

Voter dynamics on an Ising ladder: coarsening and persistence

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Abstract

Coarsening and persistence of Ising spins on a ladder is examined under voter dynamics. The density of domain walls decreases algebraically with time as $t^{-1/2}$ for sequential as well as parallel dynamics. The persistence probability decreases as $t^{-\theta_s}$ under sequential dynamics and as $t^{-\theta_p}$ under parallel dynamics where $\theta_p = 2\theta_s \approx 0.88$. Numerical values of the exponents are explained. The results are compared with the voter model on one- and two-dimensional lattices, as well as Ising model on a ladder under zero-temperature Glauber dynamics.

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1. Introduction

Voter dynamics is a simple stochastic dynamics for Ising spins on a lattice [1]. It does not require a Hamiltonian or a heat bath for its specification. In voter dynamics, the state of a spin at (discrete) time $t + 1$ is equal to the state of one of its nearest neighbours (chosen randomly) at the previous time t . It is not obvious if this dynamics should evolve a random configuration of spins into an ordered configuration in the long-time limit. However, analytic solutions of the model show that an ordered state is reached on lattices of dimensionality $d \leq 2$, but not on lattices of higher dimensionality $d \geq 3$. Although a ferromagnetically ordered state of spins is realized in $d = 1$ as well as $d = 2$, the approach to the ordered state is qualitatively different in the two cases. For example, the persistence probability decays algebraically as $t^{-\theta}$ in $d = 1$, but in $d = 2$ it decays as $\exp[-\text{constant}(\log t)^2]$ [2]. The existence of apparently universal features (independent of the initial configuration) characterizing the approach to a steady state, and their likeness to some non-equilibrium phenomena observed in nature has generated a good deal of interest in the study of voter dynamics.

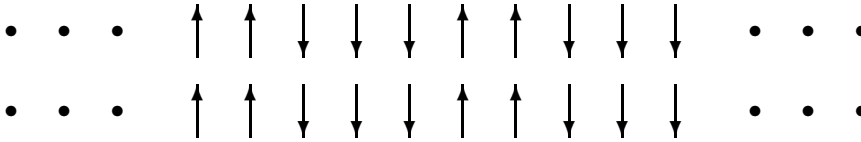
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It is easy to understand the tendency of the voter dynamics to produce an ordered state, if we imagine replacing the spins with balls of different colours. Consider a large but finite lattice of N points, and N balls of N different colours. Initially, at $t = 0$, each lattice point is occupied by a ball whose colour is unique to that lattice point. Now pick a ball at random, repaint it with the colour of one of its neighbours, and put it back in its place. Repeat this process next time and so on. At each time, the number of colours on the lattice is reduced by one or zero. Therefore, after sufficiently long time the entire lattice will be populated by balls of the same colour. We may expect this tendency irrespective of the dimensionality of the lattice as long as the lattice is finite, and the number of time steps is sufficiently large. But the limits $N \rightarrow \infty$ and $t \rightarrow \infty$ do not commute. If the limit $N \rightarrow \infty$ is taken before $t \rightarrow \infty$, a mapping of the voter model onto a system of coalescing random walkers on a d -dimensional lattice brings out the d -dependent behaviour mentioned in the preceding paragraph.

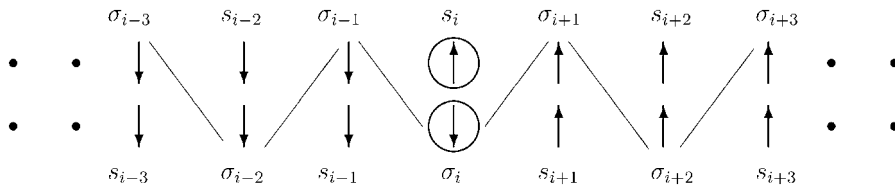
On account of its simplicity, amenability to analysis, and rich behaviour, voter dynamics has emerged as a paradigm for a broad class of many-body stochastic systems. It has been studied extensively on regular d -dimensional hypercubic lattices as well as networks and inhomogeneous lattices [3–6]. Variants of the voter model [7] have been applied for modelling complex social phenomena [8] and some non-equilibrium physical phenomena including spinodal decomposition [9] and catalytic reactions [10]. Non-equilibrium physical phenomena belong to the realm of thermal physics, and are usually studied in their simplest form by using Glauber dynamics of Ising spins [11]. The similarity in the behaviour of a system under voter and Glauber's dynamics is surprising at first sight. Glauber dynamics is based on ideas of equilibrium thermal physics. Glauber's spin-flip rates are chosen to satisfy the Boltzmann distribution and detailed balance in thermal equilibrium. The same transition rates are then exported outside the realm of equilibrium statistical mechanics with the assumption that they would bring a non-equilibrium system to equilibrium. On the other hand, concepts of a Hamiltonian, Boltzmann's distribution, or detailed balance do not exist in voter dynamics. Voter dynamics is based on ideas of random walks and coarsening without surface tension. Yet, voter dynamics in $1d$ is identical to zero-temperature Glauber dynamics of the $1d$ Ising model. In higher dimensions as well the two dynamical schemes often produce similar caricatures of a system. It is interesting to speculate if these similarities are fortuitous or perhaps Glauber dynamics contains more than what is minimally required to describe non-equilibrium phenomena such as coarsening and persistence.

