RINGS OVER WHICH SOME CLASSES OF MODULES ARE INJECTIVE: A SURVEY

ABSTRACT

BY

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ABSTRACT

It is a well-known result in ring theory that a ring R is semi-simple (in the sense of Bourbaki) if and only if every left R-module is injective. Sarath and Varadarajan defined *left-DSI* rings as rings such that every direct sum of simple left R-modules is injective. The class of semi-simple rings is thus contained in the class of left DSI-rings. Chapter 1 deals with the classes of rings mentioned above. The following important results have been proved in this chapter.

1.2.2 Proposition. Every left DSI-ring is left Noetherian.

1.2.3 Proposition. A strongly regular ring R is a left DSI-ring if and only if R is a direct product of finitely many skew-fields.

1.2.4 Corollary. A commutative ring R is a DSI-ring if and only if it is a direct sum of finitely many fields.

Michler and Villamayor studied the class of rings over which every simple left module is injective. These rings were known as left V-rings . Following are some of the results on left V-rings and left V-modules included in Chapter 1.

1.1.1 Proposition. The following properties of the ring R are equivalent

1)R is a left V-ring.

2)Every left R-module has radical zero.

3)Every cyclic left R-module has radical zero.

4)Every left ideal of R is an intersection of maximal left ideals of R.

1.1.3 Proposition. Factor rings of left V-rings are left V-rings.

1.1.4 Proposition. If M is a left V-module then $Rad(_RM) = 0$.

1.1.7 Proposition. Following conditions are equivalent for a left R-module M

(1)M is a left V-module.

(2)$Rad(_RM/K) = 0$ for all submodules K of M.

1.1.8 Proposition. Let R be a commutative ring . Following conditions are equivalent

(1)R is regular

(2)R is a V-ring

1.1.18 Theorem. A ring R is strongly regular if and only if R is a left V-ring in which every maximal left ideal is two-sided.

The concept of left V-rings led authors such as Ming, Alin and Armendariz, Sarath and Varadarajan, Song and Yin to define *left pV-rings*, *left fV-rings*, *left V'-rings*, *left pV'-rings*, *left fV'-rings* and *left GPV-rings*.

A ring R is a left pV-ring (resp. left fV-ring) if every simple left R-module is p-injective (resp. f-injective). We clearly have the following implications:

left V-ring $\Rightarrow$ left fV-ring $\Rightarrow$ left pV-ring

In the second chapter, some properties of left pV-rings and left fV-rings have been studied. Some important results are given below.

2.1.2 Corollary. If R is a left pV-ring then the centre of R is von Neumann regular.

2.1.5 Proposition. Let R be a left pV-ring. If a maximal left ideal I of R is a left annihilator then I is a direct summand of R.

2.1.6 Proposition. Let R be a left pV-ring. Then

(i)Every left ideal of R is an idempotent

(ii)Every non-zero left ideal of R contains a maximal left sub-ideal.

(iii)$Rad(_RR) = 0$.

2.1.15 Proposition. Following conditions are equivalent for a ring R:-

(i)R is a left pV-ring.

(ii)If K is a maximal left subideal of a principal left ideal P of R then $K^* \neq P^*$, where K^* denotes the intersection of all maximal left ideals of a ring R containing the left ideal K of R.

2.1.16 Corollary. Let R be a left pV-ring. For any left ideal I of R, either $I = I^*$ or I^* is not principal.

2.1.17 Proposition. If R is a regular ring then R is a left pV-ring.

2.2.1 Proposition. Following conditions are equivalent

1)R is a left fV-ring.

2)If K is a maximal left subideal of a finitely generated left ideal F of R then $K^* \neq F^*$.

In the last chapter, properties of left V'-rings, left pV'-rings, left fV-rings and left GPV-rings have been studied. A ring R is a left V'-ring (resp. a left fV'-ring, a left pV'-ring) if every simple and singular left R-module is injective (resp. f-injective, p-injective). A ring R is a left GPV- ring if every simple left R-module is YJ-injective. Again we have the following implications:

$$\text{left V}'\text{-ring} \Rightarrow \text{left fV}'\text{-ring} \Rightarrow \text{left pV}'\text{-ring}$$

A ring R is a T-ring if every nonzero R-module has nonzero socle. Some important results on T-rings , left V'-rings and GPV-rings are given below.

3.1.2 Theorem. For a T-ring R, the following are equivalent

1)R is a left V'-ring

2)$Z(_RR) = 0$ and $Rad(_RR/I) = 0$ for any essential left ideal I of R.

3.1.6 Theorem. Let R be a commutative T-ring. Then R is a left V'-ring if and only if R is regular.

A ring R is a *MELT* ring if every maximal essential left ideal of R is an ideal.

3.2.2 Proposition. If R is a *MELT*, left pV'-ring then $Z(_RR) = 0$.

3.2.9 Proposition. Let R be a left pV'-ring. Then for any $b \in Z(_RR)$, $R = RbR + l(b)$.

3.2.12 Lemma. If R is a left pV'-ring then for every element $b \in R$, there exists a left ideal K of R such that $R = (RbR + l(b)) \oplus K$.

3.2.16 Proposition. If R is a left pV'-ring then

(i)$Z(_RR) \cap N = 0$, where N is the Jacobson radical of R.

(ii)$R = RcR$ for every nonzero divisor c of R.

(iii)Every essential left ideal of R is idempotent.

3.2.17 Proposition. Let R be a left pV'-ring. If R is semiprime then every left ideal of R is idempotent.

3.4.3 Lemma. In a left GPV-ring, $Rad(_RR) = 0$.

Let R be a ring and L be a left ideal of R. L is a *generalized weakly ideal (briefly a $GW-ideal$)* if for any $a \in L$, there exists a positive integer n such that $a^nR \subseteq L$. This can be similarly defined for a right ideal of R.

3.4.11 Theorem. Following conditions are equivalent for a ring R

1)R is strongly regular.

2)R is a left GPV-ring whose every maximal left ideal is a GW ideal.

3)R is a left GPV-ring whose every maximal right ideal is a GW ideal.

3.4.15 Proposition. If R is a left GPV-ring such that every maximal left ideal of R is an ideal of R then R is reduced.

3.4.16 Corollary.If R is a left GPV-ring such that every maximal left ideal of R is an ideal of R then R is fully left and right idempotent.

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CERTIFICATE

I certify that the dissertation entitled "RINGS OVER WHICH SOME CLASSES OF MODULES ARE INJECTIVE": A SURVEY" submitted by Miss Wannarisuk Nongbsap, in partial fulfilment of the requirement of the degree of the Master of Philosophy in Mathematics is the outcome of a study undertaken by the candidate.

I certify that the sources from which the ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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January, 2010

DECLARATION

I, Wannarisuk Nongbsap, hereby declare that the subject matter in this dissertation is the record of the work done by me, that the contents of this dissertation do not form a basis of the award of any previous degree to me or to anybody else and that this dissertation has not been submitted by me for any research degree in any other university or institute.

This dissertation is being submitted to the North-Eastern Hill University for the degree of Master of Philosophy in Mathematics.

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********GOD BLESS YOU ALL********

PREFACE

In this dissertation, we first start with a chapter which deals with some basic results and definitions.

It is a well-known result in ring theory that a ring R is semi-simple (in the sense of Bourbaki) if and only if every left R-module is injective. Sarath and Varadarajan ([1]) defined *left- DSI* rings as rings such that every direct sum of simple left R-modules is injective. The class of semi-simple rings is thus contained in the class of left DSI-rings. Chapter 1 deals with the classes of rings mentioned above. In this chapter, we prove that a strongly regular ring R is a left DSI-ring if and only if R is a direct product of finitely many skew-fields([1]). Michler and Villamayor ([14]) studied the class of rings over which every simple left module is injective. These rings were known as *left V-rings* . In the same chapter, there is a result proved by Kaplansky which states that a commutative ring R is a left V-ring if and only if R is von Neumann Regular. Sarath and Varajadaran ([1]) proved that a ring R is strongly regular if and only if R is a left V-ring in which every maximal left ideal is two-sided. Examples of left V-rings and left DSI-rings have been recorded.

The concept of left V-rings led authors such as Ming, Alin and Armendariz, Sarath and Varadarajan, Song and Yin to define *left pV-rings*, *left fV-rings*, *left V'-rings*, *left pV'-rings*, *left fV'-rings* and *left GPV-rings*.

A ring R is a left pV-ring (resp. left fV-ring) if every simple left R-module is p-injective (resp. f-injective). We clearly have the following implications:

left V-ring $\Rightarrow$ left fV-ring $\Rightarrow$ left pV-ring

In the second chapter, some properties of left pV-rings and left fV-rings have been studied. R. Y. C. Ming ([7]) proved that R is a left pV-ring $\Leftrightarrow$ for a maximal left subideal K of a principal left ideal P of R, $K^* \neq P^*$ (where K^* is the intersection of all maximal left ideals of R containing K) and the same result holds in case of a left fV-ring.

In the last chapter, properties of left V'-rings, left pV'-rings, left fV'-rings and left GPV-rings have been studied. A ring R is a left V'-ring (resp. a left fV'-ring, a left pV'-ring) if every simple and singular left R-module is injective (resp. f-injective, p-injective). Again we have the following implications:

$$\text{left V}'\text{-ring} \Rightarrow \text{left fV}'\text{-ring} \Rightarrow \text{left pV}'\text{-ring}$$

A ring R is a T-ring if every nonzero R-module has nonzero socle. Ming ([12]) proved that for a T-ring R, R is a left V'-ring if and only if the left singular submodule, $Z(_RR)$, of R is zero and $Rad(_RR/I) = 0$ for any essential left ideal I of R. A ring R is a *MELT* if every maximal essential left ideal of R is an ideal. Ming ([7]) proved that if R is a *MELT*, left pV'-ring then $Z(_RR) = 0$. Ming [13] proved that in a left GPV-ring, the Jacobson Radical, $Rad(_RR) = 0$.

As mentioned before, the concept of left V-rings was generalised to left fV-rings, left pV-rings, left V'-rings, left fV'-rings, left pV'-rings and left GPV-rings. It is interesting to study the relations between these classes of rings and their properties, for example in terms of Jacobson radical and singular submodules. This dissertation is a survey of these results.

Contents

0. Preliminaries

1. Left V-rings and left DSI-rings

2. Left pV-rings and left fV-rings

3. Left V'-rings and left GPV-rings

Chapter 0

Preliminaries

As mentioned in the preface, this chapter records basic definitions and results which will be of use in the later chapters. By a ring R, we mean a ring with identity. Unless mentioned, a module means a unitary left R-module.

0.0.1 Definition. A ring R is *von Neumann regular* or simply *regular* if $\forall a \in R$, there exists $b \in R$ such that $a = aba$. In this case, a is a *regular* element and b a *1-inverse* of a.

0.0.2 Proposition. *If R is a regular ring then every principal left ideal of R is generated by an idempotent and conversely.*

0.0.3 Definition. A ring R is called *fully (left or right) idempotent* if every ideal(left or right) of R is idempotent.

0.0.4 Definition. Let R be a ring.The *left Jacobson radical* of R , denoted by $J(_RR)$ or $Rad(_RR)$ is the intersection of all maximal left ideals of R. Similarly the *right Jacobson radical* $J(R_R)$ or $Rad(R_R)$ is the intersection of all maximal right ideals of R.

0.0.5 Remark. We always have $Rad(_R R) = Rad(R_R)$. We use the common notations $J = J(R) = Rad(_R R)$.

0.0.6 Definition. A ring R is *left (right) weakly regular* if $I = I^2$, for every left(right) ideal I of R.

0.0.7 Definition. A ring is *semi-primitive* if its Jacobson radical is zero.

0.0.8 Definition. A ring R is *strongly regular* if $\forall a \in R$, there exists $b \in R$ such that $a = a^2 b$.

0.0.9 Proposition. *If a ring R is strongly regular then R is reduced.*

0.0.10 Proposition. *If a ring R is reduced and $ab = 0$ for some $a, b \in R$ then $ba = 0$.*

0.0.11 Proposition. *If a ring R is strongly regular then R is regular.*

0.0.12 Proposition. *If a ring R is reduced and regular then R is strongly regular.*

0.0.13 Proposition. *If a ring R is strongly regular then $\forall a \in R$, there exists $b \in R$ such that $a = ba^2$.*

0.0.14 Proposition. *If R is a regular domain then R is a division ring.*

0.0.15 Proposition. *If R is a regular ring then $Rad(_R R) = 0$.*

0.0.16 Proposition. *Zorn's Lemma:-Let S be a non-empty partially ordered set. If every chain of S has an upper bound in S then S has at least one maximal element.*

0.0.17 Definition. Let M be a module over a ring R . Then the *Jacobson radical* of M is the intersection of all maximal submodules of M denoted by $J(_RM)$ or $Rad(_RM)$. If no maximal submodule exists then $Rad(_RM) = M$.

0.0.18 Definition. A left R-module M is *semi-simple* if every submodule of M is a direct summand of M.

0.0.19 Definition. A left R-module S is a *simple* module if $S \neq 0$ and the only submodules of S are 0 and S.

0.0.20 Proposition. *Let R be a ring. If M is a semi-simple left R-module then $Rad(_RM) = 0$.*

0.0.21 Proposition. *Let J be the Jacobson radical of a ring R. Then the following conditions are equivalent.*
(i)$xS = 0$, for every simple left R-module S
(ii)$f(x) = 0$,for every left R-homomorphism $f : R \longrightarrow S$ and for every simple left R-module S.
(iii)$x \in J$
(iv)$\forall r \in R$, $1 - rx$ is a unit in R.

0.0.22 Definition. Let M be a module over a ring R. A submodule L of M is *essential* or *large* if whenever $L \cap N = 0$, for some submodule N of M, then $N = 0$. We denote this by $L \trianglelefteq M$.

0.0.23 Definition. A left ideal of a ring R is *large* if it is a large submodule of R regarded as a module over itself.

0.0.24 Definition. A ring R is an *ELT* if every essential left ideal of R is an ideal.

0.0.25 Definition. A ring R is a *MELT* if every maximal essential left ideal of R is an ideal.

0.0.26 Proposition. *A finite intersection of essential submodules is essential.*

0.0.27 Proposition. *Let M be a module over a ring R . Let K and N be submodules of M such that $K \leq N \leq M$ then $K \trianglelefteq M$ implies that $N \trianglelefteq M$.*

0.0.28 Proposition. *Let M be a module over a ring R. A submodule K of M is large in M if and only if $\forall x \neq 0$, $x \in M$, there exists $r \in R$, $r \neq 0$ such that $0 \neq rx \in K$.*

0.0.29 Proposition. *Let M be a module over a ring R and let N be a submodule of M then the poset $\sum = \{K \leqslant M : K \cap N = 0\}$ has a maximal element.*

With notations as in proposition 0.0 29, we have the following definition.

0.0.30 Definition. Let M be a module over a ring R and N be its submodule. A submodule K of M which is a maximal element of $\sum$ is an *M-complement* of N.

0.0.31 Proposition. *Let M be a module over a ring R and N be its submodule. Let K be an M-complement of N then $N \oplus K = N + K$ is large in M.*

0.0.32 Definition. Let M be a module over a ring R. Let S be a subset of M. The set $l(S) = \{r \in R : rm = 0, \forall m \in S\}$ is the *left annihilator* of S. The *right annihilator* of S is similarly defined. If $S = \{m\}$, we denote $l(S)$ by $l(m)$.

0.0.33 Definition. A left R module M is a *faithful left module* if its left annihilator is zero.

0.0.34 Definition. A ring R is *primitive* if it has a simple faithful module.

0.0.35 Definition. Let M be a module over a ring R. The set $Z(M) = \{m \in M : l(m) \trianglelefteq R\}$ is the *singular* submodule of M. If $Z(M) = M$ then M is singular. If $Z(M) = 0$ then M is *non-singular.*

0.0.36 Definition. Let R be a ring. Let $b \in R$. The set $l(b) = \{r \in R : rb = 0\}$ is the *left annihilator* of b and the set $r(b) = \{s \in R : bs = 0\}$ is the *right annihilator* of b.

0.0.37 Definition. Let R be a ring. The set $Z(_RR) = \{b \in R : l(b)$ is an essential left ideal of $R\}$ is the *singular left ideal* of R.

0.0.38 Definition. Let R be a ring.The set $Z(R_R) = \{b \in R : r(b)$ is an essential left ideal of $R\}$ is the *singular right ideal* of R.

0.0.39 Proposition. *If R is a commutative ring then $Z(_RR) = Z(R_R)$.*

0.0.40 Definition. A ring R is *semiprime* if it has no nonzero nilpotent left ideal.

0.0.41 Definition. A ring R is a *prime* ring if there are no nonzero two sided ideals A and B of R such that $AB = 0$. In other words, if $AB = 0$ then either $A = 0$ or $B = 0$.

0.0.42 Definition. A ring R is *left Noetherian* if it satisfies the following three equivalent conditions:

(i)Every non-empty set of left ideals in R has a maximal element.

(ii)Every ascending chain of left ideals in R is stationary.

(iii)Every left ideal in R is finitely generated.

0.0.43 Definition. A ring R is *left Artinian* if it satisfies the following equivalent conditions:

(i)Every non-empty set of left ideals in R has a minimal element.

(ii)Every descending chain of left ideals in R is stationary.

0.0.44 Remark. Right Noetherian and right Artinian rings are similarly defined.

0.0.45 Proposition. *A left(right) Noetherian and regular ring is semi-simple artinian.*

0.0.46 Definition. A sequence $M_1 \longrightarrow M_2 \longrightarrow M_3 ... \longrightarrow M_i \longrightarrow ... \longrightarrow M_n$ of left R modules and left R-homomorphisms $f_i : M_i \longrightarrow M_{i+1}$ is *exact* at M_i if $Ker(f_i) = Im(f_{i-1})$, $\forall i$, $2 \leq i \leq n-1$. The sequence is *exact* if it is exact at every M_i.

0.0.47 Definition. If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of left R-modules and homomorphisms then the sequence is a *short exact sequence.*

0.0.48 Definition. Let $\alpha : M \longrightarrow M'$ be a left R-monomorphism. An exact sequence $0 \longrightarrow M' \longrightarrow M$ is a *split exact sequence* or is *splittable* or α is a *split homomorphism* if there exists an R-homomorphism $\gamma : M \longrightarrow M'$ such that $\gamma_o \alpha$=Identity homomorphism. Here γ is a split of α.

0.0.49 Definition. Let $\beta : M \longrightarrow M'$ be an onto homomorphism. An exact sequence $M \longrightarrow M'' \longrightarrow 0$ of left R-modules is a *split exact sequence* or is *splittable* or β is a *split epimorphism* if there exists an R-homomorphism $\delta : M'' \longrightarrow M$ such that $\beta_0\delta$=Identity homomorphism. Here δ is a split of β.

0.0.50 Definition. Let P be a left module over a ring R. We say P is projective over R if for every R-epimorphism $\beta : M \longrightarrow N$ and every R-homomorphism $g : P \longrightarrow N$, there exists a left R-homomorphism $f : P \longrightarrow M$ satisfying $\beta_o f = g$.

0.0.51 Proposition. *Every module is a factor module of a projective module.*

0.0.52 Proposition. *Direct summands of projective modules are projective.*

0.0.53 Proposition. *A left ideal generated by an idempotent is projective as a module over R.*

0.0.54 Definition. A module M over R is *divisible* if $M = cM$ for every non-zero divisor c of R.

0.0.55 Proposition. *Any quotient module of a divisible module is divisible.*

0.0.56 Definition. Let R be a ring and E be a left R-module. Then E is *injective* if given a row exact sequence $0 \longrightarrow M' \longrightarrow M$ and a left R-homomorphism $f : M' \longrightarrow E$, there exists a left R-homomorphism $g : M \longrightarrow E$ such that $g_o\alpha = f$, where $\alpha : M' \longrightarrow M$ is a monomorphism.

0.0.57 Definition. Let M be a left R-module. Then *socle* of M , denoted by $Soc(M)$ is given by $Soc(M) = \sum S_i$, where S_i is a simple submodule of M.

0.0.58 Definition. A ring R is a T-ring if every nonzero R-module has nonzero socle.

0.0.59 Proposition. *Baer's Theorem:-Let E be a module over a ring R. Then E is injective if and only if for every left ideal A of R, any left R-homomorphism from A to E can be extended to a left R-homomorphism from R to E.*

0.0.60 Proposition. *Let E be an injective module over a ring R such that E is a submodule of a module M over R then E is a direct summand of M.*

0.0.61 Definition. Let M be a module over a ring R. Then the *injective hull* of M is the smallest injective module over R containing M.

0.0.62 Proposition. *If every left ideal of a ring R is a direct summand of R then every left R-module is injective over R.*

0.0.63 Proposition. *Every injective left R-module is divisible.*

0.0.64 Definition. A ring R is *left semi-hereditary* if every finitely generated left ideal of R is a projective module.

0.0.65 Definition. A ring R is *hereditary* if every submodule of a projective module of R is projective.

0.0.66 Definition. A ring R is a *left V-ring* if every simple left R-module is injective.

0.0.67 Definition. Let M be a left module over a ring R. A left module E over R is *M-injective* if for every submodule N of M, every left R-homomorphism from N to E extends to one from M to E.

0.0.68 Definition. Let R be a ring and M be a left R-module. Then M is a *V-module* if every simple left R-module is M-injective.

0.0.69 Definition. Let R be a ring. A left R-module M is *p-injective* if for any principal left ideal P of R , every left R-homomorphism of P into M extends to one of R into M.

0.0.70 Definition. A ring R is a left pV-ring if every simple left R-module is p-injective.

0.0.71 Definition. Let R be a ring. A left R-module M is *f-injective* if for any finitely generated left ideal I of R , every left R-homomorphism of I into M extends to one of R into M.

0.0.72 Definition. A ring R is a left *fV-ring* if every simple left R-module is f-injective.

0.0.73 Definition. Let R be a ring. A left R-module M is *YJ-injective* if for any $0 \neq a \in R$, there exists a positive integer n with $a^n \neq 0$ such that any left R-homomorphism from Ra^n to M extends to one from R to M.

0.0.74 Definition. A ring R is a left GPV- ring if every simple left R-module is YJ-injective.

0.0.75 Definition. Let R be a ring and L be a left ideal of R. L is a *generalized weakly ideal (briefly a* $GW-ideal$*)* if for any $a \in L$, there exists a positive integer n such that $a^nR \subseteq L$.

0.0.76 Definition. Let R be a ring and K be a right ideal of R. K is a *generalized weakly ideal (briefly a* $GW-ideal$*)* if for every $a \in L$, there exists a positive integer n such that $Ra^n \subseteq K$.

0.0.77 Definition. A ring R is a left *pV'-ring* if every simple singular left R-module is p-injective.

0.0.78 Definition. A ring R is a left *fV'-ring* if every simple singular left R-module is f-injective.

0.0.79 Definition. A ring R is a left *DSI-ring* if every direct sum of simple left modules over R is injective.

0.0.80 Definition. A ring R is called an *MTE* ring if for any proper essential left ideal E of R, every maximal left subideal is an ideal of E.

0.0.81 Definition. A ring R is an *ALD (almost left duo) ring* if for any proper essential left ideal E of R, every complement or maximal left subideal is an ideal of E and E is an ideal of R.

