LUSTERNIK-SCHNIRELMANN CATEGORY AND COBORDISM :
A SURVEY

ASHISH KUMAR DAS.
DEPARTMENT OF MATHEMATICS
NORTH-EASTERN HILL UNIVERSITY.

Submitted in Partial Fulfilment of the requirement of the
Degree of Master of Philosophy.

TO

Date........

NORTH-EASTERN HILL UNIVERSITY
SHILLONG
MAY 1990

CERTIFICATE

I certify that the dissertation entitled "Lusternik - Schnirelmann category and cobordism : A survey" submitted by Mr. Ashish Kumar Das in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

Prof. S.S. Khare
Supervisor
Mathematics Department
North-Eastern Hill University
SHILLONG.

Shillong,
May 15, 1990.

Math

NEHU Library
Acc. No 102430
Acc. by
Date 24/10/92 28/3/92
Transcribed by

DS
515.73
DAS

ACKNOWLEDGEMENTS

With deep sense of gratitude I wish to express my sincere thanks to Prof.S.S. Khare, Department of Mathematics, North-Eastern Hill University, for his invaluable help and guidance in the preparation of this dissertation.

I express my profound gratitude to Dr.H.K. Mukerjee, Dr.N.J. Dev, Dr.M.B. Rege, Dr.P. Saikia and all other faculty members of the Department of Mathematics, N. E .H.U., for giving M.Phil. courses and seminars and also for providing me with lots of tools needed for the preparation of this dissertation.

I am greatly indebted to all students, research colleagues and the office staff of the Department of Mathematics, N.E.H.U., for extending all possible help to me and for their co-operation.

I am very much thankful to Mr.V.T. James for typing my entire dissertation with utmost care and devotion.

Finally, I am grateful to my family members, my Mother in particular, for giving me constant encouragement and providing me with unbelievable opportunities to continue my studies.

Shri Ashish Kumar Das.

PREFACE.

The notion of category of a space, which is nearly sixty years old, was originally introduced by Lusternik and Schnirelmann during study of the theory of critical points. The category of a space X, cat X, is the smallest positive integer n such that the space X may be covered by n open subsets of X which are contractible in X.

In early 40's R.H. Fox [F.1] made an extensive study of this notion; in particular, towards the relationships between the categories and the standard topological invariants like homotopy type, homotopy group etc. because the categories seem to have a large measure of independence from these invariants. Later in 1978 I.M. James [J.1] made a very good survey on category.

The original notion of category was generalised in a number of ways. One of the most fruitful of these is the notion of the category of a map, due to Berstein and Ganea [B.1] in early 60's, ofcourse the concept goes back to Fox, 1941. The category of a map $f:X \to Y$, cat f, is the smallest positive integer n such that X may be covered by n open sets $U_i \subset X$ with $f|U_i \simeq 0$. Berstein and Ganea have also studied the relationship between cat f and other generalisations like genus of a map, n-dimensional category and category of a cohomology class.

Way back in around 1934, Lusternik and Schnirelmann [L.1] obtained a lower bound of the number of critical points of a real valued function f on a closed manifold M, in terms of the category of M. In fact they have shown that $\text{cat } M \leqslant$ no of critical points of f. In fact this was the reason why a number of differential topologists were attracted towards category. After quite sometime around 1966 R.S. Palais [P.1] extended this result for Banach manifolds. Later on at the end of 60's, F. Takens, [T.2] , studied the circumstances when the above inequality turns into an equality.

Just a few years back, in around 1986, Clapp and Puppe [C.1] introduced and studied in detail a generalisation of the notion of Lusternik-Schnirelmann category, namely the notion of $\mathcal{A}$-category, which gives good informations about the topology of the set of critical points of a differentiable function.

One of the significant extensions of the notion of category is the Equivariant Lusternik-Schnirelmann category. In around 1985, E.Fadell [F.2] made a detailed study on what he called G-category, G being a compact Lie group. If X is an invariant subset of a G-manifold M then $G\text{-cat}_M X$ is the smallest positive integer n such that X can be covered by n G-categorical open sets, where by a G-categorical set we mean an invariant set $U \subset M$ with an equivariant homotopy $H: U \times I \longrightarrow M$ such that $H_o : U \longrightarrow M$ is the inclusion and $H_1 : U \longrightarrow M$ has image in a single orbit. Fadell has shown that the number of critical orbits of a C^1 - functional on a G manifold is greater than or equal to the G-category of the manifold. He has also introduced a more general invariant, namely relative cohomological index and has shown how it can be used to enumerate the number of critical orbits of a C^1- functional on a G-manifold.

Recently, in 1989, J.R. Ramsay [R.1] extended Lusternik-Schnirelmann category theory to a relative G-category theory. This theory is related to the equivariant cohomological index theory of Fadell and Husseini [F.3] .

About twenty years back, at the fag end of 60's, M.V. Mielke [M.2] studied the cobordism properties of manifolds with small category and showed that, 'An n-dimensional manifold M with cat $M \leq 3$, and with $n \equiv 3 \bmod 4$, is a boundary. The classification upto cobordism of manifolds with category less than or equal to 3 was done by H.Singh [S.4] in 1988. In the same year he [S.5] attempted a similar classification of manifolds with category less than or equal to 4. However, to our knowledge, complete classification is not done for M^n , with cat $M^n \leq 4$, so far.

In the chapter I, the notion of Lusternik-Schnirelmann category of topological space has been introduced, which includes the definitions due to James [J.1] , Fox [F.1] and

Whitehead, and it has been studied in detail.

In chapter II, we introduce the notions of category of a map, genus of a map, n-dimensional category and category of a cohomology class, as given in [B.1], and we have tried to relate all these notions with one another. Ofcourse we have studied the notion of category of a map in somewhat detail.

One of the main objectives of the chapter III is to prove the following celebrated theorem, [P.1],

THEOREM:- If $f:M^n \longrightarrow \mathbb{R}$ is a function with f and M^n satisfying certain specific conditions, then cat $M^n \leq$ no. of critical points of f.

Next we introduce the notion of $\mathcal{A}$-category as given by Clapp and Puppe [C.1] and prove the analogue of the above theorem. The notion of $\mathcal{A}$-category is mainly used to study the topology of the set of critical points of f.

In the chapter IV, we introduce the notion of G-cat X fora G-space X and prove that if M^n is a G-manifold with $f: M^n \longrightarrow \mathbb{R}$ a differentiable map then no. of critical orbits is greater than or equal to G-cat M^n, [F.2]. Next the notion of G-cup length has been introduced and the inequality 'G-cup length of $M^n <$ G-cat M^n' is established for a G-manifold with cohomologically free action. One notices that G-cup length satisfies all but two properties satisfied by G-cat M^n namely subadditivity and continuity. Accordingly the concept of $\text{Index}_a X$ is defined for a G-space X and an element a in H*(BG). It has been shown that Index_a X is more powerful tool than G-cat M^n with $\text{Index}_a X \leq$ G-cat X, if the action of G is cohomologically free. Next we introduce a more general invariant; namely, relative cohomological index and study its properties. The idea of relative cohomological index is applied to study the critical orbits of a map $f: S^n \longrightarrow \mathbb{R}$ in the set $S^n - A$, where G acts on S^n as a subgroup of o(n+1), A is an invariant subset of S^n with relative cohomological index of (S^n, A) is $k < \infty$. Further the notions of relative G-category G-cat_Y (X, A) and G-cat^*_Y (X,A) have been defined for a G-pair (X,A) and a G-space Y and then elementary properties have been stuied, [R.1]. Next the notion of G-Index_Y (X, A) has been defined to give a lower bound for G-cat^*_Y (X,A) and computation of G-cat* (X,A) for some special examples have been done, [R.1].

In [M.2], Mielke showed that an n-dimensional closed manifold M^n with cat $M^n \leq 3$ and with $n \equiv 3 \mod 4$ is boundary.

In the last chapter, we first described the notions of cob-cat (M^n) and Poincare' algebra of M^n and then using these two notions and their properties, the bordism classification has been given in the following form:

THEOREM. [S.4] If M^n is an n-dimensional manifold with cob-cat $M^n \leq 3$ then M^n either bounds or one of the following is true,

(i) $n = 2^s$, $s \geq 1$ and M is bordant to $(RP^2)^{2^{s-1}}$

(ii) $n = 2^r(2^s+1)$, $r \geq 0$, $s \geq 2$ and M is bordant to $(\tilde{M})^{2^r}$

with $\tilde{M} = \coprod_{j=0}^{2^{s-2}} P(1, 2^{s-1}-2j) \times N_j$, where N_j is the degree $4j$ term in the formal inverse $(1+CP^2+CP^4+\dots)^{-1}$ and $P(1,m)$ is the Dold manifold $S^1 \times CP^m/Z_2$ and $\coprod$ denotes disjoint union.

The bordism classification for M^n with cat $M^n \leq 4$ is much more harder. We have given bordism classification for M^n with cob-cat $(M^n) \leq 4$ for some cases, [S.5].

Some of the important results are as below

THEOREM [S.5]. If M is 2^t - dimensional closed manifold with cob-cat $(M) \leq 4$, $t \geq 1$, then M is either boundary or is bordant to $(RP^2)^{2^{t-1}}$.