The present paper makes a comparative study of the voter dynamics and zero-temperature Glauber dynamics of Ising spins on a ladder using sequential as well as parallel dynamics. A ladder comprises two chains of Ising spins placed next to each other in such a way that each spin has three neighbours at equal distance. The ladder (visualized by imagining bonds between nearest neighbours) is thus one step away from a $1d$ lattice towards a $2d$ lattice. Spin ladders are used for modelling several physical systems and are of current interest in the field of low-dimensional correlated electron systems [12]. However, our primary motivation for studying voter dynamics on a ladder does not arise from a direct interest in a physical system but rather from the fact that a ladder is the simplest topology that distinguishes voter dynamics from zero-temperature Glauber dynamics. In the same vein, we study parallel as well as sequential dynamics. So far the voter model and its variants have been studied mainly with sequential dynamics although social processes such as voting are essentially parallel processes in the sense that a large population votes together. It is therefore interesting to examine if new features emerge in the behaviour of a model under parallel dynamics. In the case examined here, it turns out that the fraction of voters who never change their vote up to time t decreases much faster in parallel dynamics than in sequential dynamics.

Zero-temperature Glauber dynamics on a ladder is remarkably different from the one- and two-dimensional cases. In $1d$ and $2d$, it shows coarsening and algebraic decay of persistence with d -dependent persistence exponents. There is no coarsening or algebraic decay of persistence on a ladder. The reason is simple. Coarsening and decay of persistence on a linear or square lattice is related to the random motion of zero field sites, i.e. sites that have as many up neighbours as down. This is not possible on a lattice with an odd coordination number. On a ladder, under zero-temperature Glauber dynamics, an initially random configuration of spins gets stuck into a fixed point configuration of the type



The two rows of arrows in the above figure represent segments of spins along the two legs of the ladder. In a fixed point configuration the nearest-neighbour spins across the width of the ladder are aligned parallel to each other, and there are ferromagnetic domains of various lengths along the length of the ladder. The minimum length of a stable domain is two lattice spacings. There is a huge degeneracy of fixed point states. Therefore, under zero-temperature sequential dynamics, an initial random configuration quickly settles into a nearby fixed point. Parallel dynamics gives qualitatively similar results but produces limit cycles of period two in addition to fixed point states. An example of a limit cycle of period two is given below. In this example, spins labelled s_i and σ_i (with circles drawn around them to aid the eye) flip at every time step while other spins remain fixed. The labelling scheme and the diagonal lines are explained in the following section.



The zero-temperature fixed points or limit cycles become metastable at a low-temperature T because spins on a domain wall can flip at a small rate even if it increases their energy. The lifetime of a metastable state is of the order of $\exp(2J/k_B T)$ where $2J$ is the energy barrier on a domain wall, and $k_B T$ is the thermal energy. There is no corresponding metastability in the voter dynamics.

2. Coarsening

We represent Ising spins on a ladder as a $2 \times N$ matrix labelled by $S_{j,i} = \pm 1$; $j = 1, 2$; $i = 1, 2, \dots, N$; and consider voter dynamics that evolves a random initial configuration of spins $\{S_{j,i}(t = 0)\}$ according to the following rule:

$$S_{j,i}(t + 1) = S_{j,i-1}(t) \text{ or } S_{k,i}(t) \text{ or } S_{j,i+1}(t); \quad \text{each with probability } \frac{1}{3};$$

$$k = j + 1 \text{ mod } 2. \tag{1}$$

In other words, $S_{j,i}(t + 1)$ takes one of three equally likely values; $S_{j,i-1}(t)$, $S_{k,i}(t)$ or $S_{j,i+1}(t)$. Periodic boundary conditions are imposed, i.e. $S_{j,i+N} = S_{j,i}$. The dynamics can

be implemented sequentially (updating one site at a time) or in parallel (updating all sites simultaneously). We study both cases. There are similarities as well as differences in the two cases. In both cases, the dynamics orders an initially disordered lattice. The rate at which ordering occurs is comparable if time is appropriately rescaled so that N steps of sequential dynamics correspond to a single step of parallel dynamics. In the case of sequential dynamics, all spins get aligned parallel to each other in the limit $t \rightarrow \infty$. In the case of parallel dynamics, the lattice falls into two sub-lattices. Each sub-lattice gets aligned ferromagnetically in the limit $t \rightarrow \infty$, but the two sub-lattices may be aligned parallel or anti-parallel to each other. In case they are aligned anti-parallel to each other, the dynamics ends in a limit cycle of period two rather than a fixed point.