0.0.82 Proposition. *Let $\{M_i\}_{i \in I}$ be any family of modules over a ring R with I an infinite set. Suppose there are elements $x_i \in M_i, a_i \in R$ and a well ordering $<$ on I satisfying*
(i)$a_i x_i \neq 0$ for all $i \in I$
(ii)$a_i x_j = 0$ whenever i is strictly less than j
Then $\oplus_{i \in I} M_i$ is not a direct summand of $\prod_{i \in I} M_i$.

0.0.83 Definition. A ring R is *zero insertive (briefly ZI)* if for $a, b \in R, ab = 0$ implies $aRb = 0$.

0.0.84 Definition. A ring R is *zero commutative (briefly ZC)* if for $a, b \in R, ab = 0$ implies $ba = 0$.

0.0.85 Definition. A ring R is called *strongly left bounded (briefly left SLB)* if every nonzero left ideal of R contains a nonzero two-sided ideal of R.

0.0.86 Definition. A ring R is a *2-primal ring (also called an N-ring)* if its prime radical coincides with the set of nilpotent elements in R.

0.0.87 Definition. A ring R is *left weakly continuous* if $J(R) = Z_l(R), R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$.

0.0.88 Definition. A left ideal I is *reflexive* if $xRy \subseteq I$ implies that $yRx \subseteq I$ for $x, y \in R$. A ring R is *reflexive* if 0 is a reflexive ideal.

0.0.89 Definition. A left ideal I is *idempotent reflexive* if $xRe \subseteq I$ implies that $eRx \subseteq I$ for $x \in R$ and for $e \in I(R)$. A ring R is an *idempotent reflexive ring* if 0 is an idempotent reflexive ideal.

0.0.90 Definition. A ring R is *abelian* if every idempotent of R is central.

0.0.91 Definition. An element $a \in R$ is a *left weakly regular element* if $a \in RaRa$.

Chapter 1

Left V-rings and Left DSI-rings

In this chapter, we study rings (called left V-rings) whose simple left R modules are injective. We also study left DSI-rings, satisfying the condition that every direct sum of simple left R modules is injective. In this chapter, we also discuss relations betweeen left V-rings and regular rings and a section on V-modules is included.

1.1 Study of left V-rings and left V-modules

The following theorem is due to Michler and Villamayor (See [14], p-186, Theorem 2.1)

1.1.1 Theorem. *The following properties of the ring R are equivalent*
1)R is a left V-ring.
2)Every left R-module has radical zero.
3)Every cyclic left R-module has radical zero.

4)Every left ideal of R is an intersection of maximal left ideals of R.

Proof. 1)⇒2)

Let M be a left R-module. Let $x \in Rad(_RM)$ such that $x \neq 0$. Consider the set $S = \{N \leq M : x \notin N\}$. We order this set by inclusion. Since $0 \leq M$ and $x \notin 0$ therefore $0 \in S$ and hence $S \neq \phi$. So S is a non-empty partially ordered set. Let (N_α) be a chain of submodules in S so that for each pair of indices α, β we have either $N_\alpha \subseteq N_\beta$ or $N_\beta \subseteq N_\alpha$ Let $U = \cup N_\alpha, \alpha \in I$ where I is an indexing set. Let $x, y \in U$ then $x \in N_\alpha$ and $y \in N_\beta$ for some α and β in I. Without any loss, we assume that $N_\alpha \subseteq N_\beta$ then $x, y \in N_\beta$ therefore $x - y \in N_\beta$ (since N_β is a submodule of M) and hence $x - y \in U$. So $(U, +)$ is a subgroup of M. Let $r \in R$ and $n \in U$ then $n \in N_\alpha$ for some $\alpha \in I$. so $rn \in N_\alpha$ and hence $rn \in U$. So U is a submodule of M. Also since $x \notin N_\alpha$, $\forall \alpha \in I$ (since $N_\alpha \in S$) so $x \notin U$. Therefore $U \in S$ is an upper bound of the chain. Hence, by Zorn's lemma, S has a maximal element Y(say). Let $D = \cap S_i$, where $S_i \leq M$ such that $Y \subset S_i$ with $i \in J$, some indexing set. Then clearly $Y \subset D$.

Claim 1:-$x \in D$.

For if $x \notin D$ then $x \notin S_i$ for some $i \in J$ and hence $S_i \in S$ but $Y \subset D$ and $Y \in S$. This contradicts that Y is a maximal element of S. So our supposition was wrong and therefore the claim.

Claim 2:-Y is a maximal submodule of D.

Let X be a submodule of D such that $Y \subseteq X \subseteq D$. If $Y \subset D$ then $D \subseteq X$(since X is one of the $S_i's$ containing Y properly). Hence $D = X$, so Y is a maximal left submodule of D. Hence D/Y is simple. Since R is a left V-ring so D/Y is injective. Hence $D/Y \leq \oplus M/Y$.

Therefore there exists a submodule K of M such that $M/Y = D/Y \oplus K/Y$.

Claim 3:-$x \notin K$.

For if $x \in K$ then $x + Y \in K/Y$. So $x + Y \in D/Y \cap K/Y = Y$. This implies $x \in Y$ which is impossible (since $Y \in S$). Hence $x \notin K$.

Claim 4:-Y is a maximal submodule of M

Since $x \notin K$ and Y being a maximal element of S implies that $K = Y$. Hence $K/Y \cong 0$. So $M/Y = D/Y$. Since D/Y is simple, we see that M/Y is also simple. Hence Y is a maximal submodule of M.

Now $Rad_R(M) \subseteq Y$. Since $x \notin Y$ we have $x \notin Rad_R(M)$ which contradicts our assumption in the beginning. Hence $Rad(_RM) = 0$.

2)$\Rightarrow$3) is clear.

3)$\Rightarrow$4)

Let A be a left ideal of R then R/A is a cyclic left R-module (generated by $1 + A$). Hence $Rad(_RR/A) = 0$.

Let $Max(R)$ be the set of all maximal left ideals of R. Let $\Im = \{V \in Max(R) : V \supseteq A\}$. By definition, $Rad(_RR/A) = \cap(V/A), V \in \Im$. So we get $0 = Rad(_RR/A) = \cap(V/A) = (\cap V)/A$ implying therefore that $A = \cap V$.

4)$\Rightarrow$1)

Let S be a simple R-module and A be a left ideal of R. Let $f : A \longrightarrow S$ be an R-homomorphism. Since $Kerf$ is a left ideal of R so $Kerf = \cap V$, $V \in Max(R)$ where $Max(R)$ is the set of all maximal left ideals of R. So there exists a maximal left ideal M of R such that $Kerf \subseteq M$ but $A \nsubseteq M$. Since S is simple, either $f = 0$, in which case it can be extended or f is onto. If f is onto then by the fundamental theorem of module homomorphism $A/Kerf \cong S$. Since S is simple, $Kerf$ becomes a maximal left ideal of A.

Now, $Kerf \subseteq M \cap A \subseteq A$. This yields either $M \cap A = A$ or $M \cap A = Kerf$. The former case cannot arise as $A \not\subseteq M$. Hence the latter equality holds.
We next observe that since M is a maximal left ideal of R, the inclusion $M \subseteq M + A \subseteq R$ implies either $M = M + A$ or $M + A = R$. But $M = M + A$ gives a contradiction as $A \not\subseteq M$. So $M + A = R$. Now $R/M = (M+A)/M \cong A/(M \cap A) = A/Kerf \cong S$. Since there exists an R-module homomorphism $g : R \longrightarrow R/M$ defined by $g(x) = x + M$, $\forall x \in R$ we find that f can be extended to $g \cdot R \longrightarrow S$ (since $R/M \cong S$). Hence by Baer's theorem, S is injective. □

1.1.2 Corollary. *If R is a left V-ring then every left ideal of R is idempotent*

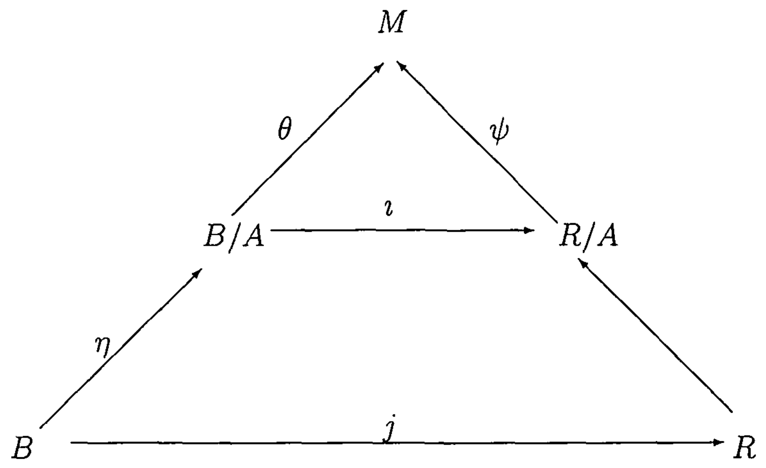
Proof. Let A be a left ideal of R. Suppose $A^2 \neq A$. Since A^2 is also a left ideal of R so it is an intersection of maximal left ideals of R. So there exists a maximal left ideal M of R such that $A^2 \subseteq M$ but $A \not\subseteq M$. Now M is maximal and $M \neq M + A$ (otherwise $A \subseteq M$) implies that $M + A = R$. So $1 = m + a$, for some $m \in M$ and some $a \in A$. Since $a \in A$, we have $a = am + a^2 \Rightarrow a \in M \Rightarrow 1 \in M \Rightarrow M = R$ which is impossible. So our assumption was wrong and hence $A^2 = A$.

□

1.1.3 Proposition. *Factor rings of left V-rings are left V-rings.*

Proof Let R be a left V-ring, A be a two sided ideal of R and M be a simple left R/A module
We show that M is injective over R/A .

Let B/A be a left ideal of R/A. Let us consider the following figures



Since M is injective over R, we see that there exists a left R-homomorphism $\phi : R \longrightarrow M$ such that $\Theta \circ \eta = \phi \circ \jmath$. Let $\psi : R/A \longrightarrow M$ be defined by $\psi(r+A) = \phi(r)$, $\forall r+A \in R/A$. As A is contained in $Ker\phi$, ψ is well defined.

Let $b + A \in B/A$ then

$$(\psi \circ \imath)(b + A) = \psi(b + A) = \phi(b) = (\phi \circ \jmath)(b) = (\Theta \circ \eta)(b) = \Theta(b + A)$$

Hence M is injective over R/A. So R/A is a left V-ring □

1.1.4 Proposition. *If M is a left V-module then $Rad(_RM) = 0$.*

Proof. Let $0 \neq m \in Rad(_RM)$. Since Rm is a finitely generated nonzero module over R so it has a maximal submodule μ. Now Rm/μ is a simple left R module and M is a left V-module so Rm/μ is M-injective. Hence for the canonical homomorphism $\eta : Rm \longrightarrow Rm/\mu$, there exists an R-homomorphism $g : M \longrightarrow Rm/\mu$ such that g extends η. Since η is onto , g is also onto. Hence $M/Kerg \cong Rm/\mu$ which shows that $Kerg$ is a maximal submodule of M. This implies that $m \in kerg$. So $g(m) = \mu \Rightarrow \eta(m) = \mu$

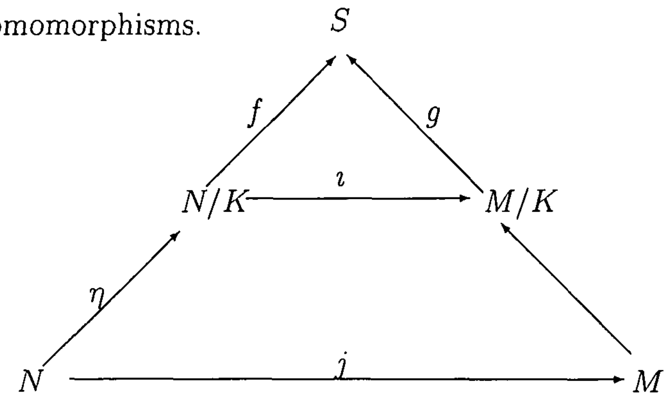
(since g extends η) $\Rightarrow m + \mu = \mu \Rightarrow m \in \mu \Rightarrow Rm = \mu$ which is not possible. Hence $Rad(_RM) = 0$

$\square$

Following are some of the results on left V-modules.

1.1.5 Proposition. *Factor modules of left V-modules are left V-modules.*

Proof. Let M be a left V-module and K be a submodule of M. Let S be a simple left R module. Let us consider the following figure of left R-modules and R-homomorphisms.

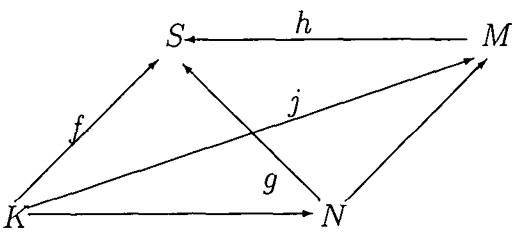


Since M is a left V-module, there exists a left R-homomorphism $h : M \longrightarrow S$ such that $h \circ j = f \circ \eta$. Let $g : M/K \longrightarrow S$ be defined by $g(m + K) = h(m)$. Then g is well defined. For if $x + K, y + K \in M/K$ such that $x + K = y + K$ then $x - y \in K \Rightarrow f(x - y + K) = 0 \Rightarrow f(\eta(x - y)) = 0$ $\Rightarrow (f \circ \eta)(x - y) = 0 \Rightarrow (f \circ \eta)(x) = (f \circ \eta)(y) \Rightarrow (h \circ j)(x) = (h \circ j)(y)$ $\Rightarrow h(j(x)) = h(j(y)) \Rightarrow h(x) = h(y)$.

Let $n + K \in N/K$, where N is a submodule of M. Then $g(n + K) = h(n) = (h_o j)(n) = (f_o \eta)(n) = f(\eta(n)) = f(n + K)$ showing that g extends f. Hence M/K is a left V-module. $\square$

1.1.6 Proposition. *Submodules of left V-modules are left V-modules*

Proof. Let M be a left V-module and N be its submodule. Let K be a submodule of N and S be a simple left R-module. Let us consider the following figure of left R-modules and R-homomorphisms.



Since M is a left V-module, there exists a left R-homomorphism $h : M \longrightarrow S$ such that $h \circ j = f$. Let $g : N \longrightarrow S$ be defined by $g(n) = h(n)$, $\forall n \in N$. Then g is well defined. For if $x, y \in N$ such that $x = y$ then $h(x) = h(y)$ which implies that $g(x) = g(y)$. Let $k \in K$. Now $g(k) = h(k) = (h \circ j)(k) = f(k)$ showing that g extends f. Hence N is a left V-module. □

1.1.7 Proposition. *Following conditions are equivalent for a left R-module M*

(1)M is a left V-module.

(2)Rad($_RM/K$) = 0 for all submodules K of M.

Proof. (1)⇒(2). If M is a left V-module then M/K is also a left V-module so by the proposition 1.1.5 $Rad(_RM/K) = 0$

(2)⇒(1). Let S be a simple left R-module. Let N be a submodule of M and let $f : N \longrightarrow S$ be a left R-homomorphism. If $f = 0$ then we are through. Suppose not, then $N/Kerf \cong S$. Let $K = Kerf$. Since $Rad(_RM/K) = 0$, there exists a maximal submodule N' of M such that $N' \supseteq K$. If not then

$Rad(_RM/K) = \bigcap\limits_{\phi} N'/K = M/K$, a contradiction. Again, we claim that there exists a maximal submodule N' of M such that $N' \supseteq K$ but $N' \not\supseteq N$. If not then $\{N' \leq M : N'$ maximal in M and $N' \supseteq K\} = \{N' \leq M : N'$ maximal in M and $N' \supseteq N\}$. This will imply that $K = N$, which is absurd. Hence the claim.

Now $K \subseteq N \cap N'$. So we have $N \cap N'/K$ is a submodule of N/K. Since N/K is simple, we have $N \cap N'/K = 0$ or $N \cap N'/K = N/K$. From the second case, we have $N \cap N' = N$. This implies that $N \subseteq N'$, a contradiction. Hence $N \cap N' = K$.

Now since N' is maximal, $M = N + N'$ which implies that

$$M/N' = N + N'/N' \cong N/(N \cap N') = N/K \cong S ....(1)$$

Let η be the natural map from M to M/N' and let θ be a left R-homomorphism from M/N' to S obtained via (1). Also let $h : N/K \longrightarrow S$ be a left R-homomorphism obtained via (1). Let $n \in N$ then

$$(\theta \circ \eta)(n) = \theta(n + N') = h(n + K) = f(n)$$

This shows that $\theta \circ \eta$ extends f. So S is M-injective. Hence M is a left V-module.

□

In the following proposition, given by Kaplansky, we can see that a commutative regular ring is equivalent to a V-ring.

1.1.8 Proposition. *Let R be a commutative ring . Following conditions are equivalent*

(1)R is regular

(2)R is a V-ring

Proof. (1)$\Rightarrow$(2) Let A be a left ideal of R. Since R is regular so R/A is regular. This shows that $Rad(_{R/A}R/A) = 0$ which implies that $Rad(_RR/A) = 0$. Hence R is a left V-ring.

(2)$\Rightarrow$(1) Let $a \in R$, then Ra is a left ideal of R. Since R is a left V-ring, Ra is idempotent. Hence $Ra = RaRa$. So $a = \sum r_i a s_i a$, $1 \leq i \leq m$. $\Rightarrow a = \sum a r_i s_i a, 1 \leq i \leq m$. This shows that $a = axa$, where $x = \sum r_i s_i, 1 \leq i \leq m$. Hence R is regular.

□

Following are some important propositions and remarks which will lead us to an example of a ring which is regular but not a left V-ring.

1.1.9 Remark. Let K be a field and let $W = K[x]$. Let $R = \{\theta : W \longrightarrow W : \theta$ is a $K-$ homomorphism $\}$. Then $\theta \in I(R)$ iff $\theta|_{\theta(W)} = Id_{\theta(W)}$.

1.1.10 Proposition. *Let V be a vector space over a field K. Let $x, y \in V$ then there exists a linear transformation $f : V \longrightarrow V$ such that $f(x) = y$.*

Proof. If $x = 0$, we can take $y = 0$ and hence we take f to be the zero transformation. If $x \neq 0$ then x is linearly independent and so $\{x\}$ can be extended to a basis $\{x\} \cup \beta$ of V. We take $f(x) = y$ and $f(z) = 0\ \forall z \in \beta$ and extend it linearly. □

1.1.11 Proposition. *If $W = K[x]$ where K is a field then $R = \{\theta : W \longrightarrow W : \theta$ is a $K-$ homomorphism $\}$ is regular.*

Proof. Let $\theta \in R$ and let $A = \theta(W)$ then A is a subspace of W. So there exists a subspace B of W such that $W = A \oplus B$. Let $e : W \longrightarrow W$ be defined by $e(a+b) = a$, $\forall a \in A, b \in B$. Then e is a well-defined K-homomomorphism and also $e^2(a+b) = e(e(a+b)) = e(a) = e(a+0) = a = e(a+b)$. So e is an idempotent element of R. Now for any $w \in W$, $\theta(w) \in A$. By remark 1, $(e \circ \theta)(w) = e(\theta(w)) = \theta(w)$. Hence $\theta \in eR$. Again for any $w \in W$, $e(w) = e(a+b) = a = \theta(v)$, for some $a \in A, b \in B, v \in W$. So by the previous proposition, there exists $\phi : W \longrightarrow W$ so that $\phi(w) = v$. Therefore $e(w) = \theta(\phi(w)) = (\theta \circ \phi)(w)$. This shows that $e \in \theta R$. So $eR = \theta R$. This implies that every principal right ideal in R is generated by an idempotent. Hence R is regular. □

1.1.12 Proposition. *Let W be a vector space over a field K and let $R = \{\theta : W \longrightarrow W : \theta$ is a $K-$ homomorphism $\}$. Then W is a simple left R module.*

Proof. Let $0 \neq v \in W$ then v being linearly independent can be extended linearly to a basis $\{v\} \cup \beta$ of W. Let $w \in W$ and let $f : W \longrightarrow W$ be defined by $f(v) = w$ and $f(x) = 0$, $\forall x \in \beta$ and we extend f linearly. Then $f \in R$ and since $f(v) = w$, we see that $w \in Rv$. This implies that $W = Rv$. So W is simple. □

Following is an example due to [1] to show that a regular ring need not be a left V-ring.

1.1.13 Example. Let W be an infinite-dimensional vector space over a field K and let $R = \{\theta : W \longrightarrow W : \theta$ is a $K-$ homomorphism $\}$ then by the previous remark and propositions, R is regular and W is simple over R. Now our claim is that R is not a left V-ring.

Suppose W has a countable infinite base $\{v_n\}_{n\geq 1}$ over K. For each integer $n \geq 1$, let $W_n = W$ and $g_n : W_n \longrightarrow W$ be the identity map of W. Let $g : \oplus_{n\geq 1} W_n \longrightarrow W$ be defined by

$$g(x_1, x_2, ..., x_{n+1}, ..., 0, 0, ...) = \sum_{i=1}^{l} g_i(x_i) = \sum_{i=1}^{l} x_i$$

Suppose W is injective over R, then there is an extension $h : \Pi_{n\geq 1} W_n \longrightarrow W$ of g. Let $q : R \longrightarrow \Pi_{n\geq 1} W_n$ be defined by

$$q(\theta) = (\theta(v_1), \theta(v_2), ..., \theta(v_{n+1}), ..., 0, ...0, ...)$$

Then q is a left R-homomorphism. Let $f : R \longrightarrow W$ be the map $f = h \circ q$ and let $w = f(1)$. Then for any $\theta \in R$, we have

$$f(\theta) = f(\theta.1) = \theta f(1) = \theta w = \theta(w)$$

Now, for any integer $n \geq 1$, let F_{n+1}=vector subspace of W spanned by $v_k, k \geq n+1$ and let $I_n = \{\theta \in R : \theta(v_k) = 0 \text{ for } k \geq n+1\}$. Then for any $\theta \in I_n$, $q(\theta) = (\theta(v_1), \theta(v_2), ..., \theta(v_n), ..., 0, ...0, ...) \in W_1 \oplus W_2 \oplus ... \oplus W_n$. Since $h|_{\oplus_{n\geq 1} W_n} = g$, we get

$$f(\theta) = (h \circ q)(\theta) = h(q(\theta)) = g(q(\theta)) = g(\theta(v_1), \theta(v_2), ..., \theta(v_n), ..., 0, ...0, ...)$$

$$= g_1(\theta(v_1)) + g_2(\theta(v_2)) + ... + g_n(\theta(v_n)) = \theta(v_1) + \theta(v_2) + ... + \theta(v_n) = \theta(v_1 + v_2 + ... + v_n)$$

.

Let $\phi_n : W \longrightarrow W$ be determined by $\phi_n(v_i) = v_i, \forall 1 \leq i \leq n$ and $\phi_n(v_k) = 0$, for $k \geq n+1$. Then $\phi_n \in I_n$. So $f(\phi_n) = \phi_n(\sum_{i=1}^{n} v_i)$. But $f(\phi_n) = \phi_n(w)$. So $\phi_n(w - \sum_{i=1}^{n} v_i) = 0$.

