SOME PROBLEMS AND APPLICATIONS OF COMPUTATIONAL ALGEBRAIC TOPOLOGY – A SURVEY

ABSTRACT

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Local to global problem is prevalent in various branches of engineering, physical and biological sciences (applied mathematics). Very few calculus based tools are proving to be sufficient. Algebraic and differential topology on the other hand has been dealing with such local-to-global problems for more than hundred years.

Algebraic topology has classically helped in studying (mathematical) geometric objects by associating algebraic invariants with these. Now Algebraic topology has an important role to play in analysing the geometric objects presented by numerical and experimental data.

"Computational Algebraic Topology" has evolved in the recent past to solve many problems of this type arising in engineering, physical and biological sciences. Application of homology to these kinds of problems is new and is in a primitive state. One variant of homology theory especially evolved for such computational purposes is "Cubical Homolgy"

Cubical homology has played important role in analyzing image data and numerically generated data. More sophisticated ideas from topology is needed for further work on understanding today's complex geometric challenges like identifying and classifying geometric properties and abnormalities in a geometric object (medical imaging), and recognizing them even in the presence of small perturbations.

All these require the ability to efficiently compute algebraic topological quantities starting with experimental or numerical data, which are purely combinatorial in nature.

The most significant application of cubical homology will be realized by those who understand its fundamental concepts. The book of Kaczynski, Mishaikow and Mrozek [1], is a a very good source for this and we have adapted it in chapters 2-5.

Even though Cubical homology has many advantages over the simplicial homology, it is a bit rigid (it is suited for pixellated, or voxellated images), and to broaden the horizon of its applicability it has to be adapted for working with polyhedral objects too, because geometrical surfaces are quite often represented by a triangulation.

We will however continue to need simplicial homology coming from "Čech complex" and "Vietoris' Rips Complex" for many types of applications.

Now we will list a small number of applications in which the homology theories mentioned above have played important role and are going to play still more important role:

(i) Blanket coverage of a given domain by covers using sensor networks,

(ii) Hole detection of covers of a given domain defined by sensor networks,

(iii) Determination of shape and structure of high-dimensional data sets (shape reconstruction),

(iv) Robot motion planning on a graph (robotics),

(v) Rigourous verification of chaos in dynamical systems,

Most of these applications are found in ([1], [2], [3], [4]). By its basic property of topological (and even homotopy type) invariance homology groups are considered to be robust tools which can determine the nature of a geometric domain or its subdomain (often given by a data set, called point cloud) irrespective of small perturbation (so called noise).

To define any type of homology of a domain, all one has to do is to associate an appropriate (simplicial, cubical, or any other) "chain complex" to it and define homology. Thanks to the Eilenberg-Steenrod uniqueness theorem of homology, it does not depend on the way the complex is defined. This gives us a lot of freedom to choose the complex which best suits a given situation, and more importantly which is amenable to effective computation using computers.

In problems of type (i) Blanket coverage of a given domain by covers using sensor networks (refer to [4]) where we need to cover a domain (a field, forest, ocean) by neighbourhoods of a finite number of sensors or nodes (these sensors could perform various sensing jobs like video surveillance, detection of radiological or biological hazards, motion detection etc.). Homological methods permit choosing a minimal class of these sensors without the need to know their orientation or location or the need to measure distances. (This has significance in the security and surveillance problems).

If the position of the sensors (*nodes*) are fixed, like in the communication towers of mobile phone networks the coverage problem is simple.

However if we are faced with situations where there is no means to determine relative position of the sensors (*nodes*), then apart from probalistic methods (which need very strong assumptions on the uniformity and density of the random distribution of the sensors), one can use homological methods. Homological methods can even handle situations in which the environment has moved the sensors to arbitrary (unknown) positions.

For obtaining complete or partial solutions to the blanket coverage problem, one needs to impose some minimal conditions on these sensors.

We describe them below for a planar domain:

Assumption 0.0.1. [4] Let D be a compact connected planar domain with connected polygonal boundary ∂D. Let $\chi \subset D$ be the set of sensors (*nodes*). Let $\chi_f \subset \partial D$ be the sensors lying on the vertices of the polygonal boundary (*fence nodes*). We assume that

1. Each node of χ broadcast its unique ID number. Each node of χ can detect the ID of any node within broadcast radius r_b.

2. Nodes have radially symmetric covering domains of cover radius (disk neighbourhoods with centres at the nodes and of radius) $r_c \geq r_b/\sqrt{3}$.

3. Each fence nodes $v \in \chi_f$ knows the ID of its neighbours on ∂D, and these neighbours both lie within distance r_b from v.

Definition 0.0.2. [4] The *network graph* of the above system is a combinatorial graph, Γ, in which vertex set is the set of labeled sensors (nodes), and (undirected) edges correspond to pairs of nodes that are within mutual broadcasting range (within distance r_b). By assumption the fence nodes form a cycle $\mathcal{F} \subset \Gamma$.

The problem at hand is to determine whether the set $\mathcal{U}$ of the disks $B_{r_c}(x)$ of radius r_c around the nodes x of χ is a cover of D.

The input for the problem is the pair $(\Gamma, \mathcal{F})$.

Now the nerve of the cover $\mathcal{U}$ gives the Čech complex $\mathcal{C}(\mathcal{U})$ of $\cup\{U|U \in \mathcal{U}\}$.

Result 0.0.3. Under the above assumptions (0.0.1)) of the coverage area (also see chapter 6 (6.1.1)) $\cup\{U|U \in \mathcal{U}\}$ contains the domain D if and only if the fence 1-cycle $\mathcal{F}$ is null-homologous in the Čech complex $\mathcal{C}(\mathcal{U})$.

Theoretically this is a very satisfying result however it is not possible to compute the Čech complex of $\cup\{U|U \in \mathcal{U}\}$ from the network graph Γ alone. Precise distances between nodes are required to determine the higher-dimensional simplices of $\mathcal{C}(\mathcal{U})$. All we have is two radii r_b (broadcast radius) and r_c (coverage radius). It is not possible to derive the Čech complex $\mathcal{C}(\mathcal{U})$ of r_c-disks around nodes from the network graph Γ defined using r_b. It is not even possible to recover the homotopy type of $\mathcal{C}(\mathcal{U})$.

Note however that the convex hull in $\mathbb{R}^2$ of any triple of nodes which are pairwise within the broadcast radius r_b is contained in $\cup\{U|U \in \mathcal{U}\}$. In the extreme case when these nodes are pairwise at a distance exactly r_b form an equilateral triangle in $\mathbb{R}^2$ that is contained in $\cup\{U|U \in \mathcal{U}\}$ (recall U are disks of radius r_c around nodes) only if $r_c \geq r_b/\sqrt{3}$.

Definition 0.0.4. [4] Let $\mathcal{R}$ be defined to be the largest simplicial complex whose 1-skeleton is the network graph Γ. Any $k+1$ nodes of χ defines a k-simplex if these nodes are pairwise within distance r_b. We call $\mathcal{R}$ a *Rips complex* (also referred to as *Vietoris-Rips complex* or *flag complex*).

The Rips complex need not capture the topology of $\cup\{U|U \in \mathcal{U}\}$, but it does contain enough topological information of $\cup\{U|U \in \mathcal{U}\}$.

Theorem 0.0.5 ([5]). *For a set of nodes χ in a domain $D \subset \mathbb{R}^2$ satisfying assumptions of (0.0.1), the sensor cover $\mathcal{U}$ covers D if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ such that $\partial\alpha \neq 0$.*

This is a sufficient condition for coverage, but not a necessary condition. Small perturbation of nodes may permit coverage of the domain D but the topology of $(\mathcal{R}, \mathcal{F})$ may change. The homological criterion is effective when the covers obtained by small perturbations of the nodes all give the same topology of $(\mathcal{R}, \mathcal{F})$.

In problems of type (ii) Hole detection of covers of a given domain defined by sensor networks we consider situations where nodes change position as a function of time. We assume further that **the fence nodes are fixed** (such a situation might arise with sensors used to detect a forest fire, with a ring of fixed nodes outside the forest and nodes within the forest are moved around by environmental forces (eg. animals)). There may not be enough sensors to cover the domain but the effort is to eventually "squeeze" out any holes.

Assumption 0.0.6. [4] Assume that the network communication graph is updated at certain time intervals $0 = t_1 < ... < t_i < ... < t_N = 1$, giving an ordered sequence of communication graphs $\Gamma_i, i = 1, 2, ..., N$. These give the corresponding Rips complexes $\mathcal{R}_i$. We further assume that

1. If two nodes are within broadcast radius at time steps t_i and t_{i+1}, then they remain so for all $t_i \leq t \leq t_{i+1}$.
2. Nodes can go off-line or come on-line, represented by deleting or inserting the nodes in the appropriate graph Γ_i.
3. Fence nodes remain fixed and on-line.

Definition 0.0.7. [4] We amalgamate the sequence of Rips complexes into a single simplicial cell complex $\mathcal{A}R$ as follows: For each $i = 1, 2, ..., N-1$, let $\mathcal{R}_i \cap \mathcal{R}_{i+1}$ denote the largest labeled subcomplex common to $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$. We define the *amalgamated Rips complex* to be the quotient of the disjoint union $\bigsqcup_i \mathcal{R}_i$ obtained by identifying $\mathcal{R}_i \cap \mathcal{R}_{i+1} \subset \mathcal{R}_i$ with $\mathcal{R}_i \cap \mathcal{R}_{i+1} \subset$

$\mathcal{R}_{i+1}$ for each i. Note that $\mathcal{F}$ is a subcomplex of each $\mathcal{R}_i$ and thus is identified with a well defined cycle $\mathcal{F} \subset \mathcal{A}R$.

Theorem 0.0.8 ([5]). *Consider a set of mobile nodes $\chi(t)$ in a domain $D \subset \mathbb{R}^2$ satisfying assumptions of (0.0.1), and (0.0.6) (see also chapter 7 (7.1.1)). Any continuous curve $p : [0,1] \to D$ must have $p(t) \in \cup\{U(t) \mid U(t) \in \mathcal{U}(t)\}$, where $\mathcal{U}(t)$, is the sensor cover at time t for some $0 \leq t \leq 1$ if there exists $[\alpha] \in H_2(\mathcal{A}R, \mathcal{F})$ such that $\partial[\alpha] \neq 0$.*

We now consider a situation where **there are no fixed fence nodes but nodes in the interior can sense if they are near the boundary ∂D and can register themselves as fence nodes**. (For example a very coarse range-finder can detect the presence of a wall within a set distance, without necessarily knowing the distance to the wall).

We consider therefore a system of stationary nodes which can detect the presence of the boundary of the domain ∂D within some fixed *fence radius* r_f. This choice of system leads to a collection of fence nodes $\chi_f \subset \chi$ which spans a *fence subcomplex* $\mathcal{F} \subset \mathcal{R}$, the maximal simplicial complex generated by the fence nodes and edges between them. For getting a simple homological criterion in this case one can use what is known as *Persistent homology*. In this case we use the follwing assumptions:

Assumption 0.0.9. [4] 1. Each node of χ broadcast its unique ID number. Each node of χ can detect the ID of any node within broadcast radius r_s via *strong* signal, or within a larger broadcast radius r_w via a *weak* signal, where $r_w \geq r_s\sqrt{10}$.

2. Nodes have radially symmetric covering domains of cover radius (disk neighbourhoods with centres at the nodes and of radius) $r_c \geq r_s/\sqrt{2}$.

3. Nodes lie in a compact domain $D \subset \mathbb{R}^d$ and can detect the presence of the boundary ∂D within a *fence detection radius* r_f.

4. The restricted domain $D \backslash C$ is connected, where

$$C = \{x \in D \mid |x - \partial D| \leq r_f + r_s/\sqrt{2}\}.$$

5. The fence detection hypersurface $\{x \in D \mid |x - \partial D| \leq r_f\}$ has internal injectivity radius at least r_s.

Such a system gives rise to a pair of Rips complexes, $\mathcal{R}_s$ and $\mathcal{R}_w$, computed at the strong and weak radii respectively, each with fence subcomplexes $\mathcal{F}_s \subset \mathcal{R}_s$ and $\mathcal{F}_w \subset \mathcal{R}_w$.There is a natural inclusion map of pairs

$$\iota : (\mathcal{R}_s, \mathcal{F}_s) \hookrightarrow (\mathcal{R}_w, \mathcal{F}_w), \tag{1}$$

Theorem 0.0.10 ([6]). *For a set of nodes χ in a domain $D \subset \mathbb{R}^d$ satisfying assumptions of (0.0.9), the sensor cover $\mathcal{U}$ covers the restricted domain $D \backslash C$ if the induced homomorphism*

$$\iota_* : H_d(\mathcal{R}_s, \mathcal{F}_s) \to H_d(\mathcal{R}_w, \mathcal{F}_w)$$

is nonzero.

This theorem works because of a squeezing theorem of Čech complex. Let $\mathcal{C}_\epsilon(\chi)$ denote the Čech complex of the cover of χ by balls of radius $\epsilon/2$. Let $\mathcal{R}_\epsilon(\chi)$ denote the Rips complex of the network graph having set of vertices χ and edges between vertices within distance ϵ in $\mathbb{R}^d$.

Theorem 0.0.11 ([6]). *Let $\chi \subset \mathbb{R}^d$. Given $\epsilon' < \epsilon$, there is a chain of inclusions*

$$\mathcal{R}_{\epsilon'}(\chi) \subset \mathcal{C}_\epsilon(\chi) \subset \mathcal{R}_\epsilon(\chi), \quad \textit{if } \frac{\epsilon}{\epsilon'} \geq \sqrt{\frac{2d}{d+1}}.$$

Moreover, this ratio is the smallest for which the inclusions hold in general.

Remark 0.0.12 ([4]). 1. Rips complex is computable, but does not give accurate representation of the cover. The Čech complexe gives the exact homotopy type of the union of the cover, but is not computable with the coarse information available from the network graph. The last theorem removes this difficulty.

2. The technique of comparison between Rips complexes at two different scales is a simple instance of a more general thoery of persistence homology ([7], [8], [9]). This concerns homological properties of a nested family of topological spaces, and is related to spectral sequences. The subject is heavily driven by applications in computational geometry and nonlinear data analysis. Persistent homology is the algebraic topology for the twenty-first century.

3. The homological coverage criteria surveyed here are the beginning of a larger foray of topological ideas in the theories of networks and sensing.

4. Homological methods may allow engineers to focus on designing simpler sensors that are useful in security network, and determine the minimal sensing needed to solve a global problem. Hence it promises to have significant impact on the way systems and sensors are developed and deployed.

Next we consider problems of type (iii) Determination of shape and structure of high-dimensional data sets (shape reconstruction) (refer [10]) Given $X \subset \mathbb{R}^d$. The homology $H_*(X)$ of X contains a lots of information about its shape, like number of components , or holes present in it. Finer features like how many corners, or edges (in general singular points) are present in it can also be determined by associating a new space $T(X)$ called the *tangent complex* of X (defined in [10]), and then calculating the homology of $T(X)$, $H_*(T(X))$. As is well known that the task of calculating $H_*(X)$ or $H_*(T(X))$ gets simplified if X and $T(X)$ can be triangulated with finite number of simplices. Some times these spaces are presented in the form of solutions of a finite system of algebraic equations or inequalities. In such situations it is feasible to determine the set of singular points. If on the other hand only a discrete (finite but a large) set of sample points $N \subset X$ (referred in the literature as *point cloud data*) is given from which one has to get information (Homology, etc.) about X or $T(X)$, then what does one do?

As mentioned above one can associate Čech complex and Rips complex with the points of N as nodes (or vertices) and using open balls of radius ϵ, say, around these points. But the sample points N may not be dense enough to cover X with ϵ-balls. One would also not like to have very large number of points in N to get over this coverage problem, because it will become computationally unviable. One can keep the data set N fixed and vary ϵ. If ϵ is too small then Rips complex $\mathcal{R}_\epsilon$ is a discrete set and if ϵ is too large then $\mathcal{R}_\epsilon$ will be a single simplex. The Grothendieck programme suggests that the topology of a given space is framed in the mappings to or from that space. With this perspective as guide one considers the Persistent homology as earlier. It captures the homological features which persists over a range of parameters $[\epsilon, \epsilon']$.

In [9], Zomorodian and Carlson uses the classification of modules over a polynomial ring (with field coefficient) to compute persistent homology and correlate it with the birth and death of topological features of X.

Remark 0.0.13. Persistent homology was used in [11] to find significant features hidden in a large data set of *pixellated natural images* compressed onto 7-dimensional sphere; most notable is a persistent Klein bottle.

Remark 0.0.14. Since we shall be concentrating on the literature for the first three problems we will not go into details of the last two problems (**(iv) Robot motion planning on a graph (robotics)**, and **(v) Rigourous verification of chaos in dynamical systems**), but conclude with a brief account of each with the hope that in future we may take up these problems also.

The so called "*configuration spaces*" of points give a useful model of autonomous agents (robots) in an environment (see [12], also [13]). Problems of robot motion planning, coordination, cooperation and assembly are directly related to the topological and geometric properties of configuration spaces, including their braid groups. These are amenable to the homology of "cubical complexes".

"Morse index" of a (smooth) Morse function has been generalized to the so called "*Conley index*" which unlike Morse index is not an integer but a homology type of a space. For a Morse function it is a sphere of the same dimension as is the Morse index. It can be defined for non smooth, non-gradient vector fields. For experimentally-generated data on the dynamics of a magneto-elastic ribbon in an oscillatory magnetic field, a Conley index approach gives a rigorous proof that the experimental system is "chaotic" (has positive "topological entropy").

We shall now come to chapter wise description of the materials covered in the dissertation.

In chapter 1 we will give a quick recollection of the well known concepts of chain complex and its homology (in section 1.1), then we proceed to recall the definitions of geometric and combinatorial simplicial complexes (in section 1.2, and section 1.3). We end the chapter by recalling homology of a simplicial complexes (in section 1.4) (see [1], [45]).

In chapter 2 we will introduce cubical sets (in section 2.1), which is analogous to simplicial complex but more conducive to the calculation of homology of sets which are represented as union of pixels and voxels (useful in feature identification, refer to [1] for details). We define cubical chain complexes in $\mathbb{R}^d$ and in a cubical set in section 2.2 and section 2.3 respectively. In the last section 2.4 we give some computation of zero-dimensional homology group $H_0(X)$ of a cubical set.

In chapter 3, usual properties of a homology theory like homotopy, exactness, excision (not explicitly) and dimension properties are given for the cubical homology. In section 3.1 elementary collapse and deformations in cubical set are introduced. In section 3.2 we introduce reduced homology, and acyclic space. In the last section 3.3 exact homology sequence of cubical pairs and Mayer-Vietoris sequence is breifly introduced (see [1] for more details).

Simplicial homology has been used by mathematicians over the last several decades and the computations involving them is also quite familiar to most topologists. Cubical homology on the other hand is a new construct not familiar to many. We are interested in the relationship between cubical homology and the simplicial homology, and also in applying cubical homology to some concrete problem of the real world. This will be done in chapter 4. In the first section 4.1 we will give a comparison between the two complexes, in the next section 4.2 we will discuss about cubical homology of topological polyhedra. In the last section 4.3 we will give a brief discussion about some applications of Cubical homology (see [1] for details).

Chapter 5 is devoted to the algorithms for computing cubical homology of cubical sets using computers.

We compute homology groups of finite cubical sets by associating a chain complex of free abelian groups ($\mathbb{Z}$–modules) and homomorphism. Then we compute the images and kernels of the chain homomorphism and their quotients give homology groups.

All these can be done step-wise using suitable algorithms and linear algebra. The purpose of this chapter is to give concrete algorithm of each of these steps leading to the computation of homology group (refer to [1] for details).

In section 5.1 we outline the steps for developing an algorithm which takes input a free chain complex and gives output its homology. Section 5.2 is devoted to a finite sequence of algorithms which take as input an integer matrix (associated with the boundary operator) and eventually returns its Row echelon form, and thus the kernel and image of the boundary operator. Section 5.3 is devoted to a finite number of algorithms which takes input an integer matrix and returns its Smith normal form. In the last section 5.4 we use the algorithms developed in the previous sections to evolve algorithms which takes input a cubical set and sequentially through these algoritms returns the rank of its homology group.

In Chapter 6 we consider the problem of blanket coverage of planar domain as discussed above.

In section 6.1 we consider the coverage problem with special assumption for the sensors on the boundary of the domain and prove Theorem (6.1.9). In section 6.2 we consider the situation when the coverage fails with a given coverage radius. It is then described how by a modification of coverage radii one can regain coverage of the domain (Hole Repair). Section 6.3 considers several different situations where assumptions A1-A3 have to be modified, like (i) Network in an unbound domain, (ii) Network in a disconnected domain, (iii) Domain D with disconnected boundary ∂D, (iv) restriction on the condition A1 (Opaque Boundary), and derives coverage criteria in each case.

Chapter 7 is a continuation of the coverage problems considered in the last chapter and considers more varied situations of coverage of planar domains and also coverage of 3-dimensional domains. In section 7.1 we consider coverage problems in which Broadcast and Coverage radii varies with nodes. In section 7.2 we consider Coverage of 3 dimensional cylindrical regions. The last section 7.3 considers Coverage Problem in which nodes change position with time - two constructions are discussed, one is of Stacked Rips complex, and the other is of Amalgamated Rips complex.

In the discussion of the previous chapters about variants of the hypotheses on nodes leading to coverage of a planar domain we have in some way or the other retained the restriction on the fence nodes (condition **A3**).

However a more realistic condition will not put such a stringent restriction on its fence nodes, even there may not be any nodes on the fence (the boundary ∂D), instead, nodes which are lying nearby the fence can register themselves as fence nodes.

This situation is considered in Chapter 8 and a new concept of "Persistent homology" is used.

In section 8.1 we consider Coverage problem of a domain without well defined boundary. In section 8.2 we give a brief introduction to Persistent homology.

Chapter 9 deals with Persisitent Algorithm and its use in the feature recognition. We shall give a brief introduction to the work of A. Zomorodian and G. Carlsson on the persistent algorithm ([9]) which computes persistent homology of a filtered complex. We also indicate its use in the work of E. Çarlsson, G. Carlsson, and V. de Silva in feature identification of geometric objects ([10]). Section 9.1 describes the Development of the persistent algorithm. In section 9.2 we give applications of tangent complex and persistent homology in feature identification. In the last section 9.3 we list some open problem on the various works presented in this dissertation.

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SOME PROBLEMS AND APPLICATIONS OF COMPUTATIONAL ALGEBRAIC TOPOLOGY – A SURVEY

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CERTIFICATE

I certify that the dissertation entitled "SOME PROBLEMS AND APPLICATIONS OF COMPUTATIONAL ALGEBRAIC TOPOLOGY - A SURVEY" submitted by Mr. Sainkupar Mn Mawiong in partial fulfilment of the requirement of the degree of Master of Philosophy in Mathematics is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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DECLARATION

I, Sainkupar Mn Mawiong, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

This dissertation is being submitted to the North-Eastern Hill University for the degree of Master of Philosophy in Mathematics.

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Sainkupar Mn Mawiong

PREFACE

Local to global problem is prevalent in various branches of engineering, physical and biological sciences (applied mathematics). Very few calculus based tools are proving to be sufficient. Algebraic and differential topology on the other hand has been dealing with such local-to-global problems for more than hundred years.

Algebraic topology has classically helped in studying (mathematical) geometric objects by associating algebraic invariants with these. Now Algebraic topology has an important role to play in analysing the geometric objects presented by numerical and experimental data.

"Computational Algebraic Topology" has evolved in the recent past to solve many problems of this type arising in engineering, physical and biological sciences. Application of homology to these kinds of problems is new and is in a primitive state. One variant of homology theory especially evolved for such computational purposes is "Cubical Homolgy"

Cubical homology has played important role in analyzing image data and numerically generated data. More sophisticated ideas from topology is needed for further work on understanding today's complex geometric challenges like identifying and classifying geometric properties and abnormalities in a geometric object (medical imaging), and recognizing them even in the presence of small perturbations.

All these require the ability to efficiently compute algebraic topological quantities starting with experimental or numerical data, which are purely combinatorial in nature.

The most significant application of cubical homology will be realized by those who understand its fundamental concepts. The book of Kaczynski, Mishaikow and Mrozek [1], is a a very good source for this and we have adapted it in chapters 2-5.

Even though Cubical homology has many advantages over the simplicial homology, it is a bit rigid (it is suited for pixellated, or voxellated images), and to broaden the horizon of its applicability it has to be adapted for working with polyhedral objects too, because geometrical surfaces are quite often represented by a triangulation.

We will however continue to need simplicial homology coming from "Čech complex" and "Vietoris' Rips Complex" for many types of applications.

Now we will list a small number of applications in which the homology theories mentioned above have played important role and are going to play still more important role:

(i) Blanket coverage of a given domain by covers using sensor networks,

(ii) Hole detection of covers of a given domain defined by sensor networks,

(iii) Determination of shape and structure of high-dimensional data sets (shape reconstruction),

(iv) Robot motion planning on a graph (robotics),

(v) Rigourous verification of chaos in dynamical systems,

Most of these applications are found in ([1], [2], [3], [4]). By its basic property of topological (and even homotopy type) invariance homology groups are considered to be robust tools which can determine the nature of a geometric domain or its subdomain (often given by a data set, called point cloud) irrespective of small perturbation (so called noise).

To define any type of homology of a domain, all one has to do is to associate an appropriate (simplicial, cubical, or any other) "chain complex" to it and define homology. Thanks to the Eilenberg-Steenrod uniqueness theorem of homology, it does not depend on the way the complex is defined. This gives us a lot of freedom to choose the complex which best suits a given situation, and more importantly which is amenable to effective computation using computers.

In problems of type (i) Blanket coverage of a given domain by covers using sensor networks (refer to [4]) where we need to cover a domain (a field, forest, ocean) by neighbourhoods of a finite number of sensors or nodes (these sensors could perform various sensing jobs like video surveillance, detection of radiological or biological hazards, motion detection etc.). Homological methods permit choosing a minimal class of these sensors without the need to know their orientation or location or the need to measure distances. (This has significance in the security and surveillance problems).

If the position of the sensors (*nodes*) are fixed, like in the communication towers of mobile phone networks the coverage problem is simple.

However if we are faced with situations where there is no means to determine relative position of the sensors (*nodes*), then apart from probalistic methods (which need very strong assumptions on the uniformity and density of the random distribution of the sensors), one can use homological methods. Homological methods can even handle situations in which the environment has moved the sensors to arbitrary (unknown) positions.

For obtaining complete or partial solutions to the blanket coverage problem, one needs to impose some minimal conditions on these sensors.

We describe them below for a planar domain:

Assumption 0.0.1. [4] Let D be a compact connected planar domain with connected polygonal boundary ∂D. Let $\chi \subset D$ be the set of sensors (*nodes*). Let $\chi_f \subset \partial D$ be the sensors lying on the vertices of the polygonal boundary (*fence nodes*). We assume that

1. Each node of χ broadcast its unique ID number. Each node of χ can detect the ID of any node within broadcast radius r_b.

2. Nodes have radially symmetric covering domains of cover radius (disk neighbourhoods with centres at the nodes and of radius) $r_c \geq r_b/\sqrt{3}$.

3. Each fence nodes $v \in \chi_f$ knows the ID of its neighbours on ∂D, and these neighbours both lie within distance r_b from v.

Definition 0.0.2. [4] The *network graph* of the above system is a combinatorial graph, Γ, in which vertex set is the set of labeled sensors (nodes), and (undirected) edges correspond to pairs of nodes that are within mutual broadcasting range (within distance r_b). By assumption the fence nodes form a cycle $\mathcal{F} \subset \Gamma$.

The problem at hand is to determine whether the set $\mathcal{U}$ of the disks $B_{r_c}(x)$ of radius r_c around the nodes x of χ is a cover of D.

The input for the problem is the pair $(\Gamma, \mathcal{F})$.

Now the nerve of the cover $\mathcal{U}$ gives the Čech complex $\mathcal{C}(\mathcal{U})$ of $\cup\{U | U \in \mathcal{U}\}$.

Result 0.0.3. Under the above assumptions (0.0.1)) of the coverage area (also see chapter 6 (6.1.1)) $\cup\{U | U \in \mathcal{U}\}$ contains the domain D if and only if the fence 1-cycle $\mathcal{F}$ is null-homologous in the Čech complex $\mathcal{C}(\mathcal{U})$.

Theoretically this is a very satisfying result however it is not possible to compute the Čech complex of $\cup\{U|U \in \mathcal{U}\}$ from the network graph Γ alone. Precise distances between nodes are required to determine the higher-dimensional simplices of $\mathcal{C}(\mathcal{U})$. All we have is two radii r_b (broadcast radius) and r_c (coverage radius). It is not possible to derive the Čech complex $\mathcal{C}(\mathcal{U})$ of r_c-disks around nodes from the network graph Γ defined using r_b. It is not even possible to recover the homotopy type of $\mathcal{C}(\mathcal{U})$.

Note however that the convex hull in $\mathbb{R}^2$ of any triple of nodes which are pairwise within the broadcast radius r_b is contained in $\cup\{U|U \in \mathcal{U}\}$. In the extreme case when these nodes are pairwise at a distance exactly r_b form an equilateral triangle in $\mathbb{R}^2$ that is contained in $\cup\{U|U \in \mathcal{U}\}$ (recall U are disks of radius r_c around nodes) only if $r_c \geq r_b/\sqrt{3}$.

Definition 0.0.4. [4] Let $\mathcal{R}$ be defined to be the largest simplicial complex whose 1-skeleton is the network graph Γ. Any $k+1$ nodes of χ defines a k-simplex if these nodes are pairwise within distance r_b. We call $\mathcal{R}$ a *Rips complex* (also referred to as *Vietoris-Rips complex* or *flag complex*).

The Rips complex need not capture the topology of $\cup\{U|U \in \mathcal{U}\}$, but it does contain enough topological information of $\cup\{U|U \in \mathcal{U}\}$.

Theorem 0.0.5 ([5]). *For a set of nodes χ in a domain $D \subset \mathbb{R}^2$ satisfying assumptions of (0.0.1), the sensor cover $\mathcal{U}$ covers D if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ such that $\partial\alpha \neq 0$.*

This is a sufficient condition for coverage, but not a necessary condition. Small perturbation of nodes may permit coverage of the domain D but the topology of $(\mathcal{R}, \mathcal{F})$ may change. The homological criterion is effective when the covers obtained by small perturbations of the nodes all give the same topology of $(\mathcal{R}, \mathcal{F})$.

In problems of type (ii) Hole detection of covers of a given domain defined by sensor networks we consider situations where nodes change position as a function of time. We assume further that **the fence nodes are fixed** (such a situation might arise with sensors used to detect a forest fire, with a ring of fixed nodes outside the forest and nodes within the forest are moved around by environmental forces (eg. animals)). There may not be enough sensors to cover the domain but the effort is to eventually "squeeze" out any holes.

Assumption 0.0.6. [4] Assume that the network communication graph is updated at certain time intervals $0 = t_1 < ... < t_i < ... < t_N = 1$, giving an ordered sequence of communication graphs $\Gamma_i, i = 1, 2, ..., N$. These give the corresponding Rips complexes $\mathcal{R}_i$. We further assume that

1. If two nodes are within broadcast radius at time steps t_i and t_{i+1}, then they remain so for all $t_i \leq t \leq t_{i+1}$.
2. Nodes can go off-line or come on-line, represented by deleting or inserting the nodes in the appropriate graph Γ_i.
3. Fence nodes remain fixed and on-line.

Definition 0.0.7. [4] We amalgamate the sequence of Rips complexes into a single simplicial cell complex $\mathcal{AR}$ as follows: For each $i = 1, 2, ..., N-1$, let $\mathcal{R}_i \cap \mathcal{R}_{i+1}$ denote the largest labeled subcomplex common to $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$. We define the *amalga-*

mated Rips complex to be the quotient of the disjoint union $\bigsqcup_i \mathcal{R}_i$ obtained by identifying $\mathcal{R}_i \cap \mathcal{R}_{i+1} \subset \mathcal{R}_i$ with $\mathcal{R}_i \cap \mathcal{R}_{i+1} \subset \mathcal{R}_{i+1}$ for each i. Note that $\mathcal{F}$ is a subcomplex of each $\mathcal{R}_i$ and thus is identified with a well defined cycle $\mathcal{F} \subset \mathcal{AR}$.

