RINGS OVER WHICH SOME CLASSES OF MODULES ARE FLAT: A SURVEY

ABSTRACT

BY

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DEPARTMENT OF MATHEMATICS

SUBMITTED

IN PARTIAL FULFILMENT OF THE

REQUIREMENT OF THE DEGREE OF

MASTER OF PHILOSOPHY

IN

MATHEMATICS

TO

NORTH-EASTERN HILL UNIVERSITY

SHILLONG - 793022, INDIA

FEBRUARY, 2010

ABSTRACT

John von Neumann defined a regular ring as a ring with the property that for each $a \in R$, there exists some $b \in R$ such that $a = aba$. In order to distinguish these rings from the regular Noetherian rings of commutative algebra, non commutative ring theorists have added von Neumann's name as a modifier. Motivated by the coordinatization of projective geometry which was being reworked at that time in terms of lattices, in 1936, von Neumann introduced regular rings as an algebraic tool for studying certain lattices. Since then, regular rings are extensively studied. It is well known that the following conditions are equivalent for a ring R: (i) R is von Neumann regular; (ii) Every left (right) R-modules are flat; (iii) Every cyclic left (right) R-modules are flat. By weakening the condition (iii), a ring R is called a left SF-ring if all simple left R-modules are flat. Ramamurthy initiated the study of SF-rings and asked the question whether a left SF-ring is necessarily regular. This question is still open. However, much work is done in the positive direction by many authors over the last three and a half decades. The main purpose of this dissertation is a survey of the von Neumann regularity of SF-rings.

Now we give a brief description of the contents of each chapter of our dissertation.

The first chapter is introductory in nature. In this chapter, we have

studied basic notions like tensor product, flat modules, singular submodule of a module, socle of a module, semiprimitive ring, rings of fractions, etc.

We begin the second chapter by including few preliminary results on regular rings followed by a survey of the work done by different authors over these classes of rings. Some of the results in this chapter are:

Proposition 2.1.10. If a ring R is regular, then $M_n(R)$ is regular for each n.

Proposition 2.1.12. For a ring R, following conditions are equivalent:

(1) R is regular.

(2) Every principal right ideal of R is generated by an idempotent.

(3) Every finitely generated right ideal of R is generated by an idempotent.

Proposition 2.1.14. The following properties of a ring R are equivalent:

(1) R is a reduced regular ring.

(2) Every principal right ideal of R is generated by a central idempotent.

(3) R is regular and every right ideal of R is two sided.

(4) R is strongly regular.

Lemma 2.2.4. Following conditions are equivalent for a ring R:

(1) R is regular.

(2) Every left R-module is p-injective.

(3) Every cyclic left R-module is p-injective.

Proposition 2.4.3. Every strongly regular ring is unit regular.

Theorem 2.4.13. Every unit regular ring is an elementary divisor ring.

Proposition 2.4.15. A regular ring R is unit regular if and only if $aR + bR = R$ for some a, $b \in R$ implies there is some $t \in R$ such that $a + bt$ is a unit.

Proposition 2.5.3. An ELT right weakly regular ring is regular.

Proposition 2.5.7. Let R be a reduced ring. Then R is left weakly regular if and only if R is right weakly regular.

Proposition 2.5.8. The centre of any left (right) weakly regular ring is regular.

Proposition 2.6.4. The following conditions are equivalent for a ring R with centre C:

(1) R is regular.

(2) C is regular and R/MR is regular for each maximal ideal M of C.

(3) R_M is regular for each maximal ideal M of C.

Example 2.7.4. There exists a ring R such that
$\{\text{ideals of } R\} \subsetneq \{\text{W} - \text{ideals of } R\} \subsetneq \{\text{GW} - \text{ideals of } R\}$.

Theorem 2.8.8. The following are equivalent for a ring R:

(1) R is strongly regular.

(2) R is a left GP-V-ring whose every maximal left ideal is a GW-ideal.

(3) R is a left GP-V-ring whose every maximal right ideal is a GW-ideal.

Proposition 2.9.7. The following conditions are equivalent for a ring R:

(1) R is a strongly regular ring.

(2) R is a generalized regular ring whose left annihilator of any element is a weakly right ideal of R.

(3) R is a generalized regular ring whose right annihilator of any element is a weakly left ideal of R.

Proposition 2.9.10. A ring R is strongly regular if and only if R is a left P-V-ring and every maximal left ideal of R is a weakly right ideal of R.

In chapter three, the topics discussed are: Characterizations of flat modules in terms of exact sequences; some injective properties of flat modules and vice versa; SGPF rings. Some important results in this chapter are:

Proposition 3.1.2. The following properties of a module $_RF$ are equivalent:

(1) F is flat.

(2) Every exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow F \longrightarrow 0$ is pure.

(3) There is a pure exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow F \longrightarrow 0$ where M is a flat module.

Corollary 3.1.6. Let R be a ring and I be a left ideal of R. Then R/I is flat left R-module if and only if for all $a \in I$, there exists some $b \in I$ such that $a = ab$.

Proposition 3.2.4. Let A be a maximal right ideal of R which is two sided sided, then following are equivalent:

(1) R/A is left R-flat.

(2) R/A is right R-injective.

(3) R/A is pR-complete for each p in R

Proposition 3.2.12. Let R be a right SPI-ring. Then

(1) For each $a \in R$, there is an x in RaR such that $a = ax$.

(2) For each ideal A in R, (R/A) is left R-flat.

(3) For each maximal right ideal M of R which is two sided, R/M is right injective.

Proposition 3.3.4. Let $f : R \longrightarrow R'$ be a ring homomorphism and S be a left R'-module which is left R-flat. Then S is left R'-flat.

Theorem 3.3.6. Let R be a left SGPF ring. If every maximal left ideal of R is a GW-ideal, then $R/J(R)$ is strongly regular.

Theorem 3.3.8. Let R be a left SGPF ring. If every maximal right ideal of R is a GW-ideal, then $R/J(R)$ is strongly regular.

The fourth chapter is devoted to the study of SF-rings and their relations to closely related classes of rings like regular rings. The important results in this chapter are:

Proposition 4.1.1. Let R be a ring such that the left annihilator of any element of R is also a right ideal. If R is a right SF-ring, then R is regular.

Proposition 4.1.3. The centre of any right (left) SF-ring is von Neumann regular.

Proposition 4.1.8. Let R be a ring with centre C. Consider the following conditions:

(1) R is a left SF-ring.

(2) For each maximal ideal m of C the localization R_m ($= S^{-1}R$, where $S = C - m$) is a left SF-ring.

Then (1) implies (2). If R is finitely generated as a C-algebra, then (2) implies (1).

Proposition 4.1.17. A semilocal left SF-ring is semisimple.

Theorem 4.2.10. Let R be a left or right quasi-duo ring. Then the following are equivalent:

(1) R is a left V-ring.

(2) R is a left SPI-ring.

(3) R is a left weakly regular ring.

(4) R is a left SF-ring.

(5) R is a right V-ring.

(6) R is a right SPI-ring.

(7) R is a right weakly regular ring.

(8) R is a right SF-ring.

(9) R is a regular ring.

(10) R is a strongly regular ring.

Theorem 4.3.6. For a left SF-ring R, the following are equivalent:

(1) R is semisimple.

(2) R is left or right Noetherian.

(3) $R/J(R)$ is semisimple.

(4) R satisfies ${}^{\perp}$PACC.

(5) R satisfies $\text{PDCC}^{\perp}$.

(6) R satisfies left PACC.

Theorem 4.4.8. The following conditions are equivalent for a ring R:

(1) R is strongly regular.

(2) R is strongly left bounded left SF-ring.

(3) R is strongly right bounded left SF-ring.

(4) R is strongly left bounded right SF-ring.

(5) R is strongly right bounded right SF-ring.

Lemma 4.5.5. If R is a left or right SF-ring. If $R/Z(R_R)$ is a reduced ring, then R is strongly regular.

Theorem 4.5.6. For a ring R, the following conditions are equivalent:

(1) R is strongly regular.

(2) R is a left SF, PCLZ-ring.

(3) R is a right SF, PCLZ-ring.

(4) R is a PCLZ-ring whose every maximal left ideal is p-injective.

(5) R is a PCLZ-ring whose every maximal right ideal is p-injective.

Theorem 4.6.6. If R is a left SF-ring whose every complement left ideal is a W-ideal, then R is strongly regular.

Theorem 4.7.7. If R is a left SF-ring whose maximal essential right ideals are GW-ideals, then R is regular.

Theorem 4.8.24. The following conditions are equivalent:

(1) A ring R is strongly regular.

(2) R is a left SF-ring and $r(a)$ is a quasi-ideal for every $a \in \{x \in R : x^2 = 0\}$.

Theorem 4.9.21. The following conditions are equivalent for a ring R:

(1) R is left self-injective regular with non-zero socle.

(2) R is a right SF-ring containing a non-singular injective maximal left ideal.

(3) R contains an injective maximal left ideal which is von Neumann regular.

Theorem 4.9.25. The following conditions are equivalent for a ring R:

(1) R is strongly regular.

(2) R is a ZI-ring whose simple left modules are flat.

Proposition 4.9.26. Let R be a right SF-ring. Then any left non-zero divisor is right invertible in R; Consequently R coincides with its classical right (and left) quotient ring

Proposition 4.9.27. Let R be a right SF-ring which has a finite number of maximal right ideals whose product is contained in $J(R)$, then
$Z({}_RR) = J(R) = 0$.

Theorem 4.9.30. The following conditions are equivalent.

(1) R is a division ring.

(2) R is a left uniform ring whose simple right modules are flat.

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CERTIFICATE

I certify that the dissertation entitled "RINGS OVER WHICH SOME CLASSES OF MODULES ARE FLAT : A SURVEY" submitted by Mr. Tikaram Subedi in partial fulfilment of the requirement of the degree of Master of Philosophy in Mathematics is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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Date : February 3, 2010

NORTH-EASTERN HILL UNIVERSITY

February, 2010

DECLARATION

I, Tikaram Subedi, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

This dissertation is being submitted to the North-Eastern Hill University for the degree of Master of Philosophy in Mathematics.

Signature of the Candidate

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Signature of the Head

Signature of the Supervisor

ACKNOWLEDGEMENT

This work was carried out under the supervison of Dr. Ardeline M. Buhphang, Department of Mathematics, North-Eastern Hill University, Shillong. I sincerely convey my heartfelt gratitude to her for her excellent guidance and constant inspiration for the completion of my work. Without her guidance and persistent help, this dissertation would not have been possible.

I would like to thank Dr. P. K. Saikia, Department of Mathematics, N.E.H.U. for giving me a course as per the requirement of the M. Phil programme. I am thankful to Prof. M B Rege, Dr. A. K. Das, and Sir A T. Singh, Department of Mathematics, N.E.H.U for their help and suggestions.

I am also thankful to Prof. H. K. Mukerjee, the head of the department and all other faculty members of the Department of Mathematics, N.E.H U. for their constant encouragement.

I would like to thank Prof. Xiao Guangshi, Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, China, for sending me a paper through e-mail.

I am also grateful to all the office staff in the Department of Mathematics for their sincere co-operation.

It gives me immense pleasure to thank all the research scholars of the Department of Mathematics N.E.H.U for their encouragement and co-operation throughout the course I would like to extend my special thanks and offer my best wishes to Bharat, Bineeta, Sarujinee, Rajat and Kshitiz for their suggestions and contributions to the typing of my dissertation.

I heartily recognize the hard work of my father (Late) Shree. Tilachand Subedi and mother (Late) Smt. Saraswati Subedi towards my education. I take this opportunity to thank them for their beloved support, encouragement and blessings throughout my life. I extend my immense thanks to my brothers, sisters, sister-in-laws, nephews and nieces for their kind co-operation, support and believe in me.

I wish to avail myself of this opportunity to express a sense of gratitude to all my colleagues, relatives and friends for their constant support, encouragement and help in the best possible manner.

Also primordial to mention here is God's care, blessings and guidance in all my undertakings.

This dissertation is dedicated to my mother Smt. Saraswati Subedi.

Tikaram Subedi

PREFACE

John von Neumann defined a regular ring as a ring with the property that for each $a \in R$, there exists some $b \in R$ such that $a = aba$. In order to distinguish these rings from the regular Noetherian rings of commutative algebra, non commutative ring theorists have added von Neumann's name as a modifier. Motivated by the coordinatization of projective geometry which was being reworked at that time in terms of lattices, in 1936, von Neumann introduced regular rings as an algebraic tool for studying certain lattices. Since then, regular rings are extensively studied. It is well known that the following conditions are equivalent for a ring R: (i) R is von Neumann regular; (ii) Every left (right) R-modules are flat; (iii) Every cyclic left (right) R-modules are flat. By weakening the condition (iii), a ring R is called a left SF-ring if all simple left R-modules are flat. Ramamurthy ([16]) initiated the study of SF-rings and asked the question whether a left SF-ring is necessarily regular. This question is still open. However, much work is done in the positive direction by many authors over the last three and a half decades. The main purpose of this dissertation is a survey of the von Neumann regularity of SF-rings.

Now we give a brief description of the contents of each chapter of our dissertation.

The first chapter is introductory in nature. In this chapter, we have studied basic notions like tensor product, flat modules, singular submodule of a module, socle of a module, semiprimitive ring, rings of fractions, etc.

We begin the second chapter by including few preliminary results on regular rings followed by a survey of the work done by different authors over these classes of rings. Some of the results in this chapter are: If a ring R is regular, then $M_n(R)$ is regular for each n; A ring R is regular if and only if every principal left ideal of R is generated by an idempotent; A ring R is strongly regular if and only if R is a reduced regular ring; A ring R is regular if and only if every left R-module is p-injective if and only if every cyclic left R-module is p-injective ([5]); A ring R is regular if and only if every principal left ideal of R is the left annihilator of an element of R and every left cyclic singular R-module is p-injective if and only if every left cyclic semiprimitive R-module is p-injective ([6]); A strongly regular ring is a unit regular ring; Every unit regular ring is an elementary divisor ring ([9]); A ring R is regular if and only if R_M is regular for each maximal ideal M of C, the centre of R ([11]); An ELT fully right idempotent ring is regular ([12]); The centre of a left (right) weakly regular ring is regular ([8]); A reduced ring R is left weakly regular if and only if it is right weakly regular; A ring R is strongly regular if and only if R is a left GP-V-ring whose every maximal left (right) ideal is a GW-ideal ([14]); A ring R is strongly regular if and only if R is a generalized regular ring such that the left (right) annihilator of every element of R is a weakly right (left) ideal ([13]); A ring R is strongly regular if and only if R is a left P-V-ring whose every maximal left ideal is a weakly right ideal ([13]).

In chapter three, the topics discussed are: Characterizations of flat modules in terms of exact sequences; Some injective properties of flat modules and vice versa; SGPF rings. Some important results in this chapter are: A module

$_RF$ is flat if and only if every exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow F \longrightarrow 0$ is pure; If I is a left ideal of a ring R, then R/I is a flat left R-module if and only if for every $a \in I$, there exists some $b \in I$ such that $a = ab$; If A is a maximal right ideal of R which is two sided, then R/A is left R-flat if and only if R/A is right R-injective ([16]); If R is a left SGPF ring whose every maximal left (right) ideal is a GW-ideal, then $R/J(R)$ is strongly regular ([14]).

The fourth chapter is devoted to the study of SF-rings and their relations to closely related classes of rings like regular rings. The important results in this chapter are: A homomorphic image of a left SF-ring is left SF ([17]); A commutative left SF-ring is regular ([16]); A left (right) quasi-duo left SF-ring is strongly regular ([17]); A semilocal left SF-ring is semisimple ([17]); An MERT left SF-ring is regular ([19]); A strongly left (right) bounded left SF- ring is strongly regular ([18]); If R is a left SF-ring, then R is semisimple if and only if R is left or right Noetherian if and only if $R/J(R)$ is semisimple ([19]); In a left SF-ring, $Z(R_R) \subseteq J(R)$ ([19]); A PCLZ left (right) SF-ring is strongly regular ([20]); If R is a left SF-ring whose every complement left (right) ideals are W-ideals, then R is strongly regular ([24]); If R is a left SF-ring whose every maximal essential right ideals are GW-ideals, then R is regular ([24]); A left SF-ring R is strongly regular if and only if $r(a)$ is a quasi-ideal for every $a \in \{x \in R : x^2 = 0\}$ ([26]); R is a right SF-ring containing a non-singular injective maximal left ideal if and only if R is left self-injective regular with non-zero socle if and only if R contains an injective maximal left ideal which is von Neumann regular ([21]); A ring R is ZI left SF-ring if and only if R is strongly regular ([28]); In a left SF-ring R, any right non-zero divisor is left invertible in R ([27]); If R is a right SF-ring which has a finite number of maximal right ideals whose product is contained in $J(R)$, then $J(R) = Z(_RR) = 0$ ([27]); A ring R is a left SF, right uniform ring if and only if R is a division ring ([7]).

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Chapter 1

Preliminaries

In this chapter, we collect all the basic definitions and basic results which will be of use in the later chapters. By a ring we mean a ring with identity and by an ideal we mean a two sided ideal. Unless mentioned, a module will usually mean a left R-module which is unitary.

1.1 Tensor product and flat modules

In this section, we study some basic definitions and results related to tensor product of modules and flat modules.

Definition 1.1.1. Let R be a ring and let there be given modules L_R and ${}_RM$. Let G be an abelian group. A map $\phi : L \times M \longrightarrow G$ is *bilinear* if for all $r \in R$, x, $x' \in L$, y, $y' \in M$ we have,

$\phi(x + x', y) = \phi(x, y) + \phi(x', y)$,

$\phi(x, y + y') = \phi(x, y) + \phi(x, y')$,

$\phi(xr, y) = \phi(x, ry)$.

Definition 1.1.2. A *tensor product* of modules L_R and ${}_RM$ is an abelian group T together with a bilinear map $\tau : L \times M \longrightarrow T$ such that for every abelian group G and bilinear map $\phi : L \times M \longrightarrow G$ there exist a unique homomorphism $\alpha : T \longrightarrow G$ satisfying $\alpha\,\tau = \phi$.

Proposition 1.1.3. *The tensor product of L_R and ${}_RM$ exists.*

Proof. Let F denote the free $\mathbb{Z}$ module on the set $L \times M$, that is,
$F = \{\sum n_i(x_i, y_i) | n_i \in \mathbb{Z}, x_i \in L, y_i \in M$ and the summation is finite$\}$.
Let D be the subgroup of F generated by all the elements of the following type
$(x + x', y) - (x, y) - (x', y)$.
$(x, y + y') - (x, y) - (x, y')$.
$(xr, y) - (x, ry)$, where $x, x' \in L$, y, $y' \in M. r \in R$.
Let $T = F/D$. Define $\tau : L \times M \longrightarrow T$ by $\tau(x, y) = (x, y) + D$. Now for all $r \in R$, x, $x' \in L$, y, $y' \in M$ we have $(x + x', y) - (x, y) - (x', y) \in D$. Then $(x + x', y) + D = \{(x, y) + D\} + \{(x', y) + D\}$.
Therefore $\tau(x + x', y) = \tau(x, y) + \tau(x', y)$.
Similarly $\tau(x, y + y') = \tau(x, y) + \tau(x, y')$.
Also $(xr, y) - (x, ry) \in D$. Therefore $(xr, y) + D = (x, ry) + D$. So $\tau(xr, y) = \tau(x, ry)$. So τ is bilinear. Let $\varphi : L \times M \longrightarrow G$ be a bilinear map into an abelian group G. Define $\beta : F \longrightarrow G$ by linear extension of ϕ. Then $\beta(D) = 0$. So $D \subseteq \ker(\beta)$. Then β induces a homomorphism $\alpha : T \longrightarrow G$ defined by $\alpha\left(\sum n_i(x_i, y_i) + D\right) = \beta(\sum n_i(x_i, y_i))$. Then for all $(x, y) \in L \times M$,

$(\alpha\tau)(x, y) = \alpha(\tau(x, y)) = \alpha((x, y) + D) = \phi(x, y)$.

Also α is unique for if there exists $\alpha' : T \longrightarrow G$ such that $\alpha\tau = \alpha'\tau$, then $\alpha((x, y) + D) = \alpha(\tau(x, y)) = \alpha\tau(x, y) = \alpha'\tau(x, y) = \alpha'((x. y) + D)$. □

Proposition 1.1.4. *If (T, τ) and (T', τ') are tensor products of L_R and ${}_RM$, then there exists an isomorphism $\alpha : T \longrightarrow T'$ such that $\alpha\tau = \tau'$.*

Proof. Since τ' is bilinear and (T, τ) is tensor product, there exists a homomorphism $\alpha : T \longrightarrow T'$ such that $\alpha\tau = \tau'$. Again τ is bilinear implies there exists $\alpha' : T' \longrightarrow T$ such that $\alpha'\tau' = \tau$. Then $\alpha'\alpha\tau = \alpha'\tau' = \tau = I_T\tau$. By uniqueness it follows that $\alpha'\alpha = I_T$. Similarly $\alpha\alpha' = I_{T'}$. Hence α is an isomorphism. □

Remark 1.1.5. (i) Because of the unicity (upto isomorphism) we will speak of *the* tensor product of L and M and denote it by $L \otimes_R M$ or simply by $L \otimes M$. We also denote $(x. y) + D \in L \otimes_R M$ by $x \otimes y$.

(ii) An element of $L \otimes_R M$ looks like $\sum(x \otimes y)$ with $x \in L$. $y \in M$, where the sum is finite, satisfying the relations, $(x + x') \otimes y = x \otimes y + x' \otimes y$,

$x \otimes (y + y') = x \otimes y + x \otimes y'$,

$xr \otimes y = x \otimes ry$.

Proposition 1.1.6. *$R \otimes_R M \simeq M$ for any left R-module M.*

Proof. Define $\tau : R \times M \longrightarrow M$ by $\tau(r, x) = rx$. Then τ is bilinear. Let $\phi : R \times M \longrightarrow G$ be any bilinear map into an arbitrary abelian group G. Define $\alpha : M \longrightarrow G$ by $\alpha(x) = \phi(1, x)$. Then α is homomorphism as $\alpha(x + y) = \phi(1, x + y) = \phi(1, x) + \phi(1, y) = \alpha(x) + \alpha(y)$.

Also $\alpha\tau(r,x) = \alpha(\tau(r,x)) = \alpha(rx) = \phi(1,rx) = \phi(r,x)$.

Also $\alpha\tau = \phi$. The couple (M,τ) is thus the tensor product of R and M. □

Proposition 1.1.7. ([1], Chapter I, Proposition 8.4) *If L_i $(i \in I)$ are right R-modules and M is a left R-module, then* $\left(\bigoplus_I L_i\right) \otimes_R M \simeq \bigoplus_I (L_i \otimes_R M)$.

The following proposition is an example in section 8 in Chapter I of [1].

Proposition 1.1.8. *Let A be a right ideal of R and M is a left R-module. Then $(R/A) \otimes M \simeq M/AM$.*

Remark 1.1.9. (1) Let $\lambda : L \longrightarrow L'$ and $\mu : M \longrightarrow M'$ be homomorphisms of right R-modules respectively left R-modules. We assert that there is a induced homomorphism $\alpha : L \otimes_R M \longrightarrow L' \otimes_R M'$. Define
$\phi : L \times M \longrightarrow L' \otimes_R M'$ by $\phi(x,y) = \lambda(x) \otimes \mu(y)$. Then ϕ is bilinear map. So there exists a unique homomorphism $\alpha : L \otimes_R M \longrightarrow L' \otimes_R M'$ such that $\alpha(x \otimes y) = \lambda(x) \otimes \mu(y)$. The homomorphism α is denoted by $\lambda \otimes \mu$.
(2) It is not in general true that if $f : L \longrightarrow L'$ is a monomorphism of right R-modules and M is a left R-module, then $f \otimes 1 : L \otimes M \longrightarrow L' \otimes M$ is a monomorphism. For example, the inclusion map $f : {}_{\mathbb{Z}}\mathbb{Z} \longrightarrow {}_{\mathbb{Z}}\mathbb{Q}$ is a monomorphism but $f \otimes 1 : \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z}$ is not a monomorphism as $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$ but $\mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} = 0$ for if $\frac{m}{n} \in \mathbb{Q}$, then
$\frac{m}{n} \otimes \bar{1} = \frac{2m}{2n} \otimes \bar{1} = \frac{m}{2n} \otimes \bar{2} = \frac{m}{2n} \otimes 0 = 0.$

Definition 1.1.10. A left R-module M is *flat* if for every monomorphism $f : L \longrightarrow L'$ of right R-modules, $f \otimes 1 : L \otimes M \longrightarrow L' \otimes M$ is a monomorphism.

Example 1.1.11. Every projective module is flat. In particular, every free module is flat.

1.2 Essential submodules

In this section, some basic properties of essential submodules and essential monomorphisms are studied.

Definition 1.2.1. :- A submodule L of a left R-module M is *essential (or large)* if for any submodule K of M. $L \cap K = 0$ implies $K = 0$.

Notation 1.2.2. :- If L is an essential submodule of M, then we write $L \trianglelefteq M$ or $L \leq_e M$.

Example 1.2.3. :-(i) For any left R-module M, $M \trianglelefteq M$.
(ii) 0 is never an essential submodule of any non-zero left R-module M.
(iii) If V is a vector space, $L \trianglelefteq V$ implies $L = V$ (since every subspace of V is a direct summand of V). More generally, if M is semisimple and $L \trianglelefteq M$, then $L = M$.

Proposition 1.2.4. *All the non-zero submodules of* ${}_{\mathbb{Z}}\mathbb{Z}$ *are essential.*

Proof. Let $L = m\mathbb{Z}$, where $0 \neq m \in \mathbb{Z}$, be a non-zero submodule of $\mathbb{Z}$. Let K be a non-zero submodule of $\mathbb{Z}$. Then $K = n\mathbb{Z}$ for some $0 \neq n \in \mathbb{Z}$ and $L \cap K = [m, n]\mathbb{Z} \neq 0$. This implies that L is an essential submodule of $\mathbb{Z}$. □

Proposition 1.2.5. $\mathbb{Z}$ *is an essential submodule of* ${}_{\mathbb{Z}}\mathbb{Q}$.

Proof. Let K be a non-zero submodule of $\mathbb{Q}$ such that $\mathbb{Z} \cap K = 0$. Let $0 \neq \frac{m}{n} \in K$. Then $m = n\frac{m}{n} \in \mathbb{Z} \cap K = 0$. Therefore $\frac{m}{n} = 0$ which is a contradiction. □

Proposition 1.2.6. *Intersection of any two essential submodules is essential.*

Proof. Let L and K be two essential submodules of a left R-module M. Let N be a submodule of M such that $(L \cap K) \cap N = 0$. This implies $K \cap N = 0$, as L is essential. Again as K is essential, we get $N = 0$. Therefore $L \cap K$ is essential. □

Corollary 1.2.7. *Finite intersection of essential submodules of M is essential.*

Remark 1.2.8. Arbitrary intersection of essential submodules of a module M need not be essential.

For example, for every $n \in \mathbb{N}$, $L_n = n\mathbb{Z}$ is an essential submodule of ${}_{\mathbb{Z}}\mathbb{Z}$. But $0 = \bigcap_{n \in \mathbb{N}} L_n$ is not essential.

Proposition 1.2.9. *If $f : M \longrightarrow N$ be a left R-module homomorphism. If $L \trianglelefteq N$, then $f^{-1}(L) \trianglelefteq M$.*

Proof. Let W be a submodule of M such that $f^{-1}(L) \bigcap W = 0$. Let $0 \neq w \in W$. Then $w \notin f^{-1}(L)$. Therefore $f(w) \notin L$. Let $0 \neq w' = f(w)$. Then $Rw' \neq 0$ which implies $Rw' \bigcap L \neq 0$ (since L is essential). So there exists some $r \in R$ such that $rw' \neq 0$, $rw' \in L$, that is, $rf(w) \in L$. This implies that $f(rw) \in L$. Then $rw \in f^{-1}(L) \bigcap W = 0$. Then $rw' = rf(w) = f(rw) = f(0) = 0$, contradicting $rw' \neq 0$. This proves that $f^{-1}(L)$ is essential. □

Corollary 1.2.10. *If L is an essential submodule of a left R-module M and $a \in M$. Then $I = \{r \in r : ra \in L\}$ is an essential left ideal of R.*

Proof. Define $f : R \longrightarrow M$ by $f(r) = ra$. Then f is a left R-module homomorphism. Thus by Proposition 1.2.9, $f^{-1}(L)$ is essential, that is, I is essential. □

Proposition 1.2.11. *If $L \trianglelefteq M$ and S is a minimal submodule of M, then $S \subseteq L$.*

Proof. $L \cap S$ is a submodule of S. Since S is minimal, $L \cap S = S$ or 0. If $L \cap S = S$, then $S \subseteq L$. If $L \cap S = 0$, then $S = 0$, as L is essential. But this is a contradiction to the minimality of S. Thus $S \subseteq L$. □

Definition 1.2.12. A monomorphism $f : K \longrightarrow M$ of left (right) R-module is *essential* if $\mathrm{im}(f) \trianglelefteq M$.

Proposition 1.2.13. *For a submodule K of M the following conditions are equivalent.*

(1) $K \trianglelefteq M$.

(2) The inclusion map $\imath : K \longrightarrow M$ *is essential.*

(3) For every submodule N and for each $h \in Hom(M, N)$, $\ker(h) \cap K = 0$ *implies* $\ker(h) = 0$.

Proof. (1) implies (2) is trivial.

(1) implies (3) is trivial.

(3) implies (1):- Let L be a submodule of M such that $K \bigcap L = 0$. Let $\eta : M \longrightarrow M/L$ be the natural map. Then $\ker \eta = L$. So $\ker \eta \bigcap K = 0$. This implies $L = \ker \eta = 0$. □

Corollary 1.2.14. *A monomorphism $f : L \longrightarrow M$ is essential if and only if for all homomorphism h, hf is monic implies h is monic.*

Proof. Let hf be monic and $0 \neq x \in \ker h$. Then $Rx \neq 0$. As $f(L)$ is essential submodule of M, $Rx \bigcap f(L) \neq 0$. Let $0 \neq z \in R$ such that $zx \in f(L)$. Then $zx = f(y)$ for some $y \in L$. Then $0 = zh(x) = h(zx) = h(f(y) = hf(y)$. This implies $y = 0$, as hf is monic. Then $zx = 0$, a contradiction. So $\ker h = 0$. So h is monic.

Converse is trivial. □

Proposition 1.2.15. *Let $\alpha : B \longrightarrow C$ and $\beta : C \longrightarrow D$ be mononorphism of left R-modules. Then $\beta\alpha$ is essential if and only if α and β are essential.*

Proof. Let α and β be essential. Let $K \cap \text{im}\,(\beta\alpha) = 0$ for some submodule K of D. Let $K_1 = \beta^{-1}(K)$ and $x \in \text{im}\,(\alpha) \cap K_1$. Then $\beta(x) \in \text{im}\,(\beta\alpha) \cap K = 0$. As β is one-one, $x = 0$. This implies $\text{im}\,(\alpha) \cap K_1 = 0$. This implies that $K_1 = 0$ as $\text{im}\,(\alpha)$ is essential. Then β is one-one implies $K = 0$. Hence $\beta\alpha$ is essential.

Conversely, let $\beta\alpha$ be essential. Now $\text{im}\,(\beta) \supseteq \text{im}\,(\beta\alpha)$. So β is essential. Let $\text{im}\,(\alpha) \cap K = 0$. Then $\beta(\text{im}(\alpha) \cap K) = 0$. Since β is one-one, we get, $\beta(\text{im}\ \alpha) \cap \beta(K) = 0$ which further implies that $\text{im}(\beta\alpha) \cap \beta(K) = 0$. Then $\beta(K) = 0$. Then as β is one-one, we get $K = 0$. So α is essential. □

Proposition 1.2.16. *A submodule K of a left R-module M is essential if and only if for each $0 \neq x \in M$ there exists some $r \in R$ such that $0 \neq rx \in K$.*

Proof. Let K be an essential submodule of M. Let $0 \neq x \in M$. Then $Rx \neq 0$. So $Rx \bigcap K = 0$. This implies $0 \neq rx \in K$.

Conversely, let L be a submodule of M such that $K \bigcap L = 0$. Let $L \neq 0$ and $0 \neq x \in L$. By hypothesis, there exists $0 \neq r \in R$ such that $0 \neq rx \in K$. But $rx \in K \cap L = 0$ which is absurd. So $L = 0$ and hence K is essential. □

Proposition 1.2.17. *Suppoose that $K_1 \leqslant M_1 \leqslant M$, $K_2 \leqslant M_2 \leqslant M$ be modules and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$ if and only if $K_1 \trianglelefteq M_1$ and $K_2 \trianglelefteq M_2$.*

Proof. Suppose $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$ and K_1 is not essential in M_1. Then $(K_1 \oplus K_2) \cap L_1 = 0$ for some submodules $0 \neq L_1$ of M_1. If $K_1 \oplus K_2 \cap L_1 \neq 0$, then let $0 \neq l_1 \in K_1 \oplus K_2 \cap L_1$. Let $l_1 = k_1 + k_2$ where $k_1 \in K_1$ and $k_2 \in K_2$. Then $k_2 = l_1 - k_1 \in M_1 \cap M_2 = 0$. So $l_1 \in K_1 \cap L_1 = 0$. Then $l_1 = 0$, contradicting $l_1 \neq 0$. So $K_1 \trianglelefteq M_1$. Similarly $K_2 \trianglelefteq M_2$.

Conversely, suppose $K_1 \trianglelefteq M_1$ and $K_2 \trianglelefteq M_2$. Let $0 \neq x_1 + x_2 \in M_1 + M_2$ where $x_1 \in M_1$ and $x_2 \in M_2$. Since $K_1 \trianglelefteq M_1$, there exists $r_1 \in R$ such that $0 \neq r_1x_1 \in K_1$. If $r_1x_2 \in K_2$, then by independence,
$0 \neq r_1x_1 + r_1x_2 = r_1(x_1 + x_2) \in K_1 \oplus K_2$. If $r_1x_2 \notin K_2$ then since $K_2 \trianglelefteq M_2$ there exists some $r_2 \in R$ such that $0 \neq r_2r_1x_2 \in K_2$ and we have
$0 \neq r_2r_1x_1 + r_2r_1x_2 \in K_1 \oplus K_2$. Thus $K_1 \oplus K_2 \trianglelefteq M_1 \oplus M_2$. □

Proposition 1.2.18. *Let M be a left R-module and N a submodule of M. Then $\mathcal{F} = \{K : K \leqslant M \text{ and } K \cap N = 0\}$ has a maximal element.*

Proof. $\mathcal{F} \neq \phi$ as $0 \in \mathcal{F}$. Order $\mathcal{F}$ by inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Let $A = \bigcup_{K \in \mathcal{C}} K$. Then $A \in \mathcal{F}$ and A is an upper bound of $\mathcal{C}$. Therefore by Zorn's lemma, $\mathcal{F}$ has a maximal element. □

Definition 1.2.19. Let N be a submodule of a left R-module M. Then $\mathcal{F} = \{K : K \leqslant M \, and \, K \cap N = 0\}$ has a maximal element. A maximal element N' of $\mathcal{F}$ is an *M-complement* of N.

Proposition 1.2.20. *If N' be an M-complement of a left R-module M, then*

(1) $N \oplus N' \trianglelefteq M$ *(2)* $\frac{N \oplus N'}{N'} \trianglelefteq \frac{M}{N'}$

Proof. Suppose K is a submodule of M such that $(N \oplus N') \cap K = 0$. Then $N \cap K = 0$. Then N' is an M - complement of N implies $K \subseteq N'$. This implies $(N \oplus N') \cap K = K$. Therefore $K = 0$. This proves (1).

Let L/N' be a non-zero submodule of M/N'. Then $L \supsetneq N'$. Then N' is an M- complement of N implies $L \cap N \neq 0$. Then $(N \oplus N') \cap L \supsetneq N'$. So $\frac{L}{N'} \cap \frac{N \oplus N'}{N'} \neq 0$. This completes the proof of (2). □

1.3 Semisimple modules and rings

Here, we study some basic properties of semisimple modules and rings.

Definition 1.3.1. A left R-module M is *semisimple* if every submodule of M is a direct summand of M. A ring R is *semisimple* if ${}_RR$ is semisimple.

Example 1.3.2. (1) Every vector space over a division ring is semisimple.
(2) $M_n(D)$ is a semisimple ring for every division ring D and for every $n > 0$.

Proposition 1.3.3. *A ring R is semisimple if and only if it has no proper essential left ideals.*

Proof. Let R be a semisimple ring and A be a proper left ideal of R. Then there exists a left ideal B of R such that $A \oplus B = R$. As $A \neq R$, $B \neq 0$. So A cannot be essential.

