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LETTER TO THE EDITOR

Spin glass dynamics in the spherical model

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Abstract. A phenomenological Langevin-like equation for spin glass dynamics is proposed and solved. The equation reproduces the static spherical model results for spin glass in the long time limit. The magnetisation, order parameter and on-site correlations are found to decay exponentially above the transition temperature and algebraically at and below it, as $t \to \infty$. A closed-form expression for the dynamic susceptibility valid at all temperatures is also obtained.

The dynamics of spin glass systems with infinite range random exchange interactions has been studied recently by several authors (Kinzel and Fischer 1977, Kirkpatrick and Sherrington 1978, Ma and Rudnick 1978). Kinzel and Fischer (KF) have employed the Glauber model for this purpose and have obtained results valid at and much above the spin glass transition temperature T_Q. The KF calculation has been improved upon by Kirkpatrick and Sherrington (KS) by taking into account the Onsager reaction field neglected by KF. These authors (KS) have also studied this model by Monte Carlo techniques. Ma and Rudnick (1978) (MR) have considered a time-dependent Ginzburg–Landau model (TDGL) and have obtained results for $T \leqslant T_Q$. In this Letter we propose and solve a model of spin glass dynamics which in the infinite time limit reproduces the equilibrium behaviour of the spherical model (Kosterlitz *et al* 1976). We obtain closed-form expressions for various dynamic quantities valid at all temperatures for long times. Our result for the on-site correlation agrees qualitatively with the calculations of KF, KS and MR in the appropriate limits. This qualitative agreement between different calculations, based on Glauber, TDGL and spherical models, is interesting but not altogether unexpected since these are all basically mean-field calculations.

Our spin glass system is characterised by the spherical model Hamiltonian

$$\mathscr{H} = \sum_{\lambda} (P_{\lambda}^2/2I) + \tfrac{1}{2} \sum_{\lambda} (2\mu - J_{\lambda}) S_{\lambda}^2. \qquad (1)$$

The above Hamiltonian is written in a generalised Fourier space whose basis is the set of orthonormalised eigenfunctions of a large $N \times N$ random interaction matrix. The eigenvalues J_{λ} of this matrix have the distribution (Mehta 1967)

$$P(J_{\lambda}) = (N/2\pi\sigma^2)[(2\sigma)^2 - J_{\lambda}^2]^{1/2} \qquad J_{\lambda}^2 < 4\sigma^2$$
$$= 0 \qquad \text{otherwise.} \qquad (2)$$

The quantities S_{λ} and P_{λ} are the components of a spin and its conjugate momentum in the above basis, I is the moment of inertia of spin, and μ is the usual spherical field.

The equation of motion for the variables S_λ can be obtained from the Hamiltonian (1). Over time periods long compared with the characteristic time of the fast decaying P_λ modes, S_λ obey the Langevin-like equation (Ma 1976)

$$\tau(\mathrm{d}/\mathrm{d}t)S_\lambda = -\beta(\partial\mathscr{H}/\partial S_\lambda) + \zeta_\lambda(t). \tag{3}$$

The first term on the RHS is the friction term and is the noise generated by the eliminated fast modes with the properties

$$\langle\zeta_\lambda(t)\rangle = 0$$

$$\langle\zeta_\lambda(t)\zeta_\lambda(t')\rangle = 2\tau\delta(t-t').$$

The parameter τ is a constant and $\beta = (k_\mathrm{B}T)^{-1}$. Equation (3) is the basic equation characterising the dynamics of the spherical model. It can be easily solved to give (with $\tau = 1$ for convenience)

$$\langle S_\lambda(t)\rangle = \langle S_\lambda(\infty)\rangle + a_\lambda \exp[-\beta(2\mu - J_\lambda)t] \tag{4}$$

where $a_\lambda = \langle S_\lambda(0)\rangle - \langle S_\lambda(\infty)\rangle$ and μ is given by (Kosterlitz *et al* 1976)

$$\begin{aligned} x \equiv 2\beta(\mu-\sigma) &= (1-\beta\sigma)^2 \qquad && (\beta < \sigma^{-1}) \\ &= 0 && \text{otherwise.} \end{aligned} \tag{5}$$

For the correlation one obtains

$$\langle S_\lambda(0)S_\lambda(t)\rangle = \langle S_\lambda(0)S_\lambda(\infty)\rangle + b_\lambda \exp[-\beta(2\mu - J_\lambda)t] \tag{6}$$

where

$$b_\lambda = \langle S_\lambda^2(0)\rangle - \langle S_\lambda(0)S_\lambda(\infty)\rangle.$$

It should be noted that while $\langle S_\lambda(0)\rangle$ in equation (4) is an arbitrary initial value, the quantity $\langle S_\lambda^2(0)\rangle$ in equation (6) is not. The value of $\langle S_\lambda^2(0)\rangle$ is related uniquely to the equilibrium susceptibility of the system via the fluctuation–dissipation theorem (Suzuki and Kubo 1968). We now present our results.

(i) Magnetisation. We define $M(t) = \sum_i\langle S_i(t)\rangle$. Clearly, for the interactions given by equation (2), $M(\infty) = 0$. However, if the system is given a non-zero initial magnetisation $M(0)$, it relaxes as follows:

$$M(t) = M(0)(\beta\sigma t)^{-1}I_1(2\beta\sigma t)\exp(-2\beta\mu t)$$

where I_1 denotes a modified Bessel function in the standard notation. For long times it gives

$$M(t) \sim M(0)(\beta\sigma t)^{-3/2}\exp(-xt).$$

In view of equation (5), $M(t)$ decays exponentially above $k_\mathrm{B}T_\mathrm{Q} = \sigma$ and as $t^{-3/2}$ at and below it.

