concept of α^*-compactness behaves relative to Alexander's subbase theorem and product spaces.

EXAMPLE.1.(G,S&W). Let X be an infinite set. For each $a \in X$ & each $n \in N$ ewe define $S_{na} : X \to I$ by $S_{na}(x) = \alpha - \alpha/n$ if $x=a$ & $S_{na}(x) = 0$ if $x \neq a$ where $0 < \alpha \leq 1$ is fixed. Then $\Sigma = \{S_{na} : n \in N$ & $a \in X\}$ is a subbase for a unique fuzzy topology T which is generated by Σ, on X.

For each $a \in X$ let $U_a = \bigcup_{n=1}^{\infty} S_{na}$. Then $\{U_a : a \in X\}$ is an α^*-shading of X that has no finite α^*-subshading. Therefore there is no Alexander subbase theorem for the concept of α^*-compactness.

EXAMPLE.2.(G,S&W). Let X be any nonempty set. Let Y be a nongenerated bounded closed interval of real numbers. Let $\alpha\in]0,1]$. For each $0 < \beta < 1$ define $V_\beta : X \to I$ by $V_\beta(x) = \alpha(1-\beta)$ for all $x \in X$.

Let $\mathcal{T}$ be the usual topology on Y and for each $U\in\mathcal{T}$ define $W_U : X \to I$ by $W_U(x) = \alpha$ if $x \in U$ & $W_U(x) = 0$ if $x \in X - U$.

Let T_X be the fuzzy topology on X with subbase $\{V_\beta : 0 < \beta < 1\}$ and let T_Y be the fuzzy topology on Y with subbase $\{W_U : U\in \mathcal{T}\}$.

Clearly (X,T_X) is α^*-compact.

Since, $(Y,\mathcal{T})$ is compact in ordinary sense. Hence by Heine - Borel theorem (Y,T_Y) is α^*-compact.

However, the product fuzzy space $(X\times Y, T_X \times T_Y)$ is not α^*-compact because, for each $y \in Y$ let $B(y,\beta) = \{ z \in Y : |z-y| < \beta\}$ be the open ball in Y of radious β & define $B_y = \cup\{V_\beta \times W_{B(y,\beta)} : 0 < \beta < 1\}$ where $V_\beta \times W_U = \pi_1^{-1}(V_\beta) \times \pi_2^{-1}(W_U)$. Then B_y is an open fuzzy set in $X \times Y$ and B_y has value $\geq \alpha$ precisely on $X \times \{y\}$. Since Y is infinite.

Therefore, $\{B_y : y \in Y\}$ is an α^*-shading of $X \times Y$ thathas no finite sub α^*-shading

Now we introduce star product of fuzzy spaces.

Let (X,T) and (Y,U) be two fts. Let $A \in T$ & $B \in U$ Define a$A_* B : X \star Y \to I$ byby

$(A_* B)(x,y) = A(x)\, B(y)$ for all $(x,y) \in X \times Y$

Thus $(A_* B)$ is a fuzzy set on $X \times Y$.

Consider a fuzzy topology Γ on $X \times Y$ which has a subbase consisting of all fuzzy sets of the form $A * B$ where $A \in T$ & $B \in U$.

When $X \times Y$ is equipped with this fuzzy topology, we denote it by $X * Y$ & call it the star fuzzy product of the fuzzy spaces X and Y.

<u>THEOREM</u>.4.5.12.(G,S&W 1978(28)).If (X,T) & (Y,U) are θ-compact fts's then the star fuzzy product $X * Y$ is also θ-compact.

Finally we deal with <u>ONE POINT COMPACTIFICATION</u>.

Let (X,T) be an fts and $\alpha \in]0,1[$ .Let H_α be the collection of all crisp closed subsets of X that are α-compact as subspaces of (X,T).

Choose any object $\omega \notin X$ and define $X^* = X \cup \{\omega\}$. For each $k \in H_\alpha$ define $k_\omega : X^* \to I$ by $k_\omega(x) = 1$ if $x = \omega$ & $k_\omega(x) = 1 - k(x)$ if $x \in X$.

and for each $U \in T$ we define $U^* : X^* \to I$ by $U^*(x) = 0$ if $x = \omega$ and $U^*(x) = U(x)$ if $x \in X$

Let T^*_α denote the fuzzy topology on X^* having the collection $\Sigma_\alpha = \{U^* : U \in T\} \cup \{k_\omega : k \in H_\alpha\}$ as a subbase.

Clearly (X,T) is a crisp subspace of (X^*, T^*_α)

<u>THEOREM</u>.4.5.13. (G,S&W 1978(28)). Let $\alpha \in I$. Using the above notations (X^*_α, T^*_α) is α-compact. Moreover X is dense in (X^*, T^*_α) iff (X,T) is not α-compact.

<u>DEFINITION</u>.4.5.13.(G,S&W 1978(28)). If (X,T) is a fts, $\alpha \in I$ and (X,T) is not α-compact, then the space (X^*, T^*_α)constructed above is called a one point α-compactification of (X,T).

.6.FUZZY UNIT INTERVAL AND FUZZY REAL LINE.

In this part we develope fuzzy version of the unit interval and the real line. The closed set $[0,1]$ and the real line are denoted by I and R respectively.

DEFINITION.4.6.1.(G,S&W 1978(28)). The fuzzy unit interval $I(I)$ is the set of all monotonic decreasing maps $\lambda : R \to I$ satisfying
(1) $\lambda(t) = 1$ for $t < 0$, $t \in R$ (2) $\lambda(t) = 0$ for $t > 1$, $t \in R$ after the identification of $\lambda, \mu : R \to I$ if for every $t \in R$, $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(T+)$, where $\lambda(t-) = \operatorname{Inf}_{s<t} \lambda(s)$ & $\lambda(t+) = \operatorname{Sup}_{s>t} \lambda(s)$

We define a fuzzy topology on $I(I)$ by taking as a subbase $\{L_t, R_t\}$ $t \in R$, where L_t, R_t are fuzzy sets on $I(I)$ defined by $L_t(\lambda) = (\lambda(t-))^c$ and $R_t(\lambda) = \lambda(t+)$ for all $\lambda \in I(I)$.
This topology is called the usual topology for $I(I)$ and $\{L_t : t \in R\}$ and $\{R_t : t \in R\}$ and called the left hand and right hand topology.
Note that if we replace I by $\{0,1\}$then the fuzzy unit interval and its topology reduce to the unit interval & its usual topology.
Note that the notation has not distinguished between the map $\lambda : R \to I$ and the equivalence class $I(I)$ containing λ. This causes no difficulty since we are only interested inthe limit of the class at $t \in R$ which is exactly the same for each member of the class.
Since, $L_a \cap L_b = L_{a \wedge b}$ & $R_a \cap R_b = R_{a \vee b}$, it follows that $\{R_a, L_b, R_a \cap L_b : a, b \in R\}$ is a base for the usual topology on $I(I)$
One also observes that $\{R_a \cap L_b : a, b \in R\}$ is another base for the usual topology on $I(I)$.

DEFINITION.4.6.2.(G,S&GW 1978(28)). The fuzzy real line $R(I)$ is the set of all monotonic decreasing maps $\lambda : R \to I$ satisfying
(1) $\operatorname{Sup}\{\lambda(t) : t \in R\} = 1$ & (2) $\operatorname{Inf}\{\lambda(t) : t \in R\} = 0$ after the identification as in the previous definition.

The fuzzy topology on $R(I)$ with $\{L_t, R_t : t \in R\}$ as subspace is called the usual fuzzy topology for $R(I)$.

We may embed the real line in the fuzzy real line by identifying $r \in R$ with the map $\lambda_r : R \to I$ defined by $\lambda_r(t) = 0$ if $t > r$ and $\lambda_r(t) = 1$ if $t < r$.

Note that $R(I)$ and its usual topology reduce to R and its usual topology for $\{0,1\}$ taken in place of I. Also note that $I(I)$ is a subspace of $R(I)$. Furthermore, $\{\bigcup_{n=1}^{\infty} L_n\}$ & $\{\bigcup_{n=1}^{\infty} R_n\}$ are 1^*-shadings of $R(I)$. We need the following definition.

<u>DEFINITION.</u> 4.6.3.(G,S&W 1978(28)).A subfamily Γ of T is said to be a cover of the fts (X,T) iff $\bigcup_{C \in \Gamma} C = 1$ An fts (X,T) is said to be compact iff every open cover has a finite subcover.

For the sake of convenience we term these as W-cover & W-compact.

<u>THEOREM.</u>4.6.4.(G,S&W 1978(28)). The fuzzy unit interval $I(I)$ is (1) W-compact, (2) α-compact for all $\alpha \in I$ & (3) 1^*-compact.

<u>THEOREM.</u>4.6.5.(G,S&W 1978(28)). The fuzzy real line is not W-compact.

The proofs of these theorems are left.

* * *

CHAPTER - 5

SEPARATION AXIOMS AND CONNECTEDNESS IN FUZZY STRUCTURE.

In this chapter we discuss various separation axioms and connectedness in fuzzy set structure. In (28), Steinlage, Gantner and Warren have introduced the Hausdorff separation axioms for only crisp (ordinary) points. In (27), Srivastava, Lal and Srivastava have considered the fuzzy points in the definition of Hausdorff space. In separate papers, Ming and Ming (21) and Sarkar (26) have presented the Hausdorff separation axioms in terms of fuzzy points.

The sections 1,2,3,4 are devoted to the study of different separation axioms and the remaining sections deal with connectedness in fts's.

We begin with the Hausdorff separation axioms as given by Steinlage and others.

.1.HAUSDORFF SPACE AS DEFINED BY STEINLAGE,GANTNER & WARREN.

Consider a classical set X and a fuzzy topology T on the set X.

DEFINITION.5.1.1.(Steinlage, Gantner and Warren 1978 (28)). The fts (X,T) is called Hausdorff if $x,y \in X$ and $x \neq y$ imply that there are $U,V \in T$ with $U(x) = 1 = V(y)$ and $U \cap V = \underline{0}$.

THEOREM.5.1.2.(S , G & W 1978 (28)). Let S be a crisp subspace of a Hausdorff space (X,T) .

(1) If $0 \lneqq \alpha < 1$ and if S is α-compact then S is T-closed in X .

(2) If $0 < \alpha \leq 1$ and if S is α^*-compact then S is T-closed in X.

Proof : (1) We shall show that S^c is T-open.

Let $x \in S^c$. It is required to show that there exists $U \in T$ with $U(x) = 1$ and $U \subset S^c$.

For each $y \in S$ we can find U_y and V_y in T such that $U_y(x) = 1 = V_y(y)$ and $U_y \cap V_y = \underline{0}$.

Thus $\{ V_y/S : y \in S \}$ is an α-shading of S (Here V_y/S means the restriction of V_y to S.). But S is α-compact.

Therefore this collection has a finite α-subshading

$\{V_{y_1}/S , V_{y_2}/S , \ldots\ldots\ldots\ldots V_{y_m}/S\}$.

Let $U = U_{y_1} \cap U_{y_2} \cap \ldots\ldots\ldots\ldots\ldots \cap U_{y_m}$.

Then $U(x) = 1$ and $U \cap (V_{y_1} \cap V_{y_2} \cap \ldots\ldots\ldots\ldots \cap V_{y_m}) = \underline{0}$.

For each $z \in S$, there exists a K with $V_{y_k}(z) > \alpha \geqslant 0$. So $U(z) = 0$.

Thus $U \subseteq S^c$ which implies that S^c is open & hence S is closed in X .

The proof of the other part is similar.

DEFINITION.5.1.3.(S , G & W 1978 (28)). Let (X,T) be an fts and $\alpha \in I$

If A is a fuzzy set in X then SuppA is the crisp subset of X defined by $\text{Supp}A = \{x \in X : A(x) > 0\}$.

We call (X,T) locally α-compact if for each point $x \in X$ there exists an open fuzzy set U such that $U(x) = 1$ and $\text{Supp}\overline{U}$ is α-compact as a crisp subspace of X.

THEOREM.5.1.4.(S , G & W 1978 (28)). Let (X,T) be a locally α-compact Hausdorff space and (X^*,T^*_α) a one point α-compactification of (X,T). Then (X^*,T^*_α) is Hausdorff where $\alpha \in I$.

Proof : Let $x,y \in X$ and $x \neq y$. Then there are open fuzzy sets U & V in (X,T) such that $U(x) = 1 = V(y)$ and $U \cap V = \underline{0}$.

Hence $U^*(x) = 1$, $V^*(y) = 1$ and $U^* \cap V^* = \underline{0}$ in (X^*,T^*)

Now suppose that $x \in X$. Then there exists $U \in T$ such that $U(x) = 1$ and $K = \text{Supp}\overline{U}$ is α-compact.

Then K is closed subspace of (X,T) (by theorem 5.1.2).

Therefore $K_\omega \in T^*_\alpha$. So $U^*(x) = 1$, $K_\omega(\omega) = 1$ and $U^* \cap K_\omega = \underline{0}$,

Hence (X^*,T^*_α) is Hausdorff.

§ .2. HAUSDORFF SPACE AS DEFINED BY SRIVASTAVA, LAL & SRIVASTAVA.

According to Srivastava, Lal and Srivastava (27) a fuzzy point x_α $(0 < \alpha < 1)$ belongs to a fuzzy set A if $\alpha < A(x)$ and $x_\alpha(y) < A(y)$ if $y \neq x$. and they have followed the definition of fuzzy point given by Wong. In this fuzzy structure two fuzzy points x_α and y_β are said to be distinct iff their supports x and y are distinct.

DEFINITION.5.2.1.(Srivastava , Lal & Srivastava 1981 (27)). An fts (X,T) is said to be fuzzy Hausdorff iff for any two distinct fuzzy points x_α and y_β in X , there exists disjiont U and V in T with $x_\alpha \in U$ and $y_\beta \in V$.

REMARK.5.2.2.(S , L & S) Using Wong's definition of 'belonging' it is impossible to find two disjiont fuzzy sets U & V in an fts (X,T) "separating " two distinct fuzzy points p and q in X.

For, if $U \cap V = \underline{0}$ we get $(U \cap V)(x) = 0$ for all $x \in X$, which implies that $\mathrm{Inf}\{U(x), V(x)\} = 0$ i.e. $U(x) = 0$ or $V(x) = 0$.

But this is a contradiction to either $p = x_\alpha \in U$ or $p = x_\alpha \in V$.

THEOREM.5.2.3.(S , L & S 1981(27)). Let (X,T) be an fts. Then the following are equivalent.

(1) (X,T) is a fuzzy Hausdorff space.

(2) $\Delta_X = \{(x,x) : x \in X\}$ is fuzzy closed in X x X .

(3) For any two F-continuous functions $f,g : (Y,U) \to (X,T)$ the set $A = \{y \in Y : f(y) = g(y)\}$ is F-closed in (Y,U).

(4) If $f : (Y,U) \to (X,T)$ is an F-continuous function then the graph of f i.e. the fuzzy set $\{(y,f(y)) : y \in Y\}$ is fuzzy closed in $(Y \times X, U \times T)$.

Proof : (1)$\Rightarrow$(2) : It is sufficient to show that $X \times X - \Delta_X$ is fuzzy open in X x X .

Let p be any fuzzy point in $X \times X - \Delta_X$ with support (x_1, x_2) .

Consider two fuzzy points r and s of X given by

$r(x_1) = p(x_1,x_2) = s(x_2) = t$,(say) . Therefore $0 < t < 1$.

As $r \neq s$, there are disjoint $U, V \in T$ such that $r \in U$ and $s \in V$.

Consider the set $U \times V$.

$[U \times V](x_1,x_2) = \mathrm{Inf}\ \{U(x_1), V(x_2)\}$.

Now $r \in U \Rightarrow r(x_1) < U(x_1)$ and $s \in V \Rightarrow s(x_2) < V(x_2)$.

So $t = \mathrm{Inf}\ \{U(x_1)\ ,\ V(x_2)\} = [U \times V](x_1,x_2)$ which implies that $p \in U \times V$.

Further, for any $(x_1',x_2') \in X \times X$, $[U \times V](x_1',x_2') = \mathrm{Inf}\ \{U(x_1'), V(x_2')\}$.

So $[U \times V](x_1',x_2') = 0$ if $x_1' \neq x_2'$ because $[U \times V](x_1',x_2') = [U \cap V](x_1') = 0$

(since $U \cap V = \underline{0}$)

This ensures that $[U \times V](x_1',x_2') \leq (X \times X - \Delta_X)(x_1',x_2')$ $\forall$ $(x_1',x_2') \in X \times X$.

i.e. $U \times V \subseteq X \times X - \Delta_X$.

We have thus shown that given any fuzzy point p in $X \times X - \Delta_X$ there is an open fuzzy set $U \times V$ in $X \times X - \Delta_X$ such that $p \in U \times V \subseteq X \times X - \Delta_X$.

Hence $X \times X - \Delta_X$ is fuzzy open .

(2) $\Rightarrow$ (3) : Consider the function $(f,g) : (Y,U) \to (X \times X, T \times T)$ given by $(f,g)(y) = (f(y),g(y))$ for all $y \in Y$.

To show that this is F-continuous, it is sufficient to show that the inverse image of each basic open fuzzy set in $X \times X$ is fuzzy open in Y.

Take any basic open fuzzy set, say, $U \times V$ in $X \times X$.

Then $[(f,g)^{-1}(U \times V)](y) = [U \times V](f(y),g(y)) = \mathrm{Inf}\{U(f(y)), V(g(y))\}$

$= \mathrm{Inf}\ \{[f^{-1}(U)](y), [g^{-1}(V)](y)\} = [f^{-1}(U) \cap g^{-1}(V)](y)$

where $y \in Y$.

Thus $[(f,g)^{-1}](U \times V) = f^{-1}(U) \cap g^{-1}(V)$ which is an F-open set in Y.

Therefore (f,g) is F-continious .

Now $A = [(f,g)^{-1}](\Delta_X)$ is fuzzy closed in (Y,U) as Δ_X is F-closed.

(3) $\Rightarrow$ (4) : Since $\pi_X : (Y \times X, U \times T) \to (X,T)$ and $\pi_Y : (Y \times X, U \times T) \to (Y,U)$ and the function f are F-continuous, it is clear that $\overline{f} = f \circ \pi_Y$ is also F-continuous.

Now consider the set

$$\{(y,x) \in Y \times X : \pi_X(y,x) = \bar{f}(y,x)\} = \{(y,x) : \pi_X(y,x) = (f \circ \pi_Y)(y,x)\}$$
$$= \{(y,x) : x = f(y)\} = \{(y,f(y)) : y \in Y\} .$$

By hypothesis, $\{(y,x) \in (Y \times X)\}:\pi_X(y,x)=\bar{f}(y,x)\}$ is F-closed in $Y \times X$. Therefore $\{(y,f(y) : y \in Y\}$ is F-closed in $Y \times X$.

(4) $\Rightarrow$ (1) : Consider the identity function on (X,T) . Using the condition (4) we can say that the set $\Delta_X = \{(x,x) : x \in X\}$ is F-closed in $(X \times X, T \times T)$ i.e. $X \times X - \Delta_X$ is fuzzy open.

Let $p = x_\alpha$ and $q = y_\beta$ be two distinct fuzzy points in X. Then $x \neq y$. So $(x,y) \in X \times X - \Delta_X$.

Consider a fuzzy point r in $X \times X$ such that $r(x,y) = \text{Max}\,\{p(x),q(y)\}$. Clearly $r \in X \times X - \Delta_X$. Hence there exists a basic open fuzzy set $U \times V$ such that $r \in U \times V \subset X \times X - \Delta_X$.

Further it is clear that $p \in U$ and $q \in V$. Also, since $U \times V \subset X \times X - \Delta_X$, $(U \times V)(x,x) = 0$ for all $x \in X$ which implies that $U \cap V = \underline{0}$.

Thus U and V are two disjoint open fuzzy sets in (X,T) containing the fuzzy points p and q resprctively. Hence (X,T) is Hausdorff.

THEOREM.5.2.5.(S , L & S 1981 (27)). Let $(X,\mathcal{T})$ be a topological space. Then $(X,\mathcal{T})$ is Hausdorff iff $(X,\omega(\mathcal{T}))$ is fuzzy Hausdorff.

Proof : ($\Rightarrow$) Suppose that $(X,\mathcal{T})$ is Hausdorff.

Let $p = x_\alpha$ and $q = y_\beta$ be two distinct fuzzy points in X. Then $x \neq y$. Using Hausdorffness of X, we can find two disjoint $\mathcal{T}$-open sets in X such that $x \in U$ and $y \in V$.

Now consider the characteristic functions χ_U and χ_V of U & V respectively from $(X,\mathcal{T})$ to I_r. Clearly χ_U and χ_V are continuous and belong to $\omega(\mathcal{T})$. Also, since $x \in U$, $\chi_U(x) = 1$. Hence $p \in U$. Similarly , $q \in V$. Thus we have obtained two disjoint open fuzzy sets U and V in $(X,\omega(\mathcal{T}))$

containing p and q respectively.

($\Leftarrow$) Suppose that $(X,\omega(T))$ is fuzzy Hausdorff.

Let x and y be two distinct points in X. Consider two distinct fuzzy points $p = x_\alpha$ and $q = y_\beta$.

Since $(X,\omega(T))$ is fuzzy Hausdorff, we can find two disjoint open fuzzy sets U and V in X such that $p \in U$ and $q \in V$. Let $p(x) = r$ and $q(y) = s$. Then $p(x) < U(x)$ and $q(y) < V(y)$ imply that $x \in U^{-1}(]r,1])$ & $y \in V^{-1}(]s,1])$. Now $]r,1]$ and $]s,1]$ are open in the topology given in I. Therefore $U^{-1}(]r,1])$ and $V^{-1}(]s,1])$ are open in (X,T).

Moreover, $U^{-1}(]r,1]) \cap V^{-1}(]s,1]) = \underline{0}$, For, if not, then let $z \in U^{-1}(]r,1]) \cap V^{-1}(]s,1])$.

Now, $z \in U^{-1}(]r,1]) \Rightarrow U(z) > r$ and $z \in V^{-1}(]s,1]) \Rightarrow V(z) > s$.