Conversely, let R has no proper essential left ideals. Let A be a left ideal of R and B be an R-complement of A. Then $A \oplus B$ is an essential left ideal of R. This implies $A \oplus B = R$, as R has no proper essential left ideal. Therefore R is semisimple. □

Proposition 1.3.4. *Let R be a ring such that every simple left R-module is projective. Then R is semisimple.*

Proof. Let A be a left ideal of R. By Zorn's lemma, the set $\mathcal{F}=\{B \leq R : A \cap B = 0\}$ has a maximal element B_0. If $A \oplus B_0 \neq R$ there exists a maximal left ideal L of R containing $A \oplus B_0$. Then R/L is projective. Thus the sequence $0 \longrightarrow L \longrightarrow R \longrightarrow R/L \longrightarrow 0$ splits. Then there exists a left ideal K of R such that $L \oplus K = R$. Let $A \oplus B_0 \neq L$. Since L is maximal, $K \neq 0$. Therefore there exists some $0 \neq z \in K$. Then $L \cap K = 0$ implies $L \cap Rz = 0$. But this implies $A \oplus B_0 \cap Rz = 0$ as $A \oplus B_0 \subseteq L$. Let $y \in A \cap (B_0 \oplus Rz)$. Then $y = b_0 + rz$ for some $b_0 \in B, r \in R$. Then $rz = y - b_0 \in (A \oplus B_0) \cap Rz = 0$. Then $y = b_0 \in A \cap B_0 = 0$. Then $B_0 + Rz \in \mathcal{F}$. But $B_0 \subsetneq B_0 + Rz$. This contradicts that B_0 is a maximal element of $\mathcal{F}$. Thus $A \oplus B_0 = L$. Therefore $A \oplus B_0 \oplus K = R$. Therefore R is semisimple. □

1.4 Injective envelope

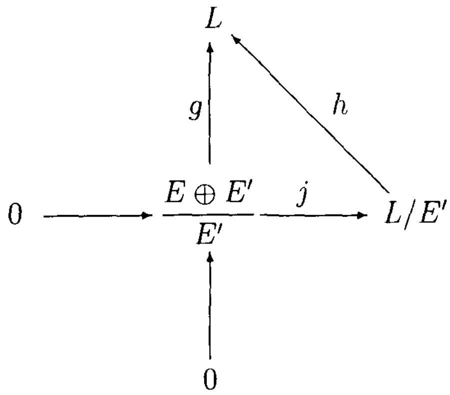
In this section, an injective envelope of a module is defined and its existence and uniqueness is proved.

Definition 1.4.1. Let M be a left R-module, E be an injective left R-module and $i : M \longrightarrow E$ is an essential monomorphism. Then (E, i) is an *injective envelope* of M.

Example 1.4.2. $({}_{\mathbb{Z}}\mathbb{Q}, i)$ is an injective envelope of ${}_{\mathbb{Z}}\mathbb{Z}$ where i is the inclusion map.

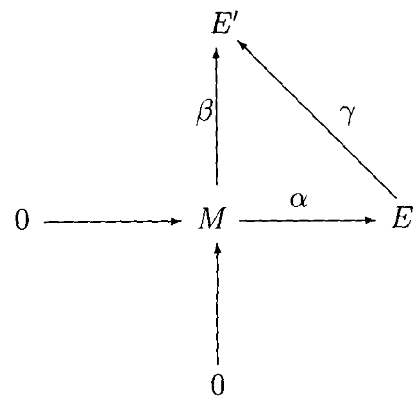
Proposition 1.4.3. *Every module has an injective envelope. It is unique to within isomorphism.*

Proof. Let M be a left R-module. Then there exists an injective left R-module L such that $M \subseteq L$. Let $\mathcal{F}=\{N \leq L : M \trianglelefteq N\}$. Order $\mathcal{F}$ by inclusion. Then $\mathcal{F}$ being a non empty partial ordered inductive set has a maximal element E. Let E' be an L-complement of E. Then $\frac{E \oplus E'}{E'} \trianglelefteq \frac{L}{E'}$. Define $g : \frac{E \oplus E'}{E'} \longrightarrow E$ by $g(x) = e$ if $x = (e+e'+E')$, $e \in E$. $e' \in E'$. Then g is an isomorphism. Then using the injectivity of L we have a commutative diagram with exact rows and column as follows:



Since $g\left(\dfrac{E \oplus E'}{E'}\right) = E \trianglelefteq L$ and $hj = g$ it follows that h is essential. So $M \trianglelefteq E = \operatorname{im} g = h\left(\dfrac{E \ominus E'}{E'}\right) \trianglelefteq h\left(\dfrac{L}{E'}\right)$. This implies $M \trianglelefteq h\left(\dfrac{L}{E'}\right)$. Then by maximality of E we have $E = h\left(\dfrac{L}{E'}\right)$. But this implies $\dfrac{E \oplus E'}{E'} = \dfrac{L}{E'}$, as h is monic. This implies $L = E \oplus E'$ [If $x \in L$, then $x + E' \in L/E' = \dfrac{E \oplus E'}{E'}$. So $x + E' = e + e' + E'$ for some $e \in E, e' \in E'$. Then $x - e \in E'$. Then $x = e + x - e \in E \oplus E'$]. Since a direct summand of an injective module is injective, E is injective. This proves the existence of injective envelope.

Uniqueness:- Let (E, α) and (E', β) be two injective envelope of ${}_RM$. Since E' is injective, there exists a homomorphism $\gamma : E \longrightarrow E'$ such that the following diagram is commutative.



Since β is essential, γ is essential. Since E is injective, $E = Im(\gamma) \oplus A$ for some submodule A of E'. This implies $A = 0$, as γ is essential. So $E \simeq E'$. □

1.5 Closed submodules

Here, closed submodules and independent left ideals are studied. An important characterization of a closed submodule is given.

Definition 1.5.1. A submodule L of a module M is a *closed submodule* of M if L has no proper essential extension inside M, that is, if the only solution of the form $L \leq_e K \leq M$ is $K = L$.

Example 1.5.2. (1) 0 and M are always closed submodule of any left (right) R-module M.

(2) Every direct summand of a module M is a closed submodule of M.

Proof. Let L be a direct summand of a left R-module M. Then there exist a submodule N of M such that $L \oplus N = M$. Suppose $L \leq_e K \leq M$, then

$N \cap K \leq K$. Now $L \cap (N \cap K) \subseteq L \cap N = 0$. Hence $N \cap K = 0$ (since $L \leq_e K$). This implies $K \subseteq L$. Hence $K = L$. □

Proposition 1.5.3. ([31], Proposition 1.4) *Let M be a left R-module and $L \leq M$, then following conditions are equivalent:*

(1) L is a closed submodule of M.

(2) L is an M-complement of some $N \leq M$.

(3) If K is any M-complement of L, then L is an M-complement of K.

(4) $L \leq T \leq_e M$ implies $T/L \leq_e M/L$.

Proof. (1) implies (4):- Let S/L be a submodule of M/L such that $T/L \cap S/L = 0$. Then $T \cap S = L$. Since $T \leq_e M$ we have $T \cap S \leq_e M \cap S$. [For if $A \leq M \cap S$ such that $(T \cap S) \cap A = 0$, then $(S \cap A) \cap T = 0$. This gives $S \cap A = 0$ (since $T \leq_e M$). But since $A \subseteq M \cap S \subseteq S$, so $A \cap S = A$. Hence $A = 0$]. That is, $L \leq_e S$. But since L is closed submodule of M, we have $S = L$. Hence $S/L = 0$.

(4) implies (3) :- If K is any M-complement of L, then $K \cap L = 0$. So L can be enlarged to an M-complement L' of K. Now, $(K \oplus L) \bigcap L' = L \oplus (K \cap L') = L$. Hence $\frac{K \oplus L}{L} \bigcap \frac{L'}{L} = 0$. Also we have $L \leq L \oplus K \leq_e M$. Hence by hypothesis, $\frac{L \oplus K}{L} \leq_e \frac{M}{L}$. Hence $\frac{L'}{L} = 0$, that is, $L = L'$ is an M-complement of K.

The implications (3) implies (2) is trivial

(2) implies (1) :- Suppose $L \leq_e K \leq M$. Since $(K \cap N) \cap L = L \cap N = 0$ we have $K \cap N = 0$. Then L is an M - complement of N and $N \leq K$ implies $K = L$. Hence L is a closed submodule of M. □

Definition 1.5.4. Two left ideals L and K of a ring R are *independent* if $L \cap K = 0$.

Example 1.5.5. (1) If $L = 0$, then for any left ideal K of R, L and K are independent.

(2) If $R = M_2(\mathbb{R})$,
$L = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$, $K = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$. Then L and K are independent left ideals of R.

1.6 Radical of a module

In this section, basic definitions and results related to the radical of a module or a ring are discussed. An important results in this section is that the radical of a module M is equal to the sum of all small submodules of M.

Definition 1.6.1. The *Jacobson radical* of a left R-module M is the intersection of all maximal submodules of M and is denoted by $Rad(M)$.

Definition 1.6.2. The *Jacobson radical* of a ring R denoted by $J(R)$ or *Rad(R)*, is the Jacobson radical of R considered as a left R-module.

Example 1.6.3. (1) $Rad(\mathbb{Z}) = 0$.

(2) $Rad(F) = 0$ for any field F.

(3) $Rad(M_n(F)) = 0$ for any field F and for any n.

Definition 1.6.4. The *annihilator* of a left R-module M is denoted by $ann(M)$ and is defined as $ann(M) = \{r \in R : rM = 0\}$.

Definition 1.6.5. Let M be a left (right) R-module and $m \in M$. The set $l(m) = \{r \in R : rm = 0\}$ ($r(m) = \{r \in R : mr = 0\}$) is the *left (right) annihilator* of m.

Example 1.6.6. $ann({}_{\mathbb{Z}}\mathbb{Q}) = 0$

Proposition 1.6.7. *For any left R-module M, $ann(M)$ is an ideal of R.*

Proof. Let $x, y \in ann(M)$, $r \in R$. Then $xM = 0 = yM$. Therefore $(x - y)M = 0$. Also $rxM = 0$. Thus $rx \in ann(M)$. Again $xrM \subseteq xM = 0$. Therefore $xr \in ann(M)$. Thus $ann(M)$ is an ideal of R. □

Proposition 1.6.8. *Let M be a left R-module and $m \in M$. Then $l(m)$ is a left ideal of R.*

Proof. Let $x, y \in l(m)$, $r \in R$. Then $xm = 0 = ym$. Therefore $(x - y)m = xm - ym = 0$ and $rxm = 0$. This proves that $l(m)$ is a left ideal of R. □

We can similarly prove the following

Proposition 1.6.9. *Let M be a right R-module and $m \in M$. Then $r(m)$ is a right ideal of R.*

Proposition 1.6.10. *For $y \in R$ the following conditions are equivalent.*

(1) $y \in Rad(R)$.

(2) $1 - xy$ *is left invertible for any* $x \in R$.

(3) $yS = 0$ *for any simple left R-module S.*

Proof. (1) implies (2):- Assume $y \in Rad(R)$. If for some $x \in R, 1 - xy$ is not left invertible, then there exists some maximal left ideal L of R such that $R(1 - xy) \subseteq L$. As $xy \in L$, we get $1 \in L$, a contradiction.
(2) implies (3):- Let S be a simple left R-module and assume $ys \neq 0$ for some $s \in S$. Then $Rys = S$. In particular, $s = xys$ for some $x \in R$. So $(1 - xy)s = 0$. Then $(1 - xy)$ is left invertible implies $s = 0$, a contradiction.
(3) implies (1):- For any maximal left ideal M, R/M is a simple left R-module. Therefore $y(R/M) = 0$ which implies $y \in M$. Then $y \in Rad(R)$. ☐

Corollary 1.6.11. *$Rad(R) = \bigcap ann(S)$, where S ranges over all simple left R-module. In particular, $Rad(R)$ is an ideal of R.*

Proposition 1.6.12. *For any ring R, $J(R)$ is the largest two sided ideal of R consisting of elements a such that $1 - a$ is invertible.*

Proof. If I is a two sided ideal such that $1 - a$ is invertible for every $a \in I$, then $I \subseteq J(R)$ by Proposition 1.6.10. It remains to show that $a \in J(R)$ implies $1 - a$ is invertible. By Proposition 1.6.10, there exists some $c \in R$ such that $c(1 - a) = 1$. Then $1 - c = -ca \in J(R)$. So there exists some $c' \in R$ such that $c'(1 - (1 - c)) = 1$, that is, $c'c = 1$. Hence c is invertible and so is $1 - a$. ☐

Remark 1.6.13. The characterization of $J(R)$ obtained in Proposition 1.6.12 is left right symmetric and accordingly we may conclude:

Corollary 1.6.14. *For any ring R, $J(R)$ is the intersection of all maximal right ideals of R.*

Definition 1.6.15. A ring R is *semiprimitive* if $J(R) = 0$.

Example 1.6.16. $\mathbb{Z}$ is a semiprimitive ring.

Definition 1.6.17. A submodule N of a left R-module M is *superfluous (or small)* if for every submodule L of M, $N + L = M$ implies $L = M$.

Notation 1.6.18. If N is a small submodule of M, then we write $N << M$.

Example 1.6.19. 0 is a small submodule of any left R-module M.

Proposition 1.6.20. *For any left R-module M, Rad(M) is equal to the sum of all small submodules of M.*

Proof. Let $\mathcal{F}$ be the collection of all small submodules of M and $T = \sum_{S \in \mathcal{F}} S$. Let N be a maximal submodule of M. Then for every $S \in \mathcal{F}$. $S \subseteq N$ (If not, then $S + N = M$ implies $N = M$, a contradiction). Then $T \subseteq Rad(M)$. For the reverse inclusion it suffices to show that for any $m \in Rad(M)$, Rm is a small submodule of M. Let N be any submodule of M such that $N + Rm = M$ and $m \notin N$. Then M/N is a non-zero cyclic module and hence has a maximal submodule N'/N. N' is a submodule of M containing N. Then N' is a maximal submodule of M and we must have $m \notin N'$ contradicting that $m \in Rad(M)$. Therefore $m \in N$ and $M = N + Rm = N$. □

1.7 Quasi-Duo, ELT, ERT, MELT, MERT rings

In this section, left (right) quasi-duo rings, ELT rings, MELT rings, MERT rings are defined. Also we give an example of an MELT (MERT) ring which is not left (right) quasi-duo.

Definition 1.7.1. A ring R is *left (right) quasi-duo* if every maximal left (right) ideal of R is an ideal. A ring R is *quasi-duo* if it is both left and right quasi-duo.

Example 1.7.2. (1) Every commutative ring is quasi-duo.
(2) $UT_2(\mathbb{Q})$, the ring of upper triangular matrices over $\mathbb{Q}$ is a quasi-duo ring as left ideals of this ring are
0, $I_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}$, $I_2 = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}$,
$I_3 = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Q} \right\}$, $I_4 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Q} \right\}$, R and maximal left ideals are I_3 and I_4 both of which are ideals of R. Also right ideals of this ring are
$0, I_2$, I_3, I_4, R and $K = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{Q} \right\}$ and maximal right ideals are I_3 and I_4 both of which are ideals of R.

Definition 1.7.3. A ring R is an *ELT (ERT)* ring if every essential left (right) ideal of R is an ideal.

Example 1.7.4. (1) Every commutative ring is an $ELT\,(ERT)$ rings.
(2) $M_2(\mathbb{Q})$, the ring of 2×2 matrices over $\mathbb{Q}$ is an $ELT\,(ERT)$ ring as it is semisimple and hence has no proper essential left (right) ideals.

Definition 1.7.5. A ring R is an $MELT\,(MERT)$ ring if every maximal essential left (right) ideal is an ideal.

Example 1.7.6. (1) Every commutative ring is an $MELT\,(MERT)$ ring.
(2) $M_2(\mathbb{Q})$, the ring of 2×2 matrices over $\mathbb{Q}$ is an $MELT\,(MERT)$ ring which is not left (right) quasi-duo.

1.8 Socle

Here, some basic properties of the socle of a module are studied. It is proved that: Left (right) socle of a ring R is an ideal of R; The socle of a module M is equal to the intersection of all large submodules of M. An example of a ring is given whose left socle is not equal to the right socle.

Definition 1.8.1. For any left R-module M, the *socle* of M is the sum of all simple submodules of M.

Notation 1.8.2. The socle of a module M is denoted by $soc(M)$.

Example 1.8.3. M is semisimple if and only if $soc(M) = M$.

Proposition 1.8.4. *For any ring R, $soc({}_RR)$ is an ideal of R.*

Proof. Let I be a minimal left ideal of R and $r \in R$.
Define $f : I \longrightarrow R$ by $f(a) = ar$. This implies $f(I) = Ir$ is either zero or a minimal left ideal of R, as I is simple. So in either case, $Ir \subseteq soc({}_RR)$. So $\mathrm{soc}({}_RR)$ is an ideal of R. □

Proposition 1.8.5. *For any left R-module M, $soc(M) = \bigcap\limits_{L \trianglelefteq M} L$.*

Proof. Let $H = \bigcap_{L \trianglelefteq M} L$, $T \leq M$ be simple and $L \trianglelefteq M$. Then $T \cap L = 0$ or T. If $T \cap L = 0$, then $L \trianglelefteq M$ implies $T = 0$, a contradiction. So $T \cap L = T$, that is, $T \subseteq L$. Then $\text{soc}(M) \subseteq H$. Let $N \leq H$ and N' be an $M-$ complement of N. Then $N \oplus N' \trianglelefteq M$. Then $N \leq H \leq N \oplus N'$.
Therefore $H = H \cap (N \oplus N') = N \oplus (H \cap N')$. Then H is semisimple. Thus $H \subseteq soc(M)$. Therefore $soc(M) = H$. □

Example 1.8.6. Consider $R = UT_2(\mathbb{Q})$. With the same notations as in Example 1.7.2 (2), we see that I_1 and I_2 are the only minimal left ideals of R. Therefore $soc(_RR) = I_1 + I_2 = I_4$. Also the only minimal right ideals of R are I_2 and K. Therefore $soc(R_R) = I_2 + K = I_3 \neq soc(_RR)$

1.9 Singular submodule of a module

Here, we study some basic properties of singular submodule of a module and left (right) singular ideal of of a ring and prove that left (right) singular ideal of a ring R is an ideal of R .

Proposition 1.9.1. *Let M be a left R-module. The set*
$Z(M) = \{m \in M : l(m)$ is an essential left ideal of $R\}$ is a submodule of M.

Proof. Let $x \in Z(M)$, $y \in Z(M)$, $r \in R$. Then $l(x) \trianglelefteq R$, $l(y)) \trianglelefteq R$. Let K be a left ideal of R such that $l(x - y) \cap K = 0$. Since $l(x) \cap\ l(y) \subseteq l(x - y)$, we get $l(x) \cap l(y) \cap K = 0$. Then $K = 0$. Let L be a left ideal of R such that $l(rx) \cap T = 0$. If $T \neq 0$, then $T \nsubseteq l(rx)$. Then there exists some $a \in T$ such that $arx \neq 0$. Then $ar \neq 0$ and $ar \notin l(x)$. As $l(x)$ is essential,

$Rar \cap l(x) \neq 0$. Let $0 \neq z \in RaR \cap l(x)$. Let $z = r'ar$ for some $r' \in R$. Then $(r'a)(rx) = (r'ar)x = 0$. So $r'a \in l(rx) \cap T = 0$. Then $z = 0$, a contradiction. So $T = 0$. Then $l(rx) \trianglelefteq R$. $\square$

Definition 1.9.2. The *singular submodule* of a left R-module M is the set $Z(M) = \{m \in M : l(m)$ is an essential left ideal of $R\}$.
M is *non-singular* if $Z(M) = 0$ and M is *singular* if $Z(M) = M$.

Proposition 1.9.3. *Let $f : M \longrightarrow N$ be a left R-homomorphism. Then $f(Z(M)) \leq Z(N)$.*

Proof. Let $m \in Z(M)$ and K be a left ideal of R such that $l(f(m)) \cap K = 0$. If $K \neq 0$, then there exists some $0 \neq a \in K$. Since $m \in Z(M)$, we get $l(m) \cap Ra \neq 0$. Let $0 \neq z \in l(m) \cap Ra$. Then $zm = 0$. So $f(zm) = 0$, that is, $zf(m) = 0$ implying $z \in l(f(m)) \cap K = 0$, a contradiction. So $K = 0$. Then $f(Z(M)) \subseteq Z(N)$. $\square$

Corollary 1.9.4. *For any ring R, $Z(_RR)$ is an ideal of R.*

Proof. Let $r \in R$. Define $f : R \longrightarrow R$ by $f(a) = ar$. By Proposition 1.9.3, $f(Z(_RR)) \subseteq Z(_RR)$, that is, $Z(_RR)r \subseteq Z(_RR)$. This implies that $Z(_RR)$ is an ideal of R. $\square$

Definition 1.9.5. For any ring R, $Z(_RR)$ $(Z(R_R))$ is the *left (right) singular ideal* of R. R is *left (right) non-singular* if $Z(_RR) = 0$ $(Z(R_R) = 0)$.

1.10 Nil ideals, nilpotent ideals and semiprime rings

Here, we study nil ideals, nilpotent ideals and a semiprime rings. A characterization of a semiprime ring is given. It is proved that: In a semiprime ring left socle is equal to right socle; If a semiprime ring satisfies ascending chain conditions on right annihilator ideals, then it has no non-zero left or right nil ideals; If a ring R satisfies ascending chain condition on right annihilators, then the right singular ideal is nilpotent.

Definition 1.10.1. A left ideal N of a ring R is *nilpotent* if $A^n = 0$ for some $n > 0$.

Example 1.10.2. $A = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}$ is a nilpotent left ideal of $UT_2(Q)$ as $A^2 = 0$.

Definition 1.10.3. A right (left) ideal A of a ring R is *nil* if every element of A is nilpotent.

Example 1.10.4. (1) Every nilpotent ideal is nil.
(2) If $R = \mathbb{Z}[x_1, x_2, \ldots]/(x_1^2, x_2^3, x_3^4, \ldots)$. Then the ideal generated by $\overline{x_1}$, $\overline{x_2}, \ldots$ is nil but not nilpotent.

Definition 1.10.5. A ring R is *semiprime* if it has no non-zero nilpotent left ideal.

Example 1.10.6. Any semiprimitive ring is semiprime.

Proposition 1.10.7. *The following conditions are equivalent:*

(1) R is a semiprime ring.

(2) R has no non-zero nilpotent ideals.

(3) R has no non-zero principal left ideals.

Proof. (2) implies (1):- Let $0 \neq A$ be a left ideal of R such that $A^n = 0$. Then $(AR)^n = 0$. As AR is an ideal of R and R is semiprime so $AR = 0$. Therefore $A = 0$, a contradiction.

(1) implies (3) is trivial.

(3) implies (2):- Let A be an ideal of R such that $A^n = 0$ and $a \in A$. Then $(Ra)^n R = (Ra)(Ra) \ldots (Ra)R = (RRa)(RRa) \ldots (RRa)R$
$= (RaR)(RaR) \ldots (RaR) = (RaR)^n \subseteq A^n = 0$. So $(Ra)^n = 0$. Then $Ra = 0$. Therefore $a = 0$. Thus $A = 0$ and R is a semiprime ring. □

Lemma 1.10.8. (**Brauer's Lemma**)*: Let S be a minimal left ideal in a ring R. Then we have either $S^2 = 0$ or $S = Re$ for some idempotent $e \in S$*

Proof. Assume $S^2 \neq 0$. Then, $Sa \neq 0$ for some $a \in S$. Therefore $Sa = S$. Choose $e \in S$ such that $a = ea$. Then the set $I = \{r \in S : ra = 0\}$ is a left ideal of R and $I \subsetneq S$. As S is simple, $I = 0$. On the other hand we have $e^2 - e \in S$ and $(e^2 - e)a = e^2a - ea = e(ea) - ea = ea - ea = 0$. Thus $e^2 - e \in I = 0$. Therefore $e^2 = e$. Since, $e \neq 0$ and $e \in S$, we have $0 \neq Re \subseteq S$. By S is a minimal left ideal, we have $S = Re$. □

Corollary 1.10.9. *If S is a minimal left ideal in a semiprime ring R, then $S = Re$ for some idempotent $e \in S$.*

Lemma 1.10.10. *Let R be a semiprime ring and $a \in R$. If Ra is a minimal left ideal, then aR is a minimal right ideal.*

Proof. We have to prove that if A is a non-zero left ideal of R, such that $A \subseteq aR$, then $A = aR$ It is sufficient to show that for any non-zero element $ar \in aR$, $a \in arR$. Since R is semiprime, $arsar \neq 0$ for some $s \in R$. Let $\phi : Ra \longrightarrow Ra$ be the left homomorphism defined by $\phi(x) = xrsa$. Then $\phi(a) \neq 0$. As Ra is simple, ϕ is an isomorphism. Let ψ denote the inverse of ϕ. Then $a = \psi\phi(a) = \psi(arsa) = ar\psi(sa) \in arR$. Thus aR is a minimal right ideal. □

Corollary 1.10.11. *If R is semiprime ring, then $Soc(R_R) = Soc({}_RR)$.*

Remark 1.10.12. If R is not semiprime, then the conclusion of the Lemma 1.10.10 may no longer hold. For example, consider $R = UT_2(\mathbb{Q})$, the ring of upper triangular matrices over $\mathbb{Q}$

Then $R\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q}\right\}$ is a minimal left ideal of R.

But $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Q}\right\}$ is not minimal as it contains non-zero right ideal $I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q}\right\}$. The ring R is not semiprime as $I^2 = 0$.

Proposition 1.10.13. *If R is a semiprime ring, then $soc({}_RR)$ $(soc(R_R))$ is fully left (right) idempotent.*

Proof. Sufficient to show that for all $a \in soc({}_RR), Ra = RaRa$. Now $a \in soc({}_RR)$ implies there exists some left ideal L of R such that $RaRa \oplus L =$

$soc(_RR)$. If $a \in L$, then for all $x \in R$, $axa \in L \cap RaRa = 0$ which contradicts that R is semiprime. Thus $a \in RaRa$. □

Lemma 1.10.14. *If R is a semiprime ring, satisfies ACC on right annihilator ideals, then R has no right or left nil ideals $\neq 0$.*

Proof. It is sufficient to show that R has no non-zero principal right or left nil ideals. Since Ra is a nil if and only if aR is nil it is sufficient to show that R has no non-zero principal left nil ideals. Suppose $0 \neq Ra$ is nil. Set $\mathcal{F} = \{r(x) : 0 \neq x \in Ra\}$. Then $\mathcal{F} \neq \phi$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Since R satisfies ACC on right annhilators, $\bigcup_{r(x)\in\mathcal{C}} r(x) = r(d)$ for some $r(d) \in \mathcal{C}$. Thus $r(d)$ is an upper bound of $\mathcal{C}$ and $r(d) \in \mathcal{F}$. Thus by Zorn's lemma, we see that $\mathcal{F}$ has a maximal element $r(b)$. Let $c \in R$, then $cb \in Ra$. If $cb \neq 0$, then there exists a least positive integer $k > 1$ such that $(cb)^k = 0$. Then $(cb)^{k-1} \neq 0$. Since $r(b) \subseteq r((cb)^{k-1})$, by maximality of $r(b)$ we have $r(b) = r((cb)^{k-1})$. Thus $cb \in r(cb)^{k-1} = r(b)$. Then $bcb = 0$. Since c is arbitrary, $bRb = 0$. Then $(RbR)^2 = 0$. As R is semiprime, $RbR = 0$. This implies $b = 0$ which is a contradiction. Thus $Ra = 0$. This completes the proof. □

Lemma 1.10.15. *If R satisfies ACC on right annihilators, then the right singular ideal is nilpotent.*

Proof. Let $Z = Z(R_R)$. We shall show that the ascending chain $r(Z) \subseteq r(Z^2) \subseteq \ldots$ would be strictly increasing if Z is not nilpotent. Suppose $Z^n \neq 0$ for some $n > 1$.

Set $\mathcal{F}= \{r(x) : x \in Z,\ Z^{n-1}x \neq 0\}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element $r(a)$. For each $b \in Z$, $r(b) \cap aR \neq 0$. Thus there exists some

$c \in R$ such that $ac \neq 0$, $bac = 0$. This shows that $r(ba)$ strictly contains $r(a)$. Thus by maximality of $r(a)$, we have $Z^{n-1}ba = 0$. Since $b \in Z$ is arbitrary, we have $Z^n a = 0$, so $a \in r(Z^n)$. But as $a \notin r(Z^{n-1})$ we have $r(Z^{n-1}) \subsetneq r(Z^n)$. So if $Z^n \neq 0$ for each n we get an strictly accending chain $r(Z) \subsetneq r(Z^2) \subsetneq r(Z^3) \subsetneq \ldots$ which is a contradiction to the hypothesis. □

Lemma 1.10.16. *If N is a nilpotent ideal of R, then $r(N)$ is an essential left ideal of R.*

Proof. If $N = 0$, then $r(N) = R$ is an essential left ideal. Suppose $N \neq 0$ and let m be the least positive integer such that $N^m = 0$. There exists a left ideal K of R such that $L = r(N) \oplus K$ is an essential left ideal of R. Then $N^{m-1}K \subseteq K \cap r(N) = 0$. Then $N^{m-2}K \subseteq K \cap r(N) = 0$. Finally we get $NK = 0$, that is, $K \subseteq r(N)$. This implies $K = 0$. Therefore $L = r(N)$ is an essential left ideal of R. □

1.11 The rings of fractions

Here, we study some basic definitions and results related to left or right ring of fraction, total ring of fraction of a ring R as well as prove Goldie's theorem.

Definition 1.11.1. Let R be a ring and S be a multiplicatively closed subset of R. A *right ring of fraction of R* with respect to S is a ring $R[S^{-1}]$ together with a ring homomorphism $\phi : R \longrightarrow R[S^{-1}]$ satisfying
F1: $\phi(s)$ is invertible for every $s \in S$.
F2: Every element of $R[S^{-1}]$ has the form $\phi(a)\phi(s)^{-1}$ for some $a \in R$, $s \in S$.

F3: $\phi(a) = 0$ if and only if $as = 0$ for some $s \in S$.
Similarly we define a *left ring of fraction* $[S^{-1}]R$.

Proposition 1.11.2. ([1], Chapter II, Proposition 1.1) *When $R[S^{-1}]$ exists, it has the folllowing universal property: for every ring homomorphism $\psi : R \longrightarrow R'$ such that $\psi(s)$ is invertible in R' for every $s \in S$ there exists a unique homomorphism $\sigma : R[S^{-1}] \longrightarrow R'$ such that $\sigma\phi = \psi$.*

Corollary 1.11.3. ([1], Chapter II, Corollary 1.2) *$R[S^{-1}]$ is unique upto isomorphism.*

Corollary 1.11.4. ([1], Chapter II, Corollary 1.3) *If both $R[S^{-1}]$ and $[S^{-1}]R$ exists they are naturally isomorphic.*

Proposition 1.11.5. ([1], Chapter II, Proposition 1.4) *Let S be a multiplicatively closed subset of R. $R[S^{-1}]$ exists if and only if S satisfies ·*
S1: If $s \in S$ and $a \in R$, then there exists some $b \in R$ and $t \in S$ such that $sb = at$.
S2: If $sa = 0$ with $s \in S$ and $a \in R$, then $at = 0$ for some $t \in S$.
When $R[S^{-1}]$ exists it has the form $R[S^{-1}] = R \times S/\sim$ where $\sim$ is the equivalence relation defined as $(a, s) \sim (b, t)$ if there exists $c, d \in R$ such that $ac = bd$ and $sc = td \in S$.

Definition 1.11.6. : A subset S of a ring R is a *right denominator set* if it is a multiplicatively closed set and satisfies both $S1$ and $S2$ in Proposition 1.11.5. Similarly a subset S of R is defined to be a *left denominator set.*

Example 1.11.7. If R is a commutative ring, then every multiplicatively closed subset of R is both right and left denominator set.

Proposition 1.11.8. *Assume R satisfies ACC on right annihilators. If S is a multiplicatively closed subset of R and satisfies S1, then S is a right denominator set.*

Proof. Suppose $sa = 0$ with $s \in S$, $a \in R$. There exists a positive integer n such that $r(s^n) = r(s^k)$ for all $k \geq n$. Since R satisfies S1, there exists some $b \in R$, and $t \in S$ such that $s^n b = at$. Then $s^{n+1}b = sat = 0$. So $b \in r(s^{n+1}) = r(s^n)$. Hence $at = s^n b = 0$. Thus $S2$ is satisfied and hence S is a right denominator set. □

Definition 1.11.9. Let S_{reg} denote the set of all non-zero divisors of a ring R. Then $R[{S_{reg}}^{-1}]$ is the *classical right quotient ring of R (or the total right ring of fractions of R)* and is also denoted by ${Q_{cl}}^r(R)$. Similarly we define the *classical left quotient ring (or the total left ring of fraction) ${Q_{cl}}^l(R)$* of R.

Proposition 1.11.10. *${Q_{cl}}^r(R)$ exists if and only if R satisfies the right ore condition, that is, for every a and s in R with s a non-zero divisor there exists b and t in R with t non-zero divisor such that $sb = at$.*

Proof. Trivial, since $S2$ is trivially satisfied in this case □

Definition 1.11.11. If R has no non-zero divisors, the right ore conditions states that $aR \cap sR \neq 0$ for all non-zero element a and s of R. This is the same as saying that $A \cap B \neq 0$ for all non-zero right ideals A and B of R. A ring without zero divisor and satisfying the right ore condition is a *right ore domain.*

Proposition 1.11.12. *The classical right quotient ring of a right ore domain is a skew field.*

Proof. Let $0 \neq \phi(a)\phi(s)^{-1} \in Q_{cl}{}^r(R)$ where R is a right ore domain. Then a is a non-zero divisor, so $\phi(a)^{-1}$ exists. Now $\phi(a)\phi(s)^{-1}\phi(s)\phi(a)^{-1} = 1 = \phi(s)\phi(a)^{-1}\phi(a)\phi(s)^{-1}$. This proves that $Q_{cl}{}^r(R)$ is a skew field. □

Proposition 1.11.13. *Every right Noetherian ring without zero divisor is a right ore domain.*

Proof. Let a and b be any two non-zero element of a right Noetherian ring R. Let $A_n = bR + abR + a^2bR + \ldots + a^nbR$. Since R is right Noetherian, there exists a least positive integer n such that $A_k = A_n$ for all $k \geq n$. Then $bR + abR + \cdots + a^nbR = bR + abR + \ldots + a^nbR + a^{n+1}bR$. Then $a^{n+1}b = bc_0 + abc_1 + \ldots + a^nbc_n$ for some $c_0, c_1, \ldots, c_n \in R$. Then $bc_0 = a(a^nb - bc_1 - \cdots - a^{n-1}bc_n) \neq 0$ (for if $bc_0 = 0$, then $a^nb - abc_1 - \cdots - a^{n-1}bc_n = 0$. Since $a \neq 0$. Then $A_n = A_{n-1}$. But this will contradict the choice of n). Then $bR \cap aR \neq 0$. So R is a right ore domain. □

Definition 1.11.14. A ring R is a *ring of quotients* if every non-zero divisor of R is invertible, that is, R is its own classical left and right ring of quotient.

Example 1.11.15. Every division ring is a ring of quotient.

Lemma 1.11.16. *Suppose R satisfies DCC on principal right ideals The following properties of an element a of R are equivalent:*

(1) a is invertible.

(2) a is a non-zero divisor.

(3) $r(a) = 0$.

Proof. Obviously (1) implies (2) implies (3).

(3) implies (1) :- We have $aR \supseteq a^2R \supseteq a^3R \supseteq \ldots$. Since R satisfies DCC on principal right ideals, $a^nR = a^{n+1}R$ for some $n \geq 1$. Then $a^n = a^{n+1}b$ for some $b \in R$. This implies $a^n(1 - ab) = 0$. Then $a^{n-1}(1 - ab) \in r(a) = 0$. Then $a^{n-1}(1 - ab) = 0$. Similarly $a^{n-2}(1 - ab) = 0$ and so on. Finally we get $1 - ab = 0$, that is, $ab = 1$. Then $aba = a$. This implies $a(1 - ba) = 0$. So $1 - ba = r(a) = 0$. Thus $ab = ba = 1$. That is, a is invertible. □

Definition 1.11.17. Let R be a ring of quotient and R' be a subring of R. Then R' is said to be *right order in* R if R is the clasical right quotient ring of R'.

Definition 1.11.18. A ring R is said to have a *finite right rank* if R does not contain any right ideal A of form $A = A_1 \oplus A_2 \oplus \ldots$ with $0 \neq A_i$ are right ideals for each i.

Example 1.11.19. A right Noetherian ring has finite right rank.

Proof. Let R be a right Noetherian ring. Suppose A is a right ideal of R such that $A = A_1 \oplus A_2 \oplus \ldots$ for some non-zero right ideals $A_1, A_2, \ldots$. Then $A_1 \subseteq A_1 \oplus A_2 \subseteq A_1 \oplus A_2 \oplus A_3 \subseteq \ldots$. Since R is right Noetherian, $\bigoplus_{i=1}^{n} A_i = \bigoplus_{i=1}^{n+1} A_i$ for some n. Then $A = \bigoplus_{i=1}^{n} A_i$ which is a contradiction. So a right Noetherian ring has finite right rank. □

Lemma 1.11.20. *If R has a finite right rank and $r(s) = 0$, then sR is an essential right ideal of R.*

Proof. Let A be a non-zero ideal of R such that $sR \cap A = 0$. Then $s^nA \neq 0$ for each n, since $r(s) = 0$. Also if $x \in s^nA \cap s^{n+1}A$, then $x = s^na = s^{n+1}b$

for some $a \in A, b \in A$. Then $s^n(a - sb) = 0$. This implies $a - sb = 0$, since $r(s) = 0$. Then $a = sb \in A \cap sR = 0$. Then $x = 0$. Hence we get a direct sum $A \oplus sA \oplus s^2A \oplus \ldots$ within A which contradicts that R has a finite right rank. Thus sR is essential right ideal of R □

Lemma 1.11.21. *A right ideal I is a right annihilator if and only if $I = r(l(I))$.*

Proof. Suppose $I = r(X)$ for some $X \subseteq R$. Then $X \subseteq l(I)$. Therefore $I = r(X) \supseteq r(l(I)) \supseteq I$. □

Definition 1.11.22. A ring R is a *prime ring* if there are no non-zero two sided ideal A and B such that $AB = 0$.

Example 1.11.23. (1) Any domain is a prime ring.
(2) Any simple ring is a priume ring.

Lemma 1.11.24. *Suppose R has a classical right quotient ring R'. Let S denote the set of all non-zero divisors of R. Then for each*
$q_1, q_2, \ldots, q_n \in R'$, there exists $a_1, a_2, \ldots . a_n \in R$, and $x \in S$ such that $q_i = a_i x^{-1}$.