(ii) Order parameter. An Edwards–Anderson-like (Edwards and Anderson 1975) order parameter may be defined as $Q^2(t) = \sum_i\langle S_i\rangle^2$. In contrast with the magnetisation, $Q^2(\infty)$ is $1 - (T/T_\mathrm{Q})$ for $T < T_\mathrm{Q}$ and zero otherwise (Kosterlitz *et al* 1976). We get

$$Q^2(t) = Q^2(\infty) + [Q^2(0) - Q^2(\infty)](2\beta\sigma t)^{-1}I_1(4\beta\sigma t)\exp(-4\beta\mu t).$$

Again for long times, $Q^2(t)$ relaxes to its equilibrium value exponentially above T_Q, and as $t^{-3/2}$ at and below it.

(iii) On-site correlations. The expression $G(t) = \sum_i \langle S_i(0)S_i(t)\rangle$ gives the on-site unequal correlation. In calculating this quantity, an account has to be taken of the fluctuation dissipation theorem which dictates that

$$\langle S_\lambda^2(0)\rangle = \beta^{-1}\chi_\lambda(\omega = 0) = [\beta(2\mu - J_\lambda)]^{-1} \tag{7}$$

where $\chi_\lambda(\omega)$ is the dynamic shattered susceptibility of the system. We obtain

$$G(t) = G(\infty) + \sum_{\lambda \neq \lambda_m} [\beta(2\mu - J_\lambda)]^{-1} \exp[-\beta(2\mu - J_\lambda)t] \tag{8}$$

where $G(\infty) = \beta^{-1}\chi_{\lambda_m}(0)$. For $T \leqslant T_Q$, the integral in equation (8) yields

$$G(t) = G(\infty) + N(\beta\sigma)^{-1}[I_0(2\beta\sigma t) + I_1(2\beta\sigma t)] \exp(-2\beta\sigma t). \tag{9}$$

For $T > T_Q$, the integral in equation (8) can be evaluated in the long time limit and gives

$$G(t) = G(\infty) + N(\beta\sigma)^{-3/2}\{(\pi t)^{-3/2} \exp(-xt) - \sqrt{x}\,\mathrm{erf}[(xt)^{1/2}]\}. \tag{10}$$

For the long time decay, we obtain from equations (9) and (10), respectively $G(t) \sim t^{-1/2}$ at and below T_Q and

$$G(t) \sim t^{-3/2} \exp(-xt) \text{ above } T_Q.$$

This behaviour agrees with the calculations of MR, KS and KF in the appropriate cases. One should note that the general result (10) has a crossover behaviour. As one approaches T_Q from above, one has to wait a longer and longer time $t \sim x^{-1} \sim (T - T_Q)^{-2}$ to see the exponential decay characteristic of the paramagnetic phase.

(iv) Dynamic susceptibility. The dynamic susceptibility $\chi(\omega)$ is obtained from the correlation function by the use of the fluctuation–dissipation theorem:

$$\begin{aligned} \chi(\omega) &= \chi'(\omega) + \mathrm{i}\chi''(\omega) \\ &= \chi(0) - \mathrm{i}\omega\beta \int_0^\infty \mathrm{d}t \exp(-\mathrm{i}\omega t)\langle S_i(0)S_i(t)\rangle. \end{aligned}$$

We obtain

$$\chi'(\omega) = (2\beta\sigma^2)^{-1}\{2\beta\mu - (A + B)^{1/2}\} \tag{11a}$$

$$\chi''(\omega) = (2\beta\sigma^2)^{-1}\{\omega - (A - B)^{1/2}\} \tag{11b}$$

where, for $T > T_Q$,

$$\mu = (2\beta)^{-1}(1 + \beta^2\sigma^2)$$

$$A = \tfrac{1}{2}[\{(1 + \beta\sigma)^4 + \omega^2\}\{(1 - \beta\sigma)^4 + \omega^2\}]^{1/2}$$

$$B = \tfrac{1}{2}\{(1 - \beta^2\sigma^2)^2 - \omega^2\}$$

and, for $T \leqslant T_Q$,

$$\mu = \sigma$$

$$A = \tfrac{1}{2}(\omega^4 + 16\beta^2\sigma^2\omega^2)^{1/2}$$

$$B = -\tfrac{1}{2}\omega^2.$$

In the high-frequency limit, equations (11a) and (11b) give

$$\chi(\omega) \sim \beta(1 + \sigma^2\beta^2)\omega^{-2} - \mathrm{i}\beta\omega^{-1} \qquad (T > T_Q)$$

and

$$\chi(\omega) \sim 2\beta^2\sigma\omega^{-2} - \mathrm{i}\omega^{-1}\beta \qquad (T \leqslant T_\mathrm{Q}).$$

In the low-frequency limit, they give

$$\chi(\omega) - \chi(0) \sim -\beta(1 - \beta^2\sigma^2)^{-2}[\omega^2 - \mathrm{i}\omega(1 - \beta^2\sigma^2)] \qquad (T > T_\mathrm{Q})$$

and

$$\chi(\omega) - \chi(0) \sim -(2\beta\sigma^2)^{-1}\omega^{1/2}\{1 + \mathrm{i}(2\beta\sigma)^{1/2}\} \qquad (T \leqslant T_\mathrm{Q}).$$

A more detailed study of the low-frequency dynamic susceptibility reveals a crossover similar to the one mentioned earlier in connection with the on-site correlations.

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