Theorem 0.0.8 ([5]). *Consider a set of mobile nodes $\chi(t)$ in a domain $D \subset \mathbb{R}^2$ satisfying assumptions of (0.0.1), and (0.0.6) (see also chapter 7 (7.1.1)). Any continuous curve $p : [0,1] \to D$ must have $p(t) \in \cup\{U(t) \mid U(t) \in \mathcal{U}(t)\}$, where $\mathcal{U}(t)$, is the sensor cover at time t for some $0 \leq t \leq 1$ if there exists $[\alpha] \in H_2(\mathcal{AR}, \mathcal{F})$ such that $\partial[\alpha] \neq 0$.*

We now consider a situation where **there are no fixed fence nodes but nodes in the interior can sense if they are near the boundary ∂D and can register themselves as fence nodes.** (For example a very coarse range-finder can detect the presence of a wall within a set distance, without necessarily knowing the distance to the wall).

We consider therefore a system of stationary nodes which can detect the presence of the boundary of the domain ∂D within some fixed *fence radius* r_f. This choice of system leads to a collection of fence nodes $\chi_f \subset \chi$ which spans a *fence subcomplex* $\mathcal{F} \subset \mathcal{R}$, the maximal simplicial complex generated by the fence nodes and edges between them. For getting a simple homological criterion in this case one can use what is known as *Persistent homology.* In this case we use the follwing assumptions:

Assumption 0.0.9. [4] 1. Each node of χ broadcast its unique ID number. Each node of χ can detect the ID of any node within broadcast radius r_s via *strong* signal, or within a larger broadcast radius r_w via a *weak* signal, where $r_w \geq r_s\sqrt{10}$.

2. Nodes have radially symmetric covering domains of cover radius (disk neighbourhoods with centres at the nodes and of radius) $r_c \geq r_s/\sqrt{2}$.

3. Nodes lie in a compact domain $D \subset \mathbb{R}^d$ and can detect the presence of the boundary ∂D within a *fence detection radius* r_f.

4. The restricted domain $D \backslash C$ is connected, where

$$C = \{x \in D \mid | x - \partial D | \leq r_f + r_s/\sqrt{2}\}.$$

5. The fence detection hypersurface $\{x \in D \mid | x - \partial D | \leq r_f\}$ has internal injectivity radius at least r_s.

Such a system gives rise to a pair of Rips complexes, $\mathcal{R}_s$ and $\mathcal{R}_w$, computed at the strong and weak radii respectively, each with fence subcomplexes $\mathcal{F}_s \subset \mathcal{R}_s$ and $\mathcal{F}_w \subset \mathcal{R}_w$.There is a natural inclusion map of pairs

$$\iota : (\mathcal{R}_s, \mathcal{F}_s) \hookrightarrow (\mathcal{R}_w, \mathcal{F}_w), \tag{1}$$

Theorem 0.0.10 ([6]). *For a set of nodes χ in a domain $D \subset \mathbb{R}^d$ satisfying assumptions of (0.0.9), the sensor cover $\mathcal{U}$ covers the restricted domain $D \backslash C$ if the induced homomorphism*

$$\iota_* : H_d(\mathcal{R}_s, \mathcal{F}_s) \to H_d(\mathcal{R}_w, \mathcal{F}_w)$$

is nonzero.

This theorem works because of a squeezing theorem of Čech complex. Let $\mathcal{C}_\epsilon(\chi)$ denote the Čech complex of the cover of χ by balls of radius $\epsilon/2$. Let $\mathcal{R}_\epsilon(\chi)$ denote the Rips complex of the network graph having set of vertices χ and edges between vertices within distance ϵ in $\mathbb{R}^d$.

Theorem 0.0.11 ([6]). *Let $\chi \subset \mathbb{R}^d$. Given $\epsilon' < \epsilon$, there is a chain of inclusions*

$$\mathcal{R}_{\epsilon'}(\chi) \subset \mathcal{C}_\epsilon(\chi) \subset \mathcal{R}_\epsilon(\chi), \quad \text{if } \frac{\epsilon}{\epsilon'} \geq \sqrt{\frac{2d}{d+1}}.$$

Moreover, this ratio is the smallest for which the inclusions hold in general.

Remark 0.0.12 ([4]). 1. Rips complex is computable, but does not give accurate representation of the cover. The Čech complexe gives the exact homotopy type of the union of the cover, but is not computable with the coarse information available from the network graph. The last theorem removes this difficulty.

2. The technique of comparison between Rips complexes at two different scales is a simple instance of a more general thoery of persistence homology ([7], [8], [9]). This concerns homological properties of a nested family of topological spaces, and is related to spectral sequences. The subject is heavily driven by applications in computational geometry and nonlinear data analysis. Persistent homology is the algebraic topology for the twenty-first century.

3. The homological coverage criteria surveyed here are the beginning of a larger foray of topological ideas in the theories of networks and sensing.

4. Homological methods may allow engineers to focus on designing simpler sensors that are useful in security network, and determine the minimal sensing needed to solve a global problem. Hence it promises to have significant impact on the way systems and sensors are developed and deployed.

Next we consider problems of type (iii) Determination of shape and structure of high-dimensional data

sets (shape reconstruction) (refer [10]) Given $X \subset \mathbb{R}^d$. The homology $H_*(X)$ of X contains a lots of information about its shape, like number of components , or holes present in it. Finer features like how many corners, or edges (in general singular points) are present in it can also be determined by associating a new space $T(X)$ called the *tangent complex* of X (defined in [10]), and then calculating the homology of $T(X)$, $H_*(T(X))$. As is well known that the task of calculating $H_*(X)$ or $H_*(T(X))$ gets simplified if X and $T(X)$ can be triangulated with finite number of simplices. Some times these spaces are presented in the form of solutions of a finite system of algebraic equations or inequalities. In such situations it is feasible to determine the set of singular points. If on the other hand only a discrete (finite but a large) set of sample points $N \subset X$ (referred in the literature as *point cloud data*) is given from which one has to get information (Homology, etc.) about X or $T(X)$, then what does one do?

As mentioned above one can associate Čech complex and Rips complex with the points of N as nodes (or vertices) and using open balls of radius ϵ, say, around these points. But the sample points N may not be dense enough to cover X with ϵ-balls. One would also not like to have very large number of points in N to get over this coverage problem, because it will become computationally unviable. One can keep the data set N fixed and vary ϵ. If ϵ is too small then Rips complex $\mathcal{R}_\epsilon$ is a discrete set and if ϵ is too large then $\mathcal{R}_\epsilon$ will be a single simplex. The Grothendieck programme suggests that the topology of a given space is framed in the mappings to or from that space. With this perspective as guide one considers the Persistent homology as earlier. It captures the homological features which persists over a range of parameters $[\epsilon, \epsilon']$.

In [9], Zomorodian and Carlson uses the classification of modules over a polynomial ring (with field coefficient) to compute persistent homology and correlate it with the birth and death of topological features of X.

Remark 0.0.13. Persistent homology was used in [11] to find significant features hidden in a large data set of *pixellated natural images* compressed onto 7-dimensional sphere; most notable is a persistent Klein bottle.

Remark 0.0.14. Since we shall be concentrating on the literature for the first three problems we will not go into details of the last two problems (**(iv) Robot motion planning on a graph (robotics)**, and **(v) Rigourous verification of chaos in dynamical systems**), but conclude with a brief account of each with the hope that in future we may take up these problems also.

The so called "*configuration spaces*" of points give a useful model of autonomous agents (robots) in an environment (see [12], also [13]). Problems of robot motion planning, coordination, cooperation and assembly are directly related to the topological and geometric properties of configuration spaces, including their braid groups. These are amenable to the homology of "cubical complexes".

"Morse index" of a (smooth) Morse function has been generalized to the so called "*Conley index*" which unlike Morse index is not an integer but a homology type of a space. For a Morse function it is a sphere of the same dimension as is the Morse index. It can be defined for non smooth, non-gradient vector fields. For experimentally-generated data on the dynamics of a magneto-elastic ribbon in an oscillatory magnetic field, a Conley index approach gives a rigorous proof that the experimental system is "chaotic" (has positive "topological entropy").

We shall now come to chapter wise description of the materials covered in the dissertation.

In chapter 1 we will give a quick recollection of the well known concepts of chain complex and its homology (in section 1.1), then we proceed to recall the definitions of geometric and combinatorial simplicial complexes (in section 1.2, and section 1.3). We end the chapter by recalling homology of a simplicial complexes (in section 1.4) (see [1], [45]).

In chapter 2 we will introduce cubical sets (in section 2.1), which is analogous to simplicial complex but more conducive to the calculation of homology of sets which are represented as union of pixels and voxels (useful in feature identification, refer to [1] for details). We define cubical chain complexes in $\mathbb{R}^d$ and in a cubical set in section 2.2 and section 2.3 respectively. In the last section 2.4 we give some computation of zero-dimensional homology group $H_0(X)$ of a cubical set.

In chapter 3, usual properties of a homology theory like homotopy, exactness, excision (not explicitly) and dimension properties are given for the cubical homology. In section 3.1 elementary collapse and deformations in cubical set are introduced. In section 3.2 we introduce reduced homology, and acyclic space. In the last section 3.3 exact homology sequence of cubical pairs and Mayer-Vietoris sequence is breifly introduced (see [1] for more details).

Simplicial homology has been used by mathematicians over the last several decades and the computations involving them is also quite familiar to most topologists. Cubical homology on the other hand is a new construct not familiar to many. We are interested in the relationship between cubical homology and the simplicial homology, and also in applying cubical homology to some concrete problem of the real world. This will be done in chapter 4. In the first section 4.1 we will give a comparison between the two complexes, in the next section 4.2 we will discuss about cubical homology of topological polyhedra. In the last section 4.3 we will give a brief discussion about some applications of Cubical homology (see [1] for details).

Chapter 5 is devoted to the algorithms for computing cubical homology of cubical sets using computers.

We compute homology groups of finite cubical sets by associating a chain complex of free abelian groups ($\mathbb{Z}-$modules) and homomorphism. Then we compute the images and kernels of the chain homomorphism and their quotients give homology groups.

All these can be done step-wise using suitable algorithms and linear algebra. The purpose of this chapter is to give concrete algorithm of each of these steps leading to the computation of homology group (refer to [1] for details).

In section 5.1 we outline the steps for developing an algorithm which takes input a free chain complex and gives output its homology. Section 5.2 is devoted to a finite sequence of algorithms which take as input an integer matrix (associated with the boundary operator) and eventually returns its Row echelon form, and thus the kernel and image of the boundary operator. Section 5.3 is devoted to a finite number of algorithms which takes input an integer matrix and returns its Smith normal form.

In the last section 5.4 we use the algorithms developed in the previous sections to evolve algorithms which takes input a cubical set and sequentially through these algoritms returns the rank of its homology group.

In Chapter 6 we consider the problem of blanket coverage of planar domain as discussed above.

In section 6.1 we consider the coverage problem with special assumption for the sensors on the boundary of the domain and prove Theorem (6.1.9). In section 6.2 we consider the situation when the coverage fails with a given coverage radius. It is then described how by a modification of coverage radii one can regain coverage of the domain (Hole Repair). Section 6.3 considers several different situations where assumptions A1-A3 have to be modified, like (i) Network in an unbound domain, (ii) Network in a disconnected domain, (iii) Domain D with disconnected boundary ∂D, (iv) restriction on the condition A1 (Opaque Boundary), and derives coverage criteria in each case.

Chapter 7 is a continuation of the coverage problems considered in the last chapter and considers more varied situations of coverage of planar domains and also coverage of 3-dimensional domains. In section 7.1 we consider coverage problems in which Broadcast and Coverage radii varies with nodes. In section 7.2 we consider Coverage of 3 dimensional cylindrical regions. The last section 7.3 considers Coverage Problem in which nodes change position with time - two constructions are discussed, one is of Stacked Rips complex, and the other is of Amalgamated Rips complex.

In the discussion of the previous chapters about variants of the hypotheses on nodes leading to coverage of a planar domain we have in some way or the other retained the restriction on the fence nodes (condition **A3**).

However a more realistic condition will not put such a stringent restriction on its fence nodes, even there may not be any nodes on the fence (the boundary ∂D), instead, nodes which are lying nearby the fence can register themselves as fence nodes. This situation is considered in Chapter 8 and a new concept of "Persistent homology" is used.

In section 8.1 we consider Coverage problem of a domain without well defined boundary. In section 8.2 we give a brief introduction to Persistent homology.

Chapter 9 deals with Persisitent Algorithm and its use in the feature recognition. We shall give a brief introduction to the work of A. Zomorodian and G. Carlsson on the persistent algorithm ([9]) which computes persistent homology of a filtered complex. We also indicate its use in the work of E. Carlsson, G. Carlsson, and V. de Silva in feature identification of geometric objects ([10]). Section 9.1 describes the Development of the persistent algorithm. In section 9.2 we give applications of tangent complex and persistent homology in feature identification. In the last section 9.3 we list some open problem on the various works presented in this dissertation.

List of Symbols

$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of positive integers (or set of natural numbers)
$\mathbb{Z}^+$	$\mathbb{N} \cup \{0\}$ (set of nonnegative integers)
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
$H \subseteq G$	H is a subset of G
$H \subset G$	H is a proper subset of G
$H \trianglelefteq G$	H is a normal subgroup of G
$\|G : H\|$	index of H in G
G/N	factor group
HK	$\{hk \mid h \in H,\ k \in K\}$
$H \times K$	direct product of the groups H and K
$H \oplus K$	direct sum of the groups H and K
aH	$\{ah \mid h \in H\}$, left coset of H
$G \cong H$	G and H are isomorphic
$\langle a \rangle$	$\{a^n \mid n \in \mathbb{Z}\}$, the cyclic group generated by a
$\langle a_1, a_2, \cdots, a_n \rangle$	subgroup generated by $a_1, a_2, \cdots, a_n$

$o(g)$	order of g
$\|G\|$	order of the group G
$[x, y]$	$xyx^{-1}y^{-1}$, the commutator of x and y
$[H, K]$	$\langle\{[x, y] \mid x \in H, y \in K\}\rangle$
	of all irreducible characters of G
$\ker\chi$	$\{g \in G \mid \chi(g) = \chi(1)\}$, kernel of χ
$\det(A)$	determinant of A
$\mathrm{tr}(A)$	trace of A
$\mathrm{Ker}\,\phi$	kernel of the homomorphism ϕ
$GL(n, \mathbb{F})$	group of $n \times n$ non-singular matrices over the field $\mathbb{F}$
$M_n(\mathbb{F})$	algebra of $n \times n$ matrices over $\mathbb{F}$
$SL(n, \mathbb{F})$	$\{A \in M_n(\mathbb{F}) \mid \det(A) = 1\}$
G^n	direct product of n copies of G
$\prod_{i=1}^{n} G_i$	direct product of $G_1, G_2, \ldots, G_n$
$\bigoplus_{i=1}^{n} G_i$	direct sum of $G_1, G_2, \ldots, G_n$
$\mathrm{diag}(G \times G)$	$\{(g, g) \mid g \in G\}$, diagonal of $G \times G$
$C = \{(C_q, \partial_q)\}_{q \in \mathbb{Z}}$	Chain complex
C_q	Free $\mathbb{R}-$ module
$\partial_q : C_q \to C_{q-1}$	is a $\mathbb{R}-$ homomorphism
$Z_q(C)$	$q^{th}-$ cycles
$B_q(C)$	$q^{th}-$ boundaries
$H_q = \dfrac{Z_q(C)}{B_q(C)}$	$q^{th}-$ homology

I	Elementary interval
Q	Elementary cube
$\mathcal{K}^d$	The set of all elementary cubes in $\mathbb{R}^d$
$\mathcal{K}$	The set of all elementary cubes
emb Q	embedding number of Q
dim$Q$	dimension of Q
$\mathcal{K}(X)$	Cubical complex of X
$\hat{\mathcal{K}}_k^d$	The set of all elementary k-dimensional chains of $\mathbb{R}^d$
C_k^d	The group of k-dimensional chains of $\mathbb{R}^d$
$\|c\|$	Support of the chain c
$< c_1, c_2 >$	Scalar product of the chains c_1 and c_2
$c_1 \Diamond c_2$	Cubical product of c_1 and c_2
$cc_X(x)$	Connected component of x in X
$ecc_X(x)$	Edge connected component of x in X
$\mathcal{K}_{max}(X)$	Maximal face in X

Contents

Chapter 1

Homology of Simplicial Complexes

In this chapter we will give a quick recollection of the well known concepts of chain complex and its homology then we proceed to recall the definitions of geometric and combinatorial simplicial complexes. We end the chapter by recalling homology of a simplicial complexes (see [1]).

1.1 Abstract Chain Complex and its homology

Definition 1.1.1. ([46]) Let R be a commutative ring with identity. A chain complex $C = \{(C_q, \partial_q)\}_{q\in\mathbb{Z}}$, is a sequence of pair $\{(C_q, \partial_q)\}$, where C_q is a free $\mathbb{R}$- module $\forall q$ and $\partial_q : C_q \rightarrow C_{q-1}$ is a $\mathbb{R}$-homomorphism $\forall q$, such that $\partial_q o \partial_{q+1} = 0 \; \forall q$

In most cases $C_q = 0$ if $q < 0$. An element of C_q is said to have dimension q. $Z_q(C)$, $B_q(C)$ are submodules of C_q defined by

$Z_q(C) = \operatorname{Ker} \partial_q$ called *q-cycles*,

$B_q = \operatorname{Im} \partial_{q+1}$ called *q-boundaries*.

Definition 1.1.2. ([46]) The q-th *homology* module of C is defined by $H_q(C) = \dfrac{Z_q(C)}{B_q(C)}$.

By construction, $H_q(C)$ is an R-module. If $z \in Z_q(C)$ we write $\overline{z}$ for its class in $H_q(C)$.

Define a chain map $f : C \to C'$ to be a sequence of homomorphism $f_q : C_q \to C_{q'}$ such that $\partial'_q o f_q = f_{q-1}\, o \partial_q$.

$$\begin{array}{ccc} C_q & \xrightarrow{f_q} & C_{q'} \\ \downarrow \partial_q & & \downarrow \partial'_q \\ C_{q-1} & \xrightarrow{f_{q-1}} & C'_{q-1} \end{array}$$

A chain map $f : C \to C'$ sends cycles to cycles and boundaries to boundaries. Hence f induces a well-defined homomorphism

$H_q(f) : H_q(C) \to H_q(C')$,
$H_q(f)(\overline{z}) = \overline{f_q(z)}$.

Thus H defines a functor from the category of chain cmplexes over R and chain maps to the category of R-modules and homomorphisms.

Definition 1.1.3. ([46]) A *chain homotopy* between the chain maps

$f = \{f_q : C_q \to C'_q\}$ and $g = \{g_q : C_q \to C'_q\}$ is a sequence $D = \{D_q : C_q \to C'_{q+1}\}$ of homomorphism such that $\partial'_{q+1} D_q + D_{q-1}\partial_q = f_q - g_q$.

We write $f \simeq g$. ($If\ C_q = 0\ for\ q < 0$, the equation reads $\partial'_1 D_0 = f_0 - g_0$.)

Proposition 1.1.4. *([46]) Chain homotopic maps induce equal maps in homology.*

Proof. Let $\overline{z} \in H_q(C)$ be a class with representative $z \in Z_q(C)$. then $f_q(z) - g_q(z) = \partial'_{q+1} D_q(z) \in B_q(C)$. Hence $H_q(f)(\overline{z}) = H_q(g)(\overline{z})$. □

1.2 Simplicial complex (Geometric definition)

We recall now the definition of (geometric) simplicial complex by giving a series of definitions (see [45]):

Definition 1.2.1. ([45]) Let $S = \{p_0, p_1, \ldots, p_k\} \subseteq \mathbb{R}^d$. A *linear combination of elements of* S is $x = \sum_{i=0}^{k} \lambda_i p_i$,for some $\lambda_i \in \mathbb{R}$. An *affine combination* is a linear combination with $\sum_{i=0}^{k} \lambda_i = 1$. A *convex combination* is an affine combination with $\lambda_i \geq 0$,for all i. The set of all convex combinations is the *convex hull.*

Definition 1.2.2. ([45]) A set S is *linearly (affinely) independent* if no point in S is a linear (affine) combination of the other points in S.

Definition 1.2.3. ([45]) A *k-simplex* is the convex hull of $k+1$ affinely independent points $S = \{v_0, v_1, \ldots, v_k\}$. The points in S are the *vertices* of the simplex.

Definition 1.2.4. ([45]) Let σ be a $k-$ simplex defined by $S = \{v_0, v_1, \ldots, v_k\}$. A simplex τ defined by $T \subseteq S$ is a *face* of σ (denoted $\tau \leq \sigma$) and has σ as a *coface* of τ (denoted $\tau \geq \sigma$). In particular, $\sigma \leq \sigma$ and $\sigma \geq \sigma$.

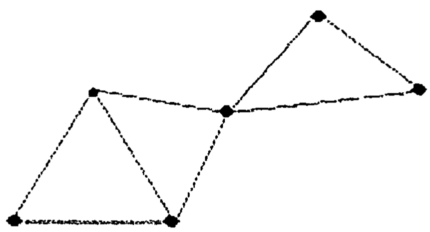
Definition 1.2.5. ([45]) **(simplicial complex)** A *(finite) simplicial complex* K is a finite set of simplices such that

1. $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$,

2. $\sigma, \sigma^{'} \in K \Rightarrow \sigma \bigcap \sigma^{'} \leq \sigma, \sigma^{'}$ or $\sigma \bigcap \sigma^{'} = \phi$.

The *dimension* of K is $\dim K = max\{\dim \sigma \mid \sigma \in K\}$. The *vertices* of K are the zero-simplices in K. A simplex is *principal* if it has no proper coface in K (i.e $\sigma \leq \bar{\sigma} \Rightarrow \sigma = \bar{\sigma}$).

Here, *proper* has the same definition as for sets. So, a simplicial complex is a collection of simplices that fit together nicely, as shown in Figure 1.1, as opposed to simplices in 1.2.



Figurc 1.1:

Remark 1.2.6. ([45]) (i) *size of a simplex*: For a k-simplex σ
Number of 0-simplices= $k + 1 = \binom{k+1}{1}$
Number of 1-simplices= $\binom{k+1}{2}$

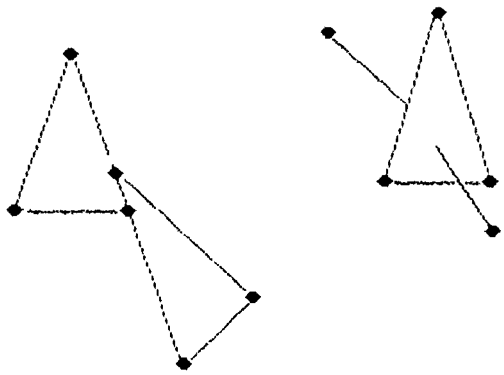


Figure 1 2

Number of l-simplices = $\binom{k+1}{l+1}$
So on (see the table below).
Size of a k-simplex is defined to be the sum $\binom{k+1}{1} + \cdots + \binom{k+1}{k+1}$
(ii) We define empty set ϕ on a (-1)-simplex.

(iii) If we add 1 to the left of each row of the table then we get Pascal's triangle.See figure 1.3.

k/l	0	1	2	3
0	1	0	0	0
1	2	1	0	0
2	3	3	1	0
3	4	6	4	1
4	?	?	?	?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Figure 1.3 Pascal's triangle

1.3 Simplicial Complex (Abstract Definition)

We now recall the purely combinatorial definition of simplicial complex, without any reference to an ambient space (see [45]).

Definition 1.3.1. ([45]) **(abstract simplicial complex)** An *abstract simplicial complex* is a set K, together with a collection $\mathcal{S}$ of subsets of K called *(abstract) simplices* such that:

1. For all $v \in K, \{v\} \in \mathcal{S}$. We call the sets $\{v\}$ the vertices of K.

2. If $\tau \subseteq \sigma \in \mathcal{S}$,then $\tau \in \mathcal{S}$

When it is clear from context what $\mathcal{S}$ is, we refer to K as a complex. We say σ is a k-simplex (or a simplex of *dimension* k) if the number of elements of σ, $|\sigma| = k + 1$. If $\tau \subseteq \sigma$, τ is a face of σ, and σ is a *coface* of τ. Note that the definition automatically allows for ϕ as a (-1)-simplex.The abstract definition affirms the notion that topology only cares about how the simplices are connected, and not how they are placed within a space. We now relate this abstract set-theoretic definition to the geometric one by extracting the combinatorial structure of a (geometric)simplicial complex.

Definition 1.3.2. ([45]) **(vertex scheme of a geometric simplicial complex)** Let K be a (geometric) simplicial complex with vertices V and let $\mathcal{S}$ be the collection of all subsets $\{v_0, v_1, \ldots, v_k\}$ of V such that the vertices $v_0, v_1, \ldots, v_k$ span a simplex of K. The collection $\mathcal{S}$ is called the *vertex scheme* of K. Clearly, the set K and the collection $\mathcal{S}$ together form an abstract simplicial complex. It allows us to compare simplicial complexes easily, using isomorphisms between sets.

Definition 1.3.3. ([45]) Let K_1, K_2 be abstract simplicial complexes with vertices V_1, V_2 and subset collection $\mathcal{S}_1$, $\mathcal{S}_2$ respectively. An isomorphism between K_1, K_2 is a bijection $\varphi : V_1 \to V_2$, such that the sets in S_1 and S_2 are the same under the renaming of the vertices by φ and its inverse.

Theorem 1.3.4. *([45]) For every abstract simplicial complex $(K, \mathcal{S})$, $\mathcal{S}$ is the vertex scheme of some simplicial complex. Two simplicial complexes are isomorphic iff their vertex schemes are isomorphic as abstract simplicial complexes.(refer to [45] for proof)*

Definition 1.3.5. ([45]) **(Goemetric Realization)** If the simplices $\mathcal{S}$ of an abstract simplicial complex K_1 is isomorphic with the vertex scheme $\mathcal{S}$ of the simplicial comlex K_2, we call K_2 a *geometric realization* of K_1. It is uniquely determined up to an isomorphism, linear on the simplices.

Remark 1.3.6. ([45]) A (finite) abstract simplicial complex is purely a combinatorial object, which can be easily stored and manipulated in a computer. For a (finite) geometric simplicial complex the realization map into the ambient space can approximately be represented in a computer using floating point representation.

1.4 Chain complex associated with a simplicial complex

Definition 1.4.1. ([1]) Two ordering $(v_0, v_1, \dots, v_n)$ and $(v_{p_0}, v_{p_1}, \dots, v_{p_n})$ of vertices of an n-simplex S are said to have the same *orientation* if one can get one from the other by an even permutation This defines an equivalence relation on the set of all orderings of vertices of S. An *oriented simplex* $= [v_0, v_1, \ldots, v_n]$ is an equivalence class of the ordering $(v_0, v_1, \ldots, v_n)$ of vertices of a simplex $S = conv\{v_0, v_1, \ldots, v_n\}$. If S and T are geometric simplices, the corresponding oriented simplices are denoted respectively by σ and τ.

It is easy to see that for $n > 0$ the above equivalence relation divides the set of all orderings into two equivalence classes. Hence we may say that the ordering that are not in the same equivalence class have the *opposite orientation.* We shall denote the pairs of opposite oriented simplices by σ, σ' or τ, τ'. An oriented simplicial complex is a simplicial complex S with one of the two equivalence classes chosen for each simplex of S. The orientation of a simplex and its faces may be done arbitrarily; they do not need to be related.

Example 1.4.2. ([1]) Let S be a triangle in $\mathbb{R}^2$ spanned by vertices v_1, v_2, v_3. Then the orientation equivalence class $\sigma = [v_1, v_2, v_3]$ contains the orderings $(v_1, v_2, v_3), (v_2, v_3, v_1), (v_3, v_1, v_2)$ and the opposite orientation σ' contains $(v_1, v_3, v_2), (v_2, v_1, v_3)$, (v_3, v_2, v_1).

Definition 1.4.3. ([1]) (i) Let S^n be the set of all oriented n-simplices of S. Let $\mathbb{Z}(S^n)$ be the free abelian group generated by S^n; this can also be defined as the set of all functions $c : S^n \to \mathbb{Z}$, generated by the characteristic functions $\hat{\sigma} : S^n \to \mathbb{Z}$, of $\sigma \in S^n$ which sends σ to 1 and rest of the elements of S^n to 0.

(ii) The group of *n-chains* denoted by $C_n(S)$ is the subgroup of $Z(S^n)$ consisting of those functions c that satisfy the identity $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same n-simplex. We put $C_n(S) = 0$ if S contains no n-simplices.

Proposition 1.4.4. *([1]) The group $C_n(\mathcal{S})$ is a free abelian group generated by functions $\tilde{\sigma} = \hat{\sigma} - \hat{\sigma'}$ given by the formula*

$$\tilde{\sigma}(\tau) = \begin{cases} 1 & \text{if } \tau = \sigma, \\ -1 & \text{if } \tau = \sigma', \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma, \sigma', \tau \in S^n$ and σ, σ' are opposite orientations of the same simplex. This set of generators is not a basis since $\tilde{\sigma'} = -\tilde{\sigma}$ for any pairs σ, σ'. A basis is obtained by selecting one $\tilde{\sigma}$ from each pair of oriented simpices with mutually opposite orientations.

Remark 1.4.5. ([1]) The *simplicial boundary operator* $\partial_k : C_k(\mathcal{S}) \to C_{k-1}(\mathcal{S})$ is defined on any basic element $[v_0, v_1, \ldots, v_n]$ by the formula

$$\partial_k[v_0, v_1, \ldots, v_n] = \sum_{i=0}^{n}(-1)^i[v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$$

Thus $C(\mathcal{S}) = \{C_k(\mathcal{S}), \partial_k\}_{k\in\mathbb{Z}}$ has the structure of a chain complex. The homology of that chain complex is the sequence of abelian groups

$H_*(\mathcal{S}) = \{H_n(\mathcal{S})\} = \{\ker \partial_n / \mathrm{im}\ \partial_{n+1}\}$, as defined earlier.

Chapter 2

Cubical Homology

In this chapter we will introduce cubical sets, which is analogous to simplicial complex but more conducive to the calculation of homology of sets which are represented as union of pixels and voxels.(useful in feature identification).(refer to [1] for details)

2.1 Cubical sets and Cubical complex

Definition 2.1.1. ([1]) An *elementary interval* is a closed interval $I \subset \mathbb{R}$ of the form $I = [l, l+1]$ or $I = [l, l]$ for some $l \in \mathbb{Z}$. To simplify the notation, we write $[l] = [l, l]$ for an interval that contains only one point. Elementary intervals that consist of a single point are *degenerate*, while those of lenght 1 are *nondegenerate*.

Example 2.1.2. ([1]) The interval $[2, 3]$, $[-15, -14]$, and $[7]$ are all examples of elementary intervals. On the other hand,$[\frac{1}{2}, \frac{3}{2}]$ is not an elementary interval since the boundary points are not integers. Similarly, $[1, 3]$ is not an elementary interval since the length of the interval is greater than 1.

Definition 2.1.3. ([1]) An *elementary cube* Q is a finite product of elementary intervals, that is, $Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d$ where each I_i is an elementary interval. The set of all elementary cubes in $\mathbb{R}^d$ is denoted by$\mathcal{K}^d$. The set of all elementary cubes is denoted by $\mathcal{K}$, namely $\mathcal{K} = \bigcup_{d=1}^{\infty} \mathcal{K}^d$

Definition 2.1.4. ([1]) Let $Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d$ be an elementary cube. The *embedding number* of Q is denoted by *emb* Q and is defined to be d since $Q \subset \mathbb{R}^d$. The interval I_i is referred to as the ith *component* of Q and is written as $I_i(Q)$. The *dimension* of Q is defined to be the number of nondegenrate components in Q and is denoted by $\dim Q$. Observe that if *emb* $Q = d$, then $Q \in \mathcal{K}^d$. We also let $\mathcal{K}_k = \{Q \in \mathcal{K} | \dim Q = k\}$ and $\mathcal{K}_k^d = \mathcal{K}_k \cap \mathcal{K}^d$.