In the process of reaching a fixed point or a limit cycle, the system develops domains of up spins, down spins and spins in an anti-ferromagnetic arrangement. As time progresses the domains become larger and fewer in number in a self-similar way. In other words the system coarsens [13]. In order to understand the coarsening quantitatively, it is convenient to divide the ladder into two sub-lattices at the beginning itself. It simplifies the notation, and helps to discuss parallel dynamics with greater clarity. Thus in place of $2N$ spins $S_{j,i}$, we consider two sets of N spins denoted by $\{s_i\}$ and $\{\sigma_i\}$, $i = 1, 2, \dots, N$. The new spins lie on two one-dimensional chains zig-zagging through each other such that $S_{1,i} = s_i$; $S_{2,i} = \sigma_i$; $S_{1,i+1} = \sigma_{i+1}$; $S_{2,i+1} = s_{i+1}$ and so on. This is shown schematically in the second picture in the previous section. The diagonal lines are drawn to aid the eye in visualizing one of the two sub-chains (the σ chain). In the new notation, $\{s_i(t+1)\}$ are determined by $\{\sigma_i(t)\}$ and $\{\sigma_i(t+1)\}$ by $\{s_i(t)\}$.

It is convenient to define correlation functions $C_n^s(t)$ and $C_n^\sigma(t)$ for spins separated by n units on each sub-lattice at time t .

$$C_n^s(t) = \frac{1}{N} \sum_i \langle s_i(t) s_{i+n}(t) \rangle; \quad C_n^\sigma(t) = \frac{1}{N} \sum_i \langle \sigma_i(t) \sigma_{i+n}(t) \rangle. \quad (2)$$

The angular brackets denote average over sufficiently large number of initial random configurations $\{s_i(0)\}$ and $\{\sigma_i(0)\}$. The two sub-lattices are statistically similar, and for most purposes one can drop the superscripts on C_n . Thus, under parallel dynamics, $C_n(t+1)$ is given by

$$C_n(t) = \frac{1}{9} [C_{n-2}(t-1) + 2C_{n-1}(t-1) + 3C_n(t-1) + 2C_{n+1}(t-1) + C_{n+2}(t-1)]. \quad (3)$$

The first term on the right-hand side of the above equation can be understood as follows: $s_i(t)$ is equal to $\sigma_{i+1}(t-1)$ with probability $1/3$ and $s_{i+n}(t)$ is equal to $\sigma_{i+n-1}(t-1)$ with probability $1/3$. Thus $\langle s_i(t) s_{i+n}(t) \rangle$ is equal to $\langle \sigma_{i+1}(t-1) \sigma_{i+n-1}(t-1) \rangle$ or $C_{n-2}(t-1)$ with probability $1/9$. Other terms in the equation are obtained similarly. Successive iterations of equation (3) yield $C_n(t)$ in terms of correlations $C_{n-2t}(0)$, $C_{n-2t+1}(0)$, $C_{n-2t+2}(0)$, \dots , $C_{n+2t}(0)$ in the initial state. The equation is linear, and therefore we can easily write a closed form expression for $C_n(t)$ in terms of the initial correlations. Before we do so, let us note some general features of the above equation. We note that $C_{n-2}^* = C_{n-1}^* = C_n^* = C_{n+1}^* = C_{n+2}^* = 1$ is a fixed point solution of equation (2) in the limit $t \rightarrow \infty$. We also note that, by definition, $C_0(t) = 1$, for all t . Therefore, a reduced correlation function, $c_n(t) = C_0(t) - C_n(t)$, also satisfies the same equation as $C_n(t)$.

$$c_n(t) = \frac{1}{9} [c_{n-2}(t-1) + 2c_{n-1}(t-1) + 3c_n(t-1) + 2c_{n+1}(t-1) + c_{n+2}(t-1)]. \quad (4)$$

Equation (4) has a fixed point solution $c_{n-2}^* = c_{n-1}^* = c_n^* = c_{n+1}^* = c_{n+2}^* = 0$, and we shall use this equation to study the approach to the fixed point. A closed form expression for $c_n(t)$ in terms of correlations in the initial state can be obtained by several methods. The easiest method is to write a few iterations of the equation explicitly, i.e. write $c_n(t)$ explicitly

in terms of appropriate correlations at time $t - 2, t - 3$, etc and then generalize the formula by induction. We get

$$c_n(t) = \frac{1}{9^t} \left[\sum_{l,r,d}^{t=l+r+d} \binom{l+r+d}{l,r,d} c_{n-2l+2r}(0) \right]. \tag{5}$$