Proof. For each q_i ($i = 1, 2, \ldots, n$) there exists $b_i \in R$ and $x_i \in S$ such that $q_i = b_i x_i^{-1}$. Since S is a right ore set, there exists $c_1, c_2, \ldots, c_n \in R$ and $x \in S$ such that $x = x_1c_1 = x_2c_2 = \cdots = x_nc_n$. Since for each i, both x and x_i are invertible in R', c_i is invertible in R' and $x^{-1} = c_i^{-1}x_i^{-1}$ Then $q_i = b_i x_i^{-1} = b_i c_i x^{-1} = a_i x^{-1}$ where $a_i = b_i c_i$. □

Theorem 1.11.25. (Goldie):- *The following properties of a ring R are equivalent:*

(1) *R is a right order in a semisimple ring*

(2) *R has finite right rank, satisfies ACC on right annihilators and is a semiprime ring*

(3) *A right ideal of R is essential if and only if it contains a non-zero divisor*

Proof (1) implies (2) - Let R' denote the semisimple right quotient ring of R If R does not have a finite right rank, it contains a right ideal of the form $A_1 \oplus A_2 \oplus$ where each A_i are non-zero right ideals Choose a non-zero element $a_i \in A_i$ for each i Since R' is semisimple, R' is right Noetherian Thus the right ideals a_1R', a_2R', cannot be all independent Therefore there exists $q_1, q_2, \quad, q_n \in R'$ such that $a_1q_1 + a_2q_2 + \quad + a_nq_n = 0$, but $a_nq_n \neq 0$ There exists $b_1, b_2, \quad, b_n, x \in R$ with x a non-zero divisor such that $q_i = b_ix^{-1}$ Then $a_1b_1 + a_2b_2 + \quad + a_nb_n = (a_1q_1 + a_2q_2 + \quad + a_nq_n)x = 0$ But $A_1, A_2, \quad, A_n$ are independent, so $a_nb_n = 0$ But $a_nb_n = a_nq_nx$ So $a_nq_n = a_nb_nx^{-1} = 0$ which is a contradiction So R has finite right rank

Let $I_1 \subseteq I_2 \subseteq$ be an ascending chain of right annihilators in R Set $J_n = l_R(I_n)$ Then $J_1 \supseteq J_2 \supseteq$ is a descending chain of left annihilators in R and each $I_n = r_R(J_n)$ by Lemma 1 11 21
In R' we have $r_{R'}(J_1) \subseteq r_{R'}(J_2) \subseteq$ Since R' is right Noetherian (since R' is semisimple) there exists a positive integer m such that $r_{R'}(J_n) = r_{R'}(J_m)$ for all $n \geq m$ Consequently $I_n = r_R(J_n) = R \cap r_{R'}(J_n) = I_m$ Thus R satisfies ACC on right annihilators

It remains to show that R is a semiprime ring Let N be any ideal of R such that $N^2 = 0$ Set $L = l_R(N)$, then L is an essential right ideal of R

Next we show that LR' is an essential right ideal of R'. Given any non-zero right ideal B of R', choose a non-zero element $b \in B$ and write $b = ax^{-1}$ for some $a, x \in R$ with $a \neq 0$ and x a non-zero divisor. Then $a = bx$ lies in $R \cap B$. Thus $R \cap B$ is a non-zero right ideal of R. Consequently $R \cap B \cap L \neq 0$ and hence $B \cap LR' \neq 0$. Thus LR' is an essential right ideal of R'. As R' is semisimple, we get $LR' = R'$. Hence there exists $y_1, y_2, \ldots . y_n \in L$ and $q_1, q_2, \ldots, q_n \in R'$ such that $y_1q_1 + y_2q_2 + \cdots + y_nq_n = 1$. There exists $a_1, a_2, \ldots, a_n, x \in R$ with x a non-zero divisor such that $q_i = a_ix^{-1}$. Then $x = (y_1q_1 + y_2q_2 + \cdots + y_nq_n)x = (y_1a_1 + y_2a_2 + \cdots + y_na_n) \in L$. Hence $xN = 0$. Since x is a non-zero divisor, $N = 0$. Thus R is a semiprime ring.
(2) implies (3):- Suppose A is a right ideal of R which contains a non-zero divisor s. Then $r(s) = 0$. Since R has finite right rank we have by Lemma 1.11.20, sR is an essential right ideal of R. So A is an essential right ideal.

Conversely, suppose A is an essential right ideal of R. We have to find a non-zero divisor in A. Since R satisfies ACC on right annihilators and A is not nil ideal by Lemma 1.10.14, there exists $a_1 \neq 0$ in A such that $r(a_1) = r(a_1^2)$. If $A \cap r(a_1) \neq 0$ we continue and choose $a_2 \in A \cap r(a_1)$ such that $a_2 \neq 0$ and $r(a_2) = r(a_2^2)$. If $A \cap r(a_1) \cap r(a_2) \neq 0$, we go and get $0 \neq a_3 \in A \cap r(a_1) \cap r(a_2)$ and so on. At each step we obtain a direct sum $a_1R \oplus a_2R \oplus \cdots \oplus a_kR$. This is proved by induction. Suppose $a_1R \oplus a_2R \oplus \cdots \oplus a_{k-1}R$ is direct and let $x \in a_kR \cap \bigoplus_{i=1}^{k-1} a_iR$. Then $x = a_kb$ for some $b \in R$. Then $a_kb = a_1b_1 + a_2b_2 + \ldots. + a_{k-1}b_{k-1}$ for some $b_1, b_2, \ldots, b_{k-1} \in R$. For every $i < k$, $a_k \in r(a_i)$ so $a_ia_k = 0$. Hence we have $a_1b_1 + a_2b_2 + \ldots. + a_{k-1}b_{k-1} = 0$. That is, $x = 0$. Thus we get a direct sum $\bigoplus_{i=1}^{k} a_iR$. But R has a finite rank so the process must stop at some stage say at k^{th} stage. Then $A \cap r(a_1) \cap r(a_2) \cap \cdots \cap r(a_k) = 0$. As A is essential , we get $r(a_1) \cap r(a_2) \cap \cdots \cap r(a_k) = 0$. Let $c = a_1 + a_2 + \cdots + a_k$. Let $x \in r(c)$. Then $a_1 x + a_2 x + \cdots + a_k x = 0$. This implies $a_i x = 0$ for every $i = 1, 2, \ldots.., k$ (Since $\bigoplus_{i=1}^{k} a_i R$ is direct). Then $x \in r(a_i)$ for every $i = 1, 2, \ldots.., k$. Thus $x \in r(a_1) \cap r(a_2) \cap \cdots \cap r(a_k) = 0$. Thus $r(c) = 0$. We also show that $l(c) = 0$. If $x \in l(c)$ and $y \in cR$. Then $y = cu$ for some $u \in R$. So $xy = xcu = 0$ (Since $x \in l(c)$). So $y \in r(x)$. So $cR \subseteq r(x)$. Since cR is essential right ideal of R, $r(x)$ is essential right ideal of R. Therefore $l(c) \subseteq Z$, where Z is right singular ideal of R. Therefore by Lemma 1.10.15, $l(c) = 0$.

(3) implies (1) : Let s be a non-zero divisor and a an arbitrary element of R. By hypothesis sR is essential right ideal. Then $I = \{c \in R \mid ac \in sR\}$ is essential right ideal. Thus I contains a non-zero divisor (by hypothesis). Thus $at = sb$ for some $b \in R$. Thus R has a classical right quotient ring R'. Remains to show that R' is semisimple. It suffices to show that R' has no proper essential right ideals. Consider an essential right ideal I of R'. We claim that $I \cap R$ is an essential right ideal of R. Given any non-zero element $b \in R$, there exists some $q \in R'$ such that bq is a non-zero element of I. Write $q = ax^{-1}$ for some a, $x \in R$ with x a non-zero divisor. Then $ba = bqx \in I \cap R$ is a non-zero element of $I \cap R$ proving our claim. By hypothesis, there exists a non-zero divisor $y \in I \cap R$. Since y is invertible in R' we conclude that $I = R'$. Thus R' has no proper essential right ideals as desired. □

Chapter 2

Von Neumann Regular Rings

These classes of rings were introduced by John von Neumann in 1936 for studying modular lattices. Since then, these rings are extensively studied. It is seen that modules over these classes of rings have some common properties. In this chapter, basically we study some properties of regular rings and their characterizations via central localizations, injectivity, flatness, weakly left (right) ideals. We also study W-ideals and GW-ideals.

2.1 Basic results

Here, we explore some basic properties of regular and strongly regular rings. Some of the important results are: If R is regular, then $M_n(R)$ is regular for each n; A ring R is regular if and only if every left (right) R-module is flat if and only if every cyclic left (right) R-module is flat; A ring R is strongly regular if and only if R is regular and left (right) duo if and only if R is regular and reduced.

Definition 2.1.1. A ring R is *regular (von Neumann)* if for every $a \in R$ there exists some $x \in R$ such that $a = axa$.

Definition 2.1.2. A ring R is *strongly regular* if for every $a \in R$ there exists some $x \in R$ such that $a = xa^2$.

Example 2.1.3. Every division ring is regular as well as strongly regular.

Proposition 2.1.4. *Arbitrary direct product of strongly regular rings is strongly regular.*

Proof. Let $\{R_i : i \in I\}$ be a family of strongly regular rings. Let $(a_i)_{i\in I} \in \prod_I R_i$. For each $i \in I$, R_i is strongly regular, so there exists some $b_i \in R_i$ such that $a_i = b_i a_i^2$. Therefore $(a_i)_{i\in I} = (b_i)_{i\in I}\left((a_i)_{i\in I}\right)^2$. Therefore $\prod_I R_i$ is strongly regular. □

Proposition 2.1.5. *If R is a regular domain, then R is a division ring*

Proof. Let $0 \neq a \in R$. Since R is regular, there exists some $x \in R$ such that $a = axa$. Hence $a(1 - xa) = 0$. Then R is a domain implies $1 - xa = 0$. Then $xa = 1$. Hence R is a division ring □

Proposition 2.1.6. *Every strongly regular ring is reduced.*

Proof. Let R be a strongly regular ring. Let $a \in R$ such that $a^2 = 0$. Since R is strongly regular, there exists some $b \in R$ such that $a = ba^2$. Then $a = ba^2 = 0$. Thus R is reduced. □

Definition 2.1.7. An element a of a ring R is *regular* if $a = axa$ for some $x \in R$.

Example 2.1.8. Every element of a von Neumann regular ring is regular.

Lemma 2.1.9 (**Mccoy's lemma**). *Let a and b be elements of a ring R such that $aba - a$ is regular, then a is regular.*

Proof. $aba - a$ is regular implies there exists some $x \in R$ such that

$$aba - a = (aba - a)x(aba - a).$$

Then

$$a(-baxab + bax + xab - x + b)a = a,$$

which implies a is regular. □

Proposition 2.1.10. *If a ring R is regular, then $M_n(R)$ is regular for each n.*

Proof. Suppose R is regular. We first prove that $M_2(R)$ is regular. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. Since R is regular, there exists some $r \in R$ such that $c = crc$. Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} arc - a & ard - b \\ 0 & crd - d \end{pmatrix}.$$

So by Mccoy's lemma, if matrices of the type $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is regular, then each

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ is regular.

Again by regularity of R, there exists some $x, y \in R$ such that $axa = a, dyd = d$. Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}-\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}=\begin{pmatrix} 0 & axb+byd-b \\ 0 & 0 \end{pmatrix}.$$

Hence by Mccoy's lemma, if matrices of the type $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is regular, then matrices of the type $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is regular. Again, there exists some $z \in R$ such that $b = bzb$. Then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

Hence matrices of the type $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is regular for each $b \in R$. Hence each $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is regular. Then $M_2(R)$ is regular.

Now we consider the case $n = 2^k$. $k \geqslant 1$. Since $M_{2^k}(R) = M_2\left(M_{2^{k-1}}(R)\right)$, by $M_2(R)$ is regular and by induction, it follows that $M_{2^k}(R)$ is regular.

Now we consider the case $n \neq 2^k$.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{nn} \end{pmatrix} \in M_n(R)$. Choose a positive integer k

such that $2^k > n$. Let $m = 2^k$.

Let $A' = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} & \vdots & \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ & & 0 & & \vdots & 0 \end{pmatrix} \in M_m(R)$. Since $M_m(R)$ is regular, there exists some $B' \in M_m(R)$ such that $A' = A'B'A'$. Let $B \in M_n(R)$ be the matrix whose entries are equal to the corresponding entries of B'. Then $ABA = A$. Therefore R is regular.

□

Proposition 2.1.11. *If $M_n(R)$ is regular for some natural number n, then R is regular.*

Proof. Let $a \in R$. Consider $A = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_n(R)$. As $M_n(R)$ is regular, there exists some $B = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix} \in M_n(R)$ such that $ABA = A$.

Then $ax_{11}a = a$. Thus R is regular. □

Proposition 2.1.12. *For a ring R, following conditions are equivalent:*

(1) R is regular.

(2) Every principal right ideal of R is generated by an idempotent.

(3) Every finitely generated right ideal of R is generated by an idempotent.

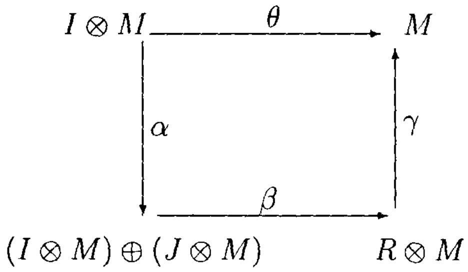
(4) Every left R-module is flat

(5) Every cyclic left R-module is flat.

Proof. (2) implies (1) :- Let $a \in R$. Then by hypothesis, $aR = eR$ for some idempotent e. Therefore $e = ax$ and $a = ey$ for some $x \in R$ and $y \in R$. Therefore $a = ey = eey = axey = axa$. So (2) implies (1).
(1) implies (2):- Let aR be a principal right ideal of R. Since R is regular, $a = axa$ for some $x \in R$. Now $ax = axax$. Then ax is an idempotent and $aR = axR$. So (1) implies (2).
(1) implies (3):- Suffices to show that if $a \in R$, $b \in R$, then $aR + bR$ is principal right ideal. There exists an idempotent e such that $aR = eR$.
We claim that $aR+bR = eR+(1-e)bR$. We have $aR = eR \subseteq eR+(1-e)bR$. Also $b = eb + (1-e)b \in eR + (1-e)bR$. Thus $aR + bR \subseteq eR + (1-e)bR$. Also $eR = aR \subseteq aR + bR$ and $(1-e)bR \subseteq bR \subseteq aR + bR$. Then $eR + (1-e)bR \subseteq aR + bR$. Therefore $aR + bR = eR + (1-e)bR$. Now R is regular implies there exists some $c \in R$ such that $(1-e)b = (1-e)bc(1-e)b$. Then $f = (1-e)bc$ is idempotent and $ef = 0$. Also $f \in (1-e)bR$ and $(1-e)b = (1-e)bc(1-e)b \in fR$ implies $fR = (1-e)bR$. Let $g = f(1-e)$. Then $g^2 = f(1-e)f(1-e) = f(1-e) = g, eg = ge = 0$. Also $g \in fR$, $f =$

$f(1-e)f \in gR$ implies $gR = fR$. We claim that $eR+gR = (e+g)R$. Clearly $(e+g)R \subseteq eR+gR$. Let $x \in eR+gR$. Then $x = er+gs$ for some $r, s \in R$. Then $x = (e+g)(er+gs) \in (e+g)R$. Thus $eR+gR = (e+g)R$. Therefore $aR+bR = eR+bR = eR+(1-e)bR = eR+fR = eR+gR = (e+g)R$.

(3) implies (4):- Let M be a left R-module and I be a finitely generated right ideal of R. Consider the map $\theta : I \otimes M \longrightarrow M$ such that $\theta(a \otimes f) = af$. Since I is generated by idempotent, I is a direct summand of R. So there exists a right ideal J of R such that $I \oplus J = R$. Now consider the following diagram



where $\alpha(a \otimes m) = a \otimes m$, $\beta(a \otimes m + b \otimes m') = am + bm'$, $\gamma(r \otimes m) = rm$, where $a \in I$, $b \in J$, $m \in M$, $m' \in M$. Then $\gamma\beta\alpha = \theta$. Again since α is monomorphism, β is isomorphism and γ is isomorphism it follows that θ is a monomorphism. Hence M is flat.

(4) implies (5) is trivial.

(5) implies (1):- Let $a \in R$ and $I = aR$, $J = Ra$. By hypothesis, R/J is flat. Therefore $\theta : I \otimes R/J :\longrightarrow R/J$ such that $\theta(a \otimes (r+J)) = ar+J$, $a \in I$, $r \in R$ is a monomorphism. Consider the composition $I/IJ \xrightarrow{\phi} I \otimes R/J \xrightarrow{\theta} R/J$ where $\phi(a + IJ) = a \otimes (1 + J)$. Then ϕ is an isomorphism. So $\theta\phi$ is a

monomorphism. Then $\ker\theta\phi = \{IJ\}$. But
$\ker\theta\phi = \{a + IJ : \theta\phi(a + IJ) = J\} = \{I \cap J\}$. Thus $I \cap J = IJ$. That is, $aR \cap Ra = (aR)(Ra) = aRa$. Then $a \in aRa$. So (5) implies (1). □

Proposition 2.1.13. *Every idempotent in a reduced ring is central.*

Proof. Let e be an idempotent element and a be any element of R. Then $\{ea(1-e)\}^2 = ea(1-e)ea(1-e) = 0$. Therefore $ea = eae$. Similarly $ae = eae$. Hence $ea = ae$ and e is central. □

Proposition 2.1.14. *The following properties of a ring R are equivalent.*

(1) R is reduced regular ring.

(2) Every principal right ideal of R is generated by a central idempotent.

(3) R is regular and every right ideal of R is two sided.

(4) R is strongly regular.

Proof. (1) implies (2) follows from Proposition 2.1.12 and Proposition 2.1.13.
(2) implies (3):- By Proposition 2.1.12, R is regular. We need only to prove that every right ideal of R is two sided. Let I be a right ideal of R and $a \in I$, $r \in R$. Then $aR = eR$ for some central idempotent e. Then $a = ex$ for some $x \in R$. This implies $ra = rex = erx \in eR = aR \subseteq I$. Thus every right ideal of R is two sided, that is, (2) implies (1).
(3) implies (4):- Let $a \in R$. Since R is regular, there exists some $x \in R$ such that $a = axa$. Also xaR is two sided ideal implies $axa = xab$ for some $b \in R$. Therefore $a = axa = xab = xaxab = xaa = xa^2$. Thus R is strongly regular,

that is, (3) implies (4).

(4) implies (1):- R is reduced by Proposition 2.1.6. Let $a \in R$. By hypothesis there exists some $x \in R$ such that $a = xa^2$. This implies $a - xa^2 = 0$. Then $(1 - xa)a = 0$. As R is reduced, we get $a(1 - xa) = 0$. Then $a = axa$. Therefore R is regular. Therefore (4) implies (1). □

Remark 2.1.15. From Proposition 2.1.14, it follows that a ring R is strongly regular if and every if for all $a \in R$ there exists some $x \in R$ such that $a = a^2x$.

Proposition 2.1.16. *The endomorphism ring of a semisimple module is regular.*

Proof. Let M be a semisimple module and $\alpha : M \longrightarrow M$ a linear map. Then $M = \ker\alpha \oplus K$ and $M = \mathrm{im}(\alpha) \oplus N$ for some submodules K and N of M. Let $\theta = \alpha|_K$. We claim that $\theta : K \longrightarrow \mathrm{im}(\alpha)$ is an isomorphism. Clearly θ is onto. Let $x \in \ker\theta$. Then $0 = \theta(x) = \alpha(x)$ which implies that $x \in \ker\alpha \cap K = 0$ and so $x = 0$. Thus θ is monomorphism and hence θ is an isomorphism proving our claim.

Let $\beta : \mathrm{im}(\alpha) \longrightarrow K$ be the inverse of θ. Then β can be extended to $\xi : M \longrightarrow M$ by putting $\xi(N) = 0$. We claim that $\alpha\xi\alpha = \alpha$

If $x \in M$, then $x = y + z$ for some $y \in \ker\alpha$, $z \in K$. So

$\alpha\xi\alpha(x) = \alpha\xi(\alpha(x)) = \alpha\xi(\alpha(y + z)) = \alpha\xi(\alpha(z)) = \alpha\xi\alpha(z) = \alpha(z) = \alpha(x)$.

This proves the claim. Then the proposition is proved. □

Proposition 2.1.17. *Every strongly regular ring is a subring of a product of skew fields.*

Proof. Let $\{M_i : i \in I\}$ be the family of maximal right ideals of a strongly regular ring R. Then each M_i is two sided ideal and R/M_i is a skew field.

Consider the ring homomorphism $\theta : R \longrightarrow \prod_I R/M_i$ defined by $\theta(r) = (r + M_i)_{i \in I}$. Now $\ker\theta = \{r \in R : (r + M_i)_{i\in I} = (M_i)_{i\in I}\} = \bigcap M_i$. If $\ker\theta \neq 0$, then there exists an idempotent $0 \neq e \in \ker\theta$. Since e belongs to every maximal right ideal, $1 - e$ cannot belong to any maximal right ideal. Therefore $(1 - e)b = 1$ for some $b \in R$. Then $e(1 - e)b = e$ implying $0 = e$ which is a contradiction. Therefore R is a subring of $\prod_I R/M_i$. □

Proposition 2.1.18. *If every element x of a ring R satisfies $x^n = x$ for some fixed $n > 1$, then R is regular.*

Proof. If $n = 2$. Then $aR = a^2R$ for all $a \in R$. Then R is regular. If $n > 2$, then for all $a \in R$, $a = a^n = a\,a^{n-2}\,a$ which implies R is regular. □

Proposition 2.1.19. *Every homomorphic image of a regular ring is regular.*

Proof. Let R be a regular ring and $f : R \longrightarrow R'$ be an onto ring homomorphism. Let $b \in R'$. Since f is onto, there exists some $a \in R$ such that $b = f(a)$. Now R is regular implies there exists some $x \in R$ such that $a = axa$. Then $f(a) = f(a)f(x)f(a)$, that is, $b = bf(x)b$. This implies R' is regular. □

Corollary 2.1.20. *Every factor ring of a regular ring is regular.*

Remark 2.1.21. (1) If R is a ring and R/I is regular for some ideal I of R, then R need not be regular. For example, $\mathbb{Z}/3\mathbb{Z}$ is regular but $\mathbb{Z}$ is not.

(2) A subring of a regular ring need not be regular. For example, $\mathbb{Q}$ is regular but $\mathbb{Z}$ is not regular.

Proposition 2.1.22. *The centre of a regular ring is regular.*

Proof. Let R be a regular ring and C be its centre. Let $a \in C$ As R is regular, there exists some $r \in R$ such that $a = ara$. We claim that $a^2r^3 \in C$ and $a = a(a^2r^3)a$. Let $x \in R$. Then
$x(a^2r^3) = x(a^2r)r^2 = (a^2r)xr^2 = (ra^2)xr^2 = r(xa^2r)r = r^2a^2xr = a^2r^3x$.
Also $a(a^2r^3)a = a(a^2r)(r^2a) = a(ar^2)a = (a^2r)(ra) = ara = a$. □

2.2 Regular rings and p-injective modules

Here, we characterize regular and strongly regular rings via p-injective modules.

Definition 2.2.1. Let M be a left R-module and I be a left ideal of R. Then M is *I-complete* if any left R - homomorphism from I to M can be extended to a left R - homomorphism from R to M. Also M is *pR-complete* or *p-injective* if M is Ra-complete for any principal left ideal Ra of R.

Example 2.2.2. For any field F, ${}_FF$ is I-complete for any left ideal I of F.

Proposition 2.2.3. ([5], Proposition 1) *For a reduced ring R, following conditions are equivalent:*

(1) R is von Neumann regular.

(2) Every principal left ideal of R is a left annihilator of some element of R.

Proof. (1) implies (2):- Let $a \in R$. Then there exists some $b \in R$ such that $a = aba$. Let $e = ba$. Then $e = ba = baba = e^2$. Clearly $Re \subseteq Ra$. Also

$Ra = Raba \subseteq Rba = Re$. This shows that $Ra = Re$. We claim that $Ra = l(1-e)$. Let $x \in Ra$. Then $x = za$ for some $z \in R$. This implies that $x(1-e) = x - xe = za - zae = za - zababa = za - za = 0$. Hence $Ra \subseteq l(1-e)$. Now if $x \in l(1-e)$, then $x - xe = 0$. Hence $x = xe = xbaba \in Ra$. Therefore $l(1-e) \subseteq Ra$. Hence the claim. So (1) implies (2).

(2) implies (1):- Let c be a non-zero divisor of R. Let $s \in R$ such that $Rc = l(s)$. Then $cs = 0$ which further implies $s = 0$, as c is a non-zero divisor. So $Rc = R$. Hence c is left invertible. Thus every non-zero divisor of R is left invertible.

Let $0 \neq a \in R$. If a is a non-zero divisor, then $a = aba$ where b is the left inverse of a. If a is not a zero divisor, let $Ra = l(b), b \in R$. Then $b \neq 0$. We claim that $c = a + b$ is a non-zero divisor.

Suppose $cy = 0$ for some $y \in R$, then $(a+b)y = 0$ which implies $ay = -by \in r(a) \cap r(b)$ (since $a \in l(b)$ implies $ab = 0$ which also implies that $ba = 0$, R being reduced). Let $w \in r(a) \cap r(b)$, then $w = za$ for some $z \in R$ (since $r(b) = l(b) = Ra$). So $aza = aw = 0$ which implies that $(za)^2 = zaza = 0$. Since R is reduced, $za = 0$. Therefore $r(a) \cap r(b) = 0$. This implies that $ay = -by$. Hence $y \in r(a) \cap r(b) = 0$. Hence the claim.

Now $ca = (a+b)a = aa + ba = a^2$. Then $a = da^2$ where d is the left inverse of c. This implies $(a-ada)^2 = (a-ada)(a-ada) = a^2 - a^2da - ada^2 + ada^2da = a^2 - a^2da - a^2 + a^2da = 0$. As R is reduced, we get $a - ada = 0$. So $a = ada$. Hence R is regular. □

Lemma 2.2.4. ([5], Lemma 2) *Following conditions are equivalent for a ring R:*

(1) R is regular.

(2) Every left R-module is $p-injective$.

(3) Every cyclic left R-module is $p-injective$.

Proof. (1) implies (2):- Let M be a left R-module, Rb be a principal left ideal of R and $f : Rb \longrightarrow M$ be a left R-module homomorphism. Since R is regular, $b = bcb$ for some $c \in R$. Let $cb = y \in M$. Then for any $a \in R$, $f(ab) = f(abcb) = abf(cb) = aby$ which implies that M is p-injective. Thus (1) implies (2).

(2) implies (3) is trivial.

(3) implies (1):- For any $b \in R$, consider the identity map $i : Rb \longrightarrow Rb$. Since Rb is p-injective, there exists $c \in Rb$ such that $i(ab) = abc$ for all $a \in R$. Then $b = i(b) = bc$. Since $c \in Rb$, $c = db$ for some $d \in R$. This shows that $b = bdb$. Thus R is regular and (3) implies (1).

□

Proposition 2.2.5. ([5], Proposition 3) *For a ring R, the following conditions are equivalent:*

(1) R is strongly regular.

(2) Every simple left R-module is $p-injective$ and every left ideal of R is two-sided.

Proof. (1) implies (2):- Follows from Lemma 2.2.4 and Proposition 2.1.14.

(2) implies (1):- Let $a \in R$. We shall show that $l(a) + Ra = R$. If not, then there exists a maximal left ideal M of R such that $l(a) + Ra \subseteq M$. Since

R/M is simple, so by hypothesis, R/M is p-injective.

Define $f : Ra \longrightarrow R/M$ by $f(xa) = a + M$. Then f is well-defined. For if $xa = x'a$, then $x - x' \in l(a) \subseteq M$ which implies that $x + M = x' + M$, that is, $f(xa) = f(x'a)$. Also f is left R-module homomorphism. So there exists a left R - homomorphism $g : R \longrightarrow R/M$ which extends f. Therefore $1 + M = f(a) = g(a) = ag(1) = a(b + M) = ab + M$ for some $b \in R$. So $1 - ab \in M$. Then $ab \in M$ (since M is two sided) implies $1 \in M$, a contradiction. Therefore $l(a) + Ra = R$. Then $xa + d = 1$, for some $x \in R$, $d \in l(a)$. Then $xa^2 + da = a$. This implies $a = xa^2$. So R is strongly regular. □

Corollary 2.2.6. ([5], Corollary 4) *If R is commutative, then R is regular if and only if every simple left R-module is $p-injective$.*

Lemma 2.2.7. ([6], Lemma 1) *Let R be a ring whose every simple singular left R-module is $p-injective$, then for every $b \in R$, there exists a left ideal K of R such that $R = (RbR + l(b)) \oplus K$.*

Proof. Let $b \in R$, then there exists a left ideal K of R such that $(RbR + l(b)) \oplus K$ is an essential left ideal of R.

Claim:- $R = (RbR + l(b)) \oplus K$. If not, then,

there exists a maximal left ideal L of R containing $(RbR + l(b)) \oplus K$.——(*)

So R/L is a simple left R-module. Therefore $Z(R/L) = L$ or R/L. If $Z(R/L) = L$, then $l(1 + L) = L$ is not an essential left ideal of R which contradicts (*). So $Z(R/L) = R/L$ implying that R/L is singular. Therefore the simple singular left R-module R/L is p-injective. Let $f : Rb \longrightarrow R/L$ be given by $f(rb) = r + L$ for all $r \in R$. Then f is well-defined. For if $rb =$

$r'b$, r, $r' \in R$, then $r - r' \in l(b) \subseteq L$. So $r + L = r' + L$, that is, $f(rb) = f(r'b)$. Clearly f is a left R-homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow R/L$ extending f. Hence $1 + L = f(b) = g(b) = bg(1) = bc + L$ for some $c \in R$. This implies $1 - bc \in L$. Therefore $bc \in RbR \subseteq L$ implies $1 \in L$ which contradicts that L is a maximal left ideal of R Hence the claim. □

The following result gives a generalisation of Proposition 2.2.5.

Theorem 2.2.8. ([6], Theorem 2) *Following conditions are equivalent for a ring R:*

(1) R is strongly regular.

(2) Every simple singular left R-module is p − injective and every left ideal of R is two-sided.

Proof. (1) implies (2) by Proposition 2.2.5.
Assume (2). Let $a \in R$. Then $R = (RaR + l(a)) \oplus K$ for some left ideal K of R which implies that $(Ra + l(a)) \oplus K = R$ [since every left ideal of R is two-sided, so $aR \subseteq Ra$ which implies that $RaR \subseteq Ra$ and so $Ra = RaR$]. Therefore $KRa \subseteq K \cap Ra = 0$ which implies $K = 0$. So $Ra + l(a) = R$. This implies that $1 = xa + y$ for some $x \in R$ and some $y \in l(a)$ and hence $xa^2 + ya = a$. As $ya = 0$, $xa^2 = a$, which proves that R is strongly regular. □

Proposition 2.2.9. ([6], Proposition 3) *If R is a ring whose simple singular left R-module is p − injective, then*

(1) $Z({}_RR) \cap J(R) = 0$.

(2) $R = RcR$ *for every non-zero divisor c of R.*

(3) Every essential left ideal of R is idempotent.

Proof. Proof of (1):- Let $x \in Z({}_RR) \cap J(R)$. Then $l(x)$ is an essential left ideal of R. By Lemma 2.2.7, there exists a left ideal K of R such that $(l(x) + RxR) \oplus K = R$. Since $l(x)$ is essential so $K = 0$. Therefore $l(x) + RxR = R$. So there exists $a \in l(x)$ and $d \in RxR$ so that $a + d = 1$. Since $d \in RxR \subseteq J(R)$, so $1 - d$ is a unit. So there exists some $v \in R$ so that $v(1-d) = 1$, that is, $va = 1$. Thus $x = vax = 0$. Hence $Z({}_RR) \cap J(R) = 0$.

Proof of (2):- Let c be a non-zero divisor. By Lemma 2.2.7, there exists a left ideal K of R such that $(l(c) + RcR) \oplus K = R$. Since c is a non-zero divisor, $l(c) = 0$ which implies that $RcR \oplus K = R$. Therefore $cK \subseteq RcR \cap K = 0$. Then c is a non-zero divisor implies $K = 0$. Therefore $RcR = R$.

Proof of (3):- Let I be an essential left ideal of R and $b \in I$. Then $IR + l(b)$ is essential left ideal of R. Suppose $IR + l(b) \neq R$, then there exists a maximal left ideal L of R containing $IR + l(b)$. Then R/L is p-injective. Define $f : Rb \longrightarrow R/L$ by $f(rb) = r + L, \forall r \in R$. Then f is well-defined. For if $rb = r'b$, $r, r' \in R$, then $r - r' \in l(b) \subseteq L$. So $r + L = r' + L$. Then $f(rb) = f(r'b)$. Clearly f is a left R-homomorphism. So there exists a left R-homomorphism $g : R \longrightarrow R/L$ extending f. Hence $1 + L = f(b) = g(b) = bg(1) = bc + L$ for some $c \in R$. This implies $1 - bc \in L$. Then $bc \in RbR \subseteq L$ implies $1 \in L$ which contradicts that L is a maximal left ideal of R. Hence $IR + l(b) = R$. So $1 = u + d$ for some $u \in IR$ and some $d \in l(b)$. This implies that $b = ub \in I^2$. So $I = I^2$. □

Proposition 2.2.10. ([6], Theorem 9) *Following conditions are equivalent for a ring R:*

(1) R is regular.

(2) Every principal left ideal of R is a left annihilator of an element of R and every cyclic singular left R-module is p-injective.

(3) Every principal right ideal of R is a right annihilator and every cyclic singular left R-module is p-injective.

(4) Every left cyclic semiprimitive R-module is p-injective.

Proof. (1) implies (2), (3) and (4) follows from Lemma 2.2.4.
Assume (2). For any $b \in R$, $Rb = l(t)$ for some $t \in R$ and there exists a left ideal K of R so that $l(t) \oplus K$ is an essential left ideal of R. Let $N = l(t) \oplus K$. Then R/N is cyclic and singular and hence is p-injective. Let
$f : Rt \longrightarrow R/N$ be defined by $f(rt) = r + N$ for all $r \in R$. Then f is well-defined. For if $rt = r't$, r, $r' \in R$, then $r - r' \in l(t) \subseteq N$. So $r + N = r' + N$. Clearly f is a left R-homomorphism. So there exists $g : R \longrightarrow R/N$ extending f. Hence $1 + N = f(t) = g(t) = tg(1) = t(c + N) = tc + N$ for some $c \in R$. So $1 - tc \in N$. Therefore $1 - tc = xb + y$ for some $x \in l(t) = Rb$ and some $y \in K$. This implies that $b - btc = bxb + by$. Then $b = bxb + by$ [since $b \in l(t))$]. Then $by = b - bxb \in l(t) \cap K = 0$. Therefore $b = bxb$ which proves that R is regular. Hence (2) implies (1).
Assume(3). For every $a \in R$, $aR = r(S)$ for some subset S of R. Now $l(a) \oplus K$ is essential in R for some left ideal K of R. Then $R/(l(a) \oplus K)$ is singular and hence is p-injective by hypothesis Define
$f : Ra \longrightarrow R/(l(a) \oplus K)$ by $f(r) = r + l(a) \oplus K$. Then f is well-defined for if $ra = r'a$, r, $r' \in R$, then $r - r' \in l(a) \subseteq l(a) \oplus K$. So $r + l(a) \oplus K =$

$r' + l(a) \oplus K$, that is, $f(ra) = f(r'a)$. Also f is left R - homomorphism. Thus there exists a left R - homomorphism $g : R \longrightarrow R/(l(a) \oplus K)$ which extends f. Then $1 + l(a) \oplus K = f(a) = g(a) = ag(1) = a(b + l(a) \oplus K)$ for some $b \in R$. Then $1 - ab \in l(a) \oplus K$. So $1 - ab = t + k$..............(1)
for some $t \in l(a)$, $k \in K$. Then for any $s \in S$, $s(1 - ab) = s(t + k)$. Then $aR = r(S)$ implies $s = st + sk$. Then $sk = s - st \in l(a) \cap K = 0$. So $Sk = 0$. Then $k \in r(S) = aR$. If $k = ac$, $c \in R$. Then from (1), $a - aba = ta + ka = ka = aca$. This implies $a = aba + aca = a(b + c)a$. Thus R is regular. Hence (3) implies (1).
Assume (4). Then every simple left R-module is $p - injective$. We prove that every principal left ideal I of R is semiprimitive. Then (4) implies (1) as in Lemma 2.2.4. For any $0 \neq b \in I$, by Zorn's lemma, the set of all subideals K of Rb such that $b \notin K$ has a maximal member M. Then Rb/M is simple p-injective. Therefore the canonical homomorphism $f : Rb \longrightarrow Rb/M$ can be extended to $g : R \longrightarrow Rb/M$. Let $h = g \mid_I$. Then $I/\ker h \simeq Rb/M$. Then $\ker h$ is a maximal subideal of I. Since $\ker h \cap Rb = \ker f$, $b \in Rb$, $b \notin \ker f$ so $b \notin \ker h$. Thus I is semiprimitive. Hence (4) implies (1). □

2.3 Self-injective regular rings

In this section, we study some properties of a self-injective ring and obtain a condition for a self-injective ring to be regular.

Definition 2.3.1. A ring R is *left (right) self-injective* if $_RR$ (R_R) is injective.

Example 2.3.2. Every semisimple ring R is left and right self-injective.

