A SURVEY OF SOME APPLICATIONS OF RATIONAL HOMOTOPY THEORY

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Angom Tiken Singh

PREFACE

This dissertation is an outcome of an effort to learn and understand the beautiful connection between geometric properties of a topological object and its algebraic properties contained in its homotopy and homology groups. The classical Chern-Hopf's conjecture, which states that nonnegatively curved manifolds have nonnegative Euler characteristics, and Gromov's conjecture, which states that the sum of Betti numbers of a compact nonnegatively curved m-manifold is $\leq 2^m$ ([8],[12]) are instances in elucidating this connection. More recently Halperin made a conjecture that any closed nonpositively curved manifold is elliptic, that is a simply connected cell complex all but finitely many of whose homotopy groups are finite. ([15]). This last conjecture generalize the previous two and was an outcome of insight gained by Halperin after working for two decades in the field of rational homotopy theory. Consequently the first step in reaching the goal to understand and investigate the connection between geometry and algebra was to learn rational homotopy theory.

Rational homotopy theory was brought to the notice of the mathematical community by the seminal work of Dennis Sullivan in 1970 ([30],[29]). He drew insight for this work from the book "Geometric Integration Theory" by Whitney. Sullivan's Rational homotopy theory [30] has provided the method to arrive at the above and many similar results. Sullivan has proved a polyhedral version of de Rham theorem which establishes an isomorphism between de Rham cohomology of PL-de Rham forms and ordinary cohomology with rational coefficients. Sullivan went on to find a minimal model for the de Rham complex, of a simply connected PL-manifold , which has the same cohomology. This minimal model is quite a computable object and it helps in extracting intricate rational homotopy information of spaces from their computable rational cohomology (see [14], [1], [4]).

It turns out that objects with rich geometric structures are formal, i.e. the rational homotopy of such objects are formal consequences of their rational cohomology (see [6]). Homogeneous spaces, symmetric spaces, Kähler manifolds are examples of such objects. For formal spaces one can establish equivalence of the rational Lusternik-Schnirelmann category with other variants like cone length & Toomer's invariant and get a lot of insight into such spaces.

The purpose of the dissertation is to first give a quick introduction to all the relevant concepts required to describe Sullivan's work and its apllications alluded to above, and finally to give a survey of the recent results on the application of rational homotopy theory and conclude to with some open problems.

We now give a chapterwise outline of the dissertation.

In Chapter 1 we give a quick introduction to simplicial complexes, simplicial homology and singular homology of spaces; we also define Betti numbers and Euler characteristics.

Chapter 2 introduces differentiable manifolds, differentiable maps, local coordinates, tangent vectors at a point, tangent space at a point, differential of a smooth map at a point, tangent and cotangent bundles, vector fields and differential 1-forms.

We continue to study higher differential forms and exterior derivative in chapter 3 which begins with a review of exterior algebras, and which ends with elements of vector analysis and some applications.

In chapter 4 we define de Rham cohomology of a smooth manifold; we study its homotopy invariance, particularly Poincaré's lemma, submanifolds and orientation of manifolds are introduced and a glimpse of the classification of compact orientable surfaces is given. The chapter concludes with the definition of simplicial cohomology.

Chapter 5 begins with smooth triangulation of smooth manifolds, it states and gives a sketch of a proof of de Rham's theorem for smooth manifolds. In the last section forms on simplicial complexes are introduced and a P.L. version of de Rham's theorem has been stated and a sketch of its proof given.

In chapter 6 we first give a quick introduction of Serre spectral sequence of an orientable fibration; next we give a brief introducion to obstruction theory, Eilenberg-MacLane spaces, principal $K(\pi, n)$-fibrations, Postnikov towers and conclude the chapter with rational homotopy theory for simply connected spaces.

In chapter 7 we introduce differential graded algebras, minimal models, their construction, Hirsch extension, homotopy theory and obstruction theory of D.G.A.'s. In the final section we demonstrate the parallel between homotopy theory of D.G.A. of forms on a simplicial complex and its rational homotopy theory, more specifically we draw parallel between Q-Postnikov towers and minimal D.G.A.'s of a simplicial complex.

In chapter 8 we give computation of minimal models and rational homotopy for selected examples.

In the final chapter 9 we define Kähler manifold, L-S category, cone length

and Toomer's invariant and give a survey of some recent works on the analysis of these as applications of rational homotopy theory. We conclude the dissertation with a few open questions.

Contents

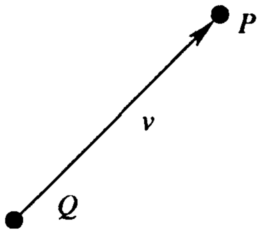
Chapter 1

Simplicial & singular homology

1.1 Simplicial complexes

1.1.1 Definition. An affine space of dimension n over $\mathbb{R}$ is a set E on which the additive group $\mathbb{R}^n$ operates simply transitively.

Thus for each pair of points P and Q in E there is a unique vector v in $\mathbb{R}^n$ from Q to P:



We write $v = P - Q$ and $P = Q + v$. However, the expression $P + Q$ is meaningless in this context.

Let t be a real number. We define $tP + (1-t)Q$ to be the unique point S such that $S - Q = t(P - Q)$ (this being a vector equation). If $P \neq Q$, the set of all such points for all $t \in \mathbb{R}$ is the line through P and Q (by definition). More generally, given points $P_0, \dots, P_r$ and real numbers $a_0, \dots, a_r$ such that $a_0 + \cdots + a_r = 1$, we can define the point

$$\sum_{i=0}^{r} a_i P_i$$

as the unique S such that

$$S - P_0 = \sum_{r=1}^{r} a_i(P_i - P_0).$$

If $P_0, \dots, P_r$ are *independent* (meaning the vectors $P_1 - P_0, \dots, P_r - P_0$ are linearly independent), the set of all such S is an affine space of dimension r called the span of $P_0, \dots, P_r$. Each point S in the span has a unique set of coordinates $(a_0, \dots, a_r)$ called its *barycentric coordinates* relative to $P_0, \dots P_r$. These coordinates are arbitrary except for the equation $a_0 + \cdots + a_r = 1$.

1.1.2 Definition. A function from one affine space E to another E' will be called an *affine map* if

$$f(tP + (1-t)Q) = tf(P) + (1-t)f(Q)$$

for all points P, Q and real numbers t.

If $P_0, \dots, P_n$ are independent points which span E, the affine map is uniquely determined by its effect on these points, since

$$f(\sum_{0}^{n} a_i P_i) = \sum_{0}^{n} a_i f(P_i) \quad \text{where} \quad \sum_{0}^{n} a_i = 1$$

We now specialize to affine concepts in a vector space.

1.1.3 Definition. Let V be a vector space over $\mathbb{R}$ and let C be a subset of V. C is *convex* if

$$c_1, c_2 \in C \Rightarrow tc_1 + (1-t)c_2 \in C$$

for all $t \in I = [0,1]$.

1.1.4 Definition. A set $\{v_0, v_1, \ldots, v_k\}$ of vectors in a vector space V is *convex -independent*, or *c-independent* or *affinely independent*, if the set

$$\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$$

is linearly independent. Note that this definition does not depend on which vector is called v_0.

1.1.5 Example. In $\mathbb{R}^2$, $\{v_0, v_1, v_2\}$ is c-independent if and only if v_0, v_1, and v_2 are not collinear.

1.1.6 Theorem. *Suppose* $\{v_0, v_1, \ldots, v_k\}$ *is a c-independent set. Let* C *be the convex set generated by* $\{v_0, v_1, \ldots, v_k\}$*; that is,* C *is the smallest convex set containing* $\{v_0, v_1, \ldots, v_k\}$ *(*C *is also called the convex hull of* $\{v_0, v_1, \ldots, v_k\}$*). Then* C *consists of all vectors of the form* $\sum_{i=0}^{k} a_i v_i$*, where* $a_i \geq 0$ *for all* i *and* $\sum_{i=0}^{k} a_i = 1$*. Furthermore, each* $v \in C$ *is uniquely expressible in this form. (see [27] for a proof)*

1.1.7 Definitions. Let V be a vector space over $\mathbb{R}$. A convex set generated by c-independent vectors $\{v_0, v_1, \ldots, v_k\}$ is called a (closed) *k-simplex* and is denoted by $[v_0, v_1, \ldots, v_k]$. k is called the *dimension* of the simplex. If $v \in [v_0, v_1, \ldots, v_k]$, the coefficients a_i, with $a_i \geq 0$ and $\sum_{i=0}^{k} a_i = 1$ such that $v = \sum_{i=0}^{k} a_i v_i$, are called the *barycentric coordinates* of v. We sometimes denote by $a_i(v)$ the i^{th} barycentric coordinate of v.

1.1.8 Examples. For vectors $\{v_0, v_1\}$ in $\mathbb{R}$, the simplex $[v_0, v_1]$ is the closed interval $[v_0, v_1]$. For $\{v_0, v_1, v_2\} \subset \mathbb{R}^2, [v_0, v_1, v_2]$ is the triangle (including its interior) with vertices v_0, v_1 and v_2. The centroid of this triangle is the point with barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. For $V = \mathbb{R}^n$, the simplex $[v_0, v_1, \ldots, v_n]$ is a compact metric space (it is closed and bounded) in the relative topology.

1.1.9 Definitions. Let $\{v_0, v_1, \ldots, v_k\}$ be a c-independent set. The set $\{v \in [v_0, v_1, \ldots, v_k];\ a_i(v) > 0,\ i = 0, 1, \ldots, k\}$ is called an *open simplex* and is denoted by $(v_0, v_1, \ldots, v_k)$. If s is a c-independent set we shall denote the open simplex by (s) and the corresponding closed simplex by $[s]$.

Let $[s] = [v_0, v_1, \ldots, v_k]$ be a closed simplex. The *vertices* of $[s]$ are the points $v_0, v_1, \ldots, v_k$. The *closed faces* of $[s]$ are the closed simplices $[v_{j_0}, v_{j_1}, \ldots, v_{j_h}]$ where $\{j_0, j_1, \ldots, j_h\}$ is a nonempty subset of $\{0, 1, \ldots, k\}$. The open faces of the simplex $[s]$ are the *open simplices* $(v_{j_0}, v_{j_1}, \ldots, v_{j_h})$.

1.1.10 Remarks.

(1) A vertex is a 0-dimensional closed face. It is also an open face.

(2) An open simplex (s) is an open set in the closed simplex $[s]$. Its closure is $[s]$.

(3) The closed simplex $[s]$ is the union of its open faces.

(4) Distinct open faces of a simplex are disjoint.

(5) The open simplex (s) is the *interior* of the closed simplex $[s]$; that is, it is the closed simplex minus its proper open faces (faces$\neq (s)$).

1.1.11 Definition. A *simplicial complex* K (Euclidean) is a finite set of open simplices in some $\mathbb{R}^n$ such that

(1) if $(s) \in K$, then all open faces of $[s] \in K$;

(2) if $(s_1), (s_2) \in K$ and $(s_1) \cap (s_2) \neq \emptyset$, then $(s_1) = (s_2)$.

The *dimension* of K is the maximum dimension of the simplices of K.

1.1.12 Remarks. If K is a simplicial complex, let $|K|$ denote the point set union of the open simplices of K. Then $|K|$ is compact, and

$$|K| = \cup_{(s) \in K}(s) = \cup_{(s) \in K}[s].$$

If $[s]$ is a closed simplex, the collections of its open faces is a simplicial complex which we denote by s.

1.1.13 Definition. A *subcomplex* of a simplicial complex K is a simplicial complex L such that $(s) \in L$ implies $(s) \in K$.

1.1.14 Remark. For each $(s) \in K$, the simplicial complex s is a subcomplex of K.

1.1.15 Definition. Let K be a simplicial complex or more simply a complex. Let r be an integer less than or equal to dim K. The *r-skeleton* K^r of K is the collection $K^r = \{(s) \in K \text{—dim } s \leq r\}$.

1.1.16 Remark. The *r-skeleton* K^r is a subcomplex of K.

1.2 Simplicial homology

1.2.1 Definition. Let s be an l- simplex, with vertices $v_0, v_1, \ldots, v_l$. Two orderings $(v_{j_0}, v_{j_1}, \ldots, v_{j_l})$ and $(v_{k_0}, v_{k_1}, \ldots, v_{k_l})$ of the vertices of s are *equivalent* if $(k_0, \ldots, k_l)$ is an even permutation of $(j_0, \ldots, j_l)$. This is clearly an equivalence relation, and for $l > 0$, it partitions the orderings of $v_0, \ldots, v_l$ into two equivalence classes. Each of these classes is called an *orientation* of s. If s is 0-simplex, then there is only one class and hence only one orientation of s. An *oriented simplex* is a simplex s together with a choice of one of these equivalence classes. If $v_0, v_1, \ldots, v_l$ are the vertices of s, the oriented simplex determined by the ordering $(v_0, \ldots, v_l)$ will be denoted by $\langle v_0, v_1, \ldots, v_l \rangle$.

1.2.2 Remark. Note that an oriented 1-simplex has a sense of direction attached to it, an oriented 2-simplex has a sense of rotation attached to it and so on. In fact, each l-simplex s lies in l-dimensional plane in some $\mathbb{R}^m$. Orienting s by $\langle v_0, v_1, \ldots, v_l \rangle$ is the same as orienting the l-plane containing s by means of the ordered basis $\{v_1 - v_0, v_2 - v_0, \ldots, v_l - v_0\}$.

1.2.3 Definition. Let K be a simplicial complex, and $\mathbb{Z}$ denote the group of integers. Let $C_l(K, \mathbb{Z})$ denote the factor group of the free abelian group generated by all oriented simplices of K, modulo the subgroup generated by all elements of the form $\langle v_0, v_1, v_2, \ldots, v_l \rangle + \langle v_1, v_0, v_2, \ldots, v_l \rangle$. Thus $C_l(K, \mathbb{Z})$ is an abelian group called the *group of l- chains* of K with integer coefficients. A typical element of this group is of the form

$$\sum_{\text{s an } l-\text{simplex}} n_s \langle s \rangle \quad (n_s \in \mathbb{Z}),$$

where, for each l-simplex s, $\langle s \rangle$ is some fixed orientation of s, and where s with the opposite orientation is identified with $-\langle s \rangle$.

1.2.4 Remark. Given an arbitrary abelian group G, the group $C_l(K, G)$ of l-chains of K with coefficients in G can be defined as the set of all formal linear combinations

$$\sum_s g_s \langle s \rangle \qquad (g_s \in G)$$

subject to the identifications $-g_s\langle v_0, v_1, \ldots, v_l\rangle = g_s\langle v_1, v_0, \ldots, v_l\rangle$. (We are writing the group operation in G additively.) In particular, $C_l(K,F)$ is defined for any field F, in which case $C_l(K,F)$ is a vector space over F whose dimension equals the number of l-simplices of K.

1.2.5 Definition. Let $\langle s\rangle = \langle v_0, v_1, \ldots, v_{l+1}\rangle$ be an oriented $(l+1)$-simplex. The *boundary* $\partial\langle s\rangle$ of $\langle s\rangle$ is the l-chain defined by

$$\partial\langle s\rangle = \sum_{j=0}^{l+1}(-1)^j\langle v_0, v_1, \ldots, \widehat{v}_j, \ldots, v_{l+1}\rangle,$$

where $\widehat{\ }$ over a symbol means that the symbol is deleted.

1.2.6 Remark. Note that $\partial\langle s\rangle$ is well defined and that $\bigcup_{j=0}^{l+1}[v_0, v_1, \ldots, \widehat{v}_j, \ldots, v_{l+1}]$, the union of the faces occurring in $\partial\langle s\rangle$, is the topological boundary of $[s]$.

1.2.7 Examples. (1) $\partial\langle v_0, v_1\rangle = \langle v_1\rangle - \langle v_0\rangle$.
(2) $\partial\langle v_0, v_1, v_2\rangle = \langle v_1, v_2\rangle - \langle v_0, v_2\rangle + \langle v_0, v_1\rangle = \langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle$.

1.2.8 Definition. Let K be a simplicial complex, and let G be an abelian group. The *boundary map*

$$C_{l+1}(K,G) \xrightarrow{\partial} C_l(K,G)$$

is the group homomorphism defined by

$$\partial\left(\sum g_s\langle s\rangle\right) = \sum g_s\partial\langle s\rangle.$$

1.2.9 Lemma. *The maps*

$$C_{l+1}(K,G) \xrightarrow{\partial} C_l(K,G) \xrightarrow{\partial} C_{l-1}(K.G)$$

satisfy $\partial\partial$*(where* $\partial\partial = \partial\circ\partial$*)* $= 0$

Proof. Since $\partial\partial$ is linear, it suffices to check this on generators

$$\langle v_0, v_1, \dots, v_{l+1}\rangle$$

as follows:

$$\begin{aligned}
\partial(\partial\langle v_0,\dots,v_{l+1}\rangle) &= \partial\left[\sum_{j=0}^{l+1}(-1)^j\langle v_0,\dots,\widehat{v}_j,\dots,v_{l+1}\rangle\right] \\
&= \sum_{j=0}^{l+1}(-1)^j\partial\langle v_0,\dots,\widehat{v}_j,\dots,v_{l+1}\rangle \\
&= \sum_{j=0}^{l+1}(-1)^j\left[\sum_{i=0}^{j-1}(-1)^i\langle v_0,\dots,\widehat{v}_i,\dots,\widehat{v}_j,\dots v_{l+1}\rangle\right. \\
&\qquad \left.+\sum_{i=j+1}^{l+1}(-1)^{i-1}\langle v_0,\dots,\widehat{v}_j,\dots,\widehat{v}_i,\dots,v_{l+1}\rangle\right] \\
&= \sum_{i<j}(-1)^{i+j}\langle v_0,\dots,\widehat{v}_i,\dots,\widehat{v}_j,\dots,v_{l+1}\rangle \\
&\qquad +\sum_{i>j}(-1)^{i+j-1}\langle v_0,\dots,\widehat{v}_j,\dots,\widehat{v}_i,\dots,v_{l+1}\rangle \\
&= \sum_{i<j}[(-1)^{i+j}+(-1)^{i+j-1}]\langle v_0,\dots,\widehat{v}_i,\dots,\widehat{v}_j,\dots,v_{l+1}\rangle \\
&= 0.
\end{aligned}$$

□

1.2.10 Definition. Given K and G, let

$$\begin{aligned}
Z_l(K,G) &= \{c \in C_l(K,G) | \partial c = 0\}, \\
B_l(K,G) &= \{\partial c | c \in C_{l+1}(K,G)\}, \\
H_l(K,G) &= Z_l(K,G)/B_l(K,G).
\end{aligned}$$

Elements of $Z_l(K,G)$ are called *cycles*, and of $B_l(K,G)$ are called *boundaries*. The group $H_l(K,G)$ is called the lth ***homology group*** of K with coefficients in G.

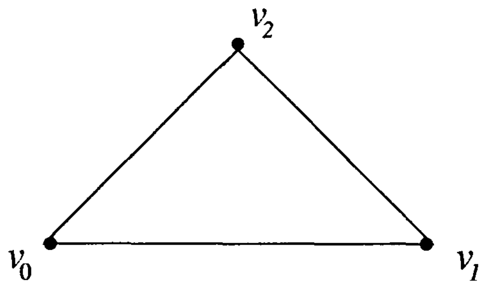
1.2.11 Remark. It turns out that the groups $H_l(K, G)$ depend only on the topology of $|K|$. If $f : |K| \to [L]$ is a homeomorphism, then there is induced an isomorphism

$$f_* : H_l(K, G) \to H_l(K, G).$$

In particular, if K_1 and K_2 are simplicial complexes with $|K_1| = |K_2|$, then they have the same homology groups.

1.2.12 Example. Let K be the 1-skeleton of a 2-simplex; so consists of three vertices v_0, v_1, v_2 and three 1-simplices $(v_0, v_1), (v_1, v_2)$ and (v_2, v_0). Then both $C_0(K, \mathbb{Z})$ and $C_1(K, \mathbb{Z})$ are isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. $C_l(K, \mathbb{Z}) = 0$ for $l > 1$. A typical element c_1 of $C_1(K, \mathbb{Z})$ is of the form

$$c_1 = m_1\langle v_0, v_1\rangle + m_2\langle v_1, v_2\rangle + m_3\langle v_2, v_0\rangle \qquad (m_1, m_2, m_3 \in \mathbb{Z})$$



Its boundary ∂c_1 is given by

$$\begin{aligned} \partial c_1 &= m_1(\langle v\rangle_1 - \langle v\rangle_0) + m_2(\langle v\rangle_2 - \langle v\rangle_1) + m_3(\langle v\rangle_0 - \langle v\rangle_2) \\ &= (m_3 - m_1)\langle v\rangle_0 + (m_1 - m_2)\langle v\rangle_1 + (m_2 - m_3)\langle v\rangle_2. \end{aligned}$$

Thus $c_1 \in Z_1(K, \mathbb{Z})$ if and only if

$$m_3 - m_1 = 0, \quad m_1 - m_2 = 0, \quad m_2 - m_3 = 0$$

that is if and only if $m_1 = m_2 = m_3$, so

$$Z_1(K,\mathbb{Z}) = \{n(\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle) | n \in \mathbb{Z}\} \cong \mathbb{Z}$$

Furthermore, $B_1(K,\mathbb{Z}) = 0$ because $C_2(K,\mathbb{Z}) = 0$. Hence

$$H_1(K,\mathbb{Z}) = Z_1(K,\mathbb{Z})/B_1(K,\mathbb{Z}) \cong \mathbb{Z}.$$

To compute $H_0(K,\mathbb{Z})$, note that a typical cycle $c_0 \in Z_0(K,\mathbb{Z})$ is of the form

$$c_0 = n_1\langle v\rangle_0 + n_2\langle v\rangle_1 + n_3\langle v\rangle_2 \quad (n_1, n_2, n_3 \in \mathbb{Z}).$$

Then $c_0 = \partial c_1$ for some

$$c_1 = m_1\langle v_0, v_1\rangle + m_2\langle v_1, v_2\rangle + m_3\langle v_2, v_0\rangle \in C_1(K,\mathbb{Z})$$

if and only if there exist (integer) solutions to the equations

$$\begin{aligned} m_3 - m_1 &= n_1 \\ m_1 - m_2 &= n_2 \\ m_2 - m_3 &= n_3 \end{aligned}$$

It is easy to check that such a solution exists if and only if $n_1 + n_2 + n_3 = 0$. Thus

$$B_0(K,\mathbb{Z}) = \{n_1\langle v\rangle_0 + n_2\langle v\rangle_1 + n_3\langle v\rangle_2 | n_1 + n_2 + n_3 = 0\}.$$

Let $\varphi : Z_0(K,\mathbb{Z}) \to \mathbb{Z}$ be the homomorphism defined by

$$\varphi(n_1\langle v\rangle_0 + n_2\langle v\rangle_1 + n_3\langle v\rangle_2) = n_1 + n_2 + n_3.$$

Then the kernel of φ is just $B_0(K,\mathbb{Z})$; thus

$$H_0(K,\mathbb{Z}) = Z_0(K,\mathbb{Z})/B_0(K,\mathbb{Z}) \cong \mathbb{Z}.$$

1.2.13 Example. Let K be the complex consisting of all the faces of a 2-simplex (v_0, v_1, v_2). Then as in Example (1.2.12),

$$H_0(K, \mathbb{Z}) \cong \mathbb{Z}.$$

Moreover, as before,

$$Z_1(K, \mathbb{Z}) = \{n(\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle) | n \in \mathbb{Z}\}.$$

Now, however,

$$C_2(K, \mathbb{Z}) = \{n\langle v_0, v_1, v_3\rangle | n \in \mathbb{Z}\},$$

so that

$$\begin{aligned} B_1(K, \mathbb{Z}) &= \{\partial(n\langle v_0, v_1, v_2\rangle) | n \in \mathbb{Z}\} \\ &= \{n(\langle v_1, v_2\rangle - \langle v_0, v_2\rangle + \langle v_0, v_1\rangle) | n \in \mathbb{Z}\} \\ &= \{n(\langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_0\rangle) | n \in \mathbb{Z}] \\ &= Z_1(K, \mathbb{Z}). \end{aligned}$$

Hence

$$H_1(K, \mathbb{Z}) = \mathbb{Z}_1(K, \mathbb{Z})/B_1(K, \mathbb{Z}) = 0.$$

Finally, since $\partial(n\langle v_0, v_1, v_2\rangle) = 0$ if and only if $n = 0$, $Z_2(K, \mathbb{Z}) = 0$, and hence

$$H_2(K, \mathbb{Z}) = 0.$$

1.3 Betti numbers and Euler characteristics

1.3.1 Definitions. Let K be a simplicial complex. The lth *Betti number* of K is the integer

$$\beta_l = \dim H_l(K, \mathbb{R}).$$

The *Euler characteristic* $\chi(K)$ of K is the integer

$$\chi(K) = \sum_{l=0}^{\dim K} (-1)^l \beta_l.$$

1.3.2 Theorem. *Let K be a simplicial complex. For each l with $0 \leq l \leq$ dim K, let α_l denote the number of l-simplices in K. Then*

$$\chi(K) = \sum_{l=0}^{\dim K} (-1)^l \alpha_l;$$

that is, $\chi(K)$ is equal to the number of vertices $-$ the number of edges $+$ the number of 2-faces $- \dots$

Proof. For each l, $0 \leq l \leq \dim K$, consider the linear map

$$C_{l-1}(K, \mathbb{R}) \xleftarrow{\partial} C_l(K, \mathbb{R}),$$

where C_{-1} is by definition the zero space. Then, by the rank and nullity theorem of linear algebra,

$$\begin{aligned} \alpha_l &= \dim C_l(K, \mathbb{R}) = \dim \ker \partial + \dim \operatorname{Im} \partial \\ &= \dim Z_l(K, \mathbb{R}) + \dim B_{l-1}(K, \mathbb{R}) \qquad (l = 0, 1, \dots, \dim K). \end{aligned}$$

Moreover,

$$\begin{aligned} \beta_l &= \dim H_l(K, \mathbb{R}) \\ &= \dim [Z_l(K, \mathbb{R}) / B_L(K, \mathbb{R})] \\ &= \dim Z_l(K, \mathbb{R}) - \dim B_l(K, \mathbb{R}). \end{aligned}$$

Thus

$$\begin{aligned}
\chi(K) &= \sum_{l=0}^{\dim K} (-)^l \beta_l \\
&= \sum_{l=0}^{\dim K} (-)^l [\dim Z_l(K,\mathbb{R}) - \dim B_l(K,\mathbb{R})] \\
&= \sum_{l=0}^{\dim K} (-)^l \dim Z_l(K,\mathbb{R}) + \sum_{l=0}^{\dim K} (-1)^{l+1} \dim B_l(K,\mathbb{R}) \\
&= \sum_{l=0}^{\dim K} (-)^l \dim Z_l(K,\mathbb{R}) + \sum_{l=1}^{\dim K} (-1)^l \dim B_{l-1}(K,\mathbb{R}) \\
&\qquad \text{(Since } \dim B_l = 0 \text{ for } l = \dim K) \\
&= \sum_{l=0}^{\dim K} (-)^l [\dim Z_l(K,\mathbb{R}) + \dim B_{l-1}(K,\mathbb{R})] \\
&\qquad \text{(since } \dim B_{-1} = 0) \\
&= \sum_{l=0}^{\dim K} (-)^l \alpha_l.
\end{aligned}$$

□

(For more information on simplicial homology see [27])

1.4 Singular homology

We take a countably infinite product $\mathbb{R}^\infty$ of copies of $\mathbb{R}$, and consider the vectors

$$\begin{aligned}
E_0 &= (0,0,\ldots,0,\ldots), \\
E_1 &= (1,0,\ldots,0,\ldots), \\
E_2 &= (0,1,\ldots,0,\ldots), \text{ etc.}
\end{aligned}$$

We identify $\mathbb{R}^n$ with the subspace having all components after the n-th equal to 0. We let, for any $q \geq 0$, Δ^q denote the q-th dimensional geometric simplex spanned by $E_0, E_1, \ldots, E_q$ (defined to be the convex hull of $E_0, E_1, \ldots, E_q$) called *the standard (geometric) q-simplex.* Thus Δ^0 is a point, Δ^1 the unit interval, Δ^2 a triangle (including its interior), Δ^3 a tetrahedron, etc.