THEOREM [S.5] Let $n = 2^r+2^{r-1}+m$, m odd, $m < 2^{r-1}$ and $m+1 = 2^t$ and $\alpha(n)$, the number of nonzero terms in the diadic expansion of n, is greater than or equal to 2. Then M^n is a boundary.

Lastly we have mentioned some of the conjectures made by H.Singh, [S.5], given as below:

(A) If M^n is an n-dimensional closed manifold with cob-cat $M^n \leq 4$ and $\alpha(n) \geq 4$, then M^n is a boundary.

(B) If cob-cat $M^n \leq 4$ and $n = 2^s + 1$ then cob-cat $(M^n) \leq 3$.

CONTENTS

Page No.

CHAPTER I L.S. CATEGORY OF A SPACE. 1

§1 Definition and elementary properties 1

§2 Bounds for category 8

§3 Categorical sequence and category of product and orbit spaces 12

§4 L.S. category and H-spaces. 20

§5 Whitehead's definition of category. 22

CHAPTER II EXTENSION OF THE NOTION OF L.S.CATEGORY TO MAPS AND COHOMOLOGY CLASSES. 27

§1 Definition and elementary properties of the category of a map. 27

§2 Category of a map with domain as a CW-complex. 29

§3 Genus of a map and the n-dimensional category. 33

§4 Category of a cohomology class 40

CHAPTER III LUSTERNIK-SCHNIRELMANN THEORY ON CRITICAL POINTS. 45

§1 Finsler manifolds, Pseudo-gradient vector fields-definition and elementary properties 45

§2 Existence of critical points 55

§3 Main theorem of L.S. Theory 71

§4 $\mathcal{A}$-category and the topology of critical sets 77

contd..

CHAPTER IV EQUIVARIANT LUSTERNIK-SCHNIRELMANN THEORY AND RELATIVE COHOMOLOGICAL INDEX THEORIES. 87

§1 Equivariant L.S. Category 87

§2 Role of G-category in the theory of critical points. 90

§3 Use of G-cohomology in the Lusternik-Schnirelmann method 95

§4 Relative cohomological index theories and the Lusternik - Schnirelmann method. 106

§5 Relative G-category and its computations. 120

CHAPTER V COBORDISM CLASSIFICATION OF MANIFOLDS WITH SMALL CATEGGORY. 141

§1 Preliminaries 141

§2 Poincare Algebra and cobcat(M). 153

§3 Lemmas for the cobordism classification of manifolds with cobordism category ≤ 3. 162

§4 Complete classification of manifolds with cobordism category ≤ 3. 171

§5 Cobordism classification of manifolds with cobordism category ≤ 4. 180

BIBLIOGRAPHY.

CHAPTER I

LUSTERNIK - SCHNIRELMANN CATEGORY OF A SPACE

In this chapter we give various versions of the Lusternik-Schnirelmann Category of a space as described by James [J.I], Fox [F.1], Whitehead [J.1] etc. and study the important properties of this notion. Our discussion here centres mainly around the estimation of the category of a space. Throughout this project, by category of a space we shall mean the one given in definition (I.1.1), unless otherwise mentioned.

§1. Definition and elementary properties

Let X be a topological space. A subset A of X is called contractible in X if the inclusion map i: A → X is nulhomotopic.

As in [J.1], let us say that a subset A of X is categorical if A is contractible in X and that a covering of X is categorical if it consists of categorical sets.

If U be any covering of X, let $|U|$ denote the cardinality of the covering i.e. $|U|$ is the number of sets present in U .

We say that X is locally categorical if there is a categorical open covering of X.

I.1.1. DEFINITION: The L.S. Category or simply the category of a topological space X is the cardinal number cat X, defined as the smallest value of $|U|$ as U runs over all possible categorical open coverings of X.

Clearly, cat X = 1 iff X is contractible. Also it is easy to see that if X is a suspension then cat $X \leq 2$; in particular, spheres have category 2.

In the light of Fox [F.1], one also defines category in a slightly more general fashion, which runs as follows:

Let $X \subset M$ be two spaces. As before, a subset A of M is said to be categorical if A is contractible in M and a covering of X by categorical subsets of M is said to be a categorical covering of X in M.

I.1.2. DEFINITION: The category of X in M, denoted $\text{cat}_M X$, is the minimum of the cardinalities of all possible categorical open coverings of X in M.

clearly, $\text{cat}_M M = \text{cat } M$.

I.1.3. PROPOSITION

(a) $\text{cat}_M X$ is an increasing function of X and a decreasing function of M in the following sense:

(i) If $X \subset Y$ then $\mathrm{cat}_M X \leq \mathrm{cat}_M Y$

(ii) If M is open in N then $\mathrm{cat}_M X \geq \mathrm{cat}_N X$

(b) $\mathrm{cat}_M X$ is subadditive in the sense that, if $X, Y \subset M$ then

$$\mathrm{cat}_M (X \cup Y) \leq \mathrm{cat}_M X + \mathrm{cat}_M Y.$$

(c) If h_t is a deformation of X in M and X is open in M then

$$\mathrm{cat}_M X \leq \mathrm{cat}_M h_1(X).$$

PROOF: (a) and (b) are trivial.

For (c), let $h_1(X) \subset B_1 \cup \text{-----} \cup B_n$, each

B_i being open and contractible in M.

If $A_i = h_1^{-1}(B_i)$ then A_i is open in X, hence in M and $X = A_1 \cup \ldots\ldots \cup A_n$. Since $h_t \mid A_i$ is a deformation of A_i in M into B_i and B_i is contractible in M, so A_i is contractible in M. Hence (c) follows. #

We now recall that a topological space X is called an absolute neighbourhood retract (ANR) if it has the property that whenever A is a closed subset of a normal space Z, every map from A to X can be extended to a map from some neighborhood of A (in Z) to X. For every n, S^n is an ANR.

I.1.4. REMARK. For spaces which are ANR as well as normal, we can use categorical closed coverings, rather than categorical open coverings in the definition (I.1.1). Similarly in the definition (I.1.2) if X is a closed subset of M and M is ANR as well as normal then we can use categorical closed coverings in place of categorical open coverings of X in M. The justification lies in the following fact:

Any open covering (finite) of a normal space admits a closed refinement, of the same cardinality, which is also a covering of the space considered (by Shrinking lemma, [M.1; p.224]) and if the original covering is categorical then so is the refinement. Also when the space is an ANR, any categorical closed set is contained in a categorical open set, [J.2; p.229] .

There is another version of the category of a space, generally known as Whitehead's definition of category, which will be studied in detail in § 5 of this chapter.

We now study a few elementary properties of the category of a space.

Recall that a topological space X is said to dominate another space Y if there exist mappings $f:X \to Y$ and $g:Y \to X$ such that $f \circ g$ is homotopic to the identity map of Y. In particular any space dominates its retracts.

I.1.5. PROPOSITION. If X dominates Y then

$$\operatorname{cat} X \geq \operatorname{cat} Y$$

PROOF. Let $f:X \to Y$ and $g: Y \to X$ be two maps such that $fog \simeq 1_Y$. Let $U \subset X$ be an open set then $V = g^{-1}(U) \subset Y$ is open and if U is contractible in X then $f|U$ is nulhomotopic, hence $fog|V$ is nulhomotopic and so V is contractible in Y, since $fog \simeq 1_Y$. Hence any categorical open covering of X pulls back to a categorical open covering of Y and hence the Proposition. #

The following corollary is immediate.

I.1.6. COROLLARY. Category is an invariant of homotopy type.

Before proceeding further, it may be worthwhile to mention that one also defines category of a space X by using the subsets of X which are contractible in themselves. The number thus defined is called the strong category of X, written Cat X.

Contrary to the corollary (I.1.6), it has been observed that strong category is not a homotopy invariant. In [F.1] Fox has given an example of a space X with $\operatorname{Cat} X = 3$ which is homotopy type of a wedge of spheres, whose strong category is 2. The minimum value of Cat X, for all spaces of the same homotopy type of X, has been studied and it has been observed that (see [T.1]) this minimum value is equal to cat X or cat X + 1, for a large class of spaces X.

Continuing with the elementary properties of the category of a space we have,

I.1.7. PROPOSITION. Category is subadditive in the sense that if $X = A \cup B$, where A, B are open subspaces of X then

$$\text{cat}\,(A \cup B) \leq \text{cat}\,A + \text{cat}\,B.$$

PROOF. If U be any open categorical subset of A (or of B) then clearly U is an open categorical subset of $X = A \cup B$. Hence any open categorical covering of A and of B together give rise to an open categorical covering of X. #

I.1.8. REMARK. From (I.1.7) using (I.1.6.) it follows that if T_f is the mapping cone of a map $f\colon W \to Y$ then $\text{cat}\,T_f \leq \text{cat}_Y + 1$ and hence it follows that if X is a CW space with a single 0-cell then cat X does not exceed the number of dimensions in which the complex has cells. For, in the sequence of skeletons

$$X^0 \subset X' \subset \ldots\ldots \subset X^K = X$$

each X^i is the mapping cone of a map

$$\bigsqcup f_\alpha : \bigsqcup S_\alpha^{i-1} \longrightarrow X^{i-1} \text{, where } f_\alpha : S_\alpha^{i-1} \longrightarrow X^{i-1} \text{ is}$$

attaching map for an i-cell (D_α^i , S_α^{i-1}).