Proposition 2.3.3. *Let E be an injective left R-module and let $S = End(_RE)$. Then $J(S) = \{\alpha \in S \mid \ker\alpha \trianglelefteq E\}$.*

Proof. Let $\alpha \in S$ such that $\ker\alpha \trianglelefteq E$. Now $\ker\alpha \cap \ker(1-\alpha) = 0$. Since $\ker\alpha \trianglelefteq E$, $\ker(1-\alpha) = 0$ which implies $E \simeq \mathrm{im}(1-\alpha)$ and hence $\mathrm{im}(1-\alpha)$ is injective left R-module. Then $\mathrm{im}(1-\alpha)$ is a direct summand of E. So $E = \mathrm{im}(1-\alpha) \oplus F$ for some left R-module F. Now $\ker\alpha \subseteq \mathrm{im}(1-\alpha)$. [For if $x \in \ker\alpha$, then $\alpha(x) = 0$ which implies $x - \alpha(x) = x$ implies $x = x - \alpha(x) = (1-\alpha)$]. Therefore $\mathrm{im}(1-\alpha) \trianglelefteq E$ and hence $E = \mathrm{im}(1-\alpha)$. So $(1-\alpha)$ is bijective which implies $\alpha \in J(S)$.

Conversely, let $\alpha \in J(S)$. Let M be a submodule of E such that $\ker\alpha \cap M = 0$. We show that $M = 0$. Consider the inclusion map $j : M \longrightarrow E$. Then $\alpha j : M \longrightarrow E$. Now αj is a monomorphism [for $\alpha j(m) = 0$ implies $\alpha(j(m)) = 0$ implies $\alpha(m) = 0$ implies $m \in \ker\alpha \cap M = 0$]. Since E is injective there exist $\theta : E \longrightarrow E$ such that $\theta\alpha j = j$. Then $\theta\alpha j(m) = m$ for every $m \in M$. This implies $(\theta\alpha - 1)(m) = 0$. As $\alpha \in J(S)$, $\theta\alpha - 1$ is invertible. So $m = 0$ and therefore $M = 0$. Thus $\ker\alpha \trianglelefteq E$. □

Corollary 2.3.4. *If R is left self-injective, then*
$J(End(_RR)) = \{\alpha \in End(_RR) \mid \ker\alpha \trianglelefteq R\}$.

Corollary 2.3.5. *If R is left self-injective, then $Z(_RR) = J(R)$.*

Proof. Let $a \in Z(_RR)$. Then $l(a) \trianglelefteq R$. Consider the map $f : R \longrightarrow R$ such that $f(r) = ra$. Then f is a well-defined left R-module homomorphism and $\ker f = \{r \in R \mid ra = 0\} = l(a) \trianglelefteq R$. Then $f \in J(End(_RR))$ and so $1 - f$ is a unit, that is, $(1-f)(R) = R$. Therefore

$$R = \{(1-f)(r) \mid r \in R\}$$

$$= \{r - ra \mid r \in R\}$$

$$= \{r(1-a) \mid r \in R\}$$

$$= R(1-a).$$

Hence $a \in J(R)$ so $Z({}_RR) \subseteq J(R)$.

Conversely, let $a \in J(R)$. Define $f : R \longrightarrow R$ by $f(r) = ra$. Then $1 - f$ is monomorphism for if $(1-f)(r) = (1-f)(r')$, then $r(1-a) = r - ra = r' - r'a = r'(1-a)$ implies $r = r'$ as $1-a$ is a unit. Also $1-f$ is epimorphism for if $r \in R$, then $(1-f)(r(1-a)^{-1}) = r$. So $f \in J(End({}_RR))$. So $1-f$ is invertible. Thus $\ker f = \{r \in R \mid f(r) = 0\} \trianglelefteq R$. This implies $\{r \in R \mid ra = 0\} \trianglelefteq R$. Therefore $l(a) \trianglelefteq R$. So $a \in Z({}_RR)$. Hence $J(R) \subseteq Z({}_RR)$. This completes the proof. □

Proposition 2.3.6. *Let ${}_RE$ be an injective module and $S = End({}_RE)$, then $S/Rad(S)$ is regular.*

Proof. Let $\alpha \in S$, then $\ker\alpha \leq E$. Let $\overline{\ker\alpha}$ denote injective envelope of $\ker\alpha$. Without loss of generality we assume that $\overline{\ker\alpha} \leq E$. Since $\overline{\ker\alpha}$ is injective, $E = \overline{\ker\alpha} \oplus E'$ for some $E' \leq E$. Let $\beta = \alpha\mid_{E'}: E' \longrightarrow \alpha(E')$. Clearly β is onto. Also β is one-one because if $u \in \ker\beta$, then $u \in \ker\alpha \cap E' = 0$. So β is an isomorphism. Since E' is injective, $\alpha(E')$ is injective. So there exist $E'' \leq E$ such that $E = \alpha(E') \oplus E''$.

Define $\gamma : E \longrightarrow E$ as follows.

If $x \in E$ such that $x = y + z$, for some $y \in E''$, $z \in \alpha(E')$, let $\gamma(x) = \beta^{-1}(z)$. We shall prove that $\ker(\alpha\gamma\alpha - \alpha) \trianglelefteq E$. Let $m \neq 0$, $m \in E$, then we have to prove that there exist $r \in R$ such that $rm \neq 0$ and $rm \in \ker(\alpha\gamma\alpha - \alpha)$. Let

$m = u + v$, $u \in \overline{\ker\alpha}$, $v \in E'$.

Case 1: $u = 0$, $v \neq 0$.

Since $v \in E'$, $\alpha(v) \in \alpha(E')$. So $\gamma(\alpha(v)) = \beta^{-1}(\alpha(v)) = \beta^{-1}(\beta(v)) = v$.

Therefore $(\alpha\gamma\alpha - \alpha)(v) = \alpha\gamma\alpha(v) - \alpha(v) = \alpha(v) - \alpha(v) = 0$.

Thus with $r = 1$, the condition $rm \neq 0$, and $(\alpha\gamma\alpha - \alpha)(rm) = 0$ are satisfied.

Case 2: $u \neq 0$. Since $\ker\alpha \trianglelefteq \overline{\ker\alpha}$, there exist some $r \in R$ such that $0 \neq ru \in \ker\alpha$. Hence $rm = r(u+v) = ru + rv \neq 0$. Also $(\alpha\gamma\alpha - \alpha)(rm) = (\alpha\gamma\alpha - \alpha)(ru) + (\alpha\gamma\alpha - \alpha)(rv) = 0$ [For if $v \in E'$, then $\alpha(v) \in \alpha(E')$. So $\gamma(\alpha(v)) = \beta^{-1}(\alpha(v)) = \beta^{-1}(\beta(v)) = v$. Therefore $\gamma\alpha(v) = v$. This implies $(\alpha\gamma\alpha)(v) = \alpha(v)$ implies $(\alpha\gamma\alpha - \alpha)(v) = 0$]. So $0 \neq rm \in \ker(\alpha\gamma\alpha - \alpha)$.

So we see that $\ker(\alpha\gamma\alpha - \alpha) \trianglelefteq E$ and hence $\alpha\gamma\alpha - \alpha \in Rad(S)$ which implies $\overline{\alpha}\,\overline{\gamma}\,\overline{\alpha} = \overline{\alpha}$. This proves that $S/Rad(S)$ is regular. □

Corollary 2.3.7. *If R is a left or right self-injective ring, then $R/Rad(R)$ is regular.*

Proof. Let R be left self-injective ring. Then $S = End({}_RR) \simeq R$. Hence $S/Rad(S) \simeq R/Rad(R)$. Since ${}_RR$ is injective, $S/Rad(S)$ is regular. This implies $R/Rad(R)$ is regular. □

Corollary 2.3.8. *A semiprimitive, left (right) self-injective ring is regular.*

Proof. Follows from Corollary 2.3.7. □

2.4 Unit regular rings

Here, some characterizations of unit regular ring and left dependent regular ring are given. Some more results in this section are: Strongly regular rings are unit regular; Every unit regular ring is an elementary divisor ring.

Definition 2.4.1. A ring R is *unit regular ring* if $\forall a \in R$, there exists a unit $x \in R$ such that $a = axa$.

Example 2.4.2. Every division ring is a unit regular ring.

Proposition 2.4.3. *Every strongly regular ring is unit regular.*

Proof. Let $a \in R$. Then there exists $x \in R$ such that $a = a^2x = xa^2$. Note that $ax = a^2xx = xaax = xa$. Let $e = xa = ax$. Then $e^2 = xaxa = xxaa = xa = e$. Also $ea = axa = a$ and $ae = axa = a$. We also have $eax = ax = e$ and $eae = ae = a$. Now
$e^2x = (xa)(xa)x = x(axa)x = xax = ex = xax = xe = xee = exe$.
We shall show that $ex + 1 - e$ is the inverse of $ea + 1 - e$.
Now $(ex + 1 - e)(ea + 1 - e) = exea + ex - exe + ea + 1 - e - eea - e + ee$
$= exa + ex - xe + a + 1 - e - a - e + e = ee + ex - ex + 1 - e = 1$.
Similarly $(ea + 1 - e)(ex + 1 - e) = 1$.
Now $a(ex + 1 - e)a = aexa + aa - aea = axa + aa - aa = axa = a$.
Thus R is unit regular. □

Lemma 2.4.4. ([9], Lemma 1) *Let a be an element of a ring R, then the following statements are equivalent:*

(1) There is a unit $u \in R$ such that $aua = a$.

(2) There is a unit $u \in R$ such that au and ua are idempotents.

(3) There is a unit $u \in R$ such that au or ua is an idempotent.

(4) There are units p and q in R such that paq is idempotent.

Proof. (1) implies (2):- Let $a \in R$, then there exists a unit $u \in R$ such that $aua = a$. This implies that $(au)^2 = (au)(au) = (aua)u = au$. Similarly $(ua)^2 = ua$.

(2) implies (3) and(3) implies (4) are obvious.

(4) implies (1):- We have $(paq)^2 = paq$. Then $(paq)(paq) = paq$. This implies $aqpa = a$. Since q and p are units, qp is a unit. So R is unit regular. □

Proposition 2.4.5. ([9], Proposition 2) *If a is an element of a ring R that satisfies any one of the conditions of the Lemma 2.4.4 and $ab = 1$ for some $b \in R$, then $ba = 1$, that is, one-sided inverse in a unit regular ring is two sided.*

Proof. If u is a unit of R such that $aua = a$, then $au = (aua)b = ab = 1$. which also implies that $ua = 1$ (since u is a unit). Hence $b = u$ is the two sided inverse of a. □

Definition 2.4.6. (Kaplansky):-A ring R is a *right Hermite ring* if for any positive integer n and $A \in M_n(R)$, there is a non-singular matrix $Q \in M_n(R)$ such that AQ is upper triangular.

Definition 2.4.7. A ring R is an *elementary divisor ring* if for every positive integer n and $A \in M_n(R)$ there are units P and $Q \in M_n(R)$ such that PAQ is a diagonal matrix.

Remark 2.4.8. Kaplansky proved in [10], that R is a right Hermite ring (respectively elementary divisor ring) if for every $A \in M_2(R)$, there is a non-singular matrix $Q \in M_2(R)$ such that AQ is upper triangular (respectively there are non-singular matrices P and Q such that PAQ is a diagonal matrix).

Remark 2.4.9. If $A \in M_n(R)$ such that all the entries of A commute with each other and $det A = 1$, then A is non-singular.

Lemma 2.4.10. ([9], Lemma 4) *Every unit regular ring R is a right Hermite ring.*

Proof. Suffices to show that for each $A \in M_2(R)$, there exists a non-singular matrix $Q \in M_2(R)$ such that AQ is upper triangular.
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since R is unit regular, there exists units u and v such that $f = cu$ and $e = dv$ are idempotents.
Let $Q_1 = diag(u, v)$. Then Q_1 is non-singular and $AQ_1 = \begin{pmatrix} au & bv \\ f & e \end{pmatrix} = A_1$ (say).
If $Q_2 = \begin{pmatrix} 1 & f-1 \\ -f & 1 \end{pmatrix}$, then all the entries of Q_2 commute with each other and $det\, Q_2 = 1$. So Q_2 is non-singular.
Now $A_1Q_2 = \begin{pmatrix} au & bv \\ f & e \end{pmatrix} \begin{pmatrix} 1 & f-1 \\ -f & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ (1-e)f & e \end{pmatrix} = A_2$ (say),
where $a_1 = au - bvf, b_1 = au(f-1) + bv$.
Again there is a unit $w \in R$ such that $f_1 = (1-e)fw$ is idempotent.
Then $Q_3 = diag(w, 1)$ is non-singular and
$A_2Q_3 = \begin{pmatrix} a_1 & b_1 \\ (1-e)f & e \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1w & b_1 \\ f_1 & e \end{pmatrix} = A_3$ (say).

Next, note that $ef_1 = e(1-e)fw = 0$ and if $g = f_1(1-e)$, then $eg = ge = 0$ and $(1-f_1e)(1+f_1e) = (1+f_1e)(1-f_1e) = 1$. Thus $Q_4 = diag(1-f_1e, 1+f_1e)$ is non-singular and

$$A_3Q_4 = \begin{pmatrix} a_1w & b_1 \\ f_1 & e \end{pmatrix} \begin{pmatrix} 1-f_1e & 0 \\ 0 & 1+f_1e \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ g & e \end{pmatrix} = A_4 \text{ (say), where}$$

$a_2 = a_1w(1-f_1e)$ and $b_2 = b_1(1+f)_1e$. Finally if we let $Q_5 = \begin{pmatrix} 1-g & 1 \\ -g & 1 \end{pmatrix}$, then Q_5 is non-singular and

$$A_4Q_5 = \begin{pmatrix} a_2 & b_2 \\ g & e \end{pmatrix} \begin{pmatrix} 1-g & 1 \\ -g & 1 \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ 0 & e+g \end{pmatrix} = AQ, \text{ which is upper tri-}$$

angular, where $a_3 = a_2(1-g) - b_2g$ and $b_3 = a_2 + b_2$
and $Q = Q_1Q_2Q_3Q_4Q_5$ is non-singular. This completes the proof. □

Lemma 2.4.11. ([9], Lemma 5) *If R is unit regular and $A \in M_2(R)$, then there are non-singular matrices P and Q such that PAQ is upper triangular and has idempotent entries.*

Proof. By Lemma 2.4.10, if $A \in M_2(R)$, then there exists a non-singular matrix Q such that AQ is upper triangular and the element in the second row and second column is idempotent. Hence to complete the proof of the lemma, it suffices to consider $A = \begin{pmatrix} a & b \\ 0 & e \end{pmatrix}$, where e is idempotent. Since R is unit regular, there are units u, v in R such that ub and $(ua)v$ are idempotents. Then $P = diag(u, 1)$ and $Q = diag(v, 1)$ are non-singular and

$$PAQ = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & e \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} uav & ub \\ 0 & e \end{pmatrix} \text{ has idempotent entries.}$$

□

Lemma 2.4.12. ([9], Lemma 6) *If R is unit regular and $A \in M_2(R)$, then there are non-singular matrices P and Q such that* $PAQ = \begin{pmatrix} e & g \\ 0 & h \end{pmatrix}$, *where e, g, h are idempotents and $eg = ge = 0$.*

Proof. By Lemma 2.4.11, it is enough to consider $A = \begin{pmatrix} e & f \\ 0 & k \end{pmatrix}$, where e, f, k are idempotents. Now $Q_1 = \begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}$ is non-singular and $AQ_1 = \begin{pmatrix} e & (1-e)f \\ 0 & k \end{pmatrix} = A_1$ (say).

Since R is unit regular, there exists a unit $w \in R$ such that $f_1 = (1-e)fw$ is idempotent. Note that $ef_1 = 0$.

Therefore $P_1 = diag(1, w^{-1})$ and $Q_2 = diag(1, w)$ are non-singular and

$$P_1 A_1 Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} e & (1-e)f \\ 0 & k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} = \begin{pmatrix} e & f_1 \\ 0 & w^{-1}kw \end{pmatrix} = A_2 \text{ (say)}.$$

Now $(1 + f_1 e)(1 - f_1 e) = 1 = (1 - f_1 e)(1 + f_1 e)$.

Therefore $P_2 = diag(1, 1 + f_1 e)$ and $Q_3 = diag(1 + f_1 e, 1 - f_1 e)$ are non-singular and

$$P_2 A_2 Q_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 + f_1 e \end{pmatrix} \begin{pmatrix} e & f_1 \\ 0 & w^{-1}kw \end{pmatrix} \begin{pmatrix} 1 + f_1 e & 0 \\ 0 & 1 - f_1 e \end{pmatrix} = \begin{pmatrix} e & g \\ 0 & h \end{pmatrix}$$

$= PAQ$ where $g = f_1(1-e), h = (1 - f_1 e)w^{-1}kw(1 + f_1 e), P = P_2 P_1$ and $Q = Q_1 Q_2 Q_3$.

Also $eg = ef_1(1-e) = 0$ and $ge = f_1(1-e)e = 0$.

Hence PAQ has the desired properties and the lemma is proved. □

Theorem 2.4.13. ([9], Theorem 3) *Every unit regular ring is an elementary divisor ring.*

Proof. By Lemma 2.4 12, it is sufficient to show that if $A = \begin{pmatrix} e & g \\ 0 & h \end{pmatrix}$, where e, g, h *are idempotents and* $eg = ge = 0$, *then there are non-singular matrices* P, Q such that PAQ is diagonal.

Now if $Q = \begin{pmatrix} 1-g & e-1 \\ g & 1-g \end{pmatrix}$, then Q is non-singular and

$$AQ = \begin{pmatrix} e & g \\ 0 & h \end{pmatrix} \begin{pmatrix} 1-g & e-1 \\ g & 1-g \end{pmatrix} = \begin{pmatrix} e+g & 0 \\ hg & h(1-g) \end{pmatrix}$$

Also the matrix $P = \begin{pmatrix} 1 & 0 \\ -hg & 1 \end{pmatrix}$ is non-singular.

And $PAQ = \begin{pmatrix} 1 & 0 \\ -hg & 1 \end{pmatrix} \begin{pmatrix} e+g & 0 \\ hg & h(1-g) \end{pmatrix} = \begin{pmatrix} e+g & 0 \\ 0 & h(1-g) \end{pmatrix}$

This completes the proof. □

Corollary 2.4.14. *If* R *is unit regular (in particular, if* R *is strongly regular) so is* $M_n(R)$.

Proof. By Theorem 2.4.13, we see that if $A \in M_n(R)$, then there are non-singular matrices P and Q such that PAQ is a diagonal matrix.

Let $PAQ = diag(a_1, a_2, ..., a_n)$. Since R is unit regular, there are units u_i such that $a_i u_i$ are idempotents for each $i = 1, 2, ... n$. Let $U = diag(u_1, u_2, . ., u_n)$, then $(PAQ)U$ is idempotent and P and QU are units. Therefore by Lemma 2.4.4, $M_n(R)$ is unit regular. □

Proposition 2.4.15. ([9], Proposition 8) *A regular ring* R *is unit regular if and only if* $aR{+}bR{=}R$ *implies there is some* $t \in R$ *such that* $a + bt$ *is a unit.*

Proof. Suppose that R is unit regular and that $aR + bR = R$, then there is a unit $u \in R$ and some $y \in R$ such that $e = au$ and $f = by$ are idempotents. Then $aR + bR = eR + fR = eR + (1-e)fR = R$. Since R is regular, there exists some $w \in R$ such that $(1-e)fw(1-e)f = (1-e)f$. Let $g = (1-e)fw(1-e)$, then $g^2 = (1-e)fw(1-e)(1-e)fw(1-e) = (1-e)fw(1-e) = g$. Also $ge = eg = 0$. Note that $gf = (1-e)f$. If $x \in (1-e)fR$, then $x = (1-e)fr$ for some $r \in R$. Then $x = gfr \in gR$. Also clearly $gR \subseteq (1-e)fR$. Thus $gR = (1-e)fR$. Therefore we have $gR + eR = R$. Therefore there exists α, β such that $e\beta + g\beta = 1$. Then $e\alpha = e$ and $g\beta = g$ (since $eg = ge = 0$). Therefore $e + g = 1$. This implies $e+(1-e)fw(1-e) = 1$. Then $au[1-efw(1-e)]+byw(1-e) = 1$. Multiplying on the right by $\gamma = (1+efw(1-e))u^{-1}$ we get $a + byw(1-e)\gamma = \gamma$ has two sided inverse $u[1-efw(1-e)]$.

Conversely, suppose R is regular and $aR + bR = R$. This implies that there exists $t \in R$ such that $a + bt$ is a unit and that there exists some $x \in R$ such that $a = axa$ and so $aR + (1-ax)R = R$. Therefore there exists some $t \in R$ such that $a + (1-ax)t$ is a unit which further means that there exists some $u \in R$ such that $[a + (1-ax)t]u = 1$. Thus we get $ax[a + (1-ax)t]ua = axa$, that is, $axaua + axtua - axaxtua = axa$. This implies $aua = a$. So R is unit regular. □

Definition 2.4.16. A ring R is a *a left dependent ring* if for every $a, b \in R$, there exists $s, t \in R$, not both both zero such that $sa + tb = 0$. Similarly a ring R is defined to be *right dependent ring.*

Proposition 2.4.17. *A unit regular ring is left dependent.*

Proof. If both a and b have left inverses in R, then $a^{-1}a + (-b^{-1}b) = 0$

with $a^{-1} \neq 0$ and $b^{-1} \neq 0$. So assume that a has no left inverse. Since R is unit regular, $a = xe$ for some unit x and idempotent e. Since a is not a unit, e cannot be 1. Let $s = (1-e)x^{-1}$, $t = 0$, then $s \neq 0$ we have $sa + tb = (1-e)x^{-1}xe = 0$. Therefore R is left dependent. □

Proposition 2.4.18. ([9], Theorem 9) *A regular ring R is left dependent if whenever a, a', b, b' are elements of R such that $aa' = bb' = 1$, then $a(1-b'b)$ fails to have a right inverse.*

Proof. Suppose R is left dependent. Let a. a', b, b' be elements of R such that $aa' = 1 = bb'$. To prove $a(1-b'b)$ fails to have right inverse. Since R is left dependent, $sa + tb = 0$ for some s, $t \in R$ not both zero. This implies $sab' + t = 0$. That is, $t = -sab'$ − − − − − − − − − − −(∗).

So $sa + tb = 0$ implies $0 = sa - sab'b = sa(1 - bb')$. Now from (*), $s \neq 0$, because if $s = 0$, then s and t both will be zero. So $a(1 - b'b)$ fails to have right inverse.

Conversely, suppose R is regular and $a, b \in R$. Suppose one of a or b say a fails to have a right inverse. Since R is regular, $a = axa$ for some $x \in R$. Now $(1-ax)a + 0.b = 0$ and $1 - ax \neq 0$. So we assume that there are a' and b' such that $aa' = 1 = bb'$. Then $\alpha = 1 - b'b$ fails to have a right inverse. By hypothesis, there exists $y \in R$ such that $\alpha = \alpha y \alpha$. We then have $(1-\alpha y)a - (1-\alpha y)abb' = (1-\alpha y)\alpha = 0$. But $1 - \alpha y \neq 0$ as α fails to have a right inverse. This proves the proposition. □

2.5 Weakly regular rings

In this section, we study left (right) weakly regular ring. The main results are : An ELT right weakly regular ring is regular; The centre of a left (right) weakly regular ring is regular; A left (right) weakly regular ring is right (left) non-singular and semiprimitive.

Definition 2.5.1. A ring R is *left (right) weakly regular* if $I = I^2$ for each left (right) ideal I of R.

Example 2.5.2. Every regular ring is left (right) weakly regular.

Followinng proposition is taken from Lemma 4 of [12].

Proposition 2.5.3. *An ELT right weakly regular ring is regular.*

Proof. Let R be an ELT right weakly regular ring and let $a \in R$. Then there exists a left ideal L of R such that $I = Ra \oplus L$ is an essential left ideal of R. As R is an ELT, I is an ideal of R. Again R is right weakly regular implies $aR = (aR)^2 = (aR)^4 = a(RaR)$. But I is an ideal of R and $a \in I$ implies $a(RaR) \subseteq aI$. Thus $a \in aR \subseteq aI = a(Ra \oplus L)$. That is, $a = a(ba + l)$ for some $b \in R$, $l \in L$. But this gives $a - aba = l \in Ra \cap L = 0$. Therefore $a = aba$ and so R is regular. □

Proposition 2.5.4. ([8], Proposition 9) *If A is a proper ideal of a right weakly regular ring, then each element of A is a left zero divisor.*

Proof. Suppose $x \in A$ such that x is not a left zero divisor. Since R is right weakly regular, $xR = xRxR$. Therefore if $y \in R$, then $xy = \sum_{i=1}^{k} xr_ixs_i$ for some $r_i \in R$, $s_i \in R$. This gives

$$xy - \sum_{i=1}^{k} xr_i xs_i = 0 = x\left(y - \sum_{i=1}^{k} r_i xs_i\right) = 0.$$

As x is not a left zero divisor, $y - \sum_{i=1}^{k} r_i xs_i = 0$. This implies $y = \sum_{i=1}^{k} r_i xs_i \in A$. Hence $A = R$ which contradicts that A is a proper ideal of R. Thus each element of A is a left zero divisor. □

Proposition 2.5.5. *A right (left) weakly regular ring is semiprimitive.*

Proof. Let $x \in J(R)$, the Jacobson radical of R. Now R is right weakly regular implies $xR = xRxR$. Hence $x = \sum_{i=1}^{k} xr_i xs_i$ for some $r_i \in R$, $s_i \in R$. Hence $x\left(1 - \sum_{i=1}^{k} r_i xs_i\right) = 0$. Since $x \in J(R)$, $1 - \sum_{i=1}^{k} r_i xs_i$ is a unit. Therefore $x = 0$. As x is arbitrary, $J(R) = 0$. □

Corollary 2.5.6. *A regular ring is semiprimitive.*

Proposition 2.5.7. *Let R be a reduced ring. Then R is left weakly regular if and only if R is right weakly regular.*

Proof. Let R be a left weakly regular ring, I be a right ideal of R and $x \in I$. Then $Rx = RxRx$. Hence $x = \sum_{i=1}^{k} r_i xs_i x$ for some $r_i \in R$, $s_i \in R$. This implies $\left(1 - \sum_{i=1}^{k} r_i xs_i\right) x = 0$. As R is reduced, we get, $x\left(1 - \sum_{i=1}^{k} r_i xs_i\right) = 0$. Hence $x = \sum_{i=1}^{k} xr_i xs_i \in I^2$. So $I = I^2$ which implies R is right weakly regular.

Converse can be proved similarly. □

Proposition 2.5.8. ([8], Proposition 12) *The centre of any left (right) weakly regular ring is regular*

Proof Let R be a left weakly regular ring and C be its centre Let $x \in C$
Then $x \in Rx = (Rx)^2 \subseteq Rx^2 = (Rx)^4$
Then $x = x(x^k y)x$ for some $y \in R$ and $k > 1$ (*)
It is enough to show that $x^k y \in C$ Let $z \in R$ Then

$$z(x^k y) = x^{k-1}(xz)y = x^{k-1}(x^{k+1}yx)zy \text{ (using (*))}$$

This shows that $x^{k-1}yz(x^{k+1}xy) = x^k yz = (x^k y)z$ Hence $x^k y \in C$ as required ☐

Proposition 2.5.9. ([8], Proposition 14 (2)) *A left (right) weakly regular ring is right (left) non-singular*

Proof Let R be a left weakly regular ring and $x \in Z(R_R)$ Then $r(x)$ is an essential right ideal of R Since R is left weakly regular, $Rx = RxRx$ So $x = \sum_{i=1}^{k} r_i x s_i x$ for some $r_i \in R$, $s_i \in R$ Let $y = \sum_{i=1}^{k} r_i x s_i$ so that $x = yx$ and therefore $yx - x = 0$ This implies that $(y-1)x = 0$ That is, $x \in r(y-1)$ Hence $xR \subseteq r(y-1)$ Now $r(y)$ is essential right ideal of R and $r(y) \bigcap r(y-1) = 0$ implies $r(y-1) = 0$ Hence $xR = 0$ Thus $x = 0$ Hence $Z(R_R) = 0$ and thus R is right non-singular
If R is right weakly regular ring we can similarly prove that R is left non-singular ☐

2.6 Central localizations of regular rings

In this section, we study the regularity of a ring via its central localizations.

Lemma 2.6.1. ([11], Lemma 1) *Let R be a ring with regular centre C and M be a maximal ideal of C. Then $MR = \ker(R \longrightarrow R_M)$, $MR \cap C = M$, and $0 \longrightarrow MR \longrightarrow R \longrightarrow R_M \longrightarrow 0$ is exact. Here the homomorphisms are natural homomorphisms.*

Proof. First we prove that $R \longrightarrow R_M$ is onto. Let $\frac{r}{s} \in R_M$. Since C is regular, $s = sts$ for some $t \in C$. So $\frac{r}{s} = \frac{rst}{sts} = rt$. Hence $rt \longmapsto \frac{r}{s}$.

Let $A = \ker(R \longrightarrow R_M)$. Then $A \cap C = \ker(C \longrightarrow C_M)$. Since C is regular, C_M is a field. Thus $A \cap C$ is a maximal ideal of C. Hence $M = A \cap C$ [For if $a \in A \cap C$, then $a \in A$ and $a \in C$. Then $as = 0 \in M$ for some $s \in S$. But since $s \notin M$, so $a \in M$. Hence $A \cap C \subseteq M$. Since $A \cap C$ is maximal ideal, $A \cap C = M$]. Thus $MR \subseteq A$.

Let $a \in A$. Then $\exists$ some $s \in S$ such that $as = 0$. Since C is regular, $\exists$ some $t \in S$ such that $s = sts$. Then $(1 - st)s = 0 \in M$. But since $s \notin M$, so $1 - st \in M$. Hence $a = (1 - st)a \in MR$. Thus $MR = A$. This completes the proof. □

Proposition 2.6.2. *Let I be a right ideal of R such that $(I/I^2)_M = 0$ for all maximal ideal of M of C, the centre of R. Then $I = I^2$.*

Proof. Let $x \in (I/I^2)$. Then $\frac{x}{1} \in (I/I^2)_M = 0$. Thus there exists some $u_M \in C_M$ such that $u_M x = 0$. If $C \neq \Sigma C u_M$, then there exists a maximal ideal L of C such that $\Sigma C u_M \subsetneq L$. But this implies $u_L = 1.u_L \in \Sigma C u_M \subseteq L$, a contradiction. Hence $C = \Sigma C u_M$. Hence $1 = r_1 u_{M_1} + r_2 u_{M_2} + \cdots + r_k u_{M_k}$ for some maximal ideals $M_1, M_2, \ldots, M_k$ of C. Thus

$x = 1.x = (r_1 u_{M_1} + r_2 u_{M_2} + \cdots + r_k u_{M_k})x = 0$. Hence $(I/I^2) = 0$ which implies $I = I^2$. □

Proposition 2.6.3. ([11], Proposition 2) *A ring R is fully right idempotent if and only if R_M is fully right idempotent for each maximal ideal M of C, the centre of R.*

Proof. Let R be a fully right idempotent. Then by Proposition 2.5.8, C is regular. So for each maximal ideal M of C, $R \longrightarrow R_M$ is onto. Thus R_M is fully right idempotent for each maximal ideal M of C.

Conversely, suppose R_M is fully right idempotent for each maximal ideal M of C. Let I be a right ideal of R Then

$0 \longrightarrow I^2 \longrightarrow I \longrightarrow I/I^2 \longrightarrow 0$ is exact........(1).

Here the homomorphisms are natural.

Since C_M is a flat C module for each maximal ideal M of C, (1) implies that

$0 \longrightarrow I^2 \otimes C_M \longrightarrow I \otimes C_M \longrightarrow (I/I^2) \otimes C_M \longrightarrow 0$ is exact. Then

$0 \longrightarrow I_M^2 \longrightarrow I_M \longrightarrow (I/I^2)_M \longrightarrow 0$ is exact. Thus

$I_M/I_M^2 \simeq (I/I^2)_M$. Therefore $(I/I^2)_M = 0$ (since R_M is fully right idempotent, $I_M = {I_M}^2$). Hence $I = I^2$ and R is fully right idempotent. □

Proposition 2.6.4. ([11], Theorem 3) *The following conditions are equivalent for a ring R with centre C:*

(1) R is regular.

(2) C is regular and R/MR is regular for each maximal ideal M of C.

(3) R_M is regular for each maximal ideal M of C.

Proof. (1) implies (2) follows from the fact that centre of a regular ring is regular and factor ring of a regular ring is regular.

(2) implies (3):- Since C is regular, $0 \longrightarrow MR \longrightarrow R \longrightarrow R_M \longrightarrow 0$ is exact. Thus $R/MR \simeq R_M$ and hence R_M is regular for each maximal ideal M of C.

(3) implies (1):- Let $a \in R$. Then $\frac{a}{1} \in R_M$ for each maximal ideal M of C. As R_M is regular, there exists some $\frac{r_M}{s_M} \in R_M$ such that $\frac{a}{1} = \frac{a}{1}\frac{r_M}{s_M}\frac{a}{1}$. Thus there exists $u_M \in C_M$ such that $(as_M - ar_M a)\, u_M = 0$. Thus

$as_M u_M = ar_M a u_M$.......................(*)

If $\sum Cs_M u_M \neq C$, then there exists a maximal ideal L of C such that $\sum Cs_M u_M \subseteq L$. So $s_M u_M \in L$ which implies $s_M \in L$ or $u_M \in L$, a contradiction. Thus $\sum Cs_M u_M = C$. Therefore

$1 = b_1 s_{M_1} u_{M_1} + b_2 s_{M_2} u_{M_2} + \cdots + b_k s_{M_k} u_{M_k}$ for some maximal ideals $M_1, M_2, \ldots, M_k$ of C and some $b_1, b_2, \ldots, b_k \in C$. Then

$$a = a\sum_{i=1}^{k} b_i s_{M_i} u_{M_i} = \sum_{i=1}^{k} b_i a s_{M_i} u_{M_i} = \sum_{i=1}^{k} b_i a r_{M_i} a u_{M_i}\, (using(*))$$

$= a\left(\sum_{i=1}^{k} b_i r_{M_i} u_{M_i}\right) a$. Hence R is regular. □

Proposition 2.6.5. ([29], Proposition 8) *The following conditions are equivalent for a ring R with centre C:*

(1) R is regular.

(2) R is semiprime ring whose essential left ideals are idempotent and for each maximal ideal M of C, R/RM is regular.

Proof. (1) implies (2) evidently.

Assume (2). For any $c \in C$, let K be a left ideal of R such that $L = (Rc + l(c)) \oplus K$ is essential left ideal of R. Then $Kc = cK \subseteq Rc \cap K = 0$ implies $K \subseteq l(c)$. Therefore $K \subseteq K \cap (Rc + l(c)) = 0$ and so $L = Rc + l(c)$ is an essential left ideal of R. So by hypothesis, $L = L^2$. Now $c \in L = L^2$ implies

$$c = \sum_{i=1}^{n} (r_i c + u_i)(s_i c + v_i), \quad r_i,\ s_i \in R,\ u_i,\ v_i \in l(c).$$

Therefore

$$c - \sum_{i=1}^{n} r_i c s_i c = \sum_{i=1}^{n} (r_i c v_i + u_i s_i c + u_i v_i) = \sum_{i=1}^{n} u_i v_i,$$

since $r_i c v_i = r_i v_i c = 0$ and $u_i s_i c = u_i c s_i = 0$. If $w \in Rc \cap l(c)$, then $w = dc$ for some $d \in R$. Therefore we get $dc^2 = wc = 0$ and so $cRdc = 0$ which again implies $(Rdc)^2 = 0$. As R is semiprime, this yields $Rdc = 0$. So $w = dc = 0$. Now

$$c - \sum_{i=1}^{n} r_i c s_i c = \sum_{i=1}^{n} u_i v_i \in Rc \cap l(c) = 0$$

$$\Longrightarrow c = \sum_{i=1}^{n} r_i c s_i c = czc, \text{ where } \quad z = \sum_{i=1}^{n} r_i s_i \in R.$$

Set $y = c^2 z^3$. Then $cyc = (czc)(zczc) = (czc)(zc) = czc = c$ and $c^2 z = zc^2 = czc = c$. For every $b \in R$, $zc^2 b = cb = bc = bc^2 z = c^2 bz$ and hence

$$z^3 c^2 b = z^2 (c^2 bz) = z(zc^2 b)z = z(c^2 bz)z = z(c^2 b)z^2 = z(c^2 b)z^2 = c^2 b z^3.$$

This shows that $yb = c^2 z^3 b = z^3 c^2 b = by$. Thus $y \in C$ and C is regular. Therefore R is regular by Proposition 2.6.4. □

2.7 W-ideals and GW-ideals

Here, we have studied weak ideals (W-ideals) and generalized weak ideals (GW-ideals). Examples of a left ideal which is not a W-ideal and a W-ideal which is not a GW-ideal are given.

Definition 2.7.1. A left ideal L of a ring R is a *weak ideal (W-ideal)* if for all $a \in L$, $a \neq 0$, there exists $n > 0$ such that $a^n \neq 0$ and $a^n R \subseteq L$. A right ideal K of R is defined similarly to be a $W-ideal$.

Definition 2.7.2. A left ideal L of a ring is a *generalized weak ideal (GW-ideal)* if for all $a \in R$, there exists some $n > 0$ such that $a^n R \subseteq L$. A right ideal K of R is defined similarly to be a GW-ideal.

Example 2.7.3. (1) All ideals are W-ideals and GW-ideals.

(2) A W-ideal is a GW-ideal.

(3) A GW-ideal need not be a W-ideal.

For example, take $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ and $\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then $L = R\alpha = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\}$ is a left ideal of R and for all $y \in L$, $y^2 = 0$. So L is a GW-ideal of R. But L is not a W-ideal as $0 \neq \alpha \in L$, $\alpha^2 = 0$. But $\alpha R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\} \nsubseteq L$.

(4) A left or right ideal of a ring R need not be a GW-ideal.

For example, let $R = UT_2(\mathbb{Q})$, the ring of upper triangular matrices over $\mathbb{Q}$.

Take $L = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Q} \right\}$. Then L is a left ideal of R but it is not a GW-ideal for if $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then for all $n > 0$,

$\alpha^n R = \alpha R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{Q} \right\} \nsubseteq L$.