If $P_0, \ldots, P_q$ are points in some affine space E, $(P_0, \ldots, P_q)$ will denote the restriction to Δ^q of the unique affine map $\mathbb{R}^q \to E$ taking E_0 into P_0,. . . ,E_q into P_q. Thus $(E_0, \ldots, E_q)$ is the identity map of Δ^q which will be denoted by δ_q.

1.4.1 Definition. Let X be a topological space. By a *singular q-simplex* in X, we mean a continuous map $\Delta^q \to X$.

1.4.2 Definitions. We define $S_q(X)$ to be the free R-module generated by all the singular q-simplexes, where R is a commutative unitary ring.

The elements of $S_q(X)$ are formal linear combinations $\sum_\sigma \nu_\sigma \sigma$ where σ runs through singular q-simplexes, and the coefficients ν_σ are from R. These sums are called *singular q-chains.*

1.4.3 Definition. For $q > 0$, we define $F_q^i : \Delta^{q-1} \to \Delta^q$, for $0 \leq i \leq q$, to be the *affine map* $(E_0 ... \widehat{E_i} ... E_q)$ where $\widehat{E_i}$ means "omit E_i".

1.4.4 Definition. For an arbitrary singular q-simplex σ in a space X we define *the i-th face* $\sigma^{(i)}$ of σ to be the singular $(q-1)$-simplex $\sigma \circ F_q^i$.

1.4.5 Definition. We define the *boundary* of a singular q-simplex σ to be the singular $(q-1)$-chain $\partial(\sigma) = \sum_{i=0}^q (-1)^i \sigma^{(i)}$.

1.4.6 Proposition. $\partial\partial = 0$. (proof is as given earlier, or see[13])

1.4.7 Definition. A singular q-chain c such that $\partial(c) = 0$ is called a *cycle*; if $c = \partial(c')$ for some $(q+1)$-chain c', c is called a *boundary*.Two q-chains which differ by a boundary are called *homologous*, written $c_1 \sim c_2$.

1.4.8 Definition. Let B_q and Z_q be the submodules of $S_q(X)$ of boundaries and cycles respectively. Then the quotient module Z_q/B_q is called *the* q-th *singular homology module* of X, denoted $H_q(X; R)$ or simply$H_q(X)$ when the reference to R is understood.

1.4.9 Example. $X = \{x\}$ be a singleton set. There is a unique singular q-simplex σ_q for each q (constant map on x). We have

$$\partial(\sigma_q) = \begin{cases} \sigma_{q-1} & q \text{ even} > 0, \\ 0 & q \text{ odd}, \end{cases}$$

$$Z_q = B_q = \begin{cases} 0 & q \text{ even} > 0, \\ S_q & q \text{ odd}, \end{cases}$$

so that $H_q = 0$ for all $q > 0$. However $Z_0 = S_0$ while $B_0 = 0$, so that $H_0 \cong R$, the isomorphism being $\nu\sigma_q \to \nu$.

(For more information about singular homology see [13])

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Chapter 2

Differentiable manifolds, vector fields and 1-forms

(The materials of this chapter has been adapted from [27])

2.1 Differentiable manifolds; definitions and examples

2.1.1 Definition. A *locally Euclidean space* of dimension n is a Hausdorff topological space X such that, for each $x \in X$, there exists a homeomorphism φ_x mapping some open set containing x onto an open set in $\mathbb{R}^n$.

2.1.2 Remark. We may, if we wish, choose each φ_x so that $\varphi_x(x) = 0$ and the image of φ_x is a ball $B_0(\varepsilon)$. We use the notation $B_x(\varepsilon)$ to denote the ball with center x and radius ε. (*Proof.* Given any φ_x homeomorphically mapping an open set U about x onto an open set in $\mathbb{R}^n$, Let $\varepsilon > 0$ be such that $B_{\varphi(x)}(\varepsilon) \subset \varphi_x(U)$. Let

$$\psi : B_{\varphi(x)}(\varepsilon) \to B_0(\varepsilon)$$

be translation by $-\varphi(x)$. Then

$$\widetilde{\varphi_x} = \psi \circ \varphi_x|_{\varphi_x^{-1}(B_{\varphi(x)}(\varepsilon))}$$

maps $\varphi_x^{-1}(B_{\varphi(x)}(\varepsilon))$ homeomorphically onto $B_0(\varepsilon)$.)

2.1.3 Example. $\mathbb{R}^n$ is locally Euclidean. For each $x \in \mathbb{R}^n$, take φ_x to be the identity map.

2.1.4 Example. $S^n = \{x \in \mathbb{R}^{n+1} | \ \|x\| = 1\}$ is locally Euclidean. Given $x \in S^n$, let $y \in S^n, y \neq x$. Then φ_x= stereographic projection from y maps $S^n - \{y\}$ homeomorphically onto $\mathbb{R}^n$.

2.1.5 Example. The Projective space P^n; (that is, the identification space of S^n obtained by identifying antipodal points of S^n) is locally Euclidean. For, since P^n is double covered by S^n, each $x \in P^n$ is contained in an open set homeomorphic to an open set in S^n that itself contains, about each of its points, an open set homeomorphic to an open set in $\mathbb{R}^n$.

2.1.6 Example. Each open subset U of a locally Euclidean space X is locally Euclidean. For if $x \in U$, let ψ_x be a homeomorphism mapping an open set about x in X onto an open set in $\mathbb{R}^n$. Take $\varphi_x = \psi_x|_{U \cap \text{domain } \psi_x}$.

2.1.7 Example. The set of all nonsingular $k \times k$ matrices forms a locally Euclidean space of dimension k^2. Each $k \times k$ matrix may be identified with a k^2-tuple by stringing out the rows in a line. The nonsingular matrices then form an open set of $\mathbb{R}^{k^2}$, namely $\det^{-1}(\mathbb{R} - \{0\})$ where $\det : \mathbb{R}^{k^2} \to \mathbb{R}$ is the determinant function.

Let the letter k stand for $0, 1, 2, \ldots, \infty, \omega$. Then in what follows C^0 means continuous; C^k for k finite means all partial derivatives of order less than or equal to k exist and are continuous; C^∞ means all partial derivatives of all orders exist and are continuous; C^ω means real analytic, that is, the function may be expressed as a convergent Taylor series in a neighborhood of each point.

2.1.8 Definition. A C^k*-differentiable manifold*(or C^k*-manifold*) of dimension n is a pair (X, Φ) where X is a Hausdorff topological space, and Φ is a collection of maps such that the following conditions hold.

(1) $\{\text{domain } \varphi\}_{\varphi \in \Phi}$ is an open covering of X,

(2) each $\varphi \in \Phi$ maps its domain homeomorphically onto an open set in $\mathbb{R}^n$,

(3) for each $\varphi, \psi \in \Phi$ with (domain φ) $\cap$(domain ψ) $\neq \emptyset$, the map $\psi \circ \varphi^{-1}$ is a C^k-map from φ(domain $\varphi \cap$ domain ψ) $\subset \mathbb{R}^n$ into $\mathbb{R}^n$,

(4) Φ is maximal relative to (2) and (3): that is, if ψ is any homeomorphism mapping an open set in X onto an open set in $\mathbb{R}^n$ such that, for each $\varphi \in \Phi$ with domain $\varphi \cap$ domain $\psi \neq \emptyset$, $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are C^k-maps from ϕ(domain $\varphi \cap$ domain ψ) and ψ(domain $\varphi \cap$ domain ψ) into $\mathbb{R}^n$, then $\psi \in \Phi$.

Note that every C^k-manifold is a locally Euclidean space and a locally Euclidean space gives rise to a C^0-manifold.

If $n = 2$ and, in condition (3), "C^k" is replaced by "complex analytic" (where $\mathbb{R}^2$ is identified with the complex numbers $\mathbb{C}^1$), (X, Φ) is called a *complex analytic manifold* of complex dimension 1 or a *Riemann surface.* Φ is then called a *complex structure* or *conformal structure* on X.

The maps $\varphi \in \Phi$ are called coordinate systems. More precisely, the map $\varphi \in \Phi$ is called a *coordinate system on the open set* (domain φ)$\subset X$. For

$x \in X$, a ***coordinate system about*** x is a coordinate system $\varphi \in \Phi$ such that $x \in \text{domain } \varphi$.

2.1.9 Remark. Each of the above Examples 2.1.3, 2.1.4, 2.1.5 and 2.1.7 of locally Euclidean spaces form the underlying space of a C^∞-manifold. Example 2.1.6 form underlying space of a C^k-manifold. Namely, if (X, Φ) is a C^k-manifold and U is an open set in X, then $(U, \Phi|_U)$ is a C^k-manifold, where $\Phi|_U = \{\varphi|_U\}_{\varphi \in \Phi}$.

2.2 Differentiable maps

2.2.1 Definitions. Let (X, Φ) be a C^k-manifold. A real-valued function $f : X \to \mathbb{R}$ is a *C^s-function* ($s \leq k$), denoted $f \in C^s(X, \mathbb{R})$, if, for each $\varphi \in \Phi$, $f \circ \varphi^{-1}$ is a C^s-function mapping the image of $\varphi \subset \mathbb{R}^n$ into $\mathbb{R}$.

Let (X, Φ) be a C^k-manifold, and $x \in X$. A real-valued function f is said to be *of class C^s ($s \leq k$) in a neighborhood of x*, denoted $f \in \mathbf{C}^s(X, x, \mathbb{R})$, if

$$U = (\text{domain } f)$$

is an open set in X containing x, and $f \in C^s(U, \mathbb{R})$, where U has the C^k-manifold structure as an open set in X.

2.2.2 Remarks. Note that we are able to define C^s-functions on X because (1) X looks locally (via the coordinate systems $\varphi \in \Phi$) like $\mathbb{R}^n$, and we know what it means for a function on $\mathbb{R}^n$ to be C^s; and (2) if $U = \text{domain } \varphi$ and $V = \text{domain } \psi$ for $\varphi, \psi \in \Phi$, with $U \cap V \neq \emptyset$, the concept of a C^s-function in a neighborhood of x in $U \cap V$ is the same relative to the coordinate system φ

as to the coordinate system ψ, because $\psi \circ \varphi^{-1}$ is a C^k-deffeomorphism and $k \geq s$

Note also that if f and g are C^s-functions in a neighborhood of x, then $f+g$ and fg (product) are C^s-functions in a neighborhood of x, where

$$\text{domain } (f+g) = \text{domain(fg)} = \text{(domain f)} \cap \text{(domain g)}.$$

2.2.3 Definition. Let (X, Φ) be a C^k-manifold, and let $\varphi \in \Phi$ be a coordinate system on $U = \text{domain } \varphi$. Let $r_j : \mathbb{R}^n \to \mathbb{R}$ be the jth coordinate function on $\mathbb{R}^n$; that is, $r_j(a_1, a_2, \ldots, a_n) = a_j$ for $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. The jth *coordinate function* of the coordinate system φ is the function $x_j : U \to \mathbb{R}$ defined by $x_j = r_j \circ \varphi$.

2.2.4 Remark. $x_j : U \to \mathbb{R}$ is a C^k-function. The n-tuple of functions $(x_1, x_2, \ldots, x_n)$ is sometimes also referred to as a coordinate system.

2.2.5 Definition. Let (X_1, Φ_1) and (X_2, Φ_2) be C^k-manifolds (not necessarily of the same dimension). A mapping $\Psi : X_1 \to X_2$ is of class C^s $(s \leq k)$, denoted $\Psi \in C^s(X_1, X_2)$, if, whenever $f \in C^s(X_2, \mathbb{R})$, the $f \circ \Psi \in C^s(X_1, \mathbb{R})$.

2.2.6 Remarks. We shall confine our attention to C^∞-manifolds. This will include, in particular, C^ω-manifolds and complex analytic manifolds of dimension 1. We shall use the word "smooth" to denote C^∞.

2.3 Tangent vectors & tangent space at a point

2.3.1 Definition. (X, Φ) be a smooth manifold and $x \in X$. A *tangent vector* at x is a map $v : C^\infty(X, x, \mathbb{R}) \to \mathbb{R}$ such that, if φ is a (fixed) coordinate system with $x \in U =$ domain of φ, then there exists an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers with the following property. For each $f \in C^\infty(X, x, \mathbb{R})$,

$$v(f) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi(x)}.$$

(Note that if $W =$ domain f, then φ and f are both defined on the open set $U \cap W$ containing x, so that $f \circ \varphi^{-1}$ is a smooth function with domain $\varphi(U \cap W) \subset \mathbb{R}^n$ containing $\varphi(x)$)

2.3.2 Remark. If $v : C^\infty(X, x, \mathbb{R}) \to \mathbb{R}$ has the property required above of a tangent vector with respect to one coordinate system φ about x, then it also has this property with respect to any other coordinate system about x. For, if ψ is another such coordinate system, then, using the chain rule,

$$\begin{aligned} v(f) &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi(x)} \\ &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i}(f \circ \psi^{-1} \circ \psi \circ \varphi^{-1})|_{\varphi(x)} \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \frac{\partial}{\partial r_j}(f \circ \psi^{-1})|_{\psi(x)} J_{ji}(\psi \circ \varphi^{-1})|_{\varphi(x)}, \end{aligned}$$

where $J(\psi \circ \varphi^{-1})$ is the Jacobian matrix of the function $\psi \circ \varphi^{-1}$. Thus

$$v(f) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_i J_{ji}(\psi \circ \varphi^{-1})|_{\varphi(x)} \right) \frac{\partial}{\partial r_j}(f \circ \psi^{-1})|_{\varphi(x)}.$$

Setting

$$b_j = \sum_{i=1}^{n} a_i J_{ji}(\psi \circ \varphi^{-1})|_{\varphi(x)},$$

we obtain

$$v(f) = \sum_{i=1}^{n} b_j \frac{\partial}{\partial r_j}(f \circ \psi^{-1})|_{\psi(x)}.$$

Thus, to check if v is a tangent vector at x, it suffices to check the required property in any one coordinate system at x.

2.3.3 Notation. Given a coordinate system φ about x, let $x_j = r_j \circ \varphi$ denote the jth coordinate function of φ. By $\partial/\partial x_j$ $(j = 1, 2, \ldots, n)$ is meant the tangent vector at x defined by

$$\frac{\partial}{\partial x_j}(f) = \frac{\partial}{\partial r_j}(f \circ \varphi^{-1})|_{\varphi(x)}$$

for $f \in C^s(X, x, \mathbb{R})$.Thus $\partial/\partial x_j$ corresponds, relative to the system φ to the n-tuple $(0, 0, \ldots, 1, \ldots, 0)$, where the 1 is in the jth spot.

2.3.4 Remark. If $x_1, x_2, \ldots, x_n$ are the coordinate functions of a coordinate system φ about x, and $y_1, y_2, \ldots, y_n$ are those of a coordinate system ψ about x, then the above computation shows that

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial}{\partial x_j}(y_i)\frac{\partial}{\partial y_i}.$$

2.3.5 Remark. A tangent vector v at $x \in X$ has the following properties. For any $f, g \in C^\infty(X, x, \mathbb{R})$ and for $\lambda \in \mathbb{R}$,

(1) $v(f + g) = v(f) + v(g)$
(2) $v(\lambda f) = \lambda v(f)$
(3) $v(fg) = v(f)g(x) + f(x)v(g)$.

These three properties say that the map $v : C^\infty(X, x, \mathbb{R}) \to \mathbb{R}$ is a *derivation*. Moreover, these properties characterize tangent vectors; that is we could have defined a tangent vector to be a map $v : C^\infty(X, x, \mathbb{R}) \to \mathbb{R}$ satisfying (1)-(3) above, and then proved that, relative to any coordinate system φ about x, $v = \sum_{i=1}^{n} a_i(\partial/\partial x_i)$ for some n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers, where x_i is the ith coordinate function of φ.

2.3.6 Remark. The set X_x of tangent vectors at x forms a vector space under the following rules of addition and scalar multiplication:

$$\begin{aligned}(v_1+v_2)(f) &= v_1(f)+v_2(f) \quad (v_1, v_2 \in X_x),\\ (\lambda v_1)(f) &= \lambda(v_1(f)) \qquad (v_1 \in X_x, \lambda \in \mathbb{R}).\end{aligned}$$

To see that $v_1 + v_2$ and λv_1 are tangent vectors at x, let φ be a coordinate system about x, with coordinate functions $(x_1, \ldots, x_n)$. Then

$$v_1 = \sum_{i=1}^{n} a_i(\partial/\partial x_i) \text{ and } v_2 = \sum_{i=1}^{n} b_i(\partial/\partial x_i)$$

for some $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. It is then easy to check that

$$\begin{aligned}v_1 + v_2 &= \sum_{i=1}^{n}(a_i+b_i)\frac{\partial}{\partial x_i},\\ \lambda v_1 &= \sum_{i=1}^{n}(\lambda a_i)\frac{\partial}{\partial x_i}.\end{aligned}$$

The map $(a_1, \ldots, a_n) \to \sum_{i=1}^{n} a_i(\partial/\partial x_i)$ gives a vector space isomorphism $\mathbb{R}^n \to X_x$, so X_x has dimension n. Moreover, it is clear that $\{\partial/\partial x_i\}_{\{i=1,\ldots,n\}}$ is a basis for X_x. The space X_x is called the *tangent space* to X at x. It is also denoted by $T(X)_x$ or by $T(X, x)$ or by $T_x(X)$.

For φ and ψ two coordinate systems at x, with coordinate functions $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively, the formula

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial}{\partial x_j}(y_i)\frac{\partial}{\partial y_i}$$

merely expresses the vector $\partial/\partial x_j$ in terms of the basis $\{\partial/\partial y_i\}_{\{1,\ldots,n\}}$. Thus the change of basis matrix from the basis $\{\partial/\partial y_i\}$ of X_x to the basis $\{\partial/\partial x_i\}$ is precisely the Jacobian matrix $((\partial/\partial x_j)(y_i))$.

2.3.7 Remark. The tangent space $T(\mathbb{R}^n, a)$ to $\mathbb{R}^n$ at a point $a \in \mathbb{R}^n$ is naturally isomorphic with $\mathbb{R}^n$ itself. The isomorphism $\mathbb{R}^n \to T(\mathbb{R}^n, a)$ is given by

$$(\lambda_1, \ldots, \lambda_n) \to \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial r_i}.$$

2.3.8 Notation. We shall henceforth omit the Φ from our notation for differentiable manifold (X, Φ). To be sure, a locally Euclidean space X may have two or more distinct differentiable structures on it (or it may have none), but we shall denote a manifold (X, Φ) merely by X and shall assume that a definite differentiable structure is given on it.

2.4 Differential of a smooth map

2.4.1 Definition. Let X and Y be smooth manifolds. Let $\Psi : X \to Y$ be a smooth map. The *differential* of Ψ at $x \in X$ is the map $d\Psi : X_x \to Y_{\Psi(x)}$ defined as follows. For $v \in X_x$ and $g \in C^\infty(Y, \Psi(x), \mathbb{R})$, $(d\Psi(v))(g) = v(g \circ \Psi)$.

2.4.2 Remark. It is easily checked that $d\Psi(v)$ is indeed a tangent vector at $\Psi(x)$. For, if φ is a coordinate system about x with coordinate functions $(x_1, \ldots, x_n)$, and τ is a coordinate system about $\Psi(x)$ with coordinate functions $(y_1, \ldots, y_m)$, then $v = \sum_{i=1}^{n} a_i(\partial/\partial x_i)$ for some real numbers a_i; and if

$g \in C^\infty(Y, \Psi(x), \mathbb{R})$, then

$$\begin{aligned}
[d\Psi(v)](g) &= v(g \circ \Psi) \\
&= \sum_i^n a_i \frac{\partial}{\partial x_i}(g \circ \Psi) \\
&= \sum_i^n a_i \frac{\partial}{\partial r_i}(g \circ \tau^{-1} \circ \tau \circ \Psi \circ \varphi^{-1})|_{\varphi(x)} \\
&= \sum_{i=1}^n a_i \sum_j^m \frac{\partial}{\partial s_j}(g \circ \tau^{-1})|_{\tau\circ\Psi(x)} \frac{\partial}{\partial r_i}(s_j \circ \tau \circ \Psi \circ \varphi^{-1})_{\varphi(x)} \\
&\qquad [(s_1, \ldots, s_m) \text{ coordinates on } \mathbb{R}^m] \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i \frac{\partial}{\partial y_j}(g) \frac{\partial}{\partial x_i}(y_j \circ \Psi) \\
&= \left[\sum_{j=1}^m v(y_j \circ \Psi) \frac{\partial}{\partial y_j}\right](g).
\end{aligned}$$

Since this holds for all $g \in C^\infty(Y, \Psi(x), \mathbb{R})$,

$$d\Psi(v) = \sum_{j=1}^m v(y_j \circ \Psi) \frac{\partial}{\partial y_j}.$$

and, in particular, $d\Psi(v)$ is a tangent vector. Furthermore, it is clear that $d\Psi$ is a linear transformation $X_x \to Y_{\Psi(x)}$. Since

$$d\Psi\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^m \frac{\partial}{\partial x_i}(y_j \circ \Psi) \frac{\partial}{\partial y_j},$$

this linear transformation $d\Psi$ has matrix

$$(d\Psi)_{ij} = \left(\frac{\partial}{\partial x_j}(y_i \circ \Psi)\right)$$

relative to the bases $\{\partial/\partial x_i\}_{i\in\{1,\ldots,n\}}$ and $\{\partial/\partial y_j\}_{j\in\{1,\ldots,m\}}$.

2.4.3 Remark. Let X, Y and Z be smooth manifolds. Let $\Psi : X \to Y$ and $\Phi : Y \to Z$ be smooth maps. Then $d(\Phi \circ \Psi) = d\Phi \circ d\Psi$.

Proof. Suppose $v \in X_x$ and $h \in C^\infty(Z, \Phi \circ \Psi(x), \mathbb{R})$. Then

$$\begin{aligned}
[d(\Phi \circ \Psi)(v)](h) &= v(h \circ (\Phi \circ \Psi)) \\
&= v((h \circ \Phi)\Psi) \\
&= d\Psi(v)(h \circ \Phi) \\
&= [d\Phi(d\Psi(v))](h) \\
&= [(d\Phi \circ d\Psi)(v)](h).
\end{aligned}$$

□

2.4.4 Remark. Let X be a smooth manifold, and let U be open in X. Then U is itself a smooth manifold. Moreover, the inclusion map $i : U \to X$ is a smooth map. Indeed, $f \in C^\infty(X, \mathbb{R})$ implies $f|_U \in C^\infty(U, \mathbb{R})$. Furthermore, the differential

$$di : T(U, u_0) \to T(X, u_0) \qquad (u_0 \in U)$$

is an isomorphism; we shall identify these two linear spaces.

2.4.5 Remark. Let X be a smooth manifold, and let $f \in C^\infty(X, \mathbb{R})$. Let us compute df. For $v \in T(X, x)$, $df(v) \in T(\mathbb{R}, f(x))$. Since $T(\mathbb{R}, f(x))$ is 1-dimensional, $df(v) = \lambda(d/dr)$ for some $\lambda \in \mathbb{R}$. To determine λ, it suffices to evaluate $df(v)$ on the coordinate function $r : \mathbb{R} \to \mathbb{R}$ as follows.

$$\lambda = \left[\lambda \frac{d}{dr}\right](r) = [df(v)](r) = v(r \circ f) = v(f).$$

Thus $df(v) = v(f)(d/dr)$. Now $T(\mathbb{R}, f(x))$ is naturally isomorphic with $\mathbb{R}$ via the isomorphism $\lambda(d/dr) \to \lambda$. Let us identify these two spaces through this isomorphism. Then $df : T(X, x) \to \mathbb{R}$ is a linear functional on $T(X, x)$; that is, df is a member of the dual space $T^*(X, x)$ and is, as such, given by

$$df(v) = v(f) \qquad (v \in T(X, x))$$

$T^*(X,x)$ is called the *cotangent space* at x

2.4.6 Definitions. Let X be a smooth manifold. A *smooth curve* in X is a map α from some (open or closed) interval $\subset \mathbb{R}$ into X. If the domain of α is a closed interval $[a,b]$, smoothness of α means that α admits a smooth extension

$$\widetilde{\alpha} : (a-\varepsilon, b+\varepsilon) \to X.$$

(Note that open intervals are open sets in $\mathbb{R}$ and hence are smooth manifolds.)

A *broken C^∞-curve* in X is a continuous map $\alpha : [a,b] \to X$ together with a subdivision of $[a,b]$ on whose closed subintervals α is a C^∞ curve.

2.4.7 Example.

$$\alpha(t) = \begin{cases} (t, t \sin 1/t) & (t \in (0,1]) \\ (0,0) & (t=0) \end{cases}$$

is not a smooth curve in $\mathbb{R}^2$ because it admits no smooth extension past 0.

2.4.8 Definition. Let $\alpha : I \to X$ (I an interval $\subset \mathbb{R}$) be smooth curve in X. The *tangent vector* to α at time t ($t \in I$), denoted by $\dot{\alpha}(t)$, is defined by

$$\dot{\alpha}(t) = d\widetilde{\alpha}\left(\left(\frac{d}{dr}\right)_t\right).$$

Note that $\dot{\alpha}(t)$ is well defined, even at the endpoints of I.

2.4.9 Remark. Given a tangent vector $v \in X_x$, let $\alpha : I \to X$ be a smooth curve whose tangent vector at time $t=0$ is v. (Such a curve may be obtained by taking a coordinate system φ about x, finding a curve (for example, the straight line) in $\mathbb{R}^n$ whose tangent vector v at time 0 is $d\varphi(v)$, and pulling this curve back to X by φ^{-1}.) Then, for $f \in C^\infty(X,x,\mathbb{R})$,

$$v(f) = \dot{\alpha}(0)(f) = d\widetilde{\alpha}\left(\left(\frac{d}{dr}\right)_0\right)(f) = \frac{d}{dr}(f \circ \widetilde{\alpha})|_0.$$

Thus $v(f)$ is the derivative of the "restriction" of f to the curve α. Moreover, two curves α_1 and α_2 have the same tangent vector v at time 0 if and only if $\alpha_1(0) = \alpha_2(0)$ and

$$\frac{d}{dr}(f \circ \widetilde{\alpha_1})|_0 = \frac{d}{dr}(f \circ \widetilde{\alpha_2})|_0.$$

for all $f \in C^\infty(X, x, \mathbb{R})$. We may use this equation to define an equivalence relation on the set of all curves α with $\alpha(o) = x$. Then we get one-to-one correspondence between equivalence classes of curves through x and tangent vectors at x. Thus, we could have defined a tangent vector at x to be such an equivalence class of curves through x.

2.5 Tangent & cotangent bundles, vector fields &differential 1-forms

2.5.1 Definition. Let X be a smooth manifold. Define

$$T(X) = \bigcup_{x \in X} T(X, x) \quad \text{and} \quad T^*(X) = \bigcup_{x \in X} T^*(X, x).$$

$T(X)$ is called the *tangent bundle* of X. $T^*(X)$ is called the *cotangent bundle* of X.

2.5.2 Definition. A *projection map* $\pi : T(X) \to X$ is defined as follows. If $v \in T(X)$, then $v \in T(X, x)$ for some (unique) $x \in X$; set $\pi(v) = x$. Similarly, there is a projection map from $T^*(X)$ onto X that we shall also denote by π.

2.5.3 Definition. A *vector field* on X is a map $V : X \to T(X)$ such that $\pi \circ V = i_X$.

2.5.4 Definition. Let V be a vector field on X and $f \in C^\infty(X, \mathbb{R})$ then we define Vf by

$$(Vf)(x) = V(x)(f).$$

2.5.5 Definition. A vector field V is *smooth* if for each $f \in C^\infty(X, \mathbb{R})$, $Vf \in C^\infty(X, \mathbb{R})$.

2.5.6 Definition. A *differential* 1*-form* on X is a map $\omega : X \to T^*(X)$ such that $\pi \circ \omega = i_X$.

2.5.7 Definition. Let ω be a differential 1-form and V be a vector field on X then we define $\omega(V)$ by $(\omega(V))(x) = \omega(x)(V(x))$.

2.5.8 Definitions. A differential 1-form ω is *smooth* if for each smooth vector field V on X,

$$\omega(V) \in C^\infty(X, \mathbb{R}),$$

We shall denote the set of all smooth vector fields on X by $C^\infty(X, T(X))$ and set of all smooth 1-forms by $C^\infty(X, T^*(X))$.