The remark (I.1.8) is quite useful due to the fact that if X is a path connected CW space then X is always homotopic to another CW space having a single 0-cell, [S.1;p.79].

Another application of (I.1.7) is that if $M = M(f,g)$ is the double mapping cylinder of $f: W \to Y$, $g: W \to Z$, then one has $\operatorname{cat} M \leq \operatorname{cat} Y + \operatorname{cat} Z$.

Now using the definition (I.1.2) we have,

I.1.9. PROPOSITION. If M is a path connected metric space and X, Y are mutually seperated (i.e. $\bar{X} \cap Y = \emptyset$, $X \cap \bar{Y} = \emptyset$) subsets of M, then

$$\operatorname{cat}_M (X \cup Y) = \max \{\operatorname{cat}_M X, \operatorname{cat}_M Y\}$$

PROOF. Since X, Y are mutually seperated, $X \subset U$ and $Y \subset V$ where U, V are open and disjoint subsets of M. Let S be a categorical covering of X in M by open subsets of U and S' be a categorical covering of Y in M by open subsets of V. Then clearly there exists a covering S'' of $X \cup Y$ which is open and categorical and $|S''| = \max \{ |S| , |S'| \}$. For example if $S = \{ U_i \}_{i=1}^{m}$, $S' = \{V_j\}_{j=1}^{n}$ (assume $m \leq n$), then take

$$S'' = \{ U_i \cup V_i \}_{i=1}^{m} \cup \{ U_m \cup V_j \}_{j=m+1}^{n} .$$

Hence $\operatorname{cat}_M (X \cup Y) \leq \max \{\operatorname{cat}_M X, \operatorname{cat}_M Y\}$

Also $X \subset X \cup Y$, so by (I.1.3)

$$\operatorname{cat}_M X \leq \operatorname{cat}_M (X \cup Y)$$

Similarly, $\operatorname{cat}_M Y \leq \operatorname{cat}_M (X \cup Y)$.

Therefore, $\mathrm{cat}_M (X \cup Y) \geq \max \{ \mathrm{cat}_M X, \mathrm{cat}_M Y \}$

Hence (I.1.9) follows. #

I.1.10 REMARK Notice that, in the proposition (I.1.9) if we replace the metric space M by a completely normal topological space, even then we will have the same conclusion.

§2. Bounds for category.

A useful lower bound, for the category of a space can be obtained from cohomology.

Recall, a ring R is called nilpotent if $R^n = 0$ for some positive integer n. The least such n is called the index of nilpotency of R, $\mathrm{nil}\, R$.

I.2.1 PROPOSITION. Consider singular cohomology theory H^* (infact any multiplicative cohomology theory) with any coefficient ring, and the corresponding reduced theory $\widetilde{H}^*$. We have then,

$$\mathrm{cat}\, X \geq \mathrm{nil}\, \widetilde{H}^*(X).$$

PROOF. If $V \subset X$ is categorical then $i^* : \widetilde{H}^*(X) \to \widetilde{H}^*(V)$ is trivial, where $i : V \to X$ is the inclusion ($i \simeq 0$). Hence the map $j^* : H^*(X,V) \to \widetilde{H}^*(X)$ is onto, by exactness. Thus if $\{V_1, V_2, \dots V_n\}$ is a categorical open covering of X (assuming $\mathrm{cat}\, X = n$) and $x_1, x_2, \dots, x_n$ are given elements of $\widetilde{H}^*(X)$, we can pull each x_i back to $\bar{x}_i \in H^*(X,V_i)$ and hence by the following commutative diagram

$$\begin{tikzcd}
H^*(X,V_1) \otimes \cdots \otimes H^*(X,V_n) \arrow[r, "\cup"] \arrow[d] & H^*(X,V_1 \cup \ldots \cup V_n) \arrow[d] \\
\tilde{H}^*(X) \otimes \cdots \otimes \tilde{H}^*(X) \arrow[r, "\cup"] & \tilde{H}^*(X)
\end{tikzcd}$$

The product $x_1 x_2 \ldots x_n$ can be pulled back

$\bar{x}_1 \ldots \bar{x}_n \in H^*(X,V_1 \cup \ldots \cup V_n)$ which is trivial group, since $V_1 \cup \ldots \cup V_n = X$. Hence $x_1 x_2 \ldots x_n = 0$.

So, $\mathrm{nil}\ \tilde{H}^*(X) \leq \mathrm{cat}\ X$. #

In terms of the <u>cuplength</u> of X the conclusion of (I.2.1) can be restated as,

$$\mathrm{Cuplength}\ (X) < \mathrm{cat}\ X$$

where the <u>cuplength</u> of X, denoted <u>cuplength(X),</u> is defined to be the largest positive integer k such that there are elements $x_1, x_2, \ldots, x_k$ in $\tilde{H}^*(X;R)$ with the cup product $x_1 x_2 \ldots x_k \neq 0$. Clearly $1+\mathrm{cuplength}\ (X) = \mathrm{nil}\ \tilde{H}^*(X)$.

Note that in the case of a manifold, the number of charts in a chart structure is an upper bound for the category (indeed strong category) of the manifold. Using this fact we have the following corollary to proposition (I.2.1).

<u>I.2.2. COROLLARY</u> $\mathrm{cat}\ P_{n-1} = n$, where $P_{n-1} = \mathbb{R}P^{n-1}$ is the real (n-1)-Projective space ($n \geq 1$).

PROOF. There is a standard chart structure consisting of the subsets $V_i \subset P_{n-1}$ where the ith coordinate is nonzero ($o \leq i \leq n-1$). Hence

$$\text{cat } P_{n-1} \leq n.$$

Also the mod 2 cohomology ring $\widetilde{H}^*(P_{n-1}; Z_2)$ is a truncated polynomial ring of height (n-1), (height is the highest possible degree of a polynomial in the ring). So it has nilpotency index n. Hence by (I.2.1)

$$\text{cat } P_{n-1} \geq n.$$

Thus, cat $P_{n-1} = n$. #

The same result for complex and quarternionic projective spaces may be proved similarly.

As an upper bound of the category of a space we have,

I.2.3 PROPOSITION. If X is a path connected, paracompact space then (assuming X to be locally categorical),

$$\text{cat } X \leq \dim X + 1.$$

PROOF. Let dim X = n. Let $\{U_i\}_{i\in I}$ be an open covering of X. without loss of generality one can assume that not more than (m+1) U_i's meet and U_i's are categorical. Let $\{g_i\}_{i\in I}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$. Now we use a well-known procedure of Milnor as follows:

For each $x \in X$, let us denote the finite set

$\{i \in I : g_i(x) > 0\}$ by $S(x)$. For each finite set $S \subset I$, let $W(S)$ be the open subset of all $x \in X$ such that $g_i(x) < g_j(x)$ for all $i \in I - S$ and $j \in S$. If $S \neq S'$ be any two finite subsets of I with the same number of elements then $W(S) \cap W(S') = \emptyset$. For every natural number k, define

$$W_k = \bigsqcup_{|S(b)|=k} W(S(b))$$

clearly $W(S(b))$ is open and is contained in the sets U_i of the original covering for all $i \in S(b)$. Now $\{W_k\}$ is also an open covering of X, infact a categorical covering, since X is pathconnected. Also note that $W_k = \emptyset$ for all $k > m+1$. Hence

$$\text{cat } X \leq m+1$$

Thus (I.2.3) is proved. #

In view of definition (I.1.2) the above proposition can be restated as

<u>(I.2.4) PROPOSITION.</u> If X is a closed subset of a space M which is paracompact, pathconnected and ANR, then

$$\text{cat}_M X \leq \dim X + 1$$

<u>PROOF.</u> Let $\dim X = m$ and let $\{U_i\}_{i \in I}$ be an open cover of X by X-open sets where each U_i is contractible in M. Also let not more than $(m+1)$ U_i's meet. Using the proof of (I.2.3) we get X-open sets W_k, $k = 1, 2, \ldots, (m+1)$ such that

$X = W_1 \cup W_2 \cup \ldots \cup W_{m+1}$ and each W_k is contractible in M. Since X is paracompact being closed subject of M, there is a closed refinement of $\{W_k\}_{k=1}^{m+1}$, having the same cardinality, which is also a categorical cover of X in M. Hence the proposition follows using the remark (I.1.4). #

From [b.1; p.95] we know that, every CW subspace Y of a CW space X has an open neighbourhood in X of which it is a strong deformation retract. Hence we have the following result:

I.2.5 RESULT. If a CW complex X is the union of k subcomplexes which are contractible in X , then cat $X \leq k$.

§3. Categorical sequence and category of product and orbit spaces.

We now give the notion of categorical sequence which will be used to find the category of product of spaces.

I.3.1. DEFINITION. For any space X we call a finite sequence $V_1 \subset V_2 \subset \cdots \subset V_n = X$ of closed subspaces of X to be categorical if each of the differences $V_i - V_{i-1}$, $1 \leq i \leq n$, is contained in an open categorical subspace of X . we take $V_0 = \emptyset$, by convention. Number of subspaces present in a categorical sequence is called the length of the sequence.