Proposition 2.1.5. *([1]) Let $Q \in \mathcal{K}_k^d$ and $P \in \mathcal{K}_{k'}^{d'}$. Then $Q \times P \in \mathcal{K}_{k+k'}^{d+d'}$.*

Definition 2.1.6. ([1]) Let $Q, P \in \mathcal{K}$. If $Q \subset P$, then Q is a *face* of P. this is denoted by $Q \preceq P$. If $Q \preceq P$ and $Q \neq P$, then Q is a *proper face* of P, which is written as $Q \prec P$.Q is a *primary* face of P if Q is a face of P and $\dim Q = \dim P - 1$.

Definition 2.1.7. ([1]) A set $X \subset \mathbb{R}^d$ is *cubical* if X can be written as a finite union of elementary cubes. If $X \subset \mathbb{R}^d$ is a cubical set, then we adopt the following notation:

$\mathcal{K}(X) = \{Q \in \mathcal{K} | Q \subset X\}$ and call it the *cubical complex of* X.

$\mathcal{K}_k(X) = \{Q \in \mathcal{K}(X) | \dim Q = k\}$.

In analogy with graphs, the elements of $\mathcal{K}_0(X)$ are the *vertices* of X and the elements of $\mathcal{K}_1(X)$ are the *edges* of X. More generally, the elements of $\mathcal{K}_k(X)$ are the *k-cubes* of X.

Proposition 2.1.8. *([1]) If $X \subset \mathbb{R}^d$ is cubical, then X is closed and bounded.*

2.2 Cubical chain complex in $\mathbb{R}^d$

To define chain complex associative to a cubical set (or complex) we need to introduce some more definition.

Definition 2.2.1. ([1]) Let I be an elementary interval. The associated *elementary cell* is

$$I^o = \begin{cases} (l, l+1) & \text{if } I = [l, l+1], \\ [l] & \text{if } I = [l, l]. \end{cases}$$

We extend this definition to a general elementary cube $Q^o = I_1^o \times I_2^o \times \cdots \times I_d^o \subset \mathbb{R}^d$ by defining the associated *elementary cell* as $Q^o = I_1^o \times I_2^o \times \cdots \times I_d^o$

Given a point in $\mathbb{R}^d$, we need to be able to describe the elementary cell or cube that contains it. For this, the following two functions are useful. Let $x \in \mathbb{R}$,

$floor(x) = max\{n \in \mathbb{Z} | n \leq x\}$

$ceil(x) = min\{n \in \mathbb{Z} | x \leq n\}$

Proposition 2.2.2. *Elementary cells have the following properties:*

(i) $\mathbb{R}^d = \cup\{Q^o | Q \in \mathcal{K}^d\}$

(ii) $A \subset \mathbb{R}^d$ bounded implies that $card\{Q^o \in \mathcal{K}^d | Q \cap A \neq \Phi\} < \infty$ where, given a set S, $card(S)$ stands for its cardinality. that is, the number of its elements.

(iii) If $P, Q \in \mathcal{K}^d$, then $P^o \cap Q^o = \Phi$ or $P = Q$.

(iv)For every $Q \in \mathcal{K}$ $clQ = Q$.

(v) $Q \in \mathcal{K}^d$ implies that $Q = \cup\{P^o | P \in \mathcal{K}^d$ such that $P^o \subset Q\}$

(vi) If X is a cubical set and $Q^o \cap X \neq \Phi$ for some elementary cube Q, then $Q \subset X$. (Proof can be found in [1])

Now we come to the definition of chain complex associated with a cubical set (complex).

With each elementary $k-cube$ $Q \in \mathcal{K}_k^d$ we associate an algebraic object $\hat{Q}$ called an *elementary* $k-chain$ of $\mathbb{R}^d$. The set of all elementary $k-chains$ of $\mathbb{R}^d$ is denoted by $\hat{\mathcal{K}}_k^d = \{\hat{Q} | Q \in \mathcal{K}_k^d\}$ and the set of all *elementary chains* of $\mathbb{R}^d$ is given by $\hat{\mathcal{K}}^d = \bigcup\limits_{k=0}^{\infty} \hat{\mathcal{K}}_k^d$.

Given any finite collection $\{\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_m\} \subset \hat{\mathcal{K}}_k^d$ of $k-$ dimensional elementary chains, we are allowed to consider sums of the form $c = \alpha_1\hat{Q}_1 + \alpha_2\hat{Q}_2 + \cdots + \alpha_m\hat{Q}_m$, where α_i are arbitrary integers. If all the $\alpha_i = 0$, then we let $c = 0$. We call c a k-chain; C_k^d is the set of these k-chains.The addition of k-chains is defined by

$\sum\alpha_i\hat{Q}_i + \sum\beta_i\hat{Q}_i = \sum(\alpha_i + \beta_i)\hat{Q}_i$.

Given an arbitrary k-chain $c = \sum\limits_{i=0}^{m}\alpha_i\hat{Q}_i$, there is an inverse element $-c = \sum\limits_{i=0}^{m}(-\alpha_i)\hat{Q}_i$ with the property that $c + (-c) = 0$. C_k^d is a free abelian group with basis $\hat{\mathcal{K}}_k^d$.

The elementary k-chains defined above is to associate algebraic analogue of elementary cubes. There is another way to view this:

For each $Q \in \mathcal{K}_k^d$, define $\hat{Q} : \mathcal{K}_k^d \longrightarrow \mathbb{Z}$ by

$$\hat{Q}(P) = \begin{cases} 1 & \text{if } P = Q, \\ 0 & \text{otherwise} \end{cases}$$

and in slight abuse of notation let $0 : \mathcal{K}_k^d \longrightarrow \mathbb{Z}$ be the zero function, namely $0(Q) = 0$ for all $Q \in \mathcal{K}_k^d$. $\hat{Q}$ is the *elementary chain dual* to the elementary cube Q. We can take finite linear combinations of these.

Definition 2.2.3. ([1]) The group C_k^d of $k-$*dimensional chains* of $\mathbb{R}^d$ is the free abelian group generated by the elementary chains of $\hat{\mathcal{K}}_k^d$. Thus the elements of C_k^d are the functions $c : \mathcal{K}_k^d \longrightarrow \mathbb{Z}$ such that $c(Q) = 0$ for all but a finite number of $Q \in \mathcal{K}_k^d$. In particular, $\hat{\mathcal{K}}_k^d$ is the basis for C_k^d. So $C_k^d = \mathbb{Z}(\hat{\mathcal{K}}_k^d)$ (This is the notation of a free abelian group generated by $\hat{\mathcal{K}}_k^d$).

If $c \in C_k^d$, then $\dim c = k$.

Proposition 2.2.4. *The map $\phi : \mathcal{K}_k^d \longrightarrow \hat{\mathcal{K}}_k^d$ given by $\phi(Q) = \hat{Q}$ is a bijection. (For proof refer to [1])*

Definition 2.2.5. ([1]) Let $c \in C_k^d$. The *support* of the chain c is the cubical set

$$|c| = \cup\{Q \in \mathcal{K}_k^d | c(Q) \neq 0\}.$$

Proposition 2.2.6. *Support satisfies the following properties:*

(i) $|c| = \Phi$ if and only if $c = 0$.

(ii) Let $\alpha \in \mathbb{Z}$ and $c \in C_k^d$; then

$$|\alpha c| = \begin{cases} \phi & \text{if } \alpha = 0, \\ |c| & \text{if } \alpha \neq 0. \end{cases}$$

(iii) If $Q \in \mathcal{K}$, then $|\hat{Q}| = Q$.

(iv) If $c_1, c_2 \in C_k^d$, then $|c_1 + c_2| \in |c_1| \cup |c_2|$

(For proof refer to [1])

Definition 2.2.7. ([1]) Consider $c_1, c_2 \in C_k^d$ where $c_1 = \sum_{i=0}^{m} \alpha_i \hat{Q}_i$ and $c_2 = \sum_{i=0}^{m} \beta_i \hat{Q}_i$. The *scalar product* of the chains c_1 and c_2 is defined as $< c_1, c_2 >= \sum_{i=0}^{m} \alpha_i \beta_i$.

Proposition 2.2.8. *The scalar product defines a mapping*
$< \cdot, \cdot >: C_k^d \times C_k^d \longrightarrow \mathbb{Z}$
$(c_1, c_2) \longmapsto < c_1, c_2 >$, *which is bilinear.*
(For proof refer to [1])

Definition 2.2.9. ([1])Given two elementary cubes $P \in \mathcal{K}_k^d$ and $Q \in \mathcal{K}_{k'}^{d'}$, set

$\hat{P} \diamondsuit \hat{Q} = \widehat{P \times Q}$.

This definition extends to arbitrary chains $c_1 \in C_k^d$ and $c_2 \in C_{k'}^{d'}$, by

$c_1 \diamondsuit c_2 = \sum_{P \in \mathcal{K}_k, Q \in \mathcal{K}_{k'}} < c_1, \hat{P} >< c_2, \hat{Q} > \widehat{P \times Q}$.

The chain $c_1 \diamondsuit c_2 \in C_{k+k'}^{d+d'}$ is called the *cubical product* of c_1 and c_2.

Proposition 2.2.10. *Let* c_1,c_2,c_3 *be any chains.*
(i) $c_1 \diamondsuit 0 = 0 \diamondsuit c_1 = 0$.
(ii) $c_1 \diamondsuit (c_2 + c_3) = c_1 \diamondsuit c_2 + c_1 \diamondsuit c_3$,*provided* $c_2, c_3 \in C_k^d$
(iii) $(c_1 \diamondsuit c_2) \diamondsuit c_3 = c_1 \diamondsuit (c_2 \diamondsuit c_3)$.
(iv) If $c_1 \diamondsuit c_2 = 0$, *then* $c_1 = 0$ *or* $c_2 = 0$
(v) $|c_1 \diamondsuit c_2| = |c_1| \times |c_1|$.
(For proof refer to [1])

Proposition 2.2.11. *Let* $\hat{Q}$ *be an elementary cubical chain of* $\mathbb{R}^d$ *with* $d > 1$. *Then there exist unique elementary cubical*

chains $\hat{I}$ and $\hat{P}$ with embedding number emb $I = 1$ and emb $P = d-1$ such that
$\hat{Q} = \hat{I} \diamondsuit \hat{P}$.
(For proof refer to [1])

2.3 Cubical chain complex in a cubical set

So far we have considered chain in $\mathbb{R}^d$. Now we start with a cubical set $X \subseteq \mathbb{R}^d$.

Proposition 2.3.1. *Let $X \subset \mathbb{R}^d$ be a cubical set. Let $\hat{\mathcal{K}}_k(X) = \{\hat{Q} | Q \in \mathcal{K}_k(X)\}$. $C_k(X)$ is the subgroup of C_k^d generated by the elements of $\hat{\mathcal{K}}_k(X)$ and is referred to as the set of $k-$chains of X.*

It can be easily checked that $C_k(X) = \{c \in C_k^d |\ |c| \subset X\}$
(For proof refer to [1])

Proposition 2.3.2. *For any $c \in C_k(X)$.*
$c = \sum_{Q \in \mathcal{K}_k(X)} < c, \hat{Q} > \hat{Q}$.
(For proof refer to [1])

Definition 2.3.3. ([1]) Given $k \in \mathbb{Z}$ the cubical boundary operator or cubical boundary map $\partial_k : C_k^d \longrightarrow C_{k-1}^d$ is a homomorphism of free abelian groups, which is defined for an elementary chain $\hat{Q} \in \hat{K}_k^d$ by induction on the embedding number d as follows.

Consider first the case $d = 1$. Then Q is an elementary interval and hence $Q = [l] \in \mathcal{K}_0^1$ or $Q = [l, l+1] \in \mathcal{K}_1^1$ for some $l \in \mathbb{Z}$. Define

$$\partial_k \hat{Q} = \begin{cases} 0 & \text{if } Q = [l], \\ \widehat{[l+1]} - \widehat{[l]} & \text{if } Q = [l, l+1]. \end{cases}$$

Now assume that $d > 1$. Let $I = I_1(Q)$ and $P = I_2(Q) \times \cdots \times I_d(Q)$. Then $\hat{Q} = \hat{I} \diamond \hat{P}$

Define

$\partial_k \hat{Q} = \partial_{k_1} \hat{I} \diamond \hat{P} + (-1)^{\dim I} \hat{I} \diamond \partial_{k_2} \hat{P}$ where $k_1 = \dim I$ and $k_2 = \dim P$. Finally , we extend the definition to all chains by linearity; that is, if

$c = \alpha_1 \hat{Q}_1 + \alpha_2 \hat{Q}_2 + \cdots + \alpha_m \hat{Q}_m$, then

$\partial_k c = \alpha_1 \partial_k \hat{Q}_1 + \alpha_2 \partial_k \hat{Q}_2 + \cdots + \alpha_m \partial_k \hat{Q}_m$

Proposition 2.3.4. *Let c and c' be cubical chains: then*

$\partial(c \diamond c') = \partial c \diamond c' + (-1)^{\dim c} c \diamond \partial c'$.

(For proof refer to [1])

Corollary 2.3.5. *([1]) If $Q_1, Q_2, \ldots, Q_m$ are elementary cubes, then*

$$\partial(\hat{Q}_1 \diamond \hat{Q}_2 \diamond \ldots \diamond \hat{Q}_m)$$
$$= \sum_{j=1}^{m} (-1)^{\sum_{i=1}^{j-1} \dim Q_i} \hat{Q}_1 \diamond \ldots \diamond \hat{Q}_{j-1} \diamond \partial \hat{Q}_j \diamond \hat{Q}_{j+1} \diamond \ldots \diamond \hat{Q}_m$$

Proposition 2.3.6. *Let $Q \in \mathbb{R}^d$ be an n-dimensional elementary cube with decomposition into elementary intervals given by $Q = I_1 \times I_2 \times \cdots \times I_d \in \mathbb{R}^d$ and let the one-dimensional intervals in this decomposition be $I_{i_1}, I_{i_2}, \ldots, I_{i_n}$, with $I_{i_j} = [k_j, k_j + 1]$. For $j = 1, 2, \ldots, n$ let*

$Q_j^- = I_1 \times \cdots \times I_{i_j - 1} \times [k_j] \times I_{i_j + 1} \times \cdots \times I_d$.

$Q_j^+ = I_1 \times \cdots \times I_{i_j - 1} \times [k_j + 1] \times I_{i_j + 1} \times \cdots \times I_d$.

denote the primary faces of Q. Then

$$\partial \hat{Q} = \sum_{j=1}^{m} (-1)^{j-1} (\hat{Q}_j^+ - \hat{Q}_j^-)$$

(For proof refer to [1])

Proposition 2.3.7. $\partial \circ \partial = 0$

(For proof refer to [1])

Proposition 2.3.8. *([1]) For any chain $c \in C_k^d$, $|\partial c| \subset |c|$. Moreover, $|\partial c|$ is contained in the $(k-1)$-dimensional skeleton of $|c|$, that is, the union of $(k-1)$-dimensional faces of $|c|$*

Proposition 2.3.9. *([1]) Let $X \subset \mathbb{R}^d$ be a cubical set. Then*
$\partial_k(\mathcal{C}_k(X)) \subset \mathcal{C}_{k-1}(X)$

Definition 2.3.10. ([1]) The boundary operator for the cubical set X is defined to be
$\partial_k^X : \mathcal{C}_k(X) \longrightarrow \mathcal{C}_{k-1}(X)$
obtained by restricting $\partial_k : C_k^d \longrightarrow C_{k-1}^d$ to $\mathcal{C}_k(X)$

Definition 2.3.11. ([1]) The *cubical chain complex* for the cubical set $X \subset \mathbb{R}^d$ is
$\mathcal{C}(X) = \{\mathcal{C}_k(X), \partial_k^X\}_{k\in\mathbb{Z}}$,
where $\mathcal{C}_k(X)$ are the groups of cubical $k-$chains generated by $\mathcal{K}_k(X)$ and ∂_k^X is the cubical boundary operator restricted to X.

The homology groups of X is defined to be the homology group of this chain complex and is denoted by $H_*(X) = \{H_k(X)\}_{k\in\mathbb{Z}}$

2.4 Computations of $0th$ Cubical homology $H_0(X)$

We compute the zero-dimensional homology group $H_0(X)$ of a cubical set. We will see that $H_0(X)$ counts the number of connected components of X (as in the usual homology)

For any topological space X and any point $x \in X$ the union of all connected subsets of X containing x is a connected subset of X. It is called the *connected component of x in X.* We denote it by $cc_X(x)$ (see [1])

Theorem 2.4.1. *For any $x, y \in X$, either $cc_X(x) = cc_X(y)$ or $cc_X(x) \cap cc_X(y) = \phi$. (well known)*

Proposition 2.4.2. *For every $x \in X$, X a cubical set, there exists a vertex $V \in \mathcal{K}_0(X)$ such that $cc_X(x) = cc_X(V)$*

(For proof refer to [1] or do it as an exercise)

Corollary 2.4.3. *([1]) A cubical set can have only a finite number of connected components.*

Definition 2.4.4. ([1]) A sequence of vertices $V_0, V_1, \ldots, V_n \in \mathcal{K}_0(X)$ is an *edge path in X* if there exist edges $E_1, E_2, \ldots, E_n \in \mathcal{K}_1(X)$ such that V_{i-1}, V_i are the two faces of E_i for $i = 1, 2, \ldots, n$. For $V, V' \in \mathcal{K}_0(X)$, we write $V \sim_X V'$ if there exist an edge path $V_0, V_1, \ldots, V_n \in \mathcal{K}_0(X)$ in X such that $V = V_0$ and $V' = V_n$. We say that X is *edge connected* if $V \sim_X V'$ for any $V, V' \in \mathcal{K}_0(X)$.

Proposition 2.4.5. 1. *Every elementary cube is edge-connected.*

2. *If X and Y are edge-connected cubical sets and $X \cap Y \neq \phi$, then $X \cup Y$ is edge-connected.*

(For proof refer to [1])

Proposition 2.4.6. *Assume that $V \sim_X V'$ for some $V, V' \in \mathcal{K}_0(X)$. Then there exist a chain $c \in \mathcal{C}_1(X)$ such that $|c|$ is connected and $\partial c = \hat{V}' - \hat{V}$.*

(For proof refer to [1])

Remark 2.4.7. For $x \in X$ we define *edge-connected component of x in X* as the union of all edge-connected cubical subsets of X that contain x. We denote it by $ecc_X(x)$.

Proposition 2.4.8. *For any $x \in X$, $\mathrm{ecc}_X(x)$ is edge-connected.*
(For proof refer to [1])

Proposition 2.4.9. *For any $x, y \in X$ either $ecc_X(x) = ecc_X(y)$ or $ecc_X(x) \cap ecc_X(y) = \phi$.*
(For proof refer to [1])

Proposition 2.4.10. *A cubical set X is connected if and only if it is edge-connected.*
(For proof refer to [1])

Proposition 2.4.11. *If X is cubical, then for every $x \in X$ its connected component $cc_X(x)$ is a cubical set.*
(For proof refer to [1])

Corollary 2.4.12. *([1]) If X is cubical, then for every $x \in X$ its connected component and edge-connected component coincide.*

Lemma 2.4.13. *Assume X is a cubical set and $X_1, X_2, \ldots, X_n$ are its connected components.If $c_i \in C_k(X_i)$ are k-dimensional chains, then*
$|\sum_{i=1}^{n} c_i| = \bigcup_{i=1}^{n} |c_i|$
(For proof refer to [1])

Theorem 2.4.14. *([1]) Let X be a cubical set. Then $H_0(X)$ is a free abelian group. Furthermore, if $\{P_i | i = 1, \ldots, n\}$ is a collection of vertices in X consisting of one vertex from each connected component of X, then*
$\{[\hat{P}_i] \in H_0(X) | i = 1, \ldots, n\}$ *forms a basis for $H_0(X)$*

Proof. Let $X_i = cc_X(P_i)$ and let $c \in Z_0(X)$. By proposition 2.4.6, $[\hat{P}] = [\hat{P}_i]$ for any $P \in \mathcal{K}_0(X_i)$. Since $Z_0(X) = C_0(X)$.

there exist integers α_P such that

$$[c] = \sum_{P \in \mathcal{K}_0(X)} \alpha_P [\hat{P}] = \sum_{i=1}^{n} \sum_{P \in \mathcal{K}_0(X_i)} \alpha_P [\hat{P}] = \sum_{i=1}^{n} (\sum_{P \sim_X P_i} \alpha_P) [\hat{P}_i].$$

This shows that the classes $[\hat{P}_i]$ generate $H_0(X)$.

It remains to show that the generators are free, that

$\sum_{i=1}^{n} \alpha_i [\hat{P}_i] = 0$

implies that all $\alpha_i = 0$. To do so put $c = \sum_{i=1}^{n} \alpha_i \hat{P}_i$, and let $[c] = 0$, we can then select a $b \in C_1(X)$ such that $c = \partial b$. Let $b = \sum_{E \in \mathcal{K}_1(X)} \beta_E \hat{E}$.

Let $b_i = \sum_{E \in \mathcal{K}_1(X_i)} \beta_E \hat{E}$

We have $c = \sum_{i=1}^{n} \alpha_i \hat{P}_i = c = \partial b = \sum_{i=1}^{n} \partial b_i$

Therefore, $0 = \sum_{i=1}^{n} (\alpha_i \hat{P}_i - \partial b_i)$

$|\alpha_i \hat{P}_i - \partial b_i| \subset X_i$.

Therefore by lemma 2.4.12

$\phi = |0| = \bigcup_{i=1}^{n} |\alpha_i \hat{P}_i - \partial b_i|$

which shows that $|\alpha_i \hat{P}_i - \partial b_i| = \phi$; that is by proposition 2.2.6, $\alpha_i \hat{P}_i = \partial b_i$.

Let $\epsilon : C_0(X) \to \mathbb{Z}$ be the group homomorphism defined by $\epsilon(\hat{P}) = 1$ for every vertex $P \in X$. Let E be an elementary edge. Then $\partial \hat{E} = \hat{V}_1 - \hat{V}_0$, where V_0 and V_1 are vertices of E. Observe that

$\epsilon(\partial \hat{E}) = \epsilon(\hat{V}_1 - \hat{V}_0)$

$$= \epsilon(\hat{V}_1) - \epsilon(\hat{V}_0)$$
$$= 1 - 1$$
$$= 0.$$

This implies that $\epsilon(\partial b_\iota) = 0$ and hence
$0 = \epsilon(\partial b_\iota) = \epsilon(\alpha_\iota \hat{P}_\iota) = \alpha_\iota \epsilon(\hat{P}_\iota) = \alpha_\iota$. □

Chapter 3

Some Properties of Cubical Homology

3.1 Elementary collapse (deformation) and Cubical Homology

Since our main aim is to compute homology using computer algorithm, size of the cubical set is a big issue. To this end we will see how the size can be reduced.

Definition 3.1.1. ([1]) Let X be a cubical set and let $Q \in \mathcal{K}(X)$. If Q is not a proper face of some $P \in \mathcal{K}(X)$, then it is a *maximal face* in X. $\mathcal{K}_{max}(X)$ is a set of maximal faces in X. A face that is a proper face of exactly one elementary cube in X is a *free face in X*.

Lemma 3.1.2. *([1]) Let X be a cubical set. Let $Q \in \mathcal{K}(X)$ be a free face in X and assume $Q \prec P \in \mathcal{K}(X)$. Then $P \in \mathcal{K}_{max}(X)$ and* $\dim Q = \dim P - 1$.

Definition 3.1.3. ([1]) Let Q be a free face in X and let P be the unique cube in $\mathcal{K}(X)$ such that Q is a proper face of P. Let

$\mathcal{K}'(X) = \mathcal{K}(X) \setminus \{Q, P\}$. Define

$$X' = \bigcup_{R \in \mathcal{K}'(X)} R$$

Then X' is a cubical space obtained from X via an *elementary collapse of P by Q*

Proposition 3.1.4. *If X' is a cubical space obtained from X via an elementary collapse of P by Q, then*

$$\mathcal{K}(X') = \mathcal{K}'(X)$$

(For proof refer to [1])

Lemma 3.1.5. *([1]) Assume X is a cubical set and X' is obtained from X via an elementary collapse of $\mathcal{P}_0 \in \mathcal{K}_k(X)$ by $Q_0 \in \mathcal{K}_{k-1}(X)$. Then*

(i) $\{c \in \mathcal{C}_k(X) | \partial c \in \mathcal{C}_{k-1}(X')\} \subset \mathcal{C}_k(X')$;

(ii) for every $c \in \mathcal{C}_{k-1}(X)$ there exists $c' \in \mathcal{C}_{k-1}(X')$ such that $c - c' \in \mathcal{B}_{k-1}(X)$.

Theorem 3.1.6. *Assume X is a cubical set and X' is obtained from X via an elementary collapse of $P_0 \in \mathcal{K}_k(X)$ by $Q_0 \in \mathcal{K}_{k-1}(X)$. then*

$$H_*(X') \cong H_*(X)$$

(For proof refer to [1])

Corollary 3.1.7. *Let $Y \subset X$ be a cubical sets. Furthermore, assume that Y can be obtained from X via a series of elementary collapses. Then*

$$H_*(Y) \cong H_*(X)$$

3.2 Reduced homology; acyclic space

Definition 3.2.1. ([1]) Let X be a cubical set. The *augmented cubical chain complex of X* is given by $\{\tilde{\mathcal{C}}_k(X), \tilde{\partial}_k\}_{k \in \mathbb{Z}}$ where

$$\tilde{\mathcal{C}}_k(X) = \begin{cases} \mathbb{Z} & \text{if } k = -1, \\ \mathcal{C}_k(X) & \text{otherwise} \end{cases}$$

and

$$\tilde{\partial}_k = \begin{cases} \epsilon & \text{if } k = 0, \\ \partial_k(X) & \text{otherwise} \end{cases}$$

Definition 3.2.2. ([1]) The homology groups $H_k(\tilde{\mathcal{C}}(X))$ are the *reduced homology groups of* X and are denoted by $\tilde{H}_k(X)$.

A chain $z \in \mathcal{C}_k(X)$ that is a cycle in $\tilde{\mathcal{Z}}_k$ is a *reduced cycle* in $\mathcal{C}(X)$. The homology class of a reduced cycle z with respect to the reduced homology is denoted by $[z]_\sim$.

Theorem 3.2.3. *Let* X *be a cubical set.* $\tilde{H}_0(X)$ *is a free abelian group and*

$$H_k(X) = \begin{cases} \tilde{H}_0(X) \bigoplus \mathbb{Z} & \textit{for } k = 0, \\ \tilde{H}_k(X) & \textit{otherwise} \end{cases}$$

Furthermore, if $\{P_i | i = 0, \ldots, n\}$ *is a collection of vertices in* X *consisting of one vertex from each connected component of* X*, then*

$\{[P_i - P_0]_\sim \in \tilde{H}_0(X) | i = 1, \ldots, n\}$ *forms a basis for* $\tilde{H}_0(X)$.

(For proof refer to [1])

Definition 3.2.4. ([1]) A cubical set X is *acyclic* if

$$H_k(X) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$$

As a consequence of this definition and the results preceding it we get the following

Corollary 3.2.5. *Let* X *be a nonempty cubical set. Then* X *is acyclic if and only if* $\tilde{H}_*(X) = 0$

Lemma 3.2.6. *Assume $Q \in \mathcal{K}^d$ and $i \in \{1, 2, \ldots, d\}$. If z is a $k-$ cycle in Q such that $< z, \hat{P} >= 0$ for every $P \in \mathcal{K}_k([l+1], i)$, then $< z, \hat{P} >= 0$ for every $P \in \mathcal{K}_k([l, l+1], i)$ where $\mathcal{K}_k([l+1], i) = \{P \in \mathcal{K}_k(Q) | I_i(P) = [l+1]\}$ and $\mathcal{K}_k([l, l+1], i) = \{P \in \mathcal{K}_k(Q) | I_i(P) = [l, l+1]\}$*

(For proof refer to [1])

Theorem 3.2.7. *All elementary cubes are acyclic.*

(For proof refer to [1])

Proposition 3.2.8. *If $K, L \subset \mathbb{R}^n$ are cubical sets, then*

$\mathcal{C}_k(K \cup L) = \mathcal{C}_k(K) + \mathcal{C}_k(L)$

(For proof refer to [1])

Theorem 3.2.9. *Assume $X, Y \subset \mathbb{R}^n$ are cubical sets. If X, Y, and $X \cap Y$ are acyclic, then $X \cup Y$ is acyclic.*

(For proof refer to [1])

Definition 3.2.10. ([1]) A *rectangle* is a set of the form $X = [k_1, l_1] \times [k_2, l_2] \times \cdots \times [k_n, l_n] \subset \mathbb{R}^n$, where k_i, l_i are integers and $k_i \leq l_i$.

Proposition 3.2.11. *A cubical set is convex if and only if it is a rectangle.*

(For proof refer to [1])

Proposition 3.2.12. *Any rectangle is acyclic.*

(For proof refer to [1])

Definition 3.2.13. ([1]) A cubical set $X \subset \mathbb{R}^d$ is *star − shaped* with respect to a point $x \in \mathbb{Z}^d$ if X is the union of a finite number of rectangles each of which contains the point x.

Proposition 3.2.14. *Let X_i, for $i = 1, \ldots, n$ be a collection of star-shaped sets with respect to the same point x. Then*
$\bigcup_{i=1}^{n} X_i$ *and* $\bigcap_{i=1}^{n} X_i$ *are star-shaped.*
(For proof refer to [1])

Proposition 3.2.15. *Every star-shaped is acyclic.*
(For proof refer to [1])

Proposition 3.2.16. *Assume that $\mathcal{Q}$ is a family of rectangles in $\mathbb{R}^d$ such that the intersection of any two of them is nonempty. Then $\bigcap \mathcal{Q}$ is nonempty.*
(For proof refer to [1])

Remark 3.2.17. Before proceeding further we would like to mention that we have not touched upon the topic of how a general continuous map $f : X \to Y$ of cubical complexes induce a homomorphism $f_* : H_*(X) \to H_*(Y)$ of their homology groups. This is not required for the applications we have chosen to present in this dissertation. All we require is the homomorphism in homology induced by the inclusion maps of cubical complexes. For, in this case the inclusion map takes elementary cubes to elementary cubes and so it readily induces a map of chain complexes which in turn gives rise to a homomorphism of homology groups. The construction for the general case is not as straight forward as in the ordinary homology theory, or as in the case of inclusion maps, and it can not be described in a few pages. The interested reader is referred to ([1]) for this construction.

3.3 Exact homology sequence of cubical pairs; Mayer-Vietoris sequence

In this section we give a very brief account of the homology of "cubical pairs", a long exact sequence of homology groups associated with a cubical pair (X, A), and the Mayer Vietoris sequence of homology groups which help in calculating homology group of a cubical set in terms of homology of its two "well behaved" subsets. Refer to ([1]) for more details.

Definition 3.3.1. Let X be a cubical set and $A \subseteq X$ be a cubical subset. We call (X, A) a cubical pair.

Definition 3.3.2. For a cubical pair (X, A), we define *relative chains of X modulo A* to be the elements of the quotient group:

$$C_k(X, A) = C_k(X)/C_k(A)$$

If $c \in C_k(X)$, we denote its coset $c + C_k(A)$ in $C_k(X, A)$ by $[c]_A$.

Definition 3.3.3. Define $\partial_k^{(X,A)} : C_k(X, A) \to C_{k-1}(X, A)$ by $\partial_k^{(X,A)}([c]_A) \stackrel{def}{=} [\partial_k c]_A$. $\partial_k^{(X,A)}$ is well defined and $\partial_{k-1}^{(X,A)} \circ \partial_k^{(X,A)} = 0$. So $\{C_k(X, A), \partial_k^{(X,A)}\}_{k=0,1,2,}$ form a chain complex.

Definition 3.3.4. We call $Z_k(X, A) \stackrel{def}{=} \ker \partial_k^{(X,A)}$ the relative k-cycles and

$B_k(X, A) \stackrel{def}{=} \operatorname{im} \partial_{k+1}^{(X,A)}$ the relative k-boundaries Finally we call the quotient $H_k(X, A) = Z_k(X, A)/B_k(X, A)$ the relative homology group of the pair (X, A) as in the usual case.