2.5.9 Remark. Let $f \in C^\infty(X, \mathbb{R})$. Then $df \in C^\infty(X, T^*(X))$. For if $V \in C^\infty(X, T(X))$, then $df(V) = Vf \in C^\infty(X, \mathbb{R})$.

2.5.10 Remark. $C^\infty(X, T(X))$ and $C^\infty(X, T^*(X))$ are both vector spaces over the reals under the operations of addition and scalar multiplication. For example, if $V_1, V_2 \in C^\infty(X, T(X))$, then $V_1 + V_2$ is defined by $(V_1 + V_2)(x) = V_1(x) + V_2(x)$; and $\lambda \in \mathbb{R}$, then λV_1 is defined by $(\lambda V_1)(x) = \lambda(V_1(x))$.

2.5.11 Remark. Let φ be a coordinate system on X with domain U and coordinate functions $(x_1, \ldots, x_n)$. Then the following hold.

(1) $(\partial/\partial x_i) \in C^\infty(U, T(U))$ for $i \in \{1,\dots,n\}$. $\partial/\partial x_i$ is smooth because if

$$f \in C^\infty(U,\mathbb{R}), \qquad \text{then} \qquad f \circ \varphi^{-1} \in C^\infty(\varphi(U),\mathbb{R}),$$

and, for each $x \in U$,

$$\begin{aligned} \left[\frac{\partial}{\partial x_i}(f)\right](x) &= \left[\frac{\partial}{\partial r_i}(f\circ\varphi^{-1})\right](\varphi(x)) \\ &= \left[\left[\frac{\partial}{\partial r_i}(f\circ\varphi^{-1})\right]\circ\varphi\right](x); \end{aligned}$$

that is

$$\frac{\partial}{\partial x_i}(f) = \left[\frac{\partial}{\partial r_i}(f\circ\varphi^{-1})\right]\circ\varphi \in C^\infty(U,\mathbb{R}).$$

(2) If $V \in C^\infty(U,T(U))$, then there exist functions $a_i \in C^\infty(U,\mathbb{R})$ for $i \in \{1,\dots,n\}$, such that $V = \sum_{i=1}^n a_i(\partial/\partial x_i)$. These functions a_i exist because

$$\{(\partial/\partial x_i)(x)\}_{i\in\{1,\dots,n\}}$$

is a basis for $T(X,x)$. They are smooth because $(\partial/\partial x_i)(x_j) = \delta_{ij}$, so that

$$a_j = \sum_{i=1}^n a_i\delta_{ij} = \sum_{i=1}^n a_i\frac{\partial}{\partial x_i}(x_j) = V(x_j) \in C^\infty(U,\mathbb{R}).$$

(3) If $V \in C^\infty(X,T(X))$, then $V|_U \in C^\infty(U,T(U))$ by a previous example, and $V|_U = \sum_{i=1}^n a_i(\partial/\partial x_i)$ as in (2) with $a_i \in C^\infty(U,\mathbb{R})$.

(4) $dx_j \in C^\infty(U,T^*(U))$ for $j \in \{1,\dots,n\}$ because $x_j \in C^\infty(U,\mathbb{R})$. Furthermore, $\{dx_j\}$ is at each point the dual basis to $\{\partial/\partial x_j\}$ because

$$dx_j\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}(x_j) = \delta_{ij}.$$

(5) If $\omega \in C^\infty(U,T^*(U))$, then there exist $a_i \in C^\infty(U,\mathbb{R})$ such that $\omega = \sum_{i=1}^n a_i dx_i$. These functions a_i exist because $\{dx_i\}$ is at each point a basis

for the cotangent space. They are smooth because

$$a_i = \sum_{j=1}^{n} a_j dx_j \left(\frac{\partial}{\partial x_i} \right) = \omega \left(\frac{\partial}{\partial x_i} \right) \in C^\infty(U, \mathbb{R}).$$

(6)If $f \in C^\infty(U, \mathbb{R})$, then

$$df = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(f) dx_j$$

because $df = \sum_{j=1}^{n} a_j dx_j$ for some a_j, and

$$a_i = \sum_{j=1}^{n} a_j dx_j \left(\frac{\partial}{\partial x_i} \right) = df \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i}(f).$$

Chapter 3

Exterior algebras & smooth differential forms

(The materials of this chapter has been adapted from [27])

3.1 Review of exterior algebras

Let V be an n-dimensional vector space over the reals. Then the following hold.

(1) The vector space $\Lambda^k(V^*)$ is the space of all skew-symmetric k-linear functions on V; that is each $\tau \in \Lambda^k(V^*)$ is a map $\tau : \underbrace{V \oplus \cdots \oplus V}_{k} \to \mathbb{R}$ such that for all $v_1, \ldots, v_k, v_j' \in V$, $\lambda \in \mathbb{R}$,

(i) $\tau(v_1, \ldots, v_{j-1}, v_j + v_j', v_{j+1}, \ldots, v_k)$
$= \tau(v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_k) + \tau(v_1, \ldots, v_{j-1}, v_j', v_{j+1}, \ldots, v_k)$;

(ii) $\tau(v_1, \ldots, v_{j-1}, \lambda v_j, v_{j+1}, \ldots, v_k) = \lambda\tau(v_1, \ldots, v_{j-1}, v_j, \ldots, v_k)$; and

(iii) $\tau(v_{\pi(1)}, \ldots, v_{\pi(k)}) = (-1)^{\pi}\tau(v_1, \ldots, v_k)$;

where for each element π of the permutation group S_k on k letters, and $(-1)^\pi$ equals $+1$ if π is an even permutation and equals -1 if π is an odd permutation. This third condition is equivalent to requiring that if two vectors in the argument of τ are interchanged, then the valued of τ on these vectors changes sign. The dimension of $\Lambda^k(V^*)$ is equal to the binomial coefficient $\binom{n}{k}$ for $k \leq n$; it is zero for $k > n$.

(2) If we set $G(V^*) = \sum_{k=0}^{n} \oplus \Lambda^k(V^*)$, where $\Lambda^0(V^*) = \mathbb{R}$, a product is defined on $G(V^*)$ as follows. If $\tau \in \Lambda^k(V^*)$ and $\mu \in \Lambda^l(V^*)$, their product $\tau \wedge \mu$ is the element of $\Lambda^{k+1}(V^*)$ defined by

$$\begin{aligned} &\tau \wedge \mu(v_1, \ldots, v_{k+1}) \\ &\quad = \frac{1}{(k+l)!} \sum_{\pi \in S_{k+1}} (-1)^\pi \tau(v_{\pi(1)}, \ldots, v_{\pi(k)}) \mu(v_{\pi(k+1)}, \ldots, v_{\pi(k+l)}) \end{aligned}$$

Since $G(V^*)$ is generated by such μ and τ, this multiplication extends to $G(V^*)$ by linearity, that is, by requiring that exterior multiplication $\wedge$ be distributive with respect to vector addition. This multiplication is associative and $G(V^*)$ is an algebra, with unit 1. However, this multiplication is not commutative: if $\mu \in \Lambda^k(V^*)$ and $\tau \in \Lambda^l(V^*)$, then

$$\mu \wedge \tau = (-1)^{kl} \tau \wedge \mu.$$

(3) If $\varphi_1, \ldots, \varphi_n$ is a basis for V^*, then

$$[\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}; 1 \leq i_1 < i_2 < \cdots \wedge i_k \leq n]$$

is a basis for $\Lambda^k(V^*)$. Hence the union of these sets over $k \in \{1, \ldots, n\}$, together with $1 \in \Lambda^0(V^*)$, is a basis for $G(V^*)$. It follows that the dimension of $G(V^*)$ is 2^n. If $v_1, \ldots, v_k \in V$, the value of $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ on these vectors is given by

$$(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k})(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\pi \in S_k} (-1)^{\pi} \varphi_{i_1}(v_{\pi(1)}) \cdots \varphi_{i_k}(v_{\pi(k)}).$$

(4)We note that $G(V^*)$ has the following properties:

(i) $1 \in G(V^*), V^* \subset G(V^*)$;

(ii) $G(V^*)$ is generated by 1 and V^*;

(iii) $\varphi \wedge \varphi = 0$ whenever $\varphi \in V^*$; and

(iv) dimension $G(V^*) = 2^n$.

These properties in fact characterize $G(V^*)$; that is, if $\widetilde{G}(V^*)$ is any algebra over the reals satisfying properties (i)-(iv), then $\widetilde{G}(V^*)$ and $G(V^*)$ are isomorphic (as algebras).

(Note that condition (iii) is equivalent to the condition that $\varphi_1 \wedge \varphi_2 = -\varphi_2 \wedge \varphi_1$ for all $\varphi_1, \varphi_2 \in V^*$.)

(5) If $L : V^* \to V^*$ is linear transformation, then L induces a unique algebra homomorphism $\widetilde{L} : G(V^*) \to G(V^*)$ which extends the map L. $\widetilde{L}$ preserves degrees; that is, $\widetilde{L} : \Lambda^k(V^*) \to \Lambda^k(V^*)$. In particular, $\widetilde{L} : \Lambda^n(V^*) \to \Lambda^n(V^*)$. Hence, since dim $\Lambda^n(V^*) = 1$ with generator $i_{\Lambda^n(V^*)}$ say , there exists a scalar λ such that $\widetilde{L}|_{\Lambda^n(V^*)} = \lambda i_{\Lambda^n(V^*)}$. This scalar λ is precisely det(L), the determinant of L.

(6) The algebra $G(V^*)$ is called the *Grassmann algebra*, or *exterior algebra*, of V^*. Elements of $G(V^*)$ are called forms on V. Forms in $\Lambda^k(V^*)$ are said to be of *degree k*.

3.2 Differential forms & exterior differential

Now let X be a smooth manifold. Let

$$\Lambda^k(X) = \bigcup_{x \in X} \Lambda^k(T^*(X,x)),$$

and let

$$G(X) = \bigcup_{x \in X} G(T^*(X,x)).$$

As usual, we shall denote the projection maps from these spaces onto X by π. These spaces can each be given the structure of a smooth manifold so that π is a smooth map.

3.2.1 Definition. A *k-form* on X is a mapping $\mu : X \to \Lambda^k(X)$ such that $\pi \circ \mu = id_X$.

3.2.2 Definition. Let μ be a k-form and $V_1, \ldots, V_k$ are smooth vector fields on X then we define $\mu(V_1, \ldots, V_k)$ by

$$\mu(V_1, \ldots, V_k)(x) = \mu(x)(V_1(x), \ldots, V_k(x)).$$

3.2.3 Definition. A k-form μ on X is *smooth* if whenever $V_1, \ldots, V_k$ are smooth vector fields on X then

$$\mu(V_1, \ldots, V_k) \in C^\infty(X, R).$$

3.2.4 Definition. A *differential form* on X is a mapping $\omega : X \to G(X)$ such that $\pi \circ \omega = id_X$; it is smooth if its component in $\Lambda^k(X)$ is smooth for each k. The set of smooth k-forms on X is denoted by $C^\infty(X, \Lambda^k(X))$. The set of all smooth differential forms is denoted by $C^\infty(X, G(X))$. Note that $C^\infty(X, \Lambda^k(X))$ is a vector space under pointwise addition and scalar multiplication, and that $C^\infty(X, G(X))$ is an algebra under the additional operation of pointwise exterior multiplication.

3.2.5 Remark. A 0-form on X is just a real valued function on X; it is a smooth 0-form if and only if it is a smooth function.

3.2.6 Remark. Let φ be a local coordinate system on X, with domain U and coordinate functions $(x_1, \ldots, x_n)$. Then $\{dx_1, \ldots, dx_n\}$ is a basis for $T^*(X, x)$ for each $x \in U$. Hence

$$[dx_{i_1} \wedge \cdots \wedge dx_{i_k}; i_1 < \cdots < i_k]$$

is a basis for $\Lambda^k(T^*(X, x))$ for each $x \in U$. Thus, the restriction to U of each k-form μ on X can be expressed as

$$\mu = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where each $a_{i_1 \ldots i_k}$ is a real valued function on U. Furthermore, μ is smooth if and only if, for each (φ, U), $a_{i_1 \ldots i_k} \in C^\infty(U, \mathbb{R})$. This is because

$$a_{i_1 \ldots i_k} = k! \mu \left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}} \right).$$

3.2.7 Theorem. *Let X be a smooth manifold. There exists a unique linear map $d : C^\infty(X, G(X)) \to C^\infty(X, G(X))$, called the* exterior differential, *such that the following properties hold.*

(1) $d : C^\infty(X, \Lambda^k(X)) \to C^\infty(X, \Lambda^{k+1}(X))$;

(2) $d(f) = df$ *(ordinary differential) for* $f \in C^\infty(X, \Lambda^0)$;

(3) *if* $\mu \in C^\infty(X, \Lambda^k(x))$ *and* $\tau \in C^\infty(X, G(X))$, *then*
$d(\mu \wedge \tau) = (d\mu) \wedge \tau + (-1)^k \mu \wedge d\tau$; *and*

(4) $d^2 = 0$

3.2.8 Remark. For the proof we need the following lemma, which asserts that for any exterior differential operator d, $(d\omega)(x)$ depends only on the behavior of ω in a small neighborhood of x.

3.2.9 Lemma. *Let $d : C^\infty(X, G(X)) \to C^\infty(X, G(X))$ be linear and satisfy the conditions of the theorem. Suppose $\omega \in C^\infty(X, G(X))$ is such that $\omega|_W = 0$ for some open set $W \subset X$. Then $(d\omega)|_W = 0$. Hence, if $\omega, \tau \in C^\infty(X, G(X))$ are such that $\omega|_W = \tau|_W$ for some open set W, then $(d\omega)|_W = (d\tau)|_W$.*

Proof. Suppose $\omega|_W = 0$. Let $x_0 \in W$. Let $f \in C^\infty(X, \mathbb{R})$ be such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \notin W$. Then $f\omega$ is identically zero on X, so that

$$0 = d(f\omega) = (df) \wedge \omega + f d\omega.$$

Evaluating at x_0 gives $(d\omega)(x_0) = 0$. Since this holds for all $x_0 \in W$, $d\omega|_W = 0$. If $\omega|_W = \tau|_W$, then $(\omega - \tau)|_W = 0$, so that

$$0 = [d(\omega - \tau)]|_W = [d\omega - d\tau]|_W \text{ and } d\omega|_W = d\tau|_W.$$

□

Proof. Uniqueness. Suppose $d : C^\infty(X, G(X)) \longrightarrow C^\infty(X, G(X))$ satisfies the conditions of the theorem. Let $x \in X$, and let ϕ be a local coordinate system about x with domain U and coordinate functions $(x_1, \dots, x_n)$. Let $\omega \in C^\infty(X, \Lambda^k(X))$. Then the restriction of ω to U can be expressed as

$$\omega|_U = \sum a_{i_1} \cdots {}_{i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for some $a_{i_1} \cdots {}_{i_k} \in C^\infty(U, \mathbb{R})$. Now the right-hand side of this equation is not a differential form on X, so we can not apply d to it. However, let U_1

be an open ball containing x with $\overline{U}_1$ compact and such that $\overline{U}_1 \subset U$, and let $g \in C^\infty(X, \mathbb{R})$ be such that $g(x) = 1$ for $x \in U_1$ and $g(x) = 0$ for $x \notin U$. Then $\widetilde{\omega} \in C^\infty(X, \Lambda^k(X))$, where

$$\widetilde{\omega} = \sum (ga_{i_1} \cdots {}_{i_k}) d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}).$$

Here, by gh, for $h \in C^\infty(U, \mathbb{R})$, is meant the smooth function on X defined by

$$(gh)(x) = \begin{cases} g(x)h(x) & \text{if } (x \in U) \\ 0 & \text{if } (x \notin U) \end{cases}$$

Furthermore, $\widetilde{\omega}|_{U_1} = \omega|_{U_1}$. By the lemma, $(d\omega)|_{U_1} = (d\widetilde{\omega})|_{U_1}$. Now

$$\begin{aligned} d\widetilde{\omega} &= \sum d[ga_{i_1} \cdots {}_{i_k} d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k})] \quad \text{(by linearity)} \\ &= \sum d(ga_{i_1} \cdots {}_{i_k}) \wedge d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}) \\ &\quad + \sum ga_{i_1} \cdots {}_{i_k} d(d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k})) \quad \text{(by property(3))} \\ &= \sum d(ga_{i_1} \cdots {}_{i_k}) \wedge d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}), \end{aligned}$$

since each term of the second sum is zero by properties (3) and (4). In particular, since g is identically 1 on U_1, and since $(d\omega)|_{U_1} = (d\widetilde{\omega})|_{U_1}$,

$$(d\omega)|_{U_1} = \sum_{i_1 \leq \cdots \leq i_k} \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(a_{i_1} \cdots {}_{i_k}) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Thus if d exists, its value at x on k-forms must be given by this formula. Since x was arbitrary in X, and since every differential form is a sum of k-forms, $k \in \{0, 1, \ldots, n\}$, uniqueness is established.

Existence. We first define d locally. Let ϕ be a local coordinate system on X with domain U and coordinate functions $(x_1, \ldots, x_n)$. (Note that U

is itself a smooth manifold). Define $d_U : C^\infty(U, G(U)) \longrightarrow C^\infty(U, G(U))$ as follows. For

$$\omega = \sum a_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in C^\infty(U, \Lambda^k(U)),$$

define

$$d_U\omega = \sum \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(a_{i_1 \cdots i_k}) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Extend d_U to $C^\infty(U, G(U))$ by linearity.

Then properties (1) and (2) are clearly satisfied. To verify (3) and (4), note first that each form in $C^\infty(U, G(U))$ is a sum of forms of the type $a_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. By linearity of d_U, together with distributivity of exterior multiplication with respect to addition, it suffices to check (3) and (4) on forms of this type.

Property (3). Suppose

$$\mu = a_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \text{and } \tau = b_{j_1 \cdots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

Then

$$\begin{aligned}
d_U(\mu \wedge \tau) &= d_U[a_{i_1 \cdots i_k} b_{j_1 \cdots j_l}\, dx_{i_1} \wedge \ldots dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}] \\
&= \sum_{r=1}^{n} \left[\frac{\partial}{\partial x_r}(a_{i_1 \cdots i_k}) b_{j_1 \cdots j_l} + a_{i_1 \cdots i_k} \frac{\partial}{\partial x_r}(b_{j_1 \cdots j_l}) \right] dx_r \wedge dx_{i_1} \cdots \wedge dx_{j_l} \\
&= \left(\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(a_{i_1 \cdots i_k}) dx_r \wedge dx_{i_1} \wedge \ldots dx_{i_k} \right) \\
&\qquad \wedge (b_{j_1 \cdots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}) \\
&\quad + (-1)^k (a_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\
&\qquad \wedge \left(\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(b_{j_1 \cdots j_l}) dx_r \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \right) \\
&= (d_U\mu) \wedge \tau + (-1)^k \mu \wedge d_U\tau.
\end{aligned}$$

Property (4). For $\mu = a_{i_i \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge d_{i_k}$,

$$\begin{aligned} d_U^2\mu &= d\mu\left[\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(a_{i_1}\cdots{}_{i_k})dx_r \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right] \\ &= \sum_{r,s=1}^{n} \frac{\partial}{\partial x_s}\left[\frac{\partial}{\partial x_r}(a_{i_1}\cdots{}_{i_k})\right] dx_s \wedge dx_r \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

But certainly the terms in this expression with $r = s$ are zero, since $dx_r \wedge dx_r = 0$. Moreover, for $r \neq s$, the equality of mixed partial derivatives on $\mathbb{R}^n$ implies that

$$\frac{\partial}{\partial x_s}\frac{\partial}{\partial x_r}(a_{i_1}\cdots{}_{i_k}) = \frac{\partial}{\partial x_r}\frac{\partial}{\partial x_s}(a_{i_1}\cdots{}_{i_k}),$$

so that

$$\frac{\partial}{\partial x_s}\frac{\partial}{\partial x_r}(a_{i_1}\cdots{}_{i_k})dx_s \wedge dx_r = -\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_s}(a_{i_1}\cdots{}_{i_k})dx_r \wedge dx_s;$$

thus the remaining terms match up in pairs which cancel each other.

Thus the operator d_U has Properties (1)-(4). By Uniqueness, every linear operator on $C^\infty(U, G(U))$ having these properties must be given by the above boxed formula. In particular, if U_1 is any open subset of U, then $\phi|_{U_1}$ is a coordinate system, and $D_{U_1} : C^\infty(U_1, G(U_1)) \longrightarrow C^\infty(U_1, G(U_1))$ is given in the coordinate system $\phi|_{U_1}$ by the same formula. Thus, if $\omega \in C^\infty(X, G(X))$, then

$$d_{U_1}(\omega|_{U_1}) = d_U(\omega|_U))|_{U_1}.$$

This relation enables us to define d globally by $(d\omega)|_U = d_U(\omega|_U)$ for all

$$\omega \in C^\infty(X, G(X))$$

and any coordinate neighborhood U. This d is well defined because if U and V are overlapping coordinate neighborhoods, then

$$(d_U(\omega|_U))|_{U\cap V} = d_{U\cap V}(\omega|_{U\cap V}) = (d_V(\omega|V))|_{U\cap V}.$$

Clearly, d has the required properties, since d_U has them for each U. □

3.3 Elements of vector analysis

The multilinear algebra developed above is particularly simple in the case of $n = 3$. We want to show how the classical approach of vector analysis fits into the scheme of differential forms. In order to develop the condition, we consider first the general situation in an n-dimensional vector space T.

3.3.1 Definition. A *volume element of* T is a choice of a basis in $\Lambda^n(T^*)$; since $\Lambda^n(T^*)$ is 1-dimensional, a volume element is a choice of a nonzero element in $\Lambda^n(T^*)$.

3.3.2 Example. If T is the tangent space to a manifold and $\{dx_1, \ldots, dx_n\}$ is a basis for T^*, then $dx_1 \wedge \cdots \wedge dx_n$ is a volume element of T. (Note that a volume element ω determines an isomorphism $\Lambda^n(T^*) \cong \mathbb{R}$, where $r\omega$ corresponds to r. Conversely, such an isomorphism defines a volume element ω corresponding to 1.)

3.3.3 Remark. Given a volume element ω of T, there exists a natural isomorphism $m : \Lambda^{n-1}(T^*) \longrightarrow T$ defined as follows. Recall that T is naturally isomorphic to its double dual T^{**}. Identifying T^{**} with T through this isomorphism, m will have values in T^{**}. For $\phi \in \Lambda^{n-1}(T^*)$, $m(\phi)$ is then defined by $[m(\phi)](\psi) = \lambda$, where, for $\psi \in T^*, \lambda$ is the real number such that $\phi \wedge \psi = \lambda\omega$. To show that m is an isomorphism, let $\{\phi_1, \ldots, \phi_n\}$ be a basis for T^* such that $\omega = \phi_1 \wedge \cdots \wedge \phi_n$. Then the set $\{\phi_1 \wedge \cdots \wedge \phi_{j-1} \wedge \phi_{j+1} \wedge \ldots \phi_n\}$ is a basis for $\Lambda^{n-1}(T^*)$. The value of m on these basis vectors is given by

$$m(\phi_1 \wedge \cdots \wedge \phi_{j-1} \wedge \phi_{j+1} \wedge \cdots \wedge \phi_n) = (-1)^{n+j} e_j,$$

where $\{e_1, \ldots, e_n\}$ is the basis for T dual to $\{\phi_1, \ldots, \phi_n\}$.

3.3.4 Remark. Given an inner product $<,>$ on a finite dimensional vector space T, there exists a natural isomorphism $g : T \longrightarrow T^*$ defined by

$$[g(v)](w) = <v, w> \quad (v, w \in T).$$

If $\{e_1, \ldots, e_n\}$ is a basis for T, let $g_{ij} = <e_i, e_j>$, $(i, j \in \{1, \ldots, n\})$. Then in terms of the dual $\{\phi_1, \ldots, \phi_n\}$ for T^*,

$$g(e_i) = \sum_{j=1}^{n} g_{ij}\phi_j \quad (i \in \{1, \ldots, n\}).$$

In particular, if $\{e_1, \ldots, e_n\}$ is orthonormal, then $g_{ij} = \delta_{ij}$, and

$$g(e_i) = \phi_i.$$

3.4 Application

Take $T = \mathbb{R}^n$. Then T has an inner product and a natural volume element $\omega = \phi_1 \wedge, \ldots, \wedge\phi_n$, where $\{\phi_i\}$ is the dual basis to the natural basis $\{e_i\}$ for $\mathbb{R}^n$. Thus the isomorphisms m and g are defined. Also, we have natural identification $T(\mathbb{R}^n, x) \leftrightarrow \mathbb{R}^n$ for each $x \in \mathbb{R}^n$.

(1) Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Then the *gradient* of f is the vector on $\mathbb{R}^n$ given by

$$\operatorname{grad} f = g^{-1} \circ (df).$$

Relative to the usual coordinates $(x_1, \ldots, x_n) = (r_1, \ldots, r_n)$ on $\mathbb{R}^n$,

$$\operatorname{grad} f = g^{-1} \circ (df) = g^{-1}\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j\right) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_j} \leftrightarrow \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

(2) Let V be a vector field on $\mathbb{R}^3$. Then $g \circ V$ is a 1-form and $d(g \circ V)$ is a 2-form. Now for dimension $T = 3$, $\Lambda^2(T^*) = \Lambda^{n-1}(T^*)$, so the isomorphism m maps $\Lambda^2(T^*) \to T$. Thus $m(d(g \circ V))$ is a vector field on $\mathbb{R}^3$. It is called the *curl* of V.

$$\text{curl } V = (m \circ d \circ g)(V)$$

(3) Let v_1 and v_2 be vector fields on $\mathbb{R}^3$. Then $g(v_1)$ and $g(v_2)$ are 1-forms. Their exterior product is a 2-form; its image under m is a vector field, called the cross product of v_1 and v_2.

$$v_1 \times v_2 = m(g(v_1) \wedge g(v_2)).$$

(4) Let V be a vector field on $\mathbb{R}^n$. Then $m^{-1}(V)$ is an $(n-1)$-form on $\mathbb{R}^n$. Its differential is an n-form; that is, a multiple of the volume element ω. This multiple is (up to sign) the *divergence* of V:

$$(-1)^{n-1} d \circ m^{-1}(V) = (\text{div } V)\omega.$$

3.4.1 Remark. Using these formulas, certain important formulas of vector analysis become trivial consequences of $d^2 = 0$.

A. curl grad $f = 0$ because

$$\begin{aligned} \text{curl grad } f &= m \circ d \circ g(g^{-1} \circ d(f)) \\ &= m(d^2 f) \\ &= 0. \end{aligned}$$

B. div curl $V = 0$ because

$$\begin{aligned} d \circ m^{-1}(\text{curl } V) &= d \circ m^{-1}(m \circ d \circ g(V)) \\ &= d^2(g(V)) \\ &= 0. \end{aligned}$$

Chapter 4

de Rham's cohomology of smooth manifolds and simplicial cohomology

(The materials of this chapter has been adapted from [27] ,[14] and [35])

4.1 de Rham cohomology of a smooth manifold

4.1.1 Definition. Let X be a smooth manifold. The pair $(C^{\infty}(X, G(X)), d)$ is called the de Rham complex of X.

4.1.2 Definition. Let X and Y be smooth manifolds, and let $\Psi : X \to Y$ be a smooth map. Then an *induced map* $\Psi^{*} : C^{\infty}(Y, G(Y)) \to C^{\infty}(X, G(X))$ is defined as follows. For $f \in C^{\infty}(Y, \Lambda^{0}(Y))$, $\Psi^{*}(f) = f \circ \Psi$; for $\omega \in$

$C^\infty(Y, \Lambda^k(Y))$ $(k > 0)$,

$$(\Psi^*\omega)(x)(v_1, \ldots, v_k)$$
$$= \omega(\Psi(x))(d\Psi(v_1), \ldots, d\Psi(v_k)) \;\; (v_1, \ldots, v_k \in T(X, x),\; x \in X);$$

Ψ^* is extended to $C^\infty(Y, G(Y))$ by linearity.