So $\operatorname{Inf}\{U(z),V(z)\} \neq \underline{0}$, which is a contradiction to " $U \cap V = \underline{0}$ ".

Hence, $U^{-1}(]r,1])$ and $V^{-1}(]s,1])$ are two disjoint T-open sets containing x and y respectively. This proves that (X,T) is Hausdorff.

THEOREM.5.2.6.(S , L & S 1981 (27)). If an fts (X,T) is fuzzy Hausdorff then $(X,i(T))$ is Hausdorff.

PROOF : Take any two distinct points x and y in X. Consider two distinct fuzzy points $p = x_\alpha$ and $q = y_\beta$ in X. Since (X,T) is fuzzy Hausdorff, we can get two disjoint fuzzy open sets U and V in X such that $p \in U$ and $q \in V$. Suppose that $p(x) = r$ and $q(y) = s$.

Then $r < U(x)$ and $s < V(y)$ and so $x \in U^{-1}(]r,1])$ and $y \in V^{-1}(]s,1])$.

Hence, $U^{-1}(]r,1])$ and $V^{-1}(]s,1])$ are two disjoint open sets in $(X,i(T))$ containing x and y respectively. Hence $(X,i(T))$ is Hausdorff.

THEOREM.5.2.7.(S , L & S 1981 (27)). A fuzzy subspace (A,T_A) of a fuzzy Hausdorff topological space is fuzzy Hausdorff.

THEOREM.5.2.8.(S,L,S 1981(27)). If $\{(X_\lambda,T_\lambda)\}, \lambda\in\Lambda$, is a family of fuzzy Hausdorff topological spaces, then their product (X,T) is also fuzzy Hausdorff.

PROOF : Left.

§.3. SEPARATION AXIOMS

In this section we discuss various separation axioms defined in terms of Q-nbhds. Here, we follow definitions 2.3.1 and 2.3.2 for 'fuzzy point' and 'belongingness'.

DEFINITION.5.3.1.(Ming and Ming 1980 (21)). An fts (X,T) is called a fuzzy quasi-T_0 space iff for every $x \in X$ and $\alpha \neq \beta$, $\alpha,\beta \in I$, either $x_\alpha \notin \overline{x}_\beta$ or $x_\beta \notin \overline{x}_\alpha$.

It follows that (X,T) is a quasi-T_0 if for every $x \in X$ and $0 < \alpha < \beta < 1$, $x_\beta \notin \overline{x}_\alpha$.

DEFINITION.5.3.2.(M & M 1980 (21)). An fts (X,T) is called a fuzzy T_0 space iff foe any two fuzzy points p and q with $p \neq q$, either $p \notin \overline{q}$ or $q \notin \overline{p}$.

DEFINITION.5.3.3.(M & M 1980 (21)). An fts (X,T) is called a fuzzy T_1 space iff every fuzzy point is a closed set.

THEOREM.5.3.4.(M & M 1980(21)). Let (X,T) be an fts. Then (X,T) is $T_1 \Rightarrow$ It is $T_0 \Rightarrow$ It is quasi-T_0.

PROOF : Obvious.

Every ordinary topological space vacuously satisfies condition of being quasi-T_0 and hence the quasi-T_0 separation is a particularity in fuzzy topology.

THEOREM.5.3.5.(M & M 1980 (21)). Let (X,T) be a quasi-T_0 space, $x \in X$ and $\nabla =]\rho_1,\rho_2[$ $(0 \leq \rho_1 < \rho_2 < 1)$. Then there exists $B \in T$ such that $B(x) \in \nabla$.

PROOF : Let $\alpha = 1-\rho_1$ and $\beta = 1-\rho_2$. Therefore $\alpha > \beta > 0$.

Since, (X,T) is quasi-T_0, we have $x_\alpha \notin \overline{x}_\beta$.

Thus there exists some open Q-nbhd B ($B(x) > 1-\alpha$) which is not quasi-coincident with x_β i.e. $B(x) \leq 1-\beta$. Hence $B(x) \in \nabla$.

THEOREM.5.3.6.(M & M 1980 (21)). An fts (X,T) is quasi-T_0 iff for all $x \in X$ and $\rho \in I$, there exists a $B \in T$ such that $B(x) = \rho$.

PROOF : ($\Rightarrow$) When $\rho = 0$, it suffices to take $B = \underline{0}$.

When $0 < \rho \leq 1$, take a strictly monotonic increasing sequence of positive real numbers converging to ρ. Let $V_n =]\rho_n, \rho_{n+1}]$, $n=1,2,3,\ldots.$ By theorem 5.3.5, there exists $B_n \in T$ such that $B_n(x) \in V_n$ for each n. Therefore, $B = \bigcup_{n=1}^{\infty} B_n$ is fuzzy open and $B(x) = \rho$.

($\Leftarrow$) For two fuzzy points x_α and x_β with $\alpha > \beta$, there exists from hypothesis an open set B such that $B(x) = 1-\beta > 1-\alpha$. It is clear that B is an open Q-nbhd of x_α but is not quasi-coincident with $\{x_\beta\}$. Hence it follows from theorem 2.3.18that $x_\alpha \notin \overline{x}_\beta$.

THEOREM.5.3.7.(M & M 1980 (21)). An fts (X,T) is a T_0 space iff (X,T) is quasi-T_0 and for any two distinct points x, y in X and for any $\rho\, \sigma \in I$, there exists $B \in T$ such that $B(x) = \rho$ & $B(y) > \sigma$ or $B(x) > \rho$ & $B(y) = \sigma$.

PROOF : ($\Rightarrow$) Let (X,T) be T_0 space. Then it is also quasi-T_0.

For $x \neq y$ and $\rho, \sigma \in I$, putting $\alpha = 1-\rho$ and $\beta = 1-\sigma$ we obtain two distinct fuzzy points x_α and y_β .

If $x_\alpha \notin \overline{y}_\beta$, there exists an open Q-nbhd B_1 ($B_1(x) > 1-\alpha = \rho$) which is not Q-coincident with $\{y_\beta\}$ i.e., $B_1(y) \leq 1-\beta = \sigma$.

In view of theorem 5.3.6, there is $B_2 \in T$ such that $B_2(y) = \sigma$.

Then the fuzzy open set $B = B_1 \cup B_2$ is the required one.

If $y_\beta \notin \overline{x}_\alpha$, the argument can be carried out in a similar way.

($\Leftarrow$) Assume that (X,T) is quasi-T_0 and for any $x, y \in X$ with $x \neq y$ and for any $\rho, \sigma \in I$, there exists $B \in T$ such that $B(x) = \rho$ and $B(y) > \sigma$, or $B(x) > \rho$ and $B(y) = \sigma$.

It is sufficient to consider the separation of two fuzzy points x_α and y_β with $x \neq y$. Put $\alpha = 1-\rho$ and $\beta = 1-\sigma$.

From hypothesis, we may assume that there exists $B \in T$ such that

$B(x) = \rho$ and $B(y)>\bar{\sigma}$. Then B is a Q-nbhd of y_β which is not Q-coincident with $\{x_\alpha\}$ i.e., $y_\beta \notin \bar{x}_\alpha$.

<u>THEOREM</u>5.3.8.(M & M 1980 (21)). An fts (X,T) is a T_1 space iff, for each $x \in X$ and each $\alpha \in I$, there exists $B \in T$ such that $B(x) = 1-\alpha$ and $B(y) = 1$ for $y \neq x$.

PROOF : ($\Rightarrow$) When $\alpha = 0$, it suffices to take $B = \underline{1}$.

When $\alpha > 0$, x being a fuzzy point is closed by hypothesis, then $B = (x_\alpha)^c$ is the required open set.

($\Leftarrow$) Let x_α be an arbitrary fuzzy point. Then, by hypothesis, there exists a $B \in T$ such that $B(x) = 1-\alpha$ and $B(y) = 1$ for $y \neq x$. It follows that $x_\alpha = B^c$ is closed.

<u>DEFINITION</u>.5.3.9.(M & M 1980 (21)). An fts (X,T) is called a fuzzy T_2 (Hausdorff) space iff, for any two fuzzy points p and q satisfying $\text{supp}(p) \neq \text{supp}(q)$, there exists Q-nbhds B and C of p and q, resprctively, such that $B \cap C = \underline{0}$.

This Hausdorff space will be referred to as Q-T_2 (or Q-Hausdorff).

<u>PROPOSITION</u>.5.3.10.(M & M 1980 (21)). Let (X,T) be a Q-T_2 space, then any Q-accumulation of a fuzzy point y_β in (X,T) is of the form $y_\alpha (\alpha>\beta)$.

PROOF : When $\alpha \leq \beta$, $y_\alpha \in y_\beta$.

But since any Q-nbhd of y_α can be Q-coincident with y_β atmost at y, y_α is not an Q accumulation point of y_β .

When $x \neq y$, from the property of being Q-T_2 , there exists Q-nbhds B and B_1 of x_α and y_β, respectively, such that $B \cap B_1 = \underline{0}$.

But since $B_1(y) > 1-\beta \geqslant 0$, $B(y) = 0$, i.e., B is not Q-coincident with y_β at y and hence x_α is not an Q-accumulation point of y_β .

We have thus proved that the only possible form of an Q-accumulation point of y_β is of the type y_α with $\alpha>\beta$.

Since, Q-T_2 is concerndd only with those fuzzy points with different supports, it is possible, as the following example shows, that a

Q-T_2 space need not be quasi-T_o, to say nothing of being T_1.

EXAMPLE.(M & M). Let $X = \{ y,z \}$ when $y \neq z$. Let T be the fuzzy topology on X which has

$\Xi = \{y_\lambda : \lambda \in]2/3,1]\} \cup \{z_\lambda : \lambda \in]0,1]\} \cup \{\underline{0}\}$ as a base

Obviously, (X,T) is a fuzzy Q-T_2 space. But, since there is no T-open set which takes the value 1/2 at y, in view of theorem 5.3.6., (X,T) is not quasi-T_o.

THEOREM.5.3.11.(M & M 1980 (21)). If (X,T) is both Q-T_2 and quasi-T_o then it is also T_1.

PROOF : Let y_β be an arbitrary fuzzy point. Therefore, from theorem 5.3.10., an Q-accumulation point, if any, of y_β is of the form $y_\alpha (\alpha > \beta)$. Since (X,T) is quasi-T_o, by theorem 5.3.6., there exists a $B \in T$ such that $B(y) = 1-\beta > 1-\alpha$. Thus, B is a Q-nbhd of y_α and is not Q-coincident with y_β.

Hence, y_α $(\alpha > \beta)$ cannot be an Q-accumulation point of y_β.

Therefore, y_β has no Q-accumulation point. So by theorem 2.3.20., y_β is closed. Hence, (X,T) is T_1.

THEOREM.5.3.12.(M & M 1980 (21)). The Q-derivrd set of every fuzzy set in a T_1 space is closed.

PROOF : Note that the Q-derived set of every fuzzy point in a T_1 space is obviously $\underline{0}$.

So the result follows from theorem 2.3.30.

The proofs of the following results are omitted.

THEOREM.5.3.13.(M & M 1980 (21)). Let $\{(X_\lambda,T_\lambda)\}, \lambda \in \Lambda$, be a collection of fts's, among which there is atleast one quasi-T_o space. Then the product space (X,T) is quasi-T_o.

THEOREM5.3.14.(M & M 1980 (21)). Let $\{(X_\lambda,T_\lambda)\}, \lambda \in \Lambda$, be a collection of fts's. If each (X_λ,T_λ) is T_o (resp. T_1), then the product space (X,T) is a T_0 (resp. T_1) space.

THEOREM.5.3.15.(M & M 1980 (21)). Let $\{(X_\lambda,T_\lambda)\}$, $\lambda \in A$, be a collection of fts's. The product space (X,T) is Q-T_2 space iff each coordinate space (X_λ,T_λ) is a Q-T_2 space.

When the product space enjoys any one of the properties of being qasi-T_0, T_0 and T_1, each coordinate space dees not necessarily enjoy the corresponding separation property.

For example, let $X_1 = X_2 = \{x\}$., $T_1 = \{\underline{0}\} \cup \{x_\lambda : 0 < \lambda \leq 1/2 \& \lambda = 1\}$ and $T_2 = \{\underline{0}\} \cup \{x_\lambda : 1/2 \leq \lambda \leq 1\}$.

From theorem 5.3.6., (X_1,T_1) and (X_2,T_2) are not quasi-T_0 and hence are not T_0 and T_1 spaces; but their product space is evidently a T_1 space.

However , in the light of theorem 3.1.14,we can say that if a coordinate space is quasi fts, it enjoys the same separation property as the product space does.

§.4.SEPARATION PROPERTIES AND PROPER COMPACTNESS AS DEFINED BY SARKAR.

In this section, we define the Hausdorff separation axiom and some of the other separation axioms with the help of fuzzy elements.

We consider a fuzzy point or a fuzzy singleton $p = x_\alpha$ in a set X , to be a fuzzy set in X such that

$p(x) = \alpha$ and $p(y) = 0$ for all $y \neq x$ in X , where $\alpha \in]0,1[$. x is called the support and α its value.

Also, a fuzzy point $p = x_\alpha$ is considered tobe in a fuzzy set A (denoted by $p \in A$) if and only iff $p(x) < A(x)$. So, $p \notin A \Leftrightarrow p(x) \geq A(x)$.

Areal point $z \in X$ is called a *crisp point* and is identified with its characteristic function and we say that z belongs to the fuzzy set A if $A(z) = 1$.

By points (subsets) of X, we mean both crisp and fuzzy points (subsets) and we always denote the support of a fuzzy point p by x_p .

DEFINITION.5.4.1.(Sarkar 1981 (26)). In an fts (X,T), a set A is said to be open iff for each point $p \in A$, there exists a fuzzy set $G \subset T$, such that $p \in G \subset A$.

DE. Clearly, the criterion of open sets given in theorem 1.3.7 is equivalent to this.

DEFINITION.5.4.2.(Sarkar 1981 (26)). An fts (X,T) is called $F\text{-}T_1$ iff the singletons are closed.

DEFINITION.5.4.3.(Sarkar 1981 (26)). An fts (X,T) is said to be Hausdorff or $F\text{-}T_2$ iff the following conditions hold :

If p and q are two points in X, then (1) if $x_p \neq x_q$, there exists open sets V_p and V_q , such that $p \in V_p$, $q \notin \overline{V}_p$ & $q \in V_q$, $p \notin \overline{V}_q$; (2) if $x_p = x_q$ and $p(x_p) < q(x_p)$, then there exists an open set V_p such that $p \in V_p$, but $q \notin \overline{V}_p$.

It is clear that, if (X,T) is Hausdorff according to definition 5.1.1., then (1) of definition 5.4.3 follows immediately.

THEOREM.5.4.4. (Sarkar 1981 (26)). An $F\text{-}T_2$ space is an $F\text{-}T_1$ space.

PROOF : Let p be an fuzzy point in an $F\text{-}T_2$ space(X,T). Then any point $q \in \{p\}^c$ belongs to an open set V_q such that $\{p\}^c(x_p) \geqslant \overline{V}_q(x_p)$ and so $V_q \subset \{p\}^c$.

On the other hand, if p is crisp, let $x_q \in X - x_p$ be arbitrary. If $\{q_n : n=1,2,...\}$ be a sequence of fuzzy points, where $q_{x_n} = x_q$, for all n=1,2,... and the sequence $\{q_n(x_q) : n=1,2,...\}$ is decreasing and converges to zero, then there exists a sequence of open sets $\{V_{pq_n} : n=1,2,...\}$, such that $p \in V_{pq_n}$ and $q_n \notin \overline{V}_{pq_n}$, for all n=1,2,... , as (X,T) is $F\text{-}T_2$.

SO, if $P = \bigcap_{n=\mathbb{N}} \overline{V}_{pq_n}$, then P is a closed set, where $P(x_q) = 0$, and

$P(x_p) = 1$. Hence, P^c is an open set contained in $\{p\}^c$ and containing the crisp point q(and hence any fuzzy point with support x_q).

The definition (4.1.2) of compactness given earlier does not seem very natural in an fts, especially when it is $F\text{-}T_2$, as is shown by the following proposition.

PROPOSITION.5.4.5.(Sarkar 1981 (26)). No subset of an $F\text{-}T_2$ fts can be compact(countably compact).

PROOF :Let A be a subset of the fts (X,T) such that $A(x_A) > 0$, for some $x_A \varepsilon X$. Choose a sequence $\{p_n : n=1,2,...\}$ of fuzzy points, each having support x_A, such that $p_n(x_A) \nless A(x_A)$, for all n=1,2,.. and$\{p_n(x_A) : n=1,2,... \}$ is an increasing sequence which converges to $A(x_A)$.

Then from the Hausdorff property, there exists a sequencr of open sets $\{ V_{x_{A_n}} : n=1,2,.... \}$, where $p_n \varepsilon V_{x_{A_n}}$ and $p_{n+1} \not\varepsilon \bar{V}_{x_{A_n}}$.

This sequence together with the complement of the crisp point at x_A forms an open cover of A, which has no finite subcover.

COROLLARY.5.4.6.(Sarkar 1980 (26)). Singletons in an $F\text{-}T_2$ space are not compact (countably compact).

Now let us redefine open cover and compactness.

DEFINITION.5.4.7.(Sarkar 1980 (26)). A family $\mathcal{V}= \{V_\lambda \varepsilon T : \lambda \varepsilon \Lambda\}$ is said tobe a proper open cover of the set A in the fts (X,T) iff for each $x \varepsilon X$, there exists $V_{\lambda_x} \varepsilon \mathcal{V}$, such that $V_{\lambda_x}(x) \geq A(x)$. This family $\mathcal{V}$ is called a countable (finite) proper open cover of A if Λ is countable (finite). A subfamily $\mathcal{V}_1$ of $\mathcal{V}$ is called a proper open subcover of $\mathcal{V}$ if it is a proper open cover of A in its own right.

It follows from the definition that, a proper cover of A is always a cover of A, but not conversely.

DEFINITION 5.4.8. (Sarkar 1981 (26)). A set A is properly (countably) compact in the fts (X,T) iff every (countable) proper open cover of A has a proper open finite subcover.

From the above definitions, we obtain

PROPOSITION. 5.4.9. (Sarkar 1981 (26)). Every singleton (hence a subset with finite support) in an fts is properly compact.

PROPOSITION. 5.4.10. (Sarkar 1981 (26)). Let $f : (X,T) \to (Y,U)$ be an F-continuous surjection of fts's and A a properly compact set in X. Then f(A) is a properly compact set in Y.

THEOREM. 5.4.11. (Sarkar 1981 (26)). A properly compact set in an $F\text{-}T_2$ space is closed.

PROOF : Let A be a properly compact set in an $F\text{-}T_2$ fts (X,T) and p a point in X such that $p(x_p) > A(x_p)$ ....(1).

Then , by the Hausdorff property, there exists $V_p \in T$ such that $A(x_p) < V_p(x_p)$ ....(2) and $p(x_p) \not\geq \bar{V}_p(x_p)$ ....(3).

Therefore, to each point p satisfying (1), there corresponds a family of open sets $\{ V_{pq} : x_q \in X\}$, such that $A(x_q) < V_{pq}(x_q)$ $\forall\, x_q \in X$ (if, howevee, $A(x_q) = 1$, then we must have $A(x_q) = V_{pq}(x_q)$) ...(4) and $p(x_p) \not\geq \bar{V}_{pq}(x_p)$ for all q such that $x_q \in X$. ....(5)

Hence $A(x_q) \leq \bigcup_{x_q \in X} V_{pq}(x_q) \Rightarrow A \subset \bigcup_{x_q \in X} V_{pq}$

$$\Rightarrow A \subset \bigcup_{\substack{x_{q_k} \in X \\ k=1,2,\ldots,m}} V_{pq_k} \quad \text{as A is properly compact}$$

$$\Rightarrow A \subset \bigcup_{k=1,\ldots,m} \bar{V}_{pq_k} = F_p \quad \text{say (so that } F_p \text{ is closed)}$$

$$\Rightarrow A(x_q) \leq F_p(x_q) \quad \text{for all } x_q \in X. \quad \ldots\ldots\ldots\ldots(6)$$

This is ofcourse accompanied by $p(x_p) \geqslant F_p(x_p)$ . ...(7)
Now consider all points p which satisfy (1), we get the family $\{F_p : F_p^c \in T\}$, such that (6) and (7) hold. Then clearly $A(x_q)=\bigcap_p F_p(x_q)$ for all $x_q \in X$ by (6) and (7). Hence $A = \bigcap_p F_p$ i.e., A is closed.

DEFINITION.5.4.12.(Sarkar 1981 (26)). An fts (X,T) is called a fuzzy P-space if the countable union of closed sets is closed.

THEOREM.5.4.13.(Sarkar 1981 (26)). Fuzzy Lindelof sets in a Hausdorff fuzzy P-space are closed.

PROOF : Left.

THEOREM.5.4.14.(Sarkar 1981 (26)). Properly compact sets in an F-T_1, P-space have finite supports.

PROOF : Left.

COROLLARY.5.4.15.(Sarkar 1981 (26)). Properly compact sets in a Hausdorff fuzzy p-space have finite supports.

PROOF : Left.

In ordinary topology, a closed subset of a compact (countably compact, Lindelof) set is compact (countably compact, Lindelof). But this property does not hold in an fts.

EXAMPLE. (Sarkar (26)). Let X be an uncountable set of points and T the fuzzy topology generated by the base formed by

$\underline{0}$, $\underline{1}$ = X , $\{A_\lambda : \lambda \in [1/4, 4/5]\}$ and $\{B_x : x \in X\}$, where $A_\lambda(x) = \lambda$ for all $x \in X$ and $B_x(x) = 1/4$ & $B_x(y) = 0$ for all $y \neq x$ in X

Then X is (properly) compact, (properly) countable compact, and Lindelof. But the (proper) open cover $\{B_x : x \in X\}$ of the closed set $A_{1/4}$ has no countable (proper) open subcover. So $A_{1/4}$ is neither (properly) compact nor Lindelof.