Now,

$$Ker\phi_n = \{v \in W : \phi_n(v) = 0\} = \{v = \sum_{i=1}^{t} a_i v_i \in W : \phi_n(\sum_{i=1}^{t} a_i v_i) = 0\}$$

$$= \{v \in W : \sum_{i=1}^{n} a_i v_i = 0\} = \{v \in W : v = \sum_{i \geq n+1} a_i v_i\} = F_{n+1}$$

Hence we get $w - \sum_{i=1}^{n} v_i \in F_{n+1}$ i,e

$$w = \sum_{i=1}^{n} v_i + \sum_{j=n+1}^{m} b_j v_j$$

which is impossible since w is fixed. Therefore W cannot be injective over R So R is not a left V-ring.

The following four lemmas have been given in order to arrive at an important result (theorem immediately after the lemmas) which shows the relation between a strongly regular ring and a left V-ring.

1.1.14 Lemma. *In a strongly regular ring , every principal two-sided ideal is generated by a central idempotent.*

Proof Let R be a strongly regular ring and let P be a two-sided principal ideal of R. Since P is principal, $P = \langle a \rangle$ for some $a \in P$. Since R is strongly regular so R is regular. Hence there exists $b \in R$ such that $a = aba$. Now $(ba)^2 = (ba)(ba) = b(aba) = ba$. Let $e = ba$ then $a = aba = ae$ i,e $P \subseteq Re$.

Again $e = ba \in P$. So $Re \subseteq P$. Hence $P = Re$

Now, for any $r \in R, re(1-e) = 0 \Rightarrow (1-e)re = 0$ (since R is reduced)

$\Rightarrow re - ere = 0 \Rightarrow re = ere$

Again, $(1-e)er = 0 \Rightarrow er(1-e) = 0 \Rightarrow er - ere = 0 \Rightarrow er = ere$. Hence $re = er$ i,e e is a central idempotent. □

1.1.15 Lemma. *In a strongly regular ring, all ideals are two sided. (See [1], p-297, Proposition 3.6)*

Proof. Let A be a left ideal of a strongly regular ring R. Let $a \in A$ and $r \in R$. Since R is regular, there exists $b \in R$ such that $a = aba$. Now $ar = abar \Rightarrow ar = arba$ (since ba is a central idempotent)$\in A$. This shows that A is a right ideal of R. □

1.1.16 Lemma. *If S is a simple left R-module then any non-zero element of S generates S.*

Proof. Let $0 \neq a \in S$. Let $N = Ra$. Then N is a submodule of S. Since S is simple, its submodules are 0 and S. Hence $S = Ra$. □

1.1.17 Lemma. *Let R be a ring in which every maximal left ideal is two-sided. Let S be any simple left R-module and u, v be two non zero elements of S. Then $l(u) = l(v)$. (See [1], p-297, lemma 3.3)*

Proof. From the previous lemma, $S = Ru = Rv$. Let $f : R \longrightarrow Ru$ be defined by $f(r) = ru$, $\forall r \in R$. Clearly f is a well-defined onto R-homomorphism. $Ker f = \{r \in R : ru = 0\} = l(u)$. Hence $R/l(u) \cong Ru(= S)$. This implies that $l(u)$ is a maximal left ideal of R. Similarly $l(v)$ is a maximal left ideal of R. Hence $l(u)$ and $l(v)$ are two-sided ideals of R.

Let $x \in l(u)$ then $xu = 0$. Let $v \in Rv = Ru$ so $v = yu$. Now $xy \in l(u)$ (since $l(u)$ is two-sided) $\Rightarrow xyu = 0 \Rightarrow xv = 0 \Rightarrow x \in l(v)$. Hence $l(u) \subseteq l(v)$. Similarly we can show that $l(v) \subseteq l(u)$. So $l(u) = l(v)$ □

1.1.18 Theorem. *A ring R is strongly regular iff R is a left V-ring in which every maximal left ideal is two-sided. (See [1], p-297, Theorem 3.4)*

Proof. Let R be a strongly regular ring and S be any simple R-module. Let A be any left ideal in R and $f : A \longrightarrow S$ be any left R-homomorphism. If $f = 0$ then $0 : R \longrightarrow S$ is an extension of f. Suppose $f \neq 0$ then there exists $a \in A$ such that $f(a) \neq 0$. since R is strongly regular and Ra being a principal ideal of R is a two-sided ideal of R and hence it is generated by a central idempotent $e \in R$. So $Ra = Re$. clearly $R = Re \oplus R(1-e)$ and $A \supset Ra = Re$. Let $H = A \cap R(1-e)$

Claim 1:- $A = Re \oplus H$

Clearly $Re \subset A$ and $H \subset A$ therefore $Re + H \subset A$. Let $x \in A \Rightarrow x \in R \Rightarrow x = ye + z(1-e)$ for some $x, y \in R$. $\Rightarrow z(1-e) = x - ye \in A \cap R(1-e) \Rightarrow z(1-e) \in H$. Also $Re \cap H = Re \cap A \cap R(1-e) = Re \cap R(1-e) \cap A = 0 \cap A = 0$. Let $f(e) = u \in S$. Since $Ra = Re \Rightarrow a = \lambda e$ for some $\lambda \in R$. Hence $f(a) = f(\lambda e) = \lambda f(e) = \lambda u$. Since $f(a) \neq 0$ we have $u \neq 0$. Now $u = f(e) = f(ee) = ef(e) = eu$. This implies that $eu \neq 0$. Hence $e \notin l(u)$.

Claim 2:-$f(H) = 0$

Let $h \in H$ then $h = r(1-e)$. Let $f(h) = w$. Now $ew = ef(h) = f(eh)$ and $eh = er(1-e) = re(1-e) = r(e-e) = 0$. $\Rightarrow ew = ef(h) = f(eh) = 0$. Hence $e \in l(w)$. If $w \neq 0$ then $l(u) = l(w)$ i,e $e \notin l(w)$ which is a contradiction. Hence $w = 0$ i,e $f(h) = 0$, $\forall h \in H$. Let $g : R \longrightarrow S$ be defined by $g/_{Re} = f/_{Re}$ and $g/_{R(1-e)} = 0$.

Claim 3-: $g/_A = f$.

Let $x \in A$ then $x = ye + h$ for some $y \in R$ and $h \in H$. Now $g(x) = g(ye+h) = g(ye) + g(h) = f(ye) + g(z(1-e)) = f(ye) + 0 = f(ye) + f(h) = f(ye+h) = f(x)$. Hence $g/_A = f$

Conversely:-Let $a \in R$. Let $S = \{M : M$ is a maximal left ideal of $R\}$.

Since R is a left V-ring $Ra^2 = \cap M$ where $M \in S$ Now $a^2 \in M$. Suppose $a \notin M$ then $Ra + M = R \Rightarrow ra + m = 1$ for some $r \in R$ and some $m \in M$ then $ra^2 + ma = a \Rightarrow a \in M$(since M is two-sided, $ma \in M$) which contradicts our assumption. Hence $a \in M \Rightarrow a \in \cap M \Rightarrow a \in Ra^2$. So there exists $b \in R$ such that $a = ba^2$. Hence R is strongly regular. □

1.2 Results on left DSI-rings

In the following section, some results on left DSI-rings are given and the relations between these rings with left V'-rings, left Noetherian rings and left Artinian rings are also studied. Also, an obvious result to be mentioned is that every semi-simple ring is a left DSI-ring.

1.2.1 Proposition. *Every left DSI-ring is a left V-ring.*

Proof. Let R be a left DSI-ring and S be a simple left R-module then S is a direct sum of itself and hence is injective. So R is a left V-ring. □

1.2.2 Proposition. *Every left DSI-ring is left Noetherian. (See [1])*

Proof. Suppose R is not left Noetherian then there exists an infinite ascending sequence of left ideals $I_1 \subset I_2 \subset I_3 ...$ with $I_n \neq I_{n+1}$ for any n. So there exists $a_n \in I_{n+1}$ such that $a_n \notin I_n$. Since R is a left left DSI-ring and hence a V-ring so we can find a maximal left ideal J_n of R with $J_n \supseteq I_n$ and $a_n \notin J_n$ Let $M_n = R/J_n$, then M_n is a simple left R-module. Let $\eta_n : R \longrightarrow R/J_n$ be the canonical map. Now $\eta_n(1) = 1 + J_n = x_n$(say) $a_n = x_n = a_n(1 + J_n) = a_n + J_n \neq J_n \forall n \in IN$ and for $k \gneq n, a_n \in I_{n+1} \subseteq I_k \subseteq J_k$ and $a_n x_k = a_n(1+J_k) = a_n + J_k = J_k$. Hence $\oplus_{n\geq 1} M_n$ is not a direct summand of $\prod_{n\geq 1} M_n$. Since R is a left DSI-ring, $\oplus_{n\geq 1} M_n$ is injective and is a direct summand of $\prod_{n\geq 1} M_n$. This contradiction shows that R has to be Noetherian. □

1.2.3 Proposition. *A strongly regular ring R is a left DSI-ring if and only if R is a direct product of finitely many skew-fields. (Refer [1])*

Proof. Suppose R is a strongly regular left DSI-ring then R is Noetherian. Again, since R is strongly regular, every left ideal of R is two-sided. Now, R is Noetherian and regular so it is semi-simple artinian. Since R is semi-simple, it is a direct sum of simple left ideals of R. So $R = \oplus_{i\in I} K_i$ and K_i is a simple left ideal of R.

Claim 1:-I is finite

Let L be a countable subset of I. Since R is artinian, we get

$$\bigoplus_{i\in I} K_i \supseteq \bigoplus_{i\in L} K_i \supsetneq \bigoplus_{i\in L\setminus\{i_1\}} K_i \supsetneq \bigoplus_{i\in L\setminus\{i_1,i_2\}} K_i ... \supsetneq \bigoplus_{i\in L\setminus\{i_1,i_2,\ ,i_l\}} K_i = \bigoplus_{i\in L\setminus\{i_1,i_2,\ ,i_l,i_{l+1}\}} K_i$$

Again, since R is Noetherian, we get

$$\bigoplus_{i\in L\cup\{u_1,u_2,\ ,u_{k+1}\}} K_i = \bigoplus_{i\in L\cup\{u_1,u_2,\ ,u_k\}} K_i \supsetneq ... \supsetneq \bigoplus_{i\in L\cup\{u_1,u_2\}} K_i \supsetneq \bigoplus_{i\in L\cup\{u_1\}} K_i \supsetneq \bigoplus_{i\in L} K_i$$

Therefore $L = \{i_1, i_2, ..., i_l\} \cup \{i_{k+1}, ...\} = \{i_1, i_2, ..., i_l\}$. So $L \cup \{u_1, u_2, ..., u_s\} = I$. Hence I is finite. So by claim 1, R is a direct sum of finitely many simple left ideals of R. Let K be a simple ideal of R and J be its complementary left ideal. Then $R = K \oplus J$.

Claim 2:-J is a maximal left ideal of R

Let B be a left ideal of R such that $J \subsetneq B \subsetneq R$. If $J \neq B$ then there exists a nonzero element $x \in B$ such that $x \notin J$. Since K is simple, $K = Rx$. So $R = Rx + J$. Hence $1 = rx + j$ for some $r \in R$ and some $j \in B$. This implies that $1 \in B$. So $B = R$.

Claim 3:-$R/J \cong K$

Let $f : R \longrightarrow K$ be the projection map. Then $Kerf = \{k + j \in R : k = 0\} = J$. Hence the claim and therefore K is a skew-field.

Conversely, assume that R is a direct product of finitely many skew fields. Since a skew field is strongly regular, R is strongly regular. Since finite direct product of skew fields is Artinian semi-simple hence all the modules over R are injective and so are the direct sums of simple modules over R. Hence R is a left DSI-ring. □

1.2.4 Corollary. *A commutative ring R is a DSI-ring if and only if it is a direct sum of finitely many fields.*

Proof. Any left DSI-ring is a left V-ring. Since R is commutative it follows that R is regular. Hence R is strongly regular and hence from the previous theorem the desired result can be obtained. □

The following gives an example of a V-ring which is not a left DSI-ring.

1.2.5 Example. Let $R = \prod Q$, where Q is the set of rational numbers. Since Q is regular so R is regular and since R is commutative, we see that R is a V-ring. But R is not a DSI-ring. Consider $M = \oplus Q$. Then M is not injective over R. Otherwise M will be a direct summand of R. This will imply that $M = Re$, where e is an idempotent element of R. So $e = (e_1, e_2, .., e_n, 0, 0, 0, ..)$. But there exists $t = (e_1, e_2, ..., e_n, 1, 0, 0, 0, ...) \in M$ such that $t \notin Re$. So R is a V-ring but not a DSI-ring.

Chapter 2

Left pV-rings and Left fV-rings

In this chapter, we study about left pV-rings and left fV-rings, which are two particular cases of left V-rings. So a left V-ring will automatically be a left pV-ring and also a left fV-ring.

2.1 Study of left pV-rings and p-Injectivity

In this section, we study about properties of left pV-rings and p-injectivity and an example of a left pV-ring which is not a left V-ring has also been given.

2.1.1 Proposition. *Let R be a ring whose simple left R-modules are either $p-$injective or projective. Then the centre of R is von Neumann Regular. (See [5], p-229, Lemma 2.12)*

Proof. Let C be the centre of R and let $c \in C$. Let $L = Rc + l(c)$ and let K be a left ideal of R such that $L \cap K = 0$. Then $Kc = cK \subseteq Rc \cap K \subseteq L \cap K = 0$. This implies that $K \subseteq l(c) \subseteq L \cap K = 0$, which implies that L is an essential left ideal of R. Suppose $L \neq R$, then there exists a maximal left ideal M of R so that $M \supseteq L$. This shows that M is an essential left ideal of R. So R/M cannot be projective. Therefore, by hypothesis, R/M being simple implies that R/M is p-injective. Let $f : Rc \longrightarrow R/M$ be defined by $f(rc) = r + M$, $\forall r \in R$. Then f is well defined. For if $rc = 0$ then $r \in l(c) \subseteq L \subseteq M$. So $r + M = M$ i,e $f(rc) = M$. Clearly f is a left R-homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Hence $1 + M = f(c) = g(c) = cg(1) = c(a + M) = ca + M$, for some $a \in R$. But $c \in L \subseteq M$. Hence $ac \in M$. So $1 \in M$, a contradiction. Hence $R = L$ and therefore $1 = bc + z$, for some $b \in R$ and some $z \in l(c)$. This implies that $c = bc^2 + zc \Rightarrow c = bc^2 = cbc$. Now let $d = c^2b^3$. Then $dc = ccbbbc = ccbbcb = ccbcbb = ccbb = cbcb = cb$. And $cdc = cbccb = ccb = cbc = c$. Now for any $u \in R$, $bc^2u = cu = uc = ubc^2 = c^2ub$. And $b^3c^2u = b^2bc^2u = b^2c^2ub = bbc^2ub = bc^2ubb = c^2ubbb = c^2ub^3$. Now let $d \in C$ then $du = c^2b^3u = b^3c^2u = c^2ub^3 = uc^2b^3 = ud$. Hence $d \in C$. So C is a von Neumann Regular ring. □

2.1.2 Corollary. *If R is a left pV-ring then the centre of R is von Neumann regular.*

2.1.3 Proposition. *If R is a left V-ring then R is a left pV-ring.*

Proof. Let S be a simple left R-module. Let P be a principal left ideal of R. Let $f : P \longrightarrow S$ be a left R-homomorphism. Since R is a left V-ring, there exists a left R-homomorphism $g : R \longrightarrow S$ which extends f. This implies that R is a left pV-ring. □

The converse of the above is not true. Following is an example to show that a left pV-ring need not be a left V-ring.

2.1.4 Example. Let W be an infinite-dimensional vector space over a field K and let $R = \{\theta : W \longrightarrow W : \theta \text{ is a } K- \text{ homomorphism }\}$ then (as proved in the first chapter) R is regular. So R is a left pV-ring and W is simple over R but R is not a left V-ring.

2.1.5 Proposition. *Let R be a left pV-ring. If a maximal left ideal I of R is a left annihilator then I is a direct summand of R.*

Proof. Given $I = l(S)$ for some subset S of R. Let $t \in S$ such that $t \neq 0$
Claim 1:-$I = l(t)$
Let $a \in I$ then $a \in l(S)$. So $a = l(s)$, $\forall s \in S$. In particular, for $s = t$ we get $at = 0$. This implies that $a \in l(t)$. So $I \subseteq l(t)$. Since $t \neq 0$ so $l(t) \neq R$. Since I is maximal, $I = l(t)$. Let $f : R \longrightarrow Rt$ be defined by $f(r) = rt \ \forall r \in R$. Then $Kerf = l(t)$. Hence $R/l(t) \cong Rt$. So Rt is a simple left R-module. Let $Id : Rt \longrightarrow Rt$ be the identity homomorphism. Since Rt is p-injective, there exists a left R-homomorphism $g : R \longrightarrow Rt$ extending Id. So $Id(t) = g(t)$. This implies that $t = tg(1) = tbt$ for some $bt \in Rt$. Hence $bt = btbt$. i,e bt is idempotent.
Claim 2:-$Rt = Rbt$
Clearly $Rbt \subseteq Rt$. Let $0 \neq x \in Rt$. Since Rt is simple, we have $Rt = Rx$. Therefore $t = rx$, for some $r \in R$. So $t = ryt$, for some $y \in R$. This implies that $bt = bryt \in Rt$. So $Rbt \subseteq Rt$. Hence Rt is generated by an idempotent. So Rt is a direct summand of R. Hence Rt is projective over R. Now let us consider the following exact sequence $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$. Let $h : R/I \longrightarrow R/I$ be a left R-homomorphism. Since R/I is projective, there

exists a left R-homomorphism from R/I to R. Thus we see that the above sequence is splittable. Hence I is a direct summand of R. □

2.1.6 Proposition. *Let R be a left pV-ring Then*

(i)Every left ideal of R is an idempotent

(ii)Every non-zero left ideal of R contains a maximal left sub-ideal.

(iii)$Rad(_RR) = 0$. (Refer [4], p-173, Lemma 1)

Proof Proof of (i). Suppose there exists a left ideal I of R such that $I \neq I^2$. Let $0 \neq y \in I$ such that $RyRy \subset Ry$. Consider the set $\sum = \{J \leq R : RyRy \subseteq J \subset Ry\}$. Clearly $RyRy \in \sum$. We order this set by inclusion then this set becomes a partially ordered set. Let (J_i) be a chain of left ideals in $\sum$. Let $N = \cup J_i$, $i \in I$. Then $N \in \sum$. So $\sum$ has an upper bound. By Zorn's lemma, $\sum$ has a maximal element L.

Claim .-L is a maximal left submodule of Ry.

Let M be a submodule of Ry such that $L \subseteq M \subseteq Ry$. If $M \subset Ry$ then $M \in \sum$ and L being a maximal element implies that $L = M$. Hence L is a maximal submodule of Ry. So Ry/L is a simple left R module which is p-injective. Let $f : Ry \longrightarrow Ry/L$ be defined by $f(ry) = ry + L$, $\forall r \in R$. Then f is a well defined left R-homomorphism. Since R is a left pV-ring, there exists a left R-homomorphism $g : R \longrightarrow Ry/L$ which extends f. Now $y + L = f(y) = g(y) = yg(1) = y(ay + L) = yay + L$, for some $a \in R$. Since $yay \in RyRy \subseteq L$. So $y + L = L \Rightarrow y \in L \Rightarrow Ry = L$, which is absurd. Hence the result.

Proof of (ii) Let I be a nonzero proper left ideal of R. Let $0 \neq b \in I$. Let $f : R \longrightarrow Rb$ be defined by $f(x) = xb$, $\forall x \in R$. Then f is an onto left R-homomorphism. Now $Kerf = \{x \in R : f(x) = 0\} = \{x \in R :$

$xb = 0\} = l(b)$. Hence $R/l(b) \cong Rb$. Let J be a maximal left ideal of R containing $l(b)$. Then R/J is a simple left R-module and hence p-injective. Let $w : Rb \longrightarrow R/J$ be defined by $w(xb) = x + J$, $\forall x \in R$. Then w is well-defined. For if $xb = 0 \Rightarrow x \in l(b) \subseteq J \Rightarrow x + J = J$. Clearly w is a left R-homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow R/J$ which extends w. Let $g|_I = h$.

Claim:-$I/Kerh \cong R/J$

Let $v : I \longrightarrow R/J$ be defined by $v(x) = h(x)$, $\forall x \in I$. Let $x + J \in R/J$. Since w is onto, we have g is onto and hence h is onto.. So there exists $y \in I$ such that $h(y) = x + J$ i,e $v(y) = x + J$. Hence v is onto and also $Kerv = \{x \in I : v(x) = 0\} = \{x \in I : h(x) = 0\} = Kerh$. Hence the claim and therefore $Kerh$ is a left maximal subideal of I.

Proof of (iii) Suppose there exists $0 \neq b \in Rad({}_RR)$. Now Rb is a left ideal of R, so it has a maximal left subideal I. So Rb/I is a simple left R-module and hence it is p-injective. Let $f : Rb \longrightarrow Rb/I$ be the natural homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow R/I$ which extends f. Since f is onto, g is onto. Hence we have $R/Kerg \cong Rb/I$. So $Kerg$ is a maximal left ideal of R. Hence $b \in Kerg \Rightarrow g(b) = I \Rightarrow f(b) = I \Rightarrow b + I = I \Rightarrow b \in I \Rightarrow Rb = I$, which is absurd. Hence $Rad({}_RR) = 0$ □

2.1.7 Proposition. *If R is left p-injective then every principal right ideal of R is an annihilator.*

Proof. Let $a \in R$ then aR is a principal right ideal of R.

Claim 1:-aR⊆r(l(aR)).

Let $z \in l(aR) \Rightarrow zar = 0$, $\forall r \in R \Rightarrow ar \in r(z)$, $\forall z \in l(aR) \Rightarrow aR \subseteq r(l(aR))$.

Let $0 \neq d \in r(l(aR))$.

Claim 2:-$l(a) = l(aR) = l(r(l(aR))) \subseteq l(d)$.