4.1.3 Remark. It is easy to check that, if ω is a smooth differential form, then so is $\Psi^*\omega$. It is clear that Ψ^* maps k-forms into k-forms. In fact it is easily checked that Ψ^* is an algebra homomorphism; i,e., Ψ^* is linear and

$$\Psi^*(\omega \wedge \tau) = (\Psi^*\omega) \wedge (\Psi^*\tau)$$

for all ω, τ.

4.1.4 Theorem. *Let X and Y be smooth and let $\Psi : X \to Y$ be a smooth map. Then*

$$d \circ \Psi^* = \Psi^* \circ d.$$

Proof. (1) If $f \in C^\infty(Y, \Lambda^0(Y))$, then for $v \in T(X, x)$,

$$\begin{aligned} [d \circ \Psi^*(f)](v) &= [d(f \circ \Psi)](v) \\ &= [df \circ d\Psi](v) \text{ (since } d \text{ on functions is ordinary differential)} \\ &= [\Psi^*(df)](v) \\ &= [(\Psi^* \circ d)(f)](v). \end{aligned}$$

(2) For ω a 1-form on Y of the type $\omega = df$,

$$\begin{aligned} (d \circ \Psi^*)(\omega) &= d(\Psi^*(df)) \\ &= d(\Psi^* \circ d(f)) \\ &= d(d \circ \Psi^*(f)) \qquad \text{(by(1))} \\ &= 0, \end{aligned}$$

and

$$(\Psi^* \circ d)(\omega) = \Psi^*(d\omega) = \Psi^*(ddf) = \Psi^*(0) = 0.$$

(3) Using (1) and (2), together with the fact that Ψ^* is an algebra homomorphism, the result is established in general by checking it locally on k-forms ω restricted to local coordinate neighborhoods:

$$\omega|_U = \sum a_{i_1} \cdots_{i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

□

4.1.5 Definitions. Let X be a smooth manifold. A smooth differential form ω on X is *closed* if $d\omega = 0$. A form ω is *exact* if it is the differential of another form on X; that is ω is exact if $\omega = d\tau$ for some smooth form τ. (Note that every exact form is closed, since $d^2 = 0$. The converse question is fundamental to our study.)

Let $Z^k(X, d)$ denote the vector space of closed k-forms on X. Let $B^k(X, d)$ denote the space of exact k-forms on X. Then $B^k(X, d) \subset Z^k(X, d)$ because $d^2 = 0$. Let $H^k(X, d) = Z^k(X, d)/B^k(X, d)$. $H^k(X, d)$ is called the kth *de Rham cohomology group* of X. Its dimension, which we shall see is finite for compact X, is called the kth *Betti number* of X.

4.1.6 Remark. Although these cohomology groups are defined in terms of the manifold structure of X, they are topological invariants; that is, if two manifolds are homeomorphic (by a not necessarily smooth homeomorphism), then they have isomorphic cohomology groups. In fact, these groups can be defined directly using only the topological structure of X.

4.1.7 Example. $H^0(X,d) \cong \mathbb{R}$ if X is connected. For since there is no forms of degree less than 0, $B^0(X,d) = 0$. Thus

$$H^0(X,d) = Z^0(X,d) = [f \in C^\infty(X,\mathbb{R});\ df = 0].$$

If U is any connected coordinate neighborhood of X, with coordinate functions $(x_1,\ldots,x_n)$, then $df = 0$ on U means

$$0 = df = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f)dx_i;$$

that is, $(\partial/\partial x_i)(f) = 0$ for all i. But this implies that f is constant on U. Since X is connected, and since f is constant on each connected coordinate neighborhood in X, then f must be constant on X; that is, $Z^0(X,d) =$ [constant functions on X] $\cong \mathbb{R}$.

4.1.8 Example. $H^1(S^1,d) \cong \mathbb{R}$, where S^1 is the circle. For since there are no k-forms on S^1 for $k > 1$, $Z^1(S^1,d) = C^\infty(S^1,\Lambda^1(S^1))$. Moreover,

$$B^1(S^1,d) = [df; f \in C^\infty(S^1,\mathbb{R})].$$

Now, if θ denotes the polar coordinate on S^1, then $\partial/\partial\theta$ is a nonzero vector field on S^1 and its dual 1-form $d\theta$ is a nonzero 1-form on S^1. Furthermore, $d\theta$ is not exact (in spite of the notation!) but, given any 1-form $\omega = g(\theta)\, d\theta$ on S^1, $\omega - (cd\theta)$ is exact for some $c \in \mathbb{R}$. Thus

$$Z^1(S^1,d)/B^1(S^1,d) \cong [c\, d\theta;\ c \in \mathbb{R}] \cong \mathbb{R}.$$

4.1.9 Remark. Let $\psi : X \to Y$ be smooth. Then

$$\psi^* : Z^k(Y,d) \to Z^k(X,d) \text{ and } \psi^* : B^k(Y,d) \to B^k(X,d).$$

For if ω is closed k-form on Y, then $d(\psi^*\omega) = \psi^*(d\omega) = \psi^*(0) = 0$. If $\omega = d\tau$ is exact k-form on Y, then $\psi^*(\omega) = \psi^*(d\tau) = d(\psi^*(\tau))$. Thus ψ^* induces a linear map $\widetilde{\psi}$ on cohomology, such that

$$\widetilde{\psi} : Z^k(Y,d)/B^k(Y,d) \to Z^k(X,d)/B^k(X,d);$$

that is,

$$\widetilde{\psi} : H^k(Y,d) \to H^k(X,d).$$

If $S : W \to X$ and $T : X \to Y$ are smooth, it is easy to check that $(T \circ S)^* = S^* \circ T^*$, and hence $\widetilde{(T \circ S)} = \widetilde{S} \circ \widetilde{T}$:

$$W \longrightarrow X \longrightarrow Y,$$

$$H^k(W,d) \xleftarrow{\tilde{S}} H^k(X,d) \xleftarrow{\tilde{T}} H^k(Y,d).$$

Thus we have attached to each smooth manifold X new algebraic invariants $H^k(X,d)$ such that given smooth maps between manifolds, there are induced algebraic maps between these algebraic objects. As in the case of the fundamental group, we are thus able to solve certain difficult topological problems by studying their algebraic counterparts.

4.2 Chain homotopy in de Rham complex

Now let us show that $H^k(\mathbb{R}^n, d) = 0$ for all $k > 0$. Since $\mathbb{R}^n$ is *diffeomorphic* (isomorphic as a smooth manifold) with the unit ball $B_0(1)$ about 0 in $\mathbb{R}^n$, we may as well as show that $H^k(B_0(1), d) = 0$ for all $k > 0$. For this we need the following technical lemma.

4.2.1 Lemma. *Let X be a smooth manifold. Then, for each k, consider the maps*

$$\begin{tikzcd} C^\infty(X,\Lambda^{k-1}(X)) \arrow[r, "d"] & C^\infty(X,\Lambda^{k}(X)) \arrow[r, "d"] \arrow[l, bend left, "h_{k-1}"] & C^\infty(X,\Lambda^{k+1}(X)). \arrow[l, bend left, "h_k"] \end{tikzcd}$$

Suppose there exits linear maps

$$h_j : C^\infty(X,\Lambda^{j+1}(X)) \to C^\infty(X,\Lambda^{j}(X)) \qquad (j = k-1, \text{ or } k)$$

such that $h_k \circ d + d \circ h_{k-1}$ is the identity map on $C^\infty(X,\Lambda^k(X))$. Then $H^k(X,d) = 0$; that is, every closed k-form is exact.

Proof. Suppose $\omega \in C^\infty(X,\Lambda^k(X))$ is closed. Then

$$\omega = (h_k \circ d + d \circ h_{k-1})(\omega) = h_k(d\omega) + d(h_{k-1}\omega) = d(h_{k-1}\omega).$$

□

4.2.2 Remark. If a sequence of such linear maps h_j is defined for all $j \geq 0$, the sequence h_j is called a *homotopy operator.*

4.2.3 Theorem. *(Poincaré's lemma). Let $U = B_0(1) \subset \mathbb{R}^n$. Then $H^k(U,d) = 0$ for all $k > 0$.*

Proof. We shall construct maps h_{k-1}, h_k satisfying the conditions of the lemma. This is done through an integration process. Since these maps are to be linear, it suffices to define h_{k-1} on forms $\omega = g\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$; similarly for h_k. For such ω, set

$$h_{k-1}(\omega)(x) = \left(\int_0^1 t^{k-1} g(tx)dt\right)\mu,$$

where

$$\begin{aligned}\mu &= x_{i_1} dx_{i_2} \wedge \cdots \wedge dx_{i_k} - x_{i_2} dx_{i_1} \wedge dx_{i_3} \wedge \cdots \wedge dx_{i_k} \\ &\quad + \cdots + (-1)^{k-1} x_{i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}.\end{aligned}$$

(Note that $d\mu = k dx_{i_1} \wedge \cdots \wedge dx_{i_k}$).

The map h_k is defined similarly by replacing k everywhere by $k+1$. Now, for $\omega = g\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in C^\infty(U, \Lambda^k(U))$ and $x \in U$,

$$\begin{aligned}&(d \circ h_{k-1})(\omega)(x) \\ &= d\left[\left(\int_0^1 t^{k-1} g(tx)dt\right)\mu\right] \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\int_0^1 t^{k-1} g(tx)dt\right) dx_j \wedge \mu + \left(\int_0^1 t^{k-1} g(tx)dt\right) d\mu \\ &= \sum_{j=1}^n \left(\int_0^1 t^{k-1} \frac{\partial}{\partial x_j}(g(tx))dt\right) dx_j \wedge \mu + \left(\int_0^1 t^{k-1} g(tx)dt\right) d\mu \\ &= \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial g}{\partial x_j}(tx)dt\right) dx_j \wedge \mu \\ &\qquad + k\left(\int_0^1 t^{k-1} g(tx)dt\right) dx_{i_1} \wedge \cdots \wedge dx_{i_k},\end{aligned}$$

and

$$\begin{aligned}(h_k \circ d)(\omega)(x) &= h_k\left(\sum_{j=1}^n \frac{\partial g}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\ &= \sum_{j=1}^n \left(\int_0^1 t^k \frac{\partial g}{\partial x_j}(tx)dt\right)[x_j\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} - dx_j \wedge \mu].\end{aligned}$$

Thus,

$$\begin{aligned}
&(d \circ h_{k-1} + h_k \circ d)(\omega)(x) \\
&= \left[k \left(\int_0^1 t^{k-1} g(tx) dt \right) + \sum_{j=1}^{n} \left(\int_0^1 t^k \frac{\partial g}{\partial x_j}(tx) x_j \, dt \right) \right] dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \left\{ \int_0^1 \left[k t^{k-1} g(tx) + \frac{d}{dt}(g(tx)) \right] dt \right\} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \left\{ \int_0^1 \frac{d}{dt}[t^k g(tx)] dt \right\} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= t^k g(tx)|_0^1 dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= g(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \omega(x) \qquad (\forall x \in U).
\end{aligned}$$

Since $d \circ h_{k-1} + h_k \circ d$ act as identity on such ω, it acts by linearity as identity on all k-forms. □

4.2.4 Remark. The maps h_{k-1} and h_k used in this proof were not just picked out of the air. They were constructed as follows. Given a vector space T and $v \in T$, v defines a map $i(v) : \Lambda^k(T^*) \to \Lambda^{k-1}(T^*)$ by

$$i(v)(\omega)(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1}).$$

Note that i is a bilinear map $T \otimes \Lambda^k(T^*) \to \Lambda^{k-1}(Y^*)$. This map is called *interior multiplication.* The map h_{k-1} was obtained by applying $i(x)$ to ω and averaging over the line through the origin in the direction x.

4.2.5 Remark. The above theorem is a special case of a more general result. Let U be a smooth manifold. Suppose there exists a smooth map $\Psi : U \times I_\varepsilon \to U$, where $I_\varepsilon =]-\varepsilon, 1+\varepsilon[$, such that $\Psi(u, 1) = u$ for all $u \in U$, and $\Psi(u, 0) = u_0$ for all $u \in U$; some $u_0 \in U$. Then $H^k(U, d) = 0$ for all $k > 0$. The map Ψ is a *smooth homotopy.* This theorem says that if U is smoothly homotopic to a point, then the cohomology of U is that of a point.

In the case covered by the above theorem, a smooth homotopy is given by

$$\Psi(x,t) = tx \qquad (t \in I_\epsilon;\ x \in B_0(1)).$$

Note that the above proof of Poincaré's lemma works equally well for a *star-shaped* region, that is, an open set U such that for some $x_0 \in U$, the line segment joining x_0 to any other point in U lies completely in U.

4.3 Submanifolds, orientation & classfication of compact orientable surfaces

4.3.1 Definition. A *submanifold* of a smooth manifold Y is a pair (X, ψ), where X is a smooth manifold and $\psi : X \to Y$ is an injective smooth map such that $d\psi$ is injective at each point of X.

4.3.2 Remark. Note that $\psi : X \to Y$ being injective does not imply that $d\psi$ is injective at each point. For example, the smooth map $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(x) = x^3$ is injective, and yet $d\psi(0) = 0$. Note also that (X, ψ) being a submanifold of Y does not imply that ψ is homeomorphism of X onto $\psi(X)$ with the relative topology.

4.3.3 Definition. Let V be an n-dimensional real vector space. Then $\Lambda^n(V^*)$ has dimension 1, so it is isomorphic to $\mathbb{R}$. Thus $\Lambda^n(V^*) - \{0\}$ is disconnected; it is the union of two connected components. An *orientation* of V is a choice of one of these component. An *oriented vector space* is a pair $(V, \mathcal{A})$ where $\mathcal{A}$ is an orientation of V.

4.3.4 Remark. Thus each vector space V has two possible orientations. An ordered basis $\{\phi_1, \dots, \phi_n\}$ of V^* determines an orientation of V; namely, the component of $\Lambda^n(V^*)$ in which $\phi_1 \wedge \cdots \wedge \phi_n$ lies. Given two ordered bases $\{\phi_1, \dots, \phi_n\}$ and $\{\phi'_1, \dots, \phi'_n\}$ of V^*, with $\phi' = \sum c_{ji}\phi_j$, then $\phi'_1 \wedge \dots \phi'_n = \det(c_{ij})\phi_1 \wedge \cdots \wedge \phi_n$. Hence two ordered bases determine the same orientation if and only if the determinant of the change of basis matrix is positive.In particular, if $\{\phi_1, \dots, \phi_n\}$ is an ordered basis for V^*, then the orientation determined by the basis

$$\{\phi_2, \phi_1, \phi_3, \dots, \phi_n\}$$

is deferent from the determined by $\{\phi_1, \phi_2, \dots, \phi_n\}$.

In $\mathbb{R}^2$, an orientation amounts to a sense of rotation. The orientation determined by$\{dr_1, dr_2\}$ gives the usual sense of positive rotation on $\mathbb{R}^2$ namely, so that the rotation sending $\partial/\partial r_1$ into $\partial/\partial r_2$ is one of $+\pi/2$. The orientation determined by $\{dr_2, dr_1\}$ defines the opposite sense of rotation, so that $\partial/\partial r_2 \to \partial/\partial r_1$ is orientation of $+\pi/2$. Similarly, an orientation of $\mathbb{R}^3$ amounts to choosing either the right-handed rule or the left-handed rule for cross products.

4.3.5 Definition. A smooth manifold (X, Φ) is *orientable* if there exits a subset $\Phi' \subset \Phi$ such that

(1) $\{\text{domain } \phi\}_{\phi \in \Phi'}$ is a covering of X, and

(2) If ϕ_1 and ϕ_2 are coordinate system in Φ', with domains U and V and coordinate functions $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ respectively, then the function

$\lambda : U \cap V \to \mathbb{R}$ determined by

$$dx_1 \wedge \cdots \wedge dx_n = \lambda dy_1 \wedge \cdots \wedge dy_n$$

is everywhere positive.

An *orientation* of an orientable manifold (X, Φ) is a choice of subset $\Phi' \subset \Phi$ satisfying (1) and (2) and maximal with respect to (2). An *oriented manifold* is a triple (X, Φ, Φ'), where (X, Φ) is an orientable manifold and Φ' is an orientation of (X, Φ).

4.3.6 Remark. The function λ such that

$$dx_1 \wedge \cdots \wedge dx_n = \lambda dy_1 \wedge \ldots dy_n$$

is just the Jacobian determinant of $\phi_1 \circ \phi_2^{-1}$; that is,

$$\lambda = \det \left(\frac{\partial}{\partial y_j}(x_i) \right) = \det \ d(\phi_1 \circ \phi_2^{-1}).$$

In view of this, it is easy to check that a connected orientable manifold (X, Φ) has exactly two orientations Φ' and Φ'', and that Φ is the disjoint union $\Phi' \cup \Phi''$.

4.3.7 Remark. If $|K|$ is homeomorphic to a connected compact orientable 2-dimensional manifold, then it turns out that $\beta_0 = 1$ and $\beta_2 = 1$, so that

$$\chi(K) = \beta_0 - \beta_1 + \beta_2 = 2 - \beta_1$$

or

$$\beta_1 = 2 - \chi(K).$$

Furthermore, β_1 is always even for such K. It can be shown that any such surface is homeomorphic to a sphere with a certain number of "handles" attached; $\frac{1}{2}\beta_1$ is just the number of handles.

Thus the homology groups completely determine the homeomorphism class of connected compact orientable surfaces. However, for higher dimensional manifolds, the homology groups contain comparatively little information.

4.4 Simplicial cohomology

4.4.1 Remark. We have been discussing a *homology* theory for simplicial complexes, that is, a theory arising from a sequence of groups and homomorphisms

$$\cdots \xleftarrow{\partial} C_{l-1}(K,\mathbb{R}) \xleftarrow{\partial} C_l(K,\mathbb{R}) \xleftarrow{\partial} C_{l+1}(K,\mathbb{R}) \xleftarrow{\partial} \ldots,$$

where the map ∂ lowers the dimension of chains. On the other hand, in studying de Rham *cohomology*, we used a sequence

$$\cdots \xrightarrow{d} C^\infty(X,\Lambda^{l-1}(X)) \xrightarrow{d} C^\infty(X,\Lambda^l(X)) \xrightarrow{d} C^\infty(X,\Lambda^{l+1}(X)) \longrightarrow \ldots,$$

where the map d raises dimension (degree). In order to compare these two theories, it is convenient to define a *simplicial cohomology* theory. This is done by passing to dual spaces.

4.4.2 Definition. Let K be a simplicial complex. For $0 \leq l \leq \dim K$, let

$$C^l(K) = [C_l(K,\mathbb{R})]^*.$$

Let $\partial^* : C^l(K) \to C^{l+1}(K)$ be the adjoint of the map $\partial : C_{l+1}(K,\mathbb{R}) \to C_l(K,\mathbb{R})$. Thus ∂^* is defined by

$$[\partial^*(\phi)(c)] = \phi(\partial c) \qquad (\phi \in C^l(K);\ c \in C_{l+1}(K,\mathbb{R})).$$

Then we get a sequence

$$\ldots \longrightarrow C^{l-1}(K) \xrightarrow{\partial^*} C^l(K) \xrightarrow{\partial^*} C^{l+1}(K) \longrightarrow \ldots$$

Moreover, $\partial^* \circ \partial^* = 0$ since $\partial \circ \partial = 0$. Let

$$\begin{aligned} Z^l(K) &= \{\phi \in C^l(K) | \partial^*\phi = 0\}, \\ B^l(K) &= \{\partial^*\phi | \phi \in C^{l-1}(K)\}, \\ H^l(K) &= Z^l(K)/B^l(K). \end{aligned}$$

Element of $C^l(K)$ are called *cochains*; elements of $Z^l(K)$ are *cocycles*; elements of $B^l(K)$ are *coboundaries.* The map ∂^* is the *coboundary operator.* $H^l(K)$ is the lth *cohomology group* of K.

We shall need an explicit formula exhibiting the effect of the coboundary operator ∂^*. For each oriented l-simplex $< s >$ of K, let $\phi_{<s>} \in C^l(K)$ be defined by

$$\phi_{<s>} < t >= \begin{cases} 1 & (\text{if } < t >=< s >) \\ -1 & (\text{if } < t >= - < s >) \\ 0 & (\text{if } t \neq s). \end{cases}$$

Thus, if $\{< s_1 >, \ldots, < s_m >\}$ is a basis for $C_l(K, \mathbb{R})$ (so that $\{s_1, \ldots, s_m\}$ is the set of all l-simplices of K). then $\{\phi_{<s_1>}, \ldots, \phi_{<s_m>}\}$ is the dual basis for $C^l(K)$. Since ∂^* is linear, we need only compute the effect of ∂^* on these generators $\phi_{<s>}$.

4.4.3 Lemma.

$$\partial^* \phi_{<v_0,\ldots,v_l>} = \sum_v{}' \phi_{<v,v_0,\ldots v_l>},$$

where $\sum_v'$ *denotes the sum over all vertices* $v \in K$ *such that* $(v, v_0, v_1, \ldots, v_l)$ *is an* $(l+1)$*-simplex of* K.

Proof. We need only check this formula on oriented $(l+1)$-simplices

$$<t>=<w_0, w_1, \ldots, w_{l+1}>$$

of K. If we set $<s>=<v_0, v_1, \ldots, v_l>$, the left side yields

$$\begin{aligned}(\partial^* \phi_{<s>}(<t>) &= \phi_{<s>}(\partial <t>)\\ &= \phi_{<s>}\left(\sum_{i=0}^{l+1}(-1)^i < w_0, \ldots, \hat{w}_i, \ldots, w_{l+1} >\right)\\ &= \sum_{i=0}^{l+1}(-1)^i \phi_{<s>}(< w_0, \ldots, \hat{w}_i, \ldots, w_{l+1} >).\end{aligned}$$

But each term of this sum is zero unless, for some i, $(w_0, \ldots, \hat{w}_i, \ldots, w_{l+1}) = (s)$; that is, unless (s) is a face of (t). If (s) is a face of (t), then $(t) = (v, v_0, \ldots, v_l)$ for some vertex $v \in K$, in which case either

(1) $<t>=<v, v_0, \ldots, v_l>$ and $(\partial^* \phi_{<s>})(<t>) = 1$; or

(2) $<t>= - <v, v_0, \ldots, v_l>$ and $(\partial^* \phi_{<s>})(<t>) = -1$.

Thus

$$(\partial^* \phi_{<s>})(<t>) = \begin{cases} 1 & (\text{if } <t>=<v, v_0, \ldots, v_l> \text{ for some } v)\\ -1 & (\text{if } <t>= - <v, v_0, \ldots, v_l> \text{ for some } v)\\ 0 & \text{in all other cases)}\end{cases}$$

$$= \left(\sum_v{}' \phi_{<v, v_0, \ldots, v_l>}\right)(<t>).$$

Since this holds for arbitrary $<t>$, the formula is established. □

Chapter 5

Smooth and P.L. de Rham's theorems

(The materials of this chapter has been taken from [27] ,[14] and [35])

5.1 Smooth triangulation

5.1.1 Definition. A *smoothly triangulated manifold* is a triple (X, K, h), where X is a C^{∞} manifold, K is a simplicial complex, and $h : |K| \to X$ is a homeomorphism such that for each simplex s of K, the map $h|_{[s]} : [s] \to X$ has an extension h_s to a neighborhood U of $[s]$ in the plane of $[s]$ such that $h_s : U \to X$ is a smooth submanifold.

5.1.2 Remark. If dim $X = n$, we need only require that this last condition be satisfied for each n-simplex of K, since every simplex of K is a face of an n-simplex and since restrictions of smooth maps to submanifolds are smooth.

5.1.3 Example. Let $X = S^n$. Let K be the n-skeleton of an $(n+1)$-simplex circumscribed about S^n. Let $h : |K| \to S^n$ be the radial projection. Then (X, K, h) is a smoothly triangulated manifold.

5.1.4 Remark. It can be shown that every compact smooth manifold can be smoothly triangulated. The proof is difficult and will not be presented here. Note that smoothly triangulated manifolds are compact because $|K|$ is compact for each (finite) simplicial complex K.

The goal of this section is to show that for smoothly triangulated manifolds (X, K, h), the de Rham cohomology of X is isomorphic to the simplicial cohomology of K. For this, we shall need the following facts about barycentric coordinates.

5.1.5 Definition. Let K be a simplicial complex and let v be a vertex of K. The *star* of v is the point set

$$\mathrm{St}(v) = \bigcup_{\substack{v \in [s] \\ (s) \in K}} (s)$$

5.1.6 Definition. Let K be a simplicial complex and let $v_1, \dots, v_m$ denote the vertices of K. Suppose $x \in |K|$. For $j \in \{1, \dots, m\}$, the jth *barycentric coordinate* $b_j(x)$ of x is defined as follows. If $x \notin St(v_j)$, then $b_j(x) = 0$; if $x \in St(v_j)$, then $x \in (s)$ for some simplex s having v_j as a vertex, and $b_j(x)$ is equal to the barycentric coordinate of x in s relative to the vertex v_j.

5.1.7 Remark. The following facts are easily verified.

(1) $b_j : |K| \to \mathbb{R}$ is a continuous function.

(2) $b_j(x) \geq 0$ and $\sum_{j=1}^{m} b_j(x) = 1$ for each $x \in |K|$.

(3) $x = \sum_{j=1}^{m} b_j(x) v_j$.

(4) $b_{j_0}(x) \neq 0,\ b_{j_1} \neq 0, \dots b_{j_l}(x) \neq 0$ for some $x \in |K|$ if and only if $v_{j_0}, \dots, v_{j_l}$ are the vertices of an l-simplex of K.

5.1.8 Definition. Let K be a simplicial complex, and let s be a simplex of K. The *star* of s is the union of all the open simplices (t) of K such that (s) is a face of (t).

5.1.9 Remark. (1) For $s = v$ a 0-simplex (i.e., a vertex) of K, $St(s) = St(v)$, as defined above.

(2) St(s) is an open set in $|K|$. (This is an elementary consequences of (3).)

(3) If $(s) = (v_{j_0}, \dots, v_{j_l})$ and $x \in |K|$, then $x \in St(s)$ if and only if $b_{j_i}(x) \neq 0\ \forall i = 0, 1, \dots l$

(4) If $(s) = (v_{j_0}, \dots, v_{j_l})$, then

$$|K| - St(s) = \{x \in |K|; b_{j_i}(x) = 0 \text{ for some } i \in \{0, \dots, l\}\}.$$

(5) If s_1 and s_2 are l-simplices of K with $s_1 \neq s_2$, then $[s_1] \subset |K| - St(s_2)$.