Proposition 3.3.5. *([1]) Let X be a connected cubical set and let $A \neq \emptyset$ be a cubical subset of X. Then $H_0(X, A) = 0$.*

(for proof refer to [1])

Proposition 3.3.6. *([1]) If (X, A) is a cubical pair in which X has r connected components which do not intersect A, then $H_0(X, A) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, r copies.*

(for proof refer to [1])

We will see an application of this result involving height function for an example in a later chapter.

Theorem 3.3.7. *([1]) Let X be a cubical set and $P \in \mathcal{K}_0(X)$ a chosen vertex of X. Then*

$$H_n(X, P) \cong \tilde{H}_n(X) \cong \begin{cases} H_n(X) & \text{if } n > 0 \\ \tilde{H}_0(X) & \text{if } n = 0 \end{cases}$$

and $\phi : \tilde{Z}_0(X) \to Z_0(X, P)$ defined on the basic elements $\{\hat{Q} - \hat{P} \mid Q \in \mathcal{K}_0(X) \backslash \{P\}\}$ of $\tilde{Z}_0(X)$ by $\phi(\hat{Q} - \hat{P}) \stackrel{def}{=} [Q]_P$ induces the isomorphism $\tilde{H}_0(X) \cong H_0)(X, P)$.

(for proof refer to [1])

Example 3.3.8.

$$H_k([0,1], \{0,1\}) = \begin{cases} \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

For, $C_0([0,1]) = C_0(\{0,1\}) = \mathbb{Z}\{\hat{0}\} \oplus \mathbb{Z}\{\hat{1}\}$, so $C_0([0,1], \{0,1\}) = 0$. $C_1([0,1]) = \mathbb{Z}\{\widehat{[0,1]}\}$, $C_1(\{0,1\}) = 0$, so $C_1([0,1], \{0,1\}) = \mathbb{Z}\{[\widehat{[0,1]}]_A\}$. Now $Z_1([0,1], \{0,1\}) = C_1([0,1], \{0,1\})$, because $C_0([0,1], \{0,1\}) = 0$, and $B_1([0,1], \{0,1\}) = 0$ because there are no 2-chains or higher chains. This gives the result.

Now we indicate how one gets the long homology exact sequence of the cubical pair (X, A) by giving the frame work in which it fits. This framework is drawn from homological algebra:

Definition 3.3.9. ([1]) A sequence(finite or infinite) of groups and homomorphisms

$\cdots \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to \ldots$

is exact at G_2 if $\operatorname{im}\psi_3 = ker\psi_2$

It is an *exact sequence* if it is exact at every group. If the sequence has a first or last element, then it is automatically exact at that group.

Lemma 3.3.10. *([1])* $G_1 \xrightarrow{\psi_1} G_0 \xrightarrow{\phi} 0$ *is an exact sequence if and only if* ψ_1 *is an epimorphism.*

Lemma 3.3.11. *([1])* $0 \xrightarrow{\phi} G_1 \xrightarrow{\psi_1} G_0$ *is an exact sequence if and only if* ψ_1 *is an monomorphism.*

Lemma 3.3.12. *([1]) Assume that*

$G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$

is an exact sequence. Then the following are equivalent:

1. ψ_3 *is an epimorphism.*

2. ψ_2 *is the zero homomorphism.*

3. ψ_1 *is a monomorphism.*

Definition 3.3.13. ([1]) A *short exact sequence* is an exact sequence of the form

$0 \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to 0$

Definition 3.3.14. ([1]) A short exact sequence

$0 \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to 0$

splits if there exists a subgroup $H \subset G_2$ such that $G_2 = \operatorname{im}\psi_3 \oplus H$.

Theorem 3.3.15. *([1]) Let* $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *be a short exact sequence of chain complexes. Then for each* k *there exists a homomorphism*

$\partial_* : H_{k+1}(C) \to H_k(A)$

such that

$\cdots \to H_{k+1}(A) \xrightarrow{\varphi_*} H_{k+1}(B) \xrightarrow{\psi_*} H_{k+1}(C) \xrightarrow{\partial_*} H_k(A) \to \dots$

is a long exact sequence.

The map ∂_* is called *connecting homomorphism.*

Now we try to fit our cubical pair (X, A) in the above framework. For each integer $k \geq 0$ we have a short exact sequence of chain groups

$$0 \to C_k(A) \xrightarrow{i_k} C_k(X) \xrightarrow{\pi_k} C_k(X, A) \to 0,$$

where i_k is the inclusion map and π_k is the quotient map. The reader can verify easily that i_k and π_k are the k^{th}-stems of the chain maps $i : C(A) \to C(X)$ and $\pi : C(X) \to C(X, A)$.

Once we have a short exact sequence of chain complexes

$$0 \to C(A) \xrightarrow{i} C(X) \xrightarrow{\pi} C(X, A) \to 0,$$

we can appeal to the Theorem(3.3.15) and get the following

Corollary 3.3.16. *([1]) Let* (X, A) *be a cubical pair. Then there is a long exact sequence*

$\cdots \to H_{k+1}(A) \xrightarrow{i_*} H_{k+1}(X) \xrightarrow{\pi_*} H_{k+1}(X, A) \xrightarrow{\partial_*} H_k(A) \to \dots$, *where* $i : C(A) \hookrightarrow C(X)$ *is the inclusion map and* $\pi : C(X) \to C(X, A)$ *is the quotient map.*

We conclude this section with a very important theorem from the point of view of computation of homology of a cubical set in terms of homology of some "well placed" cubical subsets. However we won't go into any details about the development of the theorem. One can see [1] or [46].

Theorem 3.3.17. *(Mayer-Vietoris sequence) Let X be a cubical space. Let A_0 and A_1 be cubical subsets of X such that $X = A_0 \cup A_1$ and let $B = A_0 \cap A_1$. Then there is along exact sequence*

$$\cdots \to H_k(B) \to H_k(A_0) \oplus H_k(A_1) \to H_k(X) \to H_{k-1}(B) \to \cdots$$

Chapter 4

Simplicial versus Cubical Homology and applications

We have discussed about Homology of Simplicial Complexes briefly in Chapter one and discussed about Cubical Homology a bit more elaborately in Chapter two. Simplicial homology has been used by mathematicians over the last several decades and the computations involving them is also quite familiar to most topologists. Cubical homology on the other hand is a new construct not familiar to many. We are interested in the relation between them and in finding out which one is suitable for which situation in tackling problem of the real world. In the first section we will give a comparison between the two complexes and in the next section we will discuss about cubical homology of topological polyhedra. In the last section we will give a brief discussion about some applications of Cubical homology.

4.1 Comparison of uses of cubes and simplices

In this section we would like to compare the two complexes. i.e the cubical complexes and the simplicial complexes (see [1]).

A. Cubical complexes have nicer properties compared to simplicial complexes in some situations:

1. Images and numerical computations naturally lead to cubical sets. Subdividing these cubes to obtain a triangulation is at this point artificial and increases the size of data significantly. For example,it requires $n!$ simplices to triangulate a single $n-$ dimensional cube.

2. Cubical complexes are very rigid. So they can be stored with minimal information. To represent an elementary cube only one vertex is sufficient, whereas to represent a simplex all the vertices need to be stored. Moreover the vertex in a cubical set X in $\mathbb{R}^2$ is shared by atmost four edges which is not so in case of Simplicial Complexes.

3. A product of elementary cubes is an elementary cube, but a product of simplices is not a simplex. For example, the product of two edges is a square, not a simplex. There is no natural projection from a higher-dimensional simplicial complex to a lower-dimensional complex.

4. To define chain complexes based on simplices requires the nontrivial concept of orientation. whereas when we define chain complexes based on cubical sets the concept of orientation is natural. More precisely, we began by writing an elementary interval as $[l, l+1]$ and not $[l+1, l]$. In other words, a linear order of real numbers imposes a choice of an orientation on each coordinate axis in $\mathbb{R}^d$. Furthermore, we have always written a product of intervals as $I_1 \times I_2 \times \cdots \times I_d$. This implicitly chooses

an ordering of the canonical basis for $\mathbb{R}^d$.

B. Looking at the above comparison it seems that cubical complexes are ideal for computational problem. But most of the Computer Graphics and visualization are based on the rendering of triangulated surfaces. Thus the basic building blocks are simplices.

1. Every cubical set can be triangulated, but every polyhedron can not be expressed as a cubical set.
2. Rigidity of cubical sets also pose some limitation.

C. Despite remark B. we can still compute cubical homology of a polyhedron, which we shall demostrate now.

Definition 4.1.1. ([1]) Given any $d \geq 0$, the *standard d-simplex* Δ^d is given by $\Delta^d = conv\{0, e_1, e_2, \ldots, e_d\}$, where 0 is the origin of coordinates in $\mathbb{R}^d$ and $\{e_1, e_2, \ldots, e_d\}$ is the canonical basis for $\mathbb{R}^d$.

Theorem 4.1.2. *([1]) Every polyhedron P is homeomorphic to a cubical set. Moreover, given any triangulation $\mathcal{S}$ of P, there exists a homeomorphism $h : P \longrightarrow X$ where X is a cubical subset of $[0,1]^d$ and $d+1$ is the number of vertices of $\mathcal{S}$, such that the restriction of h to any simplex of $\mathcal{S}$ is a homeomorphism of that simplex onto a cubical subset of X.*

Proof. (Sketch of proof:) We construct h as the following composite:

$$P \quad \overset{f\,=\,embed.}{\longrightarrow} \quad \Delta^d \quad \overset{g\,=\,homeo}{\longrightarrow} \quad [0,1]^d, \ d \text{ sufficiently large}$$

Step 1: We construct a homeomorphic embedding of P into a standard simplex in a sufficiently high-dimensional space.

Let $\mathcal{S}$ be a triangulation of P and let $\mathcal{V} = \{v_0, v_1, v_2, \ldots, v_d\}$ be the set of all vertices of $\mathcal{S}$. Let Δ^d be the standard d-simplex in $\mathbb{R}^d$. Consider the bijection f_0 of $\mathcal{V}$ onto the set $\{0 = e_0, e_1, e_2, \ldots, e_d\}$ given by $f_0(v_\imath) = e_\imath$ for $\imath = 0, 1, 2, \ldots, d$.

Given any n-simplex $S = conv\{v_{p_0}, v_{p_1}, \ldots, v_{p_n}\}$ of $\mathcal{S}$, the map $f_S : S \to \mathbb{R}^d$ defined by $f_S(\sum \lambda_\imath v_{p_\imath}) = \sum \lambda_\imath f_0(v_{p_\imath}) = \sum \lambda_\imath e_{p_\imath}$, where $\lambda_\imath$ are barycentric coordinates of a point in S, is a homeomorphism of the simplex S onto a simplex $f_S(S)$. If S and T are any two simplices of $\mathcal{S}$, $S \cap T$ is either empty or their common face. Hence, if $x \in S \cap T$, then

$f_S(x) = f_{S \cap T}(x) = f_T(x)$.

Thus the maps f_S and f_T match on intersections of simplices. Since simplices are closed and there are finitely many of them, the maps f_S can be extended to a map $f : P \to \tilde{P} = f(P) \subset \Delta^d$. Using the geometric independence of $\{0, e_1, e_2, \ldots, e_d\}$, one shows that $\tilde{P}$ is a polyhedron triangulated by $\{f(S)\}$ and f is a homeomorphism. Moreover, by its construction, f maps simplices to simplices.

Step 2: We construct a homeomorphism g of Δ^d onto $[0, 1]^d$ such that g restricted to any face of Δ^d is a homeomorphism onto a cubical subset of $[0, 1]^d$. Once we do that, it will be sufficient to take $X = g(\tilde{P})$ and define the homeomorphism h as the composition of f and g.

The idea is to keep the vertex $e_0 = 0$ constant and extend line segments in Δ^d emanating from it so that Δ^d is extended to the whole cube $[0, 1]^d$. This leads to the formula for $g : \Delta^d \to [0, 1]^d$,

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \lambda(x)x & \text{if } x \neq 0, \end{cases}$$

for $x \in \Delta^d$, where $\lambda(x)$ is a scalar function given by

$$\lambda(x) = \frac{x_1 + x_2 + \cdots + x_d}{max\{x_1, x_2, \ldots, x_d\}}$$

This map is a homeomorphism of Δ^d onto $[0,1]^d$ with the inverse given by

$$g^{-1}(y) = \begin{cases} 0 & \text{if } y = 0, \\ \frac{1}{\lambda(y)}y & \text{if } y \neq 0, \end{cases}$$

By induction on d we can show that g maps faces of $\mathcal{S}$ onto the unions of cubical faces of $[0,1]^d$. □

Example 4.1.3. Let $P = conv\{e_1, e_2\} \subset [0,1]^2$. Then P is a one dimensional face of Δ^2. The map g described above fixes the vertices and the edges are sent as follows:

$$g([e_0, e_1]) = [0,1] \times [0], \quad g([e_0, e_2]) = [0] \times [0,1]$$

$$g([e_1, e_2]) = g(P) = [0,1] \times [1] \cup [1] \times [0,1]$$

(see fig.(4.1)

4.2 Cubical Homology of polyhedra

Having established a homeomorphism from a polyhedron onto a cubical set in the last section we now proceed to define cubical homology of a polyhedron. The topological invariance (infact homotopy type invariance) of cubical homology of cubical sets suggests that given a homeomorphism $h : P \longrightarrow X$ from a polyhedron P onto a cubical set X we can define cubical homology of P as $H_*(X)$. However one needs to show that this is well defined, in the sense that given a continuous map $f : P_1 \longrightarrow P_2$ between polyhedra one gets a well defined homomorphism

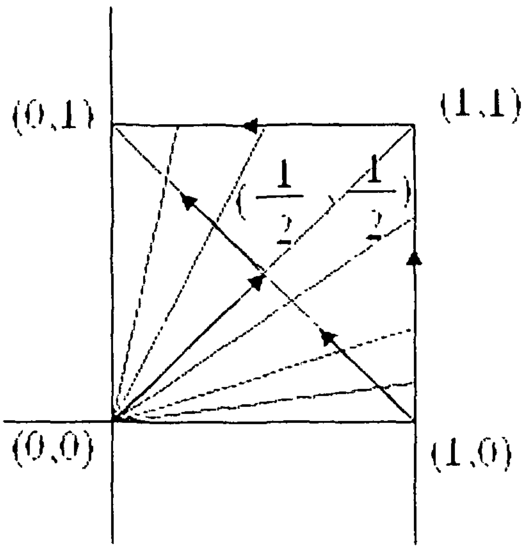


Figure 4.1: triangle-and-square

$H_*(f) : H_*(P_1) \to H_*(P_2)$.

Towards this we fix some categorical notations:

Definition 4.2.1. ([1]) The category of cubical sets Cub consist of objects which are cubical sets. if X, Y are cubical sets then the morphism from X to Y,

$Cub(X, Y) = \{f : X \to Y | f \text{ is continuous}\}$

Remark 4.2.2. ([1]) The category *Ab* has all abelian groups as objects and all group homomorphisms as morphisms.

Remark 4.2.3. ([1]) The category Ab_* called the *category of graded abelian groups*, has all sequence $A_* = \{A_n\}_{n\in\mathbb{Z}}$ of abelian groups as objects and all sequences $\{f_n\}_{n\in\mathbb{Z}}$ of group homomorphism $f_n : A_n \to B_n$ as morphisms.

Definition 4.2.4. ([1]) Given a cubical set X (an object of Cub), $H_*(X)$ is a graded abelian group (an object of Ab_*). Furthermore, given a continuous function $f : X \to Y$ (a morphism in Cub(X,Y)), $f_* : H_*(X) \to H_*(Y)$ is a graded group homomorphism (a morphism in $Ab_*(X, Y)$).

Definition 4.2.5. (i) A compact metric space K is said to be a topological polyhedron (or a representable space) if there exist a cubical set X and a homeomorphism $s : K \to X$.

(ii) *Pol*: Category of all topological polyhedra and continuous maps between them.

(iii) Every geometric polyhedron is an object of *Pol* by the result of the last section.

Definition 4.2.6. ([1]) The category *Repr.* of *representation of topological polyhedra*, denoted by Repr, is defined as follows. Its objects are all triples (K, s, X), where K is a compact space, X

a cubical set, and $s : K \to X$ a homeomorphism. If (K, s, X) and (L, t, Y) are two objects in Repr, then the morphisms from (K, s, X) to (L, t, Y) are all continuous maps from X to Y.

Definition 4.2.7. ([1]) Fix a category Cat. A *connected simple system* (CSS) in Cat is a small category $\mathcal{E}$ with objects and morphisms from Cat satisfying the property that for any two objects $E_1, E_2 \in \mathcal{E}$ there exists exactly one morphism in $\mathcal{E}(E_1, E_2)$, which is denoted by $\mathcal{E}_{E_1,E_2}$.

Since $\mathcal{E}_{E_2,E_1} o \mathcal{E}_{E_1,E_2} = \mathcal{E}_{E_1,E_1}$ and the identity must be a morphism, it follows that all morphisms in $\mathcal{E}$ are isomorphisms.

Example 4.2.8. For every topological polyhedron K we define a *CSS, Rep*(K) in *Repr.*, whose objects are representations (K, s, X) of K. If (K, s, X) , (K, t, Y) are two representation of K, then the unique morphism in *Rep*(K) $((K, s, X), (K, t, Y))$ is the map ts^{-1}.

Define *CSS(Cat)* to be a category whose objects are the connected simple system in *Cat* and whose morphism from one *CSS*, $\mathcal{E}_1$ to another*CSS*, $\mathcal{E}_2$ in *Cat* is given by $\varphi = \{\varphi_{E_1E_2} \in Cat(E_1, E_2) | E_1 \in \mathcal{E}_1 \text{ and } E_2 \in \mathcal{E}_2\}$ that satisfy
$\varphi_{E_2E_1} = \mathcal{E}_{E_2E_2'} o \varphi_{E_2E_1} o \mathcal{E}_{E_1E_1'}$ for every $E_1, E_1' \in \mathcal{E}_1$ $E_2, E_2' \in \mathcal{E}_2$ elements of φ are called representation of φ

Theorem 4.2.9. *([1]) CSS(Cat) is a category.*

Proposition 4.2.10. *([1]) Assume* $\mathbf{F} : Cat \to Cat'$ *is a functor that maps distinct objects in Cat into distinct objects in Cat' and* $\mathcal{E}$ *is a connected simple system in Cat. Then* $\mathbf{F}(\mathcal{E})$ *is a connected simple system in* Cat'*. Moreover, if* $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ *is a morphism in CSS(Cat), then* $\mathbf{F}(\varphi) : \mathbf{F}(\mathcal{E}_1) \to \mathbf{F}(\mathcal{E}_2)$ *given by*

$\mathbf{F}(\varphi) = \{\mathbf{F}(\varphi_{E_1,E_2})|E_1 \in \mathcal{E}_1, E_2 \in \mathcal{E}_2\}$
is a morphism in CSS(Cat$'$).

So with the help of the above proposition we can extend the functor $\mathbf{F} : Cat \rightarrow Cat'$ to a functor $\mathbf{F} : CSS(Cat) \rightarrow CSS(Cat')$. So we can extend the homology from cubical sets to topological polyhedra by extending the functor $H_* : Cub \rightarrow Ab_*$ to $H_* : Pol \rightarrow CSS(Ab_*)$.

We begin by extending it to $H_* : Repr \rightarrow Ab_*$ by defining

$H_*(K, s, X) = H_*(X) \times \{(K, s, X)\}$. So by the above proposition $H_* : Repr \rightarrow Ab_*$ extends to the functor $H_* : CSS(Repr) \rightarrow CSS(Ab_*)$.

If K, L are two topological polyhedra and $f : K \rightarrow L$ is a continuous map, then we define $Rep(f) : Rep(K) \rightarrow Rep(L)$ as the collection of maps.

$\{tfs^{-1}|(X, s) \in Rep(K), (Y, t) \in Rep(L)\}$.

The above collection is a morphism in CSS(Repr).

We now define the homology functor $H_* : Pol \rightarrow CSS(Ab_*)$ by

$H_*(K) = H_*(Rep(K))$

and $H_*(f) = H_*(Rep(f))$.

4.3 Some applications of Cubical homology

Having developed the notion of cubical homology of cubical sets and cubical pairs we would like to briefly indicate some situations which justifies introduction of such a concept and also to briefly indicate their use.

Example 4.3.1. Consider the Cahn-Hillard differential equation

$$\frac{\partial u}{\partial t}(x) = -\Delta(\epsilon^2 \Delta u + u - u^3)(x), \ x \in \Omega$$
$$\bar{n}.u(x) = \bar{n}.\Delta u(x) = 0, \ x \in \partial\Omega,$$

where $\bar{n}$ is the outward normal to $\partial\Omega$. (This differential equation is used in metallurgy of alloys). Take $\Omega = [0,1]^3$, $\epsilon > 0$ but small and solve numerically for $u(x,y,z,\tau)$ on a grid consisting of $128 \times 128 \times 128$ cubical elements until $t \leq \tau$, by approximating it by a set of numbers $\{u(i,j,k,\tau) \mid 1 \leq i,j,k \leq 128\}$. Consider $S = \{(x,y,z) \in \Omega \mid u(x,y,z,\tau) = 0\}$, $\epsilon = 0.1$, which is actually a triangulated surface whose simplicial homology groups are

$$H_0(S) = \mathbb{Z}, \ H_1(S) = \mathbb{Z}^{1701}, \ H_2(S) = 0.$$

Taking X to be the smallest cubical set containing S and calculating its cubical homology (using CHomP) one gets

$$H_0(X) = \mathbb{Z}, \ H_1(X) = \mathbb{Z}^{1705}, \ H_2(X) = \mathbb{Z}.$$

It differs slightly from the simplicial homology calculated above by about 0.2%. Thus the cubical homology calculation captures information of the geometry of the solution (its level surface) of the differential equation. (refer to [1] for more details).

Example 4.3.2. There are nice software packages that allow one to view and rotate three dimensional objects like human brain (as is done in MRI imaging). With sufficiently powerful computers one can easily study the surface of a complex object. If one is interested in studying the interior of the object too, then the problem of visualization becomes much more difficult. However we can invoke cubical approximation and calculate the cubical homology of the resulting cubical set. This helps in studying the nature of interior of the object; for example to determine the cavities of a brain image it is sufficient to give a binary voxel representation of the brain (see [1] for more details and some photographs). No visualization is necessary. The Betti numbers of the resulting three dimensional cubical set is given by :$\beta_0 = 1,\ \beta_1 = 3,\ \beta_2 = 1$. (see [1]).

Example 4.3.3. This example demonstrates the use of cubical homology of a cubical set and also the cubical homology of a cubical pair to distinguish two objects which are topologically same, by calculating an invariant, called "size function" to be introduced below.

Consider the following two closed curves $E_0,\ E_1$ (refer to figure 4.2). Choose points $\bar{a}_0,\ \bar{a}_1$ in the interior of the domains bounded by the curves $E_0,\ E_1$ respectively. Define continuous functions $f_i : \mathbb{R}^2 \longrightarrow [0, \infty),\ i = 0, 1$, defined by $f_i(\bar{x}) = \|\bar{x} - \bar{a}_i\|,\ i = 0, 1$. Define *level sets*

$$E_i^\alpha = \{\bar{x} \in E_i \mid 0 \leq f(\bar{x}) \leq \alpha\},\ \alpha \in [0, \infty),\ i = 0, 1.$$

For every $\beta \geq \alpha$, define an equivalence relation in $E_i^\alpha,\ i = 0, 1$ as follows: $x \overset{\beta}{\sim} y$ if x and y belong to the same connected component of $E_i^\beta,\ i = 0, 1$.

Definition 4.3.4. ([1]) The function $n : \{(\alpha, \beta) \in \mathbb{R}^2 | \beta \geq \alpha \geq 0\} \longrightarrow \mathbb{Z}$ given by $n(\alpha, \beta)$ = number of equivalence classes of $E_i^\alpha,\ i = 0, 1$ under $\overset{\beta}{\sim}$ is called the *size function* associated to the set $E_i,\ i = 0, 1$ and the function $f_i,\ i = 0, 1$.

Let $\tilde{E}_i = ch(E_i),\ i = 0, 1$ be the smallest cubical sets containing $E_i,\ i = 0, 1$, see figures (4.3, 4.4). In the figures we have placed the curves E_0 and E_1 along with the points $\bar{a}_0 = (0, 0)$

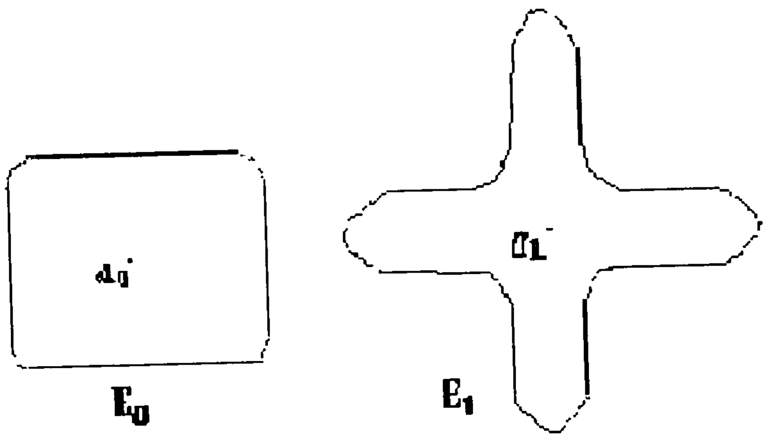


Figure 4.2: Two contours

and $\bar{a}_1 = (0,0)$ into cubical grids and used gray to indicate the sets $\tilde{E}_0$ and $\tilde{E}_1$

Since we restrict the functions f_i, $i = 0,1$ to cubical sets, we will restrict α, β to the set of nonnegative integers and $\bar{a}_0$ and $\bar{a}_1$ to $\mathbb{Z}^2$. Therefore $\tilde{E}_i^{\alpha}$, $i = 0,1$ are cubical sets for any integer α. Finally let $\tilde{n} : \{(\alpha,\beta) \in \mathbb{Z}^2 | \beta \geq \alpha \geq 0\} \to \mathbb{Z}$ be the restriction of the size function n associated with $\tilde{E}_i^{\alpha}$, $i = 0,1$ and f_i, $i = 0,1$ *to integer values of* α *and* β.

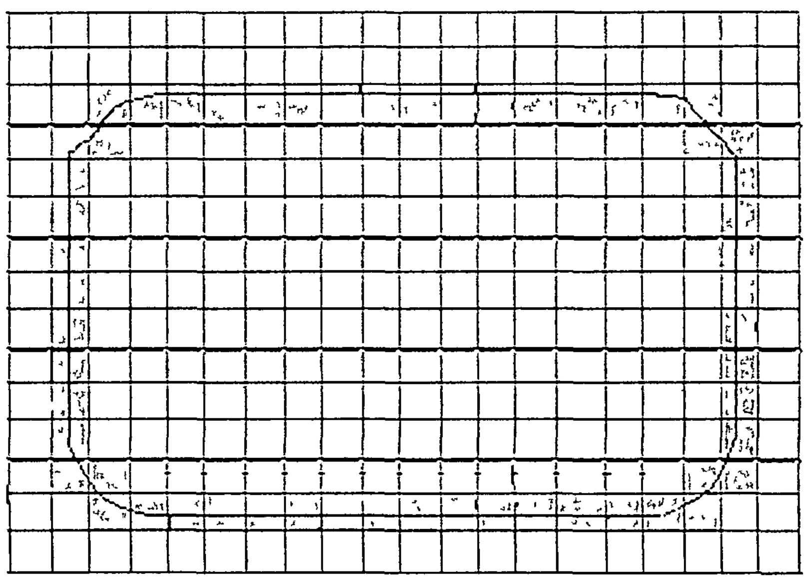


Figure 4 3 Cubical approximation of the first contour

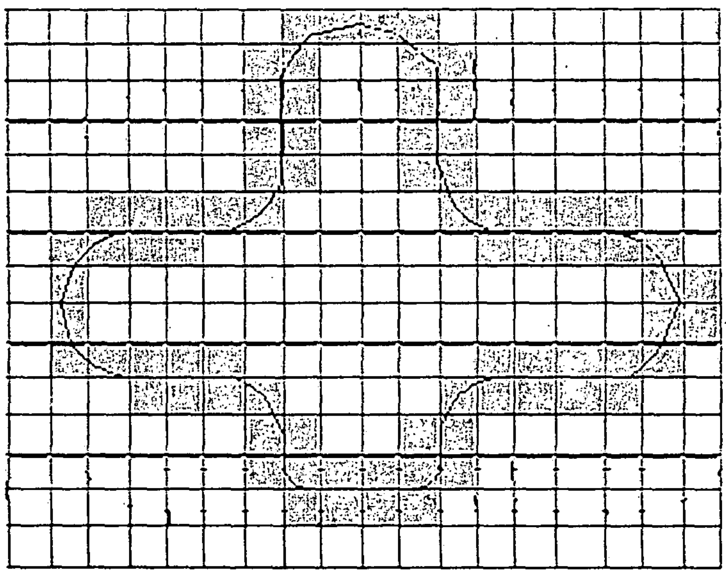


Figure 4.4: Cubical approximation of the second contour

To get a feel of the size function, set $\alpha = 4$ and $\beta = 7$. In the figures (4.5, 4.6) the sets $\tilde{E}_\iota^\alpha$ are indicated by blue shading and the sets $\tilde{E}_\iota^\beta \backslash \tilde{E}_\iota^\alpha$ are indicated by black shading. Observe that $\tilde{E}_\iota^\beta$ consists of those points in the original curve that lie in the blue- or black- shaded squares. Notice that there are no blue-shaded squares in the figure (4.5) and thus $\tilde{E}_0^4 = \phi$. Therefore $\tilde{n}_0(4, 7) = 0$.

On the other hand, figure (4.6) indicates $\tilde{E}_1^4$ consists of four components, but two of the four components are connected to the same black shaded components in the bottom, thus $\tilde{n}_1(4, 7) = 3$.

Thus the size function (considered here) provide a way of distinguishing thc given curves of different shapes, though in general one size fuction is not enough, one may have to consider a collection of size functions (see [1] for a more detailed discussion). We conclude this example by giving a formula which expresses the values of a size fuction in terms of the ranks of the 0-th cubical homology of $\tilde{E}_\iota^\beta$ and the rank of the 0-th cubical relative homology of the pair $(\tilde{E}^\beta, \tilde{E}^\alpha)$.

Proposition 4.3.5. *(see [1])*

$$\tilde{n}(\alpha, \beta) = \text{rank} H_0(\tilde{E}^\beta) - \text{rank} H_0(\tilde{E}^\beta, \tilde{E}^\alpha).$$

More application of homological techniques (independent of specific version of homology) to real world problems will be given in chapters $6 - 9$.

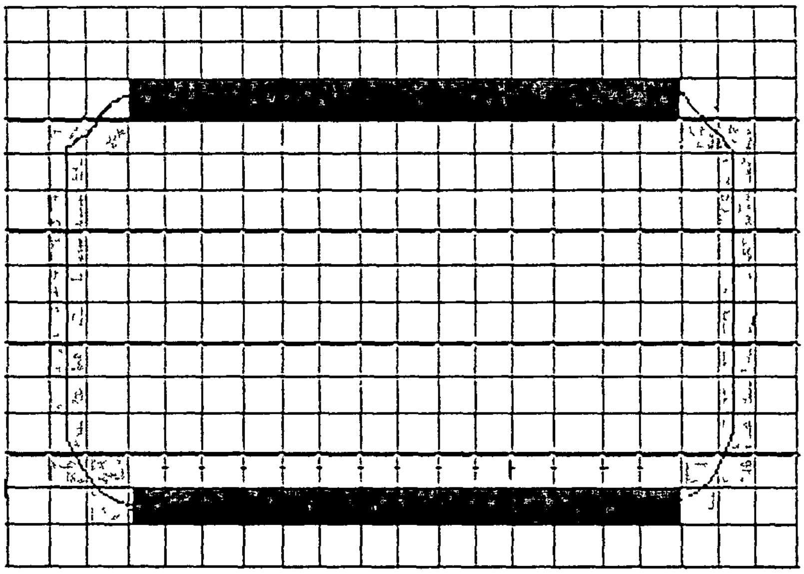


Figure 4 5 Shaded Cubical approximation of the first contour

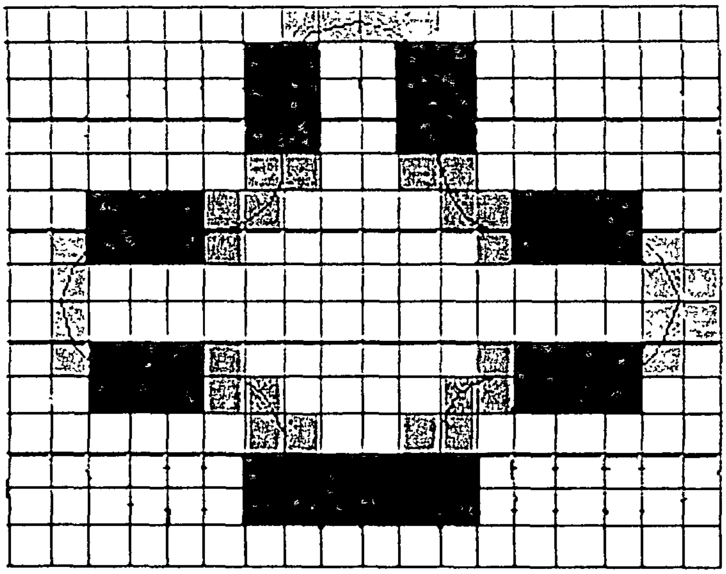


Figure 4.6: Shaded Cubical approximation of the second contour

Chapter 5

Algorithm for Computing Homology Groups

We compute homology groups of finite cubical sets by associating a chain complex of free abelian groups ($\mathbb{Z}-$modules) and homomorphism. Then we compute the images and kernels of the chain homomorphism and their quotients give homology groups.