In the above equation, the sum is over integers l, r and d such that $l + r + d = t$, and the symbol appearing immediately after the summation sign is a trinomial coefficient,

$$\binom{l+r+d}{l,r,d} = \frac{(l+r+d)!}{l!r!d!}. \tag{6}$$

Equations (5) and (6) have a simple geometrical interpretation that is natural to voter dynamics on a ladder. Imagine a backward walk in discrete time. A walker starts at site i at time t and visits those sites where the spin $s_i(t)$ has resided previously. After t steps, the walker arrives at the original address of $s_i(t)$ in the initial configuration of the system. The original address with respect to site i is specified by the number of left turns l , the number of right turns r and the number of straight steps d taken by the walker in $l + r + d = t$ steps. We can replace the single walk tracing the history of $s_i(t)$ by a set of directed random walks that always move one step closer to the initial configuration but take a left turn or a right turn or a straight step with equal probability. There are a total of 3^t such walks that can land on any site $i - t$ to $i + t$ in the initial configuration. The trinomial coefficient in equation (5) gives the number of random walks that land at a site corresponding to the set of integers $\{l, r, d\}$. A path that returns to the starting site after t steps is characterized by an equal number of left and right turns, i.e. $l = r$. The number of walks that return to the starting point are the largest in number,

$$c_n(t) \approx \frac{1}{3^t} \sum_0^{l_{\max}} \binom{t}{l,l,t-2l} c_n(0) = \frac{1}{3^t} \sum_0^{l_{\max}} \binom{2l}{l} \binom{t}{2l} c_n(0);$$

$$l_{\max} = \frac{t}{2} \quad \text{if } t \text{ even,} \quad \frac{t-1}{2} \quad \text{if } t \text{ odd.} \tag{7}$$

The two binomial coefficients appearing above result from the factorization of the trinomial coefficient, and their meaning is quite clear. The first specifies the number of ways that one can make a walk of l steps to the left, and an equal number of steps to the right. The second gives the number of ways one can take the remaining $t - 2l$ steps as straight steps in a total of t steps. As may be expected, the product of the two binomial coefficients is largest when $l = r = d = t/3$. Using Stirling's approximation for the factorials of large numbers, we find that the maximum value of the product scales as $1/t$ in the limit $t \rightarrow \infty$. In the neighbourhood of this maximum, the number of terms that contribute significantly to the sum in equation (7) is of the order of $t^{1/2}$. This follows from the fact that the distance covered by straight steps in the walk of t steps has a Gaussian distribution with mean value equal to $\frac{1}{3}t$, and variance proportional to t . Consequently, the sum of the product of the two binomial coefficients scales as $t^{-1/2}$. The correlation functions $c_n(t)$ approach zero as $t^{-1/2}$ and $C_n(t)$ defined by equation (2) approach unity with the same power law in the limit $t \rightarrow \infty$. The density of domain walls (number of domain walls per site) decreases and coarsening increases with the same power law. This is easily seen by defining the density of domain walls $\rho(t)$ at time t as

$$\rho(t) = \frac{1}{2N} \left[\sum_i \{3^2 - \{s_{i-1}(t) + s_i(t) + s_{i+1}(t)\}^2\} + \sum_i \{3^2 - \{\sigma_{i-1}(t) + \sigma_i(t) + \sigma_{i+1}(t)\}^2\} \right]. \tag{8}$$

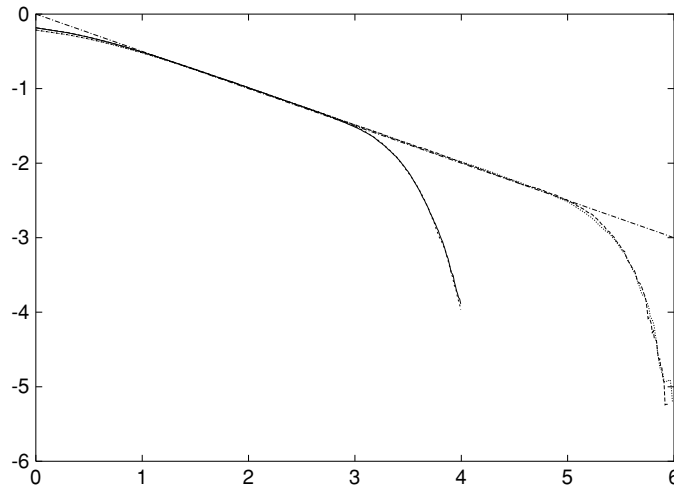


Figure 1. Coarsening: $\log \rho(t)$ versus $\log t$ for $N = 2 \times 10^2$ (continuous line) and $N = 2 \times 10^3$ (broken line) averaged over 10^4 realizations of initial random configuration. Graphs for sequential and parallel dynamics are superposed on each other and are nearly indistinguishable from each other. A line with slope $-1/2$ has been drawn for comparison.