If $y \in l(a)$ then $ya = 0$. So $yar = 0$, $\forall r \in R$. This shows that $y \in l(aR)$. If $x \in l(aR)$ then $xar = 0$, $\forall r \in R$. Taking $r = 1, xa = 0$. So $x \in l(a)$. Let $z \in r(l(aR))$ then $bz = 0$, $\forall b \in l(aR)$. This implies that $b \in l(z)$, $\forall z \in r(l(aR))$. So $l(aR) \subseteq l(r(l(aR)))$. Since $aR \subseteq r(l(aR))$, we get $l(aR) \supseteq l(r(l(aR)))$. Now let $y \in l(r(l(aR)))$ then $yt = 0 \ \forall \ t \in r(l(aR))$. Taking $t = d$, we get $y \in l(d)$. Hence the claim. Let $f : Rd \longrightarrow Ra$ be defined by $f(ra) = rd$, $\forall r \in R$. Then f is well-defined. For if $ra = 0$ then $r \in l(a) \subseteq l(d)$. So $rd = 0$. Let $j : Rd \longrightarrow R$ be the inclusion map. Then $j_o f : Ra \longrightarrow R$. Since R is p-injective, there exists a left R-homorphism $g : R \longrightarrow R$ extending $j_o f$. So $d = j(d) = j(f(a)) = (j \circ f)(a) = g(a) = ag(1) = az$ for some $z \in R$. Hence $r(l(aR)) \subseteq aR$. By claim 1, $aR = r(l(aR))$. □

2.1.8 Proposition. *Every p-injective left R-module is divisible.*

Proof. Let M be a p-injective left R-module. Let c be a regular element (i,e a nonzero divisor) of R. Clearly $cM \subseteq M$. Let $f : Rc \longrightarrow M$ be defined by $f(rc) = rm$, $\forall R \in R$ and $m \in M$. Now if $rc = sc$, $r, s \in R$ then $(r - s)c = 0$. Since c is regular, $r - s = 0$. This implies that $(r - s)m = 0$ i,e $rm = sm$. This shows that f is well defined. Clearly f is a left R-homomorphism. Since M is p-injective, there exists a left R-homomorphism $g : R \longrightarrow M$ extending f. Hence $m = f(c) = g(c) = cg(1) = cm'$, for some $m' \in M$. So $M \subseteq cM$ and therefore M is divisible. □

2.1.9 Proposition. *If a ring R is left p-injective then every left R-module is divisible.*

Proof. Since R is left p-injective, R is divisible as a module over itself. Hence for every regular element $c \in R, R = cR$. Let M be a left R-module and let $m \in M$ then $m = 1m = crm$, for some $r \in M$. This implies that $m = cm'$, where $m' = rm \in M$. This shows that $m \in cM$. Therefore $M \subseteq cM$ i,e $M = cM$. Hence M is divisible. □

2.1.10 Proposition. *Following conditions are equivalent for a ring R.*
1)Every principal left ideal of R is projective.
2)Every factor module of a p-injective module is p-injective.
3)Every factor module of an injective module is p-injective. (See [4], p-176, Remark 2)

Proof. 1)⇒2)
Let M be a p-injective left R-module and N be a submodule of M. We consider the following row exact sequence. $\eta : M \longrightarrow M/N \longrightarrow 0$, where η is the canonical homomorphism. Let $f : Ra \longrightarrow M/N$ be a left R-homomorphism. Since Ra is projective, there exists a left R-homomorphism $\theta : Ra \longrightarrow M$ such that $\theta_o\eta = f$. Again, since M is p-injective, there exists a left R-homomorphism $h : R \longrightarrow M$ extending θ. Let $g : R \longrightarrow M/N$ be such that $g = \eta_o h$. Let $x \in Ra$ then $g(x) = (\eta_o h)(x) = \eta(h(x)) = \eta(\theta(x)) = (\eta_o\theta)(x) = f(x)$. This shows that g extends f. Hence M/N is p-injective.
2)⇒3)
Let M be an injective left R-module. Then M is p-injective. So by 2), M/N is p-injective, where N is a submodule of M. Hence proved.
3)⇒1)
Let $Ra, a \in R$, be a principal left ideal of R. It is required to show that Ra is projective. Let M be a left R-module and N be a submodule of M. Let E

be the injective hull of M. Let $f : Ra \longrightarrow M/N$ be a left R-homomorphism and $k : M \longrightarrow M/N$ be the canonical homomorphism. Since E is injective so , by hypothesis, E/N is p-injective. Let $j : M/N \longrightarrow E/N$ be the inclusion map and let $F = j_o f$. Since E/N is p-injective, there exists a left R-homomorphism $w : R \longrightarrow E/N$ extending F. So we have, $\forall ba \in Ra$, $F(ba) = w(ba) = ba(w(1)) = ba(z + N) = baz + N$, for some $z \in E$. Let $\eta : E \longrightarrow E/N$ be the canonical homomorphism and let $g : Ra \longrightarrow E$ be defined by $g(ba) = baz \; \forall ba \in Ra$. Then g is a well defined left R-homomorphism. Let $F' = \eta_o g$

Claim 1:-$F' = F$

Let $ba \in Ra$ then $F'(ba) = (\eta_o g)(ba) = \eta(baz) = baz + N = F(ba)$

Claim 2:-$F(Ra) \subseteq M/N$

Let $x \in F(Ra)$ then $x = F(ba)$ for some $ba \in Ra \Rightarrow x = (j_o f)(ba) = f(ba) \in M/N$.

Claim 3:-$g(Ra) \subseteq M$

By claim 2, $\forall ba \in Ra$, $(\eta_o g)(ba) \in M/N \Rightarrow \eta(g(ba)) \in M/N \Rightarrow \eta(baz) \in M/N \Rightarrow baz + N \in M/N \Rightarrow baz \in M \Rightarrow g(ba) \in M$.

Let $h : Ra \longrightarrow M$ be defined by $h(ba) = g(ba) \; \forall ba \in Ra$.

Claim 4:-$k_o h = f$.

Let $ba \in Ra$ then $(k_o h)(ba) = k(h(ba)) = k(g(ba)) = g(ba) + N = baz + N = F(ba) = (j_o f)(ba) = f(ba)$. Hence the claim.

This implies that Ra is projective. □

2.1.11 Proposition. *The direct sum of two left p-injective modules is left p-injective.*

Proof. Let P and Q be two left p-injective modules over a ring R. It is required to show that $P \oplus Q$ is left p-injective. Let $a \in R$. Let $f : Ra \longrightarrow P \oplus Q$ be a left R-homomorphism. Let $p_1 : P \oplus Q \longrightarrow P$ and $p_2 : P \oplus Q \longrightarrow Q$ be the first and second projection maps respectively. Then $p_1 \circ f : Ra \longrightarrow P$ and $p_2 \circ f : Ra \longrightarrow Q$ are two left R-homomorphisms. Since P is p-injective, there exists a left R-homomorphism $g_1 : R \longrightarrow P$ extending $p_1 \circ f$. Again, since Q is p-injective, there exists a left R-homomorphism $g_2 : R \longrightarrow Q$ extending $p_2 \circ f$. Let $g : R \longrightarrow P \oplus Q$ be defined by $g(r) = (g_1(r), g_2(r))$, $\forall r \in R$. Then g is a well defined left R-homomorphism. Now $g(a) = (g_1(a), g_2(a)) = ((p_1 \circ f)(a), (p_2 \circ f)(a)) = (p_1(f(a)), p_2(f(a))$. Again $f(a) = (u, v)$, for some $u \in P$ and $v \in Q$. Therefore $f(a) = (p_1(u, v), p_2(u, v)) = (p_1(f(a)), p_2(f(a)) = g(a)$. This shows that g extends f. So $P \oplus Q$ is left p-injective. □

In the following part of this section, A^* will denote the intersection of all maximal left ideals of a ring R containing the left ideal A of R.

2.1.12 Lemma. *For any left ideal A of a ring R, $A \subseteq A^*$.*

Proof. By definition,$A^* = \cap M$, where M is a maximal left ideal of R containing A. Since $A \subseteq M$ for all M containing A, we have $A \subseteq \cap M = A^*$. □

2.1.13 Lemma. *For any two left ideals A and B of a ring R, if $A \subseteq B$ then $A^* \subseteq B^*$.*

Proof. Suppose $A^* \not\subseteq B^*$ then there exists $x \in A^*$ such that $x \notin B^*$. So $x \notin M$ for some maximal left ideal of R containing B. But M is also a maximal left ideal of R containing A. This implies that x does not belong to all the maximal left ideals of R containing A. So $x \notin A^*$, which is a contradiction. Hence $A^* \subseteq B^*$. $\square$

2.1.14 Lemma. *For any left ideal A of a ring R, $A^* = (A^*)^*$*

Proof. Let $Max_l(R)$ be the set of all maximal left ideals of R. Let $\Gamma_1 = \{\mu \in Max_l(R) : \mu \supseteq A\}$ and let $\Gamma_2 = \{\mu \in Max_l(R) : \mu \supseteq A^*\}$. Suppose $\mu \in \Gamma_2$ then $\mu \supseteq A^* \supseteq A$. So $\mu \in \Gamma_1$. Again, suppose $\mu \in \Gamma_1$ then $\mu \supseteq \cap \mu'_{\mu' \supseteq A} = A^*$. So $\mu \in \Gamma_2$. Hence $A^* = (A^*)^*$.

$\square$

The following theorem is due to Ming (See [7], p-14, Theorem 1)

2.1.15 Proposition. *Following conditions are equivalent for a ring R:-*
(i)R is a left pV-ring.
(ii)If K is a maximal left subideal of a principal left ideal P of R then $K^* \neq P^*$.

Proof. (i)$\Rightarrow$(ii). Suppose there exists a principal left ideal P of R and its maximal left subideal K such that $K^* = P^*$. Since P/K is a simple left module over R so P/K is p-injective. Let $f : P \longrightarrow P/K$ be the natural homomorphism then there exists a left R-homomorphism $g : R \longrightarrow P/K$ which extends f. Let $g|_{P^*} = h$ and let $H = Kerh = \{x \in P^* : h(x) = K\}$.
Claim (i):-$K \subseteq H$.
Let $k \in K$ then $k \in P^*$. So $h(k) = g(k)$(since g extends h). This implies that $h(k) = f(k) = k + K = K$. So $k \in H$. Hence the claim.
Claim(ii):-$H^* = K^*$.

Now $H \subseteq P^* = K^*$. So $H^* \subseteq (K^*)^* = K^*$. Again by claim(i), $K^* \subseteq H^*$. So $H^* = K^*$.

Claim(iii):-$Kerg \cap P^* = Kerh$.

Let $x \in Kerh \Rightarrow h(x) = K \Rightarrow g(x) = K \Rightarrow x \in Kerg \Rightarrow Kerh \subseteq Kerg$. Also $Kerh \subseteq P^*$. So $Kerh \subseteq Kerg \cap P^*$. Now let $x \in Kerg \cap P^* \Rightarrow g(x) = K$ and $x \in P^*$. Since g extends h, we have $h(x) = g(x) = K \Rightarrow x \in Kerh \Rightarrow Kerg \cap P^* \subseteq Kerh$. Hence the claim.

Claim(iv):-$(Kerh)^* \subseteq Kerg$. Let $x \in (kerh)^*$ then $x \in \cap M$, where M is a maximal left ideal of R containing $Kerh$. Since $Kerg$ is also a maximal left ideal of R containing $Kerh$. So $x \in Kerg$. This proves the claim.

So $P^* \subseteq Kerg$(from claim (iv)). This implies that $P^* = Kerh$(from claim (iii)). Hence $h(P^*) = K$. Since h extends f, we have $f(P) = K$ which implies that $P = K$, which is absurd. Hence the proof.

2)$\Rightarrow$1) Let S be a simple left R-module. Let P be a principal left ideal of R. Let $f : P \longrightarrow S$ be a left R-homomorphism. Since S is simple, $f(P) = 0$ or $f(P) = S$. If $f(P) = 0$, we get $0 : R \longrightarrow S$ which is a left R-homomorphism extending f. Suppose $f(P) = S$ then f is onto. So $P/Kerf \cong S$. Hence $Kerf$ is a maximal left ideal of P. So $(Kerf)^* \neq P^*$. Therefore there exists a maximal left ideal M of R such that $Kerf \subseteq M$ but $P \nsubseteq M$. For if every maximal left ideal M of R containing $Kerf$ also contains P then their intersection will contain P and in that case, $P^* \subseteq Kerf$. This implies that $(Kerf)^* = P^*$, which is absurd.

Claim:-$M \cap P = Kerf$.

we know that $Kerf \subseteq M \cap P \subseteq P$. Since $Kerf$ is a maximal left ideal of P, we have $Kerf = M \cap P$ for if $M \cap P = P$ then $P \subseteq M$, which is absurd.

Hence the claim.

Again $M \subseteq M + P \subseteq R$. Now $M \neq M + P$ otherwise $P \subseteq M$. So $M + P = R$. Let $g : R \longrightarrow S$ be defined by $g(m + p) = f(p)$ for $m \in M$ and $p \in P$. Suppose $m + p = m' + p'm$, $m' \in M$ and $p, p' \in P \Rightarrow m - m' = p - p' \subseteq M \cap P = Kerf \Rightarrow f(p - p') = 0 \Rightarrow f(p) = f(p')$. Hence g is a well defined left R-homomorphism which clearly extends g. So R is a left pV-ring □

2.1.16 Corollary. *Let R be a left pV-ring. For any left ideal I of R, either $I = I^*$ or I^* is not principal. (See [7], p-14, Corollary 1.1)*

Proof. Suppose there exists a left ideal I of R such that $I \neq I^*$ and $I^* = P$, where P is a principal left ideal of R. Let $\sum = \{U \leq P : I \subseteq U \subset P\}$. Then $I \in \sum$ (for if $I = P$ then $I = I^*$). We order this set by inclusion then $\sum$ becomes a Poset. By Zorn's lemma, this set has a maximal element K. Hence $K^* \neq P^*$. But $I^* \subseteq K^* \subseteq P^*$. Also $I^* = P \Rightarrow (I^*)^* = P^* \Rightarrow I^* = P^* \Rightarrow P = P^* \Rightarrow I^* = P^* \Rightarrow K^* = P^*$, which is impossible. This contradicts our assumption. Hence the proof. □

2.1.17 Proposition. *If R is a regular ring then R is a left pV-ring.*

Proof. Let $0 \neq a \in R$ and S be a simple left R-module. Let $f : Ra \longrightarrow S$ be a left R-homomorphism. Since R is regular, $Ra = Re$ where e is an idempotent element of R. This implies that Ra is a direct summand of R. So there exists a left ideal B of R such that $Ra \oplus B = R$. Let $g : R \longrightarrow S$ be defined by $g(ra + b) = f(ra)$ for $ra \in Ra$ and $b \in B$. Suppose $ra + b = r'a + b', ra, r'a \in Ra$ and $b, b' \in B \Rightarrow (r - r')a = b' - b \in Ra \cap B \Rightarrow ra = r'a \Rightarrow f(ra) = f(r'a)$.

Hence g is a well-defined left R-homomorphism which extends f. So R is a left pV-ring. □

The following corollary is due to Ming (See [7], p-14, Cor-1.2)

2.1.18 Corollary. *If R contains a principal left ideal P such that $P = I^*$ for some proper left subideal I of P, then R is not regular.*

Proof. Suppose R is regular then R is a left pV-ring. Again, since P is principal, $P \neq P^*$. Now $P = I^* \Rightarrow P^* = (I^*)^* \Rightarrow P^* = I^* \Rightarrow P^* = P$, a contradiction. Hence R is not regular. □

2.2 Some Results on left fV-rings and f-injectivity

In this section, we study about left fV-rings and left f-injectivity which are general cases of left pV-rings and left p-injectivity respectively.

2.2.1 Proposition. *Following conditions are equivalent*

1)R is a left fV-ring.

2)If K is a maximal left subideal of a finitely generated left ideal F of R then $K^* \neq F^*$. (See [7], p-15, Theorem 2)

2.2.2 Proposition. *If every maximal left ideal of R is f-injective or a left annihilator then the right annihilator of any finitely generated proper left ideal of R is nonzero.*

Proof. Let I be a finitely generated proper left ideal of R. Since $I \neq R$, there exists a maximal left ideal M of R containing I. Assume that M is

f-injective. Let $i : I \hookrightarrow M$ be the inclusion map. Then there exists a left R-homomorphism $g : R \longrightarrow M$ extending i. Let $b \in I$ then $b = i(b) = g(b) = bg(1) = bc$, for some $c \in M$. This implies that $b(1-c) = 0$. i,e $1 - c \in r(I)$. Since $c \in M$ so $c \neq 1$. Hence $1 - c \neq 0$. So $r(I)$ is nonzero. Suppose M is a left annihilator then $M = l(S)$ for some subset S of R. Since $M \neq R$, $S \neq 0$. So there exists $0 \neq x \in S$. Now $Mx = 0$ so $x \in r(M) \subseteq r(I)$(since $I \subsetneq M$). Hence $r(I)$ is nonzero. □

2.2.3 Proposition. *Let R be a semi-prime left f-injective ring. Then the following conditions are equivalent for a finitely generated left ideal F.*

1)F is generated by a central idempotent.

2)F=$l(T)$, where T is a finitely generated left ideal. (See [2], p-66, Proposition 1)

Proof. 1)⇒2) Suppose $F = Re$, where e is a central idempotent

Claim:-$F = l(R(1-e))$

Let $y \in Re$ then $y = re$, for some $r \in R$. Let $s \in R$ then

$$ys(1-e) = res(1-e) = rse(1-e) = 0$$

This implies that $y \in l(R(1-e))$ Let $x \in l(R(1-e))$ then $xr(1-e) = 0$, $\forall r \in R$. Taking $r = 1, x(1-e) = 0$ i,e $x = xe \in Re$, we prove the claim.

2)⇒1). Suppose $F = l(T)$, where T is a finitely generated left ideal of R. Then F is an ideal of R.

Claim :-$l(T) \cap T = 0$

Let $y \in l(T) \cap T$. Then $yt = 0, \forall, t \in T$ and $y \in T$. In particular, taking $t = y$ we get $y^2 = 0$. This implies that $(l(T) \cap T)^2 = 0$. Since R is semi-prime, we get $l(T) \cap T = 0$. Hence $R = r(0) = r(l(T) \cap T) = r(F \cap T) = r(F) + r(T)$.

Now $F = l(T) = r(T)$ and therefore $R = F + r(F)$. Since $F \cap r(F) = F \cap l(F) = 0$, we get $R = F \oplus r(F)$. This shows that F is a direct summand of R. Hence F is generated by a central idempotent. □

2.2.4 Proposition. *If every divisible left R-module is f-injective then R is left semi-hereditary. (See [2], p-70, Proposition 9)*

Proof. Let I be a finitely generated left ideal of R. It is required to show that I is projective. Let M be a left R-module and N be a submodule of M. Let E be the injective hull of M. Let $f : I \longrightarrow M/N$ be a left R-homomorphism and $k : M \longrightarrow M/N$ be the canonical homomorphism. Since E is injective so it is divisible. Hence E/N is divisible and therefore f-injective. Let $j : M/N \longrightarrow E/N$ be the inclusion map and let $F = j_o f$. Since E/N is f-injective, there exists a left R-homomorphism $w : R \longrightarrow E/N$ extending F. So we have, $F(b) = w(b)$, $\forall\, b \in I \Rightarrow F(b) = b(w(1)) = b(z+N) = bz+N$, for some $z \in E$. Let $\eta : E \longrightarrow E/N$ be the canonical homomorphism and let $g : I \longrightarrow E$ be defined by $g(b) = bz$, $\forall b \in I$. Then g is a well defined left R-homomorphism. Let $F' = \eta_o g$

Claim 1:$F' = F$

Let $b \in I$ then $F'(b) = (\eta_o g)(b) = \eta(bz) = bz + N = F(b)$.

Claim 2:-$F(I) \subseteq M/N$

Let $x \in F(I)$ then $x = F(b)$ for some $b \in I \Rightarrow x = (j_o f)(b) = f(b) \in M/N$.

Claim 3:-$g(I) \subseteq M$

By claim 2, $(\eta_o g)(b) \in M/N$, $\forall b \in I \Rightarrow \eta(g(b)) \in M/N$, $\forall b \in I \Rightarrow \eta(bz) \in M/N$, $\forall b \in I \Rightarrow bz + N \in M/N$, $\forall b \in I \Rightarrow bz \in M$, $\forall b \in I \Rightarrow g(b) \in M$, $\forall b \in I$.

Let $h : I \longrightarrow M$ be defined by $h(b) = g(b)$, $\forall b \in I$.

Claim 4:-$k_oh = f$

Let $b \in I$ then $(k_oh)(b) = k(h(b)) = k(g(b)) = g(b) + N = bz + N = F(b) = (j_of)(b) = f(b)$. This implies that I is projective. □

2.2.5 Theorem. *The following conditions are equivalent for a ring R whose complement left ideals are ideals.*

1)Every divisible left R-module is f-injective.

2)R is semi-hereditary and every p-injective left R-module is f-injective.

3)Every p-injective left R-module is f-injective and for every $a \in R$, there exist a central idempotent $e \in l(a)$, n elements $b_1, b_2, ..., b_n \in Ra$ and n non-zero divisors $c_1, c_2, ..., c_n \in R$ such that $a = \sum a_ib_i, a_i \in R, (1-e)a_ic_i = a$, for each $i, 1 \leq i \leq n$. (See [2], p-71, Theorem 10)

Proof. 1)⇒2)

By the previous proposition, R is left semi-hereditary and since every p-injective left R-module is divisible so by given hypothesis, every p-injective left R-module is f-injective.

2)⇒3)

By 2), it is clear that every p-injective left R-module is f-injective.

Let $a \in R$ then Ra is a finitely generated left R-module. Since R is semi-hereditary, Ra is projective over R.

Let $\beta : R \longrightarrow Ra$ be defined by $\beta(r) = ra$, $\forall r \in R$. Let $Id : Ra \longrightarrow Ra$ be the identity map. Since Ra is projective, there exists a left R-homomorphism $g : Ra \longrightarrow R$ such that $g_oId = f$. Now $Ker\beta = \{r \in R : ra = 0\} = l(a)$. Therefore the sequence $0 \longrightarrow l(a) \longrightarrow R \longrightarrow Ra \longrightarrow 0$ is exact and also splittable. Hence $l(a)$ is a direct summand of R. So $l(a) = Re$, where e is an idempotent element of R.

Claim 1:-R is reduced.

Let $a \in R$ such that $a^2 = 0$. So $a \in l(a) = Re$. Hence $a = xe$, for some $x \in R$. Since $R(1-e)$ is a compliment left ideal of Re so $R(1-e)$ (by hypothesis) is an ideal of R. Hence $(1-e)x = y(1-e)$ for some $y \in R \Rightarrow (1-e)xe = y(1-e)e = 0$. So $xe = exe$ Now $0 = ea = exe = xe = a$. Hence R is reduced.

Claim 2:-e is central

Let $x \in R$ then $(xe - exe)^2 = (1-e)xe(1-e)xe = 0$. Since R is reduced, $xe = exe$. And $(ex - exe)^2 = ex(1-e)ex(1-e) = 0$. Hence $ex = exe = xe$.

Let $c = a + e$.

Claim 3:-c is a nonzero divisor of R.