All these can be done step-wise using suitable algorithms and linear algebra. The purpose of this chapter is to give concrete algorithm of each of these steps leading to the computation of homology group (refer to [1] for details).

5.1 Computing Homology Groups of free chain complex

Let $C : \cdots \to C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \to \ldots$ be a chain complex. The purpose is to compute homology groups $H_k(C) = ker\ \partial_k / im\ \partial_{k+1}$

For this one needs:

(i) an algorithm which takes input $\ker \partial_k$ and $\operatorname{im} \partial_{k+1}$ and gives output the quotient $\ker \partial_k / \operatorname{im} \partial_{k+1}$. ("homology group of Chain complex" algorithm);

(ii) this is the particular case of an algorithm which takes input two finitely generated free abelian groups $H \subseteq G$ and gives output the quotient G/H. ("quotient group" algorithm);

(a) in this algorithm the groups H and G are inputed as matrices V and W of columns consisting of basis elements of the groups;

(b) then using an algorithm of solving linear equation $WA[i] = V[i]$ $\forall i$ one finds a matrix A ("Solve (matrix W, vector $V[i]$" algorithm);

(c) then one reduces A to smith form $(B, Q, \bar{Q}, R, \bar{R}, s, t)$ by using an algorithm called "SmithForm (A)";

(d) then $U := W \times Q$, B and s gives the generators of the free and torsion parts G/H.

We shall indicate each of the algorithms used in the above procedure and also their logical dependence on other algorithms. As is clear, everything boils down to integer matrices and linear equations, reduction to smith form etc., so we indicate the evolution of the above algorithms from the most rudimentary of the algorithms.

5.2 Algorithms leading to Row reduced Echelon form

We are now going to present the six elementary algorithms which shall be the main ingredients for all the algorithms that we are going to illustrate (see [1]):-

(ar1) rowExchange(**matrix B, int** i, j)

(ar2) rowMultiply(**matrix B, int** i)
(ar3) rowAdd(**matrix B, int** i, j, q)
(ac1) columnExchange(**matrix B, int** i, j)
(ac2) columnMultiply(**matrix B, int** j)
(ac3) columnAdd(**matrix B, int**i, j, q)

which return the matrix B to which, respectively, the (r1),(r2),(r3) elementary row operations or (c1),(c2),(c3) elementary column operations have been applied. For instance, the implementation of rowAdd may appear as follows:

Algorithm 5.2.1. *([1]) Add a multiple of a row*
function rowAdd(matrix B, int *i, j, q)*
n = **numberOfColumns(B)***;*
*B[i,l:n] = B[i,l:n] + q * B[j,l:n];*
return *B;*

and the implementation of columnAdd may appear as follows:

Algorithm 5.2.2. *([1]) Add a multiple of a column*
function *columnAdd(***matrix** *B,***int** *i, j, q)*
m=**numberOfRows***(B);*
*B[l:m,j] = B[l:m,j] + q * B[l:m,i];*
return *B;*

Algorithm 5.2.3. *([1]) Row exchange operation keeping track of bases*
function *rowExchangeOperation(***matrix** B, Q, $\bar{Q}$, **int** *i, j)*
B = *rowExchange(B, i, j);*
$\bar{Q}$ = *rowExchange*($\bar{Q}$, *i, j);*
Q = *columnExchange(Q, i, j);*
return (B, Q, $\bar{Q}$);

Similarly, when multiplying a row by -1 we obtain the following algorithm:

Algorithm 5.2.4. *([1]) Multiply a row by -1 keeping track of bases*

function *rowMultiplyOperation(***matrix** B, Q, $\bar{Q}$, **int** i*)*
B = rowMultiply(B, i);
$\bar{Q}$ = rowMultiply($\bar{Q}$, i);
Q = columnMultiply(Q, i);
return *(B, Q, $\bar{Q}$);*

Finally, when adding to a row a multiple of another row , we obtain the following algorithm:

Algorithm 5.2.5. *([1]) Add a multiple of a row keeping track of bases*

function *rowAddOperation(***matrix** B, Q, $\bar{Q}$, **int** i, j, q*)*
B = rowAdd(B, i, j, q);
$\bar{Q}$ = rowadd($\bar{Q}$, i, j, q);
Q = columnAdd(Q, i, j, -q);
return *(B, Q, $\bar{Q}$);*

Definition 5.2.6. ([1]) The *pivot position* of a nonzero vector is the position of the first nonzero element of this vector. A matrix A is in *row echelon form* if for any two consecutive rows r_i and r_{i+1}, if $r_{i+1} \neq 0$ row, then $r_i \neq 0$ row and the pivot position of r_{i+1} is greater than the pivot position of r_i.

Remark 5.2.7. ([1]) If a matrix is in row echelon form, then all its nonzero rows come first, followed by zero rows, if there are any.

Example 5.2.8. ([1]) The following matrix is in row echelon form:

$$\begin{bmatrix} 0 & 2 & 0 & 7 & 3 \\ 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows are the nonzero rows. The pivot positions of these three rows are, respectively, 2, 3 and 5.

Our aim is to develop an algorithm that transform an integer matrix $A \in M_{m,n}(\mathbb{Z})$ to row echelon form by means of elementary operations. We can do this by working on the matrix row by row. So, we will require the following concept. We say that a $m \times n$-matrix A satisfies the (k, l) *criterion of row echelon form* if the submatrix $A[1 : m, 1 : l]$ consisting of the first l columns of A is in row echelon form and the nonzero rows of this submatrix are exactly the first k rows.

Example 5.2.9. ([1]) The following matrix $\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ is not in the row echelon form, but it satisfies the (2, 3) criterion of row echelon form. It also satisfies the (2, 4) criterion of row echelon form, but it satisfies neither the (2, 2) nor the (2, 5) criterion of row echelon form.

Algorithm 5.2.10. *([1]) Partial row reduction*
function *partRowReduce(***matrix,** *B, Q,* , **int** *k, l)*
for *i = k+1* **numberOfRows***(B)* **do**

q = **floor**$(B[i,l]/B[k,l])$;
$(B,\ Q,\ \bar{Q})$ = *rowAddOperation*$(B,\ Q,\ \bar{Q},\ i,\ k,\ \text{-}q)$;
endfor;
return $(B,\ Q,\ \bar{Q})$;

We introduce the following notation concerning the subcolumn $B[k:m,l]$ of a matrix $B \in M_{m,n}(\mathbb{Z})$:

$$\alpha_{kl}(B) := \begin{cases} \min\{[i,l] \mid \mid i \in [k,m], B[i,l] \neq 0\} & \text{if } B[k:m,l] \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

Throughout this chapter this quantity will be useful for measuring the progress in decreasing the magnitudes of the entries in the subcolumns of B.

Let $A, B \in M_{m,n}(\mathbb{Z})$ be such that B satisfies the $(k-1, l-1)$ criterion of row echelon form for some $k \in \{1, 2, \ldots, m\}$, $l \in \{1, 2, \ldots, n\}$ and $B = Q^{-1}AR$, for some $\mathbb{Z}$-invertible matrices $Q \in M_{m,m}(\mathbb{Z})$ and $R \in M_{n,n}(\mathbb{Z})$. If $B[k,l] = \alpha_{kl}(B) \neq 0$, then Algorithm (5.2.10) applied to (B, Q, Q^{-1}, k, l) returns a matrix $B' \in M_{m,n}\mathbb{Z}$ and mutually inverse $\mathbb{Z}-$ invertible matrices $Q', \bar{Q}' \in M_{m,m}(\mathbb{Z})$ such that

1. The first k rows and $l-1$ columns of B and B' coincide,
2. $\alpha_{kl}(B') < \alpha_{kl}(B)$ or $B'[k+1:m,l] = 0$,
3. $B' = (Q')^{-1}AR = \bar{Q}'AR$.

Example 5.2.11. ([1]) Let $A = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 3 & 2 & 1 & 4 \\ 4 & 4 & -2 & -2 \end{bmatrix}$

and let (B, Q, $\bar{Q}$) = partRowReduce(A, $I_{3\times3}, I_{3\times3}, 1, 1$). Then

$$B = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 1 & -1 & 0 & 5 \\ 0 & -2 & -4 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \bar{Q} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Algorithm 5.2.12. *([1]) Smallest nonzero entry*
function *smallestNonzero(***vector** *v,* **int** *k)*
alpha $=$ *min* $\{$**abs** $(v[i]) \mid i$ **in** $[k:$ **length**$(v)]$ **and**$v[i] \neq 0\}$;
$i_0 = min\{i$ **in** $[k:$ **length**$(v)]$**and abs**$(v[i]) = alpha\}$;
return *(alpha,* i_0);

Algorithm 5.2.13. *([1]) Row preparation*
function *rowPrepare(***matrix** *B, Q,* $\bar{Q}$, **int** *k, l)*
$m =$ **numberOfRows***(B);*
(α, i) = smallestNonzero(B[1:m,l], k);
(B, Q, $\bar{Q}$*) = rowExchangeOperation(B, Q,* $\bar{Q}$*, k, i);*
return *(B, Q,* $\bar{Q}$*);*

Let $A, B \in M_{m,n}(\mathbb{Z})$ be such that B satisfies the $(k-1, l-1)$ criterion of row echelon form for some $k \in \{1, 2, \ldots, m\}$, $l \in \{1, 2, \ldots, n\}$ and $B = Q^{-1}AR$ for some $\mathbb{Z}$-invertible matrices $Q \in M_{m,m}(\mathbb{Z})$ and $R \in M_{n,n}(\mathbb{Z})$. If $B[k:m, l] \neq 0$, Then Algorithm (5.2.13) applied to (B, Q, Q^{-1}, k, l) returns a matrix $B' \in M_{m,n}(\mathbb{Z})$ and mutually inverse $\mathbb{Z}$-invertible matrices Q', $\bar{Q}' \in M_{m,m}\mathbb{Z}$ such that

1. the first $k-1$ rows and $l-1$ columns of B and B' coincide,
2. $B'[k, l] = \alpha_{kl}(B') = \alpha_{kl}(B) \neq 0$,
3. $B' = (Q')^{-1}AR = \bar{Q}'AR$, (see [1]).

Example 5.2.14. ([1]) Let $A = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 2 & 3 & 1 & -1 \\ 4 & 4 & -2 & -2 \end{bmatrix}$

Let $(B, Q, \bar{Q}) = rowPrepare(A, I_{3\times3}, I_{3\times3}, 1, 1)$. Then

$$B = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 3 & 2 & 1 & 4 \\ 4 & 4 & -2 & -2 \end{bmatrix}, Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \bar{Q} = Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algorithm 5.2.15. *([1]) Row reduction*
function *rowReduce(***matrix** *B, Q, $\bar{Q}$,***int** *k, l)*
m = **numberOfRows***(B);* **while***B[k+1:m, l]* $\neq 0$ **do**
(B, Q, $\bar{Q}$) = rowPrepare(B, Q, $\bar{Q}$, k, l);
(B, Q, $\bar{Q}$) = partRowReduce(B, Q, $\bar{Q}$, k, l);
endwhile;
return*(B, Q, $\bar{Q}$);*

Let $A, B \in M_{m,n}(\mathbb{Z})$ be such that B satisfies the $(k-1, l-1)$ criterion of row echelon form for some $k \in \{1, 2, \ldots, m\}$, $l \in \{1, 2, \ldots, n\}$ and $B = Q^{-1}AR$ for some $\mathbb{Z}$-invertible matrices $Q \in M_{m,m}(\mathbb{Z})$ and $R \in M_{n,n}(\mathbb{Z})$. If $B[k : m, l] \neq 0$, then Algorithm (5.2.15) applied to (B, Q, Q^{-1}, k, l) returns a matrix $B' \in M_{m,n}(\mathbb{Z})$ and mutually inverse $\mathbb{Z}$-invertible matrices Q', $\bar{Q}' \in M_{m,m}(\mathbb{Z})$ such that

1. $B' = (Q')^{-1}AR = \bar{Q}'AR$,
2. B' satisfies the (k, l) criterion of row echelon form, (see [1]).

Example 5.2.16. ([1]) Let $A = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 2 & 3 & 1 & -1 \\ 4 & 4 & -2 & -2 \end{bmatrix}$

After the first iteration of the while statement in Algorithm 5.2.15 called with arguments (A, $I_{3\times3}, I_{3\times3}, 1, 1$),

$$B = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 1 & -1 & 0 & 5 \\ 0 & -2 & -4 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \bar{Q} = Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Thus , after another application of rowPrepare,

$$B = \begin{bmatrix} 1 & -1 & 0 & 5 \\ 2 & 3 & 1 & -1 \\ 0 & -2 & -4 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \bar{Q} = Q^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

After the application of partRowReduce,

$$B = \begin{bmatrix} 1 & -1 & 0 & 5 \\ 0 & 5 & 1 & -11 \\ 0 & -2 & -4 & 0 \end{bmatrix}, Q = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}, \bar{Q}, Q^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

observe that in the example just considered we have now reduced the problem of finding a row echelon of A, to the problem of finding a row echelon reduction of

$$B[2:3, 2:4] = \begin{bmatrix} 5 & 1 & -11 \\ -2 & -4 & 0 \end{bmatrix}$$

The following algorithm is an outcome of this.

Algorithm 5.2.17. *([1]) Row Echelon*

function *rowEchelon(*matrix*B)*
m = **numberOfRows***(B);*
n = **numberOfColumns***(B);*
Q = $\bar{Q}$ = *identityMatrix(m);*
k = 0; l =1;
repeat
while $l \leq n$ **and** *B[k+1:m, l] = 0* **do** *l = l+1;*
if *l = n+1 then break endif;*
k = k+1;
(B, Q, $\bar{Q}$*) = rowreduce(B, Q,* $\bar{Q}$*, k, l);*
until *k = m;*
return*(B, Q,* $\bar{Q}$*, k);*

Algorithm 5.2.18. *([1]) Kernel-image algorithm*

function *kernelImage(***matrix** *B)*
m = **numberOfRows***(B);*
n = **numberOfColumns***(B);*
BT = transpose(B);
(B,P,$\bar{P}$*,k)=rowEchelon(BT);*

BT = *transpose(B);*
PT = *transpose(P);*
return *(PT[1:m, k+1:n],BT[1:m, 1:k]);*

Given an $m \times n$ matrix A on input. Algorithm (5.2.18) returns an $m \times (n-k)$ matrix W and an $m \times k$ matrix V such that the columns of W constitute a basis of ker A and the columns of V constitute a basis of im A (see [1]).

Example 5.2.19. ([1]) Let $A \in M_{4,3}(\mathbb{Z})$ be given by

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & -1 \\ 3 & 4 & 1 \\ 5 & 3 & -2 \end{bmatrix}$$

We will find bases for ker A and im A. Applying Algorithm (5.2.18) to A. we get the following values of the variables P, B and k:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 4 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $k = 2$. Therefore, the first two columns of $B^T - [2, 0, 4, 3]^T$ and $[0, 1, 3, 5]^T$ form a basis for im$B$ = im$A$, whereas the third column of P^T – $[1, -1, 1]^T$ is a basis of ker$A$.

5.3 Algorithm leading to reduction to Smith normal form

Now we will present an algorithm that produces a diagonal matrix with the property that the *ith* diagonal entry divides the $(i + 1)st$ diagonal entry. And the matrix which is in this form is called the *Smith normal form.* We give first an example.

Example 5.3.1. ([1]) Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix},$$

which we want to reduce to a Smith Normal Form. We need to keep track of both row and column operations. Consider the following augmented matrix:

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix},$$

When row operation are performed on A, only the upper blocks change. When column operations are performed on A, only the right-hand-side blocks change. The zero matrix in the lower left corner never changes: it is there only to complete the expression to a square matrix. At the final stage we obtain a matrix

$$\begin{bmatrix} P & B \\ 0 & R \end{bmatrix}$$

where B is in the Smith Normal Form, R is a matrix of column operations, and P is a matrix of row operations. So, in particular, we have B = P A R.

The first step is to identify a nonzero entry of A with the minimal absolute value (we will use a bold character for it) and bring it, by row and column operations, to the upper left-hand corner of the matrix B obtained from A. By exchanging column c_4 and c_5, we get

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 & 3 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that the first pivot entry containing 2 divides the entries in its column below it, but it does not divide the entries in its row on the right of it. By subtracting r_1 from r_2 and r_3, we get zero entries below the pivot 2, and then by subtracting c_4 from c_5 and c_6, we reduce the value of the entries on the right of 2.

$$\mapsto \begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 3 \\ -1 & 1 & 0 & 0 & -3 & -3 \\ -1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ -1 & 1 & 0 & 0 & -3 & -3 \\ -1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now the minimal absolute value of nonzero entries of the matrix B is 1, so we repeat the procedure. By exchanging c_4 and c_5, we bring 1 to the upper left corner and then use a series of row and column subtractions to zero out the entries below and on the right of the pivot 1.

$$\mapsto \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ -1 & 1 & 0 & -3 & 0 & -3 \\ -1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \dots \mapsto \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now the row and column of the pivot entry 1 are in the desired form. It remains to continue reductions in the 2×2 matrix

$B[2:3,2:3] = \begin{bmatrix} 6 & 0 \\ 2 & 0 \end{bmatrix}$

The minimal nonzero entry of this matrix is 2. By exchanging row r_2 with r_3 in the augmented matrix, we bring 2 to the upper right corner of B[2:3,2:3] and by subtracting $3r_2$ from r_3 we zero out the entry 6. Thus we obtain

$$\begin{bmatrix} P & B \\ 0 & R \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 2 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore the final matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is diagonal with $b_1 = 1$ dividing $b_2 = 2$. We want to know the change of basis corresponding to the row and column operations. We obtained

$$R = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

With the above example in mind, we now proceed with the formal description of the algorithm. Given an $m \times n$ matrix $A \in M_{m,n}(\mathbb{Z})$, our ultimate aim is to produce bases given by columns of some matrices $Q \in M_{m,m}(\mathbb{Z})$ and $R \in M_{n,n}(\mathbb{Z})$ such that the matrix of the homomorphism f_A in the new bases, that is, $B = Q^{-1}AR$, satisfies $B[i.j] = 0$ if $i \neq j$ and $B[i,i]$ divides $B[i+1,i+1]$. We shall do this in a recursive manner. We say that $B \in M_{m,n}(\mathbb{Z})$ is in smith form up to the kth entry if the following conditions are satisfied:

$$B = \begin{bmatrix} B[1,1] & \ldots & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & \ldots & B[k,k] & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & B[k+1:m, k+1:n] & \\ 0 & 0 & 0 & & & \end{bmatrix}$$

$B[i,i]$ divides $B[i+1, i+1]$ for $i = 1, 2, \ldots, k-1$;
and $B[k,k]$ divides $B[i,j]$ for all $i, j > k$.

For the moment let us ignore the conditions on divisibility. Then the problem is essential the same as that solve by rowReduce, except that it needs to be solved not only for the kth column, but simultaneously for the kth column and the kth row.

First we extend the algorithm *smallestNonzero* to find an entry that has the smallest nonzero magnitude for the entire submatrix $B[k:m, k:n]$ (see [1]).

Algorithm 5.3.2. *([1]) Minimal nonzero entry*

function *minNonzero(***matrix** *B,* **int** *k)*
vector *v, q;*
for *i = 1* **to numberOfRows(B)***;*
if *i < k* **then**
v[i] = q[i] = 0
else
*(v[i], q[i]) = smallestNonzero(B[i, 1:***numberOfColumns(B)]***,*
k);
endif;
endfor;
(alpha, i$_0$**)** $= smallestNonzero(v, k)$;
return *(alpha,* $i_0, q(i_0))$;

Algorithm 5.3.3. *([1]) Move minimal nonzero entry*

```
function moveMinNonzero(matrix B, Q, Q̄, R, R̄, int k)
(alpha, i, j) = minNonzero(B, k);
(B, Q, Q̄) = rowExchangeOperation(B, Q, Q̄, k, i);
(B, R, R̄) = columnExchangeOperation(B, R, R̄, k, j);
return (B, Q, Q̄, R, R̄);
```

This algorithm after having found the entry with the minimal nonzero magnitude its move it to the (k, k) position.

Algorithm 5.3.4. *([1]) Check for divisibility*

```
function checkForDivisibility(matrix B, int k)
for i = k+1 to numberOfRows(B)
for j = k+1 to numberOfColumns(B)
q = floor(B[i,j]/B[k,k]);
if q*B[k, k] ≠ B[i, j]then
return (false, i, j, q);
endif;
endfor;
endfor;
return (true, 0, 0, 0);
```

This algorithm checks if the (k, k) entry of a matrix B divides all entries in the submatrix $B[k + 1 : m, k + 1 : n]$.

Algorithm 5.3.5. *([1]) Partial Smith form algorithm*

```
function partSmithForm(matrix B, Q, Q̄, R, R̄, int k)
m = numberOfRows(B);
n = numberOfColumns(B);
repeat
(B, Q, Q̄, R, R̄) = moveMinNonzero(B, Q, Q̄, R, R̄, k);
(B, Q, Q̄) = partRowReduce(B, Q, Q̄, k, k);
```

if *B[k+1:m, k]* $\neq$ **thennext; endif:**
(B, R, $\bar{R}$) = partColumnReduce(B, R, $\bar{R}$, k, k);
if *B[k, k+1:n]* $\neq$ **thennext; endif:**
(divisible i, j, q) = checkForDivisibility(B, k):
if not *divisible* **then**
(B, Q, $\bar{Q}$) = rowAddOperation(B, Q, $\bar{Q}$, i, k, 1);
(B, R, $\bar{R}$) = columnAddOperation(B, R, $\bar{R}$, k, j, -q);
endif;
until *divisible;*
return *(B, Q, $\bar{Q}$, R, $\bar{R}$);*

Let $A, B \in M_{m,n}(\mathbb{Z})$ be integer matrices such that B is in Smith form up to the $(k-1)st$ entry for some $k \in \{1, 2, \ldots, min(m,n)\}$, $B[k : m, k : n] \neq 0$ and $B = Q^{-1}AR$ for some $\mathbb{Z}$-invertible matrices $Q \in M_{m,m}(\mathbb{Z})$ and $R \in (\mathbb{Z})$. Algorithm (5.3.5) applied to $(B, Q, Q^{-1}, R, R^{-1}, k)$ always halts. It returns a matrix B' that is in Smith form up to the kth entry. It also returns two pairs $(Q', \bar{Q}')$ and $(R', \bar{R}')$ of mutually inverse $\mathbb{Z}$-invertible matrices such that
$B' = Q'^{-1}AR' = \bar{Q}'AR'$ (see [1])

Algorithm 5.3.6. *([1]) Smith algorithm*
function *smithForm(***matrix** *B)*
m = **numberOfRows***(B);*
n = **numberOfColumns***(B);*
Q = $\bar{Q}$ = identityMatrix(m);
R = $\bar{R}$ = identityMatrix(n);
s = t = 0;
while *B[t+1:m, t+1:n]* $\neq 0$ **do**
t = t+1;
(B, Q, $\bar{Q}$, R, $\bar{R}$ = partSmithForm(B, Q, $\bar{Q}$, R, $\bar{R}$, t);

if *B[t, t]*< 0 **then**
(B, Q, $\bar{Q}$) = rowMultiplyOPeration(B, Q, $\bar{Q}$, t);
endif;
if *B[t, t] = 1* **then**
s = s + 1;
endif;
endwhile;
return*(B, Q, $\bar{Q}$, R, $\bar{R}$, s, t);*

Given a matrix $A \in M_{m,n}(\mathbb{Z})$, on input Algorithm (5.3.6) returns a matrix $B \in M_{m,n}(\mathbb{Z})$, $\mathbb{Z}$-invertible, mutually inverse matrices $Q, \bar{Q} \in M_{m,m}(\mathbb{Z})$, $R, \bar{R} \in M_{n,n}(\mathbb{Z})$, and nonnegative integers s and t such that

$B = Q^{-1}AR = \bar{Q}AR.$

Furthermore, B has the form

$$B = \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & b_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where b_i are positive integers, $b_i = 1$ for $i = 1, 2, \ldots, s$ and b_i divides b_{i+1} for $i = 1, 2, \ldots, t-1$ (see [1]).

5.4 Computing homology of cubical sets

We are now ready to give the algorithm for computing the homology of any cubical set. This algorithm uses several functions each of which will be denined using corresponding algorithms.

Algorithm 5.4.1. *([1]) Cubical homology algorithm*

```
function homology(cubicalSet K)
E = cubicalChainGroups(K);
D = boundaryOperatorMatrix(E);
H = homologyGroupOfChainComplex(D);
H = generatorOfHomology(H, E);
return H;
```

Algorithm 5.4.2. *([1]) The groups of cubical chains of a cubical set*

```
function cubicalChainGroups(cubicalSet K)
cube Q;
array[-1]of list of cube E;
while K ≠ phi do
(Q, K) = cutFirst(K);
k = dim (Q);
L = primaryFaces(Q);
K = union(K, L);
E[k-1] = union(E[k-1], L);
E[k] = join(E[k], Q);
endwhile;
returnE;
```

Algorithm 5.4.3. *([1]) Primary faces of an elementary cube*

```
function primaryFaces(cube Q)
set of cube L = ϕ;
for i = 1 to lastIndex(Q) do
if Q[i]{left}≠Q[i]{right} then
R = Q;
R[i]{left} = R[i]{right} = Q[i]{left};
L = join(L, R);
```

```
R[i]{left} = R[i]{right} = Q[i]{right};
L = join(L, R);
endif;
endfor;
return L;
```

Algorithm 5.4.4. *([1]) The matrix of the boundary operator*

```
function boundaryOperatorMatrix(cubicalChainComplex E)
array[0:] of matrix D;
for k = 0 to lastIndex(E) do
m = lastIndex(E[k-1]);
for j = 1 to lastIndex(E[k])do
c=boundaryOperator(E[k])[j])
D[k][1:m, j] = canonicalCoordinates(c, E[k-1]);
endfor;
endfor;
return D;
```

Algorithm 5.4.5. *([1]) The boundary operator of an elementary cube*

```
function boundaryOperator(cube Q)
sgn = 1;
chain c = ();
for i = 1 to lastIndex(Q) do
if Q[i]{left} ≠ Q[i]{right} then
R = Q;
if R[i]{left} = R[i]{right} = Q[i]{left};
c{R} = - sgn;
R[i]{left} = R[i]{right} = Q[i]{right};
c{R} = sgn;
sgn = - sgn;
```

endif;
endfor;
return *c;*

Algorithm 5.4.6. *([1]) Finding the coordinate vector of a chain*
function *canonicalCoordinates(chain c,* **array***[1:]***of** *cube K)*
vector *v;*
for *i = 1* **to** **lastIndex***(K)* **do**
if defined *(c{K[i]})* **then**
v[i] = c{K[i]};
else
v[i] = 0;
endif;
endfor;
return *v;*

Algorithm 5.4.7. *([1]) Homology group of chain complex*
function *homologyGroupOfChainComplex(***array** *[0:]* **of matrix** *D)*
array *[-1:]* **of matrix** *V, W;*
for *k = 0* **to lastIndex** *(D)* **do**
(W[k], V[k-1]) = kernelImage(D[k]);
endfor;
*V[***lastIndex***(D)] = 0;*
array *[0:]* **of list** *H;*
for *k = 0* **lastIndex***(D)***do**
H[k] = quotientGroup(W[k], V[k]);
endfor;
return *H;*

Algorithm 5.4.8. *([1]) Kernel-image algorithm*

```
function kernelImage(matrix B)
m = numberOfRows(B);
n = numberOfColumns(B);
BT = transpose (B);
(B, P, P̄, k) = rowEchelon(BT);
BT = transpose (B);
PT = transpose (P);
return (PT[1:m, k+1:n], BT[1:m, 1:k]);
```

Algorithm 5.4.9. *([1]) Linear equation solver*

```
function Solve(matrix A, vector b)
m = numberOfRows(A);
(B, Q, Q̄, R, R̄, s, t) = smithForm(A);
c = Q̄ * b;
vector u;
for i = 1 to t do
if B[i, i] divides c[i] then
u[i] = c[i]/B[i,i];
else
return "Failure";
endif;
endfor;
for i = t+1 to m do
if c[i] ≠ 0 then
return "Failure"
else
u[i] = 0;
endif;
endfor;
return R * u;
```

Algorithm 5.4.10. *([1])Quotient group finder*

```
function quotientGroup(matrix W, V)
n = numberOfColumns(V);
matrix A;
for i = 1 to numberOfColumns(V)
A[i] = Solve(W, V[i]);
endfor;
(B, Q, Q̄, R, R̄, s, t) = smithForm(A);
U = W * Q;
return (U, B, s);
```

Algorithm 5.4.11. *([1])Generators of homology*

```
function generatorsOfHomology(array[0:]of list H, cubicalChain-
Complex E)
for k = 1 to lastIndex(H) do
m = lastIndex(E[k]);
(U, B, s, t) = H[k];
array[0:]hash {generators, orders} HG;
for j = s + 1 to lastIndex(E[k]) do
if j ≤ t then order = "infinity";
else order = B[j, j] endif;
c = chainFromCanonicalCoordinates(U[1:m,j], E[k]);
HG[k] {generators}[j-s] = c;
HG[k] {orders[j-s] = order;
endfor;
endfor;
return HG;
```

Algorithm 5.4.12. *([1]) Reconstructing a chain from its canonical coordinates*

```
function chainFromCanonicalCoordinates(vector v,array[1:]
of cube K)
  chain c = ();
  for i := 1 to lastIndex(K) do
  if v[i] ≠ 0 then
  c{K[i]} = v[i];
  endif;
  endfor;
  return c;
```

Let X be a cubical set. Call Algorithm (5.4.1) with K containing a list of elementary cubes in $\mathcal{K}_{\max}(X)$. Then the algorithm returns an array of hashes H for which H[k]{generators} is a list $(c_1, c_2, ..., c_n)$ and H[k]{orders} is a list $(b_1, b_2, ..., b_n)$ such that $c_i \in Z_k(X)$ is a cycle of order b_i and

$$H_k(X) = \oplus_{i=1}^{n} < [c_i] > .$$

Chapter 6

Coverage of a Planar Domain by Sensor Networks, Rips and Čech complexes

Given a planar domain (e.g. a field, a forest, or an ocean), one wants to get information about the domain or scan the domain for events, objects or substance, like locating outbreak of forest fire, detecting radiological or biological hazards, detecting hidden mines, or detecting unsual or unexpected objects like individuals or vehicles, ships etc.

The method to be employed for this is to distribute a collection of sensors (or nodes) in the domain. These sensors will be capable of providing information of the domain in their neighbourhoods. Question then arises as to whether the information provided by the collection of these sensors can take care of the whole domain?

Certainly to minimize the cost for the job one would like to use a minimal collection of sensors to do the job.