The above expression has been constructed so that a site whose nearest neighbours are in the same state does not contribute to the density of domain walls in the system. Note that the site in question need not be in the same state as its neighbours. This definition takes into account independent evolutions of the two sub-lattices under parallel dynamics and treats the conclusion of the coarsening dynamics in a fixed point or a limit cycle on equal footing. The density of domain walls is related to correlation functions by the equation $\rho(t) = 2[2c_1(t) + c_2(t)]$, and vanishes as $t^{-1/2}$ in the limit $t \rightarrow \infty$. Figure 1 shows $\log \rho(t)$ versus $\log t$ for parallel and random sequential dynamics obtained by computer simulation of the voter dynamics on systems of size $N = 200$ and $N = 2000$. In each case, we have taken an average over 10^4 different realizations of the initial random configuration. A striking feature of figure 1 is that results of parallel dynamics are indistinguishable from those of sequential dynamics. This may be expected on ground that the stochastic equation (3) describes each step of sequential dynamics as well, and we have rescaled time such that $2N$ steps of sequential dynamics correspond to a single step of parallel dynamics. As expected, the power law $t^{-1/2}$ fits coarsening very well over a period of time that increases with the size of the system. Figure 1 also shows that the average coarsening for $N = 200$ and $N = 2000$ deviates from a power law and approaches zero at $t = t_{\max} \approx N^2$ in a self-similar way. This is due to the fact that the evolution of the system and consequently coarsening stops when the dynamics reaches a fixed point or a limit cycle, say at time $t = t_{\text{end}}$. The upper limit on t_{end} is approximately N^2 , but it has a broad distribution that depends on N as well as the initial configuration of the system. In the present paper, we do not go into the distribution of t_{end} , and confine ourselves only to the region where the average coarsening exhibits an algebraic power law.

Before leaving this section, we briefly present for comparison the results for coarsening in the $1d$ Ising model under voter or zero-temperature Glauber dynamics. In this case, there are two simplifications: (a) there is only one chain instead of two zig-zagging chains and (b) the backward random walk can only take a step to the right or to the left, i.e. there are no straight steps. With these simplifications, the $1d$ version of equation (4) becomes

$$c_n^{1d}(t) = \frac{1}{4}[c_{n-2}^{1d}(t-1) + 2c_n^{1d}(t-1) + c_{n+2}^{1d}(t-1)]. \quad (9)$$

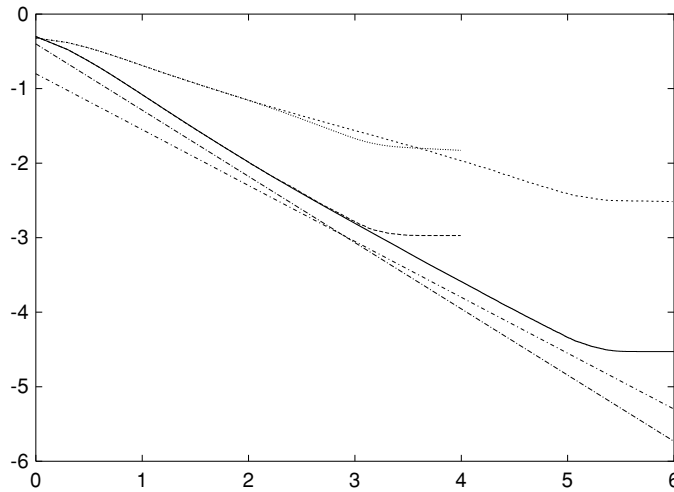


Figure 2. Persistence: $\log P(t)$ versus $\log t$ obtained from the same computer simulations as used in figure 1. The lower curves are for parallel dynamics and have approximately twice the slope of upper curves for random sequential dynamics. A single power law does not appear to fit persistence as well as it does coarsening. Two straight lines with slopes $3/4$ and $8/9$ are drawn for comparison with parallel dynamics, suggesting a crossover in the power law for persistence.

Equation (9) has the solution

$$c_n^{1d}(t) = \frac{1}{4^t} \left[\sum_{m=0}^{2t} \binom{2t}{m} c_{n-2t+2m}^{1d}(0) \right]. \tag{10}$$

As $t \rightarrow \infty$, the largest contribution on the right-hand side comes from the term $m = t$ (corresponding to an equal number of left and right turns in the random walk), and we get

$$c_n^{1d}(t) \approx \frac{1}{2^t} \binom{2t}{t} c_n^{1d}(0) \approx \frac{1}{\sqrt{\pi t}} c_n^{1d}(0). \tag{11}$$

The quantity $c_2^{1d}(t)$ is a measure of the density of domain walls (sites whose nearest-neighbour spins are aligned anti-parallel to each other) in $1d$, and as well known, it is seen to decay with the power law $t^{-1/2}$.