Again, let X be partitioned into a countable number of subsets

$\{x_n : n=1,2,\dots\}$. Then $\{ B_n : B_n = \bigcup_{x \in X_n} B_x$ and $n=1,2,\dots\}$ is a countable (proper) open cover of $A_{1/4}$ which has no (proper) finite subcover. So $A_{1/4}$ is not (properly) countably compact.

Finally, we consider some of the other separation properties of an fts and find the interrelations among them.

DEFINITION.5.4.16.(Sarkar 1981 (26)). An fts (X,T) is called regular (~~normal~~) iff for each point $p \in X$ and $V \in T$, where $p \in V$ there exists $G \in T$ such that $p \in G \subset \bar{G} \subset V$.

DEFINITION.5.4.17.(Sarkar 1981 (26)). AN fts (X,T) is said to be normal iff for every closed set K and open set V such that $K \subset V$, there exists a set $G \in T$ such that $K \subset G \subset \bar{G} \subset V$.

DEFINITION.5.4.18.(Sarkar 1981 (26)). An fts (X,T) is called $F\text{-}T_3$ $(F\text{-}T_4)$ iff it is $F\text{-}T_1$ and regular (normal).

THEOREM.5.4.19. (Sarkar 1981 (26)). An $F\text{-}T_3$ space is an $F\text{-}T_2$ space.

PROOF ; Let p ans q be two fuzzy points, where $x_p \neq x_q$ and let w be a third fuzzy point, where $x_w = x_p$ and $w(x_p) > 1 - p(x_p)$.

Then $\{w\}^c$ is open and

$$\{w\}^c(x) = 1-w(x_p) < p(x_p) \quad \text{for } x = x_p$$
$$= 1 \quad \text{otherwise.}$$

So $q \in \{w\}^c$, but $p \notin w^c$.

Now since (X,T) is regular, there exists $V_q \in T$ such that $q \in V_q \subset \bar{V}_q \subset \{w\}^c$. Obviously then, $p \notin \bar{V}_q$.

Similarly , an open set V_p can be determined such that $p \in V_p$ and $q \notin \bar{V}_p$

The other cases can be similarly handled.

THEOREM?5.4.20.(Sarkar 1981 (26)). An $F\text{-}T_4$ space is an $F\text{-}T_3$ space.

The converse results (as usual) are not true in general.

Next we prove the Urysohn's Lemma.

THEOREM.5.4.21.(Hutton 1975 (14)). (Urysohn's Lemma) An fts (X,T) is normal iff for every closed set K and open set V with $K \subseteq V$, there exists a continuous function $f : X \to I(\underline{I})$ such that for every $x \in X$ $K(x) \leq f(x)(1-) \leq f(x)(0+) \leq V(x)$.

PROOF : $(\Leftarrow)$ Since $K(x) \leq f(x)(1-) \leq f(x)(0+) \leq V(x)$, we have for any $t \in \,]0,1[$ $K(x) \leq f(x)(t+) \leq f(x)(t-) \leq V(x)$.

Now $f^{-1}(L_t^c)(x) = f(x)(t-)$ and $f^{-1}(R_t)(x) = f(x)(t+)$.

Since f is continuous, $f^{-1}(L_t^c)$ is closed and $f^{-1}(R_t)$ is open.

Hence $K \subseteq f^{-1}(R_t) \subseteq f^{-1}(L_t^c) \subseteq V$ .Thus, (X,T) is normal.

$(\Rightarrow)$ Construct $\{W_r : r \in \,]0,1[\,\}$ so that $K \subseteq W_r \subseteq V$ and $r < s \Rightarrow W_s \subseteq W_r^o$.

Define $f(x)t = W_t(x)$.

Clearly, $K(x) \leq f(x)(1-) \leq f(x)(0+) \leq V(x)$.

Now, $f^{-1}(R_t) = \bigcup_{r>t} (W_r) = \bigcup_{r>t} (W_r^o)$ is open and

$f^{-1}(L_t^c) = \bigcap_{r<t} (W_r) = \bigcap_{r<t} (\overline{W}_r)$ is closed .

Hence f is continuous .

We note that perfect normality also has a natural generalisation to fts.

DEFINITION.5.4.22. (Hutton 1975 (14)). An fts is perfectly normal if for every closed set K and open set V with $K \subseteq V$, there exists continuous function $f : X \to I(I)$ such that for every $x \in X$, $K(x) = f(x)(1-) \leq f(x)(0+) = V(x)$.

THEOREM.5.4.23.(Hutton 1975 (14)). An fts (X,T) is perfectly normal iff it is normal and every closed set is a countable intersection of open sets.

The proof is trivial consequence of theorem 5.4.21 and a generalisation of the usual topological proof .

§ .5. Q-CONNECTEDNESS.

In this section ,we discuss connectedness in fuzzy structure with the help of Q-separability.

DEFINITION.5.5.1.(Ming and Ming 1980 (21)). A fuzzy set D in an fts (X,T) is said to be disconnected iff, there exists two non-empty sets A & B in the subspace $D_0 = \text{Supp } D$ such that A and B are Q-separated and $D = A \cup B$. A fuzzy set is called connected iff it is not disconnected.

The 'connectedness' & 'disconnectedness' will be termed as 'Q-connectedness' and 'Q-disconnectedness' respectively.

DEFINITION.5.5.2.(M & M 1980 (21)). Let D be a fuzzy set in an fts (X,T). The maximal connected fuzzy set contained in D is called a component of D.

LEMMA.5.5.3. (M & M 1980 (21)). A fuzzy set D is Q-disconnected iff there are relative closed sets in the subspace D_0 such that $A \cap D \neq \underline{0}$ $B \cap D \neq \underline{0}$ $A \cap B = \underline{0}$ and $A \cup B \supset D$.

PROOF : Left.

THEOREM.5.5.4.(M & M 1980 (21)). Let D be a Q-connected set in an fts (X,T); then $(\overline{D})_X$ and $(\overline{D})_{D_0}$ are also Q-connected where, $(\overline{D})_X$ represents the closure of D in X.

PROOF : Suppose $(\overline{D})_X = E$ is Q-disconnected. Then from lemma 5.5.3., there are relative closed sets A & B in the subspace E_0 such that $A \cup B \supset E$, $A \cap E \neq \underline{0}$, $B \cap E \neq \underline{0}$ and $A \cap B = \underline{0}$.

Obviously, $A \cup B \supset D$. From the Q-connectedness of D, we may assume $A \cap D = \underline{0}$ (for the case where $B \cap D = \underline{0}$, a similar argument holds). That is, $D \subset B$. It follows that $E = E \cap E_0 = (\overline{D})_X \cap E_0 = (\overline{D})_{E_0} \subset (\overline{B})_{D_0} = B$

Since $A \cap B = \underline{0}$, $A \cap E = \underline{0}$, which is a contradiction.

The connectedness of $(\overline{D})_{D_0}$ can be proved analogously by indirect method. (At this time take $E = (\overline{D})_{D_0}$ and note that $E_0 = ((\overline{D})_{D_0})_0 = D_0$)

Now we state a few results without giving the proofs

THEOREM.5.5.5.(M & M 1980 (21)). Let Ω be a family of Q-connected fuzzy sets in an fts (X,T). If no two members of Ω are Q-separated in the subspace $(\cup\,\Omega)_0$, then $\cup\,\Omega$ is Q-connected.

THEOREM.5.5.6.(M & M 1980 (21)). Let D be a fuzzy set in an fts (X,T). Then each Q-connected fuzzy set contained in D is contained in some Q-component of D and any two distinct Q-components of D are Q-separated in the subspace D_0.

THEOREM.5.5.7.(M & M 1980 (21)). If D is a closed fuzzy set in an fts (X,T), then every Q-component of D is a closed set in X.

It is to be noted that a Q-component of a fuzzy set D need not be relative closed in D_0, which is another departure from general topology.

EXAMPLE.(M & M). Let $X = \{ x \}$. Suppose $T = \{ \underline{1}=X, \underline{0}, x_{1/3} \}$. Let $D = x_{1/3}$. Then D is of course a Q-component of D but is not a closed set in $D_0 = X$.

Now we turn our attention to product of Q-connected spaces.

PROPOSITION.5.5.8.(M & M 1980 (21)). Let T_1 and T_2 be two fuzzy topologies on X such that T_2 is finer than T_1 and T_2 is generated by the family Ω of fuzzy sets in X which consists of all the members of T_1 and a number of fuzzy sets each of which takes constant value on X. Then (X_1,T_1) is Q-connected iff (X,T_2) is Q-connected.

THEOREM.5.5.9.(M & M 1980 (21)). A product space (X,T) of a family of fts's $\{(X_\lambda,T_\lambda) ,\lambda \in \Lambda\}$ is Q-connected iff each coordinate space (X_λ,T_λ) is Q-connected.

PROOF : ($\Rightarrow$) It is not difficult to verify that the image of a fuzzy Q-connected set under an F-continuous mapping of fts's is again

Q-connected. So when (X,T) is Q-connected, it follows that (X_λ, T_λ) is also Q-connected ($\because$ $\pi_\lambda(X) = X_\lambda$ and π_λ is F-continuous.)

($\Leftarrow$) Assume that each (X_λ, T_λ) is Q-connected. From proposition 5.5.8 and theorem 3.1.14 ,it is easily seen that the section $\tilde{X}_\mu$ through a point $x \in X$ and parallel to X_μ , considered as a fuzzy subspace of (X,T), is Q-connected.

Now take a point $x = (x_\lambda) \in X$. Consider the Q-component D which contains x, where x is considered as a crisp singleton in X. From the Q-connectedness of the section just mentioned above, it is obvious that if a point $y = (y_\lambda) \in X$ is different from $x = (x_\lambda)$in only one coordinate , y is also contained in D. Moreover,if y is different from x in only a finite number of coordinates, it can be inductively proved that y is also contained in D.

Finally,we shall show that $\bar{D} = X$.

Let $z = (z_\lambda)$ be any point of X. Take an arbitrary Q-nbhd V of z, z being considered as a crisp singleton in X.

From the difinition of product topology, there is a member U , $U = \bigcap_{\lambda \in F} \pi_\lambda^{-1}(U_\lambda)$, of the defining base for T, where $U_\lambda \in T_\lambda$ $\lambda \in F$, F being a finite subset of the index set Λ , such that $U \subset V$ and U is also a Q-nbhd of z.

Then $U(z) = \min_{\lambda \in F} [U_\lambda(z_\lambda)] > 0$.

For $\lambda \notin F$, let $y_\lambda = x_\lambda$; for $\lambda \in F$, let $y_\lambda = z_\lambda$, then we get a point $y = (y_\lambda)$ in X such that y is different from x in only a finite number of coordinates. According to what we have just proved $y \in D$ i.e., $D(y) = 1$ (there y is considered as crisp singleton in X).

Since $U(y) = \min_{\lambda \in F} \{U_\lambda(y_\lambda)\} = \min_{\lambda \in F} \{U_\lambda(z_\lambda)\} > 0$, $D(y) + U(y) > 1$

So U and D are Q-coincident at y. Thus V and D are Q-coincident.at

Since V is an arbitrary Q-nbhd of z, $z \in \overline{D}$. Consequently $\overline{D} = X$. Therefore by theorem 5.5.4., Uis Q-connected and hence X is Q-connected.

§.6. α-CONNECTEDNESS.

In this section we study another type of connectedness known as α-connectedness.

Let (X,T) be an fts and $A \subset X$, $\alpha \in I$. The α-closure (α^*-closure) of A, denoted by $Cl_\alpha(A)$ ($Cl_{\alpha^*}(A)$), is defined to be $\{x \in X : U \in T$ and $U(x) > \alpha$ ($\geq \alpha$) imply $U/_A \neq \underline{0}$. The set A is said to be α-closed (α^*-closed) if $Cl_\alpha(A) \subset A$ ($Cl_{\alpha^*}(A) \subset A$)

DEFINITION.5.6.1.(Rodabaugh 1982 (22)). Let $\alpha \in I = [0,1]$. We say (X,T) is α-connected (α^*-connected) if there do not exist U and V in $T - \{\underline{0},\underline{1}\}$ such that in X $(U \cup V)(x) > 1-\alpha$ ($\geq 1-\alpha$) & $U \cap V = \underline{0}$. (X,T) is said to be α-disconnected (α^*-disconnected) if there are U and V in $T - \{\underline{0},\underline{1}\}$ such that in X $(U \cup V)(x) > \alpha$ ($\geq \alpha$) and $U \cap V = \underline{0}$.

(X,T) is said to be connected if it is 1-connected and disconnected if it is 1^*-disconnected.

REMARK.5.6.2.(Rodabaugh). If $\alpha < \beta$, then β-connected implies α-connected, and β-disconnected implies α-disconnected.

If $\alpha \leq \beta$, then β^*-connected implies α-connected and if $\alpha > \beta$, then α-connected implies β^*-connected.

DEFINITION.5.6.3.(Rodabaugh 1982 (22)). If (X,T) is an fts and $A \subset X$, then A is α-connected if A is α-connected in the fuzzy subspace topology.

DEFINITION.5.6.4.(Rodabaugh 1982 (22)). An α-component of an fts (X,T) is a maximal (with respect to inclusion) α-connected subset of X.

DEFINITION.5.6.5.(Rodabaugh 1982 (22)). An α^*-component of an fts (X,T) is a maximal (with respect to inclusion) α^*-connected subset of X.

PROPOSITION.5.6.7.(Rodabaugh 1982 (22)).Let (X,T) be an fts.& $\alpha \in I$.

(1) Unions of pairwise intersecting α-connected sets are α-connected.

(2) (X,T) is α-connected implies there is not a non-empty subset A of X such that A and A^c are α'-closed where $\alpha' = 1 - \alpha$.

(3) B is α-connected if $A \subset B \subset Cl_{\alpha'}(A)$ and A is α-connected.

(4) Each α-component is α'-closed

Similar statements hold in the α^*- case if $\alpha < 1$.

PROOF : Left.

PROPOSITION.5.6.8.(Rodabaugh 1982 (22)). Fuzzy continuity preserves α-connectivity. A fuzzy homeomorphism maps α-components onto α-components and does so fuzzy homeomorphically. Similar statements hold in the α^*-case.

THEOREM.5.6.9. (Rodabaugh 1982 (22)). Let $\{(X_\lambda,T_\lambda), \lambda \in \Lambda\}$ be a collection of fts and let (X,T) be the product fuzzy space. Then (X,T) is α-connected implies each (X_λ,T_λ) is α-connected.

Similar statement holds in the α^*-case.

PROOF : Left.

* * *

CHAPTER 6

ALGEBRAIC STRUCTURES IN FUZZY SET THEORY

In an analogous application with groups, Rosenfield (23) formulated the the elements of the theory of fuzzy groups. Anthony & Sherwood (1) modified the definitions of fuzzy subgroupoid and fuzzy subgroup. D . Foster (9) brought together the structure of a quasi fuzzy topological space and that of a fuzzy group as defined in (23), to form a combined structure, that of a fuzzy topological group. In the first three sections of this chapter we study the fuzzy groups, the redefined fuzzy groups and the fuzzy topological groups. In the remaining sections we discuss other algebraic structures like metric space in fuzzy set theory and uniformities on fuzzy topological spaces.

.1. FUZZY SUBGROUPS.

In this section we apply the concept the fuzzy sets to generalise the elementary theory of groupoids and groups.

Let X be a groupoid i.e. a set closed under a binary composition (which will be denoted multiplicatively).

DEFINITION.6.1.1. (Rosenfield, 1971 (23)). A fuzzy set A in X is called a fuzzy subgroupoid of X if for all $x, y \in X$ $A(x,y) \geq \operatorname{Min}\{A(x),A(y)\}$

DEFINITION.6.1.2. (Rosenfield, 1971 (23)). A fuzzy set A in X is called a fuzzy left ideal if $A(x,y) \geq A(y)$, a fuzzy right ideal if $A(x,y) \geq A(x)$ and a fuzzy ideal if it is a fuzzy left and right ideal (or equivalently if $A(x,y) \geq \operatorname{Max}\{A(x),A(y)\}$).

Clearly, a fuzzy (left, right) ideal is a fuzzy subgroupoid. Also for any fuzzy subgroupoid in X we have $A(x^n) \geq A(x)$ for all $x \in X$ where x^n is any composite of x's.

PROPOSITION.6.1.3.(Rosenfield 1971 (23)). For any $\alpha \in I$, $\{x : x \in X, A(x) \geq \alpha\}$

is a subgroupoid or (left, right) ideal if A is a fuzzy subgroupoid or fuzzy (left, right) ideal.

PROPOSITION.6.1.4.(Rosenfield, 1971(23)). Let $f: X \to \{0,1\}$ be into so that f is the characteristic function of a subset $Y \subseteq X$. Then f is a fuzzy subgroupoid or (left, right) ideal iff Y is a subgroupoid or (left, right) ideal respectively.

Proof : If $f : X \to \{0,1\}$ is into then, " $f(x,y) \geq \mathrm{Min}\{f(x), f(y)\}$ " is equivalent to " $f(x) = f(y) = 1 \Rightarrow f(x,y) = 1$ " i.e. to " $x, y \in Y \Rightarrow xy \in Y$ ".

Similarly " $f(xy) \geq f(y)$ " is equivalent to " $y \in Y \Rightarrow xy \in Y$ ".

PROPOSITION.6.1.5.(Rosenfield 1971 (23)). The intersection of any sets of fuzzy subgroupoids is a fuzzy subgroupoid.

Proof : Let $\{A_\lambda\}$, $\lambda \in \Lambda$, be a collection of fuzzy subgroupoids of X.

Then $(\cap A_\lambda)(xy) = \mathrm{Inf}\ \{A_\lambda(xy)\} \geq \mathrm{Inf}\ \{\mathrm{Min}\ \{(A_\lambda(x) , A_\lambda(y)\}\}$

$= \mathrm{Min}\ \{\mathrm{Inf}\ A_\lambda(x) , \mathrm{Inf}\ A_\lambda(y)\} = \mathrm{Min}([\cap A_\lambda](x), [\cap A_\lambda](y)$

DEFINITION.6.1.6.(Rosenfield 1971 (23)). The fuzzy subgroupoid (A) generated by the fuzzy set A is defined as the least fuzzy subgroupoid which contains A.

The characteristic function of a subset Y of X will be denoted by χ_Y.

PROPOSITION.6.1.7.(Rosenfield 1971 (23)). For every subset Y of X $(\chi_Y) = \chi_{(Y)}$ where (Y) is the subgroupoid generated by Y.

Proof : If $A \supset \chi_Y$ then we have $A(x) = 1$ for all $x \in Y$.

But A is a fuzzy subgroupoid. So $A(x) = 1$ where x is any composite of elements of Y . Thus $A \supset \chi_{(Y)}$.

Therefore $\chi_{(Y)} \subset$ the intersection of all such A's while conversely, $\chi_{(Y)}$ itself is such an A by proposition 6.1.4.

PROPOSITION.6.1.8.(Rosenfield 1971 (23)). The intersection or union of any sets of fuzzy (left, right) ideals is a fuzzy (left, right) ideal.

Proof : $(\cap A_\lambda)(xy) = \mathrm{Inf}(A_\lambda(xy)) = \mathrm{Inf}(A_\lambda(y)) = (\cap A_\lambda)(y)$ & similarly for union and on the right ideal.

we recall that if A is a fuzzy set and f is a function defined on X, then the fuzzy set B in f(X) defined by $B(y) = \sup_{x \in f^{-1}(y)} A(x)$ for all $y \in f(X)$ is called the image of A under f . Similarly , if B is a fuzzy set in f(X) then the fuzzy set $A = f \circ B$ in X for all $x \in X$ is called the preimage of B under f. Readily if $A = \chi_Y$ where $Y \subseteq X$, then the image of A under f is just $\chi_{f(Y)}$; and if $B = \chi_Z$ where $Z \subseteq f(X)$, then the preimage of B under f is just $\chi_{f^{-1}(Z)}$.

PROPOSITION.6.1.9.(Rosenfield 1971 (23)). A homomorphic preimage of a fuzzy subgroupoid or (left, right) ideal is a fuzzy subgroupoid or (left, right) ideal respectively.

Proof : Straightforward.

DEFINITION.6.1.10.(Rosenfield 1971 (23)). A fuzzy set A in X is said to have the Sup-property if for any subset Y of X, there exists $y_0 \in Y$ such that $A(y_0) = \sup_{y \in Y} A(y)$.

If A takes on only finitely many values (in particular, if it is a characteristic function) then it has the Sup-property.

PROPOSITION.6.1.11.(Rosenfield 1971 (23)). A homomorphic image of a fuzzy subgroupoid which has the Sup-property is a fuzzy subgroupoid. and similarly for (left, right) ideals.

Proof : Given f(x) , f(y) in f(X), let $x_0 \in f^{-1}(f(x))$ & $y_0 \in f^{-1}(f(y))$ be such that $A(x_0) = \sup_{z \in f^{-1}(f(x))} \{A(z)\}$ & $A(y_0) = \sup_{z \in f^{-1}(f(x))} \{A(z)\}$ respectively. Then

$$B(f(x)f(y)) = \sup_{z \in f^{-1}(f(x)f(y))} \{A(z)\} \geq \mathrm{Min}\{A(x_0), A(y_0)\} = \mathrm{Min}\{B(f(x)), B(f(y))\}$$

and similarly for ideals.

Next, we define fuzzy subgroup.

DEFINITION.6.1.12.(Rosenfield 1971 (23)). If X is a group then a fuzzy subgroupoid A of X is called a fuzzy subgroup of X if $A(x^{-1}) \geq A(x)$ for all $x \in X$.

It is readily verified that

PROPOSITION.6.1.13.(rosenfield 1971 (23)). χ_Y is a fuzzy subgroup iff Y is a subgroup of X .

PROPOSITION.6.1.14.(Rosenfield 1971 (23)). The intersection of any collection of fuzzy subgroups is a fuzzy subgroup.

PROPOSITION.6.1.15.(Rosenfield 1971 (23)). The fuzzy subgroup generated by the characteristic function of a set is the characteristic function of the subgroup generated by the set.

PROPOSITION.6.1.16.(Rosenfield 1971 (23)). Let A be a fuzzy subgroup of X ; then $A(x^{-1}) = A(x)$ and $A(x) \leq A(e)$ for all $x \in X$ where e is the identity element of X.