Given a smoothly triangulated manifold (X, K, h), we want to define, for each l, an isomorphism from $H^l(X, d)$ onto $H^l(X)$. To do this, note that homomorphisms $\widetilde{f}_l : H^l(X, d) \to H^l(K)$ are defined whenever there is given a sequence of linear maps $f_l : C^\infty(X, \Lambda^l(X)) \to C^l(K)$ such that $\partial^* \circ f_l = f_{l+1} \circ d$ for all l.

$$\begin{tikzcd}
\cdots \arrow[r] & C^\infty(X, \Lambda^l(X)) \arrow[r, "d"] \arrow[d, "f_l"] & C^\infty(X, \Lambda^{l+1}(X)) \arrow[r] \arrow[d, "f_{l+1}"'] & \cdots \\
\cdots \arrow[r] & C^l(K) \arrow[r, "\partial^*"] & C^{l+1}(K) \arrow[r] & \cdots
\end{tikzcd}$$

For then $f_l(Z^l(X, d)) \subset Z^l(K)$, because $d\omega = 0$ ($\omega \in C^l(X, d)$) implies that

$$\partial^*(f_l(\omega)) = f_{l+1}(d\omega) = f_{l+1}(0) = 0.$$

Also $f_l(B^l(X,d)) \subset B^l(k)$, because $\omega = d\tau$ ($\tau \in C^{l-1}(X,d)$) implies that

$$f_l(\omega) = f_l(d\tau) = \partial^*(f_{l-1}\tau) \in \operatorname{Im} \partial^*.$$

Thus f_l induces

$$\tilde{f}_l : H^l(X,d) = Z^l(X,d)/B^l(X,d) \to Z^l(k)/B^l(K) = H^l(K).$$

5.2 de Rham's theorem for smooth manifolds

We now proceed to define such a sequence of linear maps

$$\int_l : C^\infty(X, \Lambda^l(X)) \to C^l(K).$$

For $\omega \in C^\infty(X, \Lambda^l(X))$, $\int_l(\omega)$ will be a linear functional on $C_l(K)$. Thus it suffices to specify the values of $\int_l(\omega)$ on basis elements of $C_l(K)$, that is, on oriented l-simplices $< s >$. To do this, consider the smooth map $h_s : U \to X$. Then $h_s^*(\omega)$ is smooth l-form on U, an open set in the plane of $[s]$; that is, in an l-dimensional Euclidean space. We define $\int_l(\omega)(< s >)$ to be the integral of this l-form over $< s >$:

$$\int_l(\omega)(< s >) = \int_{<s>} h_s^*(\omega).$$

In other words, let $(r_1, \ldots, r_l)$ denote coordinates in the plane of $[s]$ consistent with the orientation of $< s >$; so if $< s > = < v_0, \ldots, v_l >$, let $(r_1, \ldots, r_l)$ be coordinates relative to the ordered basis $\{v_l - v_0, \ldots, v_l - v_0\}$. Then

$$h_s^*(\omega) = g dr_1 \wedge \cdots \wedge dr_l$$

for some continuous function g on U, and

$$\int_l(\omega)(< s >) = \int_{[s]} g \, dr_1 \ldots dr_l \qquad \text{(Riemann integral).}$$

Note that this integral is independent of the homeomorphism h; that is, it depends only on the point set $h([s])$ and its orientation by the change of variables for integrals.

$$Claim: \quad \partial^* \circ \int_l = \int_{l+1} \circ d.$$

This is just Stokes' theorem. For given any smooth l-form ω and oriented $(l+1)$-simplex $< s >$

$$\begin{aligned}
[\int_{l+1} \circ d(\omega)](< s >) &= \int_{<s>} (h_s)^*(d\omega) \\
&= \int_{<s>} d(h_s^*(\omega)) \\
&= \int_{\partial<s>} h_s^*(\omega) \quad \text{(By Stokes' theorem)} \\
&= \int_l (\omega)(\partial < s >) \\
&= [\partial^* \circ \int_l (\omega)] < s > .
\end{aligned}$$

Thus $\int_l$ induces a homomorphism $\widetilde{\int_l} : H^l(X, d) \to H^l(K)$.

5.2.1 Theorem. *(de Rham's Theorem). Let (X, K, h) be a smoothly triangulated manifold. Then*

$$\widetilde{\int_l} : H^l(X, d) \to H^l(K)$$

is an isomorphism for each l $(0 \leq l \leq dimX)$.

5.3 de Rham's theorem for simplicial complexes.

5.3.1 Definition. (Piecewise linear forms (P.L.forms))

We will work on a simplicial complex K. Note that K is the union of n-simplices Δ^n, where Δ^n may be thought of as

$$\Delta^n = \{(t_0, \dots, t_n) : \ 0 \leq t_i \leq 1, \ \sum_{i=0}^{n} t_i = 1\}$$

Note that $\partial\Delta^n$ is a union of $(n-1)$-simplices, and is topologically an S^{n-1}.

Consider the restriction to Δ^n of all forms in $\mathbb{R}^{n+1}$ of the form

$$\sum \phi_{i_1,\dots,i_j} dt_{i_1} \wedge \cdots \wedge dt_{i_j}$$

where the $\phi_{i_1 \dots i_j}$ are polynomials in t_i's with $\mathbb{Q}$-coefficients. There is the relation $\sum_{i=0}^{n} t_i = 1$ and the derived relation is $dt_0 + \cdots + dt_n = 0$. We call this algebra $A^*(\Delta^n)$. If $\Delta^k \subset \Delta^n$ is a face, then there is a restriction map $A^*(\Delta^n) \to A^*(\Delta^k)$.

Let K be a simplicial complex. Define

$$A^*(K) = \{(\omega_\sigma)_{\sigma\in K} : \omega_\sigma \in A^*(\sigma) \quad \text{and} \quad \omega_\sigma|\tau = \omega_\tau \ \text{ if } \tau \text{ is a face of } \sigma\}$$

Thus, $A^*(K)$ is the collection of forms (called P.L. forms), one on each simplex of K, which are compatible under restriction to faces. Clearly, wedge product and d, both defined by the corresponding operations on each simplex, give $A^*(K)$ the structure of a differential graded algebra (D.G.A). It is defined over $\mathbb{Q}$.

There is a map:

$$A^*(K) \xrightarrow{\rho} C^*(K; \mathbb{Q})$$

defined by $< \rho(\omega), \Delta^n >= \int_{\Delta^n} \omega$. This map is a map of cochain complexes by Stokes' theorem (which is valid in our setting).

5.3.2 Theorem. *(P.L. de Rham Theorem): ρ induces an algebra isomorphism on cohomology.*

We will deduce the P.L.D.R.T. from the following propositions.

5.3.3 Proposition. *(i) Let $\phi \in A^n(k)$ satisfy $d\phi = 0$, $\rho(\phi) = 0$ (i.e. $\int_{\Delta^n} \phi = 0$ for all Δ^n). Then there exists $\psi \in A^{n-1}(k)$ such that $d\psi = \phi$, $\rho(\psi) = 0$.*
(ii) $A^*(K) \xrightarrow{\rho} C^*(K)$ is onto.

Proof. (of P.L.D.R.T)(additive statement)

We have by (ii)

$$0 \longrightarrow B^*(K) \longrightarrow A^*(K) \xrightarrow{\rho} C^*(K) \longrightarrow 0.$$

The first part of the proposition says that

$$H^*(B^*(K)) = 0,$$

and thus

$$H^*_{DR}(K) \cong H^*(K; \mathbb{Q}).$$

The multiplicative statement will be considered later. ☐

We also sketch alternative proofs of smooth and P.L. de Rham,s theorem as given by [35]

Proof. Sketch of Weil's proof (for differentiable manifolds V). Construct a locally finite simple open covering $\mathcal{U} = (U_i)_{i \in I}$ of V with associated partion of unity (f_i) and nerve N. Define a differential coelement of degree (m, p) as any system $\Omega = (\omega_H) = (\omega_{i_0 \ldots i_r})$ of EDFs of degree m in $U_{|H|} = \cap_{o \leq v \leq p} U_{i_r} \neq \phi$

attached to the sequences $H = (i_0, \ldots, i_p) \subset I$, which depend alternatively on the indices $i_0, \ldots, i_p$. To Ω we have naturally two operators d and δ with $d\Omega$ and $\delta\Omega$ coelements of bidegrees $(m+1, p)$ and $(m, p+1)$ respectively. The retraction of each $U_H \neq \phi$ defines an operator I_H in U_H such that $I\Omega = (I_H \omega_H)$ is a coelement of bidegree $(m-1, p)$ satisfying the relation $\Omega = Id\Omega + dI\Omega$ for $m > 0$, with a similar relation for $m = 0$. Again, for any set $J' = J \cup \{i\} \subset I$ with $U_{J'} \neq \phi$ and ω_J an EDF in U_J, let us denote by $f_i\omega$ the EDF in U_J equal to $f_i\omega$ in $U_{J'}$, and to 0 in $U_J - U_{J'}$. The partition of unity $\{f_i\}$ defines now an operator K which associates to each coelement $\Omega = (\omega_H)$ of bidegree (m, p) with $p > 0$ the coelement $K\Omega = (\zeta_{i_0 \ldots i_{p-1}})$ of bidegree $(m, p-1)$ with

$$\zeta_{i_0 \ldots i_{p-1}} = \sum f_k \omega_{k i_0 \ldots i_{p-1}}, \tag{5.3.3}$$

the $\sum$ being extended over such $k \in I$ with $U_{(k i_0 \ldots i_{p-1})} \neq \phi$. Similarly $K\Omega$ is defined for Ω of bidegree $(m, 0)$. We have then $\Omega = K\delta\Omega + \delta K\Omega$.

It may be then easily be seen that for a closed EDF ω on V of degree m, $\Xi = \delta(I\delta)^m \omega$ will be a cocycle of dimension m in N, and conversely for a cocycle $\Xi = (\zeta_{i_0} \ldots i_m)$ of dimension m in N, $\omega = K(dK)^m \Xi$ will be a closed EDF of degree m on V. The correspondence between ω and Ξ will be establish additive isomorphism between $H(A^*(V))$ and $H^{**}_R(V)$. It is easy to varify that the isomorphism is also multiplicative as asserted. □

The above proof can be easily modified to the general case of complexes:

Proof. Proof of extended de Rham-Sullivan theorem for complexes. The complex K has a natural simple covering $\mathcal{U}$ consisting of open stars U_i of vertices v_i of K, so that the nerve of $\mathcal{U}$ concides with K. We may then define elements of bidegree (m, p) as well as operator I as before. To define the operator K, we may replace the partition of unity $\{f_i\}$ in the following manner. For each vertex v_i let t_i be the function in U_i which takes on the value $t_i = x_i$ for any point in barycentric coordinates $x_i v_i + \sum_{k=1}^{p} x_{jk} v_{jk}$ in a p-simplex of vertices $v_i, v_{j_1}, \ldots, v_{j_p}$. K is then again defined by 5.3.3 with f_k replaced by t_k. The proof then runs as before. $\square$

5.3.4 Remark. The above proof shows that the de Rham-Sullivan theorem remains true if we consider only EDFs $\sum \alpha_{i_0 \ldots i_p} dx^{i_0} \ldots dx^{i_p}$ in a simplex for which $\alpha_{i_0 \ldots i_p}$ are polynomials in the barycentric coordinates x^i with *rational* coefficients.

Chapter 6

Spectral sequences, obstruction theory, Postnikov towers & rationalization

(The materials of this chapter has been adapted from [14])

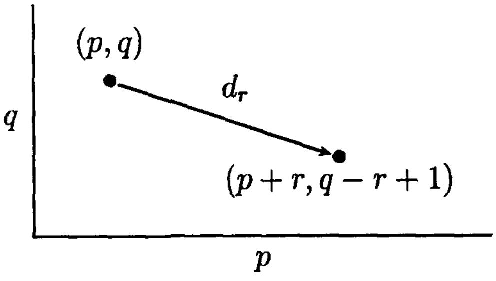
6.1 Spectral sequence of a fibration

6.1.1 Definition. A (first quadrant) spectral sequence consists of abelian groups $E_r = \{E_r^{p,q}\}_{p,q\geq 0}$ $(r \geq 0)$ and maps $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ with $d_r^2 = 0$ such that the homology $H(E_r) = E_{r+1}$. In details:

$$E_{r+1}^{p,q} = \frac{\ker\, d_r : E_r^{p,q} \to E^{p+r,q-r+1}}{\operatorname{im}\, d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}}.$$

6.1.2 Remark. The most important thing about spectral sequence is to have in mind the picture. For each term E_r, we plot the first quadrant in the

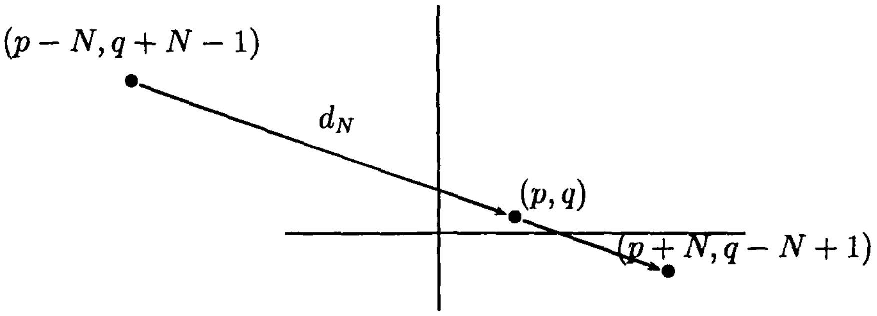
(p,q) plane and put in the group $E_r^{p,q}$ at the lattice point (p,q). Then the differentials map according to the picture



An element $\alpha \in E_r^{p,q}$ is said to *live to infinity* if $d_r\alpha = 0$ (and thus α defines an element in $E_{r+1}^{p,q}$), $d_{r+1}\alpha = 0, \dots$, etc. An element $\beta \in E_r^{p,q}$ is said to be *killed* if $d_r\beta = \cdots = d_{s-1}\beta = 0$ but $\beta = d_s\gamma$ for some $\gamma \in E^{p-s,q+s-1}$. The spectral sequence is said to *degenerate at* E_r if $E_r \cong E_{r+1} \cong \dots$ etc; it is said to be *degenerate* if $E_2 \cong E_3 \cong E_4 \cong \dots$. Spectral sequences are a necessary and useful tool.

6.1.3 Lemma. *If* $N > p, q+1$ *then* $E_N^{p,q} \cong E_{N+1}^{p,q}$.

Proof. In the picture we find



so that $d_N \equiv 0$ on $E_N^{p,q}$. □

6.1.4 Definition. We set $E_\infty^{p,q} = E_N^{p,q}$ for $N > p, q+1$ and $E_\infty^n = \oplus_{p+q=n} E_\infty^{p,q}$.

6.1.5 Lemma. *Given a spectral sequence* $\{E_r^{p,q}\}$, *then*

$$0 \longrightarrow E_2^{1,0} \longrightarrow E_\infty^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow E_\infty^2$$

is exact.

6.1.6 Theorem. *(Leray-Serre): Let $F \to E \to B$ be a fibration in which B is a connected CW complex with $\pi_1(B)$ acting trivally on the cohomology of the fiber. Then there is a spectral sequence $\{E_r^{p,q}\}$ in which*

$$\begin{aligned} E_1^{p,q} &= C^p(B, H^q(F)) \text{ and } d_1 = \text{coboundary map of } C^*(B, H^q(F)); \\ E_2^{p,q} &= H^p(B, H^q(F)); \text{ and} \\ E_N^{p,q} &= \frac{\ker\{H^{p+q}(E) \to H^{p+q}(E^{(p-1)})\}}{\ker\{H^{p+q}(E) \to H^{p+q}(E^{(p)})\}}, \quad N > p, q+1 \end{aligned}$$

6.1.7 Remarks. If we define

$$\mathcal{F}^p H^{p+q} = \ker\{H^{p+q}(E) \to H^{p+q}(E^{(p-1)})\}$$

then clearly

$$H^n(E) = \mathcal{F}^0 H^n(E) \supset \mathcal{F}^1 H^n(E) \supset \cdots \supset \mathcal{F}^{n+1} H^n(E) = 0,$$

sothat the groups $\mathcal{F}^p H^n(E)$ give a filtraion on $H^n(E)$ whose associated graded module is $\oplus_{p+q=n} E_\infty^{p,q}$. This is usally written as

$$E_r^{p,q} \Longrightarrow H^{p+q}(E)$$

and one says that the spectral sequence converges (or abuts) to $H^*(E)$.

6.2 Obstruction theory

Let (X, A) be a CW pair. Denote by $X^{(n)} \cup A$ the union of A and all the cells of $X - A$ of dimension $\leq n$. Suppose given $f_n : X^{(n)} \cup A \to Y$. We define the obstruction cochain $\widetilde{\mathcal{O}}(f_n) \in C^{n+1}(X, A, \pi_n(Y))$ as follows: If e_α^{n+1} is an oriented $(n+1)$-cell of (X, A) then its attaching map $c_\alpha : S^n \to X^{(n)} \cup A$ composed with f_n gives $f_n \circ c_\alpha : S^n \to Y$, which determines an element of $\pi_n(Y)$. If we reverse the orientation on e_n^{n+1} (and hence on ∂e_α^{n+1}), then the resulting element in $\pi_n(Y)$ changes sign. Thus, there is a well-defined homomorphism $C_{n+1}(X, A) \to \pi_n(Y)$. We denote it by $\widetilde{\mathcal{O}}(f_n)$ and call it the *obstruction cochain.*

Properties of $\widetilde{\mathcal{O}}(f_n)$:

(1) It is an invariant of the homotopy class of f_n.

(2) It is 0 if, and only if, f_n extends to a map $f_{n+1} : X^{n+1} \cup A \to Y$.

(3) It is a cocycle; i.e., $\delta\widetilde{\mathcal{O}}(f_n) : C_{n+2}(X, A) \to \pi_n(Y)$ is 0.

(4) If $g_n : X^{(n)} \cup A \to Y$ agrees with f_n on $X^{(n-1)} \cup A$, then $\widetilde{\mathcal{O}}(g_n) - \widetilde{\mathcal{O}}(f_n)$ is a coboundary.

(5) By varying the homotopy class of f_n, relative to $X^{(n-1)} \cup A$, we can change $\widetilde{\mathcal{O}}(f_n)$ by an arbitrary coboundary.

6.2.1 Theorem. *Given $f_n : X^{(X)} \cup A \to Y$ with $\pi_1(Y) = 0$, there is a cohomology class $\mathcal{O}(f_n) \in H^{n+1}(X, A; \pi_n(Y))$ constructed from the cocycle $\widetilde{\mathcal{O}}(f_n)$. This class vanishes if and only if $f_n|_{X^{(n-1)} \cup A} : X^{(n-1)} \cup A \to Y$ can be extended to a map $f : X^{(n+1)} \cup A \to Y$. Let f and $g : X \to Y$ be given and $H : (X^{(n)} \cup A) \times I \to Y$ be a homotopy between $f|X^{(n)} \cup A$ and $g|X^{(n)} \cup A$. The obstruction to extending the homotopy over $(X^{(n+1)} \cup A) \times I$*

lies in $H^{n+2}(X \times I, ((X \times \{0 \cup 1\}) \cup A \times I); \pi_{n+1}(Y))$.

By the suspension isomorphism, this group is

$$H^{n+1}(X, A; \pi_{n+1}(Y)).$$

Eilenberg-MacLane space $K(\pi, n)$: Given $n \geq 2$ and an abelian group π, up to homotopy equivalence there is exactly one CW complex $K(\pi, n)$ such that

$$\pi_i(K(\pi, n)) = \begin{cases} 0 & i \neq n \\ \pi & i = n \end{cases}$$

6.2.2 Theorem. *Let* (X, A) *be a CW-pair, and let* $f : A \to Y$ *be given. Suppose* $H^i(X, A; \pi_{i-1}(Y)) = 0$ *and* $H^{i-1}(X, A; \pi_{i-1}(Y)) = 0$ *for* $i \leq n$. *Also suppose* $\pi_i(Y) = 0$. *The first obstruction to extending* f *over* X, $\mathcal{O} \in H^{n+1}(X, A, \pi_n(Y))$ *is well-defined. It is natural with respect to maps* $\pi : (X', A') \to (X, A)$.

6.2.3 Example. Given $f : S^k \to Y$ and $g : S^l \to Y$, form $f \vee g : S^k \vee S^l \to Y$. The only obstruction to extending $f \vee g$ to a map $S^k \times S^l \to Y$ is an element in $H^{k+l}(S^k \times S^l, S^k \vee S^l; \pi_{k+l-1}(Y)) = \pi_{k+l-1}(Y)$. Since this obstruction is primary it is well defined. The obstruction, denoted $[f, g]$, is the *Whitehead product* of f and g.

There is an analogous theorem for lifting maps in a fibration.

6.2.4 Theorem. *Let* $\pi : E \to X$ *be a fibration with fiber* F. *Suppose that* $\pi_1(X)$ *acts trivially on* F *and that* $\sigma : A \to E|A$ *is a section of* π *over* A. *If* $H^i(X, A; \pi_{i-1}(F)) = 0$ *and* $H^{i-1}(X, A; \pi_{i-1}(F)) = 0$ *for* $i \leq n$, *then the first obstruction to extending* σ *on* X *lies in* $H^{n+1}(X, A; \pi_n(F))$. *It is well defined and natural.*

6.3 Relation of cohomology and Eilenberg-MacLane spaces

. There is a natural transformation $[(X,A),(K(\pi,n),*)] \to H^n(X,A;\pi)$ which assigns to any map $f:(X,A)\to(K(\pi,n),*)$ the primary obstruction to deforming f to a constant map relative to A. (Here, $*$ is the base pont of $K(\pi,n)$.) Actually, this should be viewed as an extension problem for the map $f\cup c\cup c: X\times\{0\}\cup A\times I\cup X\times\{1\}\to K(\pi,n)$ (where c denotes the constant map). Thus, the primary obstruction is well defined and lies in

$$H^{n+1}(X\times I, X\times\{0\}\cup A\times I\cup X\times\{1\};\pi)\cong H^n(X,A;\pi).$$

By the Hurewicz theorem $H_n(K(\pi,n))=\pi$ and $H_{n-1}(K(\pi,n))=0$. Thus, by the universal coefficient theorem, $H^n(K(\pi,n);\pi)=\mathrm{Hom}(\pi,\pi)$. Here, we are viewing $K(\pi,n)$ as a space together with an identification of $\pi_n(K(\pi,n))$ with π. Let $\iota\in H^n(K(\pi,n);\pi)$ be the class corresponding to the identity homomorphism. If $f:(X,A)\to(K(\pi,n),*)$, then we have $f^*\iota\in H^n(X,A;\pi)$. This defines a function

$$i:[(X,A),(K(\pi,n),*)]\to H^n(X,A,\pi).$$

6.3.1 Theorem. *$f^*\iota$ is the primary obstruction to deforming f to a constant relative to A. The association $[f]\to f^*\iota$ is a bijection for all CW-pairs (X,A).*

6.4 Principal $K(\pi,n)$-fibrations

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6.4.1 Definitions. A map $p : E \to B$ is said to be a $K(\pi, n)$-fibration if it satisfies the homotopy lifting property and if all fibers, $p^{-1}(b)$, are spaces of type $K(\pi, n)$. If B is path connected, then $p : E \to B$ is a $K(\pi, n)$-fibration provided that $p^{-1}(b)$ is a space of type $K(\pi, n)$ for some $b \in B$.

A $K(\pi, n)$-fibration is said to be principal if the action of the fundamental group of the base on the fiber is trivial up to homotopy. For each loop γ in the base B at b there is a self-homotopy equivalence $\gamma_* : p^{-1}(B) \to p^{-1}(b)$ well defined up to homotopy. The fibration is principal if all these self-equivalence are homotopic to the identity.

6.4.2 Lemma. *Let (X, A) be a CW pair, let $p : E \to X$ be a principal $K(\pi, n)$ fibration, and let $\sigma : A \to E$ be a section of E over A. There is a unique obstruction $\mathcal{O}(p, \sigma) \in H^{n+1}(X, A; \pi)$ to extending σ over all of X. Given any class $\mathcal{O} \in H^{n+1}(X, A; \pi)$ it is realized as the obstruction $\mathcal{O}(p, \sigma)$ for some principal fibration $p : E \to X$ and some section $\sigma : A \to E$.*

Proof. Since π_i (fiber) $=0$ for $i < n$, according to 6.2.4 the first obstruction to extending the section lies in $H^{n+1}(X, A; \pi)$. It is well defined and natural. Since all the higher homotopy groups of the fiber vanish $\mathcal{O}(p, \sigma)$ is the unique obstruction to extending the fiber over all of X. Given a class $\mathcal{O} \in H^{n+1}(X, A; \pi)$ there is a map

$$f_{\mathcal{O}} : (X, A) \longrightarrow (K(\pi, n+1), *)$$

so that $f^*_{\mathcal{O}}(\iota) = \mathcal{O}$. Over $K(\pi, n+1)$ we have the principal fibration

$$\begin{tikzcd}
K(\pi, n) \cong \Omega K(\pi, n+1) \arrow[r] & \mathcal{P}K(\pi, n+1) \arrow[d] \\
 & K(\pi, n+1)
\end{tikzcd}$$

Since $K(\pi, n+1)$ is simply connected this fibration is principal. If we pull back the fibration by f_O, then we get a principal $K(\pi, n+1)$ fibration over X. Any section over $*$, pulls back to a section over A. The obstruction to extending the section over $* \in K(\pi, n+1)$ to one over all of $K(\pi, n+1)$ is $\iota \in H^{n+1}(K(\pi, n+1), \pi)$. By naturality the obstruction to extending the induced section over A to one over all of X is $f_O^*(\iota) = \mathcal{O}$. This proves all classes arise as obstructions. $\square$

The results of this section are summarized in the following commutative diagram.

$$\begin{tikzcd}
{[(X,A),(K(\pi,n+1),)]} \arrow[rr, "\text{1st obstruction to deforming to a constant map}"] \arrow[dr, "\text{pull back the universal}", "\text{principal } K(\pi,n) \text{ fibration}"'] & & H^{n+1}(X,A;\pi) \\
& \{\text{Principal } K(\pi,n)\text{-fibrations over X with a section over A }\} \arrow[ur, "\text{1st obstruction}", "\text{to extending the section}"'] &
\end{tikzcd}$$

The map in the upper right hand corner is a bijection.

6.5 Postnikov towers

Given a simply connected space X, define $X_2 = K(\pi_2(X), 2)$ and define $f_2 : X \to X_2$ to be a map inducing the identity on π_2. Suppose inductively that we have

$$\begin{tikzcd}
& X_n \arrow[d] \\
& \vdots \arrow[d] \\
& X_4 \arrow[d] \\
X \arrow[uuur, "f_n"] \arrow[ur, "f_4"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & X_3 \arrow[d] \\
& X_2
\end{tikzcd}$$

a commutative diagram with:

(1) $\pi_i(X_j) = 0$ for $i > j$,

(2) $X_j \to X_{j-1}$ a principal fibration induced by some map $k^{j+1} : X_{j-1} \to K(\pi_j(X), j+1)$, and

(3) $f_j : X \to X_j$ an isomorphism on π_i for all $i \leq j$.

Consider $f_j : X \to X_j$ an inclusion. The relative homology $H_i(X_n, X)$ is a zero for $i \leq n+1$. Furthermore, $H_{n+2}(X_n, X) \cong \pi_{n+2}(X_n, X) \xrightarrow{\sim} \pi_{n+1}(X)$. By the universal coefficient theorem, $H^{n+2}(X_n, X; \pi_{n+1}(X)) =$ Hom $(\pi_{n+1}(X), \pi_{n+1}(X))$. Let $\tilde{k}^{n+1} \in H^{n+2}(X_n, X; \pi_{n+1}(X))$ be the class coressponding to the identity homomorphism. This determines a principal fibration and a lifting of f_n

$$\begin{tikzcd}
& K(\pi_{n+1}(X), n+1) \arrow[r] & X_{n+1} \arrow[d] \\
X \arrow[urr, "f_{n+1}"] \arrow[rr, "f_n"'] & & X_n
\end{tikzcd}$$

6.5.1 Lemma. *The map $f_{n+1} : X \to X_{n+1}$ meets conditions (1), (2),, (3) above.*

Remarks: · (1) For each n let X'_n be the CW complex obtained from X by inductively attaching cells of dimension $\geq n+1$ so as to kill homotopy groups in dimensions $\geq n+1$. The inclusion $X \subset X_n$ induces an isomorphism on π_i for $i \leq n$. If $f_n : X \to X_n$ is the n^{th} stage of a Postinkov tower for X, then a simple application of obstruction theory shows that f_n extends to a map $f'_n : X'_n \to X_n$. This map induces an isomorphism on all homotopy groups.

If $g_n : X \to Y_n$ is the n^{th} stage of another Postnikov tower for X, then we have a commutative diagram

$$\begin{tikzcd}
 & X_n \\
X \arrow[ur, "f_n"] \arrow[r, hook, "f'_n"'] \arrow[dr, "g_n"'] & X'_n \arrow[u] \arrow[d] \\
 & Y_n
\end{tikzcd}$$

with both vertical arrows being weak homotopy equivalence. It is easy to see that for the resulting identifications of $H^{n+2}(X_n, X; \pi_{n+1}(X))$ with $H^{n+2}(Y_n, X; \pi_{n+1}(X))$ the k-invariants for the $(n+1)^{st}$-stages of the two towers correspond. It is in this sense that the Postnikov tower is unique.

(2) If we form $\varprojlim\{X_i\}$, defined as the subspace of $\Pi_{i=2}^{\infty} X_i$ consisting of all comptible sequences, then the maps $\{f_n : X \to X_n\}$ determines $\varprojlim f_n : X \to \varprojlim\{X_i\}$. This map induces an isomorphism on all homotopy groups.

To prove this, one shows that $\pi_j(\varprojlim X_n) = \varprojlim \pi_j(X_n) = \pi_j(X_N)$ for $N \geq j$. It is not true in general for inverse systems that taking homotopy groups commutes with taking inverse limits. It is, however, true for this inverse system since $\pi_{j+1}(X_N) \to \pi_{j+1}(X_{N-1})$ is onto for all N.