Let $r \in R$ such that $rc = 0 \Rightarrow ra + re = 0 \Rightarrow ra^2 = 0 \Rightarrow ara = 0 \Rightarrow rara = 0 \Rightarrow ra = 0 \Rightarrow r \in l(a) = Re$. Now $re = 0$. So $r = r - re = r(1-e) \in R(1-e)$. So $r \in Re \cap R(1-e) = 0$. Hence the claim. Taking $n = 1$, $b_1 = a$, $c_1 = c$, $a_1 = 1$, we get $a = 1.a = a_1b_1$ and $(1-e)a_1c_1 = (1-e)(e+a) = e + a - e - ea = a$

3)$\Rightarrow$1)

Let M be a divisible left R-module.Let $f : Ra \longrightarrow M$ be a left R-homomorphism. By hypothesis, there exist $b_1, b_2, ..., b_n \in Ra$ and a central idempotent e such that $ea = 0$. Now $f(b_i) \in M$. Also, by hypothesis, there exist non-zero divisors $c_1, c_2 ,..., c_n \in R$. Since M is divisible, there exists $m_i \in R$ so that $f(b_i) = c_i m_i$. Now $f(a) = f((1-e)a) = (1-e)f(a) = (1-e)f(\sum a_i b_i) = (1-e)\sum a_i f(b_i) = (1-e)\sum a_i c_i m_i = \sum(1-e)a_i c_i m_i = a\sum m_i$. Let $g : R \longrightarrow M$ be defined by $g(r) = r\sum m_i$. Then g is a well-defined left R-homomorphism which extends f. Hence M is p-injective. By hypothesis,

M is f-injective. □

2.2.6 Corollary. *If every complement left ideal of a ring R is an ideal then the following conditions are equivalent.*

1)Every divisible left R-module is injective.

2)R is left hereditary, left Noetherian ring whose p-injective left R-modules are f-injective. (See [2], p-71, Corollary 10.1)

Proof. To show that R is left hereditary, we replace f-injective by injective and I by any ideal in the proof of proposition 2.2.4.

Let I be an ideal of R. Since R is hereditary, every left R-module is projective. So R/I is projective. Let us consider the following exact sequence $0 \longrightarrow I \hookrightarrow R \longrightarrow R/I \longrightarrow 0$, where the homomorphisms are the natural one. Let $Id : R/I \longrightarrow R/I$ be the identity homomorphism. Since R/I is projective, the above sequence splits. Hence I is a direct summand of R . So $I = Re$, where e is an idempotent element of R. This shows that I is finitely generated. Hence R is left-Noetherian.

2)$\Rightarrow$1)

Let M be a divisible left R-module and I be a left ideal of R. By hypothesis, I is generated by an idempotent. So M is p-injective and hence injective. □

2.3 More on p-Injectivity

In the following section, some properties of MTE rings , ALD rings and non singular prime rings have been used to obtain some important results on p-injectivity.

2.3.1 Proposition. *Let R be an MTE ring then a minimal left ideal of R is injective iff it is p-injective. (Refer [8], p-336, Lemma 1.1)*

Proof. If a minimal left ideal of R is injective then it is clearly p-injective. Let V be a p-injective minimal left ideal of R. Let L be a proper essential left ideal of R and let $f : L \longrightarrow V$ be a nonzero left R-homomorphism. Since V is minimal, we have $f(L) = V$. So f is onto and hence $L/Kerf \cong V$. So $K = Kerf$ is a maximal left subideal of L. Now since V is minimal, $V = Rv$, $\forall v \in V$, $v \neq$). So V is principal p-injective. Let $Id : Rv \longrightarrow Rv$ be the identity homomorphism. Since Rv is p-injective, there exists a left R-homomorphism $h : R \longrightarrow Rv$ extending Id. So $v = Id(v) = h(v) = vh(1) = vav$, for some $a \in R$. This implies that $av = avav$. So $e = av$ is an idempotent element of Rv. Clearly $Rv = Re$, hence V is a direct summand of R. Since R is projective over R so V is also projective. Hence L/K is projective over R. Let us consider the exact sequence $0 \longrightarrow K \hookrightarrow L \longrightarrow L/K \longrightarrow 0$. Let $Id : L/K \longrightarrow L/K$ be the identity homomorphism. So there exists a left R-homomorphism $w : L/K \longrightarrow L$ such that $\eta_o w = Id$ where $\eta : L \longrightarrow L/K$ is the canonical homomorphism. Hence the above sequence splits. So K is a direct summand of R. Hence $L = K \oplus U$, for some subideal U of L such that $U \cong L/K \cong V$. So $U = Re'$, where $e'^2 = e'$. Now for any $a \in L, a = k + de'$, for some $k \in K$ and some $d \in R$.

Since R is an MTE ring, K is an ideal of L. So $ke' \in K \cap U = 0$ which yields

$f(a) = f(k + de') = f(k) + f(de') = f(de') = f(ke') + f(de') = kf(e') + de'f(e') = (k + de')f(e') = af(e') = f(ae')$

Let $g : R \longrightarrow V$ be defined by $g(x) = f(xe)$, $\forall x \in R$. Then g is a well defined left R-homomorphism which extends f. Hence V is injective. □

2.3.2 Proposition. *Let R be an ALD ring. Then a simple left R-module is injective iff it is p-injective. (See [8], p-339, Proposition 2.1)*

Proof. One implication is obvious. Now let V be a p-injective simple left R-module. Then $V \cong R/M$, where M is a maximal left ideal of R. Suppose M is a direct summand of R then $M \oplus U = R$, where U is a minimal left ideal of R then $U(\cong V)$ is p-injective and following the same proof as in the previous proposition, we see that V is injective. Suppose M is not a direct summand of R then M is essential in R. Since R is an ALD ring, M is an ideal of R. Let L be a proper essential left ideal of R. Let $f : L \longrightarrow R/M$ be a nonzero left R-homomorphism.

Claim:-L is not contained in M.

Suppose $L \subseteq M$, then for any $b \in L$ and any left R-homomorphism $g : Rb \longrightarrow R/M$, which is the restriction of f to Rb, there exists a left R-homomorphism $h : R \longrightarrow R/M$ extending g (since R/M is p-injective).

So $f(b) = g(b) = h(b) = bh(1) = b(c + M) = bc + M = M$, for some $c \in R$ (since M is an ideal of R, $bc \in M$). Hence $f(L) = M$, contradicting that f is nonzero. So L cannot be contained in M. Hence the claim. Since M is maximal, $L+M = R$ and so $1 = d+u$, for some $d \in L$ and some $u \in M$. Then for any $a \in L$, we have $a = ad + au$ which implies that $au = a - ad \in L \cap M$. Since $L \cap M \subseteq M$ and $au \in L \cap M$ then following the same steps as above, we see that $f(au) = M$. Now $f(a) = f(ad + au) = f(ad) = af(d)$. Let $w : R \longrightarrow R/M$ be defined by $w(x) = xf(d)$. Then w is a well-defined left R-homomorphism which extends f. Hence R/M is injective proving that V is injective. □

2.3.3 Proposition. *Let R be a left non-singular prime ring such that for any maximal essential left ideal M, every maximal or complement left subideal is an ideal of M. If R is either left p-injective or a left V-ring then R is primitive with non zero socle. (See [8], p-338, Proposition 1.7)*

Proof. Suppose R has a zero socle. We claim that R is reduced.
Suppose not then there exists $0 \neq b \in R$ such that $b^2 = 0$. Then $b \in l(b)$. Since $b \neq 0$, $l(b) \neq R$. So there exists a maximal left ideal M of R containing $l(b)$. Let $c \in R$ such that $M \cap Rc = 0$. If $c = 0$ then M is essential in R. If $c \neq 0$ then ,since M is maximal, $M \oplus Rc = R$. This shows that Rc is simple. So $c \in Soc(R) = 0$, which is absurd. Hence $c = 0$ i,e $Rc = 0$. So M is essential in R. Again since R is left non-singular, $l(b)$ is not essential in R (since $b \neq 0$). So there exists $d \neq 0$ such that $l(b) \cap Rd = 0$. This shows that $l(b)$ is an R-complement of Rd. So $l(b)$ is an ideal of M. We have $l(b)M \subseteq l(b)$. So $RbMb \subseteq l(b)M \subseteq l(b)b$. Let $y \in l(b)b$ then $y = ab$ for some $a \in l(b)$. This shows that $ab = 0 \Rightarrow y = 0$. So $RbMb = 0$. Since R is prime, $\Rightarrow Mb = 0 \Rightarrow M \subseteq l(b) \Rightarrow M = l(b)$. Hence M is a direct summand of R, which is a contradiction since M is essential in R. Thus R is reduced. Hence R is an integral domain.
Suppose R is left p-injective. Let $0 \neq c \in R$ and let $f : Rc \longrightarrow R$ be defined by $f(rc) = r$, $\forall r \in R$. Then f is well-defined. For if $rc = 0$ then $r = 0$ (since R is an integral domain). Clearly f is a left R-homomorphism. Since R is left p-injective, there exists a left R-homomorphism $g : R \longrightarrow R$ extending f. Hence $1 = f(c) = g(c) = cg(1) = ca$ for some $a \in R$. This implies that c is right invertible. So R is a division ring. Therefore its only left ideals are 0 and R itself. So $Soc(R) = R$.

Suppose R is a left V-ring. We claim that R has no maximal essential left ideal. Suppose there exists a maximal essential left ideal L of R. Let K be a maximal left subideal of L then L/K is injective and K is an ideal of L. Let $0 \neq d \in K$ and let $h : Ld \longrightarrow L/K$ be defined by $h(ad) = a+K \; \forall a \in L$. Then h is well-defined. For if $ad = 0$ then $a = 0$ (since R is an integral domain). This implies that $a + K = K$. Clearly h is a left R-homomorphism. Since R is a V-ring, there exists a left R- homomorphism $w : L \longrightarrow L/K$ extending h. So $a + K = h(ad) = w(ad) = adw(1) = ad(c + K) = adc + K$. This implies that $a \in K \Rightarrow L \subseteq K$, a contradiction. Hence the claim.

Since R is an integral domain, it is a division ring. Hence $Soc(R) = R$. Since R is simple in R and $l(R) = 0$, it shows that R is primitive with nonzero socle. $\square$

Chapter 3

Left V'-rings and left GPV-rings

In this chapter, we study some properties of left V'-rings and rings relate to them viz, left pV'-rings and left fV'-rings. We also study left GPV-rings and record some results about their relations to the classes of rings that we have studied in the previous chapters.

Following is an interesting example of a left V'-ring

3.0.4 Example. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in Q \right\}$

Then the left ideals of R are $0, R$, $I_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in Q \right\}$,

$I_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Q \right\}$, $I_3 = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in Q \right\}$ and $I_4 = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in Q \right\}$

Now I_1 and I_4 are maximal left ideals of R. So any simple left R-module will be isomorphic to either R/I_1 or R/I_4

Again, $R/I_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} + I_1 : a \in Q \right\}$

and $R/I_4 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + I_4 : \ a \in Q \right\}$

Let $y + I_1 \in R/I_1$ then $l(y + I_1) = I_2$. But I_2 is not essential in R since $I_2 \cap I_4 = 0$ but $I_4 \neq 0$. Hence R/I_1 is not singular. Similarly, we can show that R/I_4 is not singular. Therefore there does not exist any simple singular left R-module. Hence R is trivially a left V'-ring.

3.1 Results on left V'-rings

In this section, we record some results about left V'-rings. Some properties of T-rings have been used to prove these results.

3.1.1 Lemma. *If I is a left ideal of R such that I is not an essential left ideal of R then there exists a left ideal B' of R such that $I \oplus B'$ is essential in R.*

Proof. Suppose I is not essential then there exists a non-zero left ideal B of R such that $I \cap B = 0$ Let B' be the maximal of all such B's.

Claim:-$I \oplus B'$ is essential.

If not then there exists a non-zero element x of R such that $(I \oplus B') \cap Rx = 0$. Now if $y \in I \cap (B \oplus Rx)$ then $y = b' + rx$ for some b' in B' and some r in R. $\Rightarrow rx = y - b' \in (I \oplus B') \cap Rx = 0 \Rightarrow y = b' \in I \cap B' = 0 \Rightarrow I \cap (B \oplus Rx) = 0$. Thus $I \oplus (B' \oplus Rx)$ is direct, contradicting that B' is maximal. Hence the claim. □

3.1.2 Theorem. *For a T-ring R, the following are equivalent*

1)R is a left V'-ring

2)$Z(_RR) = 0$ and $Rad(_RR/I) = 0$ for any essential left ideal I of R. (See [12], p-234, Theorem 1.1)

Proof. 1)⇒2). Assume that $Z(_RR) \neq 0$. Since R is a T-ring, $\forall$ non-zero module M of R, Soc$(M) \neq 0$. This implies that M has a simple submodule. So $Z(_RR)$ will have a simple submodule S. Therefore $Z(S) = S$. Hence S is a simple singular module opver R. So S is injective and hence a direct summand of R. So $S = Re$, where e is an idempotent element of S. This implies that $e \in Z(S) = S \subsetneq Z(_RR)$. Hence $e = 0$ and therefore $S = 0$, which is not possible. Hence $Z(_RR) = 0$.

Let I be an essential left ideal of R. Then R/I is a singular module of R and hence every submodule of R/I is singular. In particular, every simple submodule of R/I is singular. Let $0 \neq t+I \in Rad(_RR/I)$. Then $R(t+I)$ has a non-zero socle. So there exists a simple submodule S of $R(t+I)$. Since S is simple in $R(t+I)$, it is simple in R/I. So there exists a maximal submodule B of R/I such that $S \cap B = 0$. But $0 \neq S \subsetneq R(t+I) \subsetneq B$, which is absurd.

2)⇒1) Let S be a simple, singular left R-module and let I be a left ideal of R. Assume that I is essential. Let $f : I \longrightarrow S$ be a left R-homomorphism.

Claim 1:-$K = Kerf$ is an essential left ideal of R.

Suppose there exists a nonzero left ideal J of R with $K \cap J = 0$. Then $I \cap J \neq 0$. Now $K \cap (I \cap J) = (K \cap J) \cap I = 0$. Let $J' = I \cap J$ and let $f' = f|_{J'}$. $Kerf' = \{x \in J' : f(x) = 0\} \subsetneq K \cap J' = 0$. This implies that $Kerf' = 0$. Hence f' is one-one and therefore $f'(J') \cong J'$. Now $f'(J') \subseteq S$. Hence $f'(J')$ is singular and so is J'. Hence $Z(J') = J' \neq 0$ contradicting that $Z(_RR) = 0$ and therefore the claim.

Let $f'' : I/K \longrightarrow S$ be a map induced by f. If $f \neq 0$ then $f(I) = S$, so f

is onto and hence $I/K \cong S$. So K is a maximal left ideal of I. Hence I/K is a simple submodule of R/K. Since $Rad(_RR/K) = 0$, there is a maximal submodule M/K of R/K with $I/K \nsubseteq M/K$. For if $I/K \subseteq M/K$ for all maximal submodules M/K of R/K then $I/K \subseteq Rad(R/K)$. This implies that $I/K = 0$, a contradiction.

Claim 2:-$I/K \cap M/K = 0$

Let $x + K$ be a non-zero element of $I/K \cap M/K$. Then $R/K(x+K) = I/K$. Now $R/K(x + K) \subseteq M/K$. So $I/K \subseteq M/K$, which is absurd. Hence the claim.

Also since M/K is a maximal submodule of R/K, we have $I/K + M/K = R/K$ and so $I/K \oplus M/K = R/K$. Let $h : R \longrightarrow R/K$ be the natural map and $p : R/K \longrightarrow I/K$ be the projection map. Let $g : R \longrightarrow S$ be given by $g = f"ph$ and let $x \in I$ then $g(x) = (f"ph)(x) = (f"p)(x + K) = f"(x + K) = f(x)$. This implies that g extends f and hence S is injective.

Suppose I is not essential then there exists a non-zero left ideal B' of R such that $I \oplus B'$ is essential in R.

Let $\imath : I \longrightarrow I \oplus B'$ be the inclusion map.

As in the first case, there will exist $g : R \longrightarrow S$ extending $h : I \oplus B' \longrightarrow S$ and hence extending f. □

3.1.3 Lemma. *Let I be an essential left ideal of a ring R with $Z(_RR) = 0$ then I^2 is essential.*

Proof. Suppose there exists a non-zero left ideal B of R such that $I^2 \cap B = 0$ Then there exists a non-zero element b of B. So $Rb \neq 0$. Since I is essential, $I \cap Rb \neq 0$. Hence there exists a non-zero element $yb \in I$. Now $\forall z \in I, zyb \in I^2 \cap B = 0 \Rightarrow z \in l(yb) \Rightarrow I \subseteq l(yb) \Rightarrow l(yb)$ is large in R

$\Rightarrow yb \in Z(_RR) \Rightarrow yb = 0$, which is absurd. Hence I^2 is essential. $\square$

3.1.4 Theorem. *Let R be a T-ring having all singular simple R-modules injective. For any essential left ideal I of R, $I^2 = I$ and I is an intersection of maximal left ideals. Moreover $N^2 = 0$, where N is the Jacobson Radical of R. (See [12], p-234, Theorem 1.2)*

Proof. Let $x \in N \Rightarrow x \in \cap M$, where the intersection is over the family of M, maximal left ideals of $R \Rightarrow x \in M$, $\forall M$, M a maximal left ideal of R. This implies that $x + I \in M/I \subseteq Rad(R/I) = 0 \Rightarrow x \in I \Rightarrow N \subseteq I$, for every essential left ideal I of R. This implies that $N \subseteq \cap I$, for every essential left ideal I of R and this implies that $N \subseteq Soc(R)$. Suppose $N^2 \neq 0$ then there exists $x \in N$ such that $xN \neq 0$. But N is a sum of semi-simple modules of R. So there exists a simple left R-module S such that $xS \neq 0$. This implies that $x \notin N$, which is absurd. Hence $N^2 = 0$. Again since I is an essential left ideal of R , we have $Rad(_RR/I) = 0$. This implies that $\cap M/I = 0$, M/I a maximal left ideal of R/I. This implies that $(\cap M)/I = 0$, M is a maximal left ideal of R containing I. This implies that $I = \cap M$, M a maximal left ideal of R containing I. Suppose $I^2 \neq I$, then there exists $x \in I$ such that $x \notin I^2$. So there exists a maximal left ideal M of R such that $I^2 \subseteq M$ but $x \notin M$. For if $x \in M$, $\forall M$ containing I^2 then $x \in \cap M = I^2$(since I^2 is essential). Hence $R = M + Rx \Rightarrow 1 = m + rx \Rightarrow x = xm + xrx \in M + I^2 \subseteq M$. This implies that $x \in M$. Hence $I^2 = I$ $\square$

3.1.5 Lemma. *Following conditions are equivalent for a commutative ring R.*

a)R is regular.

b) $I^2 = I$ *for each essential left ideal* I *of* R.

c) $I^2 = I$ *for each left ideal* I *of* R. *(See [12], p-235, Proposition 1.5)*

Proof. a)$\Rightarrow$b). Let I be an essential left ideal of R. Clearly $I^2 \subseteq I$. Let $a \in I$ then $a \in R$. Since R is regular, there exists $b \in R$ such that $a = aba \in I^2$. Hence $I^2 = I$.

b)$\Rightarrow$c). Let I be a left ideal of R. Suppose I is not essential then there exists a nonzero left ideal B of R such that $I \cap B = 0$. If A is the maximal of all such ideals then $I \oplus A$ is essential. This implies that $I + A$ is essential. Hence $I + A = (I + A)^2 = I^2 + IA + AI + A^2 = I^2 + A^2$. Let $x \in I$ then $x = \Sigma x_i y_i + \Sigma a_i b_i x_i$, $y_i \in I$, $a_i, b_i \in A$. This implies that $x - \Sigma x_i y_i = \Sigma a_i b_i \in I \cap A = 0$. This gives $x = \Sigma x_i y_i \in I^2$. c)$\Rightarrow$a). Let $a \in R$. Then $Ra = RaRa$. So $a = \Sigma x_i a y_i a = a(\Sigma x_i y_i)a = ara$, $r = \Sigma x_i y_i$. Hence R is regular. □

3.1.6 Theorem. *Let* R *be a commutative* T*-ring. Then* R *is a left* V'*-ring if and only if* R *is regular. (See [12], p-236, Theorem 1.6)*

Proof. Suppose R is a left V'-ring then $I = I^2$ for every essential ideal of R. This implies that R is regular. Conversely, if R is regular then R is a left V-ring since R is commutative. So every simple module over R is injective. In particular, every simple left R-module which is singular is also injective. Hence R is a left V'-ring. □

3.1.7 Proposition. *Let* R *be a* T*-ring having all singular simple* R*-modules injective. If* R *is regular then* $N = 0$ *where* $N = Rad(_R R)$.

Proof. Let $a \in N$. Since R is regular, there exists $b \in R$ such that $a = aba \in N^2 = 0$. This implies that $a = 0$. Hence $N = 0$. □

3.2 Results on left pV'-rings

In this section, we study about some properties of left pV'-rings. Some properties of $MELT$-rings and the singularity of modules have been studied in order to arrive at some important results on left pV'-rings.

3.2.1 Lemma. *If* $0 \neq Z({}_RR)$ *then there exists* $0 \neq y \in Z({}_RR)$ *such that* $y^2 = 0$.

Proof. Let $0 \neq z \in Z({}_RR)$, we have $l(z)$ is essential in R. Also $Rz \neq 0$ so $l(z) \cap Rz \neq 0$. Hence there exists $0 \neq tz \in l(z)$ such that $tz^2 = 0$. Let $w = ztz$. Since $Z({}_RR)$ is two-sided, $l(w)$ is essential in R. If $w \neq 0$ then take $y = w$. So $y^2 = w^2 = ztzztz = 0$. If $w = 0$ then take $y = tz$. Here $l(y)$ will be essential in R and $y^2 = tztz = tw = 0$. Hence the proof □

3.2.2 Proposition. *If R is a MELT, left pV'-ring then* $Z({}_RR) = 0$ *(See [7], p-18, proposition 8(1))*

Proof. Suppose $Z({}_RR) \neq 0$ then there exists $0 \neq z \in Z({}_RR)$ such that $z^2 = 0$ and so $z \in l(z)$. Let L be a maximal left ideal of R containing $l(z)$.

Claim 1:-L is essential in R.

Suppose $L \cap K = 0$ for a left ideal K of R. Now $l(z) \cap K \subseteq L \cap K = 0$. So $l(z) \cap K = 0$. Since $l(z)$ is essential in R , we have $K = 0$. This proves the claim.

Claim 2:-$L \subseteq l(r + L)$.

Let $x \in L \Rightarrow xr \in L$ (since R is a MELT ring, L is two sided)$\Rightarrow xr + L = L \Rightarrow x(r + L) = L \Rightarrow x \in l(r + L)$. Hence the claim.

Claim 3:-$Z(R/L) = R/L$

Clearly $Z(R/L) \subseteq R/L$. Let $r + L \in R/L$. Suppose $l(r + L) \cap B = 0$ where B is a left ideal of R. Now $L \cap B \subseteq l(r + L) \cap B = 0$. So $L \cap B = 0$. Hence $B = 0$. This shows that $l(r + L) \in Z(R/L)$ which proves the claim.