Proof : $A(x) = A((x^{-1})^{-1}) \geq A(x^{-1}) \geq A(x)$.

Hence $A(e) = A(xx^{-1}) \geq \text{Min}\ \{A(x), A(x^{-1})\} = A(x)$.

COROLLARY.6.1.17.(Rosenfield 1971 (23)). $\{x : A(x)=A(e)\}$ is a subgroup of X where A is a fuzzy set in X.

Proof : Use proposition 6.1.3.

We shall denote this subgroup by A_e.

PROPOSITION.6.1.18.(Rosenfield 1971 (23)). $A(xy^{-1}) = A(e) \Rightarrow A(x) = A(y)$.

Proof : $A(x) = A((xy^{-1})y) \geq \text{Min}\{A(e), A(y)\} = A(y) = A((yx^{-1})x)$

$\geq \text{Min}\{A(e), A(x)\} = A(x)$.

COROLLARY.6.1.19.(Rosenfield 1971 (23)). A is constant on each coset of A_e .

COROLLARY.6.1.20.(Rosenfield 1971 (23)). If A_e has a finite index then A has the Sup-property.

PROPOSITION.6.1.21.(Rosenfield 1971 (23)). A is a fuzzy subgroup of X iff $A(xy^{-1}) \geq \text{Min}\ \{A(x), A(y)\}$ for all $x, y \in X$.

Proof : If A is a fuzzy subgroup we have $A(xy^{-1}) \geqslant \mathrm{Min}\{A(x),A(y^{-1})\} = \mathrm{Min}\{A(x),A(y)\}$.

Conversely, if $A(xy^{-1}) \geqslant \mathrm{Min}\{A(x),(A(y)\}$, let $y = x$ to obtain $A(e) \geq A(x)$ for all $x \in X$.

Hence $A(y^{-1}) = A(ey^{-1}) > \mathrm{Min}\{A(e),A(y)\} = A(y)$ and it follows that $A(xy) = A(x(y^{-1})^{-1}) > \mathrm{Min}\{A(x),A(y^{-1})\} \geqslant \mathrm{Min}\{A(x),A(y)\}$.

PROPOSITION.6.1.22.(Rosenfield 1971 (23)). A group cannot be the union of two proper fuzzy subgroups.

Proof : Let A & B be two proper fuzzy subgropps of X such that $A(x) = 1$ or $B(x) = 1$ for all $x \varepsilon X$. Let $u,v \varepsilon X$ be such that $A(u) = 1$, $A(v) < 1$, $B(u) < 1$ and $B(v) = 1$ and take uv.

If $A(uv) = 1$ then, since $A(u^{-1}) = 1$ we would have $A(v) = A(u^{-1}(uv)) \geqslant \mathrm{Min}\{A(u^{-1}),A(uv)\} = 1$, a contradiction.

A similar contradiction is obtained if $B(uv) = 1$.

PROPOSITION.6.1.23.(Rosenfield 1971 (23)). A homomorphic image or preimage of a fuzzy subgroup is a fuzzy subgroup (in the former case provided the Sup-property holds).

Proof : For preimages $A(x^{-1}) = B(f(x^{-1})) = B(x)^{-1}) \geqslant B(f(x)) = A(x)$.

For images, given $f(x) \varepsilon f(X)$ let $x_0 \varepsilon f^{-1}(f(x))$ be such that

$$A(x_0) = \sup_{z \varepsilon f^{-1}(f(x))} A(z); \text{ then } B(f(x)^{-1}) = \sup_{z \varepsilon f^{-1}((f(x))^{-1})} A(z) \geq A(x_0^{-1}) \geqslant A(x_0) = B(f(x)).$$

PROPOSITION.6.1.24.(Rosenfield 1971 (23)). The fuzzy (left, right) ideals in a group are just constant functions.

Proof : Clearly if A is a constant function from X to I then, it is a fuzzy ideal (since, $A(xy) = A(x) = A(y)$ for all x,y in X).

Conversely, let X be a group and A a fuzzy left ideal, so that $A(xy) \geqslant A(y)$ for all x,y. Putting $y = e$ we get $A(x) \geqslant A(e)$ for all $x \in X$ while putting $x = y^{-1}$ we get $A(e) \geqslant A(y)$ for all $y \in X$.

Thus $A = A(e)$ which is a constant function.

PROPOSIIION.6.1.25.(Rosenfield 1971(23)). Let G_r be the cyclic group of prime order r and A any fuzzy subgroup of G_r; then $A(x) = A(1) \leq A(0)$ for all $x \neq 0$ in G_r and conversely any such A is a fuzzy subgroup.

Proof : For any such A, $A(xy) \geq \mathrm{Min}\{A(x),A(y)\}$ is immediate since $0.0=0$ and $A(x^{-1}) \geq A(x)$ is immediate since $-0=0$.

Conversely, for any $x \neq 0$ and $y \neq 0$ in G_r, x is a sum of y's and y is a sum of x's so that $A(x) \geq A(y) \geq A(x)$.

PROPOSITION.6.1.26.(Foster 1979(9)). Let G be a fuzzy subgroup in a group X . Then for all $a \in G_e = \{x : G(x) = G(e)\}$ $\rho_a(G) = \lambda_a(G)=G$ where $\rho_a : x \to xa$ and $\lambda_a : x \to ax$

Proof : Left .

§.2. FUZZY SUBGROUPS REDEFINED.

It is observed that some mathematical structures which would intuitively seem to be 'fuzzy' do not satisfy the definitions 6.1.1. and 6.1.2. Therefore , it would seem appropriate to attempt to modify these definitions in order to include certain mathematical objects which would otherwise be excluded . One of the approaches is to replace Min with some other semigroup on I . In this section we give examples presented by Anthony & Sherwood (1) to motivate such a study and to redefine fuzzy algebraic structures to meet the requirements of the examples . At the end of the section we investigate various properties of the new definitions .

EXAMPLE 6.2.1.(Anthony & Sherwood 1979 (1)). Let X be the collection of all random variables on the probability space $(\Omega,\mathcal{A},P)$ and A be any Borel subset of the reals which is a subgroup of the reals under addition. Ofcourse , X with pointwise addition is a subgroupoid . Define a function $\sigma_A : X \to I$. by $\sigma_A(x)=P\{\omega \in \Omega : x(\omega) \in A\} = P(x^{-1}(A))$.

Now , $\sigma_A(x)$ is the probability that x is 'in' the subgroup A . the function σ_A is a natural candidate for a fuzzy subgroupoid . But σ_A does not satisfy the definition 6.1.1. To see this let $\Omega = I$, be the Borel subsets of I and P Lebesgue measure . Let $\mathbb{A}$ be the integers . Define x &y by

$x(\omega) = 1$ for $\omega\in [0,1/2]$ $\qquad$ $y(\omega) = 1/2$ for $\omega\in [\, 0,1/3\,]$

$\quad = 1/2$ for $\omega\in]1/2,1]$ $\qquad\qquad$ $= 1$ for $\omega\in\,]\, 1/3,1\,]$

Now , x & y are random variables and x+y is defined by

$(x+y)(\omega) = 3/2$ for $\omega\in [0,1/3]$, $=2$ for $\omega\in\,]1/3,1/2]$& $=3/2$ for $\omega\in]1/2,1]$

Clearly , $P(x^{-1}(\mathbb{A})) = 1/2$, $P(y^{-1}(\mathbb{A}))$ 2/3 & $P((x+y)^{-1}(\mathbb{A})) = 1/6$

Since , $\sigma_A(x+y) = 1/6 < \mathrm{Min}\ \{1/2\ ,\ 2/3\} = \mathrm{Min}\{\ \sigma_A(x)\ ,\ \sigma_A(y)\}$

so, the function σ_A is not a fuzzy subgroupoid of X according to definition 6.1.1.

EXAMPLE.6.2.2.(Anthony & Sherwood 1979 (1)). Let G be a group and Ω the family of sub groups of G . For each $x\in G$, Let $A_x = \{S\in\Omega\ :\ x\in S\}$. Let $\mathcal{A}$ be a σ-algebra of subsets of Ω such that $A_x\in\mathcal{A}$ for each x in G and P a probability measure on $(\Omega,\mathcal{A})$. Define a function $m_\Omega : G \to I$ via. $m_\Omega(x) = P(A_x)$ for each $x\in G$. The number $m_\Omega(x)$ can be interpreted as the probability that a sub group chosen at random from the collection Ω will contain x as an element . In this example , it is the 'subgroup' of G,which is 'fuzzy' in that one,is uncertain, of which a subgroup of G is under consideration . The function m_Ω is surely a reasonable candidate for a fuzzy subgroup of G and indeed, it is according to definition 6.1.12 if Ω is linearly ordered by set inclusion . However , if Ω is not linearly ordered then m_Ω need not satisfy definition 6.1.12 . For example , let $(Z,+)$ be the additive group of integers and for each positive integer n let S_n be the subgroup of integral multiples of n . If $\Omega = \{S_2,S_3,S_5\}$,$\mathcal{A}$ is the power set of Ω and P is defined on $(\Omega,\mathcal{Q})$ via. $P(S_2) = P(S_3) = P(S_5) = 1/3$ then $m_\Omega(6) = 2/3$, $m_\Omega(15) = 2/3$ and $m_\Omega(21) = 1/3$.

But $m_\Omega(21) < \text{Min}\{m_\Omega(6), m_\Omega(15)\}$ so that m_Ω is not even a fuzzy subgroupoid of Z according to definition 6.1.1.

DEFINITION.6.2.3.(Anthony & Sherwood 1979 (1)). A t-norm is a function $T : I\times I \to I$ satisfying, for each $\varepsilon,\lambda,\xi,\zeta$ in I

(1) $T(o,o) = o$, $T(\lambda,1) = \lambda = T(1,\lambda)$, (2) $T(\varepsilon,\lambda) \leq T(\xi,\zeta)$ if $\varepsilon\leq\xi$ & $\lambda\leq\zeta$

(3) $T(\varepsilon,\lambda) = T(\lambda,\varepsilon)$ and (4) $T(\varepsilon,T(\lambda,\zeta)) = T(T(\varepsilon,\lambda),\zeta)$.

Obviously the function Min defined od IxI is a t-norm. Other t-norm which are frequently used in the study of probabilistic metric spaces are T_m and Prod. defined by

$T_m(\lambda,\mu) = \text{Max}(\lambda+\mu-1,0)$ & $\text{Prod}(\lambda,\mu) = \lambda\mu$ for each λ,μ in I.

Now we redefine fuzzy algebraic structures in the following way.

DEFINITION.6.2.4.(Anthony & Sherwood 1979 (1)). A fuzzy set A of X is said to be a fuzzy subgroupoid of X with respect to a t-norm T iff for every $x,y\in X$ $A(xy) \geq T(A(x),A(y))$.

DEFINITION.6.2.5.(Anthony & Sherwood 1979 (1)). If X is a group, a fuzzy subgroupoid A of X with respect to a t-norm T, is a fuzzy subgroup of X iff for each x in X, $A(x) = A(x^{-1})$.

The following propositions show that the new definitions eliminate the problem uncovered by the examples 6.2.1 & 6.2.2.

PROPOSITION.6.2.6.(A & S 1979 (1)). The functions σ_A defined in example 6.2.1 are all fuzzy subgroups with respect to the t-norm T_m.

Proof : Let $\omega\in x^{-1}(A)\cap y^{-1}(A)$. Then $x(\omega)\in A$ & $y(\omega)\in A$.

Since A is closed under addition, so $(x+y)(\omega) = x(\omega) + y(\omega) \in A$

Thus, $\omega \in (x+y)^{-1}(A)$ ie. $x^{-1}(A)\cap y^{-1}(A) \subset (x+y)^{-1}(A)$.

$$\begin{aligned}\text{Consequently, } \sigma_A(x+y) &= P((x+y)^{-1}(A)) \geq P(x^{-1}(A)\cap y^{-1}(A))\\ &= P(x^{-1}(A))+P(y^{-1}(A))-P(x^{-1}(A)\cup y^{-1}(A))\\ &\geq \sigma_A(x) + \sigma_A(y) - 1 .\end{aligned}$$

Since, $\sigma_A(x+y) \geqslant 0$ it follows that $\sigma_A(x+y) \geqslant \mathrm{Max}\{\sigma_A(x)+\sigma_A(y)-1,\ 0\}$
$= T_m(\sigma_A(x), \sigma_A(y))$.

Therefore σ_A is afuzzy subgroupoid of X with respect to the t-norm T_m

Moreover, since A is a group, if $x \in X$, then $x^{-1}(A)=\{\omega\in\Omega\ :\ x(\omega)\in A\}$
$=\{\omega\in\Omega\ :-x(\omega)\in A\}=(-x)^{-1}(A)$

Therefore, $\sigma_A(x) = P(x^{-1}(A)) = P((-x)^{-1}(A)) = \sigma_A(-x)$ and hence σ_A is a fuzzy subgroup of X with respect to t-norm T_m .

PROPOSITION.6.2.7. (A & S 1979 (1)). The functions m_Ω defined in example 6.2.2 are all fuzzy subgroups with respect to the t-norm T_m .

Proof : Let $S \in A_x \quad A_y$. Then S is a subgroup of G , $x \in S$ and $y \in S$.
Therefore $xy \in S$ and hence $S \in A_{xy}$. Thus $A_x \cap A_y \subset A_{xy}$.
Now $m_\Omega(xy) \geqslant 0$ and $m_\Omega(xy) = P(A_{xy}) \geqslant P(A_x \cap A_y) = P(A_x)+P(A_y)-P(A_x \cup A_y)$
$\geqslant m_\Omega(x)+m_\Omega(y)-1$

Therefore, $m_\Omega(xy) \geqslant T_m(m_\Omega(x), m_\Omega(y))$ and consequently, m_Ω is a fuzzy subgroupoid of G with respect to t-norm T_m .
Note that if $S \in A_x$ then $x \in S$ and hence $x^{-1} \in S$ i.e. $S \in A_{x^{-1}}$.
Thus $A_x \subset A_{x^{-1}}$ and similarly $A_{x^{-1}} \subset A_x$. Consequently $A_x = A_{x^{-1}}$.
Hence $m_\Omega(x) = P(A_x) = P(A_{x^{-1}}) = m_\Omega(x^{-1})$ and it follows that m_Ω is a fuzzy subgroup of G with respect to t-norm T_m .

PROPOSITION.6.2.8. (A & S 1979 (1)). The functions m_Ω defined in example 6.2.2 are all fuzzy subgroups with respect to the t-norm Min if Ω is linearly ordered by set inclusion.

Proof : Let $x,y \in G$. As Ω is linearly ordered so either $A_x \subset A_y$ or $A_y \subset A_x$
Without loss of generality, suppose $A_x \subset A_y$.
If $S \in A_x$ then $S \in A_y$ and hence $y \in S$.
Since, S is a group so $xy \in S$ which implies that $S \in A_{xy}$.
Thus $A_x \subset A_{xy}$ and $P(A_{xy}) \geqslant P(A_x) \geqslant \mathrm{Min}(\ P(A_x)\ ,\ P(A_y))$.
Therefore, $m_\Omega(xy) \geqslant \mathrm{Min}\{m_\Omega(x)\ ,\ m_\Omega(y)\}$ so that m_Ω is a fuzzy subgroupoid of G with respect to Min.

Moreover, as $A_x = A_{x^{-1}}$, $m_\Omega(x) = m_\Omega(x^{-1})$ and hence m_Ω is a fuzzy subgroup of G with respect to Min.

DEFINITION.6.2.9: (A & S 1979 (1)). A t-norm T is said to be continuous if it is a continuous function (with respect to the usual topology) from $I \times I$ to I .

It is to be observed that Min, Prod, T_m are all continuous t-norms.

PROPOSITION.6.2.10. (A & S 1979 (1)). If A is a fuzzy subgroupoid of X with respect to a continuous t-norm T and if f is a homomorphism on X , then the image of A under any f is a fuzzy subgroupoid on f (X) with respect to T.

Proof : Let A^f be the image of A under a function f defined by X.

Therefore, $A^f(y) = \sup\limits_{x \in f^{-1}(y)} A(x)$ for all $y \in f(X)$.

Let y_1 , $y_2 \in f(X)$, $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$, $A_{12} = f^{-1}(y_1y_2)$ and $A_1A_2 = \{ x \in X : x = a_1a_2$ for some $a_1 \in A_1$ and $a_2 \in A_2\}$.

If $x \in A_1A_2$ then $x = x_1x_2$ for some $x_1 \in A_1$ & $x_2 \in A_2$ so that $f(x) = f(x_1) f(x_2) = y_1y_2$. Thus $A_1A_2 \subset A_{12}$

Since A is a subgroupoid of X with respect to T and $A_1A_2 \subset A_{12}$

so $A^f(y_1y_2) = \sup\limits_{x \in A_{12}} A(x) \geqslant \sup\limits_{x \in A_1A_2} A(x) \geqslant \sup\limits_{x_1 \in A_1; x_2 \in A_2} A(x_1x_2) \geqslant \sup\limits_{x_1 \in A_1, x_2 \in A_2} T(A(x_1),A(x_2))$

Now T is continuous.

Therefore, for given positive number ζ, there exists a number $\delta > 0$ such that if $x_1^* \geqslant \sup\limits_{x_1 \in A_1} A(x_1) - \delta$ & $x_2^* \geqslant \sup\limits_{x_2 \in A_2} A(x_2) - \delta$ then

$T(x_1^*, x_2^*) \geqslant T(\sup\limits_{x_1 \in A_1} A(x_1) , \sup\limits_{x_2 \in A_2} A(x_2)) - \zeta$.

Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $A(a_1) \geqslant \sup\limits_{x_1 \in A_1} A(x_1) - \delta$

$A(a_2) \geqslant \sup\limits_{x_2 \in A_2} A(x_2) - \delta$. Then $T(A(a_1),A(a_2)) \geqslant T(\sup\limits_{x_1 \in A_1} A(x_1), \sup\limits_{x_2 \in A_2} A(x_2)) - \zeta$

Consequently,

$A^f(y_1y_2) \geqslant \sup\limits_{x_1 \in A_1, x_2 \in A_2} T(A(x_1),A(x_2)) \geqslant T(\sup\limits_{x_1 \in A_1} A(x_1), \underset{x_2 \in A_2}{A(x_2)}) = T(A^f(y_1),A^f(y_2))$

and hence A^f is a fuzzy subgroupoid in $f(X)$ with respect to T.

PROPOSITION.6.2.11: (A & S 1979 (23)). If A is a fuzzy subgroup of a group X with respect to a t-norm T then $H = \{ x \in X : A(x) = 1 \}$ is either empty or is a subgroup of X.

Proof : If $x,y \in H$ then $A(xy^{-1}) \geq T(A(x),A(y^{-1})) = T(A(x),A(y)) = T(1,1) = 1$

Therefore $A(xy^{-1}) = 1$ and $xy^{-1} \in H$. Hence H is a subgroup of X.

PROPOSITION.6.2.12: (A & S 1979 (23)). If A is a fuzzy subgroup of a group X with respect to a t-norm T and if there is a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} T(A(x_n),A(x_n)) = 1$ then $A(e) = 1$ where e is the identity in X.

Proof : Let $x \in X$. Then $A(e) = A(xx^{-1}) \geq T(A(x),A(x^{-1})) = T(A(x),A(x))$

Therefore for each n, $A(e) \geq T(A(x_n),A(x_n))$.

As $1 \geq A(e) \geq \lim_{n \to \infty} T(A(x_n),A(x_n)) = 1$, it follows that $A(e) = 1$.

PROPOSITION.6.2.13.(A & S 1979 (23)). Let A be a fuzzy subgroup of a group X with respect to a t-norm T. If $A(xy^{-1}) = 1$ then $A(x) = A(y)$.

Proof : $A(x) = A((xy^{-1})y) \geq T(A(xy^{-1}),A(y)) = T(1,A(y)) = A(y) = A(y^{-1})$

$= A(x^{-1}(xy^{-1})) \geq T(A(x^{-1}),A(xy^{-1})) = T(A(x^{-1}),1) = A(x^{-1}) = A(x)$.

PROPOSITION.6.2.14.(A & S 1979 (23)). Let A be a fuzzy set in a group X and T a given t-norm. If $A(e) = 1$ and $A(xy^{-1}) \geq T(A(x),A(y))$ for all $x,y, \in X$ then A is a fuzzy subgroup of X with respect to T.

Proof : $A(y^{-1}) = A(ey^{-1}) > T(A(e),A(y)) = T(1,A(y)) = A(y)$ and similarly $A(y) \geq A(y^{-1})$. Thus $A(y) = A(y^{-1})$.

Moreover, $A(xy) = A(x(y^{-1})^{-1}) \geq T(A(x),A(y^{-1})) = T(A(x),A(y))$.

Hence A is a fuzzy subgroup of X with respect to T.

. .3. FUZZY TOPOLOGICAL SUBGROUPS .

In this section we study the combined algebraic structure known as fuzzy topological subgroup. For the definition of fuzzy subgroup we follow Rosenfield (23).

- Suppose, G is a fuzzy subgroup in a group X. Let α,β denote the mappings $(x,y) \to (xy)$ and $x \to x^{-1}$ respectively where $x,y \in X$.

The image $\alpha(G\times G)$ of the product fuzzy set $G\times G$ is given by

$$(\alpha(G\times G))(x) = \sup_{(z_1,z_2)\in\alpha^{-1}(x)} (G\times G)(z_1,z_2) = \sup_{(z_1,z_2)\in\alpha^{-1}(x)} \mathrm{Min}(G(z_1),G(z_2))$$

$$\leq \sup_{(z_1,z_2)\in\alpha^{-1}(x)} G(z_1z_2) = G(x) \quad \text{for all } x \in X.$$

Hence $\alpha(G\times G) \subset G$.

Therefore by proposition 6.1.16, $G(x) = G(x^{-1})$ for all $x \in X$.

Hence, $\beta(G) \subset G$.

Next note that if X is given a quasi fuzzy topology T, then G acquires an induced quasi fuzzy topology $T_{(G)}$. By definition $(G,T_{(G)})$ is a quasi fuzzy subspace of the quasi product fts $(X,T)\times(X,T)$.