Also if $K = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{Q} \right\}$. Then K is right ideal of R. If $\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then for all $n > 0$,

$R\beta^n = R\beta = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} : x, y \in \mathbb{Q} \right\} \nsubseteq K$. So K is not a GW-ideal of R.

(5) If R is a left (or right) duo ring, then every left (or right) ideal of R is a W-ideal and hence a GW-ideal.

Example 2.7.4. ([24], Example 1.2) There exists a ring R such that $\{Ideals\ of R\} \subsetneq \{W - ideals\ of R\} \subsetneq \{GW - ideals\ of R\}$.

Proof. Take $R = \left\{ \begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} : a,\ a_i \in \mathbb{Z}_2,\ i = 1, 2, 3, 4, 5, 6 \right\}$. Then

R is a ring. Let $L = \left\{ \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : b_i, \in \mathbb{Z}_2, i = 1, 2, 3 \right\}$. Then L is a left ideal of R. But $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin L.$$

Hence L is not an ideal of R.

If $x \in L$ we distinguish two cases:

(i) $x = \begin{pmatrix} 0 & 0 & b_2 & b_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $xR \subseteq L$ for if

$y = \begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} \in R$. Then $xy = \begin{pmatrix} 0 & 0 & b_2a & b_2a_6 + b_3a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L$.

(ii) If $x = \begin{pmatrix} 0 & 1 & b_2 & b_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then $x^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$ and $x^2 R \subseteq L$.
So L is a W-ideal of R.

Next let $T = \left\{ \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_2 \right\}$. Then T is a left ideal of R.

If $x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then $x^2 = 0$. So T is a GW-ideal of R. Also

$x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin T$. This shows that T is not a W-ideal. $\square$

Proposition 2.7.5. *For a ring R the following conditions are equivalent:*

(1) Every left ideal of R is GW.

(2) Every finitely generated left ideal of R is GW.

(3) Every principal left ideal of R is GW.

Proof. (1) implies (2) implies (3) is trivial.

(3) implies (1):- Let L be a left ideal of R and $a \in L$. By hypothesis Ra is GW-ideal. Therefore $a \in Ra$ implies there exists a positive integer n such that $a^n R \subseteq Ra \subseteq L$. This implies L is GW-ideal. ☐

Remark 2.7.6. If L is a left ideal of R such that Ra is GW-ideal for all $a \in L$, then L is GW-ideal. However, the converse is not true as is seen by the following example. Consider $R = UT_2(\mathbb{R})$, the ring of 2×2 upper triangular matrices over $\mathbb{R}$. Then R is a GW-ideal of R. But $R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$ is not a GW-ideal of R.

Proposition 2.7.7. ([23], Proposition 1) *If R be a ring and I be an ideal of R. For any left ideal K of R such that $I \subseteq K$, K is a GW-ideal of R if and only if K/I is a GW-ideal of R/I.*

Proof. Let K be a left ideal of R such that K/I is a GW-ideal of R/I. Let $a \in K$. If $r \in R$, then K/I is a GW-ideal of R/I implies there exists some $n > 0$ such that $(a+I)^n(r+I) \in K/I$. This implies $a^n r + I \in K/I$. So $a^n r + I = b + I$ for some $b \in I \subseteq K$. That is, $a^n r - b \in I$. Therefore $a^n r = a^n r - b + b \in K$. Thus K is a GW-ideal of R.

Conversely, suppose K is a left ideal of R such that $I \subseteq K$ and K is a GW-ideal of R. Let $b \in K/I$. Then $b = a + I$ for some $a \in I$. If $r \in R$, then K is a GW-ideal of R implies there exists some $n > 0$ such that $a^n r \in K$. Then $(a+I)^n(r+I) \in K/I$. This shows that K/I is a GW-ideal of R/I.

☐

Proposition 2.7.8. ([24], Proposition 1.4) *Let R be a ring and I an ideal of R. If a left ideal K of R is a W-ideal such that $I \subseteq K$, then K/I is a GW-ideal of R/I.*

Proof. Follows from Proposition 2.7.7. □

Remark 2.7.9. (1) Let R be a ring and I be an ideal of R. If K is a left ideal of R such that $I \subseteq K$ and K/I is a GW-ideal of R/I, then K need not be a W-ideal of R.

Take R and T as in Example 2.7.4 and $I = 0$, then T/I is a GW-ideal of R/I but T is not a W-ideal of R.

(2) Let I be an ideal of a ring R. Let K be a left ideal of R such that $I \subseteq K$ and K is a W-ideal of R. Then K/I need not be a W-ideal of R/I. For, let R and L as in Example 2.7.4. Let $I = \left\{ \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z}_2, \right\}$.

Then I is an ideal of R and $I \subseteq L$. But

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in I. \text{ So } \overline{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}^2 = 0.$$

But $\overline{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \; \overline{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}} = \overline{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \notin L/I.$

Hence L/I is not a W-ideal of R/I.

Lemma 2.7.10. ([14], Lemma 2.5) *Let R be a semiprimitive ring. If any maximal left (right) ideal of R is a GW- ideal, then R is reduced.*

Proof. Suppose $0 \neq a \in R$ such that $a^2 = 0$. Since R is semiprimitive, $a \notin J(R)$. So there exists a maximal left ideal M of R such that $a \notin M$. Then $M + Ra = R$ which yields $Ma = Ra$ since $a^2 = 0$. So there exists some $b \in M$ such that $a = ba$. Since M is GW-ideal, there exists a positive integer n such that $b^n a \in M$. Then $b^n a = b^{n-1}(ba) = b^{n-1}a = ... = a \in M$, a contradiction. Therefore R is reduced. □

2.8 On regular GP-V-rings

In this section, we investigate the characterizations of strongly regular rings with the condition that every maximal left (right) ideal is a GW-ideal via GP-V-ring.

Definition 2.8.1. A left R-module M is *YJ - injective* if for every $0 \neq a \in R$ there exists a positive integer n such that $a^n \neq 0$ and any left R - homomorphism from Ra^n to M extends to one from R to M.

Example 2.8.2. Every p-injective module is YJ-injective.

Definition 2.8.3. A left R-module M is *GP - injective* if for every $a \in R$ there exists a positive integer n such that every left R - homomorphism from Ra^n to M extends to one from R to M.

Example 2.8.4. Every $YJ-$ *injective* module is $GP-$ *injective.*

Definition 2.8.5. A ring R is a *left GP-V-ring* if every simple left R-module is YJ- injective.

Example 2.8.6. Every semisimple ring is a *left GP-V-ring.*

Lemma 2.8.7. ([30]) *If R is a $GP-V$-ring, then $J(R)$, the Jacobson radical of R is zero.*

Theorem 2.8.8. ([14], Theorem 2.2) *The following are equivalent for a ring R:*

(1) R is strongly regular.

(2) R is a left $GP-V$-ring whose every maximal left ideal is a GW-ideal.

(3) R is a left $GP-V$-ring whose every maximal right ideal is a GW-ideal.

Proof. (1) implies (2) and (1) implies (3) follows from Lemma 2.2.4 and Proposition 2.1.14.

(2) implies (1):- First we prove that R is reduced. If R is not reduced, then there exists $0 \neq a \in R$ such that $a^2 = 0$. So there exists a maximal left ideal M of R such that $l(a) \subseteq M$. Define $f : Ra \longrightarrow R/M$ by $f(ra) = r+M$. Then f is well-defined for if $ra = r'a$ for some $r \in R$, $r' \in R$, then $(r-r')a = 0$ which implies that $r - r' \in l(a) \subseteq M$. So $r + M = r' + M$, that is,

$f(ra) = f(r'a)$. Also f is a left R-module homomorphism. Since R is a left $GP - V$-ring and $a^2 = 0$, f can be extended to a left R-module homomorphism $g : R \longrightarrow R/M$. This implies that $1 + M = f(a) = g(a) = ag(1) = a(b + M)$ for some $b \in R$. That is, $1 - ab \in M$. Since M is a GW-ideal and $ba \in M$ (since $ba^2 = 0$ implies $ba \in l(a) \subseteq M$) there exists $n > 0$ such that $(ba)^n b \in M$. Again M is a left ideal and $b - bab \in M$ implies $(ba)^{n-1}b = (ba)^{n-1}(b - bab) + (ba)^n b \in M$. Continuing in this manner, we have, $bab \in M$. Then $b = (b - bab) + bab \in M$. Thus $ab \in M$. Hence $1 = 1 - ab + ab \in M$, contradicting that M is a maximal left ideal of R. Thus R is reduced.

Now we prove that R is strongly regular. Let $a \in R$ and $l(a) + Ra \neq R$, then there exists a maximal left ideal M of R such that $l(a) + Ra \subseteq M$. As R is a left $GP - V$-ring, the simple left R-module R/M is YJ- injective. So there exists $n > 0$ such that $a^n \neq 0$ and every left R-module homomorphism from Ra^n to R/M can be extended to a left R-module homomorphism from R to M. Define $f : Ra^n \longrightarrow R/M$ by $f(ra^n) = r + M$. Since R is reduced, $l(a^n) = l(a)$. It yields f is well-defined. Also f is a left R - homomorphism. So f can be extended to a left R - homomorphism $g : R \longrightarrow R/M$. Thus $1 + M = f(a^n) = g(a^n) = a^n g(1) = a^n(b + M)$ for some $b \in R$, that is, $1 - a^n b \in M$ and hence $b - ba^n b \in M$. Also $ba^n \in Ra$ implies $ba^n \in M$. Since M is a GW - ideal there exists some $k > 0$ such that $(ba^n)^k b \in M$. Then $(ba^n)^{k-1}b = (ba^n)^{k-1}(b - ba^n b) + (ba^n)^k b \in M$. Continuing in this manner, we have, $(ba^n)b \in M$. Thus $b = b - ba^n b + ba^n b \in M$. Then $a^n b \in M$. Hence $1 = 1 - a^n b + a^n b \in M$ which contradicts that M is a maximal left ideal of R. Thus $l(a) + Ra = R$ for all $a \in R$ which yields that

$y + xa = 1$ for some $y \in l(a)$ and $x \in R$ which again implies $xa^2 = a$. Hence R is strongly regular.

(3) implies (1):- We first prove that R is reduced. If R is not reduced, then there exists some $0 \neq a \in R$ such that $a^2 = 0$. Since R is semiprimitive, $a \notin K$ for some maximal right ideal K of R. So $K + aR = R$ which yields $aK = aR$ since $a^2 = 0$. This gives $a = ak$ for some $k \in K$. By hypothesis, K is a GW-ideal so there exists a positive integer n such that $ak^n \in K$. So $a = ak = ak^2 = ... = ak^n \in K$, a contradiction. Therefore R is reduced.

Hence $l(b) = r(b)$ is an ideal of R for every $b \in R$. Suppose $l(a) + aR \neq R$ for some $a \in R$, then $l(a) + aR \subseteq M$ for some maximal right ideal M of R. Since M is GW-ideal, $Ra^n \subseteq M$ for some $n > 0$. Hence $l(a) + Ra^nR \subseteq M$. Then $l(a) + Ra^nR \subseteq L$ for some maximal left ideal L of R. Since R is a left $GP - V$-ring, the simple left R-module R/L is YJ- injective. Therefore there exists a positive integer m such that $(a^n)^m \neq 0$ and any left R - homomorphism from $R(a^n)^m$ to R/L extends to one from R to R/L. Define $f : R(a^n)^m \longrightarrow R/L$ by $f(r(a^n)^m) = r + L$. Since R is reduced, $l(a^n)^m = l(a)$. This yields f is well-defined. Also f is left R-module homomorphism. So f can be extended to a left R - homomorphism $g : R \longrightarrow R/L$. Thus we get $1 + L = f((a^n)^m) = g((a^n)^m) = (a^n)^m g(1) = (a^n)^m(b + L)$ for some $b \in R$. Therefore $1 - (a^n)^m \in L$. But $(a^n)^m b \in Ra^nR \subseteq L$. So $1 = 1 - (a^n)^m b + (a^n)^m b \in L$, a contradiction. So $l(a) + aR = R$ for all $a \in R$. Hence $d + ax = 1$ for some $d \in l(a)$, $x \in R$. This implies $da + axa = a$. So R is regular. This together with R is reduced implies R is strongly regular. $\square$

Corollary 2.8.9. ([14], Corollary 2.3) *The following conditions are equivalent for a ring R:*

(1) R is strongly regular.

(2) R is a left quasi-duo ring whose simple right modules are $YJ-injective$.

(3) R is a left quasi-duo ring whose simple left modules are $YJ-injective$.

2.9 On generalized regular rings and weakly left (right) ideals

In this section, we study some properties of generalized regular ring as well as characterize strongly regular ring via weakly one sided ideals.

Definition 2.9.1. A ring R is *generalized regular* if every left ideal is generated by idempotents.

Example 2.9.2. (1) A semisimple ring is a generalized regular ring.

Proposition 2.9.3. *If R is a generalized regular ring, then R is left non-singular and the Jacobson radical $J(R) = 0$.*

Proof. Let $J(R) = \sum_{i \in I} Re_i$ where each e_i are idempotents. Then for each $i \in I$, $e_i \in \sum_{i \in I} Re_i = J(R)$. So $1 - e_i$ is a unit. But $e_i = e_i^2$. So $e_i(1 - e_i) = 0$. Therefore $e_i = 0$ for all i. Thus $J(R) = 0$.

Let $Z(_RR) = \sum_{i \in I} Re_i$ for some idempotents e_i. Then for each $i \in I$, $e_i \in Z(_RR)$. Thus $l(e_i) = R(1 - e_i)$ is an essential left ideal of R. But

$R(1-e_i) \oplus Re_i = R$. So $R(1-e_i)$ is essential implies $Re_i = 0$ and hence $e_i = 0$. Therefore $Z({}_RR) = 0$. □

Definition 2.9.4. An additive subgroup L of a ring R is a *weakly left ideal* of R if for every $x \in L$ and for every $r \in R$ there exists a natural number n such that $(rx)^n \in L$. The notion of a *weakly right ideal* of a ring is defined similarly.

Example 2.9.5. (1) Every left (right) ideal of R is a weakly left (right) ideal of R.

(2) Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ and $K = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\}$.
Then for all $y \in K$ and $r \in R$, $(yr)^2 = 0 \in K$. So K is a weakly right ideal of R. But K is not a right ideal of R.

(3) Let $R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & d & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ and

$L = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\}$. Then L is not a left ideal of R but L is a weakly left ideal of R as for all $y \in L$ and $r \in R$, $(ry)^2 = 0 \in L$.

Lemma 2.9.6. ([13], Lemma 2) *Let R be a semiprimitive ring. If every left annihilator of an element of R is a weakly right ideal of R, then R is reduced.*

Proof. Suppose $0 \neq a \in R$ such that $a^2 = 0$. Suppose Ra is not nil and $0 \neq ba \in Ra$ such that ba is not nilpotent. Since $a \in l(a)$ and $l(a)$ is weakly right ideal of R, $(ab)^n \in l(a)$ for some $n > 0$. So $(ab)^n a = 0$. But this yields $(ba)^{n+1} = b[(ab)^n a] = 0$ which contradicts that ba is not nilpotent. Thus Ra is nil. Hence if R is semiprimitive, then $Ra = 0$ and so $a = 0$. Therefore R is reduced. □

Proposition 2.9.7. ([13], Theorem 1) *The following conditions are equivalent for a ring R:*

(1) R is a strongly regular ring.

(2) R is a generalized regular ring whose left annihilator of any element is a weakly right ideal of R.

(3) R is a generalized regular ring whose right annihilator of any element is a weakly left ideal of R.

Proof. (1) implies (2) and (1) implies (3) are trivial.

(2) implies (1):- By Proposition 2.9.3 and Lemma 2.9.6, R is reduced. Therefore R is a normal ring (that is, all the idempotents are central). Let $a \in R$. Then R is generalized regular implies $Ra = \sum Re_i$ for some idempotents e_i. Let $a = r_1e_1 + r_2e_2 + \dots + r_ne_n$. It follows that

$Ra = R(r_1e_1 + r_2e_2 + \dots + r_ne_n) \subseteq Re_1 + \dots + Re_n \subseteq Ra$. Therefore $Ra = Re_1 + Re_2 + \dots + Re_n$. Let $f_1 = e_1 + e_2 - e_1e_2$. Since R is normal ring, we see that $f_1^2 = f_1, e_1f_1 = e_1, e_2f_1 = e_2$,

$Rf_1 = R(e_1 + e_2 - e_1e_2) \subseteq Re_1 + Re_2 = Re_1f_1 + Re_2f_1 \subseteq Rf_1$.

Therefore $Re_1 + Re_2 = Rf_1$. Also let $f_2 = f_1 + e_3 - f_1e_3$, ,

$f_{n-2} = f_{n-3} + e_{n-1} - f_{n-3}e_{n-1}, e = f_{n-2} + e_n - f_{n-2}e_n$.

By repeating the above process we have

$Ra = Rf_1 + Re_3 + Re_4 + \quad + Re_n = Rf_2 + Re_4 + \quad + Re_n = \quad = Re$

So $a = re$ and $e = ba$ for some $r, b \in R$ This gives $ae = re^2 = re = a$ Therefore $a = ea = ba^2$ Thus R is strongly regular □

Definition 2.9.8. A ring R is a *left (right) P-V-ring* or a *left (right) SPI-ring* if every simple left (right) R-module is p-injective

Example 2.9.9. Every semisimple ring is left (right) P-V-ring

Proposition 2.9.10. ([13], Theorem 2) *A ring R is strongly regular if and only if R is a left P-V-ring and every maximal left ideal of R is a weakly right ideal of R*

Proof Suppose $l(a) + Ra \neq R$ for some $a \in R$ Then there exists a maximal left ideal L of R such that $l(a) + Ra \subseteq L$ Now R/L being a simple left R-module is p - injective (as R is a left P-V- ring) Define $f \quad Ra \longrightarrow R/L$ by $f(ra) = r + L$ Then f is a left R - homomorphism Hence there exists a left R - homomorphism $g \quad R \longrightarrow R/L$ which extends f and so $1 + L = f(a) = g(a) = ag(1) = a(b + L)$ for some $b \in R$ That is, $1 - ab \in L$ Since L is a weakly right ideal of R and $a \in L$, there exists a positive integer n such that $(ab)^n \in L$ Since $1 - ab \in L$ we have

$$ab(1 - ab) = ab - (ab)^2 \in L,$$

$$(ab)^2(1 - ab) = (ab)^2 - (ab)^3 \in L,$$

$$(ab)^{n-1}(1 - ab) = (ab)^{n-1} - (ab)^n \in L$$

Therefore $(ab)^n \in L$ implies that $ab \in L$ which further gives $1 = 1 - ab + ab \in L$, a contradiction. Hence $l(a) + Ra = R$ for all $a \in R$. This implies that R is strongly regular.

Converse follows from Proposition 2.2.8. □

Definition 2.9.11. A ring R is an *LW - ring (RW- ring)* if every left (right) ideal of R is a weakly right (left) ideal of R.

Example 2.9.12. Every commutative ring is an $LW - ring$ $(RW - ring)$.

Lemma 2.9.13. ([13], Lemma 3) *Suppose R is an LW - ring (RW - ring) and I be an ideal of R. Then R/I is also an LW - ring (RW – ring)*

Proof. Let A be a left ideal of R/I. Then $A = L/I$ for some left ideal L of R such that $L \supseteq I$. Let $x + I \in L/I$ and $r + I \in R/I$. Since L is weakly right ideal of R, there exists a positive integer n such that $(xr)^n \in L$. So $(x + I)(r + I)^n \in L/I$. This implies R/I is an LW - ring. □

Theorem 2.9.14. ([13], Theorem 6) *The following conditions are equivalent for a ring R:*

(1) R is a strongly regular ring.

(2) R is an LW, left $GP - V$-ring.

(3) R is an RW, right $GP - V$-ring.

Proof. (1) implies (2) and (1) implies (3) follows from Proposition 2.1.14 and Lemma 2.2.4.

(2) implies (1):- We first prove that R is reduced. Suppose $0 \neq a \in R$ such

that $a^2 = 0$. By Zorn's lemma, there exists a left ideal L of R such that $L \subsetneq Ra$ and Ra/L is a simple left R-module. Since R is a left $GP-V$-ring, Ra/L is YJ- injective. Again since $a^2 = 0$, the canonical left R - homomorphism $\eta : Ra \longrightarrow Ra/L$ can be extended to a left R - homomorphism $f : R \longrightarrow Ra/L$. Thus $a + L = \eta(a) = f(a) = af(1) = aba + L$ for some $b \in R$. That is, $a - aba \in L$. Since R is an LW- ring and $a \in Ra$, there exists a positive integer n such that $(ab)^n \in Ra$. As $a^2 = 0$, $(ab)^n a = 0$. Therefore

$$(ab)^{n-1}(a - aba) = (ab)^{n-1}a - (ab)^n a = (ab)^{n-1}a \in L,$$

$$(ab)^{n-2}(a - aba) = (ab)^{n-2}a - (ab)^{n-1}a \in L, (ab)^{n-2}a \in L,$$

$$\vdots$$

$$ab(a - aba) = aba - (ab)^2 a \in L, aba \in L,$$

$$a = aba + (a - aba) \in L.$$

But this gives $Ra = L$ contradicting $Ra \neq L$. Thus R is a reduced ring. Suppose $Ra + l(a) \neq R$ for some $a \in R$. Then $l(a) + Ra \subseteq M$ for some maximal left ideal M of R. R/M being a simple left R-module is YJ - injective. So there exists a positive integer n such that every left R - homomorphism from Ra^n to R/M extends to one from R to R/M. Define $f : R \longrightarrow R/M$ by $f(ra^n) = r + M$. Since R is reduced, $l(a^n) = l(a)$. This yields f is well-defined. Also f is a left R - homomorphism. So f can be extended to a left R - homomorphism $g : R \longrightarrow R/M$. Thus $1 + M = f(a^n) = g(a^n) = a^n g(1) = a^n(b + M)$ for some $b \in R$ implying therefore that $1 - a^n b \in M$. Since R is an LW- ring, M is a weakly right ideal of R. So there exists a positive integer t such that $(a^n b)^t \in M$. Then

$$\begin{aligned} a^n b(1 - a^n b) &= a^n b - (a^n b)^2 \in L, \\ (a^n b)^2 (1 - a^n b) &= (a^n b)^2 - (a^n b)^3 \in L, \\ &\vdots \\ (a^n b)^{t-1}(1 - a^n b) &= (a^n b)^{t-1} - (a^n b)^t \in L. \end{aligned}$$

Thus it follows from $(a^n b)^t \in L$ that $a^n b \in M$. But $1 = 1 - a^n b + a^n b \in M$, contradicting that M is a maximal left ideal of R. Thus $l(a) + Ra = R$ for all $a \in R$. This shows that R is strongly regular. □

2.10 Some questions

With the help of Proposition 2.5.3, we can think of the following two questions:

(1) Is R regular if R is an ELT left weakly regular ring?

(2) Is R regular if R is a right weakly regular ring whose every essential left ideals are GW-ideals?

We know that a ring R is strongly regular if and only if R is left duo, regular ring (by Proposition 2.1.14). If we weaken the condition we may get the following question:

(3) Is R strongly regular if R is a regular ring whose every left ideal is a GW-ideal?

Chapter 3

Some Properties of Flat Modules

In these chapter, we have studied some characterizations of flat modules via exact sequences; Injectivity of flat modules and vice versa ; SGPF rings.

3.1 Some exact sequences and flat modules

In this section, some characterizations of flat modules via some exact sequences are discussed.

Definition 3.1.1. A short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of left R-module is *pure* if $L \otimes_R M' \longrightarrow L \otimes_R M$ is a monomorphism for every right R-module L. Similarly a short exact sequence of right R-modules is defined to be pure.

Proposition 3.1.2. ([1], Chapter I, Proposition 11.1) *The following properties of a module $_RF$ are equivalent:*

(1) F is flat.

(2) Every exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow F \longrightarrow 0$ is pure.

(3) There is a pure exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow F \longrightarrow 0$ where M is a flat module.

Proof. (1) implies (2) :- If L is a right R-module, choose an exact sequence $0 \longrightarrow K \longrightarrow H \longrightarrow L \longrightarrow 0$ with H free. We get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & K \otimes M' & \xrightarrow{\mu_1} & K \otimes M & \xrightarrow{\mu_2} & K \otimes F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_7} & & \downarrow{\scriptstyle \mu_9} & & \downarrow{\scriptstyle \mu_{11}} & & \\
0 & \longrightarrow & H \otimes M' & \xrightarrow{\mu_3} & H \otimes M & \xrightarrow{\mu_4} & H \otimes F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_8} & & \downarrow{\scriptstyle \mu_{10}} & & \downarrow{\scriptstyle \mu_{12}} & & \\
 & & L \otimes M' & \xrightarrow{\mu_5} & L \otimes M & \xrightarrow{\mu_6} & L \otimes F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and column. We prove that μ_5 is a monomorphism. Let $x \in \ker \mu_5$. Since μ_8 is onto, there exists some $y \in H \otimes M'$ such that $\mu_8(y) = x$. Now $\mu_5(\mu_8(y)) = \mu_{10}(\mu_3(y))$. Then $\mu_{10}(\mu_3(y)) = 0$. This implies

$\mu_3(y) \in \ker \mu_{10} = \operatorname{im} \mu_9$. Therefore there exists some $z \in K \otimes M$ such that $\mu_9(z) = \mu_3(y)$. Now $\mu_{11}(\mu_2(z)) = \mu_4(\mu_9(z)) = \mu_4(\mu_3(y)) = 0$. Then $\mu_2(z) \in \ker \mu_{11}$. Then μ_{11} is one-one implies $\mu_2(z) = 0$. Therefore $z \in \ker \mu_2 = \operatorname{im} \mu_1$. Then there exists some $v \in K \otimes M'$ such that $\mu_1(v) = z$. Then $\mu_3(\mu_7)(v) = \mu_9(\mu_1)(v) = \mu_9(z) = \mu_3(y)$. Then $\mu_7(v) = y$ since μ_3 is one-one. Then $0 = \mu_8\mu_7(v) = \mu_8(y) = x$. Therefore $\ker \mu_5 = 0$ and hence μ_8 is one-one.

(2) implies (3) is obvious (write F as a quotient of a free module)

(3) implies (1) :- Let $0 \longrightarrow M' \longrightarrow M \longrightarrow F \longrightarrow 0$ be a pure exact sequence of left R-modules with M flat. We have to prove that F is flat. Consider any exact sequence $0 \longrightarrow K \longrightarrow H \longrightarrow L \longrightarrow 0$ of right modules. Then we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & K \otimes M' & \xrightarrow{\mu_1} & K \otimes M & \xrightarrow{\mu_2} & K \otimes F & \longrightarrow & 0 \\
 & & \downarrow \mu_7 & & \downarrow \mu_9 & & \downarrow \mu_{11} & & \\
0 & \longrightarrow & H \otimes M' & \xrightarrow{\mu_3} & H \otimes M & \xrightarrow{\mu_4} & H \otimes F & \longrightarrow & 0 \\
 & & \downarrow \mu_8 & & \downarrow \mu_{10} & & \downarrow \mu_{12} & & \\
0 & \longrightarrow & L \otimes M' & \xrightarrow{\mu_5} & L \otimes M & \xrightarrow{\mu_6} & L \otimes F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and column. We have to prove that μ_{11} is a monomorphism.

Let $x \in \ker \mu_{11}$. Then $\mu_{11}(x) = 0$. Since μ_2 is onto, there exists some $y \in K \otimes M$ such that $\mu_2(y) = x$. Now $\mu_{11}(\mu_2(y)) = \mu_4(\mu_9(y))$. Then $0 = \mu_4(\mu_9(y))$. So $\mu_9(y) \in \ker \mu_4 = \operatorname{im} \mu_3$. Therefore there exists some $z \in H \otimes M'$ such that $\mu_3(z) = \mu_9(y)$. Now $\mu_{10}(\mu_3(z)) = \mu_5(\mu_8(z))$. Then $0 = \mu_{10}(\mu_9(y)) = \mu_5(\mu_8(z))$. Then μ_5 is one one implies $\mu_8(z) = 0$. Therefore $z \in \ker \mu_8 = \operatorname{im} \mu_7$. Then there exists some $v \in K \otimes M'$ such that $\mu_7(v) = z$. Now $\mu_3(\mu_7(v)) = \mu_9(\mu_1(v))$. Then $\mu_3(z) = \mu_9(\mu_1(v))$. Then $y = \mu_1(v)$. Therefore $x = \mu_2(y) = \mu_2(\mu_1(v)) = 0$. Thus μ_{11} is a monomorphism. □

Lemma 3.1.3. ([2], Lemma 3.14) *Consider the following commutative diagram of left R-modules with exact rows:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

Then

(1) If α, γ, f' are monic, then so is β.

(2) If α, γ, g are epic, then so is β.

(3) If β is monic and α and g are epic, then γ is monic.

(4) If β is epic and f' and γ are monic, then α is epic.

Proposition 3.1.4. *Let* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *be an exact sequence of left R-module with N flat. Then L is flat if and only if M is flat.*

Proof. Let I be a finitely generated right ideal of R. We get a commutative diagram with exact rows as follows:

$$\begin{array}{ccccc} I \otimes L & \xrightarrow{1 \otimes f} & I \otimes M & \xrightarrow{1 \otimes g} & I \otimes N \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_3} \\ L & \xrightarrow{f} & M & \xrightarrow{g} & N \end{array}$$

where $\mu_1(x \otimes y) = xy$, $\mu_2(x \otimes m) = xm$, $\mu_2(x \otimes n) = xn$, $x \in I$, $y \in L$, $m \in M$, $n \in N$.

Let L be flat. Then μ_1 is monic. Also N is flat implies μ_3 is monic. Therefore by Lemma 3.1.3, we get μ_2 is monic. Therefore M is flat.

Conversely, suppose M is flat. Then μ_2 is monic. Also N is flat implies

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is pure. Therefore $I \otimes L \xrightarrow{1 \otimes f} I \otimes M$ is a monomorphism. Therefore $f\mu_1 = \mu_2(1 \otimes f)$ is monic. This implies μ_1 is monic. Therefore L is flat. □

Lemma 3.1.5. ([2], Lemma 19.18) *Let V be a flat left R-module and K be a submodule of V. Let V' be a left R-module and*

$$0 \longrightarrow K \xrightarrow{j} V \xrightarrow{f} V' \longrightarrow 0$$

be exact. Then V' is flat if and only if for all finitely generated right ideal I of R, $IK = K \cap IV$.

Proof. Let I be a finitely generated right ideal of R. Consider the diagram

$$
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
I \otimes K & \xrightarrow{i \otimes j} & I \otimes V & \xrightarrow{i \otimes f} & I \otimes V' & \longrightarrow & 0 \\
\downarrow \mu & & \downarrow \mu_1 & & \downarrow \mu_1' & & \\
K \cap IV & \xrightarrow{j'} & IV & \xrightarrow{f|_{IV}} & V' & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
$$

where $\mu(r \otimes k) = rk$, $\mu_1(r \otimes v) = rv$, $\mu_1'(r \otimes v') = rv'$, j, j' are the inclusion map and i is identity map. The diagram is commutative and has exact rows. Since V is flat, μ_1 is an isomorphism.

Let V' be flat. Then μ_1' is a monomorphism. Therefore by Lemma 3.1.3, we have μ is an epimorphism. So $\mu(I \otimes K) = K \cap IV$. Hence $IK = K \cap IV$.

Conversely, suppose $IK = K \cap IV$. Then μ is an epimorphism. But this implies μ_1' is monic as μ_1 is monic and $i \otimes f$ is epic. Therefore V' is flat. □

Corollary 3.1.6. *Let R be a ring and I be a left ideal of R. Then R/I is flat left R-module if and only if for all $a \in I$, there exists some $b \in I$ such that $a = ab$.*

Proof. Consider the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. Let R/I be flat left R-module and $a \in I$. As R/I is flat, by Lemma 3.1.5, $(aR)I = I \cap (aR)R = I \cap aR$. Then $aI = I \cap aR$. Then $a \in I \cap aR$ implies

$a \in aI$. Therefore there exists some $b \in I$ such that $a = ab$.

Conversely, suppose for each $a \in I$ there exists some $b \in I$ such that $a = ab$. Let K be a finitely generated right ideal of R. We have to prove that $KI = I \cap KR = I \cap K$. Clearly $KI \subseteq I \cap K$. Let $x \in I \cap K$. Then $x \in I$ and $x \in K$. By hypothesis there exists some $y \in I$ such that $x = xy$. Therefore $x \in KI$. Thus $I \cap K \subseteq KI$. Hence $I \cap K = KI$. □

We can similarly prove the following:

Corollary 3.1.7. *Let R be a ring and I be a right ideal of R. Then R/I is a flat right R-module if and only if for each $a \in I$ there exists some $b \in I$ such that $a = ba$.*

Lemma 3.1.8. ([3], Lemma 3.38) *Let $0 \longrightarrow K \longrightarrow F \longrightarrow B \longrightarrow 0$ be an exact sequence of left R-module where F is free with basis $T = \{x_j : j \in J\}$. Let $v = r_1x_{j_1} + r_2x_{j_2} + \cdots + r_tx_{j_t}$ be an element of F and I_v be the right ideal generated by $r_1, r_2, \ldots, r_t$. Then B is flat if and only if for each $v \in K$, $v \in I_vK$.*

Proof. Suppose B is flat. Then $I_vK = K \cap I_vF$. Since $v \in K \cap I_vF$, $v \in I_vK$.

Conversely, let I be a left ideal of R. We have to prove that $IK = K \cap IF$. Let $x \in IK$. Then $K \subseteq F$ implies $x \in K \cap IF$. So $IK \subseteq K \cap IF$. If $v \in K \cap IF$, then by hypothesis, $v \in I_vK$. Also $v \in IF$ implies $v = r_1x_{j_1} + r_2x_{j_2} + \cdots + r_tx_{j_t}$, where $r_1, r_2, \ldots, r_t \in I$, $x_{j_1}, x_{j_2}, \ldots, x_{j_t} \in T$. Thus we have $I_v = r_1R + r_2R + \cdots + r_tR \subseteq I$ which yields $v \in I_vK \subseteq IK$. Therefore $K \cap IF \subseteq IK$. Hence we have $IK = K \cap IF$. □

Theorem 3.1.9. *(Villamayor)* ([3], Theorem 3.39) *Let* $0 \longrightarrow K \longrightarrow F \longrightarrow B \longrightarrow 0$ *be an exact sequence of left R-module where F is free. The following are equivalent:*

(1) B is flat.

(2) For every $v \in K$*, there exists a left R - homomorphism* $\theta : F \longrightarrow K$ *with* $\theta(v) = v$.

(3) For every $v_1, v_2, \ldots, v_n \in K$*, there exists a left R - homomorphism* $\theta : F \longrightarrow K$ *with* $\theta(v_i) = v_i$ *for each* $i = 1, 2, \ldots, n$.

Proof. (1) implies (2):- Let $T = \{x_j : j \in J\}$ be a basis of F and $v \in K$. Then $v = r_1x_{j_1} + r_2x_{j_2} + \cdots + r_tx_{j_t}$, where $r_1, r_2, \ldots, r_t \in R$, $x_{j_1}. x_{j_2}, \ldots, x_{j_t} \in T$. Let $I_v = r_1R + r_2R + \cdots + r_tR$. Now B is flat implies $v \in I_vK$. So $v = \sum_\lambda r_\lambda k_\lambda$, $r_\lambda \in I_v$, $k_\lambda \in K$. Now $r_\lambda \in I_v$ implies $r_\lambda = \sum_{i=1}^{k} r_i s_{\lambda_i}$ for some $s_{\lambda_i} \in R$ for each λ. Then $v = \sum_\lambda r_\lambda k_\lambda = \sum_\lambda \left(\sum_i r_i s_{\lambda_i}\right) k_\lambda = \sum_i r_i {k_i}'$ where ${k_i}' = \sum_\lambda s_{\lambda_i k_\lambda} \in K$.

Define $\theta : F \longrightarrow K$ by $\theta(x_{j_i}) = k_i'$ for each $i = 1, 2, \ldots, t$ and θ sends all other basis elements of F into 0. Then $\theta(v) = v$.

(2) implies (1) :- Let $v \in K \cap I_vF$ and $v = l_1y_1 + l_2y_2 + \cdots + l_my_m$ where $l_i \in I_v$, $y_i \in F$ for each $i = 1, 2, \ldots, m$.

Let $\theta : F \longrightarrow K$ be a left R - homomorphism such that $\theta(v) = v$. Then $v = \theta(v) = l_1\theta(y_1) + l_2\theta(y_2) + \cdots + l_m\theta(y_m) \in I_vK$. This proves that B is flat.

(3) implies (2) is trivial.

(2) implies (3):- We prove by induction on n. By hypothesis, the result is true for $n = 1$. Assume $n > 1$ and consider the elements $v_1, v_2, ..., v_n \in K$. By hypothesis, there is a map $\theta_n : F \longrightarrow K$ such that $\theta_n(v_n) = v_n$ Define $v_i' \in K$ by $v_i' = v_i - \theta_n(v_i)$ for each $i = 1, 2, \ldots, n-1$. By induction, there is a map $\theta' : F \longrightarrow K$ with $\theta'(v_i') = v_i'$ for each $i = 1, 2, \ldots, n-1$. Finally, let $\theta = 1_F - (1_F - \theta')(1_F - \theta_n)$. Then $\theta : F \longrightarrow K$. Also

$\theta(v_n) = (1_F - (1_F - \theta')(1_F - \theta_n))(v_n) = v_n - (1_F - \theta')(v_n - \theta_n(v_n))$

$= v_n - (1_F - \theta')(v_n - v_n) = v_n$.

For each $i = 1, 2, \ldots, n-1$,

$\theta(v_i) = (1_F - (1_F - \theta')(1_F - \theta_n))(v_i) = v_i - (1_F - \theta')(v_i - \theta_n(v_i))$

$= v_i - (1_F - \theta')(v_i') = v_i - v_i' + \theta'(v_i') = v_i - v_i' + v_i' = v_i$. □

Definition 3.1.10. A module ${}_RM$ is *finitely related* if there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ where F is free and both F and K are finitely generated.