Moreover, there is no means to determine the distance or relative position of the sensors, and environment may even move the position of sensors.

Probalistic methods to deal with such situation needs strong assumption on the density and uniformity of the distribution of sensors.

Homological methods however can deal with these situation with much weaker assumptions.

6.1 Coverage problem with special assumption for the sensors on the boundary of the domain

We first state the problem mathematically:

Given a planar domain D with polygonal boundary. Each sensor (node) can perform assigned jobs within a (radially symmetric) coverage disk neighbourhood. The problem of *blanket coverage* is as follows: "Can the collection of these coverage disks cover D?"

In this section we impose a few conditions on the sensors so that there is a solution or a reasonable partial solution.

Assumption 6.1.1. As earlier let D be a compact connected subset (with nonempty interior) of $\mathbb{R}^2$, with connected polygonal boundary ∂D. Let $X \subset D$ be a set of sensors (nodes,points,vertices). Let $X_f \subset \partial D$ be the sensors lying on the vertices of the polygonal boundary.

. We assume the following:

A1: There is a positive real number $r_b > 0$, called the *broadcast radius* such that within a r_b-disk nbd of a node it can detect the presence of any other node.

A2: There is a positive real number $r_c \geq r_b/\sqrt{3}$, called the *coverage radius*, such that within a r_c-disk nbd of nodes the nodes can perform assigned jobs.

A3: For each fence node the two neighbouring nodes lie within its r_b-disk nbd.

This last assumption **A3** is a *special assumption* on the fence nodes.

Definition 6.1.2. ([4]) The *network graph* of the above system is a combinatorial graph, Γ, in which vertex set is the set of labeled sensors (nodes), and (undirected) edges correspond to pairs of nodes that are within mutual broadcasting range (within distance r_b). By assumption the fence nodes form a cycle $\mathcal{F} \subset \Gamma$.

The problem at hand is to determine whether the family $\mathcal{U}$ of disks $B_{r_c}(x)$ of radius r_c around the nodes $x \in X$ is a cover of D?

The input of the problem is the pair $(\Gamma, \mathcal{F})$.

Definition 6.1.3. ([6]) Given a collection of sets $\mathcal{U} = \{U_\alpha\}$, the **Čech complex** of $\mathcal{U} = \{U_\alpha\}$, $\mathcal{C}(\mathcal{U})$, is the abstract simplicial complex whose k-simplices correspond to nonempty intersection of $k+1$ distinct elements of $\mathcal{U}$. If the cover is **good**-that is, if the cover sets and all nonempty finite intersections of cover sets are contractible-then the Čech complex $\mathcal{C}$ captures the topology of the cover:

Result 6.1.4. Under the assumptions of (6.1.1) the coverage area $\cup\{U|U \in \mathcal{U}\}$ contains the domain D if and only if the fence 1-cycle $\mathcal{F}$ is null-homologous in the Čech complex $\mathcal{C}(\mathcal{U})$.

Theoretically this is a very satisfying result however it is not possible to compute the Čech complex of $\cup\{U|U \in \mathcal{U}\}$ from the network graph Γ alone. Precise distances between nodes are required to determine the higher-dimensional simplices of $\mathcal{C}(\mathcal{U})$. All we have is two radii r_b (broadcast radius) and r_c (coverage radius). It is not possible to derive the Čech complex $\mathcal{C}(\mathcal{U})$ of r_c-disks around nodes from the network graph Γ defined using r_b. It is not even possible to recover the homotopy type of $\mathcal{C}(\mathcal{U})$.

Note however that the convex hull in $\mathbb{R}^2$ of any triple of nodes which are pairwise within the broadcast radius r_b is contained in $\cup\{U|U \in \mathcal{U}\}$. In the extreme case when these nodes are pairwise at a distance exactly r_b form an equilateral triangle in $\mathbb{R}^2$ that is contained in $\cup\{U|U \in \mathcal{U}\}$ (recall U are disks of radius r_c around nodes) only if $r_c \geq r_b/\sqrt{3}$.

Definition 6.1.5. ([5]) Given a set of point $\chi = \{x_\alpha\}$ in a metric space and a fixed $\epsilon > 0$, the **ϵ-Rips complex of χ, $\mathcal{R}_\epsilon(\chi)$**, is the abstract simplicial complex whose k-simplices correspond to unordered $(k+1)$-tuples of points in χ which are pair-wise within distance ϵ of each other.

Remark 6.1.6. [4] *r_b-Rips complex*, $\mathcal{R}$, of χ (also referred to as *Vietoris-Rips complex* or *flag complex*) is the largest simplicial complex whose 1-skeleton is the network graph Γ.

The Rips complex need not capture the topology of $\cup\{U|U \in \mathcal{U}\}$ but it does contain enough topological information of $\cup\{U|U \in \mathcal{U}\}$. We use the homology of $\mathcal{R}$ relative to $\mathcal{F}$ to obtain coverage criterion.

Lemma 6.1.7. *([5]) Any nonzero 1-cycle $\zeta \in Z_1(\mathcal{F})$ defines a nonzero element of $H_1(\partial\mathcal{D})$.*

Proof. By the definition of homology, $H_1(\mathcal{F}) = Z_1(\mathcal{F})/B_1(\mathcal{F})$. However, $B_1(\mathcal{F}) = \partial(C_2(\mathcal{F})) = 0$, since $C_2(\mathcal{F}) = 0$, $\mathcal{F}$ being a 1-dimensional simplicial complex; hence $Z_1(\mathcal{F}) = H_1(\mathcal{F}) = H_1(\partial\mathcal{D})$. ☐

Lemma 6.1.8. *([5]) A cycle $\zeta \in Z_1(\mathcal{F})$ is nonzero if and only if it has a nonzero coefficient at every fence edge.*

Proof. If ζ is a cycle, then the coefficient of ζ at any pair of adjacent edges is the same up to a sign, because $\partial\zeta$ has coefficient zero at their common vertex. Since the boundary is connected, ζ has the same coefficient at every edge of $\mathcal{F}$ up to a sign. The lemma follows immediately. ☐

Theorem 6.1.9. *([5]) For a set of nodes χ in a domain $D \subset \mathbb{R}^2$ satisfying assumptions* **A1 – A3**, *the sensor cover $\mathcal{U}_c$ contains D if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ such that $\partial\alpha \neq 0$.*

Proof. We consider the simplicial realization map $\sigma : \mathcal{R} \to \mathbb{R}^2$ which sends vertices of the abstract complex $\mathcal{R}$ to the corresponding node points of $\chi \subset D$ and which sends a k-simplex of $\mathcal{R}$ to the (potentially n singular) k-simplex given by the convex hull of the images of the vertices. By **A3**, σ takes the pair $(\mathcal{R}, \mathcal{F})$ to $(\mathbb{R}^2, \partial D)$; we therefore construct the following diagram from the long homology exact sequences of the pairs $(\mathcal{R}, \mathcal{F})$ and $(\mathbb{R}^2, \partial D)$

$$(1) \qquad \begin{array}{ccc} H_2(\mathcal{R}, \mathcal{F}) & \xrightarrow{\delta_*} & H_1(\mathcal{F}) \\ \downarrow \sigma_* & & \downarrow \sigma_* \\ H_2(\mathbb{R}^2, \partial D) & \xrightarrow{\delta_*} & H_1(\partial\mathcal{F}) \end{array}$$

Here δ_* acts on a class $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ by taking the boundary: $\delta_*[\alpha] = [\partial\alpha] \in H_1(\mathcal{F})$. It follows from the naturality of the long exact sequence that the diagram (1) is commutative: $\delta_*\sigma_* = \sigma_*\delta_*$. The homology class $\sigma_*\delta_*[\alpha]$ is the winding number of $\partial\alpha$ about ∂D.

By assumption, $\partial\alpha \neq 0$; hence, by Lemma (6.1.7), we observe that $\sigma_*\delta_*[\alpha] = \sigma_*[\partial\alpha] \neq 0$. By commutativity of diagram (1), $\sigma_*\delta_*[\alpha] \neq 0$, and thus $\sigma_*[\alpha] \neq 0$.

Assume that $\mathcal{U}$ does not cover D and choose $p \in D \backslash \cup \{U \mid U \in \mathcal{U}\}$. Since the convex hull of any collection of nodes in D which form a simplex of $\mathcal{R}$ lies within $\cup\{U \mid U \in \mathcal{U}\}$ every point in $\sigma(\mathcal{R})$ lies within $\cup\{U \mid U \in \mathcal{U}\}$, so we have that $\sigma : (\mathcal{R}, \mathcal{F}) \to (\mathbb{R}^2, \partial D)$ factors through the pair $(\mathbb{R}^2 \backslash \{p\}, \partial D)$. However, $H_2(\mathbb{R}^2\backslash\{p\}, \partial D) = 0$, for, Let $A = \mathbb{R}^2\backslash\{p\}$ and B be a small ball about p, so that $A \cap B$ is an open annulus homotopic to S^1. Let $A' = \partial D$ and $B' = \phi$. Using the relative Mayer-Vietoris sequence, we have

(2) $\cdots \to H_2(S^1) \xrightarrow{\phi_*} H_2(\mathbb{R}^2\backslash\{p\}, \partial D) \oplus 0 \xrightarrow{\psi_*} H_2(\mathbb{R}^2, \partial \mathcal{D}) \xrightarrow{\partial_*} H_1(S_1) \xrightarrow{\phi_*} \dots$

Since $(\mathbb{R}^2, \partial D)$ deformation retracts to the pair $(D, \partial D)$ fixing ∂D, we have that

(3) $H_2(\mathbb{R}^2, \partial D) \cong H_2(D, \partial D) \cong H_2(D/\partial D) \cong H_2(S^2) \cong \mathbb{R}$.

Since $p \in D$, the homomorphism ∂_* takes the generator of $H_2(\mathbb{R}^2, \partial D)$ to that of $H_1(S_1)$. (2) therefore simplifies to

(4) $\cdots \to 0 \to H_2(\mathbb{R}^2\backslash\{p\}, \partial D) \to \mathbb{R} \xrightarrow{\cong} \mathbb{R} \to \dots$

By exactness, $H_2(\mathbb{R}^2\backslash\{p\}, \partial D) = 0$ and thus $\sigma_*[\alpha] = 0$: contradiction. □

Can one get rid of the redundant members of the cover $\mathcal{U}$ and still cover D. The following corollary answers this.

Corollary 6.1.10. *([5]) If a homology class in $H_2(\mathcal{R}, \mathcal{F})$ satisfies the criterion of Theorem (6.1.9), then the restriction of $\mathcal{U}$ to those nodes which make up the representative α suffice to cover*

D, for any choice of α in the homology class.

Proof. Let $\mathcal{U}^{\alpha}$ denote the restriction of $\mathcal{U}$ to the nodes which make up the representative α. Assume that $\mathcal{U}^{\alpha}$ does not cover D and choose $p \in D \backslash \cup \{U \mid U \in \mathcal{U}^{\alpha}\}$ The fact that the convex hull of any collection of nodes in D which form a simplex of $\mathcal{R}$ lies in $\mathcal{U}$ implies that $\sigma(\mathcal{R}) \subset \cup\{U \mid U \in \mathcal{U}^{\alpha}\}$. Thus, $\sigma : (\mathcal{R}, \mathcal{F}) \rightarrow (\mathbb{R}^2, \partial D)$ again factors through the pair $(\mathbb{R}^2 \backslash \{p\}, \partial D)$, which has vanishing homology in dimension two. □

Remark 6.1.11. Since one can choose any representative α of the homology class $[\alpha]$ in this corollary, it allows us to choose α with minimum number of 0-simplices, thereby covering D with a minimal subcover.

6.2 Modification of coverage radii to regain coverage of the domain (Hole Repair)

Suppose the set of nodes satisfying **A1-A3** gives a Rips complex for which the condition of the Theorem (6.1.9) is not satisfied that is $\partial\alpha = 0$. In such a situation how can one modify the cover $\mathcal{U}_c$ to restore coverage? The following theorem ensures coverage by increasing the coverage radii slightly, and modifying the Rips complex suitably.

Theorem 6.2.1. *([5]) Consider a set of nodes χ satisfying assumption* **A1 − A3**. *Let $\Gamma = \{\gamma_i\}_1^K$ be a basis of K generators in $H_1(\mathcal{R})$ and let $N_i = \|\gamma_i\|$ for each i, where $\|.\|$ denotes length of the generator in terms of the number of nodes. Let $\mathcal{U}'$ denote the set obtained from the collection $\mathcal{U}$ by enlarging all the balls based at nodes in γ_i to balls of radius*

(5) $r_c'(i) = \frac{r_b}{2}\csc\frac{\pi}{N_i}$. *Then* $D \subset \cup\{U \mid U \in \mathcal{U}'\}$.

Proof. The quantity $r_c'(i)$ represents the minimal radius needed to cover a regular N_i-gon. We claim that this is the limiting case.

Consider the image $\mathcal{L} = \sigma(\gamma_i)$ of the loop γ_i in D. This is a (not necessarily embedded) loop in D. A point $x \in D$ is enclosed by $\mathcal{L}_i$ if $[\mathcal{L}_i]$ is nonzero in $H_1(\mathbb{R}^2\backslash\{x\}) \cong \mathbb{Z}$ (this class is the **winding number** of the loop about x). We demonstrate that covering each node of γ_i with a ball of radius $r_c'(i)$ covers any such x. For such an x it follows that one or more of the N_i edges of $\mathcal{L}$ subtends an angle at x of atleast $2\pi/N_i$; for otherwise there would exist rays originating at x which miss $\sigma(\gamma_i)$ entirely, making $\mathcal{L}_i$ contractible in $\mathbb{R}^2\backslash\{x\}$ and the winding number zero. Let ab be such an edge. Taking cosines this inequality becomes

$$(6) \qquad cos(\frac{2\pi}{N_i}) \geq \frac{|xa|^2 + |xb|^2 - |ab|^2}{2|xa||xb|} \geq 1 - \frac{r_b^2}{2|xa||xb|}$$

where we use the AM-GM inequality and the fact that $|ab| \leq r_b$ for the latter inequality. Since $cos(2\pi/N_i) = 1 - 2sin^2(\pi/N_i)$ we can rearrange to obtain $|xa||xb| \leq (r_c'(i))^2$. Thus x must lie within distance $r_c'(i)$ of the nearer of two nodes a, b as required.

We now create a modified complex $\mathcal{R}'$ obtained from $\mathcal{R}$ in the following manner. For each i, sew in an abstract 2-d disc along the loop γ_i. (If one wishes to remain in the simplicial category, one can triangulate the disc.) Next, extend the map σ to a continuous map $\sigma' : \mathcal{R}' \to \mathbb{R}^2$.

The long exact sequence yields a commutative diagram

$$
(7)\quad \begin{array}{ccccc}
H_2(\mathcal{R}', \mathcal{F}) & \xrightarrow{\delta_*} & H_1(\mathcal{F}) & \xrightarrow{\imath_*} & H_1(\mathcal{R}') \\
\downarrow \sigma'_* & & \downarrow \sigma'_* & & \downarrow \sigma'_* \\
H_2(\mathbb{R}^2, \partial D) & \xrightarrow{\delta_*} & H_1(\partial D) & \xrightarrow{\iota_*} & H_1(\mathbb{R}^2).
\end{array}
$$

Since we have filled in all the generators of $H_1(\mathcal{R})$, we have that $H_1(\mathcal{R}') = 0$ and $\delta_* : H_2(\mathcal{R}', \mathcal{F}) \to H_1(\mathcal{F})$ is onto. Exactness implies that there exists a generator $[\alpha]$ of $H_2(\mathcal{R}')$ with $\partial\alpha = \mathcal{F}$.

Assume by way of contradiction that there exists a point $p \in D \backslash \cup \{U \mid U \in \mathcal{U}'\}$. If $|\mathcal{L}_\imath| \neq 0 \in H_1(\mathbb{R}^2 \backslash \{p\})$ for any $\imath$, then $p \in \cup\{U \mid U \in \mathcal{U}'\}$ by the argument above. Therefore, assume that these homology classes vanish for all $\imath$. Since the set $\{\gamma_\imath\}$ forms a basis for $H_1(\mathcal{R})$, there exist a 2-chain ζ in $C_2(\mathcal{R})$ such that $\partial\zeta = \mathcal{F} - \sum_\imath c_\imath \gamma_\imath$ for some constants $c_\imath$. Applying σ to these 1-chains yields the equation $\partial\sigma(\zeta) = \partial D - \sum_\imath c_\imath \mathcal{L}_\imath$. This descends to an equation in $H_1(\mathbb{R}^2 \backslash \{p\})$, since p is assumed to be outside $\cup\{U \mid U \in \mathcal{U}'\}$ and $\sigma(\zeta) \subset \cup\{U \mid U \in \mathcal{U}\} \subset \cup\{U \mid U \in \mathcal{U}'\}$ by the fact that the convex hull of any collection of nodes in D which form a simplex of $\mathcal{R}$ lies in $\cup\{U \mid U \in \mathcal{U}\}$. We know that $[\partial D] \neq 0$ in $H_1(\mathbb{R}^2 \backslash \{p\})$ since $p \in D$. By Assumption that all the winding numbers of $\mathcal{L}_\imath$ about p vanish, we have that $[\partial\sigma(\zeta)] \neq 0 \in H_1(\mathbb{R}^2 \backslash \{p\})$. However, $\zeta \in C_2(\mathcal{R})$ and is an algebraic sum of 2-simplices in $\mathcal{R}$. Atleast one such 2-simplex ς of ζ must therefore satisfy $\sigma(\partial\varsigma) \neq 0 \in H_1(\mathbb{R}^2 \backslash \{p\})$, implying, that $p \in \sigma(\zeta) \subset \cup\{U \mid U \in \mathcal{U}\} \subset \cup\{U \mid U \in \mathcal{U}'\}$. Contradiction. □

6.3 Variants of the condition A1-A3

1. Network in an unbound domain: In this situation the last condition about the fence nodes is dropped, because the domain is unbounded. What kind of coverage question can then be asked? A simple answer can be given as follows.

Let γ be a 1-cycle in the Rips complex $\mathcal{R}$. We define *span of* γ, $<\gamma>$ to be the largest subcomplex of $\mathcal{R}$ generated by the nodes of γ.

Lemma 6.3.1. *Let γ be a 1-cycle in $\mathcal{R}$, such that $<\gamma>=\gamma$. Then the projection $\sigma(\gamma)$ of γ to the plane is a simple closed curve.*

Theorem 6.3.2. *For a planar network on an unbounded domain satisfying* **A1** *and* **A2** *choose a 1-cycle γ such that $<\gamma>=\gamma$. If $H_2(\mathcal{R},\gamma)$ has a generator $[\alpha]$ with $\partial\alpha \neq 0$. Then the entire domain bounded by $\sigma(\gamma)$ in $\mathbb{R}^2$ is covered by $\mathcal{U}^{\alpha}$.*

2. Network in a disconnected domain D If D is disconnected then each connected component of D can be treated seperately as earlier.

3. Domain D **with disconnected boundary** ∂D.

If D is connected but it has more than one boundary components. For example in the figure(6.1) we have $\partial^+ D$-outer boundary and $\partial^- D$-inner boundary. The fence nodes $X_f = X_f^+ \cup X_f^-$ can be partitioned accordingly, and the corresponding fence complexes can be denoted by $\mathcal{F}^+, \mathcal{F}^-$. In such a situation we have:

Theorem 6.3.3. *([5]) Let $D \subseteq \mathbb{R}^2$ whose boundary $\partial D = \partial^+ D \cup \partial^- D$ has two connected components $\partial^+ D$ (= outer boundary component) $\partial^- D$ (= inner boundary component). Let the assumptions* **A1-A3** *are satisfied. Then the sensor cover $\mathcal{U}_c$ covers D if there exists $[\alpha] \in H_2(\mathcal{R},\mathcal{F})$ such that $\partial\alpha$ is nonzero on the* **outermost boundary component**.

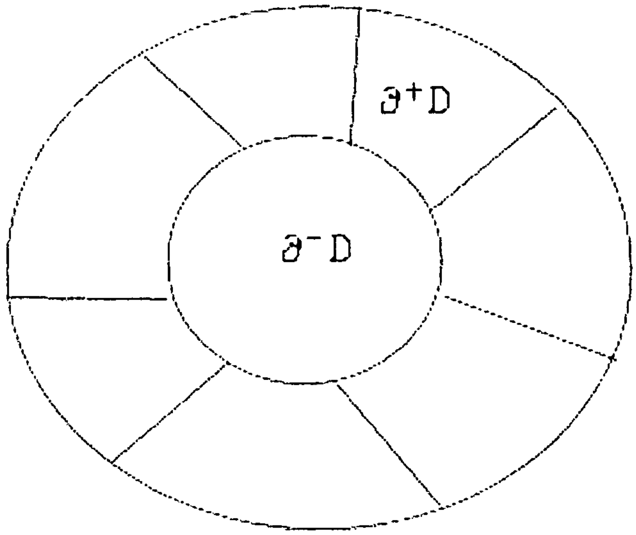


Figure 6.1: Domain with disconnected boundary

Proof. (Sketch of proof):

The condition on $[\alpha]$ is that $\delta_*[\alpha] \neq 0$ where $\delta_* : H_1(\mathcal{R}, \mathcal{F}) \to H_1(\mathcal{F}, \mathcal{F}^-) \cong H_1(\mathcal{F}^+)$ is the boundary map of the long exact sequence of the triple $(\mathcal{R}, \mathcal{F}, \mathcal{F}^-)$. One takes simplicial realization map $\sigma : (\mathcal{R}, \mathcal{F}, \mathcal{F}^-) \to (\mathbb{R}^2, \partial D, \partial^- D)$ and adapts the proof of the Theorem(6.1.9) suitably (for details refer to [5]). □

4. Restricting the condition A1 (Opaque Boundary) Suppose we replace the condition **A1** to the following:
$\mathbf{A_1'}$: There is a positive real number $r_b > 0$, called the broadcast radius, such that within its r_b-disk neighbourhood a node can detect the presence of any other node which can be joined to it by a straight line segment lying in D.

Such situation occur if the communication is through light signal and ∂D is opaque (wall) which blocks light.

In this case we have to form a *Restricted Rips complex* $\mathcal{ER}$ to include only those edges which lie within D. The relevant theorem then is:

Theorem 6.3.4. *Consider a set of nodes $X \subset D \subseteq \mathbb{R}^2$ satisfying assumptions $\mathbf{A_1'}, \mathbf{A_2}, \mathbf{A_3}$. Let $\mathcal{ER}$ is the Restricted Rips complex involving those edges which are in D. The sensor cover $\mathcal{U}_c$ covers D if $\exists\ [\alpha] \in H_2(\mathcal{ER}, \mathcal{F})$ such that $\partial\alpha \neq 0$.*

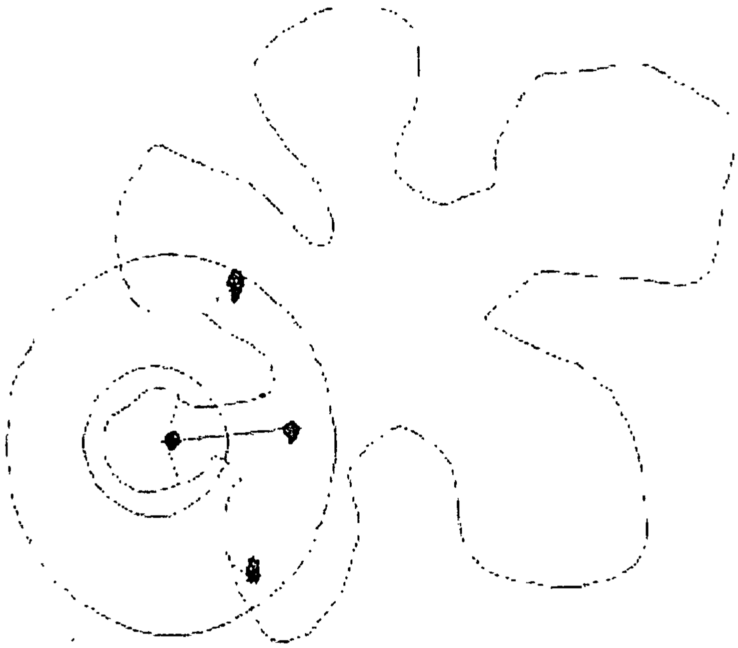


Figure 6.2: Opaque Boundary

Proof. Since $\mathcal{ER} \subset \mathcal{R}$.

$$H_2(\mathcal{ER}, \mathcal{F}) \xrightarrow{\delta_*} H_1(\mathcal{F})$$

(7) $$\downarrow \sigma_* \qquad\qquad \downarrow \sigma_*$$

$$H_2(\mathbb{R}^2, \partial D) \xrightarrow{\delta_*} H_1(\partial D)$$

is commutative. Rest of the proof is similar to the Theorem(6.1.9).

□

Chapter 7

More of coverage of planar domains and 3-dimensional domains

In the last chapter we consider a sensor network for coverage of a planar domain, where certain condition on the nodes, **A1-A3** have been imposed. We also considered some of its variants in the last section.

In all these we had only two fixed real numbers $r_b > 0$ the broadcast radius and $r_c > 0$ the coverage radius for all nodes.

Now we consider a slightly more general situation

7.1 Broadcast and Coverage radii varying with nodes - to achieve optimal coverage

The relevant condition in this situation are as follows:

One has a system of nodes $\chi = \{x_i\}$ which satisfies a modified set of assumptions:

Assumption 7.1.1. **V1**: For every node $x_i \in X$ there is associated a real number $r_b^i > 0$ called broadcast radius at x_i, such that x_i can detect the presence of any node x_j which lies in the open ball $B_{r_b^i}(x_i)$ of radius r_b^i and centre x_i. This is true for all i.

V2: For every node $x_i \in X$ there is associated a real number $r_c^i > r_b^i/\sqrt{3} > 0$, called the coverage radius at x_i

V3: Nodes χ lie in a compact connected domain $D \subset \mathbb{R}^2$ whose boundary ∂D is connected and piecewise-linear with vertices on it marked fence nodes χ_f.

V4: For each fence node $v_k \in \chi_f$ the neighbouring nodes in χ_f on either side of v_k lies in $B_{r_b^k}(v_k)$.

We modify the construction of the Rips complex as follows. For any pair of nodes x_i and x_j, there is an edge in $\mathcal{R}$ if and only if the distance between x_i and x_j in D is less than or equal to the minimum of r_b^i and r_b^j. The full complex $\mathcal{R}$ is then the maximal simplicial complex for the edge set as defined. The fence cycle $\mathcal{F}$ is defined in the same way as before, with vertex set χ_f and an edge between each pair of adjacent nodes along the fence. We define the variable-radius cover $\mathcal{U}_c$ in this context to be union of closed discs of radii r_c^i centered at node x^i.

Theorem 7.1.2. *([5]) For a set of nodes χ in a domain $D \subset \mathbb{R}^2$ satisfying the variable-radius assumptions* **V1-V4**, *the variable-radius cover $\mathcal{U}_c$ covers D if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ such that $\partial\alpha \neq 0$.*

Proof. (*Sketch of Proof*): The main ingredient of proof is to verify that even in this modified definition of communication graph and the Rips complex the following result hold:

The convex hull of any collection of nodes in D which form a simplex of $\mathcal{R}$ lies within the union of the variable radius cover $\mathcal{U}_c$ (see [5]; rest of the proof is similar to the Theorem (6.1.9)) $\square$

7.2 Coverage of 3-d cylindrical regions

Let the domain be of the form $D \times \mathbb{R}$, where $D \subseteq \mathbb{R}^2$ as earlier (compact and connected with connected polygonal boundary). We suppose that $\chi \subseteq D \times \mathbb{R}$ is the set of nodes and that the fence nodes lie in $D \times \{0\}$ and satisfy the assumption **A3**

Let $B_c(x_i)$ be a ball in $\mathbb{R}^2 \times \mathbb{R}$ of radius c and centre x_i, and $\mathcal{U}_c = \{B_c(x_i)|x_i \in \chi\}$.

The problem of *barrier coverage* asks whether one can find a path connecting $D \times \{-\infty\}$ and $D \times \{\infty\}$ which avoids all elements of $\mathcal{U}_c$? We want a criterion for a negative result.

Network graph Γ and Rips complex $\mathcal{R}$ are constructed as earlier. The fence cycle $\mathcal{F}$ is precisely $\partial D \times \{0\}$.

Theorem 7.2.1. *([5]) There do not exist any path connecting $D \times \{-\infty\}$ and $D \times \{+\infty\}$ which avoids elements of $\mathcal{U}_c$ if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ with $\partial\alpha \neq 0$.*

Proof. (*sketch of Proof*): One starts with simplicial realization map $\sigma : \mathcal{R} \to D \times \mathbb{R}$ which takes any 2-cycle to $U_\alpha = \cup\{B_c(x_i)|x_i \in \alpha\}$. As earlier we must have $\sigma_*(\alpha) \neq 0$.

Now we assume that $\exists$ a continuous path $p : \mathbb{R} \to D \times \mathbb{R} \backslash U_\alpha$ with $lim_{t \to \pm\infty} \pi o p(t) = \pm\infty$, where $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the projection map. Since every point of $\sigma(\alpha)$ lies in U_α, we have a factorization

$$\begin{array}{c}\sigma : (\alpha, \partial\alpha) \rightarrow (\mathbb{R}^2 \times \mathbb{R}, \partial D \times \{0\}) \\ \searrow \quad \nearrow \\ (\mathbb{R}^2 \times \mathbb{R} \backslash p(\mathbb{R}), \partial D \times \{0\})\end{array}$$

This leads as earlier to the fact that $\sigma_*(\alpha) = 0$ giving a contradiction (for details of the proof refer to [5]). □

7.3 Coverage Problem in which nodes change position with time

Consider a situation in which the node positions are a continuous function of time: $\chi = \chi_t \subset D$ for $t \in [0, 1]$.

Assume that the network communication graph is updated at time intervals $0 = t_1 < \ldots t_i < \cdots < t_N = 1$, giving an ordered sequence of communication graphs $\Gamma_i, i = 1, 2, \ldots, N$. These give the corresponding Rips complexes $\mathcal{R}_i$, $i = 1, \ldots N$. We now need additional condition to handle coverage problem:

Assumption 7.3.1. T1 If two nodes are connected at time steps t_i and t_{i+1}, then they remain within the broadcast radius r_b for all $t_i \leq t \leq t_{i+1}$.

T2 Nodes may go off-line or come on-line, represented by deleting the nodes in the appropriate graph Γ_i.

T3 Fence nodes always remain fixed and on-line.

Coverage problem in this case is similar to the one considered in the last section about covering a cylindrical region. If $\mathcal{U}_c = \bigcup_t \mathcal{U}_c(t)$, where $\mathcal{U}_c(t)$ is the sensor cover at time t, $\{B_c(x_i(t)) \mid x_i(t)$ is the position of the node x_i at time $t\}$, the relevant question is : can one find a continuous path in D which is contained in the complement $D \setminus \cup \{U \mid U \in \mathcal{U}_c\}$? We are interested in a negative answer.

In physical terms the problem is as follows: Let each $\mathcal{U}_c(t)$ be inadequate to cover D and leaves out a hole. Is it possible to move continuously through these holes to remain outside $\mathcal{U}_c(t)$ at all times and thereby outside $\mathcal{U}_c$?

To deal with this problem we use two constructions:

(1) Stacked Rips complex

Definition 7.3.2. ([5]) Given a sequence $\{\Gamma_i\}$ of vertex-labeled communication graphs as above, define the **stacked Rips complex,** $\mathcal{SR}$ to be the cell complex obtained from the disjoint $\coprod_i \mathcal{R}_i$ of the Rips complexes $\mathcal{R}_i$ of Γ_i by the following operation:

For each k-simplex $[v_{\alpha_1}, \ldots, v_{\alpha_{k+1}}]$ of $\mathcal{R}_i$ which is also a k-simplex on the same vertices in $\mathcal{R}_{i+1}$, connect these k-simplices by a prism $\Delta^k \times [0,1]$ with $\Delta^k \times \{0\}$ glued to $\mathcal{R}_i$ and $\Delta^k \times \{1\}$ glued to $\mathcal{R}_{i+1}$.