<u>DEFINITION</u>.6.3.1.(Foster 1977 (9)). Let X be a group and T a quasi fuzzy topology on X. Let G be a fuzzy subgroup in X and let G be endowed with the induced quasi fuzzy topology $T_{(G)}$. Then G is a fuzzy topological subgroup in X iff it satisfies the following two conditions.

(1) The mapping $\alpha : (x,y) \to xy$ of $(G,T_{(G)})\times(G,T_{(G)})$ into $(G,T_{(G)})$ is relatively quasi fuzzy continuous.

(2) The mapping $\beta : x \to x^{-1}$ of $(G,T_{(G)})$ into $(G,T_{(G)})$ is relatively quasi fuzzy continuous.

A fuzzy subgroup structure and an induced quasi fuzzy topology are said to be compatible if they satisfy the conditions (1) and (2).

<u>PROPOSITION</u>.6.3.2.(Foster 1979 (9)). Let X be a group having quasi fuzzy topology T. A fuzzy subgroup G in X is fuzzy topological subgroup iff the mapping $\alpha: (x,y) \to xy^{-1}$ of $(G,T_{(G)})\times(G,T_{(G)})$ into $(G,T_{(G)})$ is relatively quasi fuzzy continuous.

Proof($\Rightarrow$)The mapping $(x,y) \to (x,y^{-1})$ of $(G,T_{(G)})\times(G,T_{(G)})$ into itself is relatively quasi fuzzy continuous (by theorem 3.1.6). Hence, the composition $(x,y) \to (x,y^{-1}) \to xy^{-1}$ is relatively quasi fuzzy continuous.

($\Leftarrow$) By proposition 6.1.6 $G(e) \geq G(x)$ for all $x \in X$ and therefore by theorem 3.1.10 the canonical injection $y \to (e,y)$ of $(G,T_{(G)})$ into $(G,T_{(G)}) \times (G,T_{(G)})$ is relatively quasi fuzzy continuous.
The mapping $\alpha: (x,y) \to xy$ of $(G,T_{(G)}) \times (G,T_{(G)})$ into $(G,T_{(G)})$ is relatively quasi fuzzy continuous because, it is the composition of $(x,y) \to (x,y^{-1}) \to x(y^{-1})^{-1}$ of relatively quasi fuzzy continuous maps.

If G is a fuzzy topological subgroup in a group X carrying a quasi fuzzy topology T then, in general, the translations $\rho_a : x \to xa$ and $\lambda_a : x \to ax$, $a \in X$, are not relatively quasi fuzzy continuous mappings of $(G,T_{(G)})$ into itself. However we have the following special case.

<u>PROPOSITION</u>.6.3.3.(Foster 1979 (9)). Let X be a group having a quasi fuzzy topology T. Let G be a fuzzy topological subgroup in X. For each $a \in G_e = \{x : G(x) = G(e)\}$. the translations ρ_a and λ_a are relatively quasi fuzzy homeomorphisms of $(G,T_{(G)})$ into itself.
Proof : From proposition 6.1.26 we note that $\rho_a(G) = G$ and $\lambda_a(G)=G$ for all $a \in G_e$. The mapping λ_a is the composition of the injection $y \to (a,y)$ and the mapping $(x,y) \to xy$.
Since, $G(a) \geq G(y)$ for all $y \in G$, it follows from proposition 3.1.10 that the mapping $y \to (a,y)$ is a relatively quasi fuzzy continuous mapping of $(G,T_{(G)})$ into $(G,T_{(G)}) \times (G,T_{(G)})$. The mapping $(x,y) \to xy$ is relatively quasi fuzzy continuous, by hypothesis.
Hence λ_a is relatively quasi fuzzy continuous, and therefore, $\lambda_a^{-1} = \lambda_{a^{-1}}$ also. The relatively quasi fuzzy continuity of ρ_a and ρ_a^{-1} can be shown similarly.

The following proposition show that the induced quasi fuzzy topology on $f^{-1}(G)$ and subgroup structure are compatible.

PROPOSITION.6.3.4.(Foster 1979 (9)). Given groups X,Y ,a homomorphism f of X into Y and a quasi fuzzy topology $\mathcal{U}$ on Y , let X have the quasi fuzzy topology T , where T is the inverse image under f of $\mathcal{U}$ and let G be a fuzzy topological subgroup in Y . Then the inverse image $f^{-1}(G)$ of G is a fuzzy topological subgroup in X .

Proof : Left .

PROPOSITION.6.3.5.(Foster 1979 (9)). Given group X,Y , a homomorphism f of X into Y and a quasi fuzzy topology T on X , let Y have fuzzy topology $\mathcal{U}$ where $\mathcal{U}$ is the image under f of T and let G be a fuzzy topological subgroup in X . If the function G is f-invariant , then the image f(G) of G is a fuzzy topological subgroup in Y .

Proof : Left .

Given a group X carrying a quasi fuzzy topology T , and G a fuzzy topological subgroup in X , let N be a normal subgroup of X and let ϕ be the cannonical homomorphism of X onto the quotient group X/N . If the function G is constant on N , then G is ϕ -invariant and the image $\phi(G)$ is accordingly a fuzzy subgroup in X/N . We call $\phi(G)$ a quotient fuzzy subgroup and denote it by G/N .

PROPOSITION.6.3.6.(Foster 1979 (9)). Let X be a group having quasi fuzzy topology T , G a fuzzy topological subgroup in X & N a normal subgroup of X . Let the quotient group X/N be given the quasi fuzzy topology which is the image of T under the cannonical homomorphism ϕ Then if the function G is constant on N , the quotient fuzzy subgroup G/N is a quasi fuzzy topological group in X/N .

Proof : Apply proposition 6.3.5.

We refer to the above quasi fuzzy topology on the quotient group X/N as the quotient quasi fuzzy topology and to G/N as a quotient quasi fuzzy topological subgroup .

PROPOSITION.6.3.7.(Foster 1979 (9)) Let X,Y be groups & f a homomorphism of X onto Y . Let T be a quasi fuzzy topology on X , $\mathfrak{U}$ a quasi fuzzy topology on Y , and f both quasi fuzzy continuous and quasi fuzzy open . Let G be a fuzzy topological subgroup in X such that the function G is constant on the Kernel $f^{-1}(e)$ of f . Let the quotient group $X/f^{-1}(e)$ have the quotient quasi fuzzy topology . Then

(1) The fuzzy subgroups $G/f^{-1}(e)$ and f(g) are fuzzy topological subgroups in $X/f^{-1}(e)$ and Y respectively .

(2) The cannonical isomorphism $\tilde{f}$ of $X/f^{-1}(e)$ onto Y is a relative quasi fuzzy homeomorphism of $G/f^{-1}(e)$ onto f(G) .

Proof : Left .

Now we discuss the products of fuzzy topological subgroups . Let $\{X_n\}$ n = 1,2,.......,m be a finite family of groups and X the product group . For each n = 1,2,.......,m, let X_n have quasi fuzzy topology T_n and G_n be a fuzzy topological subgroup in X_n . The product fuzzy set $G = \prod_{n=1}^{m} G_n$ in X is given by

$$G(x) = \min\{G_1(x_1),\ldots\ldots,G_m(x_m)\} \quad ,\text{where } x = (x_1 ,x_2 ,\ldots\ldots,x_m).$$

It follows that G is a fuzzy subgroup in X , since for all $x,y \in X$

$$\begin{aligned} G(xy^{-1}) &= G(x_1y_1^{-1},\ldots\ldots\ldots,x_my_m^{-1})=\min(G_1(x_1y_1^{-1}),\ldots,G_m(x_my_m^{-1})) \\ &> \min\{\min\{G_1(x_1),G_1(y_1)\},\ldots,\min\{G_m(x_m),G_m(y_m)\}\} \\ &=\min\{\min\{G_1(x_1),\ldots,G_m(x_m)\},\min\{G_1(y_1),\ldots,G_m(y_m \\ &=\min\{G(x),G(y)\} \end{aligned}$$

We call G the product of the fuzzy subgroups $\{G_n\}$ n = 1,2,...,m

The product group X has associated with it the product quasi fuzzy topology . The next proposition shows that the induced quasi fuzzy topology on G and the product fuzzy subgroup structure are compatible .

PROPOSITION.6.3.8.(Foster 1979 (9)). Let $\{X_n\}$, $n = 1,2,\ldots,m$ be a finite family of groups and for each $n=1,2,\ldots,m$, let T_n be a quasi fuzzy topology on X_n and G_n a fuzzy topological sub group in X_n . Let the product group $X = \prod_{n=1}^{m} X_n$ have the product quasi fuzzy topology T . Then the product fuzzy group $G = \prod_{n=1}^{m} G_n$ is a fuzzy topological subgroup in X .

Proof : Left .

We refer to $G = \prod_{n=1}^{m} G_n$ as a product fuzzy topological subgroup . The results of propositions 6.3.6 and 6.3.8 may be combined to yield the following

PROPOSITION.6.3.9.(Foster 1979 (9)). Let $\{X_n\}$ $n=1,\ldots,m$ be a finite family of groups and for each $n=1,\ldots,m$ let T_n be a quasi fuzzy topology on X_n and N_n a normal subgroup in X_n and G_n a fuzzy topological subgroups in X_n such that G_n is constant in N_n . Let the quotient group X/N , where $N = \prod_{n=1}^{m} N_n$ and X_n/N_n $n = 1,\ldots,m$ have the respective quotient quasi fuzzy topologies and the product groups $X = \prod_{n=1}^{m} X_n$ and $\prod_{n=1}^{m} (X_n/N_n)$ the respective product quasi fuzzy topologies . Let [illegible] =

Let $G = \prod_{n=1}^{m} G_n$ be the product fuzzy topological subgroup in X. Then the canonical isomorphism f of X/N onto $\prod_{n=1}^{m} (X_n/N_n)$ is a relatively quasi fuzzy homeomorphism of the quotient fuzzy topological subgroup G/N onto the product fuzzy topological subgroup $\prod_{n=1}^{m} (G_n/N_n)$.

Proof : Left.

§.4.METRIC SPACES IN FUZZY THEORY.

This section is devoted to the study of metric spaces in fuzzy set theory. In general topology it is observed that a pseudo-quasi metric (p.q.metric) on a set may also be equivalently treated as a distance function between subsets of X. This equivalent definition is generalised to fuzzy set theory where points need not have Boolean properties and hence in which a naive generalisation of a p.q.metric is unsatisfactory.

Let X be a classical set and I the unit closed interval [0,1] . The collection of all fuzzy sets $A : X \to I$ is denoted by I^X.

<u>DEFINITION</u>.6.4.1.(Erceg 1979 (8)). A fuzzy p.q.metric on X is a map $p : I^X \times I^X \mapsto [0,\infty]$ satisfying the following conditions:

(M1) $p(\underline{0}\ , A) = \infty\ \forall\ A \in I^X\ ,\ A = \underline{0}$

$p(A\ , A) = 0$ and $p(A\ , \underline{0}) = \infty$ for all $A \in I^X$.

(M2) $p(A\ ,B) \leq p(A\ , C) + p(C\ , B)$ for all $A,B,C \in I^X$.

(M3) (1) $A \subset B \Rightarrow p(A\ , C) \geq p(B\ , C)$ for all $C \in I^X$.

(2) $p(C\ , \bigcup_\lambda A_\lambda) = \bigvee_\lambda p(C\ , A_\lambda)$ for all $C,A_\lambda \in I^X$.

(M4) Suppose $C,A_\lambda \in I^X$ for all $\lambda \in \Lambda$ and $r \in\ ]0,\infty[$. If $p(A_\lambda, B) < r \Rightarrow B \subset C$ for $B \in I^X$, $\lambda \in \Lambda$ then $p(\bigcup_\lambda A_\lambda, D) < r \Rightarrow D \subset C$ for $D \in I^X$.

<u>DEFINITION</u>.6.4.2.(Erceg 1979 (8)). For all $r \in]0,\infty[$ let $D_r : I^X \to I^X$ be defined by $D_r(A) = \bigcup \{ B : p(A,B) < r \}$. Then $\{D_r\ ;\ r > 0\}$ is called the associated neighborhood maps of p.

<u>THEOREM</u>.6.4.3.(Erceg 1979 (8)). The following statements are valid for all $r \in\ ]0,\infty[$.

(A1) $D_r(\underline{0}) = \underline{0}$. (A2) $A \subset D_r(A)$. (A3) $D_r(\bigcup_\lambda A_\lambda) = \bigcup_\lambda D_r(A_\lambda)$.

Proof : (A1) & (A2) follow from (M1) and definition 6.4.2.

Now $p(A_\mu,B) < r \Rightarrow p(\bigcup_\lambda A_\lambda,B) \underset{< r}{<} p(A_\mu,B)$ (since, $A_\mu \subset \bigcup_\lambda A_\lambda$).

So $B \subset D_r(\bigcup_\lambda A_\lambda)$ and hence $D_r(A_\mu) = \bigcup\{ B : p(A_\mu,B) < r\} \subset D_r(\bigcup_\lambda A_\lambda)$

i.e. $\bigcup_\lambda D_r(A_\lambda) \subset D_r(\bigcup_\lambda A_\lambda)$.

By (M4) $p(\bigcup_\lambda A_\lambda,C) < r \Rightarrow C \subset \bigcup_\lambda D_r(A_\lambda)$. Thus $D_r(\bigcup_\lambda A_\lambda) \subset \bigcup_\lambda D_r(A_\lambda)$.

DEFINITION.6.4.4.(Erceg 1979 (8)). If $f : I^X \to I^X$ satisfies (A1)-(A3) we define its inverse to be $f^{-1} : I^X \to I^X$ where $f^{-1}(A) = \{B : f(B) \subset A^c\}$.

It is clear that (A1) is necessary in order that f^{-1} be well-defined.

THEOREM.6.4.5.(Hutton1977 (14)). If $f : I^X \to I^X$ satisfies (A1) - (A3) then so does f^{-1}. Furthor if f and g satisfy (A1) - (A3) then

$$(1)\ f(A) \subset B \Leftrightarrow f^{-1}(B^c) \subset A^c \qquad (2)\ (f^{-1})^{-1} = f$$

$$(3)\ f \subset g \Leftrightarrow f^{-1} \subset g^{-1} \qquad (4)\ (f \circ g)^{-1} = g^{-1} \circ f^{-1}\ .$$

Proof : Straightforward.

Unless otherwise mentioned, if f satisfies (A1) - (A3), then f^{-1} will be used to denote the inverse as defined above, rather than the usual function inverse.

The next result indicates the importance of the neighborhood maps.

THEOREM.6.4.6.(Erceg 1979 (8)). If p is a fuzzy p.q.metric on X with associated neighborhood maps $\{D_r : r > 0\}$ then, for all $A, B \in I^X$

$p(A,B) = \wedge\{r : B \subset D_r(A)\}$.

Proof : Let $f(A,B) = \wedge\{r : B \subset D_r(A)\}$. Then, for all $r > p(A,B)$, $B \subset D_r(A)$ so that $f(A,B) \leqslant r$. Hence $f(A,B) \leqslant p(A,B)$.

Now for all r with $B \subset D_r(A)$

$$p(A,B) \leqslant p(A, D_r(A)) = p(A, \cup\{C : p(A,C) < r\}) = \vee\{p(A,C) : p(A,C) < r\} \quad \text{by (M3) (2)}$$

$$\leqslant r$$

Hence $p(A,B) \leqslant \{r : B \subset D_r(A)\} = f(A,B)$.

THEOREM.6.4.7.(Erceg 1979 (8)). If p is a fuzzy p.q.metric with associated neighborhood maps D_r, then $D_r \circ D_s \subset D_{r+s}$ for all $r,s>0$.

Proof : Follows directly from (M2).

Now we give a partial converse to theorem 6.4.6.

THEOREM.6.4.8.(Erceg 1979 (8)). If $\Delta = \{D_r \,;\, r \in]0,\infty[\}$ is a family of maps $D_r : I^X \to I^X$ satisfying (A1) - (A3) and such that for all $r,s \in]0,\infty[$, $D_r \circ D_s \subset D_{r+s}$ then $p : I^X \times I^X \to [0,\infty]$ defined by $p(A,B) = \wedge\{r : B \subset D_r(A)\}$ is a fuzzy p.q.metric on X.

Further, its associated neighborhood maps E_r, say, are given by $E_r = \bigcup\limits_{s<r} D_r$ i.e. $E_r(A) = \bigcup\limits_{s<r} D_s(A)$ for $A \in I^X$.

Proof : (M1) follows from the definition of p and properties (A1) & (A2) of D_r

(M2) Let $A,C,B \in I^X$.

Now for all $r > p(A,C)$, for all $s > p(C,B)$ we have $C \subset D_r(A)$ & $B \subset D_s(C)$. Hence $B \subset D_s \circ D_r(A) \subset D_{s+r}(A)$. So by definition of p, $p(A,B) \leq r+s$ i.e. $p(A,B) \leq p(A,C) + p(C,B)$.

(M3) (1) Let $A \subset B$. Then for all $r > p(A,C)$, $C \subset D_r(A)$.

So $C \subset D_r(B)$ by (A3). Hence $p(B,C) \leq r$ which gives $p(B,C) \leq p(A,C)$.

(M3) (2) Let $r > p(B, \bigcup\limits_{\lambda} A_\lambda)$. Then $\bigcup\limits_{\lambda} A_\lambda \subset D_r(B)$

Therefore $A_\lambda \subset D_r(B)$ $\forall \lambda \Rightarrow p(B,A_\lambda) \leq r$ $\forall \lambda \Rightarrow \bigvee\limits_{\lambda} p(B,A_\lambda) \leq r$

Hence $\bigvee\limits_{\lambda} p(B,A_\lambda) \leq p(B, \bigcup\limits_{\lambda} A_\lambda)$.

Let $r > \bigvee\limits_{\lambda} p(B,A_\lambda)$. Then $r > p(B,A_\lambda)$ $\forall \lambda$

Therefore $A_\lambda \subset D_r(B)$ $\forall\lambda \Rightarrow \bigcup\limits_{\lambda} A_\lambda \subset D_r(B) \Rightarrow p(B, \bigcup\limits_{\lambda} A_\lambda) \leq r$

Hence $p(B, \bigcup\limits_{\lambda} A_\lambda) \leq \bigvee\limits_{\lambda} p(B,A_\lambda)$

Let us (now) prove that $E_r = \bigcup\limits_{s<r} D_s$.

For all $B \subset D_r(A)$, $p(A,B) \leq r < s$ $\forall s > r$ and so $B \subset E_s(A)$ $\forall s>r$

Hence $D_r(A) \subset E_s(A)$ for all $s > r$ or $D_s(A) \subset E_r(A)$ for all $s < r$, so that $\bigcup\limits_{s<r} D_s(A) \subset E_r(A)$.

Now $\forall$ B such that $p(A,B) < r$ we have $p(A,B) < t$ for some $t < r$.

Hence $B \subset D_t(A) \subset \bigcup\limits_{s<r} D_s(A)$ i.e. $E_r(A) = \bigcup\{B : p(A,B) < r\} \subset \bigcup\limits_{s<r} D_s(A)$.

Finally we show that (M4) is satisfied.

Suppose $B, A_\lambda \in I^X$ $\forall\ \lambda \in \Lambda$ satisfy $p(A_\lambda, C) < r \Rightarrow C \subset B$ for $C \in I^X$, $\lambda \in \Lambda$. If $D \in I^X$ satisfies $p(\bigcup_\lambda A_\lambda, D) < r$ then
$D \subset E_r(\bigcup_\lambda A_\lambda) = \bigcup_{s<r} D_s(\bigcup_\lambda A_\lambda) = \bigcup_{s<r} \bigcup_\lambda D_s(A_\lambda) = \bigcup_\lambda E_r(A_\lambda) \subset B$ by hypothesis.

The following result is a symmetric version of the theorem 6.4.8.

THEOREM.6.4.8(a).(Erceg 1979 (8)). If $\Delta = \{D_r : r \in]0,\infty[\}$ is a family of maps $D_r : I^X \to I^X$ satisfying (A1) -(A4) such that for all $r,s \in]0,\infty[$ $D_r \circ D_s \subset D_{r+s}$, then $d : I^X \times I^X \to [o,\infty]$ defined by $d(A,B) = \{r ; B \subset D_r(A)\}$ is a fuzzy p . metric on X with neighborhood maps $E_r = \bigcup_{s<r} D_s$.

Proof : Left.

Next, we consider the topology of the fuzzy p.q.metric space.

THEOREM.6.4.9.(Erceg 1979 (8)). If p is a fuzzy p.q.metric on X with associated neighborhood maps D_r, then $\{D_r(A) : A \in I^X, r \in]0,\infty[\}$ is a base for a topology on the fuzzy space X.

(This topology will be called the topology of the fuzzy p.q.metric p)

Proof : It must be shown that the arbitrary supremums of this set together with $\underline{0}$ and $\underline{1}$ form a fuzzy topology. For this it is enough to prove that for all $A,B \in I^X$ and $t,s \in]0,\infty[$ there exists K_λ in I^X and t_λ in $]0,\infty[$, $\lambda \in \Lambda$, such that $D_r(A) \cap D_s(B) = \bigcup_{\lambda\in\Lambda} D_{t_\lambda}(K_\lambda)$.

Let $K = D_r(A) \cap D_s(B)$. If $K = \underline{0}$ then $K = D_r(\underline{o})$.

If $K \neq \underline{0}$ then $K = [\bigcup\{C_1 : p(A,C_1) < r\}] \cap [\bigcup\{C_2 : p(B,C_2) < s\}]$

$= \bigcup_{C_1}\bigcup_{C_2} \{C_1 \cap C_2 : p(A,C_1) < r \text{ and } p(B,C_2) < s\}$

$= \bigcup\{C_\lambda : \lambda \in \Lambda\}$,say, where $C_\lambda = C_{\lambda_1} \cap C_{\lambda_2}$ and $p(A,C_{\lambda_1}) < r$ and $p(B,C_{\lambda_2}) < s$

Now for all $\lambda \in \Lambda$, let $t_\lambda = (r - p(A,C_{\lambda_1})) \wedge (s - p(B,C_{\lambda_2}))$ & $K_\lambda = C_\lambda$.