If Y is a CW complex and $\phi : Y \to \varprojlim X_n$ induces an isomorphism on all homotopy groups, then the obstructions to lifting, up to homotopy, $\varprojlim f_n : X \to \varprojlim X_n$ to Y lie in $H^*(X; \pi_*(\varprojlim X_n, Y)) = 0$. Thus there is a map $\psi : X \to Y$ such that $\phi \circ \psi$ is homotopic to $\varprojlim f_n$. In particular ψ induces an isomorphism on all homotopy groups, and hence ψ is a homotopy equivalence.

This shows how to recover X, up to homotopy equivalence, from a Postnikov tower: it is the unique CW complex, up to homotopy equivalence, which maps to $\varprojlim X_n$ inducing an isomorphism on the homotopy groups.

6.6 Rational homotopy theory for simply connected spaces

We begin with a little of the theory of $\mathbb{Q}$ and $\mathbb{Q}$-vector spaces. Let A be an abelian group (usually infinitely generated). Then A may be given the structure of a $\mathbb{Q}$ vector space if and only if $A = A \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\sim} A \otimes_{\mathbb{Z}} \mathbb{Q}$. (this is equivalent to the equation $\alpha x = \beta$ having a unique solution for $\alpha \in \mathbb{Z} - \{0\}$ and $\beta \in A$.)

If $0 \to A_1 \to A_2 \to A_3 \to 0$ is a short exact sequence then so is $0 \to A_1 \otimes_{\mathbb{Z}} \mathbb{Q} \to A_2 \otimes_{\mathbb{Z}} \mathbb{Q} \to A_3 \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$.

6.6.1 Lemma. *(a) If* $0 \to A_1 \to A_2 \to A_3 \to 0$ *is a short exact sequence,* *then if two of the three terms are* $\mathbb{Q}$*-vector spaces so is the third.*

(b) If A *is an abelian group and has a composition series* $A = A_0 \supset A_1 \cdots \supset A_n \supset 0$ *with successive quotients* $\mathbb{Q}$*-vector spaces, then* A *is a* $\mathbb{Q}$*-vector space.*

(c) $H_*(X;\mathbb{Q}) \cong H_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$; $H^*(X;\mathbb{Q}) \cong Hom_{\mathbb{Z}}(H_*(X);\mathbb{Q}) \cong [H_*(X;\mathbb{Q})]^*$.

(d) If $\widetilde{H}_*(X)$ *is a* $\mathbb{Q}$ *- vector space, then* $\widetilde{H}_*(X;G)$ *is a* $\mathbb{Q}$ *vector space for many abelian group* G.

6.6.2 Corollary. $H_i(X;\mathbb{Q})$ *and* $H^i(X;\mathbb{Q})$ *are* Q*-vector spaces.*

6.6.3 Definition. A $\mathbb{Q}$*-space* is a space X satisfying:

(1) X is homotopy equivalent to a CW complex (generally having infinitely many cells in each dimension).

(2) $\pi_1(X) = 0$.

(3) $\pi_*(X)$ is a $\mathbb{Q}$-vector space for all $* \geq 1$.

Alternative definition: A space X satisfying (1) and (2) above is a Q-space if in addition (3) $\widetilde{H}_*(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space.

6.6.4 Theorem. *The two definitions of a* $\mathbb{Q}$*-space are equivalent.*

To proof it we need the following lemma.

6.6.5 Lemma. *(a)* $\widetilde{H}_*(K(\mathbb{Q},n);\mathbb{Z})$ *is a* $\mathbb{Q}$*-vector space.*

(b) $H^*(K(\mathbb{Q},2n);\mathbb{Q})$ *is a* $\mathbb{Q}$*-polynomial algebra on one generator of degree* $2n$.

(c) $H^*(K(\mathbb{Q},2n+1);\mathbb{Q})$ *is a* $\mathbb{Q}$*-exterior algebra on one generator of degree* $2n+1$.

6.6.6 Corollary. $K(\mathbb{Z},n) \to K(\mathbb{Q},n)$ *induces an isomorphism on rational cohomology, and thus on rational homology.*

6.6.7 Theorem. *Let X and $X_{(0)}$ be simply connected CW complexes with $X_{(0)}$ a $\mathbb{Q}$-space and $f : X \to X_{(0)}$. The following three conditions are equivalent:*

(a) $f_* : \widetilde{H}_*(X, \mathbb{Q}) \to \widetilde{H}_*(X_{(0)}; \mathbb{Q}) = \widetilde{H}_*(X_{(0)})$ *is an isomorphism.*

(b) $f_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(X_{(0)}) \otimes \mathbb{Q} = \pi_*(X_{(0)})$ *is an isomorphism.*

(c) *f is universal for maps of X into $\mathbb{Q}$-spaces; i.e., given $g : X \to Y_{(0)}$ with $Y_{(0)}$ a $\mathbb{Q}$-space, then g factors uniquely upto homtopy through $X_{(0)}$*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Y_{(0)} \\
X_{(0)} \arrow[ur, "h"'] &
\end{tikzcd}$$

6.6.8 Definition. Given X and $f : X \to X_{(0)}$ with $X_{(0)}$ a $\mathbb{Q}$-space and f satisfying any one the (and hence all of the) conditions of Theorem 6.6.7 above (and hence all of them), we call $f : X \to X_{(0)}$ the localization at 0 of X.

6.6.9 Theorem. *If $\phi : X \to X_{(0)}$ and $\phi' : X \to X'_{(0)}$ are localizations of X, then there is a homotopy equivalence $h : X_{(0)} \to X'_{(0)}$ such that*

$$\begin{tikzcd}
& X_{(0)} \arrow[dd, bend left, "h"] \\
X \arrow[ur, "\phi"] \arrow[dr, "\phi'"'] & \\
& X'_{(0)} \arrow[uu, bend left, "h^{-1}"]
\end{tikzcd}$$

is a homotopy commutative diagram. Moreover, h is unique up to homotopy.

6.6.10 Construction. Construction of the localization of a space

The construction of the localization of a space goes by induction on the Postnikov tower of the space. We will assume that X is a CW complex and is simply connected. The idea of the proof is to tensor both the groups and the k-invariant with $\mathbb{Q}$.

Suppose inductively that we have a localization $X_{n-1} \xrightarrow{\phi_{n-1}} X_{n-1}(0)$. Then we have a diagram:

$$\begin{tikzcd}
 & \mathcal{P} \arrow[rr] \arrow[dd] & & \mathcal{P} \arrow[dd] \\
X_n \arrow[ur] \arrow[rr, "\phi_n"] \arrow[dd] & & X_{n(0)} \arrow[ur] \arrow[dd] & \\
 & K(\pi_n(X), n+1) \arrow[rr, "i"] & & K(\pi_n(X) \otimes \mathbb{Q}, n+1) \\
X_{n-1} \arrow[ur, "k^{n+1}"] \arrow[rr, "\phi_{n-1}"'] & & X_{n-1(0)} \arrow[ur, "k^{n+1}_{(0)}"'] &
\end{tikzcd}$$

Since $K(\pi_n(X) \otimes \mathbb{Q}, n+1)$ is a $\mathbb{Q}$-space, $i \circ k^{n+1}$ factors uniquely through $X_{n-1(0)}$. Let $X_{n(0)}$ be the fibrations induced over $X_{n-1(0)}$ from $k^{n+1}_{(0)}$. Since the total composition of

$$\begin{tikzcd}
X_n \arrow[d] & & K(\pi_n(X) \otimes \mathbb{Q}, n+1) \\
X_{n-1} \arrow[r, "\phi_{n-1}"'] & X_{n-1(0)} \arrow[ur] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathcal{P} \arrow[r] & \mathcal{P} \arrow[d] \\
X_n \arrow[ur] & & K(\pi_n(X) \otimes \mathbb{Q}, n+1)
\end{tikzcd}$$

are the same map, this defines $\phi_n : X_n \to X_{n(0)}$. By the commutativity of the diagram we see that $(\phi_n)_* : \pi_*(X_n) \to \pi_*(X_{n(0)})$ is an isomorphism when tensored with $\mathbb{Q}$. This proves that $\phi_n : X_n \to X_{n(0)}$ is the localization of X_n at 0. To complete the proof we need the following lemma

6.6.11 Lemma. *Let Y be a topological space. There is a CW complex X and a map $\psi : X \to Y$ which induces an isomorphism on all homotopy groups. If $\psi' : X' \to Y$ is another such, Then there is a homotopy equivalence $h : X \to X'$ so that $\psi' \circ h$ is homotopoic to ψ.*

We apply the lemma with $Y = \varprojlim X_{n(0)}$. Let $X_{(0)}$ be a CW complex with $\psi : X_{(0)} \to Y$. The maps $X_n \to X_{n(0)}$ define a map $\varprojlim X_n \to \varprojlim X_{n(0)}$. Thus, we have

$$\begin{tikzcd}
 & & X_{(0)} \arrow[d, "\psi"'] \\
X \arrow[r, "\{\phi_n\}"] & \lim X_n \arrow[r] & \lim X_{n(0)}
\end{tikzcd}$$

The obstructions to lifting the map $X \to \varprojlim X_{n(0)}$ to a map $X \to X_{(0)}$ lie in $H^*(X ; \pi_{*-1}(F))$ where F is the homotopy-theoretic fiber of ψ. Since ψ induces an isomorphism on homotopy groups, $\pi_*(F) = 0$.

The resulting map $\phi : X \to X_{(0)}$ is a localization at 0.

6.6.12 Calculations. : (i) Since

$$H_*(K(\mathbb{Q}, 2n-1); \mathbb{Z}) = \begin{cases} \mathbb{Q} & * = 2n-1 \\ 0 & \text{otherwise} \end{cases},$$

$S^{2n-1} \to K(\mathbb{Q}, 2n-1)$ is a localization. Thus

$$\pi_*(S^{2n-1}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & * = 2n-1 \\ 0 & \text{otherwise} \end{cases}.$$

Since $\pi_(S^{2n-1})$ is always finitely generated

$$\pi_i(S^{2n-1}) \cong \begin{cases} \mathbb{Z} & i = 2n-1 \\ \text{finite group} & i \neq 2n-1 \end{cases}.$$

(ii) $S^{2n} \to K(\mathbb{Q}, 2n)$ induces an isomorphism in rational cohomology through degree $(4n-1)$. The kernel of $H^{4n}(K(\mathbb{Q}, 2n); \mathbb{Q}) \to H^{4n}(S^{2n}; \mathbb{Q})$ is generated by $(\iota)^2$. Form the principal fibration $K(\mathbb{Q}, 4n-1) \to E \to K(\mathbb{Q}, 2n)$ with k-invariant ι^2. A calculation using the Serre spectral sequence shows that $H^*(E; \mathbb{Q}) \cong H^*(S^{2n}; \mathbb{Q})$. Furthermore, we can lift $S^{2n} \to K(\mathbb{Q}, 2n)$ to a map $S^{2n} \to E$. This proves that

$$\pi_i(S^{2n}) = \begin{cases} \mathbb{Z} & i = 2n \\ \mathbb{Z} \oplus \text{finite group} & i = 4n-1 \\ \text{finite group} & i \neq 2n \text{ or } 4n-1 \end{cases}.$$

(iii) There is a map $\mathbb{C}P^n \to K(\mathbb{Q}, 2)$ which induces an isomorphism on rational cohomology through degree $2n+1$. Let $E \to K(\mathbb{Q}, 2)$ be a principal fibration with fiber $K(\mathbb{Q}, 2n-1)$ and k-invariant ι^{n+1}. As before one sees that $\mathbb{C}P^n \to E$ induces an isomorphism on rational cohomology. Thus, E is $\mathbb{C}P^n_{(0)}$.

Chapter 7

Differential graded algebras, minimal models & rational homotopy theory

(The materials of this chapter has been adapted from [14])

7.1 Differential graded algebras

In this section we shall study differential algebras in their own right. What we are doing, actually, is studying the homotopy of differential algebras. In fact, we shall construct an object (the minimal model) which should be considered the "Postnikov" tower of a differential algebra.

7.1.1 Definition. A differential graded algebra (or differential algebra for short), $\mathcal{A}^*$, is a graded vector space over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$,

$$\mathcal{A}^* = \oplus_{p\geq 0}\mathcal{A}^p,$$

having

(i) a differentiation $d : \mathcal{A}^p \to \mathcal{A}^{p+1}$ with $d^2 = 0$;

(ii) a product $\mathcal{A}^p \otimes \mathcal{A}^q \to \mathcal{A}^{p+q}$ satisfying

$$\alpha\beta = (-1)^{pq}\beta\alpha$$

(iii) $d(\alpha\beta) = d\alpha\, \beta + (-)^p \alpha d\beta$.

We use the notation D.G.A. for a differential graded algebra.

7.1.2 Example. The C^∞ de Rham complex $A^*_{DR}(M)$ of a smooth manifold and the P.L. de Rham complex $A^*_{P.L}(K)$ of a simplicial complex are D.G.A's over $\mathbb{R}, \mathbb{Q}$ respectively.

7.1.3 Example. The cohomology $H^*(X, \mathbb{Q})$ of a space is D.G.A. ($d = 0$), but the singular cochain complex $C^*(X, \mathbb{Q})$ is not (the commutativity fails).

The problem of commutative cochains: Perhaps the main genesis of the theory we are considering is the problem of commutative cochains. This was solved in an abstract manner by Quillen, and in an attempt to better understand this Sullivan was led to the P.L. forms and the connection between differential forms and homotopy type. In retrospect one can already see much of the theory in the book "Geometric Integration Theory" by Whitney; however one fundamental point was missing in that Whitney only constructs commutative cochains over $\mathbb{R}$, and as there is no way to build Postnikov towers over $\mathbb{R}$ and thus tie in the commutative cochains with homotopy type.

Let X be a simplicial complex. The usual definition of the cup-product $\alpha_p \cup \beta_q$ between a p-cochain α_p and a q-cochain β_q is

$$< \alpha_p \cup \beta_q, \Delta^{p+q} > = < \alpha_p, \text{front}\, p\text{--face of } \Delta^{p+q} > < \beta_q, \text{back}\, q\text{--face of} \Delta^{p+q} > .$$

This formula leads to the properties:

(i) $\delta(\alpha_p \cup \beta_q) = \delta\alpha_p \cup \beta_q + (-1)^p \alpha_p \cup \delta\beta_q$,

and

(ii) $\alpha_p \cup (\beta_q \cup \gamma_r) = (\alpha_p \cup \beta_q) \cup \gamma_r$.

Moreover, a somewhat grizzly computation shows that on the cohomology level we have graded commutativity

$$[\alpha_p] \cup [\beta_q] = (-1)^{pq}[\beta_q] \cup [\alpha_p.]$$

However it is palpably false that

(iii) $\alpha_p \cup \beta_q = (-1)^{pq}\beta_q \cup \alpha_p$

on the cochain level. Now we may attempt to modify the formula so as to have (i)-(iii), but all such attempt are doomed to failure since, as realized by Steenrod, the failure to find commutative cochains/$\mathbb{Z}$ is reflected in the existence of cohomology operations, such as the Steenrod squares, etc. These objections do not apply over $\mathbb{Q}$ (we have essentially proved this by calculating $H^*(K(\mathbb{Z}, n), \mathbb{Q})$), thus it is reasonable to look for commutative cochains/$\mathbb{Q}$. Before going on, it is time to precisely define what is meant by commutative cochains.

7.1.4 Definition. Commutative cochains assign functorially to each simplicial complex X a D.G.A/$\mathbb{Q}$, $C^*(X)$, satisfying (i)-(iii) above and such that

(iv) the cohomology of $C^*(X)$ is $H^*(X; \mathbb{Q})$; and

(v) given a subcomplex $Y \subset X$, we have

$$C^*(X) \longrightarrow C^*(Y) \longrightarrow 0.$$

The problem of commutative cochains is to find such a $C^*(X)$ for each simplicial complex X

7.1.5 Example. The P.L. forms $A^*_{PL}(X)$ give an explicit solution to the commutative cochain problem. The cohomology $H^*(X, \mathbb{Q})$ dose not give a solution because (v) is violated.

The argument given in this section will also show

7.1.6 Theorem. *Let $C^*(X)$ be any solution to the commutative cochain problem. Then the minimal model $\mathcal{M}$ of the D.G.A. $C^*(X)$ gives the $\mathbb{Q}$-homotopy type of X.*

So now we have come full circle. The problem of commutative cochains is equivalent to finding not only the cohomology, but also the $\mathbb{Q}$-homotopy type of a space from a cochain complex. The P.L forms explicitly solved this problem, and moreover a simple comparison theorem shows that the C^∞ forms give the $\mathbb{R}$-homotopy type of a smooth manifold.

It is interesting to note that Whitney, in his book, essentially showed that any solution to the commutative cochain problem/$\mathbb{R}$ satisfying a mild continuity condition is given by integration of suitable differential forms (the flat forms) over chains. Now almost twentyfive years later we have finally understood what he was driving at.

Given a D.G.A., $\mathcal{A}^*$, we denote by $H^*(\mathcal{A}^*)$ the cohomolgy algebra. It is again a D.G.A. with $d = 0$. We assumed throughout that $H^0(\mathcal{A}^*)$ is the ground field and that $H^1(\mathcal{A}^*) = 0$. Thus, $\mathcal{A}^*$ is, so to speak, connected and simply connected.

7.1.7 Definition. A D.G.A. $\mathcal{A}^*$ is said to be *minimal* if

(i) $\mathcal{A}^*$ is free as a graded-commutative algebra,

(ii) $\mathcal{A}^1 = 0$, and

(iii) $d(\mathcal{A}^*) \subset \mathcal{A}^+ \wedge \mathcal{A}^+$ where $\mathcal{A}^+ = \oplus_{k>0}\mathcal{A}^k$.

Condition (i) means that $\mathcal{A}^*$ is a tensor product of polynomial algebras on generators of even degrees and exterior algebras on generators of odd degrees. Condition (iii) says that d is decomposable. There is notion of minimal D.G.A.'s which do not satisfy (ii).

Given a D.G.A., $\mathcal{A}^*$, we wish to construct a minimal model, $\mathcal{M}(\mathcal{A}^*)$, for $\mathcal{A}^*$. By definition this means that $\mathcal{M}(\mathcal{A}^*)$ is a minimal D.G.A. and there is a map $\rho : \mathcal{M}(\mathcal{A}^*) \to \mathcal{A}^*$ of D.G.A.'s inducing an isomorphism on cohomology. One of the main results of this section is that every simply connected D.G.A. has a minimal model.

7.1.8 Remark. The construction of $\mathcal{M}(\mathcal{A}^*)$ is motivated by the construction of the Postnikov tower of a space. In fact, the parallel is quite precise.

Actually the fundamental property of a minimal algebra is that it is an increasing sequence of subalgebras which are nicely related, one to the next. To illuminate this we study these extensions separately. First

7.1.9 Definition. Let A be a D.G.A. A *Hirsch extension* of A is a map

$$A \to A \otimes_d \Lambda(V)_k.$$

The notation means that

(i) V is a (finite dimensional) vector space homogeneous of degree k;

(ii) $\Lambda(V)_k$ is the free graded-commutative algebra with unit generated by V (The polynomial algebra on V if k is even the exterior algebra on V if k is odd); and

(iii) $d : V \to A^{k+1}$. The differential on the full algebra is determined by $d|A$ and $d|V$. We write v for $1 \otimes v (v \in V)$ and a for $a \otimes 1 (a \in A)$

Two Hirsch extensions $A \to A \otimes_d \Lambda(V)$ and $A \to A \otimes_d \Lambda(V')$ are equivalent if there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & A \otimes_d \Lambda(V) \arrow[d, "\phi"] \\
 & A \otimes_{d'} \Lambda(V')
\end{tikzcd}$$

with ϕ an isomorphism.

7.1.10 Lemma. *$A \to A \otimes_d \Lambda(V)$ and $A \to A \otimes_d \Lambda(V')$ are equivalent if and only if there is an isomorphism $\psi : V \to V'$ so that*

$$\begin{tikzcd}
V \arrow[r, "d"] \arrow[d, "\psi"] & H^{k+1}(A) \arrow[d] \\
V' \arrow[r, "d'"] & H^{k+1}(A)
\end{tikzcd}$$

commutes.

Proof. If $\phi : A \otimes_d \Lambda(V) \simeq A \otimes_{d'} \Lambda(V')$ is an isomorphism extending the identity on A, then $\phi(v) = a_v + \psi(v)$. For ϕ to be an isomorphism, ψ must be an isomorphism. Since $\phi(dv) = d'(\phi(v)) = d'a_v + d'\psi(v)$,

$$[dv] = [d'\psi(v)] \in H^{k+1}(A).$$

Conversely, if we have $\psi : V \simeq V'$ so that $[dv] = [d'\psi(v)] \in H^{k+1}(A)$, then $dv - d'\psi(v) = da_v$ for some $a_v \in A$. We can choose a_v linearly in v. Define $\phi(v) = a_v + \psi(v)$. This defines a map $\phi : A \otimes \Lambda(V) \longrightarrow A \otimes \Lambda(V')$, which is easily seen to be an isomorphism and to commute with the differentials. □

To classify Hirsch extensions of A with a fixed vector space of new generators V, we say that two are equivalent if the isomorphism

$$\phi : A \otimes_d \Lambda(V) \longrightarrow A \otimes_{d'} \Lambda(V)$$

is the identity on A and sends v to $a_v + v$. Equivalence classes are then in a natural one-to-one correspondence with maps

$$d : V \to H^{k+1}(A),$$

or what is the same thing, the class of d

$$[d] \in H^{k+1}(A; V^*).$$

7.1.11 Proposition. *If $\mathcal{M}$ is a minimal algebra and $\mathcal{M}(n) \subset \mathcal{M}$ is the subalgebra generated by elements in degrees $\leq n$, then we have $\mathcal{M}(0) = \mathcal{M}(1) \subset \mathcal{M}(2) \subset \ldots$ with $\cup \mathcal{M}(n) = \mathcal{M}$, and with each $\mathcal{M}(n) \subset \mathcal{M}(n+1)$ being a Hirsch extension.*

Proof. Since each $\mathcal{M}(i)$ is free as an algebra, it is clear that as vector spaces

$$\mathcal{M}(n+1) \cong \mathcal{M}(n) \otimes \Lambda(V)_{n+1}.$$

Since $d(v)$ is decomposable for $v \in V_{n+1}$ and $\mathcal{M}$ has no elements of degree 1, $d(v)$ is a sum of products of elements of degree $\leq n$; i.e., $d(v) \in \mathcal{M}(n)$.

Conversely, if $\mathcal{M} = \cup_n \mathcal{M}(n)$ where $\mathcal{M}(n) \subset \mathcal{M}(n+1)$ is a Hirsch extension of degree $n+1$ and $\mathcal{M}(0)$ is the ground field, then $\mathcal{M}$ is a minimal D.G.A. (as is each $\mathcal{M}(n)$). □

7.1.12 Definition. Relative cohomology. Before beginning the actual construction of $\mathcal{M}(\mathcal{A}^*)$ we need a few basic facts about relative cohomology for a map between 2 cochain complexes. Let $C^* \xrightarrow{f} D^*$ denote a degree 0 map (i.e., $f : C^n \to D^n$) between two cochain complexes. Define

$$M_f^n = C^n \oplus D^{n-1}$$

and let $\delta : M_f^n \to M_f^{n+1}$ be given by

$$\begin{pmatrix} -\delta_c & 0 \\ f & \delta_D \end{pmatrix}$$

One checks easily that $\delta^2 = 0$. We define $H^*(C, D)$ to be $H^*(M_f^*)$. The maps $D^{*-1} \xrightarrow{-i_2} M_f^*$ and $M_f^* \xrightarrow{\pi_1} C^*$ commute with the coboundaries. The resulting maps on cohomology give a long exact sequence

$$\cdots \to H^n(C) \xrightarrow{f^*} H^n(D) \xrightarrow{-i_2^*} H^{n+1}(C, D) \xrightarrow{\pi_1^*} H^{n+1}(C) \xrightarrow{f^*} \dots$$

Clearly, this long exact sequence is functorial for commutative diagrams

$$\begin{tikzcd} C_1 \arrow[r, "f_1"] \arrow[d] & D_1 \arrow[d] \\ C_2 \arrow[r, "f_2"] & D_2 \end{tikzcd}$$

7.1.13 Theorem. Construction of the minimal model. *If A is a differential algebra (simply connected), then A has a minimal model.*

Proof. We shall construct an increasing sequence of Hirsch extensions

$$\mathcal{M}(0) \subset \mathcal{M}(1) \subset \mathcal{M}(2) \subset \dots$$

together with a maps

$$\rho_n : \mathcal{M}(n) \longrightarrow A$$

so that

$$\rho_n|\mathcal{M}(k) = \rho_k \text{ for } k \leq n.$$

The map $\rho_n : \mathcal{M}(n) \to A$ will be an n-dimensional model in the sense that:

(i) $\mathcal{M}(n)$ is minimal and generated by elements in degrees $\leq n$,

(ii) ρ_n^* is an isomorphism in degrees $\leq n$, and

(iii) ρ_n^* is an injection in degree $n+1$.

To begin with, $\mathcal{M}^*(0)$ is the ground field and ρ_0 is the map sending 1 to 1. Suppose inductively that we have constructed $\mathcal{M}(n)$ and $\rho_n : \mathcal{M}(n) \to A$ as required. The relative cohomology $H^i((M(n), A))$ vanishes for $i \leq n+1$. (This follows from the long exact sequence of the pair and conditions on ρ_n.) Let $V = H^{n+2}(\mathcal{M}(n), A)$. We will form $\mathcal{M}(n+1)$ by taking the algebra

$$\mathcal{M}(n) \otimes \Lambda(V)_{n+1}.$$

Here, $\Lambda(V)_{n+1}$ means free, graded-commutative algebra generated by the unit together with V in degree $n+1$. (Hence, $\Lambda(V)_{n+1}$ is the exterior algebra on V if $(n+1)$ is odd and the polynomial algebra on V if $n+1$ is even.) To extend the differential on $\mathcal{M}(n+1)$, by the Leibnitz rule we need only define it on V subject to the conditions that dv is closed for $v \in V$.

To define a map $\rho_{n+1} : \mathcal{M}(n+1) \to A$ extending ρ_n it is necessary only to define $\rho_{n+1}|V$ subject to the condition

$$\rho_n(dv) = d\rho_{n+1}(v) \;\; \forall v \in V.$$

Both d and ρ_{n+1} are defined by splitting the map

$$(\text{relative cocycles})^{n+2} \longrightarrow H^{n+2}(\mathcal{M}(n), A).$$

This is equivalent to choosing, linearly in v, cocycle representatives $(m_v, a_v) \in \mathcal{M}(n)^{n+2} \oplus A^{n+1}$. For (m_v, a_v) to be a cocycle means $dm_v = 0$ and $\rho_n(m_v) = da_v$. Having done this we let $d(v) = m_v$ and $\rho_{n+1}(v) = a_v$. Then

$$d^2(v) = d(m_v) = 0$$

and

$$\rho_n(dv) = \rho_n(m_v) = da_v = d(\rho_{n+1}(v)).$$

This shows that $\mathcal{M}(n+1)$ is a differential algebra and that ρ_{n+1} is a map of differential algebras. Lastly, we must show that $H^i(\mathcal{M}(n+1), A) = 0$ for $i \leq n+2$. For this we need a lemma,

7.1.14 Lemma. *(a)* $\mathcal{M}(0) = \mathcal{M}(1)$; *i.e.,* $\mathcal{M}(n)$ *has no elements of degree 1.*

(b) $H^{n+2}(\mathcal{M}(n), \mathcal{M}(n+1)) = V$.

(c) $H^{n+3}(\mathcal{M}(n), \mathcal{M}(n+1)) = 0$.

Proof. (a): Since $\widetilde{H}^0(A) = 0$ and $H^1(A) = 0$, the relative cohomology $H^2(\mathcal{M}(0), A) = 0$. Hence $\mathcal{M}(1) = \mathcal{M}(0)$. But $\mathcal{M}(n)$ and $\mathcal{M}(1)$ agree in degree 1 that $\mathcal{M}(n)$ has no elements of degree 1.