I.3.2 REMARK. Since the sets $V_i - V_{i-1}$ form a covering of X, it is clear that the existence of a categorical sequence of length n implies that cat $X \leq n$.

As a converse of the above remark we have the following proposition.

<u>I.3.3. PROPOSITION.</u> For a normal space X, if cat X = n then there is a categorical sequence of length n.

<u>PROOF</u>. If $A_1, A_2, \ldots, A_n$ be a categorical open covering of X then by using Shrinking lemma (X, normal) we have a categorical closed covering $B_1, B_2, \ldots, B_n$ for X such that $B_i \subset A_i$ for all i. Then $C_1 \subset C_2 \subset \ldots \subset C_n$ forms a categorical sequence for X, where $C_i = B_1 \cup B_2 \cup \ldots \cup B_i$ for all i. Clearly, $C_n = X$, each C_i is closed in X and $C_i - C_{i-1} \subset B_i \subset A_i$, open categorical. Thus $C_1 \subset C_2 \subset \ldots \subset C_n$ is a categorical sequence of length n. #

Using the categorical sequences we prove the following 'Product inequality' for category.

<u>I.3.4 PROPOSITION.</u> Let X,Y be pathconnected metric spaces (or let X,Y be pathconnected topological spaces such that X x Y is completely normal), then

$$\text{cat}(X \times Y) \leq \text{cat } X + \text{cat } Y - 1.$$

<u>PROOF.</u> Let cat X = m and cat Y = n. By proposition (I.3.3) there are categorical sequences

$$A_1 \subset A_2 \subset \cdots \subset A_m = X$$

and $$B_1 \subset B_2 \subset \cdots \subset B_n = Y$$

for X and Y respectively.

Consider the sequence

$$C_1 \subset C_2 \subset \text{--} \subset C_{m+n}$$

where $C_t = \bigcup_{i=1}^{t} A_i \times B_{t-i+1}$, $1 \leq t \leq m+n-1$.

Clearly each C_t is closed in X x Y and $C_{m+n-1} = X \times Y$.

Also note that,

$$C_t - C_{t-1} = \bigcup_{i=1}^{t} [(A_i - A_{i-1}) \times (B_{t-i+1} - B_{t-i})]$$

Clearly $(A_i - A_{i-1}) \times (B_{t-i+1} - B_{t-i})$ is contained in an open categorical subspace of X x Y, for each i. Furthermore, if $1 \leq i < i \leq t$, then

$$\overline{[(A_i - A_{i-1}) \times (B_{t-i+1} - B_{t-i})]} \cap [(A_{i'} - A_{i'-1}) \times (B_{t-i'+1} - B_{t-i'})]$$

$$\subset [\overline{(A_i - A_{i-1})} \cap (A_{i'} - A_{i'-1})] \times Y$$

$$\subset [A_i \cap (A_{i'} - A_{i-1})] \times Y = \emptyset$$

Also,

$$[(A_i - A_{i-1}) \times (B_{t-i+1} - B_{t-i})] \cap \overline{[(A_{i'} - A_{i'-1}) \times (B_{t-i+1} - B_{t-i})]}$$

$$\subset X \times [(B_{t-i+1} - B_{t-i}) \cap \overline{(B_{t-i'+1} - B_{t-i'})}]$$

$$\subset X \times [(B_{t-i+1} - B_{t-i}) \cap B_{t-i'+1}] = \emptyset$$

Thus, $C_t - C_{t-1}$ is the union of mutually separated sets each of which is contained in an open categorical subset of X x Y.

Hence by proposition (I.1.9), $C_t - C_{t-1}$ is contained in an open categorical subset of X x Y. Thus the sequence

$$C_1 \subset C_2 \subset -- \subset C_{m+n-1} = X \times Y$$

is a categorical sequence for X x Y of length (m+n-1). Hence by the remark (I.3.2), the proposition follows. #

Exactly same formula as given in (I.3.4) holds for the strong category also; i.e.

$$\text{Cat } (X \times Y) < \text{Cat } X + \text{Cat } Y - 1 \ ,$$

under the assumption that X and Y have the homotopy type of connected c.w. complexes, see [T.1] .

The proposition (I.3.4) can be generalised as follows:

<u>I.3.5 GENERALISATION.</u> If $X_1, X_2, --, X_k$ are path connected metric spaces (or, pathconnected topological spaces such that $X_1 \times X_2 \times --- \times X_k$ is completely normal), then

$$\text{Cat } (X_1 \times X_2 \times ---- \times X_k) \leq \sum_{i=1}^{k} (\text{cat } X_i - 1) + 1$$

<u>PROOF.</u> Follows immediately by the repeated application of Proposition (I.3.4).

In particular,

$$\text{cat}(S_1 \times S_2 \times -- \times S_k) \leq k+1,$$

since cat $S_i = 2$, where S_i are spheres.

Also nil $\tilde{H}^*(S_1 \times S_2 \times --- \times S_k) = k+1$.

102430
Date........

Hence, by (I.2.1)

$$Cat(S_1 \times S_2 \times{-}{-}\times S_k) = k+1$$

Thus the equality of (I.3.5) holds in this case. But it does not hold in general. We have the following example due to Fox:

Let p,q be mutually prime integers. Let the space X_p be formed by attaching $D^2 \subset \mathbb{R}^2$ to S^1 via the map $f:S^1 \to S^1$ such that $f(e^{i\theta}) = e^{i(\theta + \frac{2\Pi}{p})}$. Note that f is a map of degree p. Consider

$$M_p = \frac{X_p \times S^o \times I}{\sim} ,$$

where $(x,y,0)$ is identified with x and $(x,y,1)$ is identified with y for all $x \in X_p$ and for all $y \in S^o$. The space M_q is formed similarly. The spaces M_p and M_q being suspensions have category 2 and so their wedge $M_p \vee M_q$ also has category 2 . Consider now the homologies of $M_p \times M_q$ and $M_p \vee M_q$ (with coefficients in $\mathbb{Z}$).

$$H_n(M_p \vee M_q) = \begin{cases} H_n(M_p) \oplus H_n(M_q), & n > 0 \\ \mathbf{Z} , & n = 0 \end{cases}$$

and $H_n(M_p \times M_q) = \bigoplus_{r=0}^{n} H_r (M_p) \otimes H_{n-r}(M_q) \oplus \bigoplus_{r=0}^{n} Tor(H_r(M_p), H_{n-r-1}(M_q))$

Again, M_p being suspension of X_p , we have

$$H_r(M_p) = \begin{cases} \tilde{H}_{r-1}(X_p), & r > 0 \\ \mathbf{Z} , & r = 0 \end{cases}$$

Also, X_p is obtained from S^1 by attaching a 2-cell (D^2, S^1), via a map of degree p, so we have

$$\tilde{H}_t(X_p) = \tilde{H}_t(S^1), \text{ if } t \neq 1, \ t \neq 2.$$

(i.e. $H_t(X_p) = 0$, if $t \neq 1$, $t \neq 2$)

and $H_1(X_p) = \mathbf{Z}/P\mathbf{Z}$

$H_2(X_p) = 0.$

Therefore,
$$H_r(M_p) = \begin{cases} \mathbf{Z}, & r = 0 \\ 0, & r = 1 \\ \mathbf{Z}/P\mathbf{Z}, & r = 2 \\ 0, & r \geq 3 \end{cases}$$

Similarly,
$$H_r(M_q) = \begin{cases} \mathbf{Z}, & r = 0 \\ 0, & r = 1 \\ \mathbf{Z}/q\mathbf{Z}, & r = 2 \\ 0, & r \geq 3. \end{cases}$$

Hence we can easily see that

$$H_n(M_p \vee M_q) = \begin{cases} \mathbf{Z}, & n = 0 \\ 0, & n = 1 \\ \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z}, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

and
$$H_n(M_p \times M_q) = \begin{cases} \mathbf{Z}, & n = 0 \\ 0, & n = 1 \\ \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z}, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

That is the homologies of $M_p \times M_q$ and $M_p \vee M_q$ are same in every dimension.

$$\text{So, } H_n (M_p \times M_q, M_p \vee M_q) = 0 \text{ for all } n \geq 0.$$

Now, both $M_p \times M_q$ and $M_p \vee M_q$ are simply connected; so by Hurewicz isomorphism theorem [S.3; p.397] we have

$$\Pi_n (M_p \times M_q, M_p \vee M_q) = 0 \quad \text{for all} \quad n \geq 0$$

So, $M_p \times M_q$ contains $M_p \vee M_q$ as a deformation retract, [S.3; p.402] .

Hence cat $(M_p \times M_q) = 2$.

So in this case,

$$\text{cat } (M_p \times M_q) < \text{cat } M_p + \text{cat } M_q - 1$$

<u>I.3.6. REMARK.</u> There is a long standing conjecture to the effect that

$$\text{cat } (X \times S^r) = \text{cat } X + 1$$

for all spaces X and any positive integer r. Singhof [S.2] has shown recently that if M is a closed pathconnected n-dimensional piece wise linear manifold and if cat $M \geq \frac{(n+r)}{2} + 2$, then

$$\text{cat } (M \times S^r) = \text{cat } M + 1.$$

Using homotopy lifting property we prove the following proposition for the category of orbit spaces.