Example 3.1.11. A finite dimensional vector space is finitely related

Corollary 3.1.12. ([3], Corollary 3.40) *A finitely related flat module ${}_RM$ is projective.*

Proof. ${}_RM$ is finitely related implies there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ where F is free and both F and K are finitely generated. Let $v_1, v_2, ..., v_n$ be the generators of K. By Theorem 3.1.9, there is a left R - homomorphism $\theta : F \longrightarrow K$ such that $\theta(v_i) = v_i$. Then $\theta j(v_i) = \theta(v_i) = v_i = 1_K(v_i)$. This implies $\theta j = 1_K$. So the sequence splits and ${}_RM$ is projective. □

Remark 3.1.13. By using Theorem 3.1.9, Corollary 3.1.6 can be proved as follows:

Consider the exact sequence

$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. Let $_R(R/I)$ is flat. Then by Theorem 3.1.9, for each $a \in I$ there exists a homomorphism $\theta : R \longrightarrow I$ such that $\theta(a) = a$. Then $a\theta(1) = a$. So $a = ab$ where $b = \theta(1) \in I$.

Conversely, suppose $a \in I$ and there exists some $b \in I$ such that $a = ab$. Define $\theta : R \longrightarrow I$ by $\theta(r) = rb$. Then $\theta(a) = ab = b$. Thus by Theorem 3.1.9, $_R(R/I)$ is flat.

Theorem 3.1.14. *If R is left Noetherian, then every finitely generated flat left R-module M is projective.*

Proof. Since $_RM$ is finitely generated, M is isomorphic to a quotient of R^n for some $n > 0$. Let $M \simeq R^n/K$. Then the sequence

$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0$ is exact. For a left Noetherian ring R, we know that every submodule of a finitely generated left R-module is finitely generated. Then K is finitely generated. Then M is finitely related. Since a finitely related flat R-module is projective, $_RM$ is projective. □

3.2 Flat modules and injectivity

Here, the question of when certain cyclic flat modules of a ring are injective (and vice versa) is studied. The consequences of the conditions 'flat' and 'injective' on the simple modules of a ring are discussed.

Proposition 3.2.1. *If I is p-injective left ideal of a ring R, then ${}_R(R/I)$ is flat.*

Proof. Let $a \in I$. Consider the inclusion map $j : Ra \longrightarrow I$. Since I is p-injective, j can be extended to $f : R \longrightarrow I$. So $a = j(a) = f(a) = af(1) = ab$ where $b = f(1) \in I$. Therefore by Corollary 3.1.6, ${}_R(R/I)$ is flat □

Lemma 3.2.2. ([16], Lemma 1.2) *Let A be an ideal of a ring R. If R/A is left R-flat and B is any right ideal of R such that $B \subseteq A$ or $B + A = R$, then R/A is B-complete.*

Proof. Let $f : B \longrightarrow R/A$ be any right R-homomorphism. Let $B \subseteq A$ and $x \in B$. Since ${}_R(R/A)$ is flat, there exists some $y \in A$ such that $x = xy$. Then $f(x) = f(xy) = f(x)y = 0$ as y is in A. Hence $f = 0$ and hence f can be trivially extended to a right R-homomorphism $R \longrightarrow R/A$. If $B + A = R$, then there exists some $b \in B$ and $a \in A$ such that $b + a = 1$. Therefore for all $x \in B$, $bx + ax = x$. So $f(x) = f(bx) + f(ax)$. Since $ax \in A$ and ${}_R(R/A)$ is flat, there exists some $t \in A$ such that $ax = (ax)t$. Since ax is in B we get $f(ax) = f(axt) = f(ax)t = 0$. Therefore $f(x) = f(bx) = f(b)x$.
Hence f can be extended to a right R-homomorphism $R \longrightarrow R/A$. □

Lemma 3.2.3. ([16], Lemma 1.3) *Let A be a maximal right ideal of R which is two-sided. If R/A is pR-complete for each p in R, then R/A is left R-flat.*

Proof. It is sufficient to prove that $a \in aA$ for all $a \in A$. Consider the epimorphism $f : R/A \longrightarrow aR/aA$ defined by $f(r + A) = ar + aA$ for all $r \in R$. If $f \equiv 0$, then $aR = aA$. Then $a \in aA$. Suppose $f \not\equiv 0$. Then $\ker f = \{0\}$ (Since R/A is simple right R-module) and so f is an isomorphism.

Therefore $_R(R/A)$ is aR-complete implies aR/aA is aR-complete. Thus the right R-homomorphism $g : aR \longrightarrow aR/aA$ defined by $g(ar) = ar + aA$ for all $r \in R$ can be extended to a right R-homomorphism $h : R \longrightarrow aR/aA$. So $a + aA = g(a) = h(a) = h(1)a = (ar + aA)a$ for some $r \in R$. That is, $a - ara \in aA$. But $ara \in aA$ so that $a \in aA$. This completes the proof. □

Proposition 3.2.4. ([16], Proposition 1.4) *Let A be a maximal right ideal of R which is two sided sided, then following are equivalent:*

(1) R/A is left R-flat.

(2) R/A is right R-injective.

(3) R/A is pR-complete for each p in R.

Proof. (1) implies (2) :- Let R/A be left R-flat. Let $0 \neq B$ be a right ideal of R. Then $A + B = R$. Then R/A is B-complete by Lemma 3.2.2. Therefore R/A is injective right R-module.

(2) implies (1):- If $(R/A)_R$ is injective, then R/A is pR-complete for each p in R. Then by Lemma 3.2.3, R/A is left R-flat.

(2) implies (3):- Trivial.

(3) implies (1):- Follows from Lemma 3.2.3. □

Definition 3.2.5. Let A be a right ideal of a ring R. Then the set $\overline{A} = \{x \in R \mid xA \subseteq A\}$ is the *idealizer* of A in R.

Remark 3.2.6. (1) Let $x \in \overline{A}$, $y \in \overline{A}$. Then $xA \subseteq A, yA \subseteq A$. So $(x - y)A \subseteq (xA - yA) \subseteq A$. So $(x - y) \in \overline{A}$. Also $(xy)A \subseteq xA \subseteq A$. Hence $xy \in \overline{A}$. Therefore $\overline{A}$ is a subring of R.

(2) Clearly $A \subseteq \overline{A}$. Since $\overline{A}$ is a subring of R and A is a right ideal of R, A is a right ideal of $\overline{A}$. Let $x \in A$, $r \in \overline{A}$. Then $rA \subseteq A$. So $rx \in A$. This implies A is a left ideal of $\overline{A}$. Therefore A is an ideal of $\overline{A}$.

Proposition 3.2.7. ([16], Proposition 1.5) *Let A be a maximal right ideal of R. Then R/A is right R-flat $\Leftrightarrow$ $\overline{A}/A$ is left $\overline{A}$-injective $\Leftrightarrow$ $\overline{A}/A$ is $\overline{A}p$-complete for each p in $\overline{A}$.*

Proof. It is easy to see that R/A is right R flat $\Leftrightarrow$ $\overline{A}/A$ is right $\overline{A}$-flat Also we note that A is a maximal left ideal of $\overline{A}$ which is two sided. Hence applying the left analogue of Proposition 3.2.4, the proof can be completed. □

Corollary 3.2.8. ([16], Corollary 1.6) *If M is a simple left R-module over a commutative ring R, then M is flat $\Leftrightarrow$ M is injective $\Leftrightarrow$ M is pR-complete for each p in R.*

Proposition 3.2.9. ([16], Proposition 2.1) *Let each principal right ideal of a ring R be projective and A is an ideal of R such that R/A be pR-complete for each p in R. Then R/A is left R-flat R/A will also be right R-flat if, further, R contains no non-zero nilpotent elements.*

Proof. Let $a \in A$ and $g : aR \longrightarrow aR/aA$ be natural map. Let $f : R/A \longrightarrow aR/aA$ be homomorphism defined by $f(r + A) = ar + aA$. Clearly f is an epimorphism. Since aR is projective, there exists a homomorphism $h : aR \longrightarrow R/A$ such that $fh = g$. Since R/A is aR-complete, h extends to a homomorphism $\overline{h} : R \longrightarrow R/A$. Let $\overline{g} = f\overline{h}$. Now for all $x \in aR$, $\overline{g}(x) = f(\overline{h}(x)) = f(h(x)) = g(x)$. So $\overline{g}$ extends g. Therefore $a + aA = g(a) = \overline{g}(a) = \overline{g}(1)a = (ar + aA)a$ for some $r \in R$. So

$(a - ara) \in aA$. Therefore $ara \in aA$ implies $a \in aA$. Hence R/A is left R-flat.

Let R contains no nonzero nilpotent elements. Let $a \in A$. Since R/A is left R-flat there exists some $x \in A$ such that $a = ax$. So, $(a - xa)^2 = a^2 - axa - xa^2 + rara = 0$. Therefore by assumption, $a - ra = 0$ which implies $a = xa$. This implies R/A is right R-flat □

Corollary 3.2.10. ([16], Corollary 2.2) *If R is a commutative ring in which each principal ideal is projective, then any cyclic module which is pR-complete for each p in R is flat.*

Remark 3.2.11. SPI-rings have analogous, but weaker properties than regular rings as follows:

Proposition 3.2.12. ([16], Proposition 3.1) *Let R be a right SPI-ring. Then*

(1) For each $a \in R$, there is an x in RaR such that $a = ax$.

(2) For each ideal A in R, (R/A) is left R-flat

(3) For each maximal right ideal M of R which is two sided, R/M is right injective.

Proof. (1) Let $a \in R$. Suppose $a \neq ax$ for every $x \in RaR$. Then $a \neq aRaR$. Let $\mathcal{F} = \{K : K \text{ is a right ideal of } R, a \notin K, aRaR \subseteq K\}$. Order $\mathcal{F}$ by inclusion. Then $\mathcal{F} \neq \phi$ as $aRaR \in \mathcal{F}$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Let $\mathcal{B} = \bigcup_{\mathcal{D} \in \mathcal{C}} \mathcal{D}$. Then $\mathcal{B} \in \mathcal{F}$ and $\mathcal{B}$ is an upper bound of $\mathcal{C}$. Therefore by Zorn's Lemma, $\mathcal{F}$ has a maximal element K_0. Let $X = (aR + K_0)/K_0$. Now K_0 is a maximal submodule of $aR + K_0$ [for if K_1 is a submodule of $aR + K_0$ such that $K_0 \subseteq K_1 \subsetneq aR + K_0$, then $a \notin K_1$. Also $aRaR \subseteq K_0 \subseteq K_1$. So $K_1 \in \mathcal{F}$, then K_0 is a maximal element of $\mathcal{F}$ implies $K_1 = K_0$]. Therefore X is a simple right R-module and hence aR complete by hypothesis. Therefore the natural map $f : aR \longrightarrow (aR + K_0)/K_0$ can be extended to $g : R \longrightarrow (aR + K_0)/K_0$. Then $a + K_0 = f(a) = g(a) = g(1)a = (ar + K_0)a$ for some $r \in R$ which gives $(a - ara) \in K_0$. But $ara \in K_0$ implies $a \in K_0$, a contradiction. Therefore $a = ax$ for some $x \in RaR$.

(2) Let $a \in A$, from (1), there exist $x \in RaR \subseteq A$ such that $a = ax$. This implies R/A is flat left R-module.

(3) From (2), R/M is flat left R-module. This implies R/M is injective right R-module by Proposition 3.2.4. □

3.3 SGPF rings

In this section, we study some characterizations of strongly regular ring via SGPF ring.

Definition 3.3.1. A ring R is a left *SPF ring* if every simple left R-module is either p-injective or flat.

Definition 3.3.2. A ring R is a *left SGPF ring* if every simple left R-module is GP-injective or flat.

Example 3.3.3. A semisimple ring R is a left SPF (SGPF) ring.

Proposition 3.3.4. *Let $f : R \longrightarrow R'$ be a ring homomorphism and S be a left R' module which is left R - flat. Then S is left R' - flat.*

Proof. Let $g : M' \longrightarrow M$ be a monomorphism of right R' module. Define $g' : M'_R \longrightarrow M_R$ by $g'(x') = g(x')$. Then for all $x' \in M'$, $y' \in M'$, $r \in R$, $g'(x'+y') = g(x'+y') = g(x')+g(y')$. Also $g'(x'r) = g'(x'f(r)) = g(x'f(r)) = g(x')f(r) = g'(x')f(r) = g'(x')r$. Also $\ker g' = \ker g = 0$. Thus g' is a right R monomorphism. Then S is left R - flat implies

$g' \otimes Id : M' \otimes_R S \longrightarrow M \otimes_R S$ is a monomorphism.

Claim:- $g \otimes Id : M' \otimes_{R'} S \longrightarrow M \otimes_{R'} S$ is a monomorphism.

Let $\sum_{i=1}^{n} x' \otimes y \in \ker(g \otimes Id)$. Then $(g \otimes Id)\left(\sum_{i=1}^{n} x' \otimes y\right) = 0$. Then $\sum_{i=1}^{n} g(x') \otimes y = 0$. This implies $\sum_{i=1}^{n} g'(x \otimes y) = 0 = (g' \otimes Id)\left(\sum_{i=1}^{n}(x' \otimes y)\right)$.

So $\sum_{i=1}^{n}(x' \otimes y) = 0$. Thus S is left R' - flat. □

Lemma 3.3.5. ([14], Lemma 2.7) *If R is a left $SGPF$ ring and I be an ideal of R, then R/I is also a left $SGPF$ ring.*

Proof. Let $\overline{R} = R/I$ and $\overline{L}$ be a simple left $\overline{R}$ module. Then $\overline{L}$ is a simple left R-module. Since R is a left $SGPF$ ring, $\overline{L}$ is a flat left R-module or left GP-injective. If $\overline{L}$ is flat left left R-module, then $\overline{L}$ is flat left $\overline{R}$ module by Proposition 3.3.4. If $\overline{L}$ is left GP - injective, then for any $\overline{a} \in \overline{L}$ there exists a positive integer n such that any left R - homomorphism from $R\overline{a}^n$ to $\overline{L}$ extends to one from R to $\overline{L}$. Let $\overline{f} : \overline{R}\,\overline{a}^n \longrightarrow \overline{L}$ be any left $\overline{R}$ homomorphism and $\eta : Ra^n \longrightarrow \overline{R}\,\overline{a}$ be the canonical homomorphism. Now $\overline{f}$ can be viewed as an R - homomorphism and $f' = \overline{f}\eta : Ra^n \longrightarrow \overline{L}$ is a left R - homomorphism. So f can be extended to a left R - homomorphism from R to $\overline{L}$. Hence $f(a^n) = a^n\overline{b}$ for some $\overline{b} \in \overline{L}$. This implies

$\overline{f}(\overline{a^n}) = \overline{f}(\eta(a^n)) = f'(a^n) = a^n\bar{b} = \bar{a}^n\bar{b}$ which implies that $\overline{L}$ is a GP injective left $\overline{R}$ module. Therefore $\overline{R} = R/I$ is a left $SGPF$ ring. □

Theorem 3.3.6. ([14], Theorem 2.8) *Let R be a left $SGPF$ ring. If every maximal left ideal of R is a GW-ideal, then $R/J(R)$ is strongly regular.*

Proof. Let $B = R/J(R)$, then $J(B) = 0$. Let M be a maximal left ideal of B, then $M = L/J(R)$ for some maximal left ideal L of R. By hypothesis L is a GW-ideal of R and hence M is a GW-ideal of B. Therefore B is a reduced left $SGPF$ ring. If for any $a \in B$, $Ba + l(a) \neq B$, then there exists a maximal left ideal L of B such that $Ba + l(a) \subseteq L$. So B/L is a simple left B-module and hence is GP - injective or flat. Suppose B/L is GP - injective and $n > 0$ be an integer such that every left B-homomorphism from Ba^n to B/L can be extended to a left B - homomorphism from B to B/L. Consider the left B-homomorphism $f : Ba^n \longrightarrow B/L$ given by $f(ba^n) = b + L$. Since B is reduced, $l(a^n) = l(a) \subseteq L$. This yields f is well-defined. Also f is left B - homomorphism. Hence f can be extended to a left B - homomorphism $g : B \longrightarrow B/L$. So $1 + L = f(a^n) = g(a^n) = a^n g(1) = a^n c + L$ for some $c \in B$ and therefore $1 - a^n c \in L$. Since L is a GW-ideal there exists a positive integer k such that $(ca^n)^k c \in L$. Thus we get

$$(ca^n)^{k-1}c = (ca^n)^{k-1}(c - ca^n c) + (ca^n)^k c \in L.$$

Proceeding in this manner we get $ca^n c \in L$. Hence $c = c - ca^n c + ca^n c \in L$. Then $a^n c \in L$. This implies $1 = 1 - a^n c + a^n c \in L$, a contradiction. If B/L is flat, then there exists $b \in L$ such that $a = ab$. This implies $1 - b \in r(a) = l(a) \subseteq L$. Therefore $1 = 1 - b + b \in L$, a contradiction. Hence $Ba + l(a) = B$ for all $b \in B$. This implies B is strongly regular. □

Corollary 3.3.7. ([14], Corollary 2.9) *If R is a left quasi-duo left SPF ring, then $R/J(R)$ is strongly regular.*

Theorem 3.3.8. ([14], Theorem 2.10) *Let R be a left $SGPF$ ring. If every maximal right ideal of R is a GW-ideal, then $R/J(R)$ is strongly regular.*

Proof. Let $B = R/J(R)$. Then B is reduced as in Theorem 3.3.6. If $cB + l(c) \neq B$ for some $c \in B$, then $cB + l(c) \subseteq L$ for some maximal right ideal L of R. Since L is a GW-ideal and $c \in L$ there exists a positive integer n such that $Bc^n \subseteq L$. Hence $Bc^nB \subseteq L$. Therefore there exists a maximal left ideal M of B such that $Bc^nB + l(c) \subseteq M$. So B/M is a simple left B-module and hence is GP - injective or flat. Suppose B/M is GP - injective, let k be a positive integer such that every left B-module homomorphism from $B(c^n)^k$ to B/M extends to one from B to B/M. Define $f : B(c^n)^k \longrightarrow B/M$ by $f(b(c^n)^k) = b + K$. Since B is reduced, $l((c^n)^k) = l(c^n)$. This yields f is well-defined. Let $g : B \longrightarrow B/M$ be a left B-homomorphism which extends f. Then $1 + K = f(c^n)^k = g(c^n)^k = (c^n)^k g(1) = (c^n)^k(d + M)$ for some $d \in B$. So $1 - (c^n)^k d \in M$. But $(c^n)^k d \in Bc^nB \subseteq M$. Hence $1 = 1 - (c^n)^k d + (c^n)^k d \in M$ which contradicts that M is a maximal left ideal of B. If B/M is flat, then there exists $q \in M$ such that $c^n = c^n q$, then $1 - q \in r(c^n) = l(c^n) = l(c) \subseteq M$. This implies $1 = 1 - q + q \in M$ which is a contradiction. Hence $cB + l(c) = B$ for all $c \in B$. This implies B is strongly regular. □

Chapter 4

SF-Rings

It is well known that a ring R is regular if and only if every cyclic left (right) R-modules are flat. In 1975, Ramamurthy weakened the condition and asked the question whether a ring over which all simple left (right) modules are flat (that is, a left (right) SF-ring) is necessarily regular. This question is still open despite the work done in the positive direction by different authors over the last three and a half decades. In this chapter, basically we study regularity and some other properties of SF-rings.

Definition 4.0.9. A ring R is a *left SF-ring* if every simple left R-module is flat. Similarly a ring R is defined to be a *right SF-ring*.

Example 4.0.10. A von Neumann regular ring is left (right) SF-ring (by Proposition 2.1.12).

4.1 Central localizations of SF-rings

In this section, we study some properties of centre of SF-rings; 'Simple implies flat' property with respect to central localizations; Quasi perfect SF-rings; Semilocal SF-rings.

Proposition 4.1.1. ([16], Proposition 3.2) *Let R be a ring such that the left annihilator of any element of R is also a right ideal. If R is a right SF-ring, then R is regular.*

Proof. Let $a \in A$, by hypothesis, $l(a)$ is a right ideal. If $l(a) + aR \neq R$, then there exist a maximal right ideal M of R such that $l(a) + aR \subseteq M$. Since R is right SF, the simple right R-module R/M is flat. Then $a \in M$ implies there exists some $b \in M$ such that $a = ba$. Then $1 - b \in l(a) \subseteq M$. But this implies $1 \in M$, a contradiction. Therefore $l(a) + aR = R$. Therefore $1 = x + ar$ for some $x \in l(a)$ and $r \in R$ and so, $a = ax + ara$. Hence $a = ara$ (Since $ax = 0$). Thus R is regular. □

Corollary 4.1.2. *A reduced right SF-ring R is strongly regular.*

Proof. Since R is reduced, $l(a) = r(a)$ for every $a \in R$. Therefore by Proposition 4.1.1, R is regular. This implies R is strongly regular, as R is reduced. □

Proposition 4.1.3. ([16], Proposition 3.3) *The centre of any right (left) SF -ring is von Neumann regular.*

Proof. Let R be a right SF-ring and C be its centre. Let $a \in C$, then $l(a)$ is an ideal of R. By the proof of Proposition 4.1.1, $a = ara$ for some $r \in R$.

Now just as in the proof of the fact that centre of a von Neumann ring is regular, we can prove that $a^2r^3 \in C$ and $a = a(a^2r^3)a$. □

Corollary 4.1.4. *A commutative right (left) SF-ring is von Neumann regular.*

Proposition 4.1.5. ([17], Proposition 3.1) *Let R be a left SF-ring and M be a maximal left ideal of R. Then M is a flat left R-module.*

Proof. Consider the exact sequence $0 \longrightarrow M \longrightarrow R \longrightarrow R/M \longrightarrow 0$. Since R is left SF, $_R(R/M)$ is flat. Also R is flat left R-module. Therefore by Proposition 3.1.4, M is flat. □

Proposition 4.1.6. ([17], Proposition 3.2) *Let $f : R \longrightarrow R'$ be an onto homomorphism of rings. If R is a left SF-ring, then R' is also a left SF-ring.*

Proof. Let S be a simple left R' module. Then S is also a simple left R-module. But R is left SF implies $_RS$ is flat. Then $_{R'}S$ is flat by Proposition 3.3.4. Therefore R' is a left SF-ring. □

Corollary 4.1.7. *If R is a left SF-ring and I be an ideal of R, then R/I is also a left SF-ring.*

Proposition 4.1.8. ([17], Proposition 3.4) *Let R be a ring with centre C. Consider the following conditions:*

(1) R is a left SF-ring.

(2) For each maximal ideal m of C the localization $R_m (= S^{-1}R$, where $S = C - m)$ is a left SF-ring.

Then (1) implies (2). If R is finitely generated as a C-algebra, then (2) implies (1).

Proof. (1) implies (2):- We have C is regular (since centre of a left SF-ring is regular). Thus for each maximal ideal m of C the natural homomorphism $R \longrightarrow R_m$ is onto. Then R_m is a left SF-ring by Proposition 4.1.6.

Conversely, suppose R is a finitely generated C-algebra. Then $R = C[x_1, x_2, ...x_n]$ for some $x_1, x_2, \ldots, x_n \in R$.

Claim:- For any multiplicatively closed subset S of C, $Centre\,(S^{-1}R) = S^{-1}C$.

Clearly $S^{-1}C \subseteq centre\,(S^{-1}R)$. Let $\frac{x}{s} \in centre\,(S^{-1}R)$. Then for all $\frac{y}{t} \in S^{-1}R$, $\frac{x}{1}\frac{y}{t} = \frac{x}{s}\frac{sy}{t} = \frac{sy}{t}\frac{x}{s} = \frac{y}{t}\frac{x}{1}$. So $\frac{x}{1} \in centre\, S^{-1}R$. Thus for all $y \in R$, $\frac{x}{1}\frac{y}{1} = \frac{y}{1}\frac{x}{1}$. In particular for all x_i, $i = 1, 2, \ldots, n$, $\frac{x}{1}\frac{x_i}{1} = \frac{x_i}{1}\frac{x}{1}$. So for each i, there exists some $u_i \in S$ such that $u_i(xx_i - x_ix) = 0$. This implies if $u = u_1u_2 \ldots u_n$, $u(xx_i - x_ix) = 0$. Thus $uxx_i = x_iux$. Thus $R = C[x_1, x_2, ...x_n]$ implies $ux \in C$. This again gives $\frac{x}{s} = \frac{ux}{us} \in S^{-1}C$. Thus $centre\,(S^{-1}R) = S^{-1}C$. This proves the claim.

From the above claim, for each maximal ideal m of C, $centre\,(R_m) = C_m$. So whenever R_m is left SF we have C_m is regular. Therefore by Proposition 2.6.4, C is regular. Let S be a simple left R-module. Consider the left R-homomorphism $f : S \longrightarrow S_m$. As C is regular, f is onto. Again S is simple implies $\ker f = 0$. or S. But f is onto implies $\ker f \neq S$. So $\ker f = 0$. Thus $S \simeq S_m$ as left R-module. Therefore S_m is simple left R-module which implies S_m is simple as left R_m module and hence is flat as left R_m module. This implies S is left R- flat. □

Definition 4.1.9. A ring R is *left quasi-perfect* if each finitely generated flat left R-module is projective.

Example 4.1.10. A left Noetherian ring R is left quasi-perfect (by Theorem 3.1.14).

Proposition 4.1.11. ([17], Proposition 3.6) *Let R be a left quasi-perfect left SF-ring. Then R is a semisimple ring.*

Proof. Let R be a simple left R-module. As R is left quasi-perfect, S is projective. Hence R is semisimple ring. □

Corollary 4.1.12. *A left quasi-perfect, left SF-ring ring R is regular.*

Theorem 4.1.13. ([17], Theorem 3.8) *Let R be a ring which is finitely generated as a module over its centre. If R is left SF, then R is regular.*

Proof. Let m be a maximal ideal of C which is a regular ring. Then C_m is a field. Then R_m being a finite dimensional C_m algebra, is a left Noetherian ring. By Proposition 4.1.8, R_m is left SF and hence by Proposition 4.1.11, R_m is regular. Thus R is regular by Proposition 2.6.4. □

Definition 4.1.14. A ring R is *semilocal* if $R/Rad(R)$ is semisimple.

Proposition 4.1.15. *For a ring R, consider the following two conditions:*

(1) R is semilocal.

(2) R has finitely many maximal left ideals.

We have in general (2) implies (1). The converse holds if $R/Rad(R)$ is commutative.

Proof. Let $m_1, m_2, \ldots.. m_k$ be maximal left ideals of R. Then $\bigcap_{i=1}^{k} m_i = Rad(R)$. Consider the map defined by $\theta : R \longrightarrow \bigoplus_{i=1}^{k} R/m_i$ by $\theta(r) = (r + m_1, r + m_2, .. \; r + m_k)$. Since m_i and m_j are coprime (that is, $m_i + m_j = R$) for each $i \neq j$, θ is surjective. Also $\ker\theta = Rad(R)$. Thus $R/Rad(R) \simeq \bigoplus_{i=1}^{k} R/m_i$ which is semisimple. Thus $R/Rad(R)$ is a semisimple ring.

Conversely, suppose R is semilocal. Then $R/Rad(R)$ is semisimple. Since $R/Rad(R)$ is also commutative, by Wedderburn's structure theorem, $R/Rad(R) \simeq K_1 \times K_2 \times \cdots \times K_n$ for some fields K_1, K_2, ..., K_n. Therefore there exists finitely many maximal left ideals of $R/Rad(R)$ say $\mu_1/Rad(R)$, $\mu_2/Rad(R)$, ..., $\mu_l/Rad(R)$ where μ_1, μ_2, ..., μ_l are some ideals of R containing $Rad(R)$.

Claim:-μ_1, μ_2, ..., μ_l are the only maximal left ideals of R.

Suppose there exists a maximal left ideal μ of R such that $\mu \neq \mu_i$ for all $i = 1, 2, ..., l$. Then $\mu \supseteq Rad(R)$. So $\mu/Rad(R)$ is a maximal left ideal of $R/Rad(R)$. Then $\mu/Rad(R) = \mu_i/Rad(R)$ for some $i = 1, 2, ..., l$. So $\mu = \mu_i$ for some i, a contradiction. This proves the claim. Therefore R has finitely many maximal left ideal. □

Example 4.1.16. (1) Any local ring is semilocal.

(2) Any finite ring is semilocal.

Proposition 4.1.17. ([17], Proposition 3.15) *A semilocal left SF-ring is semisimple.*

Proof. Suppose R is semilocal left SF-ring. Then $R/Rad(R)$ is semisimple and therefore flat left R-module (since R is left SF-ring). Let $a \in Rad(R)$. Then $R/Rad(R)$ is flat implies there exists some $b \in Rad(R)$ such that $a = ab$. Hence $a(1 - ab) = 0$. As $b \in Rad(R)$. $1 - b$ is a unit. Therefore $a = 0$. Thus $Rad(R) = 0$ and R is a semisimple ring. □

Corollary 4.1.18. *A semilocal left SF-ring is regular.*

4.2 On quasi-duo SF-rings

In this section, some properties of left (right) quasi duo-rings are studied as well as some characterizations of left (right) quasi-duo SF-rings are given. Also the strong regularity of LW (RW) SF-rings is investigated.

Proposition 4.2.1. ([17], Proposition 4.4) *Let R be a left quasi-duo ring. Then $R/Rad(R)$ is a reduced ring.*

Proof. Let $\{m_i : i \in I\}$ be the family of maximal left ideals of R.
Let $f : R \longrightarrow \Pi R/m_i$ be canonical map. Then $\ker f = Rad(R)$. Therefore the map $g : R/Rad(R) \longrightarrow \Pi R/m_i$ given by $g(r + Rad(R)) = f(r)$ is well-defined left R monomorphism. Now R is left quasi-duo implies m_i is a maximal ideal for each i. Thus R/m_i is a division ring and hence strongly regular. Therefore $\Pi R/m_i$ is strongly regular and hence reduced. Therefore $R/Rad(R)$ is reduced. □

Corollary 4.2.2. *A semiprimitive left quasi-duo ring is reduced.*

Proposition 4.2.3. *Let R be a left or right quasi-duo ring. Then R is left weakly regular if and only if R is right weakly regular.*

Proof. Assume R is left (or right) weakly regular. Then R is semiprimitive and hence is reduced by Proposition 4.2.1. So R is right (or left) weakly regular. □

Definition 4.2.4. A ring R is a *left (right) V-ring* if every simple left (right) R-module is injective.

Example 4.2.5. Every *semisimple ring* R *is* left (right) V-ring.

Proposition 4.2.6. *A commutative regular ring is left (right) V-ring.*

Proof. Let R be a commutative regular ring and S be a simple left R-module. Then $S \simeq R/M$ for some maximal ideal M of R. Since R is regular, $(R/M)_R$ is flat. Then ${}_R(R/M)$ is injective by Proposition 3.2.4. □

Proposition 4.2.7. ([17], Proposition 4.7) *The following conditions are equivalent for a left quasi-duo ring R:*

(1) R is left weakly regular.

(2) R is a left V-ring.

(3) R is a left SPI-ring.

(4) R is strongly regular.

Proof. (1) implies (2):- Let S be a simple left R-module. Then $S \simeq R/M$ for some maximal left ideal M of R. As R is a left quasi-duo, M is an ideal of R. Let $x \in M$. As R is left weakly regular, $Rx = RxRx$. So there exists some $y \in RxR$ such that $x = yx$. So R/M is a flat right R-module. Therefore

R/M is an injective left R-module, that is, S is left R-injective and hence R is a left V-ring.

(2) implies (3) is trivial.

(3) implies (4):-We shall first show that R is semiprimitive. Suppose $Ra \neq (Ra)^2$ for some $a \in J(R)$, then $0 \neq Ra/(Ra^2)$ is a finitely generated left R-module and hence has a maximal element $\mu/(Ra^2)$. Now Ra/μ is a simple left R-module so, by hypothesis, Ra/μ is p-injective. So there exists a left R-homomorphism $f : R \longrightarrow Ra/\mu$ which extends the canonical homomorphism $\eta : R \longrightarrow Ra/\mu$. Then $a + \mu = \eta(a) = f(a) = af(1) = a(ya + \mu)$ for some $y \in R$ yielding $a + \mu = aya + \mu$. Hence $(ay - 1)a \in \mu$. But $a \in J(R)$ implies $ay - 1$ is a unit. Therefore $a \in \mu$. This contradicts $Ra \neq \mu$. Thus $Ra = (Ra)^2$ for all $a \in J(R)$. Then $a = \sum_{i=1}^{n} x_i a y_i a$ for some $x_i \in R$, $y_i \in R$. This implies $\left(1 - \sum_{i=1}^{n} x_i a y_i\right) a = 0$. Since $a \in J(R)$, $1 - \sum_{i=1}^{n} x_i a y_i$ is a unit. So $a = 0$. Thus R is semiprimitive. Thus by Corollary 4.2.2, R is reduced. Also by Proposition 3.2.12, and Corollary 3.1.6, it follows that R is left SF. Thus R is strongly regular by Corollary 4.1.2.

(4) implies (1) is trivial. □

Proposition 4.2.8. ([17], Proposition 4.8) *Let R be a left quasi-duo, right SF-ring. Then R is a left V-ring.*

Proof. Let S be a simple left R-module. Then $S \simeq R/M$ for some maximal left ideal M of R. Since R is left quasi-duo M is an ideal of R. As R is right SF, $(R/M)_R$ is flat. So ${}_R(R/M)$ is injective by Proposition 3.2.4. Hence S is injective. So R is a left V-ring. □

Proposition 4.2.9. ([17], Proposition 4.9) *Let R be a left quasi-duo left SF-ring. Then R is a right V-ring.*

Proof. Let $A = R/Rad()R$. Since R is left SF, A is a left SF-ring. Also R is left quasi-duo implies A is left quasi-duo. Again A is semiprimitive, so A is reduced. Therefore A is strongly regular and hence is right quasi-duo implying that R is right quasi-duo. Therefore by dual of Proposition 4.2 8, R is a left V-ring. □

Theorem 4.2.10. ([17], Theorem 4.10) *Let R be a left or right quasi-duo ring. Then the following are equivalent:*

(1) R is a left V-ring.

(2) R is a left SPI-ring.

(3) R is a left weakly regular ring.

(4) R is a left SF-ring.

(5) R is a right V-ring.

(6) R is a right SPI-ring.

(7) R is a right weakly regular ring.

(8) R is a right SF-ring.

(9) R is a regular ring.

(10) R is a strongly regular ring.

Proof. Let R be a left quasi-duo ring. The equivalence of (1), (2), (7), (9) and (10) follows from Proposition 4.2.3 and Proposition 4.2.7 and the trivial implications (10) implies (9) implies (3). By Proposition 4.2.8, (8) implies (1) and since (10) implies (8) trivially, we get the equivalence of (1) and (8). Now (9) implies (4) and (5) implies (6) are trivial. Also (6) implies (7) by the proof of Proposition 4.2.7. Since (4) implies (5) by Proposition 4.2.9, we have the equivalence of (4), (5), (6) and (7). This completes the proof.
The theorom can be proved similarly if R is right quasi-duo. □

Theorem 4.2.11. ([13], Theorem 3) *The following conditions are equivalent for a ring R:*

(1) R is a strongly regular ring.

(2) R is an LW left SF-ring.

(3) R is an LW right SF-ring.

Proof. (1) implies (2) and (1) implies (3) are trivial.
(2) implies (1) :- $R/J(R)$ being a semiprimitive LW-ring is reduced by Lemma 2.9.6. Also R is a left SF-ring implies $R/J(R)$ is left SF-ring. Therefore by Corollary 4.1.2, $R/J(R)$ is strongly regular. Let M be a maximal left ideal of R. Then $M/J(R)$ is a left ideal of $R/J(R)$ and thus $M/J(R)$ is an ideal of $R/J(R)$ (since an strongly regular ring is left and right duo). Thus M is an ideal of R. Therefore R is left quasi-duo. Thus R is strongly regular. Hence (2) implies (1).
(3) implies (1) can be proved similarly. □

4.3 SF-rings with certain chain conditions

Here, it is proved that with some weak chain conditions, left SF-rings are semisimple or regular.

Definition 4.3.1. A ring R *satisfies* $PDCC^{\perp}$ (descending chain conditions on principal right annihilators) if there does not exists a properly descending infinite chain: $r(x_1) > r(x_2) > \cdots > r(x_n) > \ldots$ for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq R$. Similarly a ring R is said to *satisfy* $^{\perp}PDCC$ (the descending chain conditions on principal left annihilators), $PACC^{\perp}$ (the ascending chain conditions on principal right annihilators), $^{\perp}PACC$ (the ascending chain conditions on principal left annihilators).

Example 4.3.2. (1) Every right (left) Artinian ring satisfies $PDCC^{\perp}$ ($^{\perp}PDCC$).

(2) Every right (left) Noetherian ring satisfies $PACC^{\perp}$ ($^{\perp}PACC$).

Definition 4.3.3. A ring R *satisfies left* $PACC$ (ascending chain conditions on principal left ideals) if there does not exists a properly ascending infinite chain: $Rx_1 < Rx_2 < \ldots$ for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq R$. Similarly a ring R is said to *satisfy right* $PACC$ (the ascending chain conditions on principal right ideals), left $PDCC$ (the descending chain conditions on principal left ideals), right $PDCC$ (the descending chain conditions on principal right ideals).

Example 4.3.4. (1) Every left (right) Noetherian ring satisfies left $PACC$ (right $PACC$).

(2) Every left (right) Artinian ring satisfies left $PDCC$ (right $PDCC$).