3. Persistence

Persistence and coarsening often coexist in a non-equilibrium system [14]. We therefore examine our system for persistence as well. The persistence probability $P(t)$ is the probability that a randomly picked site in the system has never flipped from its initial state up to time t . Figure 2 shows a plot of $\log P(t)$ versus $\log t$ obtained from computer simulations of the system. The data for figures 2 and 1 are taken from the same set of simulations. However, at first sight, a power law fit to persistence does not appear to be of the same quality as for coarsening. There is a regime after the initial transient period that fits a power law $t^{-\theta_s}$ for sequential dynamics and $t^{-\theta_p}$ for parallel dynamics but this regime is smaller than the corresponding regime where coarsening shows a power law. It is not very clear why the region of algebraic persistence should be noticeably shorter than it is for average coarsening. We

shall discuss it further below but we first look at the features of figure 2 that are relatively easy to understand.

On the basis of our simulations, we estimate $\theta_p = 2\theta_s \approx 0.88$. The relationship $\theta_p = 2\theta_s$ is easy to understand. Its origin lies in the fact that the ladder can be partitioned into two sub-lattices A and B that evolve similarly but independently of each other. At time $t = 0$, $A(0)$ and $B(0)$ are uncorrelated. Thereafter $A(t + 1)$ is determined by $B(t)$ and $B(t + 1)$ by $A(t)$. Persistence of a site on the lattice $A + B$ at odd times ($t = 1, 3, \dots$) is independent of its persistence at even times ($t = 2, 4, \dots$). At odd times and similarly at even times, the persistence is characterized by an exponent appropriate for sequential dynamics on a sub-lattice. Although spins on a sub-lattice are updated in parallel, the motion of domain walls is effectively the same as it would be under random sequential dynamics. The reason is that the order in which domain walls are relaxed does not matter under random sequential dynamics. A sequential relaxation process in which each domain wall on a sub-lattice is relaxed once would produce a qualitatively similar state as obtained in a single step of parallel relaxation. Therefore the probability that a given site on the ladder has been persistent at odd times scales as $t^{-\theta_s}$, and similarly the probability that it has been persistent at even times scales as $t^{-\theta_s}$. The probability that a site has been persistent at odd as well as even times, i.e. it has been persistent under parallel dynamics on the lattice $A + B$ scales as $t^{-2\theta_s}$ giving $\theta_p = 2\theta_s$. A similar effect is observed in one-dimensional Ising model under the zero-temperature parallel Glauber dynamics [15].

We give a heuristic argument for the observed value of the persistent exponent θ_p that yields $\theta_p = 8/9$ in close agreement with the numerical value. A similar argument for the $1d$ Ising model yields the exact value of the exponent $\theta_p^{1d} = 3/4$ in that case. It is therefore possible that $\theta_p = 8/9$ may be the exact value of the exponent for the ladder (in the time regime indicated below, and with the provision for a crossover behaviour at later times), but we are not in a position to assert this on the basis of our heuristic argument. It is desirable to calculate the persistence exponent for the ladder using a method similar to the one used in the case of the $1d$ chain [16], but this is outside the scope of the present paper. We note that the persistence exponent in parallel dynamics is larger than the coarsening exponent. The typical size of a coarsening domain is of the order of $t^{1/2}$, and typical separation between two persistent sites is of the order of t^{θ_p} . If $\theta_p > 1/2$, persistent sites are separated by a distance much larger than a domain. Therefore, correlations between persistent sites may be neglected. In other words, the effect of different persistent sites on each other may be neglected and one can focus on the stability of a single persistent site under the random motion of domain walls in the system. Why is the separation between persistent sites much larger in parallel dynamics than it is in sequential dynamics? The reason lies in the bi-partite nature of the lattice. The two sub-lattices $\{s_i\}$ and $\{\sigma_i\}$ evolve similarly but independently of each other. Each sub-lattice develops ferromagnetically ordered domains due to coarsening. However, domains on the two sub-lattices may be aligned parallel or anti-parallel to each other. A region where the domains on the two sub-lattices overlap anti-parallel to each other contributes zero to persistence because spins in this region flip at every time step. This not only increases separation between persistent sites but modifies their spatial distribution as well. Persistent sites in sequential dynamics tend to cluster together but in parallel dynamics they are scattered more uniformly.

We use the idea of a backward random walk mentioned after equation (6) to relate the persistence probability at time $\tau + 1$ to persistence probability at an earlier time τ . The reason for using a new label τ in place of t is as follows. We regard a long random walk of t steps as a series of 3^τ short walks of τ steps each, and replace the average of a quantity over the long walk by the average of the same quantity over 3^τ realizations of the short walk of length τ .