I.3.7. PROPOSITION If X is a pathconnected space on which a discrete group G acts freely then

$$\operatorname{cat} X \leq \operatorname{cat} (X/G).$$

PROOF. Let $U \subset X/G$ be such that the inclusion $i:U \to X/G$ is nulhomotopic, let F' be the contracting homotopy. Now consider the following diagram:

$$\begin{tikzcd}
p^{-1}(U) \times \{0\} \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "p"] \\
p^{-1}(U) \times I \arrow[r, "\tilde{F}: p \simeq c'"'] \arrow[ur, dashed, "F"'] \arrow[d, "p \times 1"'] & X/G \\
U \times I \arrow[ur, "F: i \simeq c"'] \arrow[r, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

where, $c:U \to X/G$ is a constant map given by $c(Gx) = Gx_o$, $Gx \in U$; $p:X \to X/G$ be the quotient map, $c': p^{-1}(U) \to X/G$ is also a constant map given by $c'(x) = Gx_o$, $x \in p^{-1}(U)$. $\tilde{F}$ is obtained by commuting the lower triangle and F is obtained by homotopy lifting property. (Gx denotes the orbit of x i.e. $Gx = \{gx : g \in G\}$)

Consider $F|p^{-1}(U) \times \{1\} : p^{-1}(U) \times \{1\} \to p^{-1}(Gx_o) \approx G$. where $p^{-1}(Gx_o) \approx G$ is the fibre over $Gx_o \in X/G$ and also, being discrete, is contractible in X. So $F|p^{-1}(U) \times \{1\}$ is nulhomotopic. Hence $p^{-1}(U)$ is categorical in X. Also if U is open in X/G, $p^{-1}(U)$ will be open in X. Hence the proposition follows. #

We have already seen, (I.2.2), that

$$\mathrm{cat}(S^n / \mathbb{Z}_2) = n+1 ,$$

for the antipodal action. In fact, in [K.1] , Krasnosielski has proved that

$$\mathrm{cat}\ (S^n / G) = n+1 ,$$

for all finite and nontrivial group G acting freely on S^n.

§ 4. L.S. Category and H-Spaces

Recall that a group G is called nilpotent if there is a sequence

$$G = G_1 \supset G_2 \supset --- \supset G_n = \{1\}$$

of sub groups such that $[G,G_i] \subset G_{i+1}$, $1 \leq i < n$. Such a sequence is called a central chain of length (n-1) and the minimum length of such central chains of G is called the nilpotency class of G.

If G is a topological group then, for any pointed space X, the set $\Pi(X;G)$ of pointed homotopy classes of pointed maps from X to G inherits a natural group structure from that of G. Moreover this remains true even if G is a H-space, i.e. possesses a continuous product $G \times G \longrightarrow G$ which satisfies the group axioms upto (pointed) homotopy.

I.4.1. THEOREM. Let X be a pathconnected, paracompact space of finite category. Let G be a H-space which is also an absolute neighborhood retract. Then the group $\Pi(X;G)$ is nilpotent of class < cat X. (Pathconnectedness of X may be replaced by that of G).

PROOF. Let cat X = n, then by shrinking lemma we have a closed categorical covering $\{A_1, A_2, ---, A_n\}$ of X.

$$\text{Write} \quad B_j = A_1 \cup A_2 \cup -- \cup A_j$$

Define, $\Pi_j \subset \Pi(X;G)$ to be the set of pointed homotopy classes of pointed maps of X into G which are neutral on B_j. Clearly Π_j is a subgroup of Π and

$\Pi_1 \supset \Pi_2 \supset --- \supset \Pi_n = \{1\}$. Also $\Pi_1 = \Pi$, which follows from the following fact:

Let A be a categorical closed subspace of a paracompact space X then any pointed map of X to G is pointed homotopic to a map which is neutral on A i.e. which maps A to the neutral element of G (use pathconnectedness of X or of G and [J.2; p.228]).

We claim that Π_j ($1 \leq j \leq n$) contains the commutator $[\Pi, \Pi_{j-1}]$ so that $\Pi_1 \supset \Pi_2 \supset -- \supset \Pi_n = \{1\}$ is a central chain of length (n-1).

To see this, let $a \in \Pi$ and $b \in \Pi_{j-1}$. Then the above fact with $A = A_j$ shows that a can be represented by a map

$g: X \to G$ which is neutral on A_j. Also by definition of Π_{j-1}, b can be represented by a map $h: X \to G$ which is neutral on B_{j-1}. Hence the commutator $F: X \to G$ given by

$$F(x) = g(x)\, h(x).\, g(x)^{-1}\, h(x)^{-1}, \quad (x \in X),$$

is neutral on $A_j \cup B_{j-1} = B_j$. Since F represents $[a,b]$, we have $[a,b] \in \Pi_j$, as claimed. Hence the group $\Pi(X;G)$ is nilpotent of class less than or equal to $(n-1)$. #

§5. Whitehead's definition of category.

Next we turn our attention to pathconnected spaces with base points, generally denoted by $*$.

If $*$ admits a categorical open neighborhood, we describe the space as categorically well-based.

In the n-fold topological product $\Pi^n X$ of a pointed space X with itself we consider a subspace $T^n X$, often known as the fat wedge, consisting of the n-tuples $(x_1, x_2, --, x_n)$ such that at least one of the x_i equals $*$. The space obtained from $\Pi^n X$ by collapsing $T^n X$ to a point is the n-fold smash product.

Considering the diagonal map

$\Delta_X = \Delta_{X^n} : X \to \Pi^n X$, we prove the following proposition.

I.5.1 PROPOSITION. Suppose that X is pathconnected, normal and categorically well-based. Then cat $X \leq n$ iff Δ_{X^n} can be deformed into $T^n X$.

PROOF. Let $h: X \times I \to \Pi^n X$ be a deformation of Δ_{X^n} such that $h_1 = h \mid X \times \{1\} = j \circ g$ for some $g: X \to T^n X$, where $j : T^n X \to \Pi^n X$ is the inclusion.

Write $U_i = h_1^{-1} p_i^{-1} N \subset X$ ($i = 1,2,\text{--},n$), where $p_i : \Pi^n X \to X$ denotes the ith projection and N is a categorical open neighborhood of $*$. Since $1 = p_i h_0 \simeq p_i h_1$ ($h_0 = h \mid X \times \{0\}$), and $p_i h_1$ maps U_i into N, it follows at once that U_i is categorical. Hence ($U_1, U_2, \text{--}, U_n$) constitutes a categorical open covering of X, as

$$\bigcup_i U_i = \bigcup_i h_1^{-1} p_i^{-1} N = h_1^{-1}\left(\bigcup_i p_i^{-1} N \right) \supset h_1^{-1} T^n X = X$$

Therefore cat $X \leq n$.

Conversely, suppose that $(V_1, V_2, \text{--}, V_n)$ is an open categorical covering of X and that $h_i : V_i \times I \to X$ is a contraction of V_i. Since X is pathconnected we may assume that V_i is mapped to $*$ at the end of the contraction. Since X is normal, there are closed subsets A_i of X such that ($A_1, A_2, \text{---}, A_n$) covers X, open neighborhoods W_i of A_i such that $\overline{W}_i \subset V_i$ and maps $r_i : X \to I$ such that $r_i(A_i) = 1$ and $r_i(X - W_i) = 0$. Define $d_i : X \times I \to X$ by

$$d_i(x,t) = \begin{cases} x, \text{ if } x \in X - \overline{W}_i \\ h_i(x, t r_i(x)), \text{ if } x \in V_i \end{cases}$$

and then define $d: X \times I \longrightarrow \Pi^n X$ so that $p_i \circ d = d_i$ i.e. $d \equiv (d_1, d_2, --, d_n)$. Then d is clearly a homotopy between Δ_{X^n} and a map with values in $T^n X$. #

<u>I.5.2. REMARK</u> Under these assumptions, therefore we see that cat $X \leq 2$ iff X admits a cohopf structure $\mu: X \to X \vee X$. A necessary condition for X to admit cohopf structure due to Fox is that $\Pi_1(X)$ is free, in the nonabelian sense. For if $\mu : X \to X \vee X$ is a cohopf structure then $j \circ \mu \simeq \Delta$, where $j: X \vee X \to X \times X$ is the inclusion. Hence $j_* \circ \mu_* = \Delta_*$ as shown in the following diagram, where $G = \Pi_1(X)$ and j_* denote the standard homomorphism of the free product into the direct product.

$$\begin{tikzcd}
 & G * G \arrow[d, "j_*"] \\
G \arrow[ur, "\mu_*"] \arrow[r, "\Delta_*"'] & G \times G
\end{tikzcd}$$

Now, im μ_* is free, being subgroup of a free group. Also μ_* is injective, since Δ_* is injective. Hence G is free as asserted.

By an <u>essential space</u> we mean a space which cannot be deformed into a proper subspace of itself; eg. S^n. Following is an interesting and useful result concerning essential space.