(b): Let us consider the relative cocycles of degree $(n+2)$. These are all of the form $(a, v+b)$ where $a, b \in \mathcal{M}(n)$, $v \in V$, $da = 0$, and $a = dv + db$.

Varying such a cocycle by $d(-b, 0)$ changes it to (a', v). No element of this form is exact unless $v = 0$.

Conversely, given $v \in V$ we have the cocycle (dv, v). This proves that $H^{n+2}(\mathcal{M}(n), \mathcal{M}(n+1)) = V$.

(c): Since $\mathcal{M}(n)$ has no elements of degree 1, $\mathcal{M}(n)$ and $\mathcal{M}(n+1)$ are the same in degree $(n+2)$. It follows easily from this and the fact that $\mathcal{M}(n) \subset \mathcal{M}(n+1)$ that $H^{n+3}(\mathcal{M}(n), \mathcal{M}(n+1)) = 0$. □

We have a map of pairs $(id, \rho_{n+1}) : (\mathcal{M}_n, \mathcal{M}_{n+1}) \to (\mathcal{M}_n, A)$. The corresponding map of long exact sequence , the above fact, and the five lemma prove that $\rho^*_{n+1} : H^{n+1}(\mathcal{M}(n+1)) \to H^{n+1}(A)$ is an isomorphism and that $\rho^*_{n+1} : H^{n+2}(\mathcal{M}(n+1)) \to H^{n+2}(A)$ is an injection.

This completes the inductive construction of the $\mathcal{M}(n)$ and ρ_n. Define $\mathcal{M} = \cup_n \mathcal{M}(n)$ and define $\rho : \mathcal{M} \to A$ by $\rho|\mathcal{M}(n) = \rho_n$. Since cohomology commutes with direct limit, it follows that $\rho^* : H^*(\mathcal{M}) \to H^*(A)$ is an isomorphism. Thus, $(\mathcal{M}, \rho)$ is a minimal model for A. □

7.2 Homotopy theory of D.G.A.'s

In this section we shall delve more deeply into the homotopy theory of D.G.A.'s. One consequence of this study will be to prove the uniqueness of the minimal model.

7.2.1 Definition. Homotopies. Let f and g be D.G.A. maps from A to B. A homotopy from f to g is a map

$$H : A \longrightarrow B \otimes (t, dt)$$

satisfying $H|_{\substack{t=0\\dt=0}} = f$ and $H|_{\substack{t=1\\dt=0}} = g$.

Explanation: (t, dt) represents the tensor product of polynomials on t (degree of $t = 0$) with exterior algebra on dt (degree of $dt = 1$). It is thought of as an algebra of forms on the real line, $\mathbb{R}$. The two restrictions correspond to evaluations of forms at the points $\{0\}$ and $\{1\}$. The idea for this definition comes from dualizing the usual definition on the space level.

To study homotopies we introduce an additive operator

$$\int_0^1 : B \otimes (t, dt) \to B$$

by $\int_0^1 b \otimes t^i = 0$ and $\int_0^1 b \otimes t^i dt = (-1)^{\text{degree } b} \frac{b}{i+1}$. Likewise, define

$$\int_0^t : B \otimes (t, dt) \to B \otimes (t, dt)$$

by $\int_0^t b \otimes t^i = 0$ and $\int_0^t b \otimes t^i dt = (-1)^{\text{degree } b} b \otimes \frac{t^{i+1}}{i+1}$. The following are proved directly from the definitions:

(a) *If* $\beta \in B \otimes (t, dt)$, then

$$d(\int_0^t \beta) + \int_0^t d\beta = \beta - \{(\beta|_{\substack{t=0\\ dt=0}}) \otimes 1\}.$$

(b) If $H : A \to B \otimes (t, dt)$ is a homotopy from f to g, then

$$d \int_0^1 H(a) + \int_0^1 dH(a) = g(a) - f(a).$$

Note that (b) follows from (a) by taking $\beta = H(a)$ and restricting to $t = 1$.

The basic result in the homotopy theory of Hirsch extensions is the following:

7.2.2 Proposition. Obstruction theory. *Given a diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & B \arrow[d, "\phi"] \\ A \otimes_d \Lambda(V)_k \arrow[r, "f"] & C \end{tikzcd}$$

and a homotopy $H : A \to C\otimes(t,dt)$ *from* $\phi\circ g$ *to* $f|A$, *there is an obstruction class* $\mathcal{O} \in H^{k+1}(B,C,V^*)$ *which vanishes if and only if there is an extension* $\widetilde{g} : A\otimes_d \Lambda(V)_k \to B$ *of* g *and an extension* $\widetilde{H}$ *of* H *to a homotopy from* f *to* $\phi\circ\widetilde{g}$.

Proof. For each $v \in V$, we define $\widetilde{\mathcal{O}}(v) \in B^{k+1}\oplus C^k$ by

$$\widetilde{\mathcal{O}}(v) = (g(dv), f(v) + \int_0^1 H(dv)).$$

This is the obstruction cocycle for extending g and H. First, we show that $\widetilde{\mathcal{O}}(V)$ is indeed a cocycle, and then we show that, if $\widetilde{\mathcal{O}}(v)$ is exact for all $v \in V$, the sought after extensions exits.

$$\begin{aligned} d\widetilde{\mathcal{O}}(v) &= (dg(dv), \phi\circ g(dv) - df(v) - d\int_0^1 H(dv)) \\ &= (g(d^2v), \phi\circ g(dv) - f(dv) - d\int_0^1 H(dv) - \int_0^1 dH(dv)) \\ &= (0,0). \end{aligned}$$

Let $\mathcal{O} : V \to H^{k+1}(B,C)$ be the homomorphism induced by $\widetilde{\mathcal{O}}$; i.e., $\mathcal{O}(v) = [\widetilde{\mathcal{O}(v)}]$. Such a homomorphism is the same as an element $\mathcal{O} \in H^{k+1}(B,C;V^*)$.

If $\mathcal{O}(v) = 0$ for all v, then there are relative cochains (b_v, c_v) (depending linearly on v) so that $d(b_v,c_v) = \widetilde{\mathcal{O}}(v)$. Define

$$\widetilde{g}(v) = b_v$$

and

$$\widetilde{H}(v) = f(v) + \int_0^t H(dv) + d(c_v\otimes t).$$

Let us check that these equations define maps of differential algebras $\widetilde{g} : A\otimes_d\Lambda(V)_k \to B$ extending g and $\widetilde{H} : A\otimes_d\Lambda(V)_k \to c\otimes(t,dt)$ extending H.

For this it is necessary only that $d\widetilde{g}(v) = g(dv)$ and $d\widetilde{H}(v) = H(dv)$. Clearly, $d\widetilde{g}(v) = db_v = g(dv)$. Also, by (a), $d\widetilde{H}(v) = df(v) + H(dv) - H(dv)|_{t=0} = f(dv)$. Thus $d\widetilde{H}(v) = H(dv)$. Lastly, we must show that $\widetilde{H}$ is a homotopy from f to $\phi \circ \widetilde{g}$. But

$$\widetilde{H}(v)\Big|_{\substack{t=0\\ dt=0}} = f(v)$$

and

$$\begin{aligned}
\widetilde{H}(v)\Big|_{\substack{t=1\\ dt=0}} &= f(v) + \int_0^1 H(dv) + dc_v \\
&= f(v) + \int_0^1 H(dv) + (\phi(b_v) - f(v) - \int_0^1 H(dv)) \\
&= \phi(b_v) = \phi \circ \widetilde{g}(v).
\end{aligned}$$

Conversely, if we are given any extension $\widetilde{g}$ of g and $\widetilde{H}$ of H such that $\widetilde{H}$ is a homotopy from f to $\phi \circ \widetilde{g}$, then define $\psi_{(\widetilde{g},\widetilde{H})} : V \to B^k \oplus C^{k-1}$ by $\psi(v) = (\widetilde{g}(v), \int_0^1 \widetilde{H}(v))$. One checks directly that $d\psi_{(\widetilde{g},\widetilde{H})}(v) = (g(dv), f(v) + \int_0^1 H(dv)) = \widetilde{\mathcal{O}}(v)$. □

We shall also need a relative version of this lifting property.

7.2.3 Lemma. *Suppose given*

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\phi"] \arrow[d, "i"'] & \mathcal{A} \arrow[d, "f"'] \arrow[dr, "\mu"] & \\
\mathcal{M} \otimes_d \Lambda(V)_n \arrow[r, "\psi"] & \mathcal{B} \arrow[r, "\nu"'] & \mathcal{C}
\end{tikzcd}$$

and a homotopy $H : \mathcal{M} \to \mathcal{B} \otimes (t, dt)$ *from* $\psi|_{\mathcal{M}}$ *to* $f \circ \phi$ *where*

(1) $\nu \circ f = \mu$.

(2) μ *is onto.*

(3) $\mu\circ\phi = \nu\circ\psi|_{\mathcal{M}}$

(4) $\mathcal{M} \xrightarrow{H} \mathcal{B}\otimes(t,dt) \xrightarrow{\nu\otimes 1} C\otimes(t,dt)$ *is constant (i.e.,* $(\nu\otimes 1)\cdot(H(m)) = (\nu\circ\psi(m))\otimes 1$.

Then, the obstruction cohomology class $\mathcal{O}\in H^{n+1}(\mathcal{A},\mathcal{B},V^*)$ *vanishes if and only if there is an extension* $\widetilde{\phi}$ *of* ϕ *and an extension* $\widetilde{H} : \mathcal{M}\otimes_d\Lambda(V)_n \to \mathcal{B}\otimes(t,dt)$ *of* H *to a homotopy from* ψ *to* $f\circ\widetilde{\phi}$ *satisfying*

(1) $\mu\circ\widetilde{\phi} = \nu\circ\psi$ *and*

(2) $(\nu\otimes 1)\circ\widetilde{H}$ *is a constant homotopy.*

Proof. Define $\widetilde{\mathcal{O}} : V \to$cocycles$^{n+1}(\mathcal{A},\mathcal{B})$ as before:

$$\widetilde{\mathcal{O}}(v) = (\widetilde{\phi}(dv), \psi(v) + \int_0^1 H(dv)).$$

Let $a_v \in \mathcal{A}$ be such that $\mu(a_v) = \nu(\psi(v))$. Consider $\widetilde{\mathcal{O}}(v) - d(a_v,0) = (\widetilde{\phi}(dv) - da_v, \psi(v) - f(a_v) + \int_0^1 H(dv))$. This is a cocycle in (Ker $\mu)^{n+1}\oplus$(Ker $\nu)^n$. If it is exact here, $\widetilde{\mathcal{O}}(v) - d(a_v,0) = d(\alpha_v,\beta_v)$ with $\alpha_v \in$ Ker μ and $\beta_v \in$ Ker ν. Then define $\widetilde{\phi}$ and $\widetilde{H}$ using the cochain $(a_v+\alpha_v,\beta_v)$. Checking the formulas in the above proposition one sees that $\mu\circ\widetilde{\phi} = \nu\circ\psi$ and that $(v\otimes 1)\circ\widetilde{H}$ is constant. Since μ is onto, the 5-lemma implies that H^*(Ker μ, Ker $\nu) \to H^*(\mathcal{A},\mathcal{B})$ is an isomorphism. Thus $\widetilde{\mathcal{O}}(v) - d(a_v,0)$ is exact in (Kre μ, Ker ν) if and only if $[\widetilde{\mathcal{O}}(v)] = 0$ in $H^{n+1}(\mathcal{A},\mathcal{B})$. □

7.2.4 Corollary. *Given a commutative diagram :*

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\phi"] \arrow[d, "i"'] & \mathcal{A} \arrow[d, "f"] \\
\mathcal{M}\otimes_d\Lambda(V)_n \arrow[r, "\psi"] & \mathcal{B}
\end{tikzcd}$$

such that f is onto the element $\mathcal{O} : V \to H^{n+1}(\mathcal{A},\mathcal{B})$ is the obstruction to extending ϕ to a map $\widetilde{\phi} : \mathcal{M} \otimes_d \Lambda(V)_n \to \mathcal{A}$ such that $f \circ \widetilde{\phi} = \psi$.

Proof. Applying 7.2.3 with $\mathcal{C} = \mathcal{B}$. The cohomology of Ker f is identified with the relative cohomology $H^*(\mathcal{A},\mathcal{B})$. □

7.2.5 Corollary. Application of the obstruction theory. *The relation on maps from $\mathcal{M} \to \mathcal{A}$, $\mathcal{M}$ minimal, of being homotopic is an equivalence relation.*

Proof. Let $H : \mathcal{M} \to \mathcal{A} \otimes (t_1, dt_1)$ be a homotopy from f_0 to f_1 and $J : \mathcal{M} \to \mathcal{A} \otimes (t_2, dt_2)$ be a homotopy from f_1 to f_2. Let C be the differential algebra

$$(t_1, t_2, dt_1, dt_2)/\{t_2(t_1-1) = 0,\ (t_1-1)dt_2 = t_2 dt_1 = 0\}$$

This algebra represents the form on the variety $t_2(t_1 - 1) = 0$ in the (t_1, t_2)-plane.

The homotopies H and J define a map "$H + J$": $\mathcal{M} \to \mathcal{A} \otimes \mathcal{C}$. If $H(m) = \sum a_i \otimes t_1^i + b_i \otimes t_1^i dt_1$ and $J(m) = \sum a'_j \otimes t_2^j + b'_j \otimes t_2^j dt_2$, then since $H(m)|_{t_1=1} = J(m)|_{t_2=0}$ we have $\sum_{i \geq 0} a_i = a'_0$. The formula for "$H + J$"$(m)$ is

$$\sum_{i \geq 0} a_i \otimes t_1^i + b_i \otimes t_1^i dt + \sum_{j \geq 1} a'_j \otimes t_2^j + \sum_{j \geq 0} b'_j \otimes t_2^j dt_2.$$

One checks easily that "$H + J$" is a map of D.G.A.'s. Consider the diagram

$$\begin{tikzcd}
& \mathcal{A} \otimes (t_1, t_2, dt_1, dt_2) \arrow[d, "p"] \\
\text{“}H + J\text{”}: \mathcal{M} \arrow[r] & \mathcal{A} \otimes \mathcal{C}
\end{tikzcd}$$

The obstructions to lifting "$H + J$" lie in $H^*(\mathcal{A} \otimes (t_1, t_2, dt_1, dt_2), \mathcal{C}) = 0$.

Since p is onto, 7.2.3 says that there is a map $\rho : \mathcal{M} \to \mathcal{A} \otimes (t_1, t_2, dt_1, dt_2)$ such that $p \circ \rho =$ "$H + J$". If we restrict ρ to $t_1 = t_2$ we find a homotopy from $f_0 = H|_{t_1=0}$ to $f_2 = J|_{t_2=1}$.

This proves that the relation of being homotopic is transitive. Reflexivity and symmetry are easily shown. □

7.2.6 Theorem. *Let $\phi : B \to C$ induce an isomorphism on cohomology, and let $\mathcal{M}$ be a minimal differential algebra. Then $\phi_* : [\mathcal{M}, B] \to [\mathcal{M}, C]$ is a bijection.*

7.2.7 Theorem. Uniqueness of minimal model. *If a is a differential algebra and*

$$\begin{tikzcd} & \mathcal{M} \arrow[d, "\rho"] \\ \mathcal{M}' \arrow[r, "\rho'"] & \mathcal{A} \end{tikzcd}$$

are minimal models for $\mathcal{A}$, then there is an isomorphism $I : \mathcal{M} \to \mathcal{M}'$ and a homotopy H from ρ to $\rho' \circ I$. The isomorphism I is itself determined by these conditions up to homotopy.

Proof. Applying 7.2.6 we see that there is a map $I : \mathcal{M} \to \mathcal{M}'$ satisfying $\rho' \circ I$ is homotopic to ρ, and I is well defined up to homotopy. It remains to show that any such $I : \mathcal{M} \to \mathcal{M}'$ is an isomorphism. Since, on the cochain level of cohomology, $I^* \circ (\rho')^* = \rho^*$ and $(\rho')^*$ and ρ^* are isomorphism, it follows that any such $I : \mathcal{M} \to \mathcal{M}'$ will induce an isomorphism on cohomology. To conclude the proof we have the following lemma. □

7.2.8 Lemma. *If $I : \mathcal{M} \to \mathcal{M}'$ induces an isomorphism on cohomology (with $\mathcal{M}$ and $\mathcal{M}'$ minimal), then I is an isomorphism.*

7.2.9 Corollary. *Let $\rho_A : \mathcal{M}_A \to A$ and $\rho_B : \mathcal{M}_B \to B$ be minimal models, and let $f : A \to B$ be a map of differential algebras. There is a map $\hat{f} : \mathcal{M}_A \to \mathcal{M}_B$ and a homotopy from $\rho_B \circ \hat{f}$ to $f \circ \rho_A$. The map $\hat{f}$ is determined up to homotopy by these properties.*

Proof. This is immediate from 7.2.6 applied to the following diagram:

$$\begin{tikzcd} & \mathcal{M}_B \arrow[d, "\rho_B"] \\ f \circ \rho_A : \mathcal{M}_A \arrow[ur, "\tilde{f}"] \arrow[r] & B \end{tikzcd}$$

since ρ_B induces an isomorphism on cohomology. □

7.2.10 Example. A note of caution is necessary here. On the object level a minimal model is uniquely determined. This is not true on the map level. Here is an example of a nonzero map $\mathcal{N}^* \xrightarrow{\rho} \mathcal{M}^*$ between minimal D.G.A.'s/$\mathbb{Q}$ which is homotopic to a constant. Let

$$\begin{aligned} \mathcal{N}^* &= \{\alpha^2, \beta^2, \gamma^4 : \ d\alpha = 0 = d\beta, \ d\gamma = \alpha \cdot \beta\} \\ \mathcal{M}^* &= \{\omega^2, \eta^3 : d\eta = \omega \cdot \omega\} \\ \rho(\alpha) &= \rho(\beta) = 0, \ \rho(\gamma) = \omega \cdot \omega. \end{aligned}$$

This map is homotopic to zero. One way to see this is to give explicitly the homotopy. Anticipating the connection between D.G.A.'s over $\mathbb{Q}$ and rational homotopy theory, one can give a geometric argument. On the level of spaces we have $X = S^2$ which is represented by $\mathcal{M}^*$ and Y given by

$$\begin{tikzcd} K(\mathbb{Q},4) \arrow[r] & Y \arrow[d, "\pi"] \\ & K(\mathbb{Q},2) \times K(\mathbb{Q},3). \end{tikzcd}$$

The map ρ is realized by a map $f : X \to Y$. By the Whitehead theorem, $\pi \circ f \sim$ constant, so that f is homotopic to a map of X into a fiber. This map $X \xrightarrow{f} K(\mathbb{Q}, 4)$ is given by the class $\rho(\gamma) = \omega \cdot \omega = 0$ in $H^4(X, \mathbb{Q})$, and so $f \sim$ constant.

7.3 The connection between the homotopy theory of D.G.A.'s and rational homotopy theory

We shall see how the rational homotopy theory of a simplicial complex is determined by the minimal model of D.G.A. of its P.L. forms.

7.3.1 Definition. Transgression in the Serre spectral sequence and the duality. The basis for the duality alluded to above is the duality between principal fibrations and Hirsch extensions. To understand this we study the transgression in the Serre spectral sequence. Let $p : E \to B$ be a principal fibration with fiber $K(\pi, n)$. Let $\{E_r^{p,q}, d_r\}$ be the Serre spectral sequence of this fibration with coefficients π. Then $E_2^{p,q} = 0$ for $0 < q < n$ and $E_2^{0,n} = H^n(K(\pi; n), \pi) = \mathrm{Hom}(\pi, \pi)$. As a result $E_r^{0,n} = E_2^{0,n}$ for $r \leq n$. The first nonzero differential on $E_*^{0,n}$ is $d_{n+1} : E_n^{0,n} \to E_n^{n+1,0} = H^{n+1}(B; \pi)$. This map $d_{n+1} : \mathrm{Hom}(\pi, \pi) \to H^{n+1}(B; \pi)$ is the *transgression map.* The class $d_{n+1}(\iota) \in H^{n+1}(B; \pi)$ is the *k-invariant* of the fibration. (Here ι) is the class corresponding to the identity homomorphism of π to π.) If we take $\mathbb{Q}$-coefficients instead of π coefficients, then $d_{n+1} : \mathrm{Hom}(\pi, \mathbb{Q}) \to H^{n+1}(B, \mathbb{Q})$. This map is dual to an element $[d_{n+1}] \in H^{n+1}(B; \pi \otimes \mathbb{Q})$. If π itself is a

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rational vector space (of finite dimension), then $[d_{n+1}] \in H^{n+1}(B; \pi \otimes \mathbb{Q}) = H^{n+1}(B; \pi)$ is again the k-invariant of the fibration. Thus a principal fibration where the fiber is a local Eilenberg-MacLane space (and the group is of finite dimensional over $\mathbb{Q}$) is completely determined by the homotopy group π and $[d_{n+1}] \in H^{n+1}(B; \pi)$.

On the other hand given a $\mathbb{Q}$-vector space π (of finite dimension) and an element $[d] \in H^{n+1}(B; \pi)$ there is a Hirsch extension

$$A^*(B) \otimes_d \Lambda(\pi^*)_n$$

where $d : \pi^* \to$ closed forms of A^{n+1} induces $[d] \in H^{n+1}(B; \pi)$. By lemma 7.1.10 this Hirsch extension is well defined up to equivalence. Clearly then, there is a bijective correspondence between principal fibrations over B with fiber a local Eilenberg-MacLane space and Hirsch extensions of $A^*(B)$.

Let $p : \mathcal{E} \to B$ be a fibration with B a simplicial complex. A*simplicial model* for p is a map $f : E \to B$ such that

(1) E is a simplicial complex,

(2) $f : E \to B$ is a simplicial map, and

(3) there is a map $i : E \to \mathcal{E}$ such that $p \circ i = f$, with i a homotopy equivalence.

7.3.2 Theorem. *Let $p : \mathcal{E} \to B$ be a principal fibration with fiber $K(V, n)$, V a $\mathbb{Q}$-vector space of finite dimension. Let $f : E \to B$ be a simplicial model for p. Let $A^*(B) \otimes_d \Lambda(V^*)_n$ be the corresponding Hirsch extension to p. There is a map*

$$\rho : A^*(B) \otimes_d \Lambda(V^*)_n \to A^*(E)$$

such that:

(1) $\rho|A^*(B)$ *is* f^*, *and*

(2) ρ *induces an isomorphism on cohomology.*

The given Hirsch extension is the only one, up to isomorphism, that admits such a mapping ρ.

This is the connection between principal $K(\pi, n)$ fibrations and Hirsch extensions. The theorem is proved using the Serre spectral sequence by what is, at the core, a simple argument. There are, however, enormous technical problems due to the fact that $f : E \to B$ is not a fibration. In fact simplicial maps and fibrations are rather incompatible.

There is also a converse which is easily deduced from this result.

7.3.3 Theorem. *Let* $A^*(B) \otimes_d \Lambda(V)_n$ *be a Hirsch extension with* V *a* $\mathbb{Q}$-*vector space of finite dimension. There is a principal fibration* $p : \mathcal{E} \to B$ *with fiber* $K(V^*, n)$, *and a simplicial model for* p, $f : E \to B$ *so that there is a map of D.G.A.'s*

$$\rho : A^*(B) \otimes_d \Lambda(V)_n \longrightarrow A^*(E)$$

which extends f^* *on* $A^*(B)$ *and induces an isomorphism on cohomolgy. Up to equivalence there is only one such principal fibration with local fiber.*

Let B be a simplicial complex. Consider the set of equvalence classes of all principal fibrations over B with fiber an Eilenberg-MacLane space such that Hom(π_*(fiber),$\mathbb{Q}$) is a finite dimensional rational vector space. Denote this set by $\mathcal{PF}(B)$. If, in addition, we require the homology group of the fiber itself to be a (finite dimensional) rational vector space, then denote the subset by $\mathcal{PF}_{\mathbb{Q}}(B)$. Denote the set of equivalence classes of finite dimensional

Hirsch extensions of $A^*(B)$ by $\mathcal{HE}(B)$. These three are homotopy functors of B. The above equivalences give natural transormations

$$\mathcal{H} : \mathcal{PF}(B) \longrightarrow \mathcal{HE}(B)$$

and

$$\mathcal{F} : \mathcal{HE}(B) \longrightarrow \mathcal{PF}_{\mathbb{Q}}(B).$$

The composition $\mathcal{F} \circ \mathcal{H} : \mathcal{PF}(B) \to \mathcal{PF}_{\mathbb{Q}}(B)$ is simply localization. The composition

$$\mathcal{H} \circ \mathcal{F} : \mathcal{E}(B) \longrightarrow \mathcal{HE}(B)$$

is the identity. Thus, $\mathcal{F}$ and $\mathcal{H}|\mathcal{PF}_{\mathbb{Q}}(B)$ are inverse functors.

Under these correspondences the degree of the extension is the dimension of the nontrival homotopy group in the fiber. The vector space of the extension and the homotopy group of the fiber are dual vector spaces. The k-invariant of the fibration is the map induced by d.

7.3.4 Lemma. *Suppose $f : \mathcal{A} \to \mathcal{B}$ induces an isomorphism on cohomolgy. Then f induces a bijection*

$$\mathcal{HE}(\mathcal{A}) \xrightarrow{f^*} \mathcal{HE}(\mathcal{B}).$$

7.3.5 Corollary. *If $\mathcal{M} \to A^*(B)$ is a minimal model, then we have a bijection*

$$\mathcal{HE}(\mathcal{M}) \longrightarrow \mathcal{PF}_{\mathbb{Q}}(B).$$

7.4 $\mathbb{Q}$-Postnikov towers and minimal D.G.A.'s

Having established the equivalence of Hirsch extensions with principal fibrations, we turn to the connection between $\mathbb{Q}$-Postnikov towers and minimal D.G.A.'s. Let X be a simplicial complex which is simply connected.
Let

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d] \\
\mathrm{X} \arrow[ur] \arrow[r] \arrow[dr] & X_2 \arrow[d] \\
 & \mathrm{pt}
\end{tikzcd}$$

be the rational Postinkov tower of X. Let

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3' \arrow[d] \\
X \arrow[ur] \arrow[r] \arrow[dr] & X_2' \arrow[d] \\
 & \mathrm{pt}
\end{tikzcd}$$

be a simplicial model for this tower of fibrations. Let $\mathcal{M}(0) \subset \mathcal{M}(2) \subset \mathcal{M}(3) \subset \cdots \cup \mathcal{M}(n) = \mathcal{M}$ be the minimal model for X.

7.4.1 Theorem. *$\mathcal{M}(n)$ is the minimal model for X_n'. The Hirsch extension $\mathcal{M}(n) \subset \mathcal{M}(n+1)$ corresponds to the principal fibration $X_{n+1} \to X_n$. Thus, $H^*(\mathcal{M}(n)) = H^*(X_n; \mathbb{Q})$. The indecomposables of $\mathcal{M}(n+1)$ in degree $(n+1), I^{n+1}$, are equal to $Hom(\pi_{n+1}(X), \mathbb{Q})$ and the k-invariant $k^{n+2} \in H^{n+2}(X_n; \pi_{n+1})$, when tensored with $\mathbb{Q}$ is equal to $d : I^{n+1} \to H^{n+2}(\mathcal{M}(n))$.*

(see [14])

Chapter 8

Examples and computations

(The materials of this chapter has adapted from [14])

8.1 Spheres and projective spaces

Let us consider an odd sphere S^{2n+1}. The first stage in building the minimal model for the forms on sphere S^{2n+1} is to construct an exterior algebra $\Lambda(e)$ on a generator of degree $(2n+1)$. Clearly, the cohomology of this D.G.A. maps isomorphically to $H^*(S^{2n+1})$ when we send e to a closed form on S^{2n+1} which is not exact. Thus, $\Lambda(e)$ is the minimal model for the forms on S^{2n+1}. By theorem 7.4.1 this implies that $\pi_i(S^{2n+1}) \otimes \mathbb{Q}$ is zero for $i \neq 2n+1$ and equal to $\mathbb{Q}$ for $i = 2n+1$.