Lemma 4.3.5. ([19], Lemma 1.2(1)) *Let R be a left SF-ring. For every $x \in R, Rr(x) + Rx = R$.*

Proof. If $Rr(x) + Rx \neq R$ for some $x \in R$, then there exists a maximal left ideal M of R such that $Rr(x) + Rx \subseteq M$. Now R is left SF implies R/M is flat. As $x \in M$, there exists some $y \in M$ such that $x = xy$. Then $1 - y \in r(x) \subseteq M$. Then $1 = (1 - y) + y \in M$ contadicting that M is a maximal left ideal of R. Thus $Rr(x) + Rx = R$ for all $x \in R$. □

Theorem 4.3.6. ([19], Theorem 1.3) *For a left SF-ring R, the following are equivalent:*

(1) R is semisimple.

(2) R is left or right Noetherian.

(3) $R/J(R)$ is semisimple.

(4) R satisfies ${}^{\perp}PACC$.

(5) R satisfies $PDCC^{\perp}$.

(6) R satisfies left $PACC$.

Proof. (1) implies (2) is well known.
(2) implies (3) :- Let R be left Noetherian. As R is left SF, every simple left R-modules are projective (by Theorem 3.1.14). Then R is semisimple. Suppose R is right Noetherian. Then $R/J(R)$ is right Noetherian. Let Q

denote the semisimple classical right quotient ring of $R/J(R)$. Let $ab^{-1} \in Q$ where $a \in R/J(R)$ and $b \in R/J(R)$ and b is a non-zero divisor. Then from Lemma 4.3.5, $(R/J(R))b = R/J(R)$. So $b^{-1} \in R/J(R)$. Then $R/J(R)$ coincides with its semisimple classical right quotient ring. Therefore $R/J(R)$ is semisimple.

(3) implies (1) :- If $R/J(R)$ is semisimple, then ${}_R(R/J(R)$ is semisimple. Then R is left SF implies ${}_R(R/J(R)$ is flat. Take $x \in J(R)$. Then there exists some $y \in J(R)$ such that $x = xy$. That is, $x(1-y) = 0$. Since $1-y$ is invertible as $y \in J(R)$, we have $x = 0$. Therefore $J(R) = 0$ and hence R is semisimple.

(4) implies (1) :- Let M be a maximal left ideal of R. Since R satisfies ${}^{\perp}PACC$ by Zorn's lemma, the set $\{l(1-x) : x \in M\}$ has a maximal element say $l(1-e)$.

Claim:- $M = l(1-e)$, that is, $M = Re$ and $e^2 = e$.

Let $x \in l(1-e)$. Then $x(1-e) = 0$. Thus $x = xe \in M$. Then $l(1-e) \subseteq M$. Suppose there exists some $y \in M$ such that $y(1-e) \neq 0$. Since $y(1-e) = (y - ye) \in M$ and R is left SF, there exists some $e' \in M$ such that $y(1-e) = y(1-e)e'$. So $y = y(e + e' - ee')$. Denote $f = e + e' - ee' \in M$. Since $1 - f = (1-e)(1-e')$, $l(1-e) \subseteq l(1-f)$. Again $y(1-f) = 0$, but $y(1-e) \neq 0$. So $y(1-e) \subsetneq l(1-f)$. This contradicts the maximality of $l(1-e)$. Hence the claim is proved. Then $R/M \simeq R(1-e)$ is projective. Hence R is semisimple.

(5) implies (1) :- Let M be a maximal left ideal of R.

Claim 1:- For every $x \in M$, there exists an idempotent $e_x \in M$ such that $x = xe_x$.

${}_R(R/M)$ is flat implies there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ such that $x = xx_1$, $x_1 = x_1x_2, \ldots, x_k = x_kx_{k+1}, \ldots$. This yields $r(x) \geqslant r(x_1) \geqslant r(x_2) \geqslant \ldots$. Then R satisfies $PDCC^{\perp}$ implies there exists a positive integer n such that $r(x_n) = r(x_{n+1})$. Let $e_x = x_{n+1}$. Now $x_n = x_nx_{n+1}$ implies $x_n(1 - x_{n+1}) = 0$. Therefore $1 - x_{n+1} \in r(x_n) = r(x_{n+1})$. Therefore $x_{n+1} = x_{n+1}^2$, that is, e_x is idempotent. Also $x = xx_1 = xx_1x_2 = \ldots = xx_1x_2\ldots x_n = xe_x$. This proves claim 1.

Since R satisfies $PDCC^{\perp}$ by Zorn's lemma, the set $\{r(x) : x \in M\}$ has a minimal element say $r(e)$. From claim 1, we may assume that e is an idempotent.

claim 2:- $M = Re$.

If not, then there exists some $f \in M$ such that $f(1-e) \neq 0$. Again from claim1, we may assume without loss of generality that f is an idempotent and $r(f)$ is minimal like $r(e)$. Now $f(1-e) = (f - fe) \in M$. Therefore there exists some $e' \in M$ such that $f(1-e) = f(1-e)e'$. Therefore $f = f(e + e' - ee')$. Since $e + e' - ee' \in M$, from the above assumption we have $r(e + e' - ee') = r(f)$. Now $e = e(e + e' - ee')$. Thus $r(e + e' - ee') \subseteq r(e)$ so that $r(f) \subseteq r(e)$. So $r(f) = r(e)$ by minimality of $r(e)$. This implies $r(f) = R(1-e)$. But this gives $f(1-e) = 0$, a contradiction. Thus $M = Re$.
From this claim, $R/M \simeq R(1-e)$ is projective for every maximal left ideal M of R. Thus R is semisimple.

(6) implies (1) :- Let M be a maximal left ideal of R.

Claim 1:- For every $x \in M$, there exists an idempotent $e_x \in M$ such that $x = xe_x$.

R is left SF implies ${}_R(R/M)$ is flat. Therefore $x \in M$ implies there exists a

sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ such that $x = xx_1$, $x_1 = x_1x_2, \dots, x_k = x_kx_{k+1}, \dots$. Since R satisfies left $PACC$, the sequence $Rx \subseteq Rx_1 \subseteq Rx_2 \subseteq \dots$ stops for some n. Therefore there exists some $y \in R$ such that $x_{n+1} = yx_n$. So $x_n = x_nx_{n+1} = x_nyx_n$. Denote $e_x = yx_n$. Then
$e_x^2 = (yx_n)(yx_n) = y(x_nyx_n) = yx_n = e_x$. Thus e_x is idempotent and
$x = xx_1 = xx_1x_2 = \cdots = xx_1x_2 \dots x_n = xx_1x_2 \dots x_nyx_n = xyx_n = xe_x$
So the claim 1 is proved.

Take $e \in M$ such that Re is maximal among all Rx where $x \in M$. From the above discussion we can take e to be an idempotent.

Claim 2 :- $M = Re$.

If not, then there exists some $f \in M$ such that $f \neq fe$. Without loss of generality, assume f is an idempotent and Rf is maximal like Re. Thus by exactly dualising the proof of claim 2 in (5) implies (1), we have $Re = Rf$. But $f \neq fe$ implies $f(1-e) \neq 0$. So $1-e \notin r(f) = R(1-f)$. So $1-e \in Re$. Therefore $Re \neq Rf$ which is a contradiction. Thus $M = Re$. This proves claim 2.

So every simple left R module is projective. Hence R is semisimple.

The implications (1) implies (4), (1) implies (5), and (1) implies (6) are trivial. □

4.4 On MERT SF-rings and strongly left (right) bounded SF-rings

Here, we investigate the regularity of MERT left SF-rings and strongly left (right) bounded SF-ring.

Lemma 4.4.1. ([22], Lemma 7) *Let I be a left ideal of R and $Z = Z({}_RR)$, then either $I \cap Z = 0$ or $I \cap Z$ contains a non-zero nilpotent element. Consequently, if I is reduced, then I is left non-singular R module and in that case any essential extension of ${}_RI$ in ${}_RR$ is also reduced.*

Proof. Suppose that $K = I \cap Z \neq 0$ and K is reduced. If $0 \neq k \in K$ there exists some $r \in R$ such that $0 \neq rk \in l(k)$. Then $rk^2 = 0$ implies $(krk)^2 = (krk)(krk) = krk^2rk = 0$. Since K is reduced, $krk = 0$. Then $(rk)^2 = rkrk = 0$. So $rk = 0$ which contradicts that $rk \neq 0$. So K is reduced, that is, K contains no non-zero nilpotent elements.

Now if I is reduced, then ${}_RI$ is non-singular. In this case if ${}_RE$ is an essential extension of ${}_RI$ in ${}_RR$ and $0 \neq b \in E$ such that $b^2 = 0$. Then $0 \neq rb \in I$ for some $r \in R$. Now $(brb)^2 = brb^2rb = 0$. Since $brb \in I$ and I is reduced, so $brb = 0$. Then $(rb)^2 = 0$ which implies $rb = 0$, a contradiction. Thus E is reduced. □

Corollary 4.4.2. *Suppose $Z = Z({}_RR) \neq 0$, then there exists some $0 \neq z \in Z$ such that $z^2 = 0$.*

Proof. Since $Z \neq 0$ there exists some $0 \neq a \in Z$. Then $Ra \cap Z = Ra \neq 0$. So by Lemma 4.4.1, Ra contains a non-zero nilpotent element. So there exists some $0 \neq z \in Ra$ such that $z^2 = 0$. □

Lemma 4.4.3. ([19], Lemma 1.2(2)) *If R is a left SF ring, then* $Z(R_R) \subseteq J(R)$.

Proof. Let $x \in Z(R_R)$. For any $y \in Z(R_R)$, set $u = 1 - yx$. Then $r(u) = 0$ [For if $z \in r(u) \cap r(yx)$, then $uz = 0$, $yxz = 0$. But since $u = 1 - yx$ we have $(1 - yx)z = 0$. This implies $z = 0$. Since $r(yx)$ is essential right ideal of R, we have $r(u) = 0$]. We prove that $Ru = R$. If not, then there exists a maximal left ideal M of R such that $Ru \subseteq M$. Since R is left SF, we have $_R(R/M)$ is flat. Since $u \in M$, there exist some $t \in M$ such that $u = ut$. This implies $1 - t \in r(u) = 0$. This implies $t = 1$ which is a contradiction. Thus $Ru = R$, that is, $R(1 - yx) = R$. Hence $x \in J(R)$. Thus $Z(R_R) \subseteq J(R)$. □

Lemma 4.4.4. ([19], Lemma 2.1) *Let R be a left SF-ring and $J(R)^2 = 0$, then $J(R) = Z(R_R)$*

Proof. By Lemma 4.4.3, $Z(R_R) \subseteq J(R)$. Suppose there exists some $x \in J(R)$ such that $x \notin Z(R_R)$. Then there exists some $0 \neq y \in R$ such that $r(x) \cap yR = 0$. If $y \in J(R)$, then $xy \in J(R)^2 = 0$. This implies $y \in r(x)$ which contradicts $r(x) \cap yR = 0$. Thus $y \notin J(R)$. Therefore there exists a maximal left ideal M of R such that $y \notin M$. But $xy \in J(R) \subseteq M$ implies that there exists some $m \in M$ such that $xy = xym$. So $xy(1 - m) = 0$. Thus $y(1 - m) \in r(x) \cap yR = 0$, that is, $y = ym \in M$, a contradiction. Therefore $J(R) \subseteq Z(R_R)$. Hence $J(R) = Z(R_R)$ □

Proposition 4.4.5. ([19], Proposition 2.2) *An $MERT$ left SF-ring is regular.*

Proof. Let $S = Soc(R_R)$. Then R/S is left SF Also R is $MERT$ implies R/S is right quasi-duo. Therefore R/S is strongly regular by Theorem 4.2.10. Then R/S is fully left idempotent and R is an ERT. Assume $Z(R_R) \neq 0$. Then there exists $0 \neq x \in Z(R_R)$ such that $x^2 = 0$. Take a maximal right ideal M of R such that $r(x) \subseteq M$. Then R is left SF and $x \in r(x) \subseteq M$ implies there exists some $m \in M$ such that $x = xm$. Then $x(1-m) = 0$. This implies $1 - m \in r(x) \subseteq M$. So $1 \in M$, a contradiction. Thus $Z(R_R) = 0$. Then by Lemma 4.4.4, $J(R) = 0$. This implies R is semiprime. Therefore $S = soc(R_R) = soc(_RR)$ and S is fully left idempotent. So R/S and S are fully left idempotent. Hence R is fully left idempotent. Therefore by Proposition 2.5.3, R is regular. □

Definition 4.4.6. A ring R is *strongly left bounded* if every non-zero left ideal of R contains a non-zero ideal of R. A ring R is defined similarly to be a *strongly right bounded* ring.

Example 4.4.7. Every commutative ring is left and right bounded ring.

Theorem 4.4.8. ([18], Theorem 3) *The following conditions are equivalent for a ring R:*

(1) R is strongly regular.

(2) R is strongly left bounded left SF-ring

(3) R is strongly right bounded left SF-ring.

(4) R is strongly left bounded right SF-ring.

(5) R is strongly right bounded right SF-ring.

Proof. Obviously (1) implies (2), (3), (4) and (5).

(2) implies (1) :- If $Z(_RR) \neq 0$, then there exists $0 \neq a \in Z(_RR)$. So $Ra \neq R$. Since R is strongly left bounded, there exists a non-zero ideal I of R such that $I \subseteq Ra$. This gives $r(a) \subseteq r(I)$. We now show that $r(I)+Ra = R$. If not, then there exists a maximal left ideal M of R such that $r(I)+Ra \subseteq M$. Since R is left SF, $_R(R/M)$ is flat. Therefore $a \in M$, implies there exists some $b \in M$ such that $a = ab$. Now $1-b \in r(a) \subseteq r(I) \subseteq M$. So $1 = 1-b+b \in M$, contradicting that M is a maximal left ideal of R. Hence $r(I) + Ra = R$. Therefore there exists some $d \in r(I)$ and $x \in R$ such that $d + xa = 1$. Since $I \neq 0$, there exists $0 \neq u \in I$. Then $ud = 0$ and $u = uxa$. Since $xa \in Z(_RR)$ and $Ru \neq 0$, $l(xa) \cap Ru \neq 0$. Let $0 \neq su \in l(xa) \cap Ru$. Then $su = suxa = 0$. This contradicts $su \neq 0$. Hence $Z(_RR) = 0$.

Suppose $0 \neq a \in R$, such that $a^2 = 0$. Then there exists a non-zero left ideal L of R such that $l(a) \oplus L$ is essential left ideal of R [since $Z(_RR) = 0$]. By hypothesis there exist a non-zero ideal K of R such that $K \subseteq L$. Then $KRa \subseteq K \cap Ra \subseteq K \cap l(a) = 0$. Hence $K \subseteq KR \subseteq l(a) \cap L = 0$. This contradicts that $K \neq 0$. Therefore R is reduced. As R is also left SF, R is strongly regular.

Similarly (5) implies (1).

Assume (3). Suppose there exist some $a \in R$ such that $a \notin Z(R_R)$, $a^2 \in Z(R_R)$. Then there exist a non-zero right ideal K of R such that $r(a) \bigoplus K \subseteq r(a^2)$. Since R is strongly left bounded, there exists a non-zero ideal I of R such that $I \subseteq K$. Then $a^2I \subseteq a^2K = 0$. Thus $aI \subseteq r(a) \cap I \subseteq r(a) \cap K = 0$. Hence $I \subseteq r(a) \cap K = 0$. This contradicts $I \neq 0$. This proves $R/Z(R_R)$ is reduced. As $R/Z(R_R)$ is left SF, $R/Z(R_R)$ is strongly regular.

Let L be a maximal left ideal of R, then $Z(R_R) \subseteq J(R) \subseteq L$. So $L/Z(R_R)$ is a left ideal of $R/Z(R_R)$. But since $R/Z(R_R)$ is strongly regular, $L/Z(R_R)$ is an ideal of $R/Z(R_R)$. This implies L is an ideal of R. Hence R is left quasi-duo ring. As R is also left SF, R is strongly regular.
Similarly (4) implies (1). □

4.5 On PCLZ (PCRZ) SF-rings

In this section, we discuss the regularity of PCLZ (PCRZ) left SF-ring.

Definition 4.5.1. A ring R is a *$PCLZ-$ ring* if every product of two independent closed left ideal of R is zero. Similarly a ring R is defined to be *PCRZ- ring.*

Example 4.5.2. Every field is a $PCLZ$ ring.

Proposition 4.5.3. ([20], Proposition 2) *For a ring R, the following conditions are equivalent*

(1) R is a PCLZ- ring.

(2) Every product of two independent left ideals of R is zero.

Proof. Obviously (2) implies (1).
Assume (1). Let X and Y be two independent left ideals of R. By Zorn's Lemma, the set $\{I : I$ is a left ideal of R with $X \subseteq I, I \cap Y = 0\}$ has a maximal element L. By Proposition 1.5.3, L is a closed left ideal of R. In

the same way, we can find a closed left ideal K of R such that $Y \subseteq K$ and $L \cap K = 0$. Then by hypothesis, $LK = 0$. So $XY \subseteq LK = 0$. This shows that (1) implies (2). □

Lemma 4.5.4. ([20], Lemma 8) *If R is a PCLZ-ring, $a \in R$ and $a^2 = 0$, then $a \in Z(_RR)$.*

Proof. There exist a left ideal L of R such that $l(a) \oplus L$ is essential left ideal of R. Since $a^2 = 0$, we have $a \in l(a)$. Hence $Ra \subseteq l(a)$. Thus $LRa \subseteq Ll(a) = 0$. That is, $LR \subseteq l(a)$. Therefore $L \subseteq LR \subseteq L \cap l(a) = 0$. Hence $l(a)$ is essential left ideal of R. Therefore $a \in Z(_RR)$. □

Lemma 4.5.5. ([20], Lemma 9) *If R is a left or right SF-ring. If $R/Z(_RR)$ is a reduced ring, then R is strongly regular.*

Proof. Suppose R is a right SF ring, then proof of (3) implies (1) of Theorem 4.4.8 shows that R is strongly regular.

Now suppose R is a left SF-ring. Then $R/Z(_RR)$ is also a left SF-ring. Since $R/Z(_RR)$ is also reduced by hypothesis, we have $R/Z(_RR)$ is strongly regular. Suppose $Z(_RR) \neq 0$. Then there exists $0 \neq a \in Z(_RR)$ such that $a^2 = 0$.

Claim: $r(a) + Z(_RR) \neq R$. If not, then $1 = b+c$ for some $b \in r(a)$, $c \in Z(_RR)$. Then $a = a(b+c) = ac$. Since $c \in Z(_RR)$, $l(c)$ is essential left ideal of R. Hence $l(c) \cap Ra \neq 0$. Therefore there exist $0 \neq ra \in Ra$ such that $rac = 0$. So $ra = rac = 0$, a contradiction. Therefore $r(a) + Z(_RR) \neq R$. Then there exist a maximal right ideal K of R such that $r(a) + Z(_RR) \subseteq K$. Since $R/Z(_RR)$ is strongly regular, $K/Z(_RR)$ is an ideal of $R/Z(_RR)$. Hence

K is an ideal of R. Thus there exist a maximal left ideal L of R such that $Z(_RR) + r(a) \subseteq K \subseteq L$. Since R is left SF and $a \in L$, so there exist some $d \in L$ such that $a = ad$. Then $1-d \in r(a) \subseteq L$. Then $1 = (1-d)+d \in L$, a contradiction. Hence $Z(_RR) = 0$. Therefore R is strongly regular.

□

Theorem 4.5.6. ([20], Theorem 10) *For a ring R, the following conditions are equivalent:*

(1) R is strongly regular.

(2) R is a left SF, $PCLZ$-ring.

(3) R is a right SF, $PCLZ$-ring.

(4) R is a $PCLZ$-ring whose every maximal left ideal is p-injective.

(5) R is a $PCLZ$-ring whose every maximal right ideal is p-injective.

Proof. (3) implies (1): Let $Z = Z(_RR)$. Since R is right SF, $Z \subseteq J(R)$. we claim that R/Z is reduced. If not, then there exist $a \notin Z$ such that $a^2 \in Z$. Thus there exist a left ideal L of R such that $L \neq 0$, $l(a) \oplus L \subseteq l(a^2)$ and $l(a) \oplus L$ is essential left ideal of R. Since R is a $PCLZ$-ring and $La \subseteq l(a)$, we have $L^2a = L(La) \subseteq Ll(a) = 0$. Thus $L^2 \subseteq L \cap l(a) = 0$. It follows from $l(a)L = 0$ that $(l(a) \oplus L)R \subseteq l(L) \neq R$. Hence there exists a maximal right ideal K of R such that $(l(a) \oplus L)R \subseteq K$. Since $a^2 \in Z(_RR) \subseteq J(R) \subseteq K$, R is right SF, it follows that there exists $b \in K$ such that $a^2 = ba^2$. This means that $a - ba \in l(a) \subseteq K$. But since K is a right ideal, and $b \in K$ so $ba \in K$. It then follows that $a \in K$. Therefore there exists some $d \in K$ such that $a = da$. That is, $1 - d \in l(a) \subseteq K$. Then $1 = 1-d+d \in K$, a contradiction. Thus $R/Z(_RR)$ is reduced. Hence by Lemma 4.5.5, R is strongly regular.

(2) implies (1):- Let $Z = Z(_RR)$ we shall show that R/Z is reduced. Let $a \notin Z$ such that $a^2 \in Z$.

We claim that $Z + Rr(a) \neq R$.

If not, then $1 = b + \sum r_i\ t_i$ for some $r_i \in R$, $t_i \in r(a)$. Then

$a = ba + \sum r_i t_i a$............................(*)

For each i, $(t_i a)^2 = t_i(at_i)a = 0$. So by Lemma 4.5.4, $t_i a \in Z$. Again since $ba \in Z$, it follows from (*) that $a \in Z$, which contradicts that $a \notin Z$. Thus $Z + Rr(a) \neq R$. Therefore there exists a maximal left ideal L of R such that $Z + Rr(a) \subseteq L$. Since R is a left SF, $_R(R/L)$ is flat. As $a^2 \in L$, there exists some $b \in L$ such that $a = a^2b$. So $a - ab \in r(a) \subseteq L$. Then as $ab \in L$, we get $a \in L$. Therefore there exists some $c \in L$ such that $a = ac$. So $1 - c \in r(a) \subseteq L$. Hence $1 = (1-c)+c \in L$, a contradiction. Thus R/Z is reduced and hence by Lemma 4.5.5, R is strongly regular.

(1) implies (4) and (1) implies (5) follows from Proposition 2.1.15 and Lemma 2.2.4.

(5) implies (3) and (4) implies (2) follows from Proposition 3.2.1. □

Corollary 4.5.7. ([20]) *For a ring R, following conditions are equivalent:*

(1) R is strongly regular.

(2) R is a left SF-ring such that $L \cap K = LK$ for all left ideal L, K of R.

(3) R is a right SF-ring such that $L \cap K = LK$ for all left ideals L, K of

R.

(4) *Every maximal left ideal of R is p-injective and $L \cap K = LK$ for all left ideals L, K of R.*

(5) *Every maximal right ideal of R is p-injective and $L \cap K = LK$ for all left ideals L, K of R.*

4.6 SF-rings and W-ideals

In this section, we discuss the regularity of left SF-rings via W-ideals.

Proposition 4.6.1. ([24], Proposition 2.4) *If R is a ring whose every complement left ideal is a W-ideal, then $R/Z(_RR)$ is a reduced ring.*

Proof. Suppose $a \notin Z(_RR)$ such that $a^2 \in Z(_RR)$. Then $l(a)$ is not an essential left ideal of R. Thus there exists a non-zero left ideal L of R such that $l(a) \cap L = 0$. Let $L' = L \cap l(a^2)$. Then L' is a submodule of $l(a^2)$ Since $a^2 \in Z(_RR)$, $l(a^2)$ is an essential left ideal of R. So it follows that $L' \neq 0$. Now $L' \cap l(a) = L \cap l(a^2) \cap l(a) = 0$. This shows that $l(a)$ is not left essential in $l(a^2)$. Therefore there exists a non-zero left ideal I of R such that $l(a) \oplus I$ is essential in $l(a^2)$. Let K be a complement of $l(a)$ in R such that $I \subseteq K$. Since $I \neq 0$, we can take $0 \neq b \in I$. By hypothesis K is a W-ideal of R. Hence there exists some $n > 0$ such that $b^n \neq 0$ and $b^nR \subseteq K$. Since $b^na \in K \cap l(a) = 0$, $b^n \in l(a) \cap I = 0$ contradicting $b^n \neq 0$. Thus $R/Z(_RR)$ is a reduced ring. ☐

Similarly we can prove the following:

Proposition 4.6.2. ([24], Proposition 2.5) *If R is a ring whose every complement right ideal is a W-ideal, then $R/Z(R_R)$ is a reduced ring.*

Proposition 4.6.3. ([24], Proposition 2.6) *If R is left SF-ring whose every complement left ideal is a W-ideal, then $R/Z({}_RR)$ is strongly regular.*

Proof. Follows from Corollary 4.1.2, Corollary 4.1.7 and Proposition 4.6.1. □

We also have the following proposition:

Proposition 4.6.4. *If R is a left SF-ring whose every complement right ideal is a W-ideal, then $R/Z(R_R)$ is strongly regular.*

Proposition 4.6.5. ([24], Proposition 2.7) *If R is a left SF-ring whose every complement left ideal is a W-ideal, then R is left non-singular.*

Proof. Suppose $0 \neq a \in Z({}_RR)$. Let $T = Z({}_RR) + r(a)$. If $T \neq R$, then there exists a maximal right ideal M of R such that $T \subseteq M$. By Proposition 4.6.3 $R/Z({}_RR)$ is strongly regular and hence every one sided ideal of $R/Z({}_RR)$ is an ideal. Thus M is an ideal of R. This means that there exists a maximal left ideal L of R such that $M \subseteq L$. Since R is a left SF-ring, simple left R module R/L is flat. Since $a \in T \subseteq L$ there exists some $b \in L$ such that $a = ab$. Therefore $1 - b \in r(a) \subseteq L$. But this yields $1 = (1 - b) + b \in L$, a contradiction. Therefore $T = R$. So there exists some $u \in Z({}_RR)$, $d \in r(a)$ such that $1 = u + d$. Then $a = au$ which implies $a \in l(1 - u)$. Since $l(u) \cap l(1 - u) = 0$ [if $x \in l(u) \cap l(1 - u)$, then $xu = 0$ and $x - xu = 0$. This implies $x = 0$] and $u \in Z({}_RR)$ it follows that $l(1 - u) = 0$. Then $a = 0$ which is a contradiction. Therefore $Z({}_RR) = 0$. □

Theorem 4.6.6. ([24], Theorem 2.8) *If R is a left SF-ring whose every complement left ideal is a W-ideal, then R is strongly regular.*

Proof. Follows from Proposition 4.6.3 and Proposition 4.6.5. □

Remark 4.6.7. The theorem also follows from Lemma 4.5.5 and Proposition 4.6.1.

Proposition 4.6.8. ([24], Proposition 2.10) *If R is a left SF-ring whose every complement right ideal is a W-ideal, then R is right non-singular.*

Proof. Suppose $0 \neq a \in Z(R_R)$ and $T = Z(R_R) + r(a)$. If $T \neq R$ there exists a maximal right ideal K of R such that $T \subseteq K$. Since $R/Z(R_R)$ is left SF and reduced, $R/Z(R_R)$ is strongly regular. So K is an ideal of R. Therefore there exists a maximal left ideal L of R such that $K \subseteq L$. Then R is left SF implies R/L is flat. Since $a \in Z(R_R) \subseteq L$ there exists some $b \in L$ such that $a = ab$. So $1 - b \in r(a) \subseteq L$ and hence $1 \in L$. This contradicts that $L \neq R$. Therefore $T = R$. Then $1 = u + d$ for some $u \in Z(R_R)$, $d \in r(a)$. Then $a = au$. Then $a(1 - u) = 0$. Then $u \in Z(R_R) \subseteq J(R)$ implies $a = 0$. Thus $Z(R_R) = 0$. □

Theorem 4.6.9. ([24], Theorem 2.11) *If R is a left SF-ring whose every complement right ideal is a W-ideal, then R is strongly regular.*

Proof. Follows from Proposition 4.6.4 and Proposition 4.6.8. □

Remark 4.6.10. The theorem also follows from Lemma 4.5.5 and Proposition 4.6.2.

4.7 SF-rings and GW-ideals

Here, the regularity of SF-rings via GW-ideals are discussed.

Lemma 4.7.1. ([23]) *If R is a left SF-ring whose maximal right ideals are GW-ideals, then R is strongly regular.*

Lemma 4.7.2. ([24], Lemma 3.2) *If R is a left SF-ring whose maximal essential right ideals are GW-ideals, then $J(R)^2 = 0$.*

Proof. Let $a \in J(R)$. Then there exists a right ideal K of R such that $J(R)+r(a)\oplus K$ is an essential right ideal of R. If $J(R)+r(a)\oplus K \neq R$, then there exists a maximal right ideal M of R such that $J(R)+r(a)\oplus K \subseteq M$. So M is an essential right ideal of R. We claim that $R/soc(R_R)$ is strongly regular. Let A be a maximal right ideal of $R/soc(R_R)$. Then $A = T/soc(R_R)$ for some maximal right ideal T of R containing $soc(R_R)$. Then T is an essential right ideal of R and hence by hypothesis T is a GW-ideal of R. Thus A is a GW-ideal of $R/soc(R_R)$. Hence by Lemma 4.7.1, we conclude that $R/soc(R_R)$ is strongly regular. As $soc(R_R) \subseteq M$, M is an ideal of R. Thus there exists a left ideal L of R such that $J(R)+r(a)\oplus K \subseteq M \subseteq L$. Since R is left SF, the simple left R module R/L is flat. Since $a \in J(R) \subseteq L$, there exists some $b \in L$ such that $a = ab$. So $1-b \in r(a) \subseteq L$. Then $1 \in L$ which contradicts that $L \neq R$. Thus $J(R)+r(a)\oplus K = R$. This implies there exists an idempotent $e \in R$ such that $J(R)+r(a) = eR$. Then $u+d = e$ for some $u \in J(R)$, $d \in r(a)$. Therefore $au = ae$. Since $u \in J(R) \subseteq eR$, there exists some $x \in R$ such that $u = ex$. Therefore $eu = eex = ex = u$ and so $aeu = ae$. Therefore $ae(1-u) = 0$. As $u \in J(R)$, $ae = 0$. Then $e \in r(a)$.

Therefore $J(R) \subseteq r(a)$, that is, $aJ(R) = 0$. Therefore $J(R)^2 = 0$. □

Lemma 4.7.3. ([24], Lemma 3.3) *If R is a left SF-ring whose maximal essential right ideals are GW-ideals, then $J(R) \subseteq Z(R_R)$.*

Proof. By Lemma 4.7.2, $J(R)^2 = 0$. Suppose $a \in J(R)$ such that $a \notin Z(R_R)$. Then there exists some $0 \neq b \in R$ such that $r(a) \cap bR = 0$. Now $b \notin J(R)$ [For if $b \in J(R)$, then $ab \in J(R)^2 = 0$ which implies $b \in r(a)$. Therefore $bR \subseteq r(a)$. Then $bR \cap r(a) = bR \neq 0$, a contradiction]. Therefore $b \notin M$ for some maximal left ideal M of R. But $a \in J(R)$ implies $ab \in J(R) \subseteq M$. Since R is left SF, R/M is flat. As $ab \in M$, there exists some $c \in M$ such that $ab = abc$. Thus $ab(1-c) = 0$. This implies $b(1-c) \in r(a) \cap bR = 0$. Thus $b = bc \in M$, a contradiction. Hence $J(R) \subseteq Z(R_R)$. □

Corollary 4.7.4. *If R is a left SF-ring whose maximal essential right ideals are GW-ideals, then $J(R) = Z(R_R)$.*

Proof. Follows from Lemma 4.4.3 and Lemma 4.7.3. □

Remark 4.7.5. The corollary also follows from Lemma 4.4.4 and Lemma 4.7.2.

Proposition 4.7.6. ([24], Proposition 3.4) *If R is a left SF-ring whose maximal essential right ideals are GW-ideals, then $J(R) = 0$.*

Proof. We have $R/soc(R_R)$ is strongly regular by the proof of Lemma 4.7.2. Thus $J(R/soc(R_R)) = 0$ and so $J(R) \subseteq soc(R_R)$. Let $0 \neq a \in J(R)$. By Lemma 4.7.3, $a \in Z(R_R)$. Hence $a \in J(R) \subseteq soc(R_R) \subseteq r(a) \neq R$. From the strong regularity of $R/soc(R_R)$ we have that $r(a)$ is an ideal of R. Hence there exists some maximal left ideal L of R such that $a \in J(R) \subseteq soc(R_R) \subseteq r(a) \subseteq L$ Since R is left SF, R/L is flat As $a \in L$ there exists some $b \in L$ such that $a = ab$ This shows that $1 - b \in r(a) \subseteq M$ and hence $1 \in M$, a contradiction Thus $J(R) = 0$ □

Theorem 4.7.7. ([24], Theorem 3 7) *If R is a left SF-ring whose maximal essential right ideals are GW-ideals, then R is regular*

Proof By Proposition 4 7 6, R is semiprime By Corolarry 1 10 11, $soc(_RR) = soc(R_R) = S$ (say) By Proposition 1 10 13, S is fully left idempotent Also R/S is strongly regular implies R/S is fully left idempotent and R is an ERT Since R/S and S are fully left idempotent, R is fully left idempotent Therefore by Proposition 2 5 3, R is regular □

4.8 SF-rings and quasi-ideals

In this section, we discuss some properties of quasi-ideals of a ring Also we investigate the regularity of SF-rings via quasi-ideals

Definition 4.8.1. An additive subgroup A of a ring R is a *quasi-ideal* if $ara, rar \in A$ for all $a \in A$ and $r \in R$

Example 4.8.2. (1) Every ideal of a ring R is a quasi-ideal

Proposition 4.8.3. ([20], Proposition 12) *A one sided ideal of a ring R is an ideal if and only if it is a quasi-ideal*

Proof Let L be a left ideal of R and assume that L is a quasi-ideal Let $a \in L$ and $r \in R$ Then $(1 + r)a(1 + r) \in L$, $rar \in L$ Also since L is a

left ideal, $ra \in L$. Therefore $ar = (1+r)a(1+r) - a - ra - rar \in L$. This proves that L is an ideal of R.
Converse is trivial. □

Corollary 4.8.4. ([20], Corollary 13) *For a ring R, the following conditions are equivalent:*

(1) R is strongly regular.

(2) R is a left SF-ring such that the left annihilator of any element of R is a quasi-ideal.

(3) R is a right SF-ring such that the right annihilator of any element of R is quasi-ideal.

(4) R is a left SF-ring whose every maximal left ideal is a quasi-ideal.

(5) R is a right SF-ring whose every maximal right ideal is a quasi-ideal.

Proof. (1) implies (2), (3), (4) and (5) is trivial.
(2) impies (1) and (3) implies (1) by Proposition 4.8.3 and Proposition 4.1.1.
(4) implies (1) and (5) implies (1) by Proposition 4.8.3 and Theorem 4.2.10. □

Lemma 4.8.5. ([26], Lemma 2.1) *If L is a left (right) ideal of a ring R which contains an ideal I. If L is a quasi-ideal of R, then L/I is a quasi-ideal of R/I.*

Proof. Let $a \in L$ and $r \in R$. Then $ara \in L$ and $rar \in L$ by L is a quasi-ideal of R. Thus $ara+I \in L/I$ and $rar+I \in L/I$. Thus $(a+I)(r+I)(a+I) \in L/I$

and $(r+I)(a+I)(r+I) \in L/I$ for all $r \in R$ and $a \in L$. This shows that L/I is a quasi-ideal of R/I. □

Lemma 4.8.6. ([26], Lemma 2.2) *If R is a ring whose every maximal left (right) ideal is a quasi-ideal, then R/J is reduced, where J is the Jacobson radical of R.*

Proof. Suppose $x \notin J$ such that $x^2 \in J$. Then $x \notin L$ for some maximal left ideal L of R. So $L + Rx = R$, which implies $Lx + Rx^2 = Rx$. Therefore $L + Lx + Rx^2 = R$. This implies $L + Lx = R$ by $x^2 \in J \subseteq L$. Therefore $1 - rx \in L$ for some $r \in L$. Since L is a quasi-ideal and L is a left ideal, therefore

$$x - xrx = x(1 - rx) \in L, \text{ and } xrx \in L$$

which implies $x \in L$ which contradicts $x \notin L$. This proves R/J is reduced. □

Remark 4.8.7. The Lemma trivially follows from Proposition 4 2.1 and Proposition 4.8.3.

Corollary 4.8.8. ([26], Corollary 2.3) *A ring R is strongly regular if and only if R is a regular ring whose every maximal left (right) ideal is a quasi-ideal.*

Proof. This follows from the fact that a regular ring is semiprimitive and Proposition 2.1.15 and Lemma 4.8.6. □

Corollary 4.8.9. ([26], Corollary 2.4) *A ring R is strongly regular if and only if R is a left (right) quasi-duo regular ring.*

Proof. Follows from Proposition 4.8 3 and Corollary 4.8 8. □

Theorem 4.8.10. ([26], Theorem 3 1) *The following conditions are equivalent:*

(1) A ring R is strongly regular.

(2) R is a left SF-ring whose every maximal left ideal is a quasi-ideal.

(3) R is a right SF-ring whose every maximal right ideal is a quasi-ideal.

Proof. (1) implies (2) and (1) implies (3) are obvious.
(2) implies (1):- Let $\overline{R} = R/J$. Then $\overline{R}$ is semiprimitive and hence reduced by Lemma 4.8.6. So $\overline{R}$ is strongly regular. Thus every maximal left ideal of $\overline{R}$ is an ideal and so every maximal left ideal of R is an ideal. Therefore R is strongly regular Thus (2) implies (1).
(3) implies (1) can be proved similarly. □

Remark 4.8.11. By Corollary 4.8.4, the theorem follows trivially.