I.5.3 PROPOSITION. Let $X_1, ----, X_n$ be non-singleton spaces such that $X = X_1 \times X_2 \times --- \times X_n$ is essential. Then

$$\text{cat } X > n.$$

PROOF. Let $P_i : X \to X_i$ (i = 1,2,---,n) Project X onto its ith factor. Then $p\Delta = 1d$, where $p: \Pi^n X \to X$ is the product $(p_1 \times p_2 \times ---- \times p_n)$ and $\Delta: X \to \Pi^n X$ is the diagonal map. Let $W \subset X$ denote the fat wedge, consisting of the n-tuplas $(x_1, x_2, --, x_n)$ where $x_i \in X_i$ for i = 1,2,--,n ; and at least one x_i is the base point of X_i. Then $p(T^n X) \subset W$, where $T^n X \subset \Pi^n X$ is the fat wedge as before. Suppose, to obtain a contradiction, that cat $X \leq n$ then $\Delta \simeq g$ where $g(X) \subset T^n X$. Then $1 = p\Delta \simeq pg$, where $pg(X) \subset W$ i.e. X is deformed into W , which is a proper subspace of X, since X_i are nonsingleton and so X is inessential, which is a contradiction. Hence the proposition follows. #

I.5.4. REMARK The proposition (I.5.3) along with the product inequality (I.3.4) can be used to show that

$$\text{cat}(S_1 \times S_2 \times -- \times S_n) = n+1, \quad S_i \text{ are spheres.}$$

In the light of proposition (I.5.1) we can adopt the view point taken by Whitehead and other homotopy theorists and define cat X to be the least integer n (if any) such that the diagonal Δ_{X^n} can be deformed into $T^n X$. For spaces to which the proposition (I.5.1) does not apply, this definition may not agree with the original one. But this is not a very big drawback. Most of the properties we have obtained from the old definition can also be obtained from Whitehead's definition. Though some results are more easily proved from the old definition, but for some it is the reverse, eg.

I.5.5. PROPOSITION. Let X be a finite CW space. If X is $(r-1)$-connected $(r \geq 1)$ then

$$\operatorname{cat} X \leq 1 + \frac{\dim X}{r}.$$

PROOF. If $r = 1$ then the conclusion is clear, (Proposition (I.2.3)). So assume $r > 1$.

If $\dim X \leq r-1$ then $\operatorname{cat} X = 1$ because in that case X is contractible, [S.3;p 402, Thm 13] .

So we further assume $\dim X \geq r$. Since X is $(r-1)$-connected, there exists another CW space Y such that X is homotopic to Y and the skeleton $Y^{r-1} = \{*\}$ i.e. Y has a single 0-cell and all other cells are of dimension $\geq r$. Also $\dim Y = \dim X$ or $\dim Y = 1 + \dim X$ according as $\dim X > r$ or $\dim X = r$, [S.1;p79,cor 6.14] . In the product cell structure of $\Pi^n Y$ the fat wedge $T^n Y$ is a subcomplex and the remaining cells of $\Pi^n Y$ are of dimension $\geq nr$. Hence if $\dim Y < nr$ we can deform Δ_Y into $T^n Y$, [S.3;p402,Thm 13] and so $\operatorname{cat} Y \leq n$. Thus if n is chosen such that n is the smallest integer such that $\dim Y < nr$ then clearly $\dim Y \geq (n-1)r$ i.e. $n \leq 1 + \frac{\dim Y}{r}$. So $\operatorname{cat} Y \leq n \leq 1 + \frac{\dim Y}{r}$. Also, $r > 1$, $X \simeq Y$ and $\dim Y = \dim X$ or $\dim Y = 1 + \dim X$ according as $\dim X > r$ or $\dim X = r$. Hence it follows that

$$\operatorname{cat} X \leq 1 + \frac{\dim X}{r}. \qquad \#$$

CHAPTER II

EXTENSION OF THE NOTION OF LUSTERNIK-SCHNIRELMANN CATEGORY TO MAPS AND COHOMOLOGY CLASSES.

In this chapter we define the notions of category of map, genus of a map, n-dimensional category of a space and category of a cohomology class, and discuss their elementary properties. We also study how these notions are related with each other.

§1. Definition and elementary properties of the category of a map.

II.1.1. DEFINITION. Let $f:X \to Y$ be a (continuous) map of arbitrary topological spaces. The category of f, denoted by cat f, is the least integer $k \geq 1$ with the property that X may be covered by k open subsets U_m such that the maps $f|U_m : U_m \to Y$ defined by f are nulhomotopic; if no such integer exists, we put cat $f = \infty$.

II.1.2. PROPOSITION.

(i) cat $f \leq \min\{\text{cat } X, \text{cat } Y\}$, for any map $f: X \to Y$.

(ii) cat ld = cat X, where $\text{ld}: X \to X$ is the identity map of X.

(iii) cat $(g\circ f) \leq \min\{\text{cat } f, \text{cat } g\}$, for any map $g: Y \to Z$ and $f: X \to Y$.

(iv) cat h_0 = cat h_1, if $h_t : X \to Y$ is a homotopy.

(v) cat $f = 1$ iff $f: X \to Y$ is nulhomotopic.

(vi) cat $f \geq$ nil(im f^*), where $f^*: \tilde{H}^*(Y) \to \tilde{H}^*(X)$ denotes the homomorphism induced by $f: X \to Y$ in singular cohomology with any coefficient ring.

(vii) cat f is subadditive in the sense that

cat $f \leq$ cat$(f \mid X_1)$ + cat $(f \mid X_2)$, where X_1, X_2 are open subspaces of $X = X_1 \cup X_2$ and f is a map from X to any space Y.

PROOF. Proofs of (i), (ii), (iii), (iv), (v) and (vii) are immediate from definition.

For (vi), suppose cat $f = k$. Let U_m, $1 \leq m \leq k$, be open subsets of X such that $X = \bigcup_{m=1}^{k} U_m$ and $f \mid U_m \simeq 0$ for $1 \leq m \leq k$. Now in the diagram

$$\begin{tikzcd}
H^*(X,U_m) \arrow[r, "j_m^*"] & \tilde{H}^*(X) \arrow[r, "i_m^*"] & \tilde{H}^*(U_m) \\
& \tilde{H}^*(Y) \arrow[u, "f^*"'] &
\end{tikzcd}$$

we have $i_m^* \circ f^* = 0$; so that, by exactness, for any $v_m \in \tilde{H}^*(Y)$, $f^*(v_m) = j_m^*(w_m)$ for some $w_m \in H^*(X,U_m)$, $1 \leq m \leq k$. Now the product $w_1 . w_2 ... w_k$ lies in

$H^*(X, \bigcup_1^k U_m) = H^*(X,X) = 0$. Therefore by the naturality of the cup product we have

$$
\begin{aligned}
& f^*(v_1).f^*(v_2)\text{-----} f^*(v_k) \\
= {} & j_1^*(w_1).\ j_2^*(w_2)\text{-----} j_k^*(w_k) \\
= {} & j^*(w_1.w_2.\ \text{-------}.w_k) \\
= {} & 0 ,
\end{aligned}
$$

where j^* is induced by the inclusion map $j\colon X \to (X, \bigcup_1^k U_m)$. Hence (vi) follows. #

<u>II.1.3 REMARK</u> As in the case of category of a space, when X is normal and Y is an ANR we can use closed coverings for X rather than open ones for defining cat f, where $f\colon X \to Y$ is a map.

Analogous to (I.5.1), there is a result about the category of maps also. Let Y be a pointed space, then with the notations same as in (I.5.1) we have

<u>II.1.4 PROPOSITION</u> Suppose that X is normal and that Y is pathconnected and categorically well based. Then cat $f \leq n$ iff there is a map $g\colon X \to T^nY$ such that $j \circ g \simeq \Delta_Y \circ f$, where $j\colon T^nY \subset \Pi^nY$ and $\Delta_Y : Y \to \Pi^nY$ is the diagonal map.

<u>PROOF.</u> Proof is exactly in the same line as the proof of (I.5.1) and hence omitted. #

§2. <u>Category of a map with domain as a CW-complex.</u>

<u>II.2.1. PROPOSITION.</u> If X is a CW-complex and $f\colon X \to Y$ is

an arbitrary map, then the following statements are equivalent:

(i) X may be covered by k open subsets U_m such that $f \circ g_m \simeq 0$ for any CW-complex L_m and any map $g_m : L_m \to U_m$.

(ii) There exists a CW-complex Z and maps $\phi : X \to Z$, $\psi : Z \to Y$ such that Z is the union of k self-contractible subcomplexes and $f \simeq \psi \circ \phi$.

(iii) cat $f \leq k$.

PROOF. Suppose (i) holds. There exists a simplicial complex L and a homotopy equivalence $g : |L| \to X$, [S.3; Ex.E6, p420], where $|L|$ is the associated CW-complex. In a suitable subdivision of L there are subcomplexes L_m such that, $(1 \leq m \leq k)$, $|L| = \bigcup |L_m'|$ and $|L_m| \subset g^{-1}(U_m)$. Let Z denote the CW-complex which results by attaching to $|L|$ a cone $C|L_m|$ over each $|L_m|$; thus

$$Z = C|L_1| \cup C|L_2| \cup \cdots \cup C|L_k|$$

$$\text{and} \quad C|L_m| \cap C|L_n| = |L_m| \cap |L_n|, \quad m \neq n.$$

Let $w : |L| \to Z$ denotes the inclusion map and let $g_m : |L_m| \to U_m$ be the map defined by g. We have $f \circ g_m \simeq 0$, so that $f \circ g$ extends to every $C|L_m|$. Thus we get a map $\psi : Z \to Y$ such that $\psi \circ w = f \circ g$. Let h be a homotopy inverse of g and let $\phi = w \circ h$. Then, $f \simeq f \circ g \circ h = \psi \circ w \circ h = \psi \circ \phi$ and we have (i) $\Rightarrow$ (ii).