Hence R/L is a simple singular left R-module. Let $f : Rz \longrightarrow R/L$ be defined by $f(rz) = r + L$, $\forall rz \in Rz$. Suppose $rz = sz$ for $r, s \in R \Rightarrow (r - s)z = 0 \Rightarrow r - s \in l(z) \subseteq L \Rightarrow r + L = s + L$. So f is a well-defined left R-homomorphism. Since R is a left pV'-ring, there exists a left R-homomorphism $g : R \longrightarrow R/L$ extending g. Hence

$1 + L = f(z) = g(z) = zg(1) = z(a + L) = za + L$ for some $a \in R$. Since L is a two-sided ideal and $z \in L$, we have $za \in L$. This implies that $1 \in L$, which is absurd. Hence $Z(_RR) = 0$. □

3.2.3 Lemma. *Let S be a simple left R-module such that $S \subseteq Z(_RR)$ then $Z(S) = S$.*

Proof. Clearly $Z(S) \subseteq S$. Let $s \in S$ then $s \in Z(_RR)$. Hence $s \in R$ such that $l(s)$ is essential in R. Now $s \in S$ with $l(s)$ essential in R. Hence $s \in Z(S)$. So $Z(S) = S$. □

3.2.4 Lemma. *$Z(_RR)$ has no non-zero idempotents.*

Proof. Let $0 \neq e \in Z(_RR)$ be an idempotent element. Then $l(e)$ is large in R. Since $Re \neq 0$, we have $l(e) \cap Re \neq 0$. So there exists $0 \neq ye \in l(e)$ such that $yee = 0$ i,e $ye = 0$, which is absurd. Hence $e = 0$. □

3.2.5 Lemma. *If $Z(_RR) = 0$ then $Z(_RI) = 0$ for every left ideal I of R.*

Proof. Let $x \in Z(_RI)$ then $x \in I$ such that $l(x)$ is large in R. This implies that $x \in R$ such that $l(x)$ is large in R. Hence $x \in Z(_RR)$. So $Z(_RI) = 0$. □

3.2.6 Lemma. *If I is an essential left ideal of R then R/I is a singular left R-module.*

Proof. Let $x+I \in R/I$. To show that $l(x+I)$ is essential in R. Suppose there exists $0 \neq b \in R$ such that $l(x+I) \cap Rb = 0$ Then $bx \notin I$. For if $bx \in I$ then $bx+I = I$. This implies that $b(x+I) = I \Rightarrow b \in l(x+I) \cap Rb \Rightarrow b = 0$, which is absurd. Hence $bx \notin I$. So $bx \neq 0$. Since I is essential in R, $I \cap Rbx \neq 0$. So there exists $0 \neq rbx \in I$. This implies that $rb \in l(x+I) \cap Rb \Rightarrow rb = 0 \Rightarrow rbx = 0$, which is absurd.

□

3.2.7 Lemma. *Submodules of singular modules are singular.*

Proof. Let M be a singular left R-module and N be a submodule of M. Let $n \in N$ then $n \in M$. Since $Z({}_RM) = M$, we have $l(n)$ is essential in R. So $n \in Z({}_RN)$. Hence $Z({}_RN) = N$ which implies that N is singular. □

3.2.8 Proposition. *Let R be a ring such that any essential left ideal which is an ideal of R is an essential right ideal of R. If every cyclic singular right R-module is p-injective, then R is fully right idempotent. (See [3], p-147, Proposition 15)*

Proof. Let $b \in R$. To show that $bR = (bR)^2$ Clearly $(bR)^2 \subseteq bR$. Let $E = RbR + r(RbR)$. We first show that E is an essential left ideal of R. Let K be a left ideal of R such that $E \cap K = 0$. Then $RbRK \subseteq RbR \cap K \subseteq E \cap K = 0$. This implies that $K \subseteq r(RbR)$. Now $K = K \cap r(RbR) \subseteq K \cap E = 0$. This shows that $K = 0$. Hence E is essential. Clearly E is a two sided ideal of R. Again since $r(RbR) \subseteq r(b)$ so $E \subseteq RbR + r(b)$. Since E is essential,

$A = RbR + r(b)$ is an essential right ideal of R. So R/A is a cyclic singular left ideal of R (by Lemma 3.2.6). Hence R/A is right p-injective. Let $f : bR \longrightarrow R/A$ be defined by $f(ba) = a + A$, $\forall a \in R$. Then f is well-defined. For if $ba = 0$ then $a \in r(b) \subseteq A$. S0 $a + A = 0$. i,e $f((ba) = 0$. Clearly f is a right R-homomorphism. So there exists a right R-homomorphism $g : R \longrightarrow R/A$ extending f. Hence $f(b) = g(b) \Rightarrow 1 + A = g(1)b \Rightarrow 1 + A = (d + A)b$ for some $d \in R$. This implies that $1 - db \in A \Rightarrow 1 \in A$ (since $b \in A$, $db \in A$) $\Rightarrow R = RbR + r(b) \Rightarrow 1 = \sum r_i bs_i + z$, $1 \le i \le m$, for some r_i, $s_i \in R$ and $z \in r(b) \Rightarrow b = b\sum r_i bs_i + bz \Rightarrow b = b\sum r_i bs_i \in (bR)^2$. Hence R is fully right idempotent. □

3.2.9 Proposition. *Let R be a left pV'-ring. Then for any $b \in Z(_RR)$, $R = RbR + l(b)$.*

Proof. Let $b \in Z(_RR)$ then $l(b)$ is essential in R. So $RbR + l(b)$ is essential in R. Suppose that $R \neq RbR + l(b)$ then there exists a maximal left ideal M of R containing $RbR + l(b)$. Since $RbR + l(b)$ is essential in R so L is essential in R. Hence R/M is simple singular. Since R is a left pV'-ring, R/M is p-injective. Let $f : Rb \longrightarrow R/M$ be defined by $f(rb) = r + L$, $\forall r \in R$. Then f is well-defined. For if $rb = 0$ then $r \subseteq l(b) \subseteq RbR + l(b) \subseteq M$. So $r + M = M$. Clearly f is a well defined left R-homomorphism. So there exists $g : R \longrightarrow R/M$ extending f. Hence $1 + M = f(b) = g(b) = bg(1) = b(c + M) = bc + M$, for some $c \in R$. Now $bc \in RbR \subseteq RbR + l(b) \subseteq M$. So $1 \in M$, which is absurd. Hence $R = RbR + l(b)$ □

3.2.10 Corollary. *Let R be a left pV'-ring. Then for any $b \in Z(_RR)$, Rb is idempotent.*

Proof. By the above proposition, $R = RbR + l(b)$, $\forall b \in R$. So $1 = \sum r_i b s_i + a$ for some r_i, $s_i \in R$ and some $a \in l(b)$, $1 \leq i \leq m$ This implies that $b = \sum r_i b s_i b + ab = \sum r_i b s_i b \in RbRb$. Hence $Rb \subseteq RbRb$. So Rb is idempotent. □

3.2.11 Lemma. *For every left ideal J of a ring R, there exists a left ideal K_o of R such that $J \oplus K_o$ is essential in R.*

Proof. Suppose J is a left ideal of R. Let $\Gamma = \{K : K$ is a left ideal of R satisfying $J \cap K = 0\}$ then by Zorn's lemma, Γ has a maximal element K_o. Let $J_o = J + K_o$. Suppose $J_o \cap Rb = 0$ for some $b \in R$. Let $j \in (K_o + Rb) \cap J$ then $j = s + tb$ for some $s \in K_o$ and $t \in R$. This implies that $tb = j - s \in K_o \cap Rb \subseteq J_o \cap Rb = 0$. Hence $j = s \in K_o \cap J = 0$. So by the maximality of K_o, $K_o + Rb = K_o$. This implies that $Rb \subseteq K_o \subseteq J_o \cap Rb = 0$. This implies that J_o is essential in R. Hence we have proved the result. □

3.2.12 Lemma. *If R is a left pV'-ring then for every element $b \in R$, there exists a left ideal K of R such that $R = (RbR + l(b)) \oplus K$. (See [9], Lemma 1, p-167)*

Proof. Let $b \in R$. Then $RbR + l(b)$ is a left ideal of R. So by the previous lemma, there exists a left ideal K of R such that $J = (RbR + l(b)) \oplus K$ is essential in R. Suppose $J \neq R$ then there exists a maximal left ideal L of R containing J. Since J is essential, so is L. Hence R/L is simple singular and therefore by hypothesis, it is p-injective. Let $f : Rb \longrightarrow R/L$ be defined by $f(rb) = r + L$, $\forall r \in R$. Then f is well-defined. For if $rb = 0$ then $r \in l(b) \subseteq J \subseteq L$, so $r + L = L$ and clearly f is an R-homomorphism. Now, there exists a left R-homomorphism extending f. Therefore

$$1 + L = f(b) = g(b) = bg(1) = b(a + L) = ba + L,$$

for some $a \in R$. Now $ba \in RbR \subseteq J \subseteq L$. Hence $1 \in L$, a contradiction. Therefore $J = R$ □

3.2.13 Theorem. *If R is a left pV'-ring such that every left ideal of R is two-sided then R is strongly regular. (Refer [9], Theorem 2, p-168)*

Proof. Let $b \in R$, then by the previous lemma, $R = (RbR + l(b)) \oplus K$ for some left ideal K of R. Since every left ideal of R is a right ideal, $RbR \subseteq Rb$. So $(RbR + l(b)) \oplus K \subseteq (Rb + l(b)) \oplus K$. Hence $R = (Rb + l(b)) \oplus K$ and $KRb \subseteq Rb \cap K = 0$. This implies that $Krb = 0$, $\forall r \in R$. So $Kb = 0$. Hence $K \subseteq l(b) \subseteq (Rb + l(b))$ and therefore $K \subseteq (Rb + l(b)) \cap K = 0$. Therefore $R = Rb + l(b)$. So $1 = tb + z$, for some $t \in R$ and $z \in l(b)$. This implies that $b = tb^2 + zb = tb^2$. This proves that R is strongly regular. □

3.2.14 Corollary. *If R is a left pV'-ring such that every left ideal of R is two-sided then R is reduced.*

3.2.15 Corollary. *If R is a left pV'-ring such that every left ideal of R is two-sided then R is regular.*

3.2.16 Proposition. *If R is a left pV'-ring then*
(i)$Z(_RR) \cap N = 0$, where N is the Jacobson radical of R.
(ii)$R = RcR$ for every nonzero divisor c of R.
(iii)Every essential left ideal of R is idempotent. (See [9], Proposition 3, p-168)

Proof. Proof of (i):-Let $z \in Z({}_RR) \cap N$. By the previous lemma, $R = (RzR + l(z)) \oplus K$, for some left ideal K of R. Now $l(z)$ is essential in R so $RzR + l(z)$ is essential in R. Since $(RzR + l(z)) \cap K = 0$, we have $K = 0$. So $R = RzR + l(z)$. This implies that $1 = w + d$, for some $w \in RzR$ and $d \in l(z)$. Again since N is two sided, we have $RzR \subseteq N$. So $1 - w$ is invertible. Hence there exists $v \in R$ such that $v(1 - w) = 1$ i,e $vd = 1$ i,e $0 = vdz = z$. Hence proved.

Proof of (ii):-$R = (RcR + l(c)) \oplus K$, for some left ideal K of R. Let $t \in l(c)$ then $tc = 0$. Since c is a nonzero divisor, we have $t = 0$ i,e $l(c) = 0$. Now $cK \subseteq RcR \cap K = 0$ and since c is a nonzero divisor, we have $K = 0$. Therefore $R = RcR$.

Proof of (iii):-Let I be an essential left ideal of R. Then for any $b \in I$, $J = IR + l(b)$ is essential in R. Suppose $J \neq R$, then there exists a maximal left ideal L of R containing J. Since J is essential, so is L. Hence R/L is simple singular and therefore by hypothesis, it is p-injective. Let $f : Rb \longrightarrow R/L$ be defined by $f(rb) = r + L$, $\forall r \in R$. Then f is well-defined. For if $rb = 0$ then $r \in l(b) \subseteq J \subseteq L$, so $r + L = L$ and clearly f is an R-homomorphism. Now, there exists a left R-homomorphism extending f. Therefore

$$1 + L = f(b) = g(b) = bg(1) = b(a + L) = ba + L,$$

for some $a \in R$. Now $ba \in IR \subseteq J \subseteq L$. Hence $1 \in L$, a contradiction. Therefore $J = R$. Thus $R = IR + l(b)$. This implies that $1 = u + d$, for some $u \in IR$ and $d \in l(b)$. So we have $b = ub + db = ub \in I^2$. Therefore $I = I^2$ □

3.2.17 Proposition. *Let R be a left pV'-ring. If R is semiprime then every left ideal of R is idempotent. (See [9], Proposition 6, p-169)*

Proof. Let I be a left ideal of R. Suppose there exists $b \in I$ such that $b \notin I^2$. Let B be a left ideal of Rb (considered as a rung) such that $(Rb)^2 \cap B = 0$. Now $B^2 \subseteq (Rb)^2 \cap B = 0$. Since R is semiprime, we have $B = 0$. So $(Rb)^2$ is essential in Rb. Let $\sum = \{J : J$ is a left ideal of R and $(Rb)^2 \subsetneq J \subset Rb\}$. Then by Zorn's lemma, $\sum$ has a maximal element L. So Rb/L is simple singular and hence p-injective. Let $f : Rb \longrightarrow Rb/L$ be the canonical homomorphism, then there exists $g : R \longrightarrow Rb/L$ extending f. This implies that $b + L = f(b) = g(b) = bg(1) = b(ab + L) = bab + L$, for some $a \in R$. Again, since $bab \in (Rb)^2 \subsetneq L$ we have $b \in L$, which is absurd. Hence $I = I^2$. □

3.3 Results on left fV'-rings

The following are some of the results about left fV'-rings which are also true in case of left V'-rings.

3.3.1 Theorem. *For a T-ring R, the following are equivalent*

1)R is a left fV'-ring

2)$Z(_RR) = 0$ and $Rad(R/I) = 0$ for any essential left ideal I of R. (Refer [12], p-234, Theorem 1.1)

3.3.2 Theorem. *Let R be a T-ring which is also a left fV'-ring then for any essential left ideal I of R, $I^2 = I$ and I is an intersection of maximal left ideals. Moreover $N^2 = 0$, where N is the Jacobson Radical of R. (Refer [12], p-234, Theorem 1.2)*

3.3.3 Theorem. *Let R be a commutative T-ring. Then R is a left fV'-ring if and only if R is regular. (See [12], p-236, Theorem 1.6)*

3.3.4 Proposition. *Let R be a T-ring which is also a left fV'-ring. If R is regular then $J = 0$ where $J = Rad({}_RR)$.*

3.4 Left GPV-rings and YJ-injectivity

In the following section, we look at some results on GPV-rings and in connection to this, some properties of GW-ideals have also been used.

3.4.1 Lemma. *R is a reduced ring if and only if $a^2 = 0$,for every element $a \in R$,implies that $a = 0$.*

Proof. Suppose $a^2 = 0$. This implies that $a = 0$(since R is reduced).
Conversely, let $a^n = 0$ for some positive integer n. If $n \leq 2$ then we are through. Suppose $n > 2$ then $2n - 2 > n$. Now $(a^{n-1})^2 = a^{2n-2} = 0$ (since $a^n = 0$). Hence $a^{n-1} = 0$. Again if $n - 1 \leq 2$ then we are through. So we assume that $n > 3$. Then $(a^{n-1})^2 = a^{2n-2} = 0$. This implies that $a^{n-2} = 0$. By reducing the powers of a at every stage and following the same steps as above we will finally get $a = 0$. Hence R is reduced.

□

Following is an example of a left GPV-ring.

3.4.2 Example. Let $R = Z_2XZ_2$. Then all the elements of R are idempotents. So for any nonzero $(a, b) \in R$, $(a, b)^n = (a, b) \neq 0$, for all positive integer n. Hence any left R-homomorphism from $R(a, b)^n$ to any simple left R-module S can be extended to a left R-homomorphism from R to S.

3.4.3 Lemma. *In a left GPV-ring, $Rad({}_RR) = 0$. (See [13], p-358, Lemma 2.1)*

Proof. Let $0 \neq a \in Rad(_R R)$. Then Ra is a finitely generated nonzero left R-module. So there exists a maximal submodule M of Ra. Hence Ra/M is a simple left R-module. Since R is a left GPV-ring, there exists a positive integer n such that $a^n \neq 0$ and any left R homomorphism of Ra^n to Ra/M extends to one of R to Ra/M. Let $f : Ra^n \longrightarrow Ra/M$ be defined by $f(ra^n) = ra^n + M \ \forall r \in R$. Then f is a well-defined left R-homomorphism. Since Ra/M is YJ-injective, there exists a left R-homomorphism $g : R \longrightarrow Ra/M$ extending f. Again, since Ra/M is simple, f is onto or $f = 0$. If f is onto then g is onto. So by the fundamental theorem of module homomorphism, $R/Kerg \cong Ra/M$. This shows that $Kerg$ is a maximal left ideal of R. So $a \in Kerg$ and hence $a^n \in Kerg$. This implies that $M = g(a^n) = f(a^n) = a^n + M$. This shows that $a^n \in M$.

Claim:-$a^{n-1} \in M$.

If not then $Ra = Raa^{n-1} + M$ i,e $Ra = Ra^n + M$ i,e $Ra = M$, a contradiction. Hence the claim.

Proceeding in this way, we find that $a \in M$, which is absurd. Hence f cannot be onto. So f is zero. Therefore $M = f(a^n) = a^n + M$. Hence $a^n \in M$. Following the same steps as above, we see that $Ra = M$, a contradiction. So $Rad(_R R) = 0$. □

3.4.4 Lemma. *If R is a left GPV-ring whose every maximal left ideal is a GW-ideal then R is reduced. (See [13], p-358, Theorem 2.2)*

Proof. Suppose R is not reduced then there exists $0 \neq a \in R$ such that $a^2 = 0$. Hence $l(a)$ is contained in some maximal left ideal M of R. Otherwise, $l(a) \nsubseteq M$ for all maximal left ideal M of R. So $R = l(a) + M$. Now $1 = r + m$ for some $r \in l(a)$ and some $m \in M$. So $a = ra + ma = ma$. Since M is a GW-ideal, there exists a positive integer n such that $m^nR \subseteq M$. Now $a = ma = m^2a = m^3a = ... = m^na \in M$ for all maximal left ideals M of R. So $a \in \cap M \Rightarrow a \in Rad(_RR) = 0$ which is impossible since $a \neq 0$. Again, R/M is a simple left R-module and since R is a left GPV- ring, any left R-homomorphism $f : Ra \longrightarrow R/M$ can be extended to a left R-homomorphism $g : R \longrightarrow R/M$. Define f by $f(ra) = r + M$, $\forall r \in M$. Now $1 + M = f(a) = g(a) = ag(1) = a(b + M) = ab + M$, for some $b \in R \Rightarrow 1 - ab \in M \Rightarrow ba(1 - ab) \in M \Rightarrow ba - ba^2b \in M \Rightarrow ba \in M$. So there exists a positive integer t such that $(ba)^tR \subseteq M$. This implies that $(ba)^tb \in M$. Again $1 - ab \in M$. So $b - bab \in M$. Now $(ba)^{t-1}b = (ba)^{t-1}b - (ba)^tb + (ba)^tb = (ba)^{t-1}(b - bab) + (ba)^tb \in M$. And $(ba)^{t-2}b = (ba)^{t-2}b - (ba)^{t-1}b + (ba)^{t-1}b = (ba)^{t-2}(b - bab) + (ba)^{t-1}b \in M$. Proceeding in this way we find that $bab \in M$. Thus $b = b - bab + bab \in M$, so $ab \in M$ and hence $1 \in M$ contradicting that M is maximal. Hence R is reduced □

3.4.5 Lemma. *If R is a left GPV-ring whose every maximal right ideal is a GW-ideal then R is reduced. (See [13], p-358, Theorem 2.2)*

Proof. Since R is a left GPV-ring, we have $Rad(_RR) = 0$. Suppose R is not reduced then there exists $0 \neq a \in R$ such that $a^2 = 0$ Since $a \neq 0, a \notin Rad(R)$. So $a \notin K$ for some maximal right ideal K of R. Hence $R = K + aR$. So $1 = k + ar$ for some $k \in K$ and $r \in R$. This implies that $a = ak + a^2r \Rightarrow a = ak$

Since K is a GW ideal, there exists a positive integer n such that $Rk^n \subseteq K$. Now $a = ak = ak^2 = ak^3 = ... = ak^n \in K$ which contradicts that $a \notin K$. Hence R is reduced. □

The following examples (See [13],p-358, Examples 1.2 and 1.3) show that a GW-ideal of a ring need not be an ideal and a left (or right) ideal of a ring need not be a GW-ideal.

3.4.6 Example. Let R be a set of matrices of the form

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix} : \ a, b, c, d \in Z_2 \right\}$$

Then $\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$. And $R\alpha$ is a left ideal of R.

Let $\beta \in R\alpha$ then $\beta = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Let $\gamma \in \beta^2 R$

then $\gamma = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e & f & g \\ 0 & e & h \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R\alpha$

Hence $\beta^2 R \subseteq R\alpha$. So $R\alpha$ is a GW-ideal.

Now $R\alpha$ is a set of the form

$$\left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \ a \in Z_2 \right\}$$

But $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin R\alpha$

Hence $R\alpha$ is not a right ideal of R.

3.4.7 Example. Let R be a set of matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \ a, b, c \in Z_2 \right\}$$

Suppose K is of the form

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} : \ a \in Z_2 \right\}$$

then for $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \in K$ and for any $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$,

$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & dc \end{pmatrix} \in K.$

Therefore K is a right ideal of R.

Let $\alpha \in K$ then for any positive integer n,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \notin K$$

Hence K is not a GW-ideal.

3.4.8 Example. Let R be a set of matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \quad a, b, c \in Z_2 \right\}$$

Suppose L is of the form

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : \quad a \in Z_2 \right\}$$

then for $\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \in L$ and for $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ad & 0 \\ 0 & 0 \end{pmatrix} \in L$$

Therefore L is a left ideal of R. Now for any positive integer n,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin L$$

Hence L is not a GW-ideal.