Proposition 4.8.12. ([26], Proposition 3.2) *If R is a left $SF-$ring whose every maximal essential left ideal is a quasi-ideal, then R is semiprime.*

Proof. Let $S = Soc(_RR)$. Let $\overline{M}$ be a maximal left ideal of R/S. Then $\overline{M} = M/I$ for some maximal left ideal M of R such that $S \subseteq M$. Then M is an essential left ideal of R and hence a quasi-ideal of R. Therefore by Lemma 4.8.5, $\overline{M}$ is a quasi-ideal of R/I. Hence by Theorem 4.8.10, R/S is strongly regular. Therefore
*:Every one sided ideal of R containing S is an ideal

Let $0 \neq x \in R$ with $xRx = 0$. There exists a left ideal L of R such that $RxR \oplus L$ is essential left ideal of R. Thus $S \subseteq RxR \oplus L$. By $xRx = 0$

and $RxRL \subseteq RxR \cap L = 0$, we have $RxR \subseteq r(x)$ and $L \subseteq r(x)$. Therefore $S \subseteq RxR \oplus L \subseteq r(x)$. Hence by (*), $r(x)$ is a left ideal of R. Since $x \neq 0$, $r(x) \neq R$. Therefore there exists a maximal left ideal M of R such that $RxR \oplus L \subseteq r(x) \subseteq M$. Thus the simple left R- module R/M is flat Since $x \in RxR \oplus L \subseteq M$, there exists $y \in M$ such that $x = xy$. Then $1 - y \in r(x)$. Therefore $1 = (1 - y) + y \in M$, a contradiction. Hence $x = 0$ and R is semiprime. □

Proposition 4.8.13. ([26], Proposition 3.3) *If R is a left SF-ring whose every maximal essential right ideal is a quasi-ideal, then every one sided ideal of R containing $Soc(R_R)$ is an ideal.*

Proof. Follows from the proof of Proposition 4.8.12. □

Theorem 4.8.14. ([26], Theorem 3.4) *If R is a left SF-ring whose every maximal essential left ideal is a quasi-ideal, then R is regular.*

Proof. By Proposition 4.8.12, R is semiprime. Thus $soc(R_R) = soc(_RR) = S$(say). We have R/S is strongly regular (by the proof of Proposition 4.8.12). We claim that R is regular. Let $x \in R$, then there exists a left ideal L of R such that $Rx \oplus L$ is essential left ideal of R. Then $S \subseteq Rx \oplus L$. As R/S is strongly regular, we have $Rx \oplus L$ is an ideal of R. Since R/S is strongly regular, there exists some $y \in R$ such that $x - xyx \in S$. Since R is semiprime, S is fully right idempotent. Therefore we have

$x - xyx \in (x - xyx)R(x - xyx)R = x(1 - yx)R(1 - xy)xR \subseteq xRxR$
$\subseteq x(Rx \oplus L)$.

Therefore $x - xyx = x(sx + t)$ for some $s \in R$, $t \in L$.

Then $x - xyx - xsx = xt$ Hence $x - x(y+s)x = xt \in L \cap Rx = 0$. Therefore $x = x(y+s)x$ and hence R is regular. □

Definition 4.8.15. A ring R is *normal* or *abelian* if every idempotent of R is central.

Example 4.8.16. Every commutative ring is normal.

Definition 4.8.17. An idempotent element e of R is *left (right) semicentral* in R if $Re = eRe$ $(eR = eRe)$.

Example 4.8.18. Every central idempotent is left (right) semicentral

Theorem 4.8.19. ([26], Theorem 3 5) *A ring R is strongly regular if and only if R is a regular ring whose every idempotent is left (right) semicentral.*

Proof. Let R be a regular ring whose every idempotents are left semicentral. Let $a \in R$ such that $a^2 = 0$. Then there exists some $b \in R$ such that $a = aba$. So $e = ba$ is idempotent. Since e is left semicentral, we have

$$a = ae = eae = baaba = ba^2ba = 0$$

So $a = 0$. Therefore R is reduced and hence R is strongly regular.
Converse is trivially true □

Theorem 4.8.20. ([26], Theorem 3.6) *The following conditions are equivalent:*

(1) A ring R is strongly regular.

(2) R is a left SF-ring whose every maximal essential left ideal is a quasi-ideal and every idempotent element is left semicentral

(3) R is a left SF-ring whose every maximal essential right ideal is a quasi-ideal and every idempotent element is left semicentral.

Proof. (1) implies (2) and (1) implies (3) are obvious.

(2) implies (1):- By Theorem 4.8.14, R is regular and then by Theorem 4.8.19, R is strongly regular.

(3) implies (1):- We have $R/Soc(R_R)$ is strongly regular and hence every one sided ideal of R containing $Soc(R_R)$ is an ideal. Therefore every essential right ideal of R is an ideal. We claim that R is reduced. If not, then there exists $0 \neq a \in R$ such that $a^2 = 0$. Then $r(a) \neq R$. Let M be a right ideal of R such that $r(a) \oplus M$ is essential right ideal. Then $r(a) \oplus M$ is an ideal of R. If $r(a) \oplus M \neq R$, then there exists a maximal left ideal M_0 of R such that $r(a) \oplus M \subseteq M_0$. So the simple left R-module R/M_0 is flat Since $a^2 = 0$, $a \in r(a) \oplus M \subseteq M_o$, so there exists some $b \in M_0$ such that $a = ab$ showing that $1 - b \in r(a) \subseteq M_0$. But this gives $1 \in M_0$ which is a contradiction. Hence $r(a) \oplus M = R$. Then $r(a) = r(e)$ for some idempotent $e \neq 0$. Since idempotent elements are left semicentral, $ae = eae = 0$ as $a \in r(a) = r(e)$, $ea = 0$. So $e \in r(a) = r(e)$ which implies $e = e^2 = 0$. This contradicts $e \neq 0$. Thus R is reduced. Since a reduced left SF-ring is strongly regular we conclude that R is strongly regular. □

Corollary 4.8.21. ([26], Corollary 3.7) *The following are equivalent for a ring R:*

(1) R is strongly regular.

(2) R is a normal left SF-ring whose every maximal essential left (right) ideal is a an ideal.

(3) *R is a left SF-ring whose every maximal essential left (right) ideal is an ideal and every idempotent element is left semicentral.*

Theorem 4.8.22. ([26], Theorem 3.8) *The following conditions are equivalent for a ring R:*

(1) *A ring R is strongly regular.*

(2) *R is a left SF-ring whose every maximal essential left ideal is a quasi-ideal and every complement right ideal is an ideal.*

(3) *R is a left SF-ring whose every maximal essential right ideal is a quasi-ideal and every complement right ideal is an ideal.*

Proof. (1) implies (2) and (1) implies (3) are obvious.
(2) implies (1):- By Proposition 4.8.12, R is semiprime. Therefore $Soc(R_R) = Soc(R_R) = S$ (say). By the proof of Proposition 4.8.12, R/S is strongly regular. Then every essential right ideal of R is an ideal. We claim that R is reduced. If not, then there exists $0 \neq a$ such that $a^2 = 0$. So there exists a non-zero right ideal K of R such that $r(a) \oplus K$ is essential right ideal of R. Therefore $r(a) \oplus K$ is an ideal of R. Thus $aK \subseteq K \cap r(a) = 0$, $K \subseteq r(a) \cap K = 0$. Thus $r(a)$ is an essential right ideal of R which implies $r(a)$ is an ideal of R. Since $a \neq 0$, $r(a) \neq R$. Thus there exists a maximal left ideal M of R such that $r(a) \subseteq M$. Then the simple left R module R/M is flat. Since $a \in r(a) \subseteq M$, there exists some $b \in M$ such that $a = ab$. Then $1 - b \in r(a)$ which implies $1 \in M$, a contradiction. Thus R is reduced. Since a reduced left SF-ring is strongly regular, we conclude that R is strongly regular.

(3) implies (1):- Since R is left SF. $R/Soc(R_R)$ is left SF ring. Also, every maximal right ideal of $R/Soc(R_R)$ is a quasi-ideal. Hence $R/Soc(R_R)$ is strongly regular. The remaining proof can be completed as in (2) implies (1). □

Corollary 4.8.23. ([26], Corollary 3.9) *R is strongly regular if and only if R is a left SF-ring whose every maximal essential left ideal is an ideal and every complement right ideal is an ideal.*

Theorem 4.8.24. ([26], Theorem 3.10) *The following conditions are equivalent:*

(1) A ring R is strongly regular.

(2) R is a left SF-ring and $r(a)$ is a quasi-ideal for every $a \in \{x \in R : x^2 = 0\}$

Proof. (1) implies (2) is obvious.

(2) implies (1):- Let $0 \neq a \in R$ with $a^2 = 0$. If $Rr(a) \neq R$, then there exists a maximal left ideal M of R such that $Rr(a) \subseteq M$. Note that R is a left SF-ring and $a \in r(a) \subseteq Rr(a) \subseteq M$. So R/M is flat and $a = ab$ for some $b \in M$. This shows $1 - b \in r(a) \subseteq M$. As $b \in M$, we get $1 \in M$, contradicting that M is a maximal left ideal of R. Thus $Rr(a) = R$. Hence $1 = cd$ with $c \in R$ and $d \in r(a)$. Note that $r(a)$ is a quasi-ideal. We have $c = cdc \in r(a)$. Thus $1 = cd \in r(a)$. Then $a = 0$, contradicting $a \neq 0$. Thus R is reduced. Since a reduced left SF-ring is strongly regular, we conclude that R is strongly regular. □

4.9 More on SF-rings

Here, some properties of a ring whose simple singular left (right) modules are flat are given. Regular rings with non-zero socle are studied. Some more properties related to SF-rings are discussed.

Proposition 4.9.1. ([25], Proposition 2.4) *If R is a right quasi-duo ring whose every simple singular left R module is flat, then R is reduced.*

Proof. Suppose $0 \neq x \in R$ such that $x^2 = 0$. Then $r(x) \neq R$. Therefore there exists a maximal right ideal K of R such that $r(x) \subseteq K$. By hypothesis, K is an ideal of R. Thus there exists a maximal left ideal L of R such that $K \subseteq L$. If L is not an essential left ideal of R, then there exists a non-zero left ideal A of R such that $L \cap A = 0$. So $A = Re$ for some idempotent $e \neq 0$. Then $L = R(1-e) = l(e)$. Note that $x \in r(x) \subseteq K \subseteq L = l(e)$. Thus $xe = 0$. So $e \in r(x) \subseteq L = l(e)$. So $e = e^2 = 0$. This contradicts that $e \neq 0$. So L is an essential left ideal of R. Therefore R/L is a singular left R module [For if $x + L \in R/L$, then $l(x + L) = \{r \in R : r(x + L) = L\}$ $= \{r \in R : rx \in L\}$ which is an essential left ideal of R. Thus $Z(_R(R/L)) = R/L$]. Again $_R(R/L)$ is simple. Hence by hypothesis, $_R(R/L)$ flat. Thus there exists $y \in L$ such that $x = xy$. That is, $1 - y \in r(x) \subseteq L$. Hence $1 = (1 - y) + y \in L$ which contradicts that M is a maximal left ideal. Thus R is reduced. □

Proposition 4.9.2. ([25], Proposition 2.5) *Let I be an ideal of R. If R is a ring whose every simple singular left R module is flat, then R/I is a ring whose every simple singular left R/I module is flat.*

Proof. Let $\overline{M}$ be a maximal left ideal of $\overline{R}$. Then there exists a maximal left ideal M of R containing I such that $\overline{M} = M/I$. We claim that M is essential. If M is not essential, then there exists some $0 \neq x \in R$ such that $Rx \cap M = 0$. Since $I \subseteq M$ and $x \notin M$, it follows that $I \subsetneq Rx \oplus I$. Thus $\overline{Rx \oplus I} \neq \overline{0}$. Hence $\overline{M} \cap \overline{(Rx \oplus I)} = \overline{M \cap (Rx \oplus I)} = \overline{I \oplus (M \cap Rx)} = \overline{0}$. It is a contradiction with $\overline{M}$ is an essential left ideal of $\overline{R}$. Thus M is an essential left ideal of R. By supposition, the simple singular left R module R/M is flat. Note that $\overline{R}/\overline{M} \simeq R/M$ as left R module. Therefore $\overline{R}/\overline{M}$ is flat as left R module. Therefore by Proposition 3.3.4, $\overline{R}/\overline{M}$ is flat as $\overline{R}$ module. □

Definition 4.9.3. A ring R is said to *satisfy the condition (SI)* or R is said to be a *ZI (zero insertive)-ring* if for any $x, y \in R$, $xy = 0$ implies $xRy = 0$.

Example 4.9.4. (1) Every commutative ring R satisfies (SI)

(2) Every domain R satisfies (SI)

Proposition 4.9.5. *Every reduced ring is a ring satisfying (SI).*

Proof. Let R be a reduced ring and $x, y \in R$ such that $xy = 0$. Then $yx = 0$. Let $r \in R$. Then $(xry)^2 = (xry)(xry) = xr(yx)ry = 0$.
So $xry = 0$, that is, $xRy = 0$. □

Corollary 4.9.6. *Every strongly regular ring satisfies (SI).*

Proposition 4.9.7. *If R is a ring satisfying (SI), then the left (right) annihilator of any element of R is an ideal of R.*

Proof. Let $a \in R$ and $x \in l(a)$. Then $xa = 0$. Since R satisfies (SI) $xRa = 0$, which implies $xR \subseteq l(a)$. Thus $l(a)$ is an ideal of R. □

Corollary 4.9.8. *If R is a left (right) SF-ring satisfying (SI), then R is von Neumann regular.*

Proposition 4.9.9. *A ring R is strongly regular if and only if R is a normal regular ring.*

Proof. Let R is strongly regular, then R is reduced, regular ring. We know that in a reduced ring all the idempotents are central. So it follows that R is a normal regular ring.

Conversely, if R is a normal regular ring. then every principal left ideal of R is generated by a central idempotent. Thus it follows that R is strongly regular ring. □

Proposition 4.9.10. *If R is a ring satisfying (SI), then R is a normal ring.*

Proof. Let e be an idempotent in R and $a \in R$. We have to prove that $ea = ae$.
Now, $e^2 = e$ implies $(e-1)e = 0$. So $(e-1)ae = 0$ [since R is a ring satisfying (SI)]
This implies, $eae = ae$. Again $e^2 = e$ implies $e(e-1) = 0$. So it follows that $ea(e-1) = 0$. That is, $eae = ea$. Therefore $ea = ae$. □

Lemma 4.9.11. ([25], Lemma 3.1) *The following conditions are equivalent:*

(1) A ring R is strongly regular.

(2) R is a ring satisfying (SI) and every simple singular left R module is flat.

(3) R is a ring satisfying (SI) and every simple singular right R module is flat.

Proof. (1) implies (2) and (1) implies (3) are obvious.

(2) implies (1) :- Suppose $Rx + r(x) \neq R$ for some $x \in R$. Now $r(x)$ is an ideal of R. Therefore there exists a maximal left ideal L of R such that $Rx + r(x) \subseteq L$.

We claim that L is an essential left ideal of R. If not, then there exists an idempotent $e \neq 0$ such that $L = l(e)$. Since $x \in Rx \subseteq L = l(e)$, $xe = 0$. Hence $e \in r(x) \subseteq L = l(e)$. Thus $e = e^2 = 0$ which contradicts that $e \neq 0$. Thus L is an essential left ideal of R and hence R/L is a simple, singular left R module, and hence flat by hypothesis. Therefore there exists some $y \in L$ such that $x = xy$. This gives $1 - y \in r(x) \subseteq L$. This implies $1 = (1 - y) + y \in L$, a contradiction. Hence $Rx + r(x) = R$ for all $x \in R$. This implies R is strongly regular.

(3) implies (1) can be proved similarly. □

Corollary 4.9.12. ([25], Corollary 3.2) *A ring R is strongly regular if and only if R is reduced ring whose every simple singular left (right) R module is flat.*

Proof. Follows from Lemma 4.9.11 and Proposition 4.9.5. □

Corollary 4.9.13. *A ring R is strongly regular if and only if R is a right quasi-duo ring whose every simple, singular left R module is flat.*

Proof. Follows from Proposition 4.9.1 and Corollary 4.9.12. □

Theorem 4.9.14. ([25], Theorem 3 4) *The following conditions are equivalent:*

(1) A ring R is strongly regular.

(2) R is a ring whose every simple singular left R-module is flat and every maximal right ideal is a GW-ideal.

Proof. (1) implies (2) is obvious.
(2) implies (1): Let $\overline{R} = R/J(R)$. Then $\overline{R}$ is semiprimitive. By Proposition 4.9.2, $\overline{R}$ is a ring whose every simple singular left $\overline{R}$ module is flat. Let $\overline{M}$ be a maximal right ideal of $\overline{R}$. Then $\overline{M} = M/J(R)$ for some maximal right ideal M of R By hypothesis, M is a GW ideal of R, and hence $\overline{M}$ is a GW ideal of $\overline{R}$.
Hence by Lemma 2.7.10, it follows that $\overline{R}$ is reduced. Therefore by Corollary 4.9.12, it follows that $\overline{R}$ is strongly regular. Thus every maximal right ideal of R is an ideal. Hence by Corollary 4.9.12, R is strongly regular. □

Corollary 4.9.15. ([25], Corollary 3.5) *If R is a left SF-ring whose every maximal right ideal is a GW-ideal, then R is a strongly regular ring.*

Definition 4.9.16. A left R-module M is $MP-$*injective* if for any principal left ideal P of R, any left R-monomorphism from P to M extends to a left R-homomorphism from R to M. A ring R is left $MP-$ *injective* if $_RR$ is $MP-$ *injective*.

Example 4.9.17. Every p-injective module is MP-injective.

Proposition 4.9.18. ([7], Proposition 3) *Let R be an MP-injective ring. Then*

(1) Any left non-zero divisor element of R is right invertible. Consequently, every R module is divisible.

(2) $Z({}_RR) = J$, the Jacobson radical of R.

Proof. (1). Let $c \in R$ be a left non-zero divisor. Define $f : Rc \longrightarrow R$ by $f(rc) = r$. Then f is well-defined for if $rc = r'c$, then $r - r' \in l(c) = 0$, then $r = r'$, that is, $f(r) = f(r')$. Also f is left R-homomorphism. Also $\ker f = \{rc \mid r = 0\} = \{0\}$. Thus f is a monomorphism. Since R is MP-injective, f can be extended to a left R-homomorphism $g : R \longrightarrow R$. then $1 = f(c) = g(c) = cg(1) = cy$ for some $y \in R$. Then $c = cyc$. Consequently, if c is a left non-zero divisor, then $c = cyc$ implies $1 - cy = 0$. So c is right invertible in R. Therefore $M = cM$ for every left R module M and similarly every right R module is divisible.

(2). Let $Z = Z(_R)$ and $z \in Z$. For any $a \in R$, if $u \in l(1 - za)$, then $u = uza$. So $Ru \cap l(za) = 0$ [For if $x = yu \in Ru \cap l(za)$. then $0 = yuza = yu = x$]. This implies that $Ru = 0$. Therefore $u = 0$. Hence $l(1 - za) = 0$. By (1) there exists some $y \in R$ such that $(1 - za)y = 1$. So $z \in J$ and therefore $Z \subseteq J$. Now suppose there exists some $w \in J$ such that $w \notin Z$ Then there exists a non-zero complement left ideal K such that $L = l(w) \oplus K$ is an essential left ideal of R. For any $0 \neq k \in K$, $k \notin l(w)$, that is, $Kw \neq 0$ and if $f : Rkw \longrightarrow R$ be defined by $f(rkw) = rk$, then f is well-defined for if $rkw = r'kw$, then $(rk - r'k)w = 0$. So $(rk - r'k) \in l(w) \cap K = 0$. Therefore $rk = r'k$. Also f is a monomorphism. Since R is MP-injective, there exists a left R-homomorphism $g : R \longrightarrow R$ which extends f. Hence $k = f(kw) = g(kw) = kwg(1) = kwd$ for some $d \in R$. Since $wd \in J$, $(1 - wd)v = 1$ for some $v \in R$. This implies that $k = k.1 = k(1 - wd)v = (k - kwd)v = 0$, a contradiction. Therefore $Z = J$. □

Definition 4.9.19. A left ideal I of a ring R is *von Neumann regular* if for every $a \in I$, $a \in aRa$

Example 4.9.20. If R is a von Neumann regular ring, then every left (right) ideal of R is von Neumann regular.

Theorem 4.9.21. ([21], Theorem 1) *The following conditions are equivalent for a ring R:*

(1) R is left self-injective regular with non-zero socle.

(2) R is a right SF-ring containing a non-singular injective maximal left ideal.

(3) R contains an injective maximal left ideal which is von Neumann regular.

Proof. (1) implies (2):- Let S be a simple left ideal of R. Since R is regular, there exists some idempotent e such that $S = Re$. So there exists some maximal left ideal M of R such that $M \oplus S = R$. Since ${}_RR$ is injective, M is injective. Since R is regular, R is non-singular. So M is non-singular. So R is right SF-ring containing a non-singular injective maximal left ideal. Thus (1) implies (2).

(1) implies (3) follows from proof of (1) implies (2).

(2) implies (1):- Let M be a non-singular injective maximal left ideal of R. Then $R = M \oplus U$ for some minimal left ideal U of R. We prove that ${}_RU$

is injective. Let $I \neq 0$ be a non-zero left ideal of R and $f : I \longrightarrow U$ be a non-zero left R-homomorphism. Then $I/\ker f \simeq U$. Since ${}_RU$ is projective, $I = \ker f \oplus V$ for some left ideal V of R such that ${}_RV \simeq {}_RU$. Let $0 \neq v \in V$. Let $V = Rv$. Suppose $MV = 0$. If M is not an ideal of R, then $MR = R$. So $Mv = 0$ implies $Rv = 0$ which implies $v = 0$, a contradiction. Thus M is an ideal of R. Since R is right SF, $(R/M)_R$ is flat. Therefore by Proposition 3.2.4, ${}_R(R/M)$ is injective. So ${}_RU$ is injective. Suppose $MV \neq 0$. Then there exists some $0 \neq w \in V$ such that $Mw \neq 0$. Then $V = Rw$. Combining the projection $R \longrightarrow Rw$ with f, we get $k : R \longrightarrow U$. If h is the restriction of k to M, then $h(M) = k(M)$ and $k(M) = f(Mw) \neq 0$ [otherwise $Mw \subsetneq \ker f \cap V = 0$ which is a contradiction]. Thus $h(M) = k(M) = U$ and $h : M \longrightarrow U$ yields $M/\ker h \simeq U$. Since ${}_RU$ is projective, $M \simeq \ker h \oplus M/\ker h$. This yields ${}_RU$ is injective. In any case f extends to $F : R \longrightarrow U$. Therefore $R = M \oplus U$ is left self-injective. Since R is right SF, $Z({}_RR) \subsetneq J(R) \subseteq M$. Then M is non-singular implies R is non-singular. Hence by Corollary 2.3.5, R is semiprimitive and then by Corollary 2.3.8, R is regular.

(3) implies (1):-Let M be an injective maximal left ideal of R which is von Neumann regular. Consider the case when M is an ideal of R. For any $u \in M$ there exists some $c \in R$ such that $u = ucu$. Now $u \in Mu$ [because $uc \in M$] implies $(R/M)_R$ is flat. Then ${}_R(R/M)$ is injective. The proof of (2) implies (1) shows that R must be left self-injective. Also if $0 \neq v \in J(R)$, then $v \in M$ implies $v = vwv$ for some $w \in R$. Then $J(R)$ contains a non-zero idempotent vw which is impossible. Therefore $J(R) = 0$. Hence (3) implies (1). $\square$

Proposition 4.9.22. ([21], Proposition 2) *The following conditions are equivalent for an ELT ring R:*

(1) R is regular with non-zero socle.

(2) R is a right SF-ring containing a $p-$injective finitely generated maximal left ideal.

Proof. (1) implies (2) is trivial.

Assume (2). Suppose $Z(_RR) \neq 0$. Then there exists some $0 \neq z \in Z(_RR)$ such that $z^2 = 0$. Let L be a maximal left ideal of R such that $l(z) \subseteq L$. Then R is an ELT ring implies L is an ideal of R. Since R is right SF, $(R/L)_R$ is flat Therefore there exists some $u \in L$ such that $z = uz$. Then $1 - u \in l(z) \subseteq L$. This implies $1 \in L$, a contradiction. Therefore $Z(_RR) = 0$. If M is a finitely generated p-injective maximal left ideal, then $R = M \oplus U$ for some minimal left ideal U of R The proof of Theorem 4.9.21 shows that $_RU$ is p-injective and hence R is p-injective. By Proposition 4 9 18, $J(R) = Z(_RR) = 0$. Thus R is semiprime. Hence $Soc(_RR) = soc(R_R) = S$ (say) and S is fully right idempotent. Again R is an ELT implies R/S is left quasi-duo. Then, as R/S is left SF, R/S is strongly regular. Hence R/S is fully right idempotent. Thus R is fully right idempotent. Therefore R is regular by Proposition 2.5.3. □

Proposition 4.9.23. ([28], Proposition 5) *Let R be a ring such that every maximal left ideal is either an ideal of R or an injective left R module. Assume for each maximal left ideal of R which is an ideal, $(R/M)_R$ is flat Then R is either strongly regular or left self-injective regular with non-zero socle.*

Proof. Suppose every maximal left ideal of R is an ideal and $l(a) + Ra \neq R$ for some $a \in R$. Then there exists a maximal left ideal M of R such that $l(a) + Ra \subseteq M$. By hypothesis, $(R/M)_R$ is flat. Since $a \in M$, there exists some $b \in M$ such that $a = ba$. Then $1 - b \in l(a) \subseteq M$. Then $1 \in M$, a contradiction. Thus $l(a) + Ra = R$ for all $a \in R$. Then R is strongly regular.

Suppose, not all maximal left ideal of R is an ideal. Then by hypothesis, there exists an injective maximal left ideal N of R. Therefore there exists a minimal left ideal U such that $R = N \oplus U$. By the proof of Theorem 4.9.21, it follows that R is left self-injective. It remains to show that $Z(_RR) = 0$. Suppose $Z(_RR) \neq 0$. Then there exists some $0 \neq z \in Z(_RR)$ such $z^2 = 0$. Since $l(z) \neq R$, there exists a maximal left ideal K of R such that $l(z) \subseteq K$. Now K is not injective [For if K is injective, then $R = K \oplus L$ for some non-zero left ideal L. But this is impossible as K is essential since $l(z) \subseteq K$]. Therefore by hypothesis, K is an ideal of R and $(R/K)_R$ is flat. Since $z \in l(z) \subseteq K$, there exists some $y \in K$ such that $z = yz$. This implies $1 \in K$ which is a contradiction. Thus $Z(_RR) = 0$. Therefore R is left self-injective regular with non-zero socle. □

Lemma 4.9.24. ([28], Lemma 7) *The following conditions are equivalent for a ring R:*

(1) R is fully left idempotent.

(2) R is semiprime ring such that for every essential left ideal L of R which is an ideal, $(R/L)_R$ is flat.

Proof. Assume (1). Then $J(R) = 0$ which implies R is semiprime. Let L be an essential left ideal of R which is an ideal. For any $0 \neq t \in L$, $t \in Rt =$

$RtRt$. Therefore $t = dt$ for some $d \in RtR \subseteq L$. Therefore $t \in Lt$ for every $t \in L$ which implies that $(R/L)_R$ is flat. Thus (1) implies (2).

Assume (2). For any $a \in R$, set $I = RaR + l(RaR)$. Then there exists a left ideal K of R such that $L = I \oplus K$ is an essential left ideal of R. Now $RaRK \subseteq K \cap RaR \subseteq K \cap I = 0$. Then $K \subseteq r(RaR)$. Since R is semiprime, $l(RaR) = r(RaR)$ [If $x \in l(RaR)$, then $xRaR = 0$. So $(RaRx)^2 = 0$. As R is semiprime, $RaRx = 0$ and so $x \in r(RaR)$. So $l(RaR) \subseteq r(RaR)$. Similarly $r(RaR) \subseteq l(RaR)$]. Therefore $K \subseteq l(RaR) \subseteq I$. Then $K \subseteq K \cap I = 0$. Therefore $I = L$ is an essential left ideal of R which is an ideal. By hypothesis, $(R/I)_R$ is flat. So as $a \in I$, there exists some $u \in I$ such that $a = ua$. If $u = w + c$, $w \in RaR$, $c \in l(RaR)$, then $a = wa + ca = wa \in (Ra)^2$. This proves that R is fully left idempotent. □

Theorem 4.9.25. ([28], Theorem 10) *The following conditions are equivalent for a ring R:*

(1) R is strongly regular.

(2) R is a ZI-ring whose simple left modules are flat.

Proof. (1) implies (2) evidently.

Assume (2). For any $b \in R$, $r(b)$ is an ideal of R. Suppose $Rb + r(b) \neq R$ for some $b \in R$. Then there exists a maximal left M of R such that $Rb + r(b) \subseteq M$. So ${}_R(R/M)$ is flat. As $b \in M$, there exists some $c \in M$ such that $b = bc$. That is, $1 - c \in r(b) \subseteq M$. This implies $1 \in M$, a contradiction. Thus $Rb + r(b) = R$ for all $b \in R$. This implies R is von Neumann regular. Since R is a ZI-ring every idempotent is central. Therefore R strongly regular. □

Proposition 4.9.26. ([27], Proposition 4(1)) *Let R be a right SF-ring. Then any left non-zero divisor is right invertible in R; Consequently R coincides with its classical right (and left) quotient ring*

Proof. Let $c \in R$ such that $l(c) = 0$ and suppose $cR \neq R$. Let M be a maximal right ideal of R such that $cR \subseteq M$. Then $(R/M)_R$ is flat. Since $c \in cR \subseteq M$, there exists some $d \in M$ such that $c = dc$. So $1 - d \in l(c) = 0$. Therefore $d = 1$. This contradicts that M is a maximal right ideal of R. Therefore $cR = R$. So $cu = 1$ for some $u \in R$. For any non-zero divisor c, $c = cuc$ and $1 - uc \in r(c) = 0$. Therefore $uc = cu = 1$. This proves that every non-zero divisor is invertible in R. Therefore R coincides with its classical right (left) quotient ring. □

Proposition 4.9.27. ([27], Proposition 6) *Let R be a right SF-ring which has a finite number of maximal right ideals whose product is contained in J, then $Z = J = 0$ where $Z = Z({}_RR)$.*

Proof. Let $M_1, M_2, \ldots, M_n$ be maximal right ideals such that $M_1M_2\ldots M_n \subseteq J$. Let $u \in J$. Then $u \in M_n$. Since R is right SF, $(R/M_n)_R$ is flat. Therefore there exists some $u_n \in M_n$ such that $u = u_nu$. Again $u \in M_{n-1}$ and $(R/M_{n-1})_R$ is flat. So there exists some $u_{n-1} \in M_{n-1}$ such that $u = u_{n-1}u$. Therefore $u = u_{n-1}u_nu$. Proceeding in this manner we get $u = u_1u_2\ldots u_nu$ where $u_i \in M_i$ for each $i = 1, 2, \ldots, n$. So $u_1u_2\ldots u_n \in M_1M_2\ldots M_n \subseteq J$. Therefore there exists some $v \in R$ such that $v(1 - u_1u_2\ldots u_n) = 1$. Then $u = 1u = v(1 - u_1u_2\ldots u_n)u = 0$. Therefore $J = 0$. Since R is right SF, $Z \subseteq J$. Thus $Z = 0$. □

Definition 4.9.28. A ring R is *left (right) uniform ring* if every non-zero left (right) ideal is essential.

Example 4.9.29. $\mathbb{Z}$ is left (right) uniform ring.

Theorem 4.9.30. ([7], Theorem 6) *The following conditions are equivalent:*

(1) R is a division ring.

(2) R is a left uniform ring whose simple right modules are flat.

Proof. (1) implies (2) is trivial.
(2) implies (1):- Let $a \notin Z = Z(_RR)$. Then $l(a) = 0$ and if we suppose $aR \neq R$, let M be a maximal right ideal of R containing aR. So $(R/M)_R$ is flat. Therefore there exists some $b \in M$ such that $a = ba$. This yields $1 - b \in l(a) = 0$, that is, $b = 1$, which contradicts that $M \neq R$. This proves that every proper right ideal, in particular every maximal right ideal is contained in Z. Therefore Z is the unique maximal right ideal of R. R is therefore a local ring and $Z = J$ is also the only maximal left ideal of R. Now suppose $Z \neq 0$, then there exists $0 \neq z \in Z$ such that $z^2 = 0$ and since $l(z) \subseteq Z$ (since Z is the unique maximal left ideal), $(R/Z)_R$ is flat, then $z = uz$ for some $u \in Z$. Then $1 - u \in l(z) \subseteq Z$. This implies $1 \in Z$, a contradiction. Therefore $J = Z = 0$. This proves (2) $\Rightarrow$ (1). □

Proposition 4.9.31. *If R is a right weakly regular ring, then R/K is a flat left R- module for every two sided ideal K.*

Proof. Let $x \in K$. Then $xR = xRxR$ which implies that $x = \sum xr_ixs_i$ $= x\sum r_ixs_i$ for some r_i. $s_i \in R$. Therefore $x = xz$ where $z = \sum r_ixs_i \in K$. Hence $_R(R/K)$ is flat. □

4.10 Some questions

If SF-rings are necessarily regular, then SF-rings will satisfy the properties of regular rings. Hence we think of some of the questions as follows:

(1) Is $M_n(R)$ a left SF-ring if R is a left SF-ring?

(2) Is a left SF-ring semiprimitive?

(3) Is a left SF-ring left or right non-singular?

(4) Is a left SF-ring left or right weakly regular?

Bibliography

[1] B. T. Strenstrom : *Rings of quotients- An introduction to the Methods of Ring Theory*, Springer-Verlag, New York(1975).

[2] F. W. Anderson and K.R. Fuller : *Rings and Categories of Modules (Graduate Text in Mathematics 13)*, Springer -Verlag, Berlin Heldelberg-New York, (1974).

[3] J.J. Rotman: *Notes on Homological Algebra*, van Nostrand-Reinhold, (1970).

[4] M. F. Atiyah and I. G. Macdonald: *Introduction to Commutative Algebras*, Addison Wesley, (1972).

[5] R. Yue. Chi. Ming: *On von Neumann regular rings*, Proc Edinburgh Math. Soc. 19(1974), 89-91.

[6] R. Yue. Chi. Ming: *On von Neumann regular rings - II*, Math. Scand, 39(1976), 167-170.

[7] R. Yue. Chi. Ming: *On von Neumann regular rings - VIII*, J Korean Math. Society, 19(2)(1983), 97-104.

[8] V. S. Ramamurthy: *Weakly regular rings*, Canad. Math. Bull, 16(3), (1973), 317-321.

[9] Melvin Henriksen : *On a Class of rings that are elementary divisor rings*, Archiv Der Mathematik (Basel), 24(1973), 133-141.

[10] I Kaplansky: *Elementary divisors and modules*, Trans. Amer. Math. Soc. 66(1949), 464-491.

[11] E.P. Armendariz, J. W. Fisher and S. A. Steinberg: *Central localizations of regular rings*, Proc. Amer. Math. Soc., 74(1981), 481-485.

[12] Y. Hirano and H. Tominaga: Regular rings, V-rings and their generalizations, Hiroshima Math. Journal, 9(1979), 315-321.

[13] Zhang Jule: Characterizations of strongly regular rings, Northeast Math. Journal, 10(3)(1994), 359-364.

[14] Xianmei Song and Xiaobin Yin: *Generalisations of V-rings*, Kyungpook Math. Journal, 45(2005), 357-362.

[15] R. Yue Chi Ming: *Remarks on strongly regular rings*, Portugaliae Mathematica, 44(1987), 101-111.

[16] V. S. Ramamurthy: *On the injectivity and flatness of certain cyclic modules*, Proc. Amer. Math. Soc., 48(1975), 21-25.

[17] M. B. Rege: *On von Neumann regular rings and SF-rings*, Math. Japonica, 31(6)(1986), 927-936.

[18] Zhang Jule and Du Xianneng: *Von Neumann regularity of SF-rings*, Communications in Algebra, 21(7)(1993), 2445-2451.

[19] Yufei Xiao: *One sided SF-rings with certain chain conditions*, Canad Math. Bull., 37(2)(1994), 272-277

[20] Zhang Jule and Lu Rongdian: *Strongly regular rings and SF-rings*, Northeast Math. Journal, 14(10)(1998), 61-68.

[21] R. Yue Chi Ming: *On biregularity and regularity*, Communications in Algebra, 20(3)(1992), 749-759.

[22] R. Yue Chi Ming: On Quasi-injectivity and von Neumann regularity, Mh. Math, 95(1983), 25-32

[23] Zhou H. Wang X.: *On von Neumann regular rings and SF-rings*, J. Maths. Research annd Exposition, 24(4)(2004b), 679-683.

[24] Haiyan Zhou: *Left SF-rings and regular rings*, Communications in Algebra, 35(2007), 3842-3850.

[25] Xiao Guangshi: *On Strongly regular rings and SF-rings*, Far East Journal of Mathematical Sciences, 27(3)(2007), 693-699.

[26] Xiao Guangshi: *On Strongly regular rings and SF-rings*, Far East Journal of Mathematical Sciences, 32(3)(2009), 387-396.

[27] R. Yue Chi Ming: *On injectivity and p-injectivity IV*, Bull. Korean Math. Soc., 40(2)(2003), 223-234.

[28] R. Yue. Chi Ming: *On rings close to regular and p-injectivity*, Commentationes Mathematicae Universitatis, 47(2)(2006), 203-212.

[29] R. Yue Chi Ming : *Some comments on injectivity and p-injectivity*, Acta Mathematica, Vietnamica, 36(2)(2007), 179-188.

[30] R. Yue Chi Ming : *On p-injective and generalizations*, Riv. Mat. Univ. Parma, 5(5)(1996), 183-188.

[31] K. R. Goodearl: Ring Theory: Non singular Rings and Modules, Marcel Dekker, INC., New York and Basel, 1976.

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