If $p(K_\lambda, D) < t_\lambda$ then $p(K_\lambda, D) < r - p(A,C_{\lambda_1})$.

As $C_{\lambda_1} \supset K_\lambda \Rightarrow p(C_{\lambda_1}, D) \leq p(K_\lambda, D)$ hence, $p(C_{\lambda_1}, D) + p(A, C_{\lambda_1}) < r$

Therefore by (M2) $p(A,D) < r$ and hence $D \subset D_r(A)$.

Similarly $D \subset D_s(B)$. Thus $D_{t_\lambda}(K_\lambda) \subset K$ for all $\lambda \in \Lambda$ which implies that $\bigcup_{\lambda \in \Lambda} D_{t_\lambda}(K_\lambda) \subset K$.

Now $C_\lambda = K_\lambda \subset D_{t_\lambda}(K_\lambda) \Rightarrow K = \bigcup_\lambda C_\lambda \subset \bigcup_\lambda D_{t_\lambda}(K_\lambda)$.

REMARK : Since $A \subset D_r(A)$ for $r \in]0,\infty[$, $D_r(A)$ is indeed a neighborhood of $A \in I^X$.

THEOREM.6.4.10.(Erceg 1979 (8)). In the topology of the fuzzy p.q.metric p with neighborhood maps D_r, $A^o = \bigcup\{C : D_r(C) \quad A$ for some $r > 0\}$

Proof : Let $F = \bigcup \{C : D_r(C) \subset A$ for some $r > 0\}$.

Now $A^o = \bigcup\{C : C \subset A,\ C$ is fuzzy open$\} = \bigcup\{D_r(K) : K \in I^X, r > 0$ & $D_r(K) \quad C\}$.

If $C \in I^X$ satisfies $D_r(C) \subset A$ for some $r > 0$ then $C \subset D_r(C) \subset A^o$.

Hence $F \subset A^o$.

Again if $D_r(K) \subset A$ for some $K \in I^X$, $r > 0$ and if B satisfies $p(K,B) < r$ then there exists $s \in]0,r[$ such that $p(K,B) < s$ and so $B \subset D_s(K)$.

Thus $D_{r-s}(B) \subset D_{r-s} \circ D_s(K) \subset D_r(K)$, by theorem 6.4.7, & as a result we have $D_{r-s}(B) \subset A$.

Hence $B \subset F \Rightarrow D_r(K) \subset F \Rightarrow A^o \subset F$.

THEOREM.6.4.11.(Erceg 1979 (8)). In the fuzzy p.q.metric space (X,p,D_r) (where all the symbols have usual meaning) $\bar{A} = \bigcap_{r>0} D_r^{-1}(A)$.

(Recall that D_r^{-1} is the inverse in theorem 6.4.5.)

Proof : By theorem 6.4.10, $A^o = \bigcup \{K : D_r(K) \subset A$ for some $r > 0\}$

Thus $A^o = \bigcup \{K : D_r^{-1}(A^c) \subset K^c$ for some $r > 0\}$ (by theorem 6.4.5)

$= \bigcup \{K : K \subset (D_r^{-1}(A^c))^c$ for some $r > 0\} = \bigcup_{r>0} (D_r^{-1}(A^c))^c$

Therefore by theorem 1.4.9 $\bar{A} = (\bigcup_{r>0} (D_r^{-1}(A^c)^c)^c = \bigcap_{r>0} D_r^{-1}(A)$.

DEFINITION.6.4.12.(Erceg 1979 (8)). A fuzzy pseudo metric(p.metric) on X is afuzzy p.q.metric d with neighborhood maps D_r satisfying

$D_r = D_r^{-1}$ for all $r \in]0,\infty[$ ....(A4).

THEOREM.6.4.13.(Erceg 1979 (8)). In a p.metric space (X , D_r , d)

$$\overline{A} = \bigcap_{r>0} D_r(A) = \bigcup \{B : d(A,B) = 0\} .$$

Proof : By theorem 6.4.11 $\overline{A} = \bigcap_{r>0} D_r(A)$.

Let $F = \bigcup\{B : d(A,B) = 0\}$. Clearly $F \subset \bigcap_{r>0} D_r(A)$.

Now suppose $B \subset \bigcap_{r>0} D_r(A)$. Then for all $r > 0$, $B \subset D_r(A)$, so that

$$d(A,B) \leq d(A,D_r(A)) \text{ for all } r > 0$$

$$= r \text{ for all } r > 0.$$

Thus $d(A,B) = 0$ and consequently $\bigcap_{r>0} D_r(A) = F$.

COROLLARY.6.4.14.(Erceg 1979 (8)). In a fuzzy p.metric space (X,d,D_r)

$\overline{D_r(A)} \subset D_s(A)$ for all $s > r$.

Proof : Straightforward.

REMARK. (Erceg) The reverse inequality need not hold, even in the usual set theory. Consider for example, a p.metric on two element set giving the discrete topology.

THEOREM.6.4.15.(Erceg 1979 (8)). Every fuzzy p.metric space (X,d,D_r) is normal.

Proof : Let $A,B \in I^X$ where $A = \overline{A}$ and $B = B^o$ in the pseudo-metric topology.

Therefore $\overline{A} = \bigcap_{r>0} D_r(A)$ and $A \subset B$ give $B = A \cup B = \bigcap_{r>0} (D_r(A) \cup B)$

$B^o = \bigcup \{K : D_r(K) \subset B \text{ for some } r > o\} = \bigcup \{K_\lambda : \lambda \in \Lambda \}$ say, where for all $\lambda \in \Lambda$, there exists $r_\lambda \in \,]0,\infty[$ so that $D_{r_\lambda}(K_\lambda) \subset B$.

Hence $A = A \cap B = \bigcup_{\lambda\in\Lambda} (A \cap K_\lambda)$.

Let $C = \bigcup_{\lambda\in\Lambda} D_{r_\lambda/2}(A \cap K_\lambda)$. Then $C = C^o$ and $A \subset C \subset \overline{C}$.

$$\text{Now } \overline{C} = \bigcap_{s>0} D_s(C) = \bigcap_{s>0} D_{s/2}(C) = \bigcap_{s>0} \bigcup_{\lambda\in\Lambda} (D_{s/2}(D_{r_\lambda/2}(A \cap K_\lambda)))$$

$$\subset \bigcap_{s>0} \bigcup_{\lambda\in\Lambda} D_{\frac{s}{2}+\frac{r}{2}} (A \cap K_\lambda) \subset \bigcap_{s>0} \bigcup_{\lambda\in\Lambda} D_{s \vee r_\lambda}(A \cap K_\lambda)$$

$$= \bigcap_{s>0} \bigcup_{\lambda\in\Lambda} [D_s(A \cap K_\lambda) \cup D_{r_\lambda}(A \cap K_\lambda)] \subset \bigcap_{s>0} \bigcup_{\lambda\in\Lambda} [D_s(A) \cup B]$$

$$= \underset{s>0}{} [D_s (A) \cup B] = B .$$

Next, we define conjugate fuzzy p.q.metric.

<u>THEOREM</u>.6.4.16.(Erceg 1979 (8)). Let (X,p,D_r) be a fuzzy p.q.metric space. Define $q\,; I^X \times I^X \rightarrow [0,\infty]$ by $q(A,B) = \wedge\{r : B \subset D_r^{-1}(A)\}$ .Then q is a fuzzy p.q.metric on X with associated neighborhood maps $\{D_r^{-1} : r \in]0,\infty[\}$.

Proof : Left.

<u>DEFINITION</u>.6.4.17.(Erceg 1979 (8)). The fuzzy p.q.metric q, defined above, is said to be the conjugate of p.

<u>PROPOSITION</u>.6.4.18.(Erceg 1979 (8)). In the fuzzy p.q.metric space (X,p,D_r) $\bar{A} = \cup\,\{B : q(A,B) = 0\}$.

Proof : From theorem 6.4.11 $\bar{A} = \bigcap\limits_{r>0} D_r^{-1}(A)$. But $D_r^{-1}(A) = \cup\{B : q(A,B) < r\}$

Thus $\bar{A} = \bigcap\limits_{r>0} \cup\,\{B : q(A,B) < r\} \supset \cup\,\{B : q(A,B) = 0\}$.

Consersely, if $C \subset D_r^{-1}(A)$ for all $r \in]0,\infty[$ then $q(A,C) < r \;\forall\; r \in]0,\infty[$ so that $q(A,C) = 0$. Thus $C \subset \cup\,\{B : q(A,B) = 0\}$.

Hence $\bar{A} \subset \cup\,\{B : q(A,B) = 0\}$.

REMARK.(Erceg). Theorem 6.4.13 is the special case of proposition6.4.18 when p = q .

If p and q are conjugate p.q.metrics in the usual sense then the map d defined by $d(x,y) = p(x,y) \vee q(x,y)$ is a pseudometric *in the usual sense. The following result generalises this method.*

<u>PROPOSITION</u>.6.4.19.(Erceg 1979 (8)). If (X,p,D_r) and (X,q,D_r^{-1}) are conjugate fuzzy p.q.metrics then (X,d,E_r) is a fuzzy p.metric where $E_r = \bigcup\limits_{s \leqslant r} (D_s \cap D_s^{-1})$ and $d(A,B) = \wedge\{r : B \subset (D_r \cap D_r^{-1})(A)\}$.

Proof : Since $\{D_r \cap D_r^{-1} : r \in]0,\infty[\}$ satisfy (A1) - (A4) it is enough to show that $(D_r \cap D_r^{-1}) \circ (D_s \cap D_s^{-1}) \quad D_{r+s} \circ D_{s+r}^{-1}$ and then to apply theorem 6.4.8(e).

Now $(D_r \cap D_r^{-1}) \circ (D_s \cap D_s^{-1}) \subset D_r \circ D_s \subset D_{r+s}$.

Similarly, $(D_r \cap D_r^{-1}) \circ (D_s \cap D_s^{-1}) \subset D_{r+s}^{-1}$.

Since $D_{r+s} \cap D_{r+s}^{-1}$ is the largest map satisfying (A1) - (A3) which is smaller than both D_{r+s} and D_{r+s}^{-1} , it is evident that

$(D_r \cap D_r^{-1}) \circ (D_s \cap D_s^{-1}) \subset D_{r+S} \cap D_{r+S}^{-1}$.

We conclude the section with the definition of fuzzy metric.

DEFINITION. 6.4.20.(Erceg 1999 (8)). A fuzzy pseudo metric p in (X,p,D_r) is said to be a fuzzy metric if it satisfies

$\forall A \in I^X \quad (\bigcap_{r>0} D_r)(A) = A$ ....(A5)

where $(\bigcap_{r>0} D_r : I^X \to I^X$ is the largest map satisfying (A1)-(A3) &

$(\bigcap_{r>0} D_r)(A) \subset D_s(A)$ for all $s \in]0,\infty[$.

§.5. UNIFORMITIES ON FUZZY TOPOLOGICAL SPACES.

In this section we extend the notions of quasi uniformities on topological spaces to fts. We establish results corresponding to many of the usual theorems of ordinary general topology. WE also construct a natural uniformity on the unit interval in fuzzy structure.

Consider a quasi uniformity on X in the usual topological sense. Take an element D as a subset of X × X. We may define $D : \quad \to \quad$ by

$D(V) = \{y : x \in V \text{ and } (x,y) \in D\}$

It is obvious that $V \subset D(V)$ and $D(\bigcup_\lambda V_\lambda) = \bigcup_\lambda D(V_\lambda)$ for V & V_λ in .

Conversely , given $D : \quad \to \quad$ satisfying $V \subset D(V)$ & $D(\bigcup_\lambda V_\lambda) = \bigcup_\lambda D(V_\lambda)$ we may define $D \subset X \times X$, such that D contains the diagonal, by

$D = \{(x,y) : y \in D(\{x\})\}$.

Thus in defining a quasi uniformity for a fuzzy topology, we take our basic elements of the quasi uniformity to be the elements of the set $\mathcal{Q}$ of maps $D : I^X \to I^X$ which satisfy

(A1) $D(\underline{0}) = \underline{0}$ (A2) $V \subset D(V)$ for all $V \in I^X$ and (A3) $D(\bigcup_\lambda V_\lambda) = \bigcup_\lambda D(V_\lambda)$ for $V_\lambda \in I^X$.

DEFINITION.6.5.1.(Hutton 1977 (14)). A fuzzy quasi uniformity on a set X is a subset Δ of Ω (the set of all maps satisfying (A1)-(A2)) such that

(Q1) $\Delta \neq \Phi$

(Q2) $D \in \Delta$ and $D \subset E \in \Omega \Rightarrow E \in \Delta$.

(Q3) $D \in \Delta$ and $E \in \Delta \Rightarrow D \cap E \in \Delta$.

(Q4) $D \in \Delta$ there exists $E \in \Delta$ such that $E \circ E \subset D$.

Note that this definition agrees with the usual definition when we replace $I = [0,1]$ by the set $\{0,1\}$.

It is also to be noted that (Q3) may be replaced by

(Q3') $D_1 \in \Delta$ and $D_2 \in \Delta$ there exists $D \in \Delta$ such that $D \subset D_1$ & $D \subset D_2$

Before we define the fyzzy topology generated by a quasi uniformity we state the following trivial result.

THEOREM.6.5.2.(Hutton 1977 (14)). If a map $f ; I^X \to I^X$ satisfies the interior axioms

(I1) $f(X) = X$ (I2) $f(V) \subset V$ for $V \in I^X$ (I3) $f(f(V)) = f(V)$ for $V \in I^X$ (I4) $f(V \cap W) = f(V) \cap f(W)$ for $V,W \in I^X$, then the collection $T = \{V \in I^X : f(V) = V\}$ is a fuzzy topology & $f(V) = V^o$.

DEFINITION.6.5.3.(Hutton 1977 (14)). Let (X,Δ) be a quasi uniformity Define $\text{Int} : I^X \to I^X$ by $\text{Int}(V) = \cup \{ U \in I^X : D(U) \subset V$ for some $D \in \Delta\}$.

PROPOSITION.6.5.4.(Hutton 1977 (14)). Int satisfies the interior axioms (I1) - (I4).

Proof : (I1) and (I2) are trivially satisfied & (I4) follows from(Q3) So we are to prove (I3).

If U and $V \in I^X$ and $D \in \Delta$ is such that $D(U) \subset V$, then we can find E such that $E \circ E \subset D$. So in particular, $E(E(U)) \subset V$.

Thus $E(U) \subset \text{Int}(V)$ which implies that $U \subset \text{Int}(\text{Int}(V))$.

Hence $\text{Int}(V) \subset \text{Int}(\text{Int}(V))$ and since the other inclusion follows by (I2) , we have $\text{Int}(V) = \text{Int}(\text{Int}(V))$.

DEFINITION.6.5.5.(Hutton 1977 (14)). The fuzzy topology generated by Δ is called the fuzzy topology generated by Int.

Hence, in particular, we note that $D(U)$ is a neighborhood of U in the topology generated by Δ .

LEMMA.6.5.6.(Hutton 1977 (14)). Let (X,T) be an fts. Suppose $D_n \varepsilon \Delta$ and $D_n(U)$ is a neighborhood of U for any fuzzy set U $(n=1,2)$. Then $(D_1 \cap D_2)(U)$ is a neighborhood of U.

Proof : Left.

THEOREM.6.5.7.(Hutton 1977 (14)). Every fyzzy topology is fuzzy quasi uniformisable.

Proof : Left.

THEOREM.6.5.8.(Hutton 1977 (14)). If (X,p,D_r) is a p.q.metric space then $\Delta=\{D_r : r \varepsilon]0.\infty[\}$ is a base for a quasi uniformity on the set X. Further, the fuzzy topology of the quasi uniformity is that of the fuzzy p.q metric space.

Proof : Clearly $\Phi \neq \Delta \subset \Omega$.

For (Q3) it is enough to prove that $\forall\ r,s\ \varepsilon]0,\infty[$, there exists $t\ \varepsilon]0,\infty[$ such that $D_t \subset D_r \cap D_s$. Let $t = r \wedge s$.& $A_1 \cup A_2 = A \varepsilon I^X$.

Then $D_t(A) = [D_{r\wedge s}(A_1) \cup D_{r\wedge s}(A_2)] \subset [D_r(A_1) \cup D_s(A_2)]$

However $(D_r \cap D_s)(A) = \bigcap_{A_1 \cup A_2 = A} [D_r(A_1) \cup D_s(A_2)]$which gives the required result.

Δ satisfies (Q4) since $D_{r/2} \circ D_{r/2} \quad D_r$.

The remainder of the results follows from theorem 6.4.10.

Now we define quasi uniform continuity between quasi uniform spaces.

DEFINITION.6.5.9.(Hutton 1977 (14)). Let (X,Δ) and (Y,ξ) be quasi uniform spaces. A map $f : X \rightarrow Y$ is said to be fuzzy quasi uniformly continuous if for every $E \varepsilon \xi$, there exists a $D \varepsilon \Delta$ such that $D \subset f^{-1}(E)$ i.e. for $V \varepsilon I^X$, $D(V) \subset f^{-1}(E(f(V)))$.

THEOREM.6.5.10.(Hutton 1977 (14)). Every fuzzy quasi uniformly continuous function is fuzzy continuous in the induced fuzzy topology.

Proof : Let $f : X \to Y$ be fuzzy quasi uniformly continuous.

Consider an open set V in the fuzzy topology generated by ξ.

So $V = \bigcup \{ U : E(U) \subset V$ for some $E \in \xi \}$.

If $E(U) \subset V$ then there exists a $D \in \Delta$ such that

$$D(f^{-1}(U)) \subset f^{-1}(E(ff^{-1}(U))) \subset f^{-1}(E(U)) \subset f^{-1}(V).$$

So $f^{-1}(U) \subset \text{Int } f^{-1}(V)$ & hence $\bigcup\{f^{-1}(U) : E(U) \subset V$ for some $E \in \xi\} \subset \text{Int} f^{-1}(V)$

But $f^{-1}(\bigcup_{\lambda} U_{\lambda}) = \bigcup_{\lambda} f^{-1}(U_{\lambda})$ and thus $f^{-1}(V) \subset \text{Int}(f^{-1}(V))$.

Hence $f^{-1}(V)$ is open which implies that f is fuzzy continuous.

Next we prove a theorem corresponding to the characterisation of pseudo quasi metrizability in terms of quasi uniformity.

LEMMA.6.5.11.(Erceg 1979 (8)). Let $\Sigma = \{U_n : n = 0,1,2,3,\ldots\}$ satisfying $U_0(A) = \underline{1}$ if $A \neq \underline{0}$, $U_0(A) = \underline{0}$ if $A = \underline{0}$ and $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$ for all $n = 0,1,2,3,\ldots\ldots$ Then there exists a set $\Delta \subset \Omega$, $\Delta = \{D_r : r \in]0,\infty[\}$ satisfying $D_r \circ D_s \subset D_{r+s}$ for all $r,s \in]0,\infty[$ and $U_n \subset \bigcup_{s<1/2^n} D_s \subset U_{n+1}$ for all $n \geqslant 1$.

Proof : For all $r \in]0,\infty[$ define a map $\varphi_r \in \Omega$ by

$$\varphi_r = U_n \quad \text{if} \quad r \in [1/2^{n+1} , \ 1/2^n[$$

$$_r = U_0 \quad \text{if} \quad r \geqslant 1.$$

Then $\varphi_r \circ \varphi_r \circ \varphi_r \subset \varphi_{2r}$ for all $r \in]0,\infty[$.

Now for all $r \in]0,\infty[$ define a map $D_r \in \Omega$ by

$$D_r = \bigcup \{\varphi_{r_1} \circ \varphi_{r_2} \circ \ldots\ldots\ldots \circ \varphi_{r_k} : \sum_{i=1}^{k} r_i = r\} .$$

Clearly $\varphi_r \subset D_r$ trivially for all $r \in]o,\infty[$.

Further $D_r \subset \varphi_{2r}$ because $\forall\, k \geqslant 1$, $\varphi_{r_1} \circ \varphi_{r_2} \circ \ldots\ldots \circ \varphi_{r_k} \subset \varphi_{2r}$ for all $r_1 , r_2 , \ldots\ldots, r_k$ such that $r = \sum_{i=1}^{k} r_i$.

Indeed if $k = 1$, the statement is trivial.

If $k > 1$ let j be the largest integer satisfying $\sum_{i=1}^{j} r_i \leq r/2$ and Then $\sum_{i=j+2}^{k} r_i \leq r/2$ and $r_{j+1} \leq r/2$.

By induction $\varphi_{r_1} \circ \varphi_{r_2} \circ \dots\dots \circ \varphi_{r_j} \subset \varphi_r$ and $\varphi_{r_{j+1}} \subset \varphi_r$ and

$\varphi_{r_{j+2}} \circ \varphi_{r_{j+3}} \circ \dots\dots\dots \circ \varphi_{r_k} \subset \varphi_r$.

Hence $\varphi_{r_1} \circ \varphi_{r_2} \circ \dots\dots\dots\dots \circ \varphi_{r_k} \subset \varphi_r \circ \varphi_r \circ \varphi_r \subset \varphi_{2r}$.

Since $\varphi_r \subset D_r \subset \varphi_{2r}$ $\forall\, r \in]0,\infty[$, $U_n \subset D_r \subset U_{n+1}$ $\forall\, r \in [1/2^{n+1} , 1/2^n[$.

Thus $U_n \subset \bigcup_{r<1/2^n} D_r \subset U_{n+1}$.

It is clear that $D_r \circ D_s \subset D_{r+s}$ for all $r,s \in]0,\infty[$.

PROPOSITION.6.5.12.(Erceg 1979 (8)). If Σ is the sequence of maps in Lemma 6.5.11 then there exists a fuzzy p.qmetric p on X with associated neighborhood maps E_r satisfying $U_{n+1} \subset E_{2^{-n}} \subset U_{n-1}$ $\forall\, n > 1$.

Proof : By Lemma 6.5.11 and Theorem 6.4.8.

THEOREM.6.5.13.(Erceg 1979 (8)). (P.Q Metrization Theorem).

A quasi uniform space (X,Δ) is fuzzy p.q metrizable iff Δ has a countable base.

Proof : ($\Rightarrow$) Trivial.