Next if (ii) holds then by the result (I.2.5) cat $Z \leq k$ and so by the proposition (II.1.2) we have cat $f \leq k$. So (ii) $\Rightarrow$ (iii).

Finally it is clear that (iii) $\Rightarrow$ (i). #

<u>II.2.2. PROPOSITION.</u> If X is a CW-complex and Y has continuous multiplication, then

$$\text{cat } f_1 f_2 \leq \text{cat } f_1 + \text{cat } f_2 - 1$$

for any two maps $f_m : X \to Y$, $m = 1,2$.

<u>PROOF.</u> The product map $f_1 f_2$ is equal to the composition

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{f_1 \times f_2} Y \times Y \xrightarrow{\mu} Y$$

where μ is the multiplication. By the proposition (II.2.1), there are CW-complexes Z_m and maps $\phi_m : X \to Z_m$, $\psi_m : Z_m \to Y$ such that cat $Z_m \leq$ cat f_m and $f_m \simeq \psi_m \circ \phi_m$, $m = 1,2$. Then $f_1 \times f_2$ is homotopic to the composition

$$X \times X \xrightarrow{\phi \times \phi} Z_1 \times Z_2 \xrightarrow{\psi \times \psi} Y \times Y.$$

Now, a **CW**-complex has the same homotopy type of a polyhedron, [S.3; p420] , and a polyhedron is metrizable (given a simplicial complex K, $|K|$ always has a metric). Hence, by (I.3.4),

$$\text{cat } (Z_1 \times Z_2) \leq \text{cat } Z_1 + \text{cat } Z_2 - 1.$$

Thus, by parts (iii), (iv) and (i) of Proposition (II.1.2), it follows that

$$\text{cat } f_1 f_2 \leq \text{cat } f_1 + \text{cat } f_2 - 1. \qquad \#$$

<u>II.2.3 PROPOSITION.</u> If X is a CW-complex of dimension r and if Y is a $(p-1)$-connected space then any map $f:X \to Y$ satisfies

$$\text{cat } f \leq \frac{r}{p} + 1$$

<u>PROOF.</u> If $r < p$ then clearly $f \simeq 0$, [S.3; Thm 13, p402] and so cat $f = 1$.

Suppose $r \geq p$ and let $\phi : X \to X/X^{p-1}$ denote the identification map, X^{p-1} is $(p-1)$-skeleton. Since Y is $(p-1)$-connected, $f \mid X^{p-1} \simeq 0$, [S.3; Thm 13, p402], and therefore there is a map ψ $X/X^{p-1} \to Y$ such that $f \simeq \psi \circ \phi$. Hence by parts (iv), (iii) and (i) of (II.1.2) we have

$$\text{cat} \leq \text{cat } X/X^{p-1}$$

Now X/X^{p-1} is also $(p-1)$-connected, as X/X^{p-1} has cells of dimension $\geq p$ and a cell of dimension zero only. Also

$$\dim X/X^{p-1} = r.$$

Therefore, by (I.5.5), we have

$$\text{cat } X/X^{p-1} \leq 1 + \frac{r}{p}.$$

Hence, cat $f \leq \text{cat } X/X^{p-1} \leq 1 + \frac{r}{p}$. #

§3. Genus of a map and the n-dimensional category.

II.3.1. DEFINITION. The genus of a map $f:X \to Y$ is the least integer $k \geq 1$ for which there are open subsets V_m of Y and maps $g_m : V_m \to X$ such that $Y = \bigcup V_m$ and $fog_m \simeq j_m$, where $j_m : V_m \to Y$ are inclusion maps ($1 \leq m \leq k$); if no such integer exists, genus $f = \infty$.

II.3.2. PROPOSITION.

(i) For any map $f:X \to Y$, genus $f \leq$ cat Y , if Y is pathconnected.

(ii) genus $1d_X = 1$, where $1d_X : X \to X$ is the identity map.

(iii) genus h_o = genus h_1 if $h_t : X \to Y$ is a homotopy.

(iv) genus fop = genus f = genus hof , if $p:W \to X$ and $h: Y \to Z$ are homotopy equivalences.

PROOF. (i). Let U be an open subset of Y such that $i:U \subset Y$ is nulhomotopic. Let $c: U \to X$ be a constant map, then $foc: U \to Y$ is also a constant map. Moreover, since Y is pathconnected, so any two constant maps from U to Y are homotopic. Thus $i \simeq foc$. So (i) follows.

(ii) and (iii). Easy to verify .

(iv). First equality is simple. For the second, let $V \subset Y$ be open, and $g:V \longrightarrow X$ be a map such that $fog \simeq j$, where $j: V \subset Y$ is inclusion. consider $U = k^{-1}(V)$, k is a homotopy inverse of h. Define $\bar{k}: U \to V$ by $\bar{k}(x) = k(x)$, then $j \circ \bar{k} = k|U = k \circ i$, $i: U \hookrightarrow Z$ is the inclusion. Now $f \circ g \simeq j$, so $h \circ f \circ g \circ \bar{k} \simeq h \circ j \circ \bar{k} = h \circ k \circ i \simeq i$. Hence, genus $h \circ f \leq$ genus f. Similarly, genus $f \leq$ genus $h \circ f$. #

Recall that a sequence $F \xrightarrow{i} E \xrightarrow{p} B$ of spaces and maps is a <u>fibration</u> if i defines a homeomorphism of F onto $p^{-1}(b)$ for some $b \in B$, and if for any space A, any homotopy $h_t: A \to B$ and any map $k: A \to E$ such that $p \circ k = h_o$, there is a homotopy $k_t: A \to E$ such that $k_o = k$ and $p \circ k_t = h_t$. For example, $p: \mathbb{R} \to S^1$, given by $p(t) = e^{2\pi i t}$ or in general every covering projection is a fibration.

<u>II.3.3. PROPOSITION.</u> Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a sequence of spaces and maps. If $p \circ i \simeq 0$, then cat $p \leq$ genus i. If the given sequence is a fibration and B is pathconnected then cat p = genus i.

<u>PROOF.</u> Let V be an open subset of E with inclusion map j. If $g: V \to F$ satisfies $i \circ g \simeq j$, then $p|V = p \circ j \simeq p \circ i \circ g \simeq 0$. It follows that cat $p \leq$ genus i. Next suppose that the given sequence is a fibration and let $h_t: V \to B$, $V \subset E$, be a homotopy such that $h_o = p \circ j$ and $h_1(V) = b$. Since B is pathconnected, we may assume that $b = p \circ i(F)$. Also there is a homotopy $k_t: V \to E$ such that $k_o = j$ and $p \circ k_t = h_t$, therefore, $k_1(V) \subset p^{-1}(b)$ and there is a map $g: V \to F$ given by $g(x) = i^{-1} \circ k_1(x)$ (note that i defines homeomorphism from F onto $p^{-1}(b)$). Clearly $i \circ g \simeq j$. Hence cat $p \geq$ genus i ; and so cat p = genus i. #

II.3.4 PROPOSITION. If Y is a CW-complex and $f:X \to Y$ is an arbitrary map, then the following statements are equivalent:

(i) Y may be covered by k open subsets V_m with the property that for any CW-complexes L_m and any maps $h_m : L_m \to V_m$ there are maps $g_m : L_m \to X$ such that $f \circ g_m \simeq j_m \circ h_m$, where $j_m : V_m \to Y$ are the inclusions.

(ii) genus $f \leq k$.

PROOF. (i) $\Rightarrow$ (ii). Choose a simplicial complex L and a homotopy equivalence $h: |L| \to Y$, [S.3; Ex.E6, p420]. In a suitable subdivision of L there are subcomplexes L_m such that $|L| = \bigcup |L_m|$ and $|L_m| \subset h^{-1}(V_m)$, $1 \leq m \leq k$.

Let $h_m : |L_m| \longrightarrow V_m$ be the maps defined by h and let $g_m : |L_m| \longrightarrow X$ satisfy $f \circ g_m \simeq j_m \circ h_m$. There are open subsets U_m of $|L|$ and maps $r_m : U_m \longrightarrow |L_m|$ such that $U_m \supset |L_m|$ and $t_m \circ r_m \simeq i_m$, where $t_m : |L_m| \to |L|$ and $i_m : U_m \to |L|$ are inclusions [D.1;p95] . Let d be a homotopy inverse of h.

Then,

$$d \circ f \circ g_m \circ r_m \simeq d \circ j_m \circ h_m \circ r_m = d \circ h \circ l_m \circ r_m$$

$$= d \circ h \circ i_m \simeq i_m$$

so that, genus $d \circ f \leq k$ and hence by part (iv) of (II.3.2) we get genus $f \leq k$.

The converse is trivial. #

We now define the n-dimensional category of a space X, denoted $\mathrm{cat}_n X$, using maps of arbitrary n-dimensional CW complexes.