3.4.9 Lemma. *If R is reduced then $l(a) = r(a), \forall a \in R$*

Proof. Let $x \in l(a)$ then $xa = 0$. Now $(ax)^2 = (ax)(ax) = a(xa)x = 0 \Rightarrow ax = 0 \Rightarrow x \in r(a) \Rightarrow l(a) \subseteq r(a)$. Similarly we can show that $r(a) \subseteq l(a)$. Hence $l(a) = r(a)$. □

3.4.10 Lemma. *If R is reduced then $l(a^n) = l(a)$, for all positive integer n, and $\forall a \in R$.*

Proof. Let $r \in l(a^n)$ then $ra^n = 0$. If $n = 1$ we are through. Assume $n > 1$ We have $ara^{n-1} = 0$ (since R is reduced). Now $ra^{n-1}ra^{n-1} = 0$. Hence $ra^{n-1} = 0$. So $ara^{n-2} = 0$.
Assume $n > 2$. Now $(ra^{n-2})^2 = ra^{n-2}ra^{n-2} = 0$. Hence $ra^{n-2} = 0$. Proceeding in this way we get $ra^{n-(n-1)} = 0$ i,e $ra = 0$. Hence $l(a^n) \subseteq l(a)$. Now let $s \in l(a)$ then $sa = 0$ which implies that $sa^n = 0$. So $s \in l(a^n)$. Therefore $l(a^n) = l(a)$. □

The above six lemmas lead us to the following theorem (See [13], p-358, Theorem 2.2)

3.4.11 Theorem. *Following conditions are equivalent for a ring R*
1)R is strongly regular.
2)R is a left GPV-ring whose every maximal left ideal is a GW ideal.
3)R is a left GPV-ring whose every maximal right ideal is a GW ideal.
(Refer [13], p-358, Theorem 2.2)

Proof. 1)⇒2).Let S be a simple left R module. Let $0 \neq a \in R$. Since R is strongly regular, $Ra \leqslant^{\oplus} R$. So there exists a left ideal B of R such that $Ra \oplus B = R$. Let $f : Ra \longrightarrow S$ be an R-homomorphism. Let $g : R \longrightarrow S$ be a function defined by $g(ra + b) = f(ra), \forall a \in A, \forall b \in B$. Then g is a well-defined homomorphism which extends f. So R is a left GPV ring. Let M be a maximal left ideal of R and let $a \in M$ then $a = aba$ (since R is regular). Let $r \in R$. Now $ar = abar = arba \in M$(since ba is a central idempotent) Hence $aR \subseteq M$. So M is a GW ideal.
2)⇒1)Let $a \in R$. If $l(a) + Ra = R$ then R is strongly regular otherwise there is a maximal left ideal M of R such that $l(a) + Ra \subseteq M$. Since

R/M is a simple left R-module so R/M is YJ-injective, hence there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism from Ra^n to R/M extends to a left R-homomorphism from R to R/M. Now we define a map $f : Ra^n \longrightarrow R$ by $f(ra^n) = r + M$ for any $r \in R$. Suppose $ra^n = sa^n \Rightarrow r - s \in l(a^n) = l(a) \subseteq M$. So $r + M = s + M$ i,e f is well-defined. So there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Therefore we have

$1 + M = f(a^n) = g(a^n) = a^n g(1) = a^n(b + M) = a^n b + M$ for some $b \in R$. This implies that $1 - a^n b \in M \Rightarrow b - ba^n b \in M$. Also as $a \in M$ we have $ba^n \in M$. Since M is a GW-ideal, there exists a positive integer m such that $(ba^n)^m R \subseteq M$. So $(ba^n)^m b \in M$. Now $(ba^n)^{m-1}b = (ba^n)^{m-1}b - (ba^n)^m b + (ba^n)^m b = (ba^n)^{m-1}(b - ba^n b) + (ba^n)^m b \in M$. Proceeding in this way, we get $ba^n b \in M$. So $b \in M$ and hence $1 \in M$. This contradiction shows that $R = l(a) + Ra$. This implies that $1 = x + ra$ for some $x \in l(a)$ and some $r \in R$ which implies that $a = xa + ra^2 \Rightarrow a = ra^2$. Hence R is strongly regular.

1)$\Rightarrow$3)

Let K be a maximal right ideal of R and let $a \in K$. Since R is regular, there exists $b \in R$ such that $a = aba$. Now $ra = raba = abra$ (since ab is central idempotent)$\in K$. Hence K is a GW-ideal.

3) $\Rightarrow$ 1) Let $a \in R$. Suppose $l(a) + aR = R$ for some $a \in R$ then there exists a maximal right ideal K of R containing $l(a) + aR$. Since K is a GW-ideal, $Ra^n \subseteq K$ for some positive integer n. Again K being a right ideal, we have $Ra^n R \subseteq K$. So we have the relation $l(a) + aR \subseteq K \subset R$ and there exists a maximal left ideal M of R such that $l(a) + Ra^n R \subseteq M \subset R$. Since R/M

is injective , there is a positive integer m such that $(a^n)^m \neq 0$ and any left R-homomorphism $R(a^n)^m \longrightarrow R/M$ extends to an R-homomorphism of R into R/M. Define $f : R(a^n)^m \longrightarrow R/M$ by $f(r(a^n)^m) = r + M$. Suppose $r(a^n)^m = s(a^n)^m$ then $r-s \in l((a^n)^m) = l(a) \subseteq M$. So $r+M = s+M$. Hence f is well defined. Since there exists $g : R \longrightarrow R/M$ extending f, we have $1 + M = f((a^n)^m) = g((a^n)^m) = (a^n)^m g(1) = (a^n)^m(b + M) = (a^n)^m b + M$ for some $b \in R$. But $(a^n)^m b \in Ra^nR \subseteq M$. So $1 \in M$, a contradiction. Therefore $l(a) + aR = R$ which implies that $1 = x + ar$ for some $x \in l(a)$ and some $r \in R$ which implies that $\Rightarrow a = ax + aar = a^2r$. Hence R is strongly regular. □

3.4.12 Lemma. *If R is a left quasi-duo ring then every maximal left ideal of R is a GW-ideal.*

Proof. Let M be a maximal left ideal of R. Let $a \in M$ and $r \in R$. Then $ar \in M$ (since M is two-sided). So $aR \subseteq M$. Hence M is a GW-ideal. □

The following is a corollary to the above theorem. (See [13], p-360, Corollary 2.3)

3.4.13 Corollary. *If R is strongly regular then R is a left quasi-duo ring whose simple right modules are YJ-injective.*

Proof. Let M be a maximal left ideal of R. Let $a \in M$ and $r \in R$. Since R is regular, there exists $b \in R$ such tha $a = aba$. Now $ar = abar = arba \in M$. Hence R is a left quasi-duo ring. Let S be a simple right R-module Since R is strongly regular $aR \leqslant^{\oplus} R$. So there exists a right ideal B of R such that

$aR \oplus B = R$. Let $f : aR \longrightarrow S$ be a right R-homomorphism. Let $g : R \longrightarrow S$ be a function defined by $g(ar + b) = f(ar), \forall a \in A, \forall b \in B$. Then g is a well-defined homomorphism which extends f. So S is YJ-injective. □

3.4.14 Corollary. *If R is strongly regular then R is a left quasi-duo ring whose simple left modules are YJ-injective.*

3.4.15 Proposition. *If R is a left GPV-ring such that every maximal left ideal of R is an ideal of R then R is reduced.*

Proof. Suppose there exists $0 \neq a \in R$ such that $a^2 = 0$ then $a \in l(a)$. Since $a \neq 0$, we have $l(a) \neq 0$. So there exists a left ideal M of R containing $l(a)$. Therefore R/M is singular and hence YJ-injective. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r + M$, $\forall r \in R$. Then f is well-defined. For if $ra = 0$ then $r \in l(a) \subseteq M$. So $r + M = M$ and clearly f is a left R-homomorphism. Now there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Hence $1 + M = f(a) = g(a) = ag(1) = a(c + M) = ac + M$, for some $c \in R$. Since M is an ideal of R and $a \in M$ so $ac \in M$. Therefore $1 \in M$, which is a contradiction. Hence R is reduced. □

3.4.16 Corollary. *If R is a left GPV-ring such that every maximal left ideal of R is an ideal of R then R is fully left and right idempotent.*

Proof. Let I be a left ideal of R. Let $a \in I$. We claim that $R = RaR + l(a)$. If not, then there exists a maximal left ideal M of R containing $RaR + l(a)$. Since R/M is simple so it is YJ-injective. Therefore there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism of Ra^n to R/M extends to one of R to R/M. Let $f : Ra^n \longrightarrow R/M$ be defined by $f(ra^n =$

$r + M$. Then f is well-defined. For if $ra^n = 0$ then $r \in l(a^n) = l(a)$ (since R is reduced). So $r \in M \Rightarrow r + M = M$. Clearly f is a well-defined left R-homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Hence $1+M = f(a^n) = g(a^n) = a^n g(1) = a^n(c+M) = a^n c+M$, for some $c \in R$. Again since $a \in M, a^n c \in M$ showing that $1 \in M$, a contradiction. Hence $R = RaR + l(a)$. This shows that $1 = z + t$, for some $z \in RaR$ and some $t \in l(a)$. We, therefore, get $a = za + ta = za \in RaRa$. Hence $Ra = RaRa$. So I is left idempotent. Similarly, we can show that any right ideal of R is idempotent. □

3.5 More results on YJ-injectivity

3.5.1 Lemma. *If R is reduced and if e is an idempotent element of R then e is central.*

Proof. For any $r \in R, re(1-e) = 0 \Rightarrow (1-e)re = 0$ (since R is reduced) $\Rightarrow re - ere = 0 \Rightarrow re = ere$. Again, $(1-e)er = 0 \Rightarrow er(1-e) = 0 \Rightarrow er - ere = 0 \Rightarrow er = ere$. Hence $re = er$ i,e e is a central idempotent.

□

3.5.2 Lemma. *Following conditions are equivalent for a prime ring R.*
1)R is an integral domain.
2)There exists a non zero reduced left ideal of R which is essential in a left annihilator. (See [7], p-17, Proposition 6)

Proof. Since R is an integral domain so R is reduced and clearly R is essential in $R = l(0)$. Hence 1)⇒2)

2)⇒1)

Let I be a non zero reduced left ideal of R which is essential in $l(U)$ for some subset U of R. If $0 \neq b \in I$ then $r(b) = 0$. If not then there exists $0 \neq a \in r(b)$. Then $Ra \neq 0$ and $Rb \neq 0$. Since R is prime, we have $RaRb \neq 0$. So there exists $0 \neq acb \in RaRb \subseteq I$ for some $c \in R$. Now $(acb)^2 = acbacb = 0$ (since $a \in r(b)$ so $ba = 0$) Since I is reduced, we have $acb = 0$. This contradiction proves that $r(b) = 0$ for every $0 \neq b \in I$. Now $Rb \subseteq I \subseteq l(U)$. This implies that $bU = 0$ which implies that $RbU = 0$. Since R is semi-prime, we have $U = 0$. This shows that $l(U) = R$ i,e I is essential in R. Let $v, w \in R$ such that $vw = 0$. Suppose $v \neq 0$. Since I is essential in $R, I \cap Rv \neq 0$. So there exists $y \in R$ such that $0 \neq yv \in I$. Now $yvw = 0$ so $w \in r(yv) = 0$. Hence R is an integral domain. □

With the help of the above lemma, we prove the following theorem which is due to Ming (See [6], p-310, Theorem 1)

3.5.3 Theorem. *If every non zero factor ring of a ring R is a prime ring such that it contains a non zero reduced YJ-injective left ideal which is a left annihilator then it is a division ring.*

Proof. Let B be a nonzero factor ring of R. By hypothesis, B contains a non zero reduced YJ-injective left ideal K which is a left annihilator. Clearly K is essential in K. So by the previous lemma, B is an integral domain. Let $0 \neq k \in K$. Since K is YJ-injective, there exists a positive integer n such that any left B-homomorphism of Bk^n into K extends to one of B into K. Let $j : Bk^n \longrightarrow K$ be the inclusion map. Then there exists a left R-homomorphism $g : B \longrightarrow K$ extending j. Hence $k^n = j(k^n) = g(k^n) =$

$k^n g(1) = k^n s = k^n(1-s)$ for some $s \in R$. This implies that $s = 1$ (since $K \subseteq B$ is an integral domain). Therefore $1 \in K$. Hence $B = K$. So B is YJ-injective. So for any $0 \neq c \in B$, there exists a positive integer m such that any left B-homomorphism of Bc^m into B extends to one of B into B. Let $f : Bc^m \longrightarrow B$ be defined by $f(bc^m) = b$, $\forall b \in B$. Then f is well defined left B-homomorphism. For if $bc^m = 0$ then $b \in l(c^m) = l(c)$ (since $B = K$ is reduced). Therefore $bc = 0$. Since $c \neq 0$ and B is an integral domain, we have $b = 0$. So there exists a left B-homomorphism $h : B \longrightarrow B$ extending f. Hence $f(c^m) = h(c^m) \Rightarrow 1 = c^m h(1) = c^m d$ for some $d \in B$. This shows that the inverse of c exists. So B is a division ring. □

3.5.4 Lemma. *Let M be a maximal left ideal of a ring R such that M is essential in R then R/M cannot be projective over R.*

Proof. Suppose R/M is projective. Let us consider the exact sequence $0 \longrightarrow M \hookrightarrow R \longrightarrow R/M \longrightarrow 0$ Let $Id : R/M \longrightarrow R/M$ be the identity homomorphism. Since R/M is projective, there exists a left R-homomorphism $f : R/M \longrightarrow R$ such that $\eta_o f = Id$, where η is the canonical homomorphism from R to R/M in the sequence considered above. This shows that the above sequence splits. Hence M is a direct summand of R. So there exists a left ideal K of R such that $M \oplus K = R$. So $M \cap K = 0$. Since M is essential in R, we have $K = 0$ Now $K \cong R/M$. Therefore $R/M = 0$ which implies that $R = M$, which is absurd. Hence R/M cannot be projective. □

The above lemma initialises the following proposition which is due to Ming (See [6], p-313, Proposition 2)

3.5.5 Proposition. *Suppose that every simple left R-module is either YJ-*

injective or projective. Then $Z \cap Rad({}_RR) = 0$, where Z is the singular left ideal of R.

Proof. Suppose $Z \cap Rad({}_RR)$ is a non-zero reduced ideal of R. Let $0 \neq w \in Z \cap Rad({}_RR)$ then $l(w)$ is an essential left ideal of R. Since $Rw \neq 0$ so $Rw \cap l(w) \neq 0$. Hence there exists $0 \neq bw \in l(w)$ so that $bw^2 = 0$. Now $bw \in Z \cap Rad({}_RR)$ and $wbw \in Z \cap Rad({}_RR)$. So $(wbw)^2 = wbwwbw = 0$ which implies that $wbw = 0$ (since $Z \cap Rad({}_RR)$ is reduced). Again, $(bw)^2 = bwbw = 0$ which shows that $bw = 0$, a contradiction. This proves that if $Z \cap J$ is non zero then $Z \cap Rad({}_RR)$ cannot be reduced. So there exists $0 \neq z \in Z \cap Rad({}_RR)$ such that $z^2 = 0$. Let $L = RzR + l(z)$. Since $l(z)$ is essential in R and $l(z) \subseteq L$ we see that L is essential in R.

Claim:-$L = R$

If not then there exists a maximal left ideal M of R containing L. Now $L \subseteq M$ so M is an essential left ideal of R. So, by the above lemma, R/M is non-projective. Hence, by hypothesis, R/M being simple is YJ-injective. Let $f : Rz \longrightarrow R/M$ be defined by $f(az) = a + M$ $\forall a \in R$. Then f is well defined. For if, $az = 0$ then $a \in l(z) \subseteq L \subseteq M$. So $a + M = M$ i,e $f(az) = M$. Clearly f is a left R-homomorphism. Since R/M is YJ-injective, there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Hence

$1 + M = f(z) = g(z) = zg(1) = z(c + M) = zc + M$ for some $c \in R$. Since $zc \in RzR \subseteq L \subsetneq M$. So $1 \in M$, which is absurd. Hence the claim.

So $R = RzR + l(z) \Rightarrow 1 = d + u$ for some $d \in RzR$ and some $u \in l(z)$. This implies that $z = dz + uz = dz \Rightarrow (1 - d)z = 0$. Since $d \in RzR \subseteq Rad({}_RR)$. So $1 - d$ is a unit. Hence $z = 0$, a contradiction. This implies that $Z \cap Rad({}_RR) = 0$ □

3.5.6 Proposition. *Let R be a semi-prime ring whose simple left R-modules are either YJ-injective or projective. Then R is semi-primitive. (Refer to [6], p-315, Proposition 4)*

Proof. It is required to show that $Rad(_RR) = 0$.

Claim 1:-$J = Rad(_RR)$ is reduced

If not then there exists $0 \neq c \in J$ such that $c^2 = 0$

Subclaim:-$Rc = RcRc$

Suppose not then we deduce a contradiction. Let $\sum = \{I \leqslant R : RcRc \subseteq I \subseteq Rc\}$. Then $RcRc \in \sum$ so $\sum$ is non-empty. We order this set by inclusion. Then $\sum$ is a partially ordered set. Let (I_j) be a chain of ideals in $\sum$. Let $N = \cup I_j$. Then N is an upper bound of $\sum$. Since $RcRc \subseteq I_j$, $\forall j$ in some indexing set. So $RcRc \subseteq N$ Now suppose if $Rc = N$ then $c \in I_j$ for some j. So $Rc \subseteq I_j$. This implies that $Rc = I_j$, a contradiction. So $N \in \sum$. Hence by Zorn's lemma, $\sum$ has a maximal element M. Clearly M is a maximal subideal of Rc. So Rc/M is simple. Let K be any left subideal of Rc such that $RcRc \cap K = 0$. Now $K^2 \subseteq K \cap RcRc = 0$. Since R is semi-prime, we have $K = 0$. Hence $RcRc$ is essential in Rc. Now since $RcRc \subseteq M$ so M is essential in Rc. This shows that Rc/M cannot be projective. So, by hypothesis, Rc/M is YJ-injective. Let $f : Rc \longrightarrow Rc/M$ be defined by $f(ac) = ac + M$. Then clearly f is a well defined left R-homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow Rc/M$ extending f. Hence $c + M = f(c) = g(c) = cg(1) = c(dc + M) = cdc + M$, for some $d \in R$. Since $cdc \in RcRc \subseteq M$ so $c \in M$. Hence $M = Rc$, a contradiction. So $Rc = RcRc$. This shows that $c = uc$ where $u \in RcR \subseteq J$. Therefore $(1 - u)c = 0$. Since $u \in J$ so $1 - u$ is invertible. Hence $c = 0$, which is absurd. Therefore J is reduced.

Claim 2:$J = 0$

If not, let $0 \neq v \in J$. Let $L = RvR + l(v)$. Suppose $L \cap K = 0$ for some left ideal K of R. Now $vK \subseteq RvRK \subseteq L \cap K = 0$. And $(Kv)^2 = KvKv = 0$. SO $Kv = 0$ (since R is semi-prime). This implies thar $K \subseteq l(v)$. So $K \subseteq K \cap l(v) \subseteq K \cap L = 0$. Therefore $K = 0$. So L is essential in R.

Subclaim:-$L = R$

If not then there exists a maximal left ideal M of R containing L. Now $L \subseteq M$ so M is an essential left ideal of R. So, by the above lemma, R/M is non-projective. Hence, by hypothesis, R/M being simple is YJ-injective. So there exists a positive integer m such that any left R-homomorphism of Rv^m into R/M extends to R. Let $f : Rv^m \longrightarrow R/M$ be defined by $f(av^m) = a + M$, $\forall a \in R$. Then f is well defined. For if $av^m = 0$ then $a \in l(v^m) = l(v)$ (since J is reduced)$\subseteq L \subseteq M$. So $a + M = M$ i,e $f(av^m) = M$. Clearly f is a left R-homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Hence

$1 + M = f(v^m) = g(v^m) = v^m g(1) = v^m(w + M) = v^m w + M$ for some $w \in R$. Now $v^m w \in RvR \subseteq L \subseteq M$. So $1 \in M$, which is absurd. Hence $R = L$. So $1 = s + t$, for some $s \in RvR$ and some $t \in l(v)$. This implies that $v = sv + tv = sv \Rightarrow (1 - s)v = 0$. Since $s \in RvR \subseteq J$ so $1 - s$ is invertible in R. Hence $v = 0$, a contradiction. Therefore $Rad(_R R) = 0$ □

3.5.7 Lemma. *If R is a semi-prime ring then $l(a^n) = l(a)$ for any $a \in Z(R)$ where $Z(R)$ is the centre of R. (Refer [15], p-216, Lemma 1)*

Proof. Clearly, $l(a) \subseteq l(a^n)$. Let $x \in l(a^n)$ then $xa^n = 0$. Now let $y \in (Rxa)^n$ then $y = \sum \Pi r_{ij} xa$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$ and $r_i \in R$. So $y =$

$(rx)^n a^n = rxrx...rxa^n = 0$ for some $r \in R$. This implies that $(Rxa)^n = 0$. Since R is semi-prime, we have $Rxa = 0$. So $xa = 0$ i,e $x \in l(a)$. Hence the proof. □

3.5.8 Lemma. *For any $a \in Z(R)$, if $a = ara$ for some $r \in R$ then there exists $b \in Z(R)$ such that $a = aba$. (Refer [15], p-216, Lemma 2)*

Proof Let $a \in Z(R)$, given $a = ara$ for some $r \in R$. Let $b = a^2r^3$ then $ba = a^2r^3a = aarrra = aarrar = a(ara)rr = aarr = (ara)r = ar = ra$. then $aba = (ara)ba = arara = ara = a$. Let $u \in R$ then $ra^2u = arau = au = ua = uara = ura^2 = a^2ur$ and $r^3a^2u = r^2ra^2u = r^2a^2ur = rra^2ur = ra^2urr = a^2urrr = a^2ur^3$. Now $bu = a^2r^3u = r^3a^2u = a^2ur^3 = ua^2r^3 = ub$. Hence $b \in Z(R)$ □

3.5.9 Proposition. *If R is a semi-prime ring whose simple singular left R-modules are YJ-injective then Z(R) is von Neumann regular. (Refer [15], p-216, Proposition 3)*

Proof. Claim:- $R = Ra + l(a)$, $\forall a \in Z(R)$. If not then there exists a maximal left ideal M of R containing $L = Ra + l(a)$. Suppose $L \cap K = 0$ for some left ideal K of R then $Ka = aK \subseteq Ra \cap K \subseteq L \cap K = 0$. So $K \subseteq l(a)$. Hence $K \subseteq l(a) \cap K \subseteq L \cap K = 0$. So L is essential in R which implies that M is essential in R. This implies that R/M is simple singular and hence YJ-injective. So there exists a positive integer n such that any left R-homomorphism from Ra^n to R/M extends to one of R to R/M. Let $f : Ra^n \longrightarrow R/M$ be defined by $f(ra^n) = r + M$,$\forall r \in R$. Then f is well defined, for if $ra^n = 0$ then $r \in l(a^n) = l(a) \subseteq L \subseteq M$. So $r+M = M$. Hence there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Therefore

$1 + M = f(a^n) = g(a^n) = a^n g(1) = a^n(c + M) = a^n c + M = ca^n + M$, for some $c \in R$. Since $a \in M$ so $ca^n \in M$ showing that $1 \in M$, which is absurd. Hence $R = Ra + l(a)$. So $1 = sa + t$ for some $s \in R$ and some $t \in l(a)$. This implies that $a = sa^2 + ta = asa$. So $Z(R)$ is von Neumann regular. □

3.5.10 Lemma. *If $r(a) \subseteq l(a)$, $\forall a \in R$ then $RaR + l(a)$ is essential in R. (See [15], p-217, Lemma 4)*

Proof. Let $L = RaR + l(a)$. Suppose $L \cap K = 0$, for some left ideal L of R. Now $aK \subseteq RaR \cap K \subseteq L \cap K = 0$. This implies that $K \subseteq r(a) \subseteq l(a)$. Therefore $K \subseteq l(a) \cap L \subseteq L \cap K = 0$. Hence L is essential in R. □

3.5.11 Proposition. *If $r(a) \subseteq l(a), \forall a \in R$ and if every simple singular left R-module is YJ-injective then R is reduced. (Refer [15], p-217, Lemma 5)*