We treat the time variable $t \in [0,1]$ as an extra dimension and consider the problem of evasive coverage in $D \times [0,1]$. The complex $\mathcal{SR}$ has a natural 'prism' structure: $\mathcal{SR}$ is a 1-parameter family of simplicial Rips complexes indexed by $t \in [0,1]$, these 'slices' being equal to $\mathcal{R}_i$ at t_i. We likewise consider the moving covers as a 1-parameter family in a 3-dimensional setting. If $\mathcal{U}(t)$ denotes the radius r_c cover of nodes χ_t at times t, embed the time-varying covers into $D \times [0,1]$ via the cover $\mathcal{U}(t)$ of $D \times \{t\}$. The problem of wandering loss of coverage now becomes the question of whether the complement of the union of members of $\mathcal{U}_c = \cup\, \mathcal{U}(t)$ in $D \times [0,1]$ has a 'tunnel' running from bottom ($t = 0$) to top ($t = 1$).

Theorem 7.3.3. *([5]) Consider a time-varying set of nodes χ_t in a domain $D \subset \mathbb{R}^2$ satisfying assumptions* **A1-A3** *and* **T1-T3**. *Then, for any continuous curve $p : [0,1] \to D$, $p(t)$ must lie in the union of members of $\mathcal{U}(t)$ for some $0 \leq t \leq 1$ if there exists $[\alpha] \in H_2(\mathcal{SR}, \mathcal{F} \times [0,1])$ such that $\pi_*(\partial\alpha) \neq 0$, where $\pi : \mathcal{F} \times [0,1] \to \mathcal{F}$ is the projection map.*

In practice, computing with the stacked Rips complex is incovenient. The Software we use is meant for simplicial complexes, not the more general prism complex $\mathcal{SR}$. We therefore provide a second construction

(2) Amalgamated Rips complex

Definition 7.3.4. ([5]) Given a collection of network graphs $\{\Gamma_i\}$ define the **amalgated Rips complex**, $\mathcal{AR}$, to be the space obtained from the disjoint union $\coprod_i \mathcal{R}_i$ of the Rips complexes $\mathcal{R}_i$ of Γ_i by the following operation:

For each k-simplex $[v_{\alpha_1}, \ldots, v_{\alpha_{k+1}}]$ of $\mathcal{R}_i$ which is also a k-simplex on the same vertices in $\mathcal{R}_{i+1}$, identify these simplices.

A few observations are in order. First, the amalgated Rips complex $\mathcal{AR}$ is a cell complex built from simplices. It is not, properly speaking, a [combinatorial] simplicial complex, since there may be, e.g., more than one 1-simplex connecting two vertices; hence, cells in this complex are not uniquely defined by their faces. Second, since the fence nodes are assumed stationary, the fence cycle $\mathcal{F}$ is fixed in each $\mathcal{R}_i$ and thus is identified to yield a well-defined cycle $\mathcal{F} \subset \mathcal{AR}$.

Proposition 7.3.5. *The pair* $(\mathcal{SR}, \mathcal{F}\times[0,1])$ *is homotopy equivalent to* $(\mathcal{AR}, \mathcal{F})$. *(see [5] for details)*

Corollary 7.3.6. *The homological condition of theorem (7.3.3) is satisfied if and only if* $H_2(\mathcal{AR}, \mathcal{F})$ *has a generator* α *with* $\partial\alpha \neq 0$.

Chapter 8

Coverage without restriction on fence nodes - Use of Persistence homology

In the discussion of the previous chapters about variants of the hypotheses on nodes leading to coverage of a planar domain we have in some way or the other retained the restriction on the fence nodes (condition **A3**).

However a more realistic condition will not put such a stringent restriction on its fence nodes, or even there may not be any nodes in the fence (the boundary ∂D), instead, nodes which are lying nearby the fence can register themselves as fence nodes. More precisely the situation is as follows:

One considers a set of stationary nodes which can detect the presence of the boundary of the domain ∂D within some fixed fence radius $r_f > 0$. This set is denoted by $X_f \subset X$ and is called the set of fence nodes. X_f spans a *fence subcomplex* $\mathcal{F} \subset \mathcal{R}$, the minimal simplicial complex generated by points X_f and the edges between the fence nodes. One would expect that a criterion like Theorem (6.1.9) should suffice coverage, namely,

$\exists[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ such that $\partial\alpha \neq 0$. Unfortunately, this does not suffice coverage, for example in the following diagram:

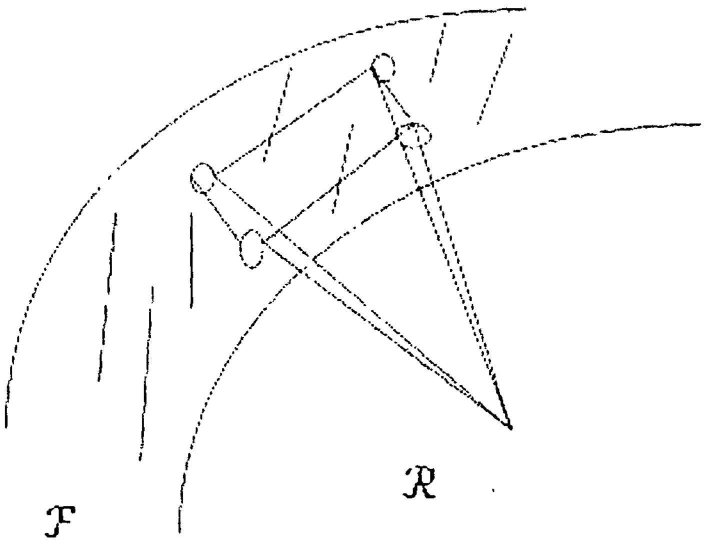


Figure 8.1:

$H_2(\mathcal{R}, \mathcal{F}) \neq 0$ but $\sigma_* : H_1(\mathcal{F}) \longrightarrow H_1(\partial D)$ is a zero map. So the commutative diagram considered earlier:

$$
\begin{array}{ccc}
H_2(\mathcal{R}, \mathcal{F}) & \xrightarrow{\delta_*} & H_1(\mathcal{F}) \\
\downarrow \sigma_* & & \downarrow \sigma_* \\
H_2(\mathbb{R}^2, \partial D) & \xrightarrow{\delta_*} & H_1(\partial D)
\end{array}
\tag{7}
$$

does not help.

To deal with such situation a new homological technique, known as persistent homology has been used. We formulate and demonstrate the corresponding coverage result using a very special instance of persistence homology in the first section. In the second section we shall give a brief introduction to persistent homology.

8.1 Coverage problem of a domain without well defined boundary

To circumvent the difficulty of obtaining homological coverage criterion as mentioned in the introduction we need to consider following condition, χ, D etc as earlier.

Assumption 8.1.1. P1: There are two real numbers $r_s > 0$, the strong broadcast radius and $r_w > 0$, the weak broadcast radius and $r_w \geq r_s\sqrt{10}$, such that for each node $x_i \in \chi$, it can detect a node x_j by strong signal if $x_j \in B_{r_s}(x_i)$ and can detect a node x_k by weak signal if $x_k \in B_{r_w}(x_i)$. (notations as earlier).

P2: Nodes have radially symmetric covering domains of cover radius $r_c \geq r_s/\sqrt{2}$. $\mathcal{U}_c = \{B_c(x) | x \in \chi\}$ the sensor cover.

P3: D is compact and if we take a collar of ∂D of radius $r_f > 0$ (fence radius), then the nodes within this collar can detect ∂D.

P4: The restricted domain $D \backslash \mathcal{C}$, where $\mathcal{C} = \{x \in D | \|x - \partial D\| \leq r_f + r_s/\sqrt{2}\} = N_{\hat{r}}(\partial D)$, $\hat{r} = r_f + r_s/\sqrt{2}$. is connected.

P5: The fence detection hypersurface $\Sigma_f = \{x \in D | \|x - \partial D\| \leq r_f\}$ has internal injectivity radius at least $r_s/\sqrt{2}$ and external injectivity radius at least r_s.

Such a system gives rise to a pair of Rips complexes, $\mathcal{R}_s$ and $\mathcal{R}_w$, computed at the strong and weak radii respectively, each with fence subcomplexes $\mathcal{F}_s \subset \mathcal{R}_s$ and $\mathcal{F}_w \subset \mathcal{R}_w$. There is a natural inclusion map of pairs

$$\iota : (\mathcal{R}_s, \mathcal{F}_s) \hookrightarrow (\mathcal{R}_w, \mathcal{F}_w), \tag{1}$$

Theorem 8.1.2 ([6]). *(Squeezing Theorem) Let $X \subset \mathbb{R}^d$. Given $\epsilon' < \epsilon$, there is a chain of inclusions*

$$\mathcal{R}_{\epsilon'}(X) \subset \mathcal{C}_\epsilon(X) \subset \mathcal{R}_\epsilon(X), \quad \textit{if } \frac{\epsilon}{\epsilon'} \geq \sqrt{\frac{2d}{d+1}}.$$

Where $\mathcal{C}_\epsilon(X)$ represents the $\hat{C}$ech complex of the cover of X by balls of radius $\epsilon/2$ Moreover, this ratio is the smallest for which the inclusions hold in general.

In the present case $\epsilon = w$, and $, \epsilon^{'} = s$.

Proof. (*Sketch of the proof*): If the balls of radius $\epsilon/2$ centered at the vertices have a common intersection then each pair of vertices is seperated by distance $\leq \epsilon$

So $\mathcal{C}_\epsilon(X) \subseteq \mathcal{R}_\epsilon(X)$.

To prove $\mathcal{R}_{\epsilon^{'}}(X) \subseteq \mathcal{C}_\epsilon(X)$ if $\frac{\epsilon}{\epsilon^{'}} \geq \sqrt{\frac{2d}{d+1}}$

We will show that whenever a collection of $(k+1)$ points $\{x_0, \ldots, x_k\}$ of $\mathbb{R}^d$ is given which have pairwise distance $\leq \epsilon^{'}$ apart, then they have a common intersection.

If $k = 1$, then $d(x_0, x_1) \leq \epsilon^{'} \leq \epsilon\sqrt{\frac{d+1}{2d}} < \epsilon$

So $B_{\frac{\epsilon}{2}}(x_0) \cap B_{\frac{\epsilon}{2}}(x_1) \neq \phi$.

One proceed by induction on k, $k \leq d$ to prove that $B_{\frac{\epsilon}{2}}(x_0) \cap \cdots \cap B_{\frac{\epsilon}{2}}(x_k) \neq \phi$.

For $k+1 \geq d+1$ one appeals to Helly's Theorem which asserts that a collection of $k+1 \geq d+2$ convex sets in $\mathbb{R}^d$ has a nonempty common intersection if the same is true for each subset of size $d+1$. (For detail refer to [5]).

□

Theorem 8.1.3 ([6]). *For a set of nodes χ in a domain $D \subset \mathbb{R}^d$ satisfying assumptions (8.1.1), the sensor cover $\mathcal{U}$ covers the restricted domain $D \backslash C$ if the induced homomorphism*

$$\iota_* : H_d(\mathcal{R}_s, \mathcal{F}_s) \to H_d(\mathcal{R}_w, \mathcal{F}_w)$$

is nonzero. In other words $[\alpha] \in H_d(\mathcal{R}_s, \mathcal{F}_s)$ which persists to $H_d(\mathcal{R}_w, \mathcal{F}_w)$

Proof. (*Sketch of the proof*): The crucial ingredients of the proof are the following lemmas:

Lemma 8.1.4. *Under the assumption of the main theorem (in particular $r_c \geq r_s/\sqrt{2}$), any collection of nodes in D which form a simplex of $\mathcal{R}_s$ has its convex hull contained in union of members of the sensor cover $\mathcal{U}_c$.*

Lemma 8.1.5. *Let p belong to the convex hull of points $x_0, x_1, \ldots, x_k \in \mathbb{R}^d$ and suppose $\epsilon' \geq \epsilon\sqrt{2d/(d+1)}$.*
If $\|x_i - x_j\| \leq \epsilon \ \ \forall i, j$ then $\|p - x_i\| \leq \epsilon'/2$ for some i.

Lemma 8.1.6. *For any collection of nodes in D which form a simplex of $\mathcal{F}_s$, its convex hull lies in $\mathbb{R}^d \backslash (D \backslash N_{\hat{r}}(\partial D))$ or else $D \backslash N_{\hat{r}}(\partial D)$ is contained in the simplex.*

Lemma 8.1.7. *Let Σ^{d-1} be a hyper surface of $\mathbb{R}^d$ and let $\chi \subset \Sigma \times (-\Delta, +\Delta)$ denote a collection of points which form a $(d-1)$-cycle γ such that $[\gamma] \in H_{d-1}(\mathcal{R}_\epsilon(\chi))$, for some $\epsilon > 0$, and γ is contained entirely in $\Sigma \times (-\Delta, +\Delta)$ If $[\gamma] = 0$ in $H_{d-1}(\Sigma \times (-\Delta, +\Delta)$, then $[\gamma] = 0$ in a ϵ'-Rips complex $\mathcal{R}_{\epsilon'}(\chi)$, where*

$$\epsilon' = \sqrt{\Delta^2 + 2\epsilon^2 \frac{d-1}{d}}$$

One begins the proof as usual with a simplicial realization map $\sigma : \mathcal{R}_s \to D$, which gives (excluding the trivial case) a map of pairs $(\mathcal{R}_s, \mathcal{F}_s) \to (\mathcal{R}^d, \mathcal{R}^d \backslash (D \backslash N_{\hat{r}}(\partial D)))$ and thereby the commutative diagram

$$\begin{array}{ccc} H_2(\mathcal{R}_s, \mathcal{F}_s) & \xrightarrow{\delta_*} & H_{d-1}(\mathcal{F}_s) \\ \downarrow \sigma_* & & \downarrow \sigma_* \\ H_d(\mathbb{R}, \mathbb{R}^d \backslash (D \backslash N_{\hat{r}}(\partial D))) & \xrightarrow{\delta_*} & H_{d-1}(\mathbb{R}^d \backslash (D \backslash (N_{\hat{r}}(\partial D))) \end{array} \quad (8)$$

Now letting $[\alpha] \in H_d(\mathcal{R}_s, \mathcal{F}_s)$ for which $i_*[\alpha] \neq 0$, one considers two cases.

Case I: $\sigma_* \delta_* [\alpha] \neq 0$.

In this case using lemma (8.1.4) and Alexander-duality one can conclude that $\mathcal{U}_c$ with $r_c > r_s/\sqrt{2}$ covers $D \backslash N_{\hat{r}(\partial D)}$, as asserted in the theorem.

Case II: $\sigma_* \delta_* [\alpha] = 0$, which is shown to be impossible.

For this one consider r_m, $r_s < r_m < r_w$ by defining

$$r_m = r_s \sqrt{\frac{7d - 5 + 2\sqrt{2d(d-1)}}{2d}}$$ and one gets a factoring of i:

$$(\mathcal{R}_s, \mathcal{F}_s) \xrightarrow{i} (\mathcal{R}_w, \mathcal{F}_w)$$

$$\searrow \qquad \nearrow$$

$$(\mathcal{R}_m, \mathcal{F}_m)$$

and thus a commutative diagram of homology of pairs

$$\begin{array}{ccccc}
H_d(\mathcal{R}_s) & \xrightarrow{j_*} & H_d(\mathcal{R}_s, \mathcal{F}_s) & \xrightarrow{\delta_*} & H_{d-1}(\mathcal{F}_s) \\
\downarrow i_* & & \downarrow i_* & & \downarrow i_* \\
H_d(\mathcal{R}_m) & \xrightarrow{j_*} & H_d(\mathcal{R}_m, \mathcal{F}_m) & \xrightarrow{\delta_*} & H_{d-1}(\mathcal{F}_m) \\
\downarrow i_* & & \downarrow i_* & & \downarrow i_* \\
H_d(\mathcal{R}_w) & \xrightarrow{j_*} & H_d(\mathcal{R}_w, \mathcal{F}_w) & \xrightarrow{\delta_*} & H_{d-1}(\mathcal{F}_w)
\end{array}$$

The hypotheses **P5** about injectivity radius of the fence hypersurface Σ_f gives a $\Delta = r_s(1 + \sqrt{\frac{d-1}{2d}})$ and combining this with hypotheses $\sigma_*\delta_*[\alpha] = 0$ tells us that the cycle $\partial\alpha$ is null homologous within $\Sigma_f \times (-\Delta, +\Delta)$. Now applying lemma (8.1.7) with $\epsilon = r$, Δ as above and

$$\epsilon' = \sqrt{\Delta^2 + 2\epsilon^2\frac{d-1}{d}} = r_s\sqrt{\frac{7d - 5 + 2\sqrt{2d(d-1)}}{2d}}$$

We get that $i_*\delta_*[\alpha] = 0 \in H_{d-1}(\mathcal{F}_m)$.

Now some arguments involving chasing the above commutative diagram gives a contradiction to the assumption that $i_*[\alpha] \neq 0$ in $H_{d-1}\mathcal{F}_w$.

Note also that the assumption $r_w > r_s\sqrt{10}$ in **P1** stems from the above inequality

$$r_w \geq r_s(\sqrt{\frac{2d}{d+1}})(\sqrt{\frac{7d - 5 + 2\sqrt{2d(d-1)}}{2d}})$$

$$= r_s\sqrt{\frac{7d-5+2\sqrt{2d(d-1)}}{d+1}}$$

□

8.2 Brief introduction to Persistent homology

Let K be a filtered simplicial complex with a filtration

$$\phi = K^0 \subseteq K^1 \subseteq ...K^m = K$$

For each $i, 1 \leq i \leq m$ we have chain groups $C_k^i \stackrel{def}{=} C_k(K^i)$, boundary operators $\partial_k^i \stackrel{def}{=} \partial_k(K^i)$, cycles $Z_k^i \stackrel{def}{=} Z_k(K^i)$, boundaries $B_k^i \stackrel{def}{=} B_k(K^i)$, homology groups $H_k^i \stackrel{def}{=} H_k(K^i)$, for all k; the superscripts are the filtration indices.

Definition 8.2.1. We define the *p-persistent k th homology group (Z-module) of* K^i as

$$H_k^{i,p} \stackrel{def}{=} Z_k^i/(B_k^{i+p} \cap Z_k^i)$$

Definition 8.2.2. The *p-persistent k th Betti number of* K^i, denoted by $\beta_k^{i,p}$ is defined as the rank of the free part of $H_k^{i,p}$.

We can alternatively define *p-persistent k th homology group of* K^i as follows:

Definition 8.2.3. Let $\eta_k^{i,p} : H_k^i \to H_k^{i+p}$ be the map of homology induced by the inclusion $K^i \subseteq K^{i+p}$. Then define

$$H_k^{i,p} \stackrel{def}{=} im\ \eta_k^{i,p}.$$

We can define persistent homology module over any ring, in particular over PID's, to make ourselves available the structure theory of such modules.

We generalize the description of the last para into the following:

Definition 8.2.4. A *persistent module* $\mathcal{M}$ is a family of R-modules M^i, together with homomorphisms $\phi^i : M^i \to M^{i+1}$, R any ring.

The homology of a filtered complex is a persistent module, where $\phi^i = \eta^{i,1}$ as described above. One can similarly define:

Definition 8.2.5. A persistent chain complex as $\mathcal{C} = \{C^i, f^i\}$, where each C^i is a chain complex and $f^i : C^i \to C^{i+1}$ is a chain map, for all i.

The chain complexes of a filtered complex gives a persistent chain complex.

Definition 8.2.6. (i) A persistent module $\mathcal{M} = \{M^i, \phi^i\}$ is of *finite type* if each M^i is finitely generated R-module and if ϕ^i are isomorphisms for $i \geq m$ for some integer m.

(ii) A persistent chain complex $\mathcal{C} = \{C^i, f^i\}$ is of *finite type* if each component of each C^i is finitely generated R-module and if f^i are isomorphisms for $i \geq m$ for some integer m.

If a filtered simplicial complex K is a finite complex then its chain complex and homology R-modules give rise to finite type persistent chain complex and persistent module.

Let $\mathcal{M} = \{M^i, \phi^i\}$ be a persistent module over a ring R. The polynomial ring $R[X]$ is a graded ring with standard grading $R[X] = \oplus_{i \geq 0} R X^i$.

Definition 8.2.7. We associate with $\mathcal{M}$ a graded $R[X]$-module $\alpha(\mathcal{M})$ as follows: $\alpha(\mathcal{M}) \stackrel{def}{=} \oplus_{\iota \geq 0} M^{\iota}$, and the action of $R[X]$ is given as follows:

$$r(m^0, m^1, ...) = (rm^0, rm^1, ...),$$

for all $r \in R$, and

$$X(m^0, m^1, ...) = (0, \phi^0(m^0), \phi^1(m^1), ...).$$

The following theorem shows that the the persistent homology of a finite filtered simplicial complex is the standard homology of a particular graded module over a polynomial ring.

Theorem 8.2.8 ([9]). *If R is a field, the correspondence α defines an equivalence of categories between the category of persistent R-modules of finite type and the category of finitely generated non-negatively graded modules over $R[X]$.*

The proof is the Artin-Rees theory of commutative algebras (see Eisenbud - Commutative algebra with a view toward algebraic geometry, Springer 1995.)

Remark 8.2.9 ([9]). (i) For R a field, the above theorem allows one to use the structure theory of finitely generated graded $R[X]$-modules (see Chapter 9 also) to give simple description of persistent homology.

(ii) If R is not a field, like $\mathbb{Z}$, there is no such simple description of persistent homology. However it suggests the possibility of interesting invariants of inductive systems.

(iii) Persistence corresponds to the differentials of spectral sequences associated with the filtered complex.

(iv) One can also explore multivariate persistence.

Chapter 9

Persisitent Algorithm and its use in the feature recognition - a brief introduction

In this chapter we shall give a brief introduction to the work of A. Zomorodian and G. Carlsson on the persistent algorithm ([9]) which computes persistent homology of a filtered complex. We also indicate its use in the work of E. Carlsson, G. Carlsson, and V. de Silva in feature identification of geometric objects ([10]).

9.1 Developing the persistent algorithm

We begin with the satements of structure theorems of **(i)** finitely generated modules over a PID and **(ii)** finitely generated graded modules over a graded PID. These have bearing on the algorithms.

Theorem 9.1.1. (i) *If R is a PID, then every finitely generated R-modules M is isomorphic to a direct sum of cyclic R-modules $M \cong R^{\beta} \bigoplus (\bigoplus_{i=1}^{m} R/d_iR)$ for $d_i \in R$, $\beta \in \mathbb{Z}$, such that $d_i | d_{i+1}$.*

(ii) *if R is a graded PID, then every finitely generated graded module M over R decomposes uniquely as* $M \cong (\bigoplus_{i=1}^{n} \sum^{\alpha_i} R) \bigoplus (\bigoplus_{j=1}^{m} \sum^{\gamma_j} R/d_j R)$ *where $d_j \in R$ are homogeneous elements so that $d_j | d_{j+1}$, α_i, $\gamma_j \in \mathbb{Z}$, and $\sum^{\alpha}$ denotes an α-shift upward in grading.*

Now from the equivalence of the categories of persistent module of finite type over a field F and the category of finitely generated non-negatively graded modules over $F[t]$ and the above structure theorem, we get that the structure of a finitely generated graded $F[t]$-module, M is $(\bigoplus_{i=1}^{n} \sum^{\alpha_i} F[t]) \bigoplus (\bigoplus_{j=1}^{m} \sum^{\gamma_j} F[t]/(t^{n_j}))$ where $(t^{n_j}) = t^{n_j}.F[t]$, other notations are as above (note that $F[t]$ is a PID).

For the purpose of developing algorithm we associate pairs of integers to the summands of the structure (of the graded modules) and formalize it in the following definition.

Definition 9.1.2. A $\mathfrak{P}$-*interval* is an ordered pair of integers (i, j), $0 \leq i < j \leq \infty$.

Let $\mathcal{S}$ = collection of finite sets of $\mathfrak{P}$-intervals.

$\mathcal{GM}(F[t])$ = collection of finitely generated graded modules over the graded PID $F[t]$.

Define $Q : \mathcal{S} \to \mathcal{GM}(F[t])$

$$Q(i,j) = \begin{cases} \sum^{i} F[t]/(t^{j-i}), & \text{if } 0 \leq i < j < +\infty \\ \sum^{i} F[t], & \text{if } 0 \leq i < j = +\infty \end{cases}$$

If $S \in \mathcal{S}$, and $S = \{(i_1, j_1), \ldots, (i_n, j_n)\}$, we define $Q(S) = \bigoplus_{l=1}^{n} Q(i_l, j_l)$.

Theorem 9.1.3. *Q is a one-to-one correspondence; consequently the isomorphism classes of Persistent modules of finite type over F is in one-to-one correspondence with S.*

We are interested in the persistent homology groups as our example of persistent module described in the last theorem.

Our input is a filtered complex

$\phi = K^0 \subseteq K^1 \subseteq \cdots \subseteq K^m = K$. $(K^m = K^{m+1} = \dots)$. $\forall k \geq 0$, $H_k(K^i)$ is a vector space over F of dim β_k^i. $\forall k \geq 0$ $\{H_k(K^i)\}_{i \geq 0}$ form a persistent module as seen earlier with homomorphism $\eta_k^i : H_k(K^i) \rightarrow H_k(K^{i+1})$ induced by inclusion. We associate a graded module over F[t] associated with this persistent module by taking the direct sum $\bigoplus\limits_{i=0}^{\infty} H_k(K^i)$. This is finitely generated, as K is a finite complex. So by structure theorem stated above we get

$$\bigoplus_{i=1}^{\infty} H_k(K^i) \cong \left(\bigoplus_{i=1}^{\lambda} \sum^{\alpha_i} F[t]\right) \bigoplus \left(\bigoplus_{j=1}^{\mu} \sum^{\gamma_j} F[t]/(t^{n_j})\right) \leftrightarrow$$

$$\leftrightarrow \{(\alpha_1, +\infty), \dots, (\alpha_\lambda, +\infty)\}$$
$$\cup\{(\gamma_1, \gamma_1 + n_1), (\gamma_2, \gamma_2 + n_2), \dots, (\gamma_\mu, \gamma_\mu + n_\mu)\}$$

by the theorem.

$(\alpha_i, +\infty)$ represents a k-cycle e which is created at α_i th stage by a simplex σ^{α_i} and which remains a cycle forever (for example, see figure 9.1)

$(\gamma_j, \gamma_j + n_j)$ represents a k-cycle e which is created at the γ_j th stage by a simplex σ^{γ_j} and which becomes a boundary at the $(\gamma_j + n_j)$ th stage, the simplex $\tau^{\gamma_j + n_j}$ caused this. (for example, see figure (9.2))

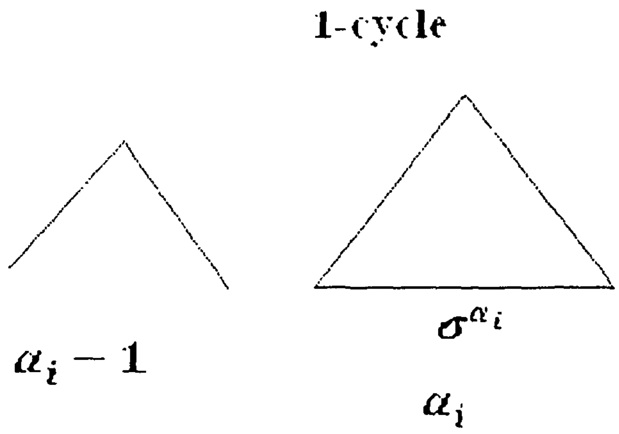


Figure 9.1: Infinite P intervals

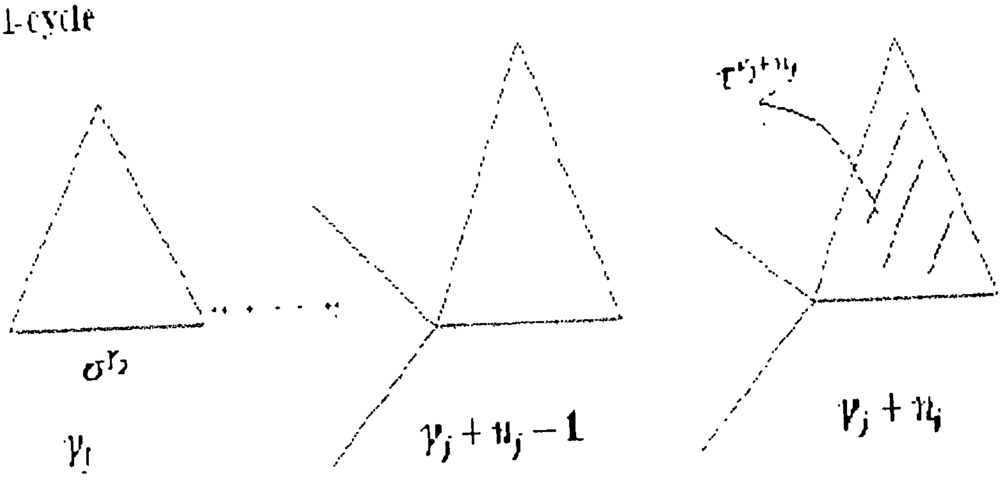


Figure 9.2: Finite P intervals

So $(\gamma_j, \gamma_j + n_j)$ is represented by a pair of simplices $(\sigma^{\gamma_j}, \tau^{\gamma_j + n_j})$. Thus $(\gamma_j, \gamma_j + n_j)$ describes a basis element for $\bigoplus_{i=0}^{\infty} H_k(K^i)$, which is given by a cycle e created at γ_j th stage and killed at $\gamma_j + n_j$ th stage.

If e is a k-cycle of K^l giving a homology class $0 \neq [e] = e + B_k^l \in H_k^l(K)$. When does $[e]$ become a basis element of $H_k^{l,p}(K) = Z_k^l/(B_k^{l+p} \cap Z_k^l)$?

As $e \notin B_k^l$ for $l < \gamma_j + n_j$, $p \geq 0$, $e \notin B_k^{l+p}$, $l + p < \gamma_j + n_j$ and $l \geq \gamma_j$. We get a triangular region in the l, p plane bounded by $p \geq 0$, $l \geq \gamma_j$, $l + p < \gamma_j + n_j$.

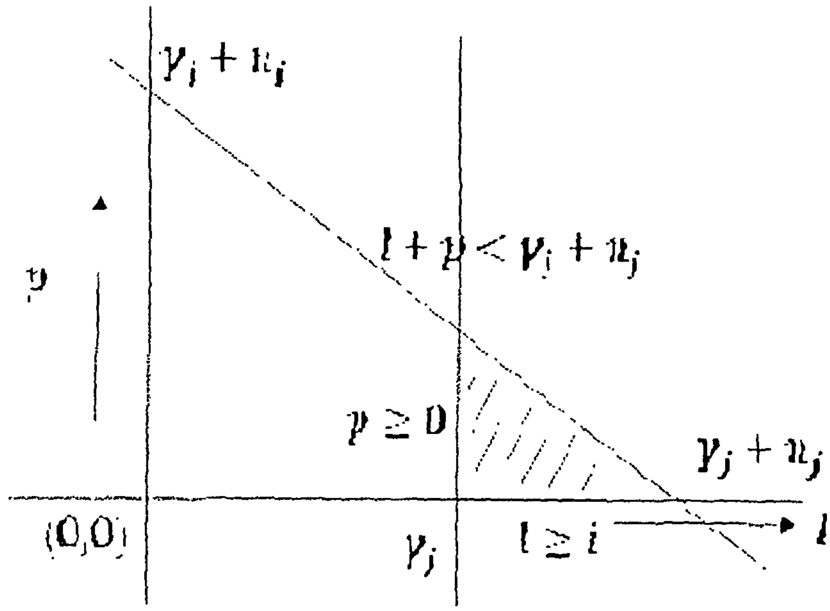


Figure 9.3: l-p-plane

Lemma 9.1.4. *Let $\mathcal{T}$ be the set of triangular regions enclosed by $\mathfrak{P}$-intervals for the k^{th}-persistent homology module. Then the persistent Betti number $\beta_k^{l,p}$* (dim *of* $H_k^{l,p}$) *is the number of triangular regions of $\mathcal{T}$ containing the point (l,p).*

Proof. (see ([9] for a proof) □

This helps us to compute the persistent homology of a filtered complex over a field by finding the corresponding $\mathfrak{P}$-intervals. The persistent algorithm that one is looking for must compute $\mathfrak{P}$-intervals for a filtered complex, over the field F, directly (without the need to compute $F[t]$ modules)

The persistent algorithm is developed using the following steps:

(1) To a filtered complex K there is associated a persistent chain complex over F. By the theorem on correspondence (9.1.3) we get a graded chain complex $\{C_k, \partial_k\}$ over $F[t]$, where $C_k = \bigoplus_{i=0}^{\infty} C_k^i$, $\partial_k = \bigoplus_{i=0}^{\infty} \partial_k^i$ etc. $t(c^0, c^1, \dots) = (0, f^0(c^0), f^1(c^1), \dots)$ etc.