Let $A^*(S^{2n})$ be the forms on S^{2n}. The first stage in building the minimal model for $A^*(S^{2n})$ is the polynomial algebra on a generator of degree $2n$, $P(x_{2n})$. Clearly, x_{2n}^2 is cohomologus to zero in $A^*(S^{2n})$. Thus, we tensor in an exterior algebra $\Lambda(y_{4n-1})$ with $dy = x^2$. The product $P(x_{2n}) \otimes_d \Lambda(y_{4n-1})$

is the minimal model. It follows that

$$\pi_i(S^{2n}) \otimes Q \cong \begin{cases} \mathbb{Q} & i = 2n, 4n-1 \\ 0 & i \neq 2n, 4n-1. \end{cases}$$

If $f : S^n \to X$ is a simplicial map (with $n > 1$ and X simply connected), then there is induced $\hat{f} : \mathcal{M}_X \to \mathcal{M}_{S^n}$ well defined upto homotopy. If we consider $\hat{f}$ in degree n, then it defines a map $I^n(\mathcal{M}_X)^* \to I^n(\mathcal{M}_{S^n})^* \cong \mathbb{Q}$. The isomorphism $I^n(\mathcal{M}_{S^n})^* \simeq \mathbb{Q}$ is via integration of the form over the fundamental cycle of S^n. The induced map $I^n(\mathcal{M}_X)^* \to \mathbb{Q}$ depends only on the homotopy class of $\hat{f}$, and hence only on the homotopy class of f. This defines a map

$$\pi_n(X) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(I^n(\mathcal{M}_X), \mathbb{Q}),$$

and hence a map

$$\pi_n(X) \otimes \mathbb{Q} \longrightarrow [I^n(\mathcal{M}_X)]^*.$$

This is exactly the duality between $I^n(\mathcal{M}_X)$ and $\pi_n(X) \otimes \mathbb{Q}$. Let us consider the minimal model for the forms on complex projective n-space, $\mathbb{C}P^n$. The first stage of the minimal model is $P(x_2)$. The class $(x_2)^{n+1}$ generates the kernel of the map $P(x_2) \to H^*(\mathbb{C}P^n; \mathbb{Q})$ Thus, the minimal model for the forms on $\mathbb{C}P^n$ is $P(x_2) \otimes_d \Lambda(y_{2n+1})$; $dy = x_2^{n+1}$. In particular

$$\pi_i(\mathbb{C}P^n) \otimes \mathbb{Q} = \begin{cases} 0 & i \neq 2, 2n+1 \\ \mathbb{Q} & i = 2, 2n+1. \end{cases}$$

One can also deduce this form the calculation of the homotopy groups of S^{2n+1} and the fibrations $S^1 \to S^{2n+1} \to \mathbb{C}P^n$.

8.2 Graded Lie algebras.

Suppose $\mathcal{L} = \oplus_n \mathcal{L}_n$ is a graded vector space (over a field of characteristic 0). Let $[,] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ be a map which is homogeneous of degree 0. We say that $(\mathcal{L}, [,])$ is *a graded Lie algebra* if

(1) $[x, y] = (-1)^{(p+1)(q+1)}[y, x]$ for $x \in \mathcal{L}_p$ and $y \in \mathcal{L}_q$ (symmetry).

(2) $[x, [y, z]] = [[x, y], z] + (-1)^{pq}[y, [x, z]]$ for $x \in \mathcal{L}_p$ and $y \in \mathcal{L}_q$ (Jacobi identity).

An ordinary Lie algebra is a graded algebra in which $\mathcal{L}_n = 0$ for $n \neq 0$.

If X is a simply connected space, then the Whitehead product is a bilinear map

$$\pi_p(X) \otimes \pi_q(X) \xrightarrow{[,]} \pi_{p+q-1}(X).$$

If we define $\mathcal{L}_n = \pi_{n+1}(X) \otimes \mathbb{Q}$, then the Whitehead product makes $\mathcal{L} = \oplus_{n \geq 1} \mathcal{L}_n$ into a graded Lie algebra.

Another source of examples is D.G.A,'s. Let $\mathcal{M}$ be a D.G.A. which is free as a graded commutative algebra on positive dimensional generators. Let $I(\mathcal{M}) = \mathcal{M}^+/\mathcal{M}^+ \Lambda \mathcal{M}^+$ where $\mathcal{M}^+$ is the ideal of elements of positive degree. Denote by $(\mathcal{M}^+)^k$ the k^{th}-power of this ideal; i.e.,

$$(\mathcal{M}^+)^k = \{\omega | \omega = \sum_i \alpha_{i1} \wedge \cdots \wedge \alpha_{ik} \text{ with } \alpha_{ij} \in \mathcal{M}^+\}.$$

If $d : \mathcal{M} \to \mathcal{M}$ is decomposable, then $d : (\mathcal{M}^+)^k \to (\mathcal{M}^+)^{k+1}$. Hence, it induces a map

$$\begin{tikzcd}
\mathcal{M}^+/(M^+)^2 \arrow[r] \arrow[d, "="] & (\mathcal{M}^+)^2/(\mathcal{M}^+)^3 \arrow[d, "="] \\
d : I(\mathcal{M}) \arrow[r] & I(\mathcal{M}) \wedge I(\mathcal{M})
\end{tikzcd}$$

The fact that $d^2 = 0$ implies that the composition

$$I(\mathcal{M}) \xrightarrow{d} I(\mathcal{M}) \wedge I(\mathcal{M}) \xrightarrow{d\wedge id + (-1)^{deg} id \wedge d} I(\mathcal{M}) \wedge I(\mathcal{M}) \wedge I(\mathcal{M}) \qquad (*)$$

is zero. Let $\mathcal{L}_n = [I^{n+1}(\mathcal{M})]^*$. Dual to d is a map $[,] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ which is homogeneous of degree 0. The fact that d maps into $I(\mathcal{M}) \wedge I(\mathcal{M})$ dualizes to the fact that $[,]$ satisfies the symmetry condition to be a graded Lie algebra bracket. The dual to $(*)$ is the Jacobi identity.

There is a connection between these two examples. If X is simply connected space, and if $\mathcal{M}_X$ is its minimal model, then the graded Lie algebras ($\pi_{*+1}(X) \otimes \mathbb{Q}$, Whitehead product) and $(\oplus I^{n+1}(\mathcal{M}_X)^*, d^*)$ are isomorphic under the map

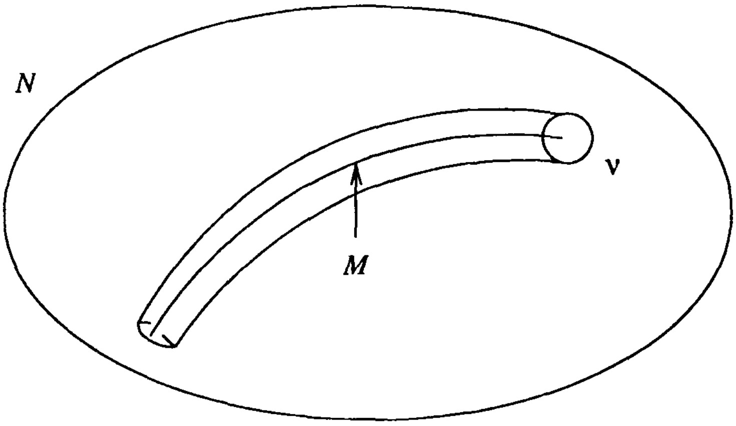
$$\pi_{n+1}(X) \otimes \mathbb{Q} \simeq [I^{n+1}(\mathcal{M}_X)]^*.$$

As an example of this let us consider $S^2 \vee S^2$. Its minimal model begins with two generators ζ_1 and ζ_2 of degree 2. We then add 3-dimensional generators to kill all 4-dimensional cohomology, and so on. The dual graded Lie algebra to this D.G.A. is the free graded Lie algebra on generators $\zeta_1^*$and ζ_2^*.

More generally, $S^{p_1} \vee \cdots \vee S^{p_k}$ has a minimal model whose indecomposables are dual to a free graded Lie algebra on generators of degrees $(p_1 - 1), \ldots, (p_k - 1)$ (assuming each $p_i \geq 2$).

8.3 The borromean rings.

There is a geometric form of Poincaré duality. It allows one to do the calculations involved in building the minimal model with sub-manifolds instead of forms. The basis for the duality is the following. Let $M^k \subset N^n$ be an embedded sub-manifold. Suppose M^k and N^n are both closed and oriented. By the tubular neighborhood theorem there is a neighborhood $\nu(M \subset N)$ which is diffeomorphic to a disk bundle over M:



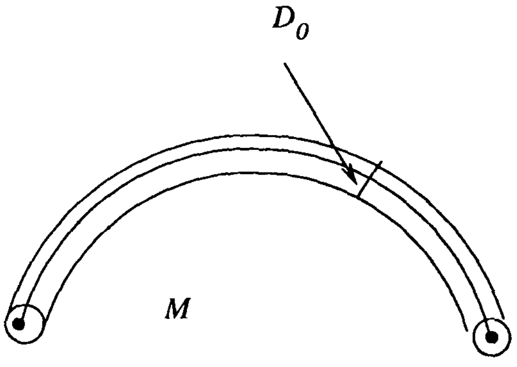
Let D_0^{n-k} be the fiber over a point $m_0 \in M$. Since M and N are oriented, D_0^{n-k} receives an orientation. By the Thom isomorphism theorem there is a unique class $U_M \in H^{n-k}(\nu(M \subset N), \partial\nu(M \subset N); \mathbb{Z})$ so that $\int_{D_0} U_M = 1$. (Here we assume that M is connected.) There is a C^∞ differential form representing U_M which vanishes identically near $\partial\nu(M \subset N)$. If we extend by 0 to $N - \nu(M \subset N)$, then we have a closed C^∞-form, $\widetilde{U_M}$, on all of N. Its cohomology class is the Poincaré dual of $[M] \in H_k(N)$.

There is a form of this duality for manifolds with boundary. If $(M, \partial M) \subset (N, \partial N)$ with M meeting ∂N normally in ∂M then the same construction yields a class $\widetilde{U_M} \in H^{n-k}(N)$ which is the Lefschetz dual of the class $[M, \partial M] \in H_k(N, \partial N)$.

Under the correspondence $M \to \widetilde{U_M}$, transverse intersection of manifolds corresponds to wedge product of forms. Thus, if M_0^k and M_1^l are transverse in N^n with intersection $M_{0,1}^{k+l-n}$, and if $\widetilde{U_0}$ and $\widetilde{U_1}$ are Thom forms for M_0

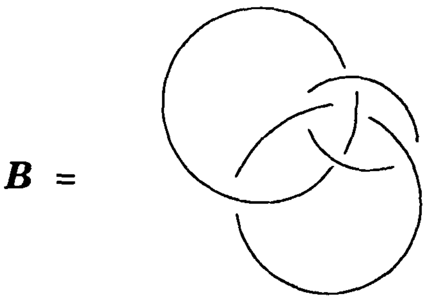
and M_1 supported in sufficiently small tubes, then $\widetilde{U_0} \wedge \widetilde{U_1}$ is a closed form supported in a tube about $M_{0,1}$ and integrating to 1 over each fiber.

The operation of finding a solution for $d\eta = \widetilde{U_M}$, given M and $\widetilde{U_M}$, corresponds to finding a sub-manifold of N whose boundary is M. Thus ,if $M^k = \partial L^{k+1}$ then there is a form $\widetilde{U_L}$ supported in a tube about L, closed outside the tube about M, integrating to 1 over fibers of $\nu(L \subset N)$ which are outside the tube about M, and so that $d\widetilde{U_L} = \widetilde{U_M}$



$\int_{D_0} \widetilde{U_L} = 1, \quad d\widetilde{U_L} = \widetilde{U_M}.$

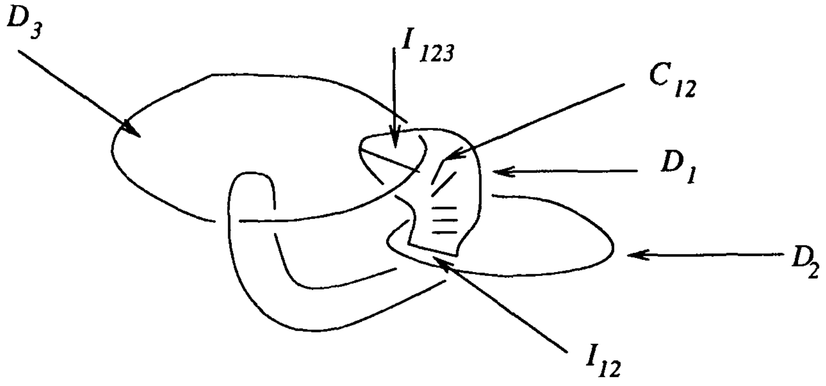
We consider now an explicit example of the operation of building the minimal model via sub-manifolds; viz. the ambient space in S^3-(borromean rings)



(Borromean rings)

We can think of this as a manifold with boundary by taking out open solid tori around each of the circles, or more simply we work with ordinary cohomolgy and proper sub-manifolds.

The first cohomology of $S^3 - \mathcal{B}$ has rank 3 and is generated by classes which are dual to 2-disks spanning the components. We choose these disks as pictured below:



Let $\widetilde{\mathcal{U}_1}$, $\widetilde{\mathcal{U}_2}$ and $\widetilde{\mathcal{U}_3}$ be the dual Thom classes in $H^1(S^3 - \mathcal{B})$. The first stage in the minimal model is $\Lambda(\widetilde{\mathcal{U}_1},\widetilde{\mathcal{U}_2},\widetilde{\mathcal{U}_3})$.

The geometric fact that the linking number of any pair is zero in $S^3 - \mathcal{B}$ means that $\widetilde{\mathcal{U}_i} \cup \widetilde{\mathcal{U}_j} = 0$ for all $i \neq j$. Clearly $\widetilde{\mathcal{U}_2} \wedge \widetilde{\mathcal{U}_3} = 0$ as a form since $D_2 \cap D_3 = \emptyset$. The form $\widetilde{U_1} \wedge \widetilde{U_2}$ is the Thom form for the interval I_{12}. To solve the equation $d\xi = \widetilde{\mathcal{U}_1} \wedge \widetilde{\mathcal{U}_2}$ we must find a proper 2-dimensional submanifold whose boundary is I_{12}. We take this to be the part of D_1 cut off by I_{12} which lies above D_2, C_{12}. To form the *Massey product*[1] $< \widetilde{\mathcal{U}_1},\widetilde{\mathcal{U}_2},\widetilde{\mathcal{U}_3} >$ we must take $n_{12} \wedge \widetilde{\mathcal{U}_3} + \widetilde{\mathcal{U}_1} \wedge n_{23}$ where $dn_{ij} = \widetilde{\mathcal{U}_i} \wedge \widetilde{\mathcal{U}_j}$. In this case $n_{23} = 0$ and n_{12} is supported near C_{12}. Thus $n_{12} \wedge \widetilde{\mathcal{U}_3} + \widetilde{\mathcal{U}_1} \wedge n_{23}$ is represented by

---

[1] Given a space X and classes $\alpha \in H^p(X)$, $\beta \in H^q(X)$, $\gamma \in H^r(X)$ that satisfy $\alpha \cup \beta = 0$ in $H^{p+q}(X)$, $\beta \cup \gamma = 0$ in $H^{q+r}(X)$, there is defined the Massey triple product $< \alpha, \beta, \gamma > \in H^{p+q+r-1}(X)/(\alpha \cdot H^{q+r-1}(X) + \gamma \cdot H^{p+q-1}(X))$.

the Thom form of $C_{12} \cap D_3$. This intersection is I_{123}. Since I_{123} is an arc with end points on different components of $\mathcal{B}$, $[I_{123}] \in H_1(S^3 - \mathcal{B})$ is nonzero. Thus,$< \widetilde{\mathcal{U}_1}, \widetilde{\mathcal{U}_2}, \widetilde{\mathcal{U}_3} > \neq 0$. If we do similar calculation for $S^3 - \mathcal{B}'$ where $\mathcal{B}'$ is three unlinked circles, then all Massey products $< \widetilde{\mathcal{U}_{i_1}}, \widetilde{\mathcal{U}_{i_2}}, \widetilde{\mathcal{U}_{i_3}} >$ are trivial. Thus, the third stages of the minimal models for the forms on $S^3 - \mathcal{B}$ and $S^3 - \mathcal{B}'$ are different. In particular $\mathcal{B}$ and $\mathcal{B}'$ are not isotopic. In fact, since the minimal models differ at the third stage this implies that $\pi_1(S^3 - \mathcal{B})/\Gamma_5$ and $\pi_1(S^3 - \mathcal{B}')/\Gamma_5$ are not isomorphic groups. The group $\pi_1(S^3 - \mathcal{B}')$ is free. The existence of nonzero Massey products in $S^3 - \mathcal{B}$ means that its fundamental group is not free.

8.4 Symmetric spaces and formality.

A space is said to be *formal* if the homotopy type of the D.G.A. of forms on the space is the same as the homotopy type of the cohomology ring of the space. Thus, if X is formal and $\mathcal{M}_X$ is a minimal model for the forms on X, then there is a map $\mathcal{M}_X \to (H^*(X), d = 0)$ which induces the identity on cohomology.

One can always define a map of cochain complexes

$$(H^*(X), d = 0) \to A^*(X) \qquad (*)$$

which induces the identity on cohomology by linearly choosing closed form representatives for each cohomology class. Usually this map will not be multiplicative. If it is possible to choose this map to be multiplicative, then $(H^*(X), d = 0)$ and $A^*(X)$ have the same minimal model. Thus, this gives algebraic topological conditions on a space which must be satisfied if there is to be a multiplicative mapping $(*)$.

If X is a Riemannian manifold, then there is a canonical map $(H^*(X), d = 0) \to A^*(X)$ which assigns to each cohomolgy class its unique harmonic representative. From the above discussion we see that for X to admit a Riemannian metric in which the wedge product of harmonic forms is harmonic it must be the case that X is formal (over $\mathbb{R}$).

There is one class of Riemannian manifolds in which wedge product of harmonic forms is harmonic. These are the Riemannian locally symmetric spaces.

Chapter 9

Survey of recent results on application of rational homotopy theory

9.1 Kähler manifolds

9.1.1 Definition. Let X be a topological space, and let $\mathcal{U}$ be an open covering of X. The covering $\mathcal{U}$ is *locally finite* if, for each $x \in X$, there exists an open set W_x containing x such that

$$\{U \in \mathcal{U};\ U \cap W_x \neq \emptyset\}$$

is a finite set.

9.1.2 Definition. A topological space X is *paracompact* if every open covering of X has a locally finite refinement; that is, if for every open covering $\mathcal{U}$, there exists a locally finite open covering $\mathcal{V}$ such that for each $V \in \mathcal{V}$ there exits a $U \in \mathcal{U}$ with $V \subset U$.

9.1.3 Remark. It can be shown that all metric spaces are paracompact. Also, every regular topological space whose topology has a countable basis is paracompact.

9.1.4 Definition. Let X be a smooth manifold. A smooth *partition of unity* on X is a pair $(\mathcal{V}, \mathcal{F})$, where $\mathcal{V}$ is a locally finite covering of X and $\mathcal{F} = \{f_V\}_{V \in \mathcal{V}}$ is a collection of smooth real-valued functions on X such that
(1) each $f_V \geq 0$,
(2) for each $V \in \mathcal{V}$, the support of f_V (=:the closure of the set $\{x \in X;\ f_V(x) \neq 0\}$) is contained in V, and
(3) $\sum_{V \in \mathcal{V}} f_V = 1$
(Note that this sum makes sense since for each $x \in X$, $f_V(x) = 0$ for all but finitely many $V \in \mathcal{V}$.)

9.1.5 Definition. A *Riemannian manifold* is a smooth manifold X, together with a map

$$<,>: X \to \bigcup_{x \in X} \{\text{inner products on } T(X,x)\}$$

such that for each $x \in X$, $<,>$ (x) (usually denoted $<,>_x$) is an inner product on $T(X,x)$, and such that $<,>$ is smooth; that is, for each pair V_1, V_2 of smooth vector fields on X, $< V_1, V_2 >$ is a smooth function, where

$$< V_1, V_2 > (x) =< V_1(x), V_2(x) >_x .$$

The map $<,>$ is called a *Riemannian structure* on X.

If X is complex manifold then a Riemannian structure $<,>$ on X is said to be *Hermitian* if $<,>_x$ is hermitian inner product $\forall x \in X$.

9.1.6 Theorem. *Let X be a paracompact smooth manifold. Then there exists a Riemannian structure on X.*

Proof. Let $(\mathcal{V}, \mathcal{F})$ be a smooth partition of unity on X such that each $V \in \mathcal{V}$ is a coordinate neighborhood. Define a Riemannian structure $<,>_V$ on each $V \in \mathcal{V}$ by

$$< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >_V = \delta_{ij},$$

where $(x_1, \ldots, x_n)$ are the coordinate functions on V. Then define $<,>$ on X by

$$<,> = \sum_{V \in \mathcal{V}} f_V <,>_V .$$

$\square$

9.1.7 Remark. The converse of the above theorem also holds; namely, every Riemannian manifold is paracompact.

9.1.8 Example. $\mathbb{R}^n$ is a Riemannian manifold. Take $\{\partial/\partial r_i\}$ as an orthonormal basis for the tangent space at each point.

9.1.9 Definition. Let X and Y be Riemannian manifolds. A map $\phi : X \to Y$ is an *isometry* if it is smooth, injective, surjective, has a smooth inverse, and is such that $d\phi$ is an isometry at each point; that is

$$< d\phi(v_1), d\phi(v_2) >_{\phi(x)} = < v_1, v_2 >_x$$

for all $v_1, v_2 \in T(X, x)$ and $x \in X$.

9.1.10 Remark. Thus an isometry preserves all the structure of a Riemannian manifold. Two manifolds are equivalent from the viewpoint of Riemannian geometry if there exists an isometry between them. Such manifolds are said to be *isometric*. Note that two manifolds can be distinct as Riemannian manifolds yet same as smooth manifolds.

An *almost complex manifold* is a smooth manifold M having an endomorphism $J : T(M) \to T(M)$ of the real tangent bundle which satisfies $J^2 = -1$. The complex tangent bundle $T_{\mathbb{C}}(M) = T(M) \otimes_{\mathbb{R}} \mathbb{C}$ splits into conjugate subbundles under the action of J

$$\begin{aligned} T_{\mathbb{C}}(M) &= T'(M) \oplus T''(M), \\ J|T'(M) &= \sqrt{-1}, \\ T''(M) &= \text{complex conjugate of } T'(M). \end{aligned}$$

On the exterior powers of the cotangent bundle there is an induced decomposition

$$\Lambda^r T_{\mathbb{C}}(M)^* = \oplus_{p+q=r}\{\Lambda^p T'(M)^* \otimes \Lambda^q \overline{T'(M)^*}\} = \oplus_{p+q=r}\Lambda^{p,q}T_{\mathbb{C}}(M)$$

Using this, a differential form on M may be uniquely written as a sum of its (p,q) components $\psi = \sum_{p,q} \psi_{p,q}$.

On any complex manifold there are positive definite Hermitian metrics

$$(\xi, \eta) = S(\xi, \eta) + \sqrt{-1}A(\xi, \eta) = (J\xi, J\eta) \quad \text{for } \xi, \eta \in T(M).$$

S and A determine one another. Because of the Hermitian symmetry, S is symmetric and A is alternating. Thus S is a Riemannian and A is a (1,1) form on M

9.1.11 Definition. The Hermitian metric is defined to be *Kähler* if $dA = 0$, and a *Kähler manifold* is a complex manifold admitting a Kähler metric.

9.1.12 Theorem. *(i) If M be a compact complex Kähler manifold, then the real homotopy type of M is a formal consequence of the cohomology ring* $H^*(M;\mathbb{R})$.

(ii) If $f : M \to N$ is a holomorphic map between compact complex Kähler manifolds, then the induced map on real homotopy types is a formal consequence of the induced map on real cohomology.

(cf. [6]),

9.1.13 Theorem. *For a simply connected finite complex M the following properties are equivalent:*

(i) the cohomology algebra of M requires at least two generators,

(ii) the Betti numbers of the space of all maps of S^1 into M are unbounded.

(cf. [25])

9.2 L.S. category, cone length, Toomer's invariant & elliptic spaces

9.2.1 Definitions. The *Lusternik-Schnirelmann category* of a topological space X, $\mathrm{cat}(X)$, is the least integer n such that X can be covered by $n+1$ open sets contractible in X. The *rational category* $\mathit{cat}_0(X)$ of a 1-connected CW complex X is the Lusternik-Schnirelmann category $\mathrm{cat}(X_0)$ of its 0-localization.

9.2.2 Definition. The *rational cone-length* $\mathrm{cl}_0(X)$ is the integer n such that there there is sequence of cofibrations

$$\Sigma^i A^i \to (X_0)_{i-1} \to (X_0)_i, \quad i = 1, \dots, n$$

Where $(X_0)_0$ is a point, $(X_0)_n$ has the homotopy type of X_0 and $\Sigma^i A^i$ is the i-fold suspension of some space A_i.

9.2.3 Definition. Let A be a (connected) chain algebra, with multiplication m and differential d, and let $\bar{m} : \bar{A} \otimes \bar{A} \to \bar{A}$ be the restriction of m where $\bar{A}$ is the augmentation ideal. The bar construction $B(A)$ is the cofree chain coalgebra $T'(s\bar{A})$ where $s\bar{A}$ is the 1-fold suspension of the graded vector space $\bar{A}$ defined by $(s\bar{A})_n = \bar{A}_{n-1}$ together with the differencial d_B whose component

$$p'd_B : T'(s\bar{A}) \to s\bar{A}$$

given by

$$p'd_B = -sds^{-1}p' + s\bar{m}(s^{-1} \otimes s^{-1})p^2.$$

9.2.4 Definition. Let $B(A)$ denote the reduced bar construction on a (connected) chain algebra A. Since the chain coalgebras $B(C_*(\Omega X;\mathbb{Q}))$ and $C_*(X;\mathbb{Q})$ are quasi-isomorphic, the filtration of $B(C_*(\Omega X;\mathbb{Q}))$ by tensor powers yields a spectral sequence, which converges to $H_*(X,\mathbb{Q})$. We then define Toomer's invariant as

$$e_0(X) = \max\{p | E^{\infty}_{p,*} \neq 0\}$$

.

9.2.5 Definition. A simply connected cell complex all but finitely many of whose homotopy groups are finite is called a *elliptic space*.

9.2.6 Definition. A simply connected space S such that $\dim H^*(S) < \infty$ will be called *rationally elliptic* if $\dim \pi_*(S) \otimes \mathbb{Q} < \infty$.

9.2.7 Theorem. *If S is a simply connected rationally elliptic space (i.e. dim $\pi_*(S) \otimes \mathbb{Q} < \infty$), and if $cat_0(S)$ is the rational L.-S. category of S, and $e_0(S)$ is the rational Toomer's invariant of S, then $cat_0(S) \geq e_0(S) \geq$ $rank(S)$.*

9.2.8 Theorem. *If X is a simply connected space whose rational cohomology has finite type and satisfies Poincaré duality, then $cat_0(X) = cl_0(X)$, where $cl_0(X)$ represents the rational cone length of X.*

(cf. [5]),

9.2.9 Theorem. *If X is a simply connected space and $H^*(X;\mathbb{Q})$ is a Poincaré duality algebra then $cat_0(X) = e_0(X)$.*

(cf. [11]).

A questions which is still open is the following

Question: Whether theorems 9.2.7, 9.2.8 and 9.2.9 are true for non-simply connected P.D. spaces (in particular for P.D. spaces with nilpotent fundamental group)

Geometric structures on topological objects put a lot of restrictions on their homotopy and homology groups. For example it has been proved that a compact homogeneous space (rich in geometric structure) is elliptic. It has further been proved that an elliptic space is homotopy equivalent to a Poincaré complex of nonnegative Euler characteristic, and the sum of its Betti numbers is $\leq 2^m$ where m is its cohomological dimension (see [8]).

The following conjecture is due to Halperin [15].

9.2.10 Conjecture. *Any closed nonpositively curved manifold is elliptic.*

There are weaker conjectures which follow from this conjecture, namely (i) classical Chern-Hopf's conjecture - nonnegatively curved manifolds have nonnegative Euler characteristics; (ii) The Gromov's conjecture - the sum of Betti numbers of a compact nonnegatively curved m-manifold is $\leq 2^m$. (see [8], [12]).

An attempt will be made to study these questions further.

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