Proof. Suppose $0 \neq a \in R$ such that $a^2 = 0$ then $a \in r(a)$. Again, suppose that $l(a) \cap K = 0$ for some left ideal K of R. Now $aK \subseteq r(a)$ (since $r(a)$ is a right ideal of R). So $aK \subseteq l(a) \cap K = 0$. This implies that $K \subseteq r(a) \subseteq l(a)$. Therefore $K \subseteq l(a) \cap K = 0$. Since $a \neq 0$ so $l(a) \neq R$. Hence there exists a maximal left ideal M of R containing $l(a)$. This implies that M is essential in R. So R/M is simple singular and hence is YJ-injective. So any left R-homomorphism from Ra to R/M extends to one of R into R/M. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r + M, \forall r \in R$. Then f is well-defined, for if $ra = 0$ then $r \in l(a) \subseteq M$. So $r + M = M$. Therefore there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Therefore
$1 + M = f(a) = g(a) = ag(1) = a(b + M) = ab + M$. Hence $1 - ab \in M$. Now $ab \in r(a) \subseteq l(a) \subseteq M$. So $1 \in M$, which is absurd. Hence $a = 0$ i,e R is reduced. □

3.5.12 Lemma. *If R is a reduced left weakly regular ring then R is weakly regular.*

Proof. Let I be a right ideal of R. Let $a \in I$ then Ra is a left ideal of R. Since R is left weakly regular, $Ra = RaRa \Rightarrow a = \sum r_i a s_i a$ for some $r_i, s_i \in R$. Let $z = \sum r_i a s_i$ then $a = za \Rightarrow (1-z)a = 0 \Rightarrow a(1-z) = 0 \Rightarrow a = az \in I^2$. Thus $I \subseteq I^2$. Hence R is weakly regular. □

3.5.13 Proposition. *If $r(a) \subseteq l(a)$, $\forall a \in R$ and if every simple singular left R-module is YJ-injective then R is a weakly regular ring. (Refer [15], p-217, Theorem 6)*

Proof. By the above proposition, R is reduced. Let $a \in R$ and let $L = RaR + l(a)$ then by the previous lemma, L is essential in R.
Claim:-$L = R$
If not then there exists a left ideal M of R containing L .This implies that M is essential in R. So R/M is simple singular and hence is YJ-injective. So there exists a positive integer n such that any left R-homomorphism from Ra^n to R/M extends to one of R to R/M. Let $f : Ra^n \longrightarrow R/M$ be defined by $f(ra^n) = r + M, \forall r \in R$. Then f is well defined. For if $ra^n = 0$ then $r \in l(a^n) = l(a)$(since R is reduced). Now $l(a) \subseteq L \subseteq M$. So $r + M = M$ and clearly f is a left R-homomorphism. Hence there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. So $1 + M = a^n g(1) = a^n(c + M) = a^n c + M$, for some $c \in R$. Therefore $1 - a^n c \in M$. Now $a^n c \in r(a) \subseteq l(a) \subseteq L \subseteq M$. So $1 \in M$, which is absurd. So $R = RaR + l(a)$. Therefore $1 = \sum r_i a s_i + t$, for some $r_i, s_i \in R, i = 1, 2, ..., m$. So $a = \sum r_i a s_i a + ta = \sum r_i a s_i a \in RaRa$. So $Ra \subseteq RaRa$. So R is a left weakly regular ring. Since R is reduced, R is a weakly regular ring. □

3.5.14 Lemma. *If M is a maximal left ideal of a ring R then M is either a direct summand of R or M is essential in R.*

Proof. Suppose M is not a direct summand of R. Let $c \in R$ such that $M \cap Rc = 0$. Suppose $c \neq 0$ then $c \notin M$. Since M is maximal, $M \oplus Rc = R$. This shows that M is a direct summand of R, which is absurd. Hence $c = 0$ i,e $Rc = 0$. So M is essential in R. □

In the following section, some properties of ZI-rings, ZC rings and SRB rings have been studied which are required for obtaining some important results on YJ-injectivity.

3.5.15 Lemma. *Every idempotent in a ZI ring is central.*

Proof. Let $e \in I(R)$ and let $x \in R$. Now $e(1-e) = 0$. Since R is ZI we have $ex(1-e) = 0$ and so $ex = exe$.
Again $(1-e)e = 0$ gives $(1-e)xe = 0$ and therefore $xe = exe = ex$. Thus e is central □

3.5.16 Lemma. *Following statements are equivalent. (See [10], p-2088, Lemma 1)*
i) R is a ZI ring
ii)For each $a \in R, l(a)($ equivalently $r(a))$ is a two-sided ideal of R.

Proof. i)⇒ii). Let $x \in l(a)$ and $r \in R$ then $xa = 0$. Since R is a ZI-ring, $xra = 0$ and so $xr \in l(a)$. So $l(a)$ is a right ideal of R and hence an ideal of R.
ii)⇒i). Let $a, b \in R$ such that $ab = 0$. This implies that $a \in l(b) \Rightarrow ar \in l(b) \forall$ $r \in R$, which implies that $aRb = 0$. Hence R is ZI. □

3.5.17 Lemma. *If R is a ZI ring then $RaR + r(a)$ is an essential left ideal of R $\forall a \in R$. (Refer [10], p-2088, Lemma 2)*

Proof. Let $a \in R$. Suppose $(RaR + r(a)) \cap I = 0$, where I is a left ideal of R. Then $aI \subseteq I \cap RaR \subseteq I \cap (RaR + r(a)) = 0$. Hence $I \subseteq r(a)$. So $I \subseteq (RaR + r(a)) \cap I = 0$. Hence $RaR + r(a)$ is essential in R. □

3.5.18 Proposition. *Let R be a ZI ring. If every simple singular left R-module is YJ-injective then R is reduced. (See [10], p-2088, Lemma 3)*

Proof. Suppose there exists $0 \neq a \in R$ such that $a^2 = 0$. Since $l(a) \neq R$, there exists a maximal left ideal M of R containing $l(a)$.

Claim :-M is an essential left ideal of R.

If not then M is a direct summand of R. So $M = Rf$, where $f \in I(R)$. Therefore $M = l(1 - f) = l(e)$ (say) where $e = 1 - f \in I(R)$ and $e \neq 0$ otherwise $M = R$, a contradiction. Now $a \in M$ so $ae = 0$. Since R is ZI, every idempotent is central. Hence $ea = 0$. Thus $e \in l(a) \subseteq M = l(e)$. This implies that $ee = 0$ i,e $e = 0$, which is absurd. Hence the claim.

Thus R/M is simple singular and hence YJ-injective. So any left R-homomorphism from Ra to R/M extends to one of R into R/M. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r + M$, $\forall r \in R$. Then f is well-defined. For if $ra = 0$ then $r \in l(a) \subseteq M \Rightarrow r + M = M$ and clearly f is a left R-homomorphism. Since R/M is YJ-injective, there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Hence $1 + M = f(a) = g(a) = ag(1) = a(c + M) = ac + M$, for some $c \in R$. Now $a \in l(a)$. Since $l(a)$ is a two-sided ideal of R so $ac \in l(a) \subseteq M$. This shows that $1 \in M$, a contradiction. Therefore $a = 0$ and so R is reduced. □

3.5.19 Theorem. *Let R be a ZI ring. If every simple singular left R-module is YJ-injective then R is reduced weakly regular. (See [10], p-2089, Theorem 4)*

Proof. We will show that $RaR + l(a) = R$, $\forall a \in R$. Suppose not then there exists $b \in R$ such that $RbR + l(b) \neq R$. Then there exists a maximal left ideal M of R containing $RbR + l(b)$. Following the same proof as in the previous proposition, we can show that M is essential in R. So R/M is YJ-injective. Therefore there exists a positive integer n such that any R-homomorphism of Rb^n into R/M extends to one of R into R/M.
Let $f : Rb^n \longrightarrow R/M$ be defined by $f(rb^n) = r + M$. Then f is well-defined. For if $rb^n = 0$ then $r \in l(b^n) = l(b)$ (since ,by the previous proposition ,R is reduced). So $r \in l(b) \subseteq RbR + l(b) \subseteq M$. Therefore $r + M = M$. Clearly f is a left R-homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Hence
$1 + M = f(b^n) = g(b^n) = b^n g(1) = b^n(c + M) = b^n c + M$ for some $c \in R$
Now $b \in RbR + l(b)$. By the previous lemma, $l(b)$ is two-sided. So $b^n c \in RbR + l(b) \subseteq M$. Hence $1 \in M$, which is absurd. So $RaR + l(a) = R \; \forall a \in R$. So R is a left weakly regular ring. Now by the previous proposition, R is reduced so R is weakly regular. □

3.5.20 Lemma. *Every ZC ring R is a ZI ring.*

Proof. Let $a, b \in R$ such that $ab = 0$. Since R is $ZC, ba = 0 \Rightarrow bar = 0$, $\forall r \in R$. Again since R is $ZC, arb = 0 \Rightarrow aRb = 0$. Hence R is ZI. □

3.5.21 Corollary. *Let R be a ZC ring. If every simple singular left R-module is YJ-injective then R is a reduced weakly regular ring. In particular,* *if R is a reduced ring whose simple singular left R-modules are YJ-injective then R is a weakly regular ring. (See [10], p-2090, Corollary 5)*

3.5.22 Lemma. *If R is a semiprime SLB ring then R is reduced. (Refer [10], p-2092, Lemma 13)*

Proof. Let $0 \neq a \in R$ such that $a^2 = 0$. Then Ra is a left ideal of R. Since R is SLB, there exists a nonzero two-sided ideal I of R contained in Ra.
Claim:-$I \cap r(a) \neq 0$
If $aI = 0$ then $I \subseteq r(a)$. So $I \cap r(a) = I \neq 0$. Suppose $aI \neq 0$ then since $a \in r(a) \Rightarrow aI \subseteq r(a)$. So $aI \subseteq I \cap r(a)$. Hence the claim.
Now $(I \cap r(a))^2 = (I \cap r(a))(I \cap r(a)) \subseteq Ir(a) \subseteq Rar(a) = 0$ Since R is semi-prime, $I \cap r(a) = 0$, contradicting the claim. Hence R is reduced. □

3.5.23 Theorem. *Let R be a SLB ring. If every simple singular left R-module is YJ-injective then R is reduced weakly regular. (See [10], p-2092, Theorem 14)*

Proof. It is enough to show that R is semi-prime. Suppose there exists a nonzero left ideal U of R such that $U^2 = 0$. Then there exists a nonzero $a \in U$ such that $a^2 = 0$. So $a \in l(a) \Rightarrow Ra \subseteq l(a)$
Claim:-$l(a)$ is an essential left ideal of R.
Suppose not then there exists a left ideal K of R such that $l(a) \oplus K$ is left essential in R. Since R is SLB, there exists a nonzero ideal I of R contained in K. Now $Ia \subseteq Ra \cap I \subseteq l(a) \cap K = 0$. This implies that $I \subseteq l(a) \cap K = 0$, a contradiction. Hence the claim.
Since $a \neq 0$, we have $l(a) \neq R$. So there exists a maximal left ideal M of R containing $l(a)$. So M is essential in R. Hence R/M, being simple

singular, is YJ-injective. Therefore any left R-homomorphism from Ra to R/M extends to one of R into R/M. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r + M$, $\forall r \in R$. Then f is well-defined. For if $ra = 0$ then $r \in l(a) \subseteq M \Rightarrow r + M = M$ and clearly f is an R-homomorphism. Since R/M is YJ-injective, there exists a left R-homomorphism $g : R \longrightarrow R/M$ extending f. Hence

$1 + M = f(a) = g(a) = ag(1) = a(c + M) = ac + M$, for some $c \in R$. Now $aca \in aRa \subseteq U^2 = 0$. Hence $ac \in l(a) \subseteq M$ and so $1 \in M$, which is a contradiction. Therefore R must be semiprime, hence R is reduced weakly regular. □

3.5.24 Lemma. *If R is a 2-primal ring then $R/P(R)$ is reduced. (See [10], p-2093, Proposition 15)*

Proof. Suppose there exists $\overline{0} \neq \overline{a} \in R/P(R)$ such that $(\overline{a})^2 = \overline{0}$. This will imply that $a^2 \in P(R)$. Since $P(R)$ coincides with the set of all nilpotent elements of R, we have $a^{2n} = 0$ for some positive integer n. Again since $\overline{a} \neq \overline{0}$ so $a \notin P(R)$ and therefore there does not exist any positive integer m such that $a^m = 0$, which is a contradiction since $a^{2n} = 0$. Hence $R/P(R)$ is reduced. □

3.5.25 Proposition. *Let R be a 2-primal ring. If every simple singular left R-module is YJ-injective then $R/P(R)$ is weakly regular. (See [10], p-2093, Proposition 15)*

Proof. Let $\overline{0} \neq \overline{a} \in \overline{R} = R/P(R)$. We will show that $L = \overline{R}\overline{a}\overline{R} + l(\overline{a}) = \overline{R}$. Suppose not then there exists a maximal left ideal M of R such that $M/P(R)$ contains L. Since $\overline{R}$ is reduced, we have $l(\overline{a}) = r(\overline{a})$ for any $\overline{a} \in \overline{R}$. Then by the previous lemma, L is an essential left ideal of R. So $M/P(R)$ must be a left essential ideal of $\overline{R}$. Therefore R/M is simple singular and hence YJ-injective. So there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism of Ra^n into R/M extends to one of R into R/M. Define $f : Ra^n \longrightarrow R/M$ by $f(ra^n) = r + M, \forall r \in R$. Then f is well-defined. For if $ra^n = 0$ then $ra^n \in P(R) \Rightarrow ra^n + P(R) = P(R) \Rightarrow (r+P(R))(a^n+P(R)) = P(R) \Rightarrow \overline{r} \in l(\overline{a}^n) \subseteq M/P(R) \Rightarrow r \in M \Rightarrow r + M = M$. Clearly f is a left R-homomorphism. Since R/M is YJ-injective, there exists $c \in R$ so that $1 + M = f(a^n = a^n c + M$ and so $1 \in M$, which is a contradiction. Therefore $R/P(R)$ is left weakly regular. Since $R/P(R)$ is reduced, it is a weakly regular ring. □

3.5.26 Lemma. *Every von Neumann regular ring is left weakly continuous. (Refer [11], p-125))*

Proof. Suppose R is a von Neumann regular ring then $Rad(_R R) = 0$. Let $a \in Z(_R R)$. Since R is regular, there exists $b \in R$ such that $a = aba \Rightarrow ab = abab \Rightarrow e = ab = abab = e^2$. So $Ra = Re$. Now $Ra = Re$ is a direct summand of R. So $Ra \oplus R(1-e) = 0$. Therefore $Ra \oplus l(e) = 0$. But $Z(_R R)$ is two-sided so $ab = e \in Z(_R R)$. Hence $l(e)$ is essential in R. So $Ra = 0$ i,e $a = 0$. Hence $Z(_R R) = 0$. Again $R/Rad(_R R) \cong R/0 \cong R$. So $R/Rad(_R R)$ is regular.

Let $e + Rad(_R R)$ be an idempotent element of $R/Rad(_R R)$. Now $(e + Rad(_R R))(e + Rad(_R R)) = e + Rad(_R R) \Rightarrow (e+0)(e+0) = (e+0) \Rightarrow e^2 = e$. Hence e is an idempotent element of R. □

3.5.27 Lemma. *If $Z(_R R)$ is reduced then $Z(_R R) = 0$. (See [11], p-125, Lemma 1)*

Proof. Suppose $Z(_RR) \neq 0$ then there exists $0 \neq z \in Z(_RR)$ such that $z^2 = 0$. Since $Z(_RR)$ is reduced, $z = 0$, which is absurd. Hence $Z(_RR) = 0$. □

3.5.28 Theorem. *For a ring R, the following statements are equivalent.*
1)R is von Neumann regular.
2)R is a left weakly continuous ring whose simple left R-modules are YJ-injective. (See [11], p-125, Theorem 2)

Proof. 1)$\Rightarrow$ 2). If R is regular then every left R-module is YJ-injective and by the previous lemma, R is left weakly continuous.

2)$\Rightarrow$ 1)Suppose $Z(_RR) \neq 0$ then by the above lemma, $Z(_RR)$ is not reduced. So there exists a nonzero $a \in Z(_RR)$ such that $a^2 = 0$.
Claim:- $Z(_RR) + l(a) = R$
If not then there exists a maximal left ideal M of R such that M contains $Z(_RR) + l(a)$. Since $l(a)$ is essential in R and $l(a) \subseteq Z(_RR) + l(a) \subseteq M$ implies that M is essential in R. Hence R/M is simple singular. So R/M is YJ-injective . Therefore any left R-homomorphism $f : Ra \longrightarrow R$ extends to a left R-homomorphism $g : R \longrightarrow R/M$. Let us define f by $f(ra) = r + M \forall$ $r \in R$. Then f is well-defined. For if $ra = 0 \Rightarrow r \in l(a) \subseteq M \Rightarrow r + M = M$. clearly f is a left r-homomorphism. Hence
$1 + M = f(a) = g(a) = ag(1) = a(c + M) = ac + M$ for some $c \in R$. Now $ac \in Z(_RR) \subseteq Z(_RR) + l(a) \subseteq M$. Hence $1 \in M$, a contradiction. Therefore the claim.

So we have $1 = s + t$, for some $s \in Z(_RR)$ and some $t \in l(a)$. This implies that $a = sa + ta = sa \Rightarrow (1 - s)a = 0$. But $s \in Z(_RR) = Rad(_RR)$. So $(1 - s)$ is invertible. Hence $a = 0$, a contradiction. Hence $Z_RR) = 0$ i,e $Rad(_RR) = 0$. Since R is left weakly continuous, $R \cong R/0 = R/Rad(_RR)$ is regular. □

3.5.29 Lemma. *Let R be an idempotent reflexive ring. If $a \in R$ is not a left weakly regular element then every maximal left ideal M of R containing $RaR + l(a)$ must be essential left ideal of R. (See [11], p-126, Lemma 6)*

Proof. Assume that a is not a left weakly regular element. Then $RaR + l(a)$ is a proper left ideal of R. For if $RaR + l(a) = R$ then $1 = z + t$, for some $z \in RaR$ and for some $t \in l(a)$. So $a = za + ta = za \in RaRa$, thus showing that a is a left weakly regular element, which is a contradiction.

Now let M be a maximal left ideal of R containing $RaR + l(a)$. If M is not essential in R then M will be a direct summand of R. So $M = Re$ for some $e \in I(R)$. Let $y \in aR(1-e)$ then $y = ar(1-e)$ for some $r \in R$. So $y = ar - are \in RaR \subseteq RaR + l(a) \subseteq M$. Now $are \in M$, so $ar \in M$. Therefore $ar = se$, for some $s \in R$. Hence $y = se(1-e) = 0$. So $aR(1-e) = 0$. Since R is idempotent reflexive, we have $(1-e)Ra = 0$. So $(1-e)a = 0 \Rightarrow 1 - e \in l(a) \subseteq M \Rightarrow 1 \in M$, a contradiction. Hence M is essential in R. □

3.5.30 Proposition. *Let R be an idempotent reflexive ring. If every simple singular left R-module is YJ-injective, then for any nonzero element $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and $RaR + l(a^n) = R$. Consequently, $Rad(_RR) = 0$. (See [11], p-126, Proposition 7)*

Proof. Suppose $a \in R$ is a left weakly regular element then $a \in RaRa$. So $a = za$, for some $z \in RaR$. This implies that $(1-z)a = 0 \Rightarrow 1 - z \in l(a) \subseteq RaR + l(a)$. Now $1 = 1 - z + z$. Since $z \in RaR \subseteq RaR + l(a)$. So $1 \in RaR + l(a)$. Therefore $R = RaR + l(a)$.

Suppose a is not a left weakly regular element then, as in the previous lemma, $RaR + l(a)$ is a proper left ideal of R.

Case 1:-Suppose a is nilpotent such that $a^m \neq 0$ and $a^{m+1} = 0$. We claim that $RaR + l(a^m) = R$. If not, there exists a maximal left ideal M containing $RaR + l(a^m)$. By the previous lemma, M must be essential in R. Therefore R/M is YJ-injective. So any left R-homomorphism of Ra^m into R/M extends to one of R into R/M. Let $f : Ra^m \longrightarrow R/M$ be defined by $f(ra^m) = r + M$, $\forall r \in R$. Then f is well-defined. For if $ra^m = 0 \Rightarrow r \in l(a^m) \subseteq RaR + l(a^m) \subseteq M \Rightarrow r + M = M$. Clearly f is a left R-homomorphism. So there exists $g : R \longrightarrow R/M$ extending f. Hence $1 + M = f(a^m) = g(a^m) = a^m g(1) = a^m(c + M) = a^m c + M$, for some $c \in R$. Now $a^m c \in RaR \subseteq RaR + l(a^m) \subseteq M \Rightarrow 1 \in M$, which is absurd. Hence the claim.

Case 2:-Suppose a is not nilpotent. Let us consider the chain $RaR + l(a) \subseteq RaR + l(a^2)$.... Let $I = \cup_{i=1}^{\infty}[RaR + l(a^i)] \neq R$. If $I \neq R$ then I is contained in a maximal left ideal M of R. Again, by the previous lemma, M is essential in R. So R/M is YJ-injective. Hence there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism of Ra^n into R/M extends to one of R into R/M. By a similar way as in the previous process, we obtain a contradiction. Therefore, we have $I = R$. Since $1 \in R, 1 \in RaR + l(a^k)$, for some positive integer k. So $R = Rar + l(a^k)$.

Finally, we assume that $Rad(_R R) \neq 0$, so there exists $0 \neq a \in Rad(_R R)$. Now there exists a positive integer n such that $a^n \neq 0$ and $RaR + l(a^n) = R$. So $1 = x + t$, for some $x \in RaR$ and some $t \in l(a^n)$. This implies that $a^n = xa^n + ta^n = xa^n \Rightarrow (1 - x)a^n = 0$. Since $x \in RaR \subseteq Rad(R)$ so $(1 - x)$

is invertible. Hence $a^n = 0$, a contradiction. So $Rad(_RR) = 0$. □

3.5.31 Corollary. *Let R be a semiprime ring or an abelian ring. If every simple singular left R-module is YJ-injective, then for any nonzero element $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and $RaR+l(a^n) = R$. (See [11], p-127, Corollary 8)*

Proof. Let $x \in R$ and $e \in I(R)$ such that $xRe = 0$. This implies that $(ReRx)^2 = 0$. Suppose R is semiprime, then $ReRx = 0$ so $eRx = 0$. Now suppose R is abelian then e is central so $eRx = 0$. Thus in either case, R is idempotent reflexive. Hence, using proposition 3.5.30 we deduce the corollary. □

3.6 Some open questions

In this section, we record some questions which we encountered during our survey in the previous chapters.

3.6.1 Definition. Let R be a ring. A left R-module M is *m-injective* if for any left ideal A of R , every left R-monomorphism of A into M extends to one of R into M.

3.6.2 Definition. If every simple left R-module is m-injective then R is a left mV-ring.

Question 1:- If R is a left mV-ring, is $Rad(_RR) = 0$?

Question 2:-If R is a left mV-ring, is R von Neumann regular?

Question 3:-In a left mV-ring R, is every left ideal of R idempotent?

Question 4:-Are factor rings of left mV-rings left mV-rings?

Question 5:-Is every m-injective left R-module divisible?

Question 6:-If every divisible left R-module is m-injective, is R semi-hereditary?

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