After this averaging process, appropriate results for the long walk of t steps are obtained by the transformation $\tau = \log_3(t/\tau)$, or $\tau \approx \log_3 t$ for $t \gg \tau$. Let us focus on two nearest neighbour spins, say σ_i and s_i connected by a rung on the ladder. The spins σ_i and s_i lie on different sub-lattices. At any time τ , $\sigma_i(\tau)$ and $s_i(\tau)$ are independent of each other. Let $P(\tau)$ be the probability that $\sigma_i(\tau)$ and $s_i(\tau)$ are persistent up to τ under parallel dynamics. This implies that $\sigma_i(\tau)$ and $s_i(\tau)$ are aligned parallel to each other. As a result of coarsening, $\sigma_i(\tau)$ and $s_i(\tau)$ are likely to be inside a ferromagnetically ordered domain on their respective sub-lattices. If the ferromagnetic domains on the two chains are not aligned parallel to each other, for example if $\sigma_{i-1}(\tau-1) = \sigma_i(\tau-1) = \sigma_{i+1}(\tau-1) = -1$ and $s_{i-1}(\tau-1) = s_i(\tau-1) = s_{i+1}(\tau-1) = 1$, then we would get $s_i(\tau) = -1$ and $\sigma_i(\tau) = 1$ contrary to our assumption that $\sigma_i(\tau)$ and $s_i(\tau)$ have never flipped up to τ . Let us focus on a specific case, say $\sigma_i(\tau) = s_i(\tau) = 1$. We now ask for the probability that $\sigma_i(\tau+1) = s_i(\tau+1) = 1$ or equivalently the probability $P(\tau+1)$ that $\sigma_i(\tau+1)$ and $s_i(\tau+1)$ are persistent up to $\tau+1$ given that they are persistent up to τ . The value of $\sigma_i(\tau+1)$ is determined by $s_{i-1}(\tau)$, $s_i(\tau)$ and $s_{i+1}(\tau)$. If $s_i(\tau) = 1$ (with probability $1/2$), the probability that $\sigma_i(\tau+1) = 1$ is to be calculated over four configurations of the spins $s_{i-1}(\tau)$ and $s_{i+1}(\tau)$. In calculating the average over configurations, each configuration occurs with equal weight but contributes differently to $\sigma_i(\tau+1)$. Keeping in mind the rules of the voter dynamics and $s_i(\tau) = 1$, the configuration $\{s_{i-1}(\tau) = 1, s_{i+1}(\tau) = 1\}$ gives $\sigma_i(\tau+1) = 1$ with probability unity. The configurations $\{s_{i-1}(\tau) = 1, s_{i+1}(\tau) = -1\}$ and $\{s_{i-1}(\tau) = -1, s_{i+1}(\tau) = 1\}$ each give $\sigma_i(\tau+1) = 1$ with probability $2/3$. Finally, the configuration $\{s_{i-1}(\tau) = -1, s_{i+1}(\tau) = -1\}$ gives $\sigma_i(\tau+1) = 1$ with probability $1/3$. Thus, the probability to get $\sigma_i(\tau+1) = 1$ equals $(1 + 2/3 + 2/3 + 1/3)/4$ or $2/3$. Similarly, the probability to get $s_i(\tau+1) = 1$ is also equal to $2/3$. Putting various probabilities together, we get

$$P(\tau+1) = \frac{1}{4} \left(\frac{2}{3}\right)^2 P(\tau) \quad \text{or} \quad P(\tau+1) - P(\tau) = -\frac{8}{9}P(\tau). \quad (12)$$

Treating τ as a continuous variable in the limit $\tau \rightarrow \infty$ and making the identification $\tau = \log t$, we get

$$\frac{dP(\tau)}{d\tau} = -\frac{8}{9}P(\tau) \quad \text{or} \quad P(\tau) = \exp\left(-\frac{8}{9}\tau\right) \quad \text{or} \quad P(t) \approx t^{-8/9}. \quad (13)$$

A similar argument yields the exact value of the persistence exponent $\theta_p^{1d} = 3/4$ in the case of $1d$ Ising model evolving under voter or zero-temperature Glauber dynamics. As mentioned earlier, there are two simplifications in the $1d$ case. We have only a single chain of spins $s_i(\tau)$, and the backward random walk can only take a step to the left or to the right. Under parallel dynamics $s_i(\tau+1)$ is determined by $s_{i-1}(\tau)$ and $s_{i+1}(\tau)$. Therefore $s_i(\tau+1)$ is independent of $s_i(\tau)$. Let $P^{1d}(\tau)$ be the probability that a site is persistent up to τ . A persistent site could be in the state $s_i(\tau) = -1$ or $s_i(\tau) = 1$ with probability equal to $P^{1d}(\tau)/2$. Let us suppose $s_i(\tau) = 1$, and ask for the probability that $s_i(\tau+1) = 1$ as well. The configuration $\{s_{i-1}(\tau) = 1, s_{i+1}(\tau) = 1\}$ gives $s_i(\tau+1) = 1$ with probability unity. The configurations $\{s_{i-1}(\tau) = 1, s_{i+1}(\tau) = -1\}$ and $\{s_{i-1}(\tau) = -1, s_{i+1}(\tau) = 1\}$ each give $s_i(\tau+1) = 1$ with probability $1/2$. The configuration $\{s_{i-1}(\tau) = -1, s_{i+1}(\tau) = -1\}$ gives $s_i(\tau+1) = 1$ with probability zero. Therefore, the probability to get $s_i(\tau+1) = 1$ equals $(1 + 1/2 + 1/2 + 0)/4$ or $1/2$. We get