($\Leftarrow$) If Δ has a countable base, say $\Xi = \{U_n : n=0,1,2,\dots\}$ we may rechoose Ξ so as to satisfy the hypothesis of lemma 6.5.11. The result follows by proposition 6.5.12.

DEFINITION.6.5.14.(Erceg 1979 (8)). Let (X,p,D_r) be a fuzzy p.q metric space and (X,Δ) be a fuzzy quasi uniform space. Then p is said to be quasi uniformly lower semicontinuous (qulsc) on (X,Δ) iff $\{D_r : r \in]0,\infty[\ \} \subset \Delta$.

THEOREM.6.5.15.(Erceg 1979 (8)). Let (X,Δ) be a quasi uniform space. Let Θ be the set of all fuzzy p.q metrics which are qulsc on (X,Δ). Then Θ generates Δ .

Proof : Let Ω be the quasi uniformity generated by Θ. Then clearly $\Omega \subset \Delta$. To prove the converse, suppose $U \in \Delta$;
Let $U_0(A) = \underline{1}$ if $A \neq \underline{0}$, $U_0(A) = \underline{0}$ if $A = \underline{0}$ and $U_1 = U$.
Define $U_n \in \Delta$ inductively so that $U_{n+1} o U_{n+1} o U_{n+1} \subset U_n$ $\forall$ $n=0,1,2,..$
Then by lemma 6.5.11 there exists a fuzzy p.q metric p on X with neighborhood maps D_r satisfying $U_n \subset D_{2^{-n}} \subset U_{n-1}$ for all $n > 1$.
Hence $p \in \Theta$ and $U \supset D_{1/4}$. Thus $U \in \Omega$

Now we define fuzzy uniform spaces.

<u>DEFINITION</u>.6.5.16.(Hutton 1977(14)). A fuzzy quasi uniformity on a set X is said to be a fuzzy uniformity if it also satisfies

(Q5) $D \in \Delta \Rightarrow D^{-1} \in \Delta$.

In section 4.6 we have already introduced fuzzy unit interval I(I) together with left hand topology L_t and right hand topology R_t. Here we construct a fuzzy uniform structure on I(I) as follows :

<u>DEFINITION</u>.6.5.17.(Hutton 1977 (14)). We define $B_r : I^X \to I^X$ by $B_r(U) = R_{t-r}$ where t is the greatest $s \in \mathbb{R}$ such that $U \subset L_s^c$

$= \cap \{R_{s-r} : U \subset L_s^c\}$.

<u>PROPOSITION</u>.6.5.18.(Hutton 1977 (14)). (1) B_r satisfies (A1)-(A3).
(2) $B_r^{-1} = \cap \{L_{s+r} : U \subset R_s^c\}$.(3) $B_{r_1} o B_{r_2} \subset B_{r_1+r_2}$.

Proof : Left;

<u>COROLLARY</u>.6.5.18.(Hutton 1977 (14)). The set $\{ B_r : r > 0\}$ is a base for a fuzzy quasi uniformity which generates the right hand topology.

Proof : Left .

<u>COROLLARY</u>.6.5.19.(Hutton 1977 (14)). The collection $\{B_r, B_r^{-1} : r > 0\}$ is a subbase for a fuzzy uniformity on I(I). The topology generated by the uniformity is the usual fuzzy topology.
(This fuzzy uniformity is called the usual fuzzy uniformity for the usual fuzzy topology on I(I)). The proof is omitted.

Now we are in a position to characterise fuzzy uniformizability.

THEOREM.6.5.20.(Hutton 1977 (14)). Let $(X,\triangle)$ be a fuzzy uniform space and $D\in\triangle$. Suppose $D(U)\subset V$. Then there exists a fuzzy uniformly continuous function $f : X \rightarrow I(I)$ such that

$U(x) \leq f(x)(1-) \leq f(x)(0+) \leq V(x)$ for $x\in X$.

Proof : Left.

CHAPTER - 7 .

APPLICATIONS OF FUZZY SETS .

Since its inception, the theory of fuzzy sets has evolved in many directions and there has been an evergrowing number of publications in which new methods of handling systems, which seem too complex or too ill-defined for conventional analysis, are presented. This rapidly developing new field has found applications in nearly every area, where human judgment and perception play a role, particularly in the realms of psychology, economics, law, medicine, decision theory, information retrieval and artificial intelligence.

In this chapter , we discuss some applications of fuzzy sets in medical diagnosis and in psychology.

.1.APPLICATION OF FUZZY SET THEORY IN MEDICAL DIAGNOSIS.

In medical science, it is seldom possible to work with exact definitions, descriptions or assertions. In medical diagnosis there is very rarely a sharp boundary between diseases and the appearance of more than one disease in the patient at the same time destroys the expected symptom patterns of disease hypothesis, which makes the diagnostic and therapeutic decision more difficult. The assignment of laboratary test results to the ranges normal or pathological is arbitrary in border line cases. The intensity of pain can only be described verbally and depends on the subjective estimation of the patient and precise relationships between symptoms, signs, test-results, findings, i.e., any observation on the patient and diagnosis can very seldom be found in descriptions of diseases.

For example, in literature the following symptoms are normally given about the disease PANCREATITIS.

(1) Acute pancreatitis is almost always connected with sickness and vomiting.

(2) Typically, acute pancreatitis begins with sudden aches in the abdomen.

Here, it is difficult to corelate the two symptoms of acute pancreatitis. Moreover, the word 'sickness' is not well defined.

A possible remedy to these problems is to generalise the concept of memberships as a binary function with the help of fuzzy set theory which was developed by L.A.Zadeh. This theory can perfectly analyse the types of diseases which might otherwise be impossible. Diagnosés are defined to be fuzzy subsets with symptoms as elements. The symptoms are combined with a degree of membership which characterizes the intensity of belonging to the fuzzy subset that represents the disease under consideration.

A few applications of fuzzy set theory in medical diagnosis are given below.

I.FUZZY RELATION BETWEEN SYMPTOMS AND DISEASES, FUZZY RELATION EQUATIONS AND ITS APPLICATION IN GROUP NURSING DIAGNOSTIC DECISION PROBLEM.

E . SANCHEZ [cf. (12), pages 437 - 444] illustrates problems of medical diagnosis based on max-min composite fuzzy relation and they correspond to three stages : determination of symptoms, of a 'medical knowledge', of diagnosis, all in the sense of degree of membership of fuzzy sets on fuzzy relations.

Binary fuzzy relations are characterized by compatibility, or membership functions defined over the cartesian product of two, possibly different, non-fuzzy sets. So a fuzzy relation R from X to Y is a fuzzy subset of X x Y, characterised by its membership function $R : X \times Y \to [0,1]$.

In a given pathology, let S be a set of symptoms, D a set of diagnosis and P a set of patients. 'Medical knowledge' is a fuzzy relation, denoted by R, from S to D expressing association between symptoms and diseases.

Let A be a fuzzy subset of S related to a patient, and R be a fuzzy relation from S to D, then the computation of the max-min composition B = A o R is assumed to describe the state of the patient in terms of diagnosis as a fuzzy subset B of D, characterised by its membership function

$$B(d) = \max_{s \in S} [\, A(s) \wedge R(s,d) \,] , \; d \in D \qquad \ldots\ldots(1)$$

If the state of a given patient p is described in terms of a fuzzy subset A of symptoms in S, then p is assumed to be assigned diagnosis in terms of fuzzy subset B of D, through a fuzzy relation R of 'medical knowledge' from S to D which is assumed to be given by a physician who can translate his own percepsion of the fuzziness involved in the degrees of associations between symptomsaand diagnosis.

Now, consider several patients belonging to P, and a fuzzy relation R from S to D. Define a fuzzy relation Q from P to S & another relation T from P to D be such that T = R o Q .....(2)

i.e., $$T(p,d) = \max_{s \in S} [Q(p,s) \wedge R(s,d)], \; (p,d) \in P \times D \qquad \ldots.(3)$$

If P is reduced to a single element, equation (2) reduces to (1). Knowing R and Q in (2), it is easy to find T.

Now the most general problem of composite fuzzy relations equations consists in finding the solutions of T = Q o R [cf.(12) pages 421 - 433[where T and Q (Problem 1) or T and R (Problem 2) are given fuzzy relations. Problem 1 and Problem 2 are dual problems and it is observed that $(Q \circ R)^{-1} = R^{-1} \circ Q^{-1}$ where R^{-1} and Q^{-1} are respectively the inverse fuzzy relations of R and Q and also $Q^{-1}(s,p) = Q(p,s)$ for all $(s,p) \in S \times P$. So $Q \circ R = T$ is equivalent to $R^{-1} \circ Q^{-1} = T^{-1}$ and a simple transformation allows one to pass from solutions in problem1 to solutions in problem2, or vice-versa.

Because of the nature of the max. and min. operators, it is seen that when a solution exists, it is not unique for both problems.

For the existence and the determination of solutions in our dual problems, the following operators are introduced.

For any elements a , b in [0,1] , define

$$\begin{aligned} a \,\alpha\, b &= 1 \quad \text{for} \quad a \leq b \\ &= b \quad \text{for} \quad a > b. \end{aligned}$$

For given fuzzy relations Q from P to S and R from S to D define $T = Q \textcircled{\alpha} R$ as a fuzzy relation from P to D by

$$(Q \textcircled{\alpha} R)(p,d) = \bigwedge_{s \in S} \left[Q(p,s) \,\alpha\, R(s,d) \right] , \text{ for all } (p,d) \in P \times D$$

where α is the operattor defined above.

Let $\mathcal{R}$ and $\mathcal{Q}$ be the family of solutions (wheh they exist) of problem1 and problem2 respectively, i.e.,

$R \in \mathcal{R}$ iff $T = Q \circ R$ (T and Q are given)

$Q \in \mathcal{Q}$ iff $T = Q \circ R$ (T and R are given).

It is seen that $\mathcal{R}$ is not void iff the fuzzy relation $\check{R} = Q^{-1} \,\alpha\, T$ is an element of $\mathcal{R}$; moreover, $\check{R}$ is the greatest element in $\mathcal{R}$. Similarly $\mathcal{Q}$ is not void iff the fuzzy relation $\check{Q} = (R \,\alpha\, T^{-1})^{-1}$ is an element of $\mathcal{Q}$; moreover, $\check{Q}$ is the greatest element in $\mathcal{Q}$.

Next, we discuss an application of the fuzzy relation method to a group nursing diagnosis problem and it is shown that the problem of finding the nurses' fuzzy diagnostic consensus decision rule constrained by the physician or the nursing supervisor will be solved by applying the method of fuzzy relation [cf. (13), 33 -45] .

It is assumed that a fuzzy decision rule is a fuzzy mapping from a certain finite set to another finite or an ordinary mapping from a certain family of fuzzy sets to another family of fuzzy sets.

Consider the group nursing care by N nurses for a set of patients. The nurses' various decision rules in the actual care depend on the nurses' own subjective uncertainty and that is fuzziness.

Assume the following finite sets :

P : a set of specific patients (patient space)
S : a set of symptoms (symptom space)
D : a set of nursing diagnoses (nursing diagnosis space)
G : a set of goals in nursing care (care goal space).

Consider, N nurses' individual fuzzy dicision rules between the above sets as follows,

$R^{(i)} : P \to S$, $i = 1,2,\ldots,N$
$L^{(i)} : S \to D$, $i = 1,2,\ldots,N$
$K^{(i)} : D \to G$, $i = 1,2,\ldots,N$

where, $R^{(i)} \in \tilde{\mathcal{P}}(P \times S)$ is the i^{th} nurse's fuzzy observation decision rule from P to S, $L^{(i)} \in \tilde{\mathcal{P}}(S \times D)$ is the i^{th} nurse's fuzzy diagnostic decision rule from S to D, and $K^{(i)} \in \tilde{\mathcal{P}}(D \times G)$ is the i^{th} nurse's fuzzy partial assessment decision rule from D to G . $\tilde{\mathcal{P}}(P \times S)$ is the family of fuzzy subsets of $P \times S$.

Next, assume the following fuzzy decision rule C by which the above decision rules are constrained,

$C : P \to G$ where $C \in \widetilde{\mathcal{P}}(P \times G)$ is the physician's or the nursing supervisor's fuzzy total assessment decision rule from the patient space P to the care goal space G.

Consider the o-composition of the fuzzy decision rules; then the N nurses' individual fuzzy total assessment decision rules from P to G can be represented by

$$R^{(i)} \circ L^{(i)} \circ K^{(i)} \in \widetilde{\mathcal{P}}(P \times G) \text{ , } i = 1,2,....,N.$$

If there exists a fuzzy decision rule $\hat{L}^{(i)}$ such that $R^{(i)} \circ L^{(i)} \circ K^{(i)} = C$, then $L^{(i)}$ is the i^{th} nurse's individual fuzzy diagnostic decision rule from S to D subject to the constraint of the physician's C.

Now, the problem (Fig - I) of finding the fuzzy diagnostic consensus decision rule $L \in (S \times D)$, in a group nursing diagnosis of N nurses, is represented as follows :

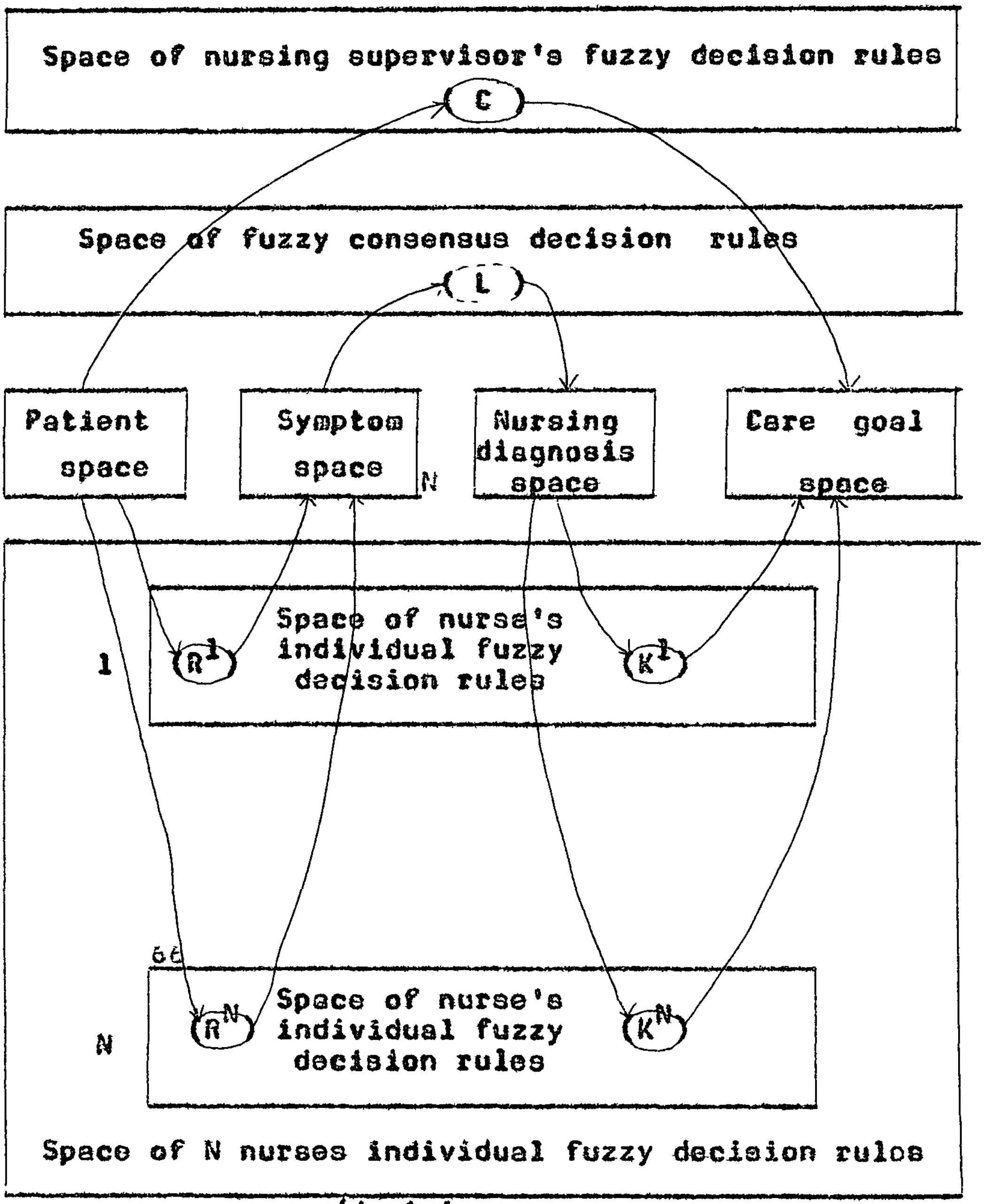
$$R^{(i)} \circ L \circ K^{(i)} = C \text{ , } i = 1,2,3,........,N \qquad .....(1)$$

where it is assumed that $R^{(i)}$, $K^{(i)}$ are given by the N nurses, C is given by the physician or the nursing supervisor, and L is the unknown fuzzy diagnostic consensus decision rule.

In general, it is very difficult to obtain L satisfying simultaneously the above N equations because, there exists no solution in most cases by reason of strict constraint of the equation (1). So consider $L = \bigcap_{i=1}^{N} L^{(i)}$ subject to $R^{(i)} \circ L^{(i)} \circ K^{(i)} = C$ ...(2) for $i = 1,2,...,N$.

If there exist the N solutions $\hat{L}^{(i)}$ satisfying (2), a fuzzy diagnostic consensus decision rule L can be obtained as follows

$$L = \bigcap_{i=1}^{N} (K^{(i)} \textcircled{α} (R^{(i)})^{-1} \textcircled{α} C)^{-1})^{-1}$$



○ : Given fuzzy /decision rule (given fuzzy relation)

( ) : Unknown fuzzy decision rule (unknown fuzzy relation)

Fig I : A schematic diagram of a moddlling for the group decision problem to find a fuzzy diagnostic consensus decision rule in a group nursing diagnosis constrained by the physician or the nursing supervisor. The solid curves with arrows denote the mapping directions of fuzzy decision rule, $R^{(I)}$ etc.

II. SYMPTOMS, SIGNS, AND TEST RESULTS AND FUZZY SUBSETS.

A

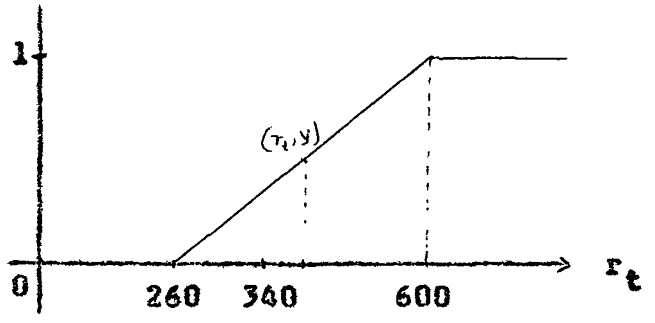
ALBIN, PEREZ-OJEDA, MOON, ESOGBUE and ELDER propose the application of fuzzy subsets to determine <u>normal</u> or <u>pathological</u> ranges as well as the boundary for <u>low</u>, <u>normal</u>, <u>high</u> or <u>normal</u>, <u>slightly decreased</u>, <u>decreased</u>, etc. for clinical or diagnostic tests.
The membership functions of these fuzzy subsets define the affiliation strength of a numerical test-result in the fuzzy subsets under consideration cf.(13), 203 - 217 .

For example, a suitable linear function for fuzzy subset Ω_t of <u>abnormal cholesterol</u> levels, expressed in milligrams per 100 milliliters of serum, is given by

$$\begin{aligned} \mu_{\Omega_t}(r_t) &= 0 && \text{for } r_t < 260 \\ &= r_t/340 - 26/34 && \text{for } 260 \leq r_t \leq 600 \\ &= 1 && \text{for } r_t > 600 \end{aligned}$$

where $r_t \in \Omega_t$ is the amount of cholesterol present in 100 milliliters of serum in a particular test t and Ω_t expresses the set including all possible test results.

The graph of the linear function Ω_t is given below.



$$\frac{y}{r_t - 260} = \frac{1}{600-260}$$

66

$$\text{or} \quad y = \frac{r_t - 260}{340}$$

$$\text{or} \quad y = \frac{r_t}{340} - \frac{26}{34}$$

Figure II

Thus, the degree of abnormality of the cholesterol test-result is reflected by the membership function $\mu_{\Omega_t'}(r_t) \in [0,1]$.

Performing the cholesterol test on a patient and getting a result of $r_t < 260$ mg/100 ml. means that the patient shows a normal cholesterol level.When a test result $r_t > 600$ mg/100 ml. is obtained the patient under consideration reveals an abnormal cholesterol level.Further, a test-result between 260 and 430 mg/100 ml. seems to be more normal than a test result between 430 and 600 mg/100 ml.

III. SYMPTOM COMBINATION AND FUZZY SUBSETS :

ALBIN presented appropriate membership functions for normal and long IVY bleeding time, for decreased, slightly decreased, and normal platelet count, for normal, slightly long, and long thrombin time, etc. It is said that physicians normally try to fit patients to certain prototypes of disease. As a first appooximation to such an approach, disease prototypes can be defined as shown in table-1 [cf.(13), 203 - 217]

Table-1 Disease prototype definition for the hemorrhagic disorders .6

Disorder	S_1 IVY Bleeding time	S_2 Platelet count	S_3 Quick time or Prothrombin time	S_4 Partial Thromboplastin time	S_5 Thrombin time
D_1.Thrombocytopenia	Long	Decreased	Normal	Normal	Normal
D_2.Von-Willebrand's Disease	Long	Normal	Normal	Normal or Long	Normal

When blood escapes from blood vessels, then the disease Thrombocytopenia and Von-Willebrands occur. The definition of fuzzy membership functions for long S_1, decreased S_2, normal S_3, etc., allows the calculation of degrees of membership in the fuzzy subset long S_1, decreased S_2, normal S_3, etc. given symptom values measured on the patient. Fuzzy disease membership values of patients can now be computed by using the Max difinition for fuzzy set union and Min difinition for fuzzy set intersection.