(2) We choose a pair of homogeneous bases $\{e_j\}$ for C_k and $\{\hat{e}^i\}$ for C_{k-1}. With respect to these bases we write the matrix of $\partial_k : C_k \to C_{k-1}$, $M_k(i,j)$. Since $\partial_k(e_j) = \sum_{i=1}^{\dim C_{k-1}} M_k(i,j)\hat{e}_i$, if $e_j \in C_k^\lambda$, i.e if *deg* $e_j = \lambda$, then the degree of each $(M_k(i,j)\hat{e}_i)$ should be λ. So *deg* $M(i,j)$ + *deg* $\hat{e}_i = \lambda$. Thus $M(i,j)$ (as elements of $F[t]$) must have degree satisfying: *deg* e_j = *deg* $M(i,j)$ + *deg* $\hat{e}_i$.

(3) Since standard bases of C_k's are homogeneous bases we can start with a standard basis $\{e_j\}$ of C_k and a homogeneous basis $\{e_i\}$ of Z_{k-1} and represent the matrix of ∂_k as M_k.

(4) Then we reduce the matrix using the following elementary row operations

1. Exchange row i and row j.

2. Multiply row i by -1.

3. Replace row i by (row i)+ q(row j), $j \neq i$, where q is an integer.

One can use similar elementary column operation also. These elementary operation correspond to change of bases as follows:

(a) A column operation of type 3 replace e_i with $e_i + qe_j$ in C_k.

(b) A row operation of type 3 replaces $\hat{e}_j$ with $\hat{e}_j - q\hat{e}_i$ in C_{k-1}.

Repeated application of these operations replace the matrix M_k of ∂_k w.r.t $\{e_j\}$, $\{\hat{e}_i\}$ with $\tilde{M}_k$, which is in the Smith normal form:

$$\begin{bmatrix} b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{l_k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$l_k = \text{rank}\tilde{M}_k = \text{rank}M_k$, $b_i \geq 1$, $b_i | b_{i+1}$, $1 \leq i < l_k$.

One can side by side get new bases $\{e_j'\}$, $\{\hat{e}_i'\}$ of C_k and Z_{k-1} giving matrix $\tilde{M}_k$ of ∂_k.

Once we have obtained the Smith normal form we get information about the homology of $\{C_k, \partial_k\}$ (which is infact the persistent homology) as explained below: If

$$H_k(C) = (\bigoplus \sum^{\alpha_i} R) \bigoplus (\bigoplus \sum^{\gamma_j} R/(d_j R))$$

then d_j will be the diagonal entries $b_j > 1$.

However we need not reduce the matrix to Smith normal form. Reducing it to a column echelon form, for example,

$$\begin{bmatrix} b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & b_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & b_3 & 0 & 0 & 0 & 0 \\ * & * & * & b_4 & 0 & 0 & 0 \\ * & * & * & * & b_5 & 0 & 0 \end{bmatrix}$$

is sufficient, because the pivots in column echelon form are the same as the diagonal elements in the Smith normal form. So we get:

Theorem 9.1.5. *Let $\tilde{M}_k$ be the column-echelon form of the matrix of ∂_k relative to bases $\{e_j\}$ of C_k and $\{\hat{e}_i\}$ of Z_{k-1}. If i^{th} row has pivot $\tilde{M}_k(i,j) = t^n$, it contributes $\sum^{deg \hat{e}_i} F[t]/t^n$ to H_{k-1}. Otherwise, it contributes $\sum^{deg \hat{e}_i} F[t]$. Thus $(deg\ \hat{e}_i, deg\ \hat{e}_i + n)$ and $(deg\ \hat{e}_i, \infty)$ are the $\mathfrak{P}$-intervals for H_{k-1}.*

Remark 9.1.6. The basis $\{\hat{e}_i\}$ of Z_{k-1} are constructed by induction on k, at every stage $l < k$ one considers the column echelon form of the matrix ∂_l w.r.t. transformed bases $\{e_j\}$ of C_l and $\{\hat{e}_i\}$ of Z_{l-1}. The basis elements corresponding to columns containing pivot elements are the ones representing B_{l-1}. The basis elements corresponding to non pivot columns represent the basis of Z_l.

This induction procedure also shows that

Lemma 9.1.7. *To represent ∂_k relative to the standard basis of C_k and the basis computed inductively for Z_k, delete rows of M_k that correspond to pivot columns of $\tilde{M}_{k-1}$. (for proof refer [9])*

The above discussion leads to the following algorithm

Algorithm 9.1.8. *([9]) Remove pivot rows*

chain REMOVEPIVOTROWS (σ)
{
$k = \dim\sigma$; $d = \partial_k\sigma$;
Remove unmarked terms in d;
while $(d \neq \phi)$
{
$i =$ *maxindex d;*
if $T[i]$ *is empty,* **break**;
Let q *be the coefficient of* σ^i *in* $T[i]$;
$d = d - q^{-1}T[i]$;
}
return d;
}

This algorithm **REMOVEPIVOTROWS** takes a simplex σ^j as input. Then it checks whether the boundary chain d of σ^j corresponds to a zero or pivot column. If the chain is empty, it corresponds to a zero column and we mark σ^j: its column is a basis element for Z_k and the corresponding row should not be eliminated in the next dimension. Otherwise, the chain corresponds to a pivot column and the term with the maximum index $i =$ maxindex d is the pivot. It uses Lemma (9.1.7) to eliminate all terms involving unmarked simplices to get a representation in terms of the basis for Z_{k-1}. The rest of the algorithm is Gauss elimination in the order of decreasing degree. The term with the maximum index $i = maxd$ is a potential pivot. If $T[i]$ is non-empty, a pivot already exists in that row, and we use the inverse of its coefficient to eliminate the row from our chain. Otherwise, we have found a pivot and our chain is our pivot column.

Algorithm 9.1.9. *([9]) Compute Intervals*

```
COMPUTEINTERVALS(K)
{
for k=0 to dim(K) L_k = φ;
for j=0 to m − 1
{
d = REMOVEPIVOTROWS (σ^j);
if (d = φ) Mark σ^j;
else
{
i = maxindex d; k = dimσ^i;
Store j and d in T[i];
L_k = L_k ∪ {(deg σ^i, deg σ^j)}
}
}
for j=0 to m-1
}
if σ^j is marked and T[j] is empty
}
k = dimσ^j; L_k = L_k ∪ {(deg σ^j, ∞)}
}
}
}
```

This algorithm **COMPUTEINTERVALS** stores the list of $\mathfrak{P}$-intervals for H_k in L_k. It takes simplex σ^j as input. Then with the help of algorithm **REMOVEPIVOTROWS** its checks whether the boundary chains of σ^j corresponds to a zero or pivot column. If the chain is empty, its corresponds to a zero column and we mark σ^j: Its column is a basis element for Z_k and the corresponding row should not be eliminated in the next dimension. Otherwise, the chain corresponds to a pivot column and the term with the maximum index i = maxindex d is the pivot; index j and chain d are stored, representing the column $T[i]$. According to Theorem (9.1.5) it gives $\mathfrak{P}$-interval $(deg\ \sigma^i, deg\ \sigma^j)$. It continues until all the simplexes of the filtration are exhausted. We then perform another pass through the filtration in search of infinite $\mathfrak{P}$-intervals: marked simplices whose slot is empty.

Example 9.1.10. ([9]) We now consider a simple concrete example of a filtered complex as given in figure (9.4) to see the above calculation.

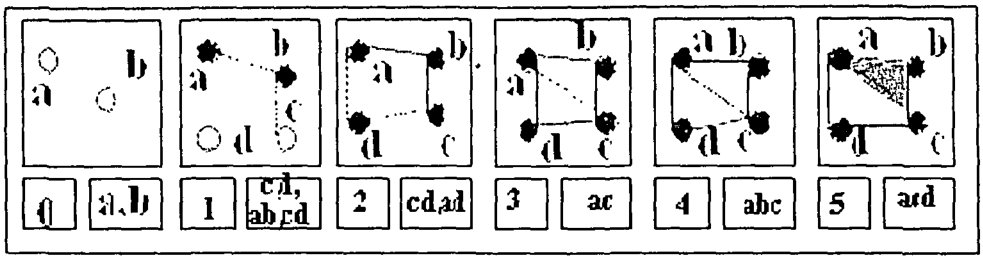


Figure 9.4: A filtered complex

The table in figure (9.5) gives the degrees of the simplices in the filtered complex.

$$M_1 = \left[\begin{array}{c|ccccc} & ab & bc & cd & ad & ac \\ \hline d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & \mathrm{t}^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & \mathrm{t}^2 & \mathrm{t}^3 \end{array}\right], \quad \widetilde{M_1} = \left[\begin{array}{c|ccccc} & cd & bc & ab & z_1 & z_2 \\ \hline d & t & 0 & 0 & 0 & 0 \\ c & t & 1 & 0 & 0 & 0 \\ b & 0 & t & t & 0 & 0 \\ a & 0 & 0 & t & 0 & 0 \end{array}\right]$$

a	b	c	d	ab	bc	cd	ad	ac	abc	acd
0	0	1	1	1	1	2	2	3	4	5

Figure 9.5: Degree of simplices

$$M_2 = \left[\begin{array}{c|cc} & abc & acd \\ \hline ac & t & t^2 \\ ad & 0 & t^3 \\ cd & 0 & t^3 \\ bc & t^3 & 0 \\ ab & t^3 & 0 \end{array}\right], \quad \widetilde{M_2} = \left[\begin{array}{c|cc} & abc & acd \\ \hline z_2 & t & t^2 \\ z_1 & 0 & t^3 \end{array}\right]$$

The table in figure (9.6) gives the data structure after running the algorithms on the filtration.

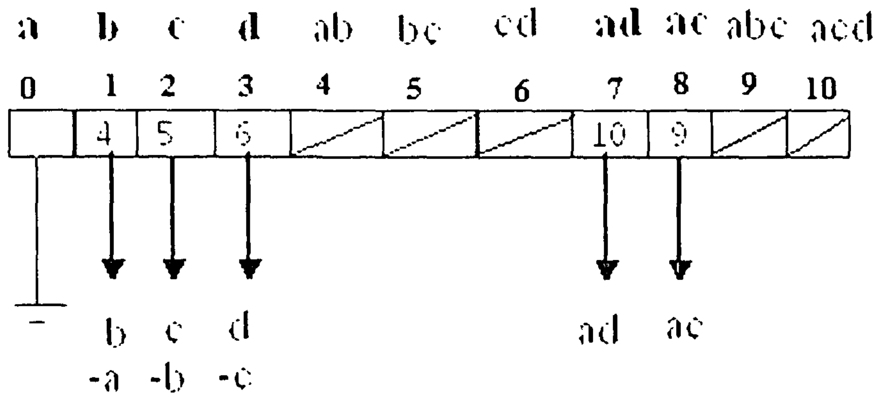


Figure 9.6: Data Structure

The marked 0-simplices $\{a, b, c, d\}$ and 1-simplices $\{ad, ac\}$ generate $\mathfrak{P}$-intervals

$L_0 = \{(0, \infty), (0, 1), (1, 1), (1, 2)\}$, and $L_1 = \{(2, 5), (3, 4)\}$, respectively.

9.2 Application of persistent homology in feature identification

Consider geometric objects of the following types (see figures 9.7, 9.8)

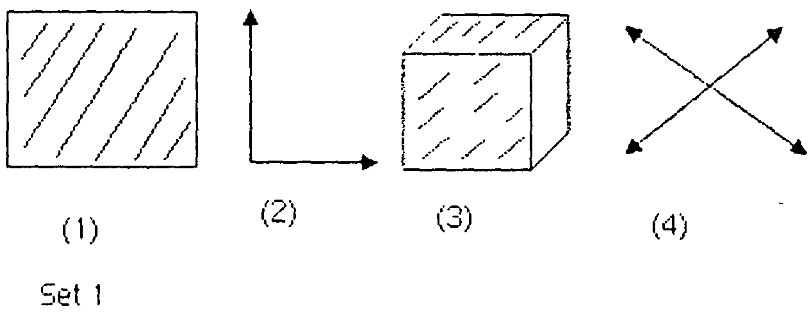


Figure 9.7: figures of set 1

There are interior points (smooth points) there are edges (boundary point), there are vertices (corners), there are points of intersection. Let us call points which are not smooth as singular points.

By a continuous deformation first three figures can be converted to smooth objects.

but the fourth one can not be converted in this fashion.

Singular points in first three examples are called *topologically standard.*

Singular point in the fourth example is topologically non standard. Since topological invariants like homology Betti numbers etc. remain unchanged under continuous deformations, we can not distinguish the corresponding figures in Set 1 and Set 2. Using topological invariants.

Let X be any of the geometric objects of Set 1, more generally let $X \subseteq \mathbb{R}^n$. One associates a now object $T(X) \subseteq X \times \mathbb{R}^n$ with X, called the *"tangent complex"* of X. It has the property that it is sensitive to the local smooth structure of X. So if we take homeomorphic spaces of figure (9.9), their tangent complexes are topological distinct.

This construction allows us to identify local features of a geometric object.

We will give a very brief introduction to tangent complex now (refer to [10]).

Definition 9.2.1. Let $X \subseteq \mathbb{R}^n$, the open tangent complex, $T^0(X) \subseteq X \times S^{n-1}$ of X is defined as:

$$T^0(X) = \{(x, v) \in X \times S^{n-1} | lim_{t \to 0+} \frac{d(x+tv, X)}{t} = 0\},$$

$$\text{where } d(\xi, X) = \inf_{x \in X} d(\xi, x).$$

The closed tangent complex $T(X)$ of X is defined as $T(X) =$ closure of $T^0(X)$ in $X \times S^{n-1}$.

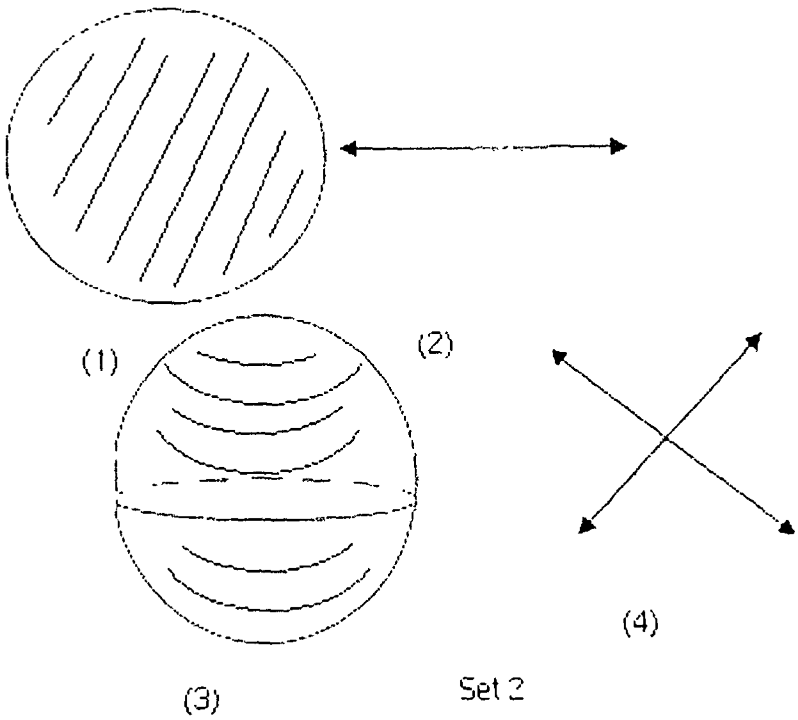


Figure 9.8: figures of set 2

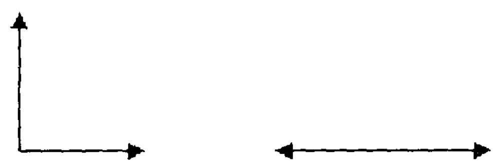


Figure 9.9 These homeomorphic spaces have distinct tangent complexes

Definition 9.2.2. We define the projection map

$p : T(X) \to X$

by $p(x, v) = x$

$T_x(X) = p^{-1}(x)$, fibre at x. There is also a projection

$q : T(X) \to S^{n-1}$

$q(x, v) = v$.

Proposition 9.2.3. *If $X \subseteq \mathbb{R}^n$, $x \in X$ is a smooth point i.e X is locally euclidean at x or $\exists$ an open nbd U of x in $\mathbb{R}^n$ and a smooth homeomorphism $Q : U \cap X \to \mathbb{R}^m$ such that $Q(x) = 0$, and $RQ(\xi)$ is of rank m $\forall \xi \in U \cap X$. Then $T_x(X) \cong S^{m-1}$*

We consider some simple examples.

Example 9.2.4. Let $X_1 = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ (the x-axis in xy-plane), then

$T(X_1) = \mathbb{R} \times \{0\} \times \{\bar{e}_1\} \sqcup \mathbb{R} \times \{0\} \times \{-\hat{e}_1\}$, where $\hat{e}_1 = (1, 0)$, $-\bar{e}_1 = (-1, 0)$.

For determining $T^0(X_1) \subseteq X_1 \times S^1$ we consider pairs (x, v). Then

$$lim_{t \to 0+} \frac{d(x + tv, X_1)}{t} = \begin{cases} 0 & \text{if } v = \pm \bar{e}_1 \\ \sin\theta & \text{if } \bar{v} = \bar{v}_\theta \end{cases}$$

See the figure (9.10)

So by definition $T^0(X_1) = \{(x, \bar{e}_1) | x \in X_1\} \cup \{(x, -\bar{e}_1) | x \in X_1\}$

$= \mathbb{R} \times \{0\} \times \{\bar{e}_1\} \sqcup \mathbb{R} \times \{0\} \times \{-\bar{e}_1\}$

$T(X_1) = T^0(X_1)$

has two components as indicated.

Example 9.2.5. Let $X_2 = \mathbb{R}_+ \times \{0\} \cup \{0\} \times \mathbb{R}_+$ Then one can calculate as above the tangent complex of X_2 and it is given by

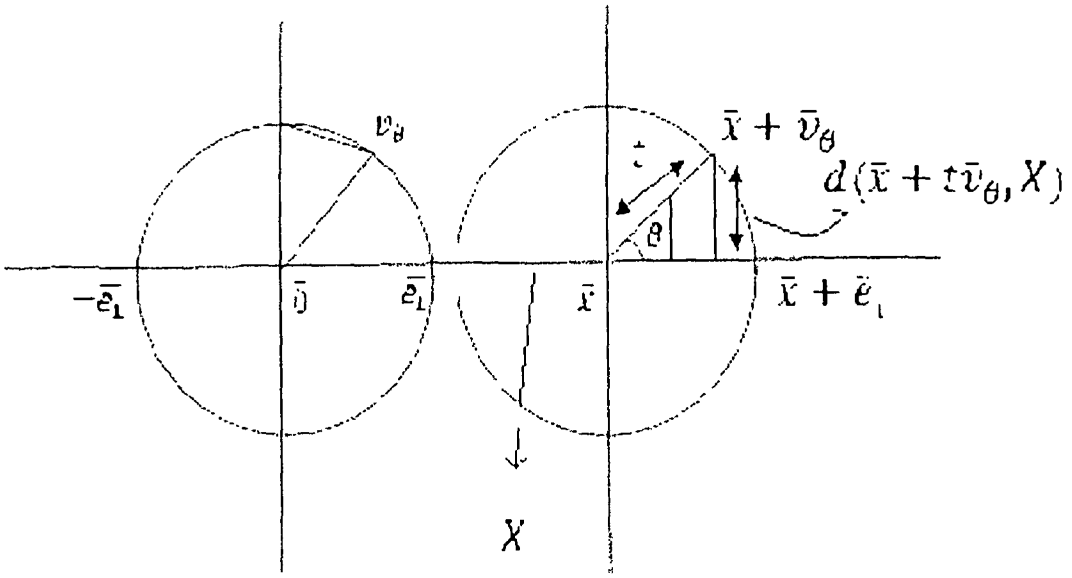


Figure 9.10: Tangent complex calculation of Real line

$T(X_2) = \mathbb{R}_+ \times \{0\} \times \{\bar{e}_1\} \sqcup \mathbb{R}_+ \times \{0\} \times \{-\bar{e}_1\} \sqcup \{0\} \times \mathbb{R}_+ \times \{\bar{e}_2\} \sqcup \{0\} \times \mathbb{R}_+ \times \{-\bar{e}_2\}$ where $\bar{e}_2 = (0,1)$. So it is disjoint union of four connected components.

See figure (9.11)

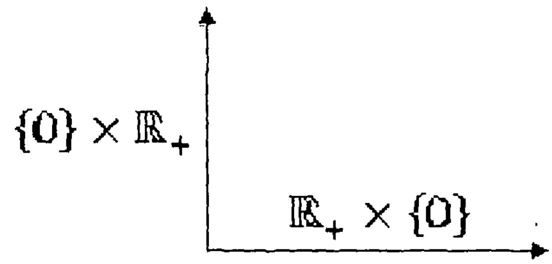


Figure 9.11: To calculate tangent complex of union of positive x and y axes

Remark 9.2.6. Now topologically X_1 and X_2 are equivalent but their tangent complexes are distinct, distinguished by the number of connected components.

Example 9.2.7. $X_3 = \{0\} \times \mathbb{R}^2 \subseteq \mathbb{R}^3$ (yz-plane in $\mathbb{R}^3$) Then one can calculate that $T(X_3)$ is connected and is homotopy type of a *circle*

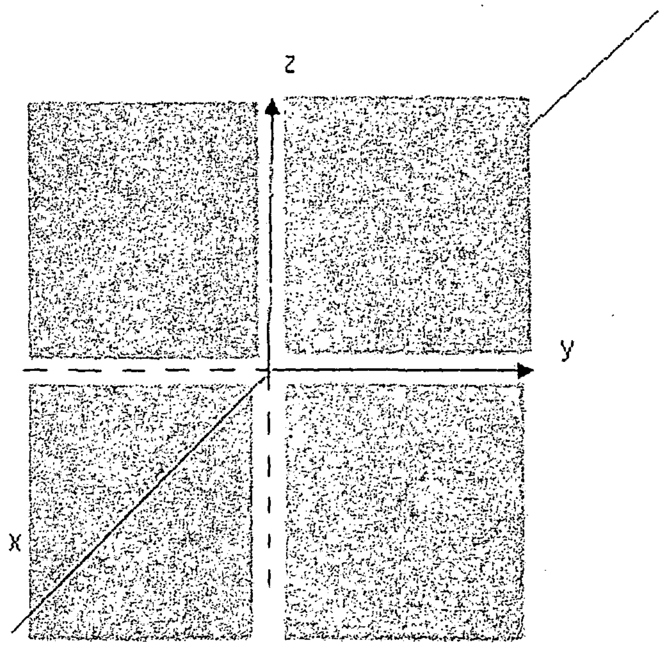


Figure 9.12: y z - plane in $\mathbb{R}^3$

Example 9.2.8. $X_4 = \mathbb{R}_+ \times \{0\} \times \mathbb{R} \cup \{0\} \times \mathbb{R}_+ \times \mathbb{R}$, (union of *half xz-plane and half xy plane*, meeting at the edge z-axis)

$T(X_4)$ = homotopy type or bouquet of three circles though X_3 and X_4 are topologically equivalent and their tangent complexes $T(X_3)$, $T(X_4)$ are both connected but their first homology groups differ.

Coming to real world problems, X is not given (or perceived by computer) as a nice geometrical figure. Instead it is given as a finite (but sufficiently large) set N of (sample) points, referrred in the literature as *"point cloud data"* (see [10]). From N we have to get information of X, $T(X)$, Homology etc.

Following questions arise:

1. How to calculate $H(X)$. where X is only represented by a discrete subset $N \subset X$?

2. How to construct $T(X)$, given only a discrete subset $N \subset X$ and where the definition of $T(X)$ involves taking limit $t \longrightarrow 0+$?

Using Čech complex and Rips complex construction one can tackle question 1.

For question 2, E.Carlson,G.Carlson and Vinde Silva in (see [10]) "using local components analysis at small number of base points in the complex obtain an approximation to the tangent space T_x at these points: then they sample the unit spheres in these tangent spaces uniformly to obtain a point cloud in $\mathbb{R}^n \times S^{n-1}$." Homology of the tangent complex can be found using Rips complex of this point cloud data.

One further questions remains:

3. Čech complex can recover homotopy type of the union (as mentioned in earlier chapter) provided the sampling is ad-

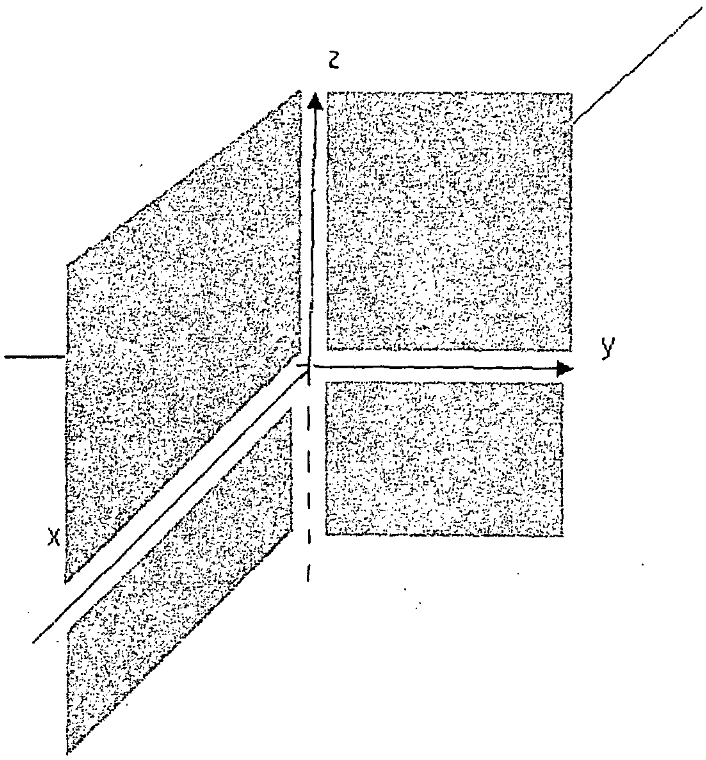


Figure 9.13: Union of half x z - plane and half y z - plane

equate. Rips complex though weak can use this property of Čech complex using the squeezing theorem referred to in earlier chapters.

One can associate Čech complex and Rips complex with the points of N as nodes (or vertices) using open balls of radius ϵ, say, around these points. But the sample points N may not be dense enough to cover X with ϵ-balls. One would also not like to have very large number of points in N to get over this coverage problem, because it will become computationally unviable. One can keep the data set N fixed and vary ϵ. If ϵ is too small then Rips complex $\mathcal{R}_\epsilon$ is a discrete set and if ϵ is too large then $\mathcal{R}_\epsilon$ will be a single simplex. The Grothendieck programme suggests that the topology of a given space is framed in the mappings to or from that space. With this perspective as guide one considers the *Persistent homology* to capture the homological features which persists over a range of parameters $[\epsilon, \epsilon']$ [10].

Now the last important question is how to use homological methods to locate singular points without prior knowledge of their location?

Proposition 9.2.9. *Let $X \subseteq S^{n-1} \subseteq \mathbb{R}^n$. Let $CX \subset \mathbb{R}^n$ denote a cone on X with vertex p. Then $T_p(CX) \hookrightarrow T(CX)$ is a homotopy equivalence.*

Thus if there is a cone like neighbourhood of a cone like point, we can calculate homology of its tangent complex. The algorithmic method proceeds as follows:

(1) Enclose $X\backslash$ (a singular curve say) in a bounded rectangular region where one is looking for a singular point and compute $H_*(T(X))$. If $H_*(T(X)) \cong H_*(S^1)$ then the rectangular region does not contain any singular point. Stop the algorithm (see

figure (9.14)).

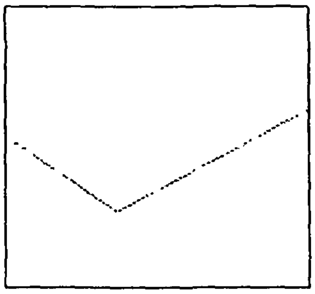


Figure 9.14: First step in identifying the singular point

(2) If $H_*(T(X)) \not\cong H_*(S^1)$ subdivide the rectangular region into four subrectangular regions as in figure (9.15). Find $T(X)$ in each of these subrectangular regions. Discard those subrectangular regions in which $H_*(T(X)) \cong H_*(S^1)$ (see figure (9.15)).

(3) Subdivide the subrectangular region in which $H_*(T(X)) \not\cong H_*(S^1)$ as above and continue the process.

In the limiting case we locate the singular point or numerically come sufficiently close to the singular point (see figure (9.16)).

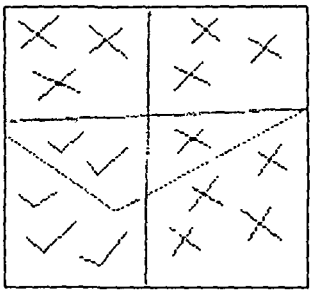


Figure 9.15: Second step in identifying the singular point

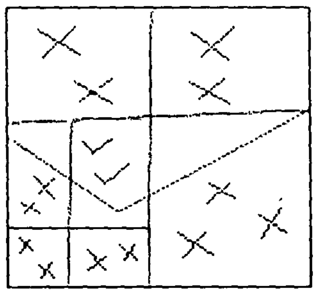


Figure 9.16: Third step in identifying the singular point

9.3 Some open Problems

To quote Edelsbruner and Harer, ([7]), "In spite of its short history, persistent homolgy has already lead to a number of interesting results and connected problems from seemingly distant fields. To substantiate this view we briefly mention developments that are related to persistence and we draw a speculative bigger picture by expressing where we believe persistent homology might lead us":

1. *Morse-Smale complexes and simlpification :* Simplification of the Morse-Smale decomposition of a 2 or 3 dimensional manifold using persistent homology. Simplification of the decomposition by adjusting the function is a more difficult problem. A controlled adjustment of a piecewise linear function on a 2-manifold has been described in [50]. The problem for manifolds of dimension three and higher is still open.

2. *Coverage by sensor networks:* Here the central problem is deciding whether a collection of relatively primitive sensors with limited domains of observation cover a given region. De Silva and Ghrist ([5]) use Vietoris-Rips complexes and their homology to decide this question under rather weak assumptions on what we know about the location of the sensors. Using persistence characterizations in terms of homology can be made robust to fluctations in the distribution of sensors and gaps in the coverage. Following are some concrete questions on coverage problems (see [5]):

2(i). Construct an effective homological coverage criterion which is distributed, allowing nodes with limited computational capabilities to compute local homology, and agree on global coverage (see [54]).

2(ii). By changing the bound in (6.1.1) to $r_c \geq r_b$, the homological criterion verifies 3-coverage in planar network (a simple exercise). Is it possible to verify k-coverage for any k via homology ? One wants to impose as few restrictions on r_c as possible.

2(iii). In practice, coverage and communication domains are not radially symmetric: elliptic or conical shapes are closer to reality in many cases. Is it possible to construct a homological coverage criterion for sensors whose communication and / or coverage domains are not radially symmetric ? What additional capabilities do the sensors require in order to handle such asymmetry ?

3. *Machine learning.* A related question is about the connection between persistent homology and machine learning. Manifold learning is very much part of that discipline and obviously connects to topological ideas and questions of robustness addressed by persistence (see ([7])).

4. *Dynamical systems.* It would be interesting to extend persistence from gradient fields to general smooth vector fields defined on manifolds. We refer to [1] for an account of discrete methods and combinatorial algorithms in the field. The connection to the idea of persistence is still unclear.

5. *Multidimensional Persistence.* To which extent the theory of persistence presented here can be generalized to multivariate situation (see [23])

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