$$P^{1d}(\tau+1) = \frac{1}{4}P^{1d}(\tau) \quad \text{or} \quad P^{1d}(\tau+1) - P^{1d}(\tau) = -\frac{3}{4}P^{1d}(\tau) \quad \text{or} \quad P^{1d}(t) \approx t^{\theta_p^{1d}} \approx t^{-3/4}. \quad (14)$$

The value $\theta_p^{1d} = 3/4$ corresponds to the exact result for the persistence exponent $\theta_s^{1d} = 3/8$ for the one-dimensional Ising model under sequential dynamics [16]. However,

the derivation of the exact result is based on the limit $t \rightarrow \infty$ in a finite system with periodic boundary conditions. Our equation relating $P^{1d}(\tau)$ to $P^{1d}(\tau + 1)$ under parallel dynamics is intended over time scales $\log t \ll N$. The number of backward time steps τ should be large but not so large that backward walks from each persistent site end in overlapping regions in the initial state of the system. Thus the agreement of our result with the exact answer for Ising spins may be fortuitous but it is interesting in view of the difficulty of the exact calculation. We do not expect the same argument to reproduce the exact values of the persistence exponent for the one-dimensional q -state Potts model for $q > 2$ [16]. In this case the exact values of the exponents correspond to distances between persistent sites under parallel dynamics that are larger than the length of the chain.

4. Conclusion

Voter dynamics on an Ising ladder exhibits coarsening and persistence that is qualitatively similar to the one observed in the one-dimensional Ising model under the zero-temperature Glauber dynamics. In both cases, coarsening scales as $t^{1/2}$ in the limit $t \rightarrow \infty$, although the technical details are somewhat different. This behaviour is easily understood in view of the diffusive nature of coarsening. For a finite system of N spins, the longest time over which coarsening is observed scales as N^2 . However, in a typical numerical simulation of the system, a fully ordered state may be reached in a time much less than the upper limit mentioned above. The time taken by the dynamics to put the system into a completely ordered state and thus to come to a stage where it stops evolving depends significantly on the initial configuration of the system. It is not clear exactly what features of the initial random configuration are responsible for a broad spectrum of lifetimes of coarsening. Given an initial random configuration, we are not in a position to predict how long it would take the dynamics to reach a fixed point or a limit cycle. Numerical simulations show that coarsening when averaged over a sufficiently large number of initial random configurations of the system begins to deviate from the algebraic law $t^{1/2}$ several decades before reaching $t_{\text{end}} \approx N^2$.

Persistence also shows an algebraic power law $t^{-\theta}$, but when compared with coarsening in numerical simulations, the power law for persistence is observed over a smaller regime. Unlike the coarsening exponent, the persistent exponent θ depends on what type of dynamics is used, parallel or sequential. The exponent for the parallel dynamics θ_p is twice the exponent θ_s for sequential dynamics. The relationship $\theta_p = 2\theta_s$ is easily understood in view of the bi-partite nature of the ladder lattice. We have given a heuristic argument to suggest $\theta_p = 8/9$, a value that is close to the numerically observed value. Numerical simulations show considerable persistence in the system even after the exponent deviates from the value mentioned above. It is possible that in the limit $t \rightarrow \infty$, the persistent exponent crosses over to the corresponding exponent for the one-dimensional Ising model under zero-temperature Glauber dynamics. If this were so, it would mean that voter dynamics on a ladder is in the same universality class of non-equilibrium phenomena as the one-dimensional Ising model under Glauber dynamics. However, this issue is difficult to decide numerically because it is computationally intensive. It already takes several days to generate the data shown in figures 1 and 2. We expect that with increasing size of the ladder the exponents $\theta_p = 8/9$ and $\theta_s = 4/9$ will be observed over longer decades of time. We have suggested an explanation for the observed values of the exponents over time scale $\log t \ll N$. Numerical results leave open the possibility that at longer times the persistence exponent under voter dynamics on a ladder may cross over to the persistence exponent in the one-dimensional Ising model under zero-temperature Glauber dynamics.

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