The standard time for IVY bleeding time to be long is taken to be 12 min. The standard number for platelet count to be called decreased is taken to be $60000/\text{mm}^3$. the standard time for prothrombin time to be normal is taken to be 12 sec. The standard time for partial thromboplastin time to be normal is taken to be 37 sec. and the standard time for thrombin time to be normal is taken to be 18 sec. Now we define

$$\mu_{D_1}(p) = \mu_{\underline{\text{long}}\ S_1}(12\text{ min}) \wedge \mu_{\underline{\text{decreased}}\ S_2}(60000/\text{mm}^3) \wedge \mu_{\underline{\text{normal}}\ S_3}(12\text{ sec}) \wedge \mu_{\underline{\text{normal}}\ S_4}(37\text{ sec}) \wedge \mu_{\underline{\text{normal}}\ S_5}(18\text{ sec}) .$$

For example, in a particular experiment if IVY bleeding time is found to be 12·5 min., number of platelet/ cu.mm to be 58000/cu.mm, prothrombin time to be 12·5 sec., partial thromboplastin time to be 38·5 sec. and thrombin time to be 19 sec., then

$$\begin{aligned}\mu_{D_1}(p) &= \mu_{\underline{\text{long}}\ S_1}(12\text{ min}) \wedge \mu_{\underline{\text{decreased}}\ S_2}(60000/\text{mm}^3) \wedge \mu_{\underline{\text{normal}} S_3}(12\text{ sec}) \\ &\quad \wedge \mu_{\underline{\text{normal}}\ S_4}(37\text{ sec}) \wedge \mu_{\underline{\text{normal}}\ S_5}(18\text{ sec}) \\ &= (12/12{\cdot}5) \wedge (58/60) \wedge (12/12{\cdot}5) \wedge (37/38{\cdot}5) \wedge (18/19) \\ &= {\cdot}96 \wedge {\cdot}96 \wedge {\cdot}96 \wedge {\cdot}96 \wedge {\cdot}95 \\ &= {\cdot}95 .\end{aligned}$$

In this example , patient p fits the prototype definition of D_1 almost exactly and if additionally computed fuzzy disease membership values $\mu_{D_i}(p)$, where $i \neq 1$, are much smaller than $\mu_{D_1}(p)$, diagnosis D_1 = Thrombocytopenia is presumably the correct diagnosis for patient p.

APPLICATIONS OF FUZZY SETS IN PSYCHOLOGY.

Fuzzy set theory can help psychology with new concepts to use as building blocks for improved theories and in return, psychology can offer not only continuing challenges and test problems, but methods of experimentation as well. Fuzzy sets are relevant, useful and possibly necessary to explain certain psychological findings. Thus it is more fruitful to introduce the notions of fuzzy set theory when the need for them arises in the development of psychological conceptualizations then to seek out psychological problems for potiential applications of fuzzy set theory.

Manfred Kochen illustrates an application of fuzzy sets in psychology and the work reported here originated during the course of developing a new model of cognitive learning (cf.(33), 395 - 407).

According to Kochen, cognative learning is viewed as an algorithm which forms, revises and uses a system of representation for recognising and coping with an increasing variety of opportunities and traps.This view led to such an hypothesis as :
" If a problem-solver practices with tasks requiring shifts of representation, he is likely to perform better in solving an ill-defined problem than one who has no prior practice or one who has prior practice with welldefined problems not requiring representational shifting " (Barde,1973).

To test such an hypothesis, a new experimental technique has been developed in which human subjects (college students) are asked questions that would help them to recognise, formulate and perform a task that the experimenter has in mind and has created for them. Certain actions and words are prespecified but not known to the subject. The subject's use of these is interprited as indicative of representational shifting.

When some people are asked how strongly they believe that " x is a large number ", they behave as if they had fixed a threshold or decision criterion d that enables them to say, consistently, that if $x \geq d$ they agree, and if $x < d$ they disagree. If they are asked to mark on a scale, such as the strength of their belief in the above statement then the mark they place on the scale (agreement-disagreement scale) might be distributed uniformly over the right half of the scale whenever $x \geq d$ and it might be distributed uniformly over the left half if $x < d$. Let us call such people "thresholders".

Another type of person might try to place his mark close to the agreement side of the scale according to how large he thinks x is. This depends critically on the sample that is presented, for if he has "used up" the scale by placing a mark close to "agree", his response to the largest number just presented, and now an even as larger one is presented, he will be "squeezed".The strength of belief of such people in " $x \gg 5$ " resembles their estimate of x. Let us call such people " estimatops ".

There is another kind of people who might place there marks near "agree strongly" or"disagree strongly" when they feel strongly and place no mark when they are uncertain. The set of numbers is then divided into three classes : those that such a type of persons consider not large ;those they consider large ; those they do not consider either large or not large. Let us term these people " reliables " or conservatives, as in the psychological literature.

Undoubtedly there are other possible kinds of people. Fuzzy set theory applied to psychology might be interpreted to suggest the general hypothesis that most people are "estimators" rather than "thresholders" or "reliables". If enough people in a sample behave as if their strength of belief varies nearly continuously with the stimulus variable in the statement to be believed then this hypothesis would be supported, and psychological reality of fuzzy sets would be made more evident.

If, on the other hand, too many people in a sample behave as if they use a threshold-decision criterion - as would be implied by most models of decision-making used in decision theory and mathematical statistics - then the psychological reality of fuzzy sets would be in doubt and other concepts more plausible.

Suppose we want to measure how a person would behave in response to an instruction "move far to the right". The most obvious and natural thing to do is to ask him to move and observe where he goes. This should be done repeatedly with the same person, withdetractors between repeated instructions to avoid the effect of the person's recalling or trying to be consistent. If he distributes the distances that he moves uniformly between some minimum distance and some upper limit of of possible distances that he could move, we infer that he used a threshold : the minimum distance. The cumulative frequency is a straight line from the threshold to the maximum. The resulting curve is viewed as his characteristic or grade-of-membership curve.

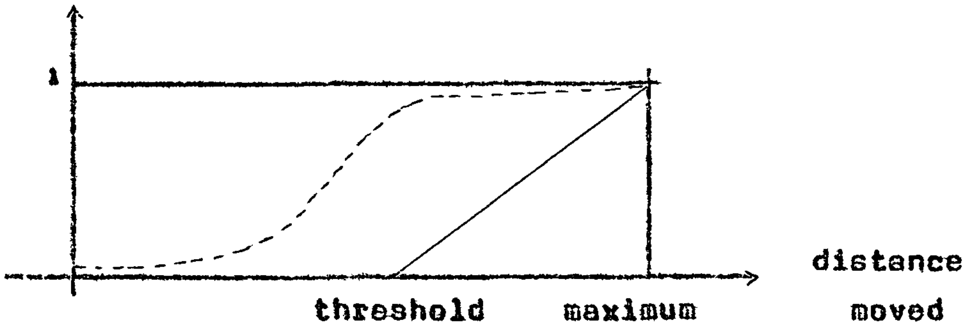


FIGURE * 1

If the person is an estimator then he distributes the distances he moves according to a skew or Bell-shaped curve. In this case the cumulative frequency has a typical S-shape as shown by the dashed line in Figure * 1.

It is common in psychological experiments to regard the subjects in a random sample of people as interchangeable. If the dependent variable that is observed has a bimodal distribution, that might indicate that the sample was drawn from two populations, such as thresholders and estimators. Figure*2 shows the pattern of a bimodal distribution. If the population of estimators dominates, the general hypotheses about the psychological reality of fuzzy sets appears to be supported.

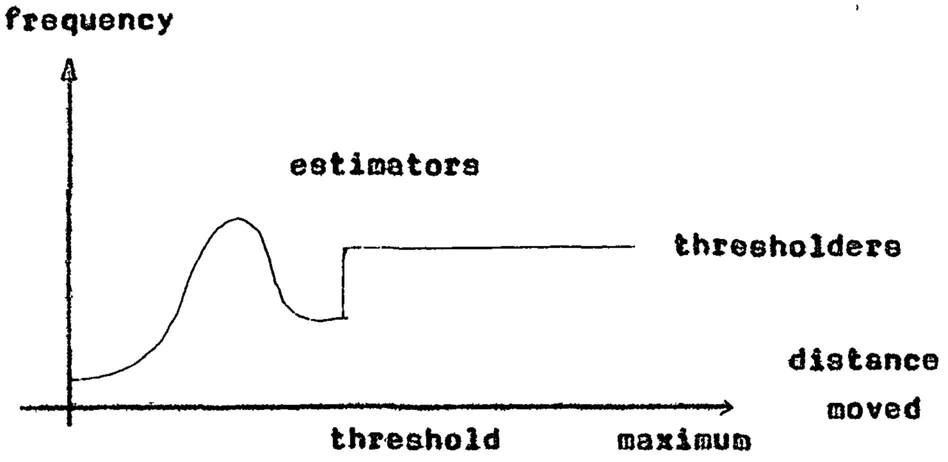


FIGURE # 2

Now we describe an experiment to find out how different people interpret words like 'far', 'close' etc. in specific context.

In this experiment 24 college freshmen were taken as subjects. The experiment was of two sections.

In the first section, they were given the following instructions : " You are to put an X above the point on the line which is ____________ to right of the 0.
For example : Put an X above the point on the line which is little to the right of the 0.

FIGURE#3

The subjects were then given 42 instructions as in the above example for the first part of the experiment. The blank was filled in by "far", "very far", "not so far", "not so close", "very close", or nothing. Two different line lengths were used to explore the effect of context in this sense. Two positions of the 0 were used with the smaller line to see if that made any difference. Each instruction was presented twice and the order of presenting the 42 instructions was randomised.

In the second part of the experiment, each subject was given 42 pictures in random order such as Figure*2 above, and asked to assign to each diagram one of the following 7 statements:

1. X is very far to the right of 0.
2. X is far to the right of 0.
3. X is not so far to the right of 0.
4. X is to the right of 0.
5. X is not so close to the right of 0.
6. X is close to the right of 0.
7. X is very close to the right of 0.

The lines in the diagrams were again of two different lengths and two different positions of the 0 in the shorter line were used. The diagrams had fixed X's at 7 distances from 0.So there were 7 x 3 i.e., 21 diagrams. Each was presented twice.

RESULTS.

Consider first the distribution of the responses from the 24 subjects when told to place an X far to the right of 0 on the long line. The distances were measured by the number of quarter ~~in~~ inches. The results were :

Distance(in 1/4 " units):	7	8	9	10	11	12	13	Total
Number of subjects who moved that distance :	1	3	13	6	10	8	7	48

This is shown in Figure*4. The dip at 10 is too sharp to be due to chance. A bimodal distribution seems to be present. The maximum distance was 13, and the right-hand part of the distribution resembles a uniform distribution with a threshold at 11. It might be the case that we drew our sample from a population in which 1+3+13+6 or 23 out of 48 or about half, were estimators, and 10+8+7 or 25 out of 48 or the other half, were thresholders. It does not support the belief that most people behave in the way conceptualised by fuzzy set theory. Just about as many people seem to behave in the way conceptualized by decision theory.

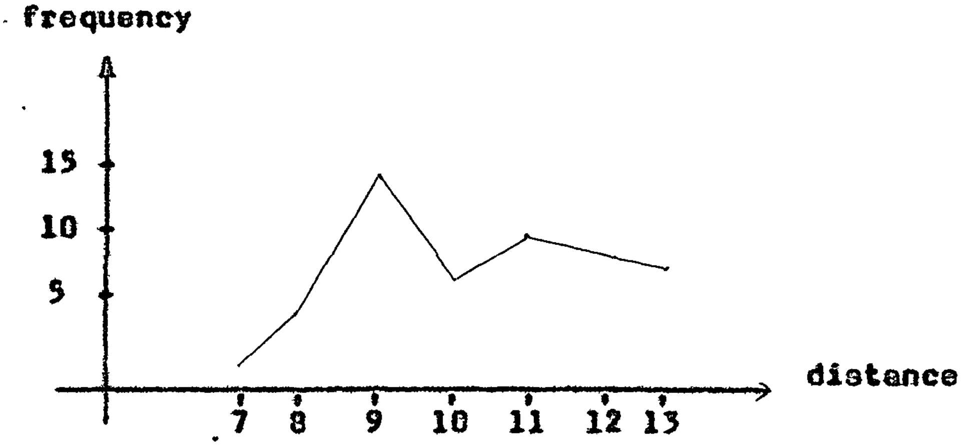


Figure * 4

REcall that each person responded to the same diagram twice at different times, with distractions in between. It is interesting to note that many people responded the same way or differently both times. This is given below.

Difference in distance moved on both trials :	0	1	2	3	4	5	Total
Number of subjects who obtained that difference :	4	7	5	2	4	2	24

Again we have a bimodal distribution. It seems as if some people in the same category come from a population which estimates consistently, while others are very inconsistent. It is noted that the distances chosen by the 4 perfectly consistent subjects were 9,9,9 and 11; that is, three of them came from the "estimator" population, and the fourth may have.

From the remaining data in part-1, the following conclusions can be drawn. Recall that the responses were the distance from 0 at which X was placed.

1. The responses or distances decreased according to the following order of the stimuli in the verbal instruction : very far, far, not so close, ∅, not so far, close, very close. The responses were consistent (transitive). Here, ∅ refers to the absence of any adjective, as in the instruction " put an X above the point on the line which is ______ to the right of the 0 ". (The blank was left blank). The reversal of "not so close" and "not so far" is perhaps a little surprising, but understandable, because "not so close" is semantically similar to "far". (It is actually ambiguous.)

2. The response to "very far" is the maximum length of the line, independently of line length or the 0's location. Similarly, the response to "very close" is an X right next to 0, independently of the line length or the 0's location. There is very little variance over the subjects.

3. The variances of the responses to diagrams in which the 0 is at the center of the line is less than the variance when the 0 is

to the left of the center. The latter diagram gives a longer maximum distance over which to distribute the X's. To test whether this is due to the eccentric location of 0 or the length, we can compare the responses on the long line with the responses on the shorter line with the 0-entered. The difference is not very significant. Hence, it seems to be in the eccentricity of the 0 that accounts for the increased variance.

4. The variance is greatest when the blank is not filled in i.e., for the Ø stimulus. Indeed the variance increases as we move from either extreme (very far, very close) towards Ø.

5. The length of the line does not effect the responses for extreme stimuli, such as very far and very close, but the mean response to other stimuli is scaled down. The ratio of the line lengths was 9/13. The mean responses were

Stimulus		far	not so close	Ø	not so far	close
Long line (0 as center)	:	10·0	6·5	4·5	4·3	2·4
Short line (0 as center)	:	6·6	4·9	3·3	3·4	2·1
Ratio	:	0·66	0·75	0·73	0·81	0·89

Atleast for 'far' the responses shrank by ·66 from the long to the short line, which were in a ratio about ·69 . It seems as if the subjects scaled down in direct proportion to the line-lengths.

From the data in part-2, the following conclusions can be drawn. Here the response is the selection of the statement such as " X is far to the right of 0 " from 7 such statements about marked lines presented as stimuli.

6.

6. More subjects select 'far' and 'close', while fewer subjects select 'very far' and 'very close'.

7. Very few subjects select 'not so far' and 'not so close', and the blank.,Ø, is used least often of all.

The data shows remarkable uniformity for a psychological experiment, even though 2 subjects place X's to the left of 0 in some cases.

CONCLUSIONS :

One out of two people seem to behave, when asked to place an X far to the right of a mark on a line, as if they interprete "far distances" as a fuzzy set with a grade-of membership assignable to "$d \varepsilon F$" that increases continuously with d. Measuring that grade-of-membership by observing how frequently they place the X at distance d from the mark appears to be useful for connecting fuzzy set theory with psychology. Context, in the form of a line of limited length, affects the response in nearly direct proportion to the line-lengths. If the line length increases indefinitely then this conclusion may fail to hold. The linearity of the relation between the response and the context (line-length) is p probably local.

Most of these conclusions are hypotheses supported by evidence. More experimentation is required to establish them more firmly and to delimit the range of variables over which they hold. On the whole, fuzzy set theory seems appropriate for conceptualizing certain aspects of the behaviour of perhaps half the population.

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REFERENCES

1. Anthony, J.M. and Sherwood, H – " Fuzzy groups redefined " JMAA, 69 (1979) , 124 – 130.

2. Bellman, R , Kalaba, R and Zadeh, L.A —— " Abstraction and patter pattern classification " , JMAA, 13 (1976) , 1 – 7.

3. Capocelle, R.M and De-Luca, A —— " Fuzzy sets and decision theory " , Inform and control, 23 (1973), 446-473.

4. Chang, C.L —— " Fuzzy topological spaces " , JMAA, 24 (1968), 182 – 190.

5. Christoph, F.T —— " Quotient fuzzy topology and local compactness " , JMAA, 57 (1977), 497 – 504.

6. Conrad, F —— " Fuzzy topological concepts " , JMAA, 74 (1980) , 432 – 440.

7. Dugundji, J —— Topology , Prentice Hall of India, New Delhi, 1975.

8. Erceg, M.A —— " Metric space in fuzzy set theory " , JMAA, , 269 (1979) , 205 – 230.

9. Foster, D.H —— " Fuzzy topological group " , JMAA, 67 (1979) , 549 – 564.

10. Goguen, J ——

    (a) " L – fuzzy sets " , JMAA , 18 (1967), 145-174.

    (b) " Fuzzy Tychonoff theorem " , JMAA ,43 (1973) 734 – 742.

JMAA : Journal of Mathematical Analysis & Applications.

11. Gupta, M.M and Sanchez, E ——— Fuzzy information and decision processes , North Holland publishing company Amsterdam-New York-Oxford.

12. Gupta, M.M , Ragade, R.K and Yager, R.R ——— Advances in fuzzy set theory and applications , North-Holland publishing company , Amsterdam-New York-Oxford(1979).

13. Gupta, M.M and Sanchez, L ——— Approximate reasoning in decision analysis , North-Holland publishing Co. Amsterdam-New York-Oxford (1982).

14. Hutton, B ———

    (a)"Normality in fuzzy topological spaces" , JMAA , 50 (1975) , 74 - 79 .

    (b) " Uniformities in fuzzy topological spaces " , JMAA , 58 (1977) , 559 - 571 .

15. Kaufmann, A ——— Introduction to the theory of fuzzy subsets Volume-1 , Academic press.

16. Kelley, J.L ——— General Topology , Van Nostrand, Princeton N.J (1955) .

17. Liu, D.B and Katsaras, A.K ——— " Fuzzy vector spaces and fuzzy topological vector spaces " , JMAA , 58 (1977) 135 - 146 .

18. Lou, S.P and Pan, S.H ——— " Fuzzy structure " , JMAA , 76 (1980) , 631 - 642 .

19. Lowen, R ———

    (a) " Topological floues " , C.R. Acad Sci. Paris, 278 (1974) , 925 - 928 .

(b) " Fuzzy topological spaces and fuzzy compactness JMAA , 56 (1976) , 621 - 633 .

(c) " Initial and final fuzzy topologies and the fuzzy Tychonoff Theorem " , JMAA , 58 (1977) 11 - 21 .

(d) " A comparision of different compactness notations in fuzzy topological spaces " , JMAA , 64 (1978) , 446 - 454 .

(e) " Convergence in fuzzy topological spaces, general topology and its application ", JMAA, 10 (1979) , 147 - 160 .

20. Nazaroff, G.J ——— " Fuzzy topological polysystem " , JMAA, 4 (1973) , 478 - 485 .

21. Pao-Ming, Pu and Ying-Ming, Liu ———

(a) " Fuzzy topology-I ; Neighborhood structure of a fuzzy point; Moore-Smith convergence " , JMAA, 76 (1980) , 571 - 599 .

(b) " Fuzzy topology-II ; Product and Quotient spaces" JMAA , 77 (1980) , 20 - 37 .

22. Rodabaugh, S.E ——— " Connectivity and the L-fuzzy unit interval " , Rocky Mountain Journal of Mathematics , 12 (1982) , 113 - 121 .

23. Rosenfield, A ——— " Fuzzy groups " , JMAA , 35 (1971) 512 - 517 .

24. Ruspini, S.H ——— " A fast method for probabilistic and fuzzy cluster analysis using enociation measures" in Proc. Sixth Hawaii International Conf. on System sciences . Western Periodical, North Hollywood, C,A , (1973).

25. Santos, E.S ——— " Fuzzy algorithms " , Inform and Control, 17 (1970) , 326 - 339 .

26. Sarkar, M ——— " On fuzzy topological spaces " , JMAA, 79 (1980) ,384 - 394 .

27. Srivastava, R , Lal, S.N and Srivastava, A.K ——— " Fuzzy Housdorff topological spaces " , JMAA , 81 (1981) 497 - 506 .

28. Steinlage, R.C , Gantner, T.E and Warren, R.H ——— " Compactness in fuzzy topological spaces " , JMAA , 62(1978), 547 - 562 .

29. Warren, R,H ———

    (a) " Continuity of mappings of fuzzy topological spaces " , Notices Amer. Maths Soc. 21 ,No4(1974).

    (b) " Boundary of a fuzzy set " , Indiana Math. Journal 26 (1977) , 191 - 197 .

    (c) " Neighborhoods, bases and continuity in fuzzy topological spaces " , Rocky Mountain Journal of Mathematics , 8 (1978) , 459 - 470 .

30. Weiss, M.D ——— " Fixed points, separation and induced topologies for fuzzy sets " , JMAA , 50 (1975) , 142 - 150 .

31. Wong, C.K ———

(a) " Covering properties of fuzzy topological spaces " JMAA , 43 (1973) , 697 - 704 .

(b) " Product and Quotient theorems " , JMAA , 45 (1974) , 512 - 521 .

(c) " Categories of fuzzy sets and fuzzy topological spaces " , JMAA , 53 (1976) , 704 - 714 .

32. Zadeh, L.A ——— " Fuzzy sets " , Inform. and Control, 8 (1965) , 338 - 353 .

33. Zadeh, L.A , King-Sun, Fu , Tanaka, K and Shimura, M ——— " Fuzzy sets and their applications to cognative decisions and processes " , Academic Press Inc. (1975) N.Y , S.F , London .

34. Guojun, W ——— " A new compactness defined by fuzzy nets " JMAA , 94 (1983) , 1 - 23 .