II.3.5 DEFINITION For any $n \geq 0$, $\mathrm{cat}_n X$ is the least integer $k \geq 1$ with the property that X may be covered by k open n-categorical subsets, i.e. open subsets U_m such that, for any map $h: L \to U_m$ of any CW complex L of dimension $\leq n$, $j_m \circ h \simeq 0$ where $j_m : U_m \to X$ is the inclusion. If no such integer exists, set $\mathrm{cat}_n X = \infty$.

II.3.6. REMARK. $\mathrm{cat}_n X \leq \mathrm{cat}\, X$, for all $n \geq 0$.

II.3.7. THEOREM Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of connected CW complexes.

Let $n \geq 1$. If

$\Pi_q(X) = 0$ for $q \leq n$ and $f_q: \Pi_q(X) \approx \Pi_q(Y)$ for $q > n$,

$\Pi_q(Z) = 0$ for $q > n$ and $g_q: \Pi_q(Y) \approx \Pi_q(Z)$ for $q \leq n$,

then genus f = $\mathrm{cat}_n Y$ = cat g .

PROOF. Let V be an open subset of Y with the inclusion map $j: V \to Y$. Suppose that V is n-categorical. Let L be an arbitrary CW complex with n-skeleton L^n and let $h: L \to V$ be an arbitrary map. Use of mapping cylinder of f will enable us to assume that f is an inclusion map, [S.1; p40]. Further, we have $\Pi_q(Y,X) = 0$ for $q > n$, from the hypothesis. The assumption on V implies that $j \circ h_{|} L^n \simeq 0$. Hence there is a map $d: L \to X$, such that $j\circ h \simeq f \circ d$, obtained in the following manner:

We shall prove that there is a map $d_{n+1}: L^{n+1} \longrightarrow X$ such that $joh \mid L^{n+1} \simeq fod_{n+1}$ and $d_{n+1} \mid L^n$ is a null map. Using exactly the same technique one can show that if there is a map $d_m: L^m \to X$, $m \geq n$, such that $joh \mid L^m \simeq fod_m$ then there is a map $d_{m+1}: L^{m+1} \to X$ such that $joh \mid L^{m+1} \simeq fod_{m+1}$; moreover $d_{m+1} \mid L^m = d_m$ and the homotopy D_{m+1} from $joh \mid L^{m+1}$ to fod_{m+1} is an extension of the homotopy D_m from $joh \mid L^m$ to fod_m joined with the stationary homotopy $I_m: fod_m \simeq fod_m$ i.e. $D_{m+1} \mid L^m \times [0,1] = D_m * I_m$. Thus we can define $d: L \to X$ by $d(x) = d_i(x)$ if $x \in L^i$ and $D: L \times [0,1] \to Y$ as $D(x,t) = D_i * I_i(x,t)$ if $x \in L^i$, where d_i and D_i have their respective meaning as above and $I_i: fod_i \simeq fod_i$ is the stationary homotopy. For $i \leq n$, assume d_i and D_i to be null map and null homotopy respectively. Then clearly $D: joh \simeq fod$.

Now since (L, L^n) is a cofibered pair and also $joh \mid L^n \simeq 0$ there exists a map $\bar{h}: L \to Y$ with $joh \simeq \bar{h}$ such that $\bar{h}$ maps L^n to a single point (in X). Let F be the family of all triples (M, k, K) such that M is a subcomplex of L^{n+1} containing L^n, $k: M \to X$ is a map such that $fok \mid L^n = \bar{h} \mid L^n$ and $K: M \times I \to Y$, $I = [0,1]$, is a homotopy from $\bar{h} \mid M$ to fok relative to L^n. We give a partial ordering to F by saying $(M', k', K') \leq (M'', k'', K'')$ if and only if $L^n \subseteq M' \subseteq M'' \subseteq L^{n+1}$, $k'' \mid M' = k'$ and $K'' \mid M' \times I = K'$.

clearly $F \neq \emptyset$ and every chain in F has an upper bound in F. Hence we may apply Zorn's lemma to conclude that F has a maximal element (M,k,K). It is now enough to show that $M = L^{n+1}$. If $M \neq L^{n+1}$ there is an $(n+1)$-cell e in $L^{n+1} - M$.. Consider the composition

$$(D^{n+1}, S^n) \xrightarrow{\phi_e} (L^{n+1}, L^n) \xrightarrow{\bar{h}} (Y,X)$$

where ϕ_e is the characteristic map. Since $\Pi_{n+1}(Y,X) = 0$, there is a map $\theta : D^{n+1} \to X$ and a homotopy $H: D^{n+1} \times I \to Y$ from $\bar{h} \circ \phi_e$ to $f \circ \theta$ relative to S^n.

Define, $k' : M \cup e \longrightarrow X$

$$\text{by} \quad k'(x) = \begin{cases} k(x) & \text{if } x \quad M \\ \theta(y) & \text{if } x = \phi_e(y) \text{ for some } y \in D^{n+1}. \end{cases}$$

and $K' : (M \cup e) \times I \longrightarrow Y$

$$\text{by} \quad K'(x,t) = \begin{cases} K(x,t) & \text{if } x \in M \\ H(y,t) & \text{if } x = \phi_e(y) \text{ for some } y \in D^{n+1} \end{cases}$$

Clearly $k' | M = k$ and K' is a homotopy from $\bar{h} | M \cup e$ to $f \circ k'$ relative to L^n with $K' | M \times I = K$ so that $(M,k,K) \leq (M \cup e, k', K')$. Thus we get a contradiction to the maximality of (M,k,K). Hence $M = L^{n+1}$ and we choose $d_{n+1} = k$. For the next step, we start with a map $\tilde{h} : L \to Y$ such that $j \circ h \simeq \tilde{h}$ and $\tilde{h} | L^{n+1} = f \circ d_{n+1}$. Since f is an inclusion, $\tilde{h}$ maps L^{n+1} inside X. Then we change F suitably and proceed.

Hence, coming back to the theorem, we get by (II.3.4),

$$\text{genus } f \leq \text{cat}_n\, Y \quad .$$

Next suppose that $goj \simeq 0$. Let L be an arbitrary CW-complex of dimension $\leq n$ and let $h: L \to Y$ be an arbitrary map. Without altering the homotopy type of Y we may assume that g is a fibre map, [S.3; p99] , with fibre F and inclusion map $i: F \to Y$. Since $gojoh \simeq 0$ and Z is connected, there is a map $d: L \to F$ such that $iod \simeq joh$ (see the proof of (II.3.3)). Using the exact homotopy sequence of the fibration $F \xrightarrow{i} Y \xrightarrow{g} Z$ and the fact that $g_q: \Pi_q(Y) \approx \Pi_q(Z)$, $q \leq n$, we have $\Pi_q(F) = 0$, $q \leq n$; so that $d \simeq 0$, since $\dim L \leq n$. Therefore $joh \simeq 0$ and hence $\text{cat}_n\, Y \leq \text{cat}\, g$.

Finally, since $\Pi_q(X) = 0$, $q \leq n$, the n-skeleton X^n of X is contractible in X, [S.3; p402, Thm 13] , and so $gof \mid X^n \simeq 0$. Also since $\Pi_q(Z) = 0$ for $q > n$, we have (by the same induction technique used in the first part of this proof) a map $d: X \to Z$ whose image is a single point such that $d \simeq gof$, so that $gof \simeq 0$. So by (II.3.3) we get $\text{cat}\, g \leq \text{genus}\, f$. Thus the theorem is completely proved. #

<u>II.3.8 PROPOSITION.</u> *Let $g: Y \to Z$ be a map of connected CW-complexes, and consider $n \geq 1$. If $\Pi_q(Z) = 0$ for $q > n$ and if the homomorphism $g_n: \Pi_n(Y) \to \Pi_n(Z)$ is trivial then $\text{cat}\, g \leq \text{cat}_{n-1}\, Y$.*

<u>PROOF.</u> Let V be an open $(n-1)$-categorical subset of Y.

Let L be an arbitrary CW complex with m-skeleton L^m, and let $h: L \to Y$ be an arbitrary map such that $h(L) \subset V$. Since $h \mid L^{n-1} \simeq 0$, so $goh \mid L^{n-1} \simeq 0$ and hence, by the same induction technique used in the proof of (II.3.7), we have $goh \simeq 0$. The only difference to be noticed is that here we don't have $\Pi_n(Z) = 0$. But since $g_n: \Pi_n(Y) \longrightarrow \Pi_n(Z)$ is a trivial map, the composition

$$(D^n, S^{n-1}) \xrightarrow{\phi} (L^n, L^{n-1}) \xrightarrow{goh'} (Z, .)$$

will be homotopic to a null map from D^n to Z relative to S^{n-1} where ϕ is a characteristic map for an n-cell and $h': L \to Y$ is such that $h \simeq h'$ and $h' \mid L^{n-1}$ is a null map. Thus the induction technique may be applied to show that $goh \mid L^n \simeq 0$. To show that $goh \mid L^m \simeq 0$ for all $m > n$, we can apply the induction technique without any difficulty, because $\Pi_q(Z) = 0$ for $q > n$.

Hence by (II.2.1) we have cat $g \leq \text{cat}_{n-1}$ Y. #

§ 4. The category of a cohomology class.

Let X be an arbitrary space, G an abelian group, and $n \geq 1$ an integer.

II.4.1 DEFINITION. For any cohomology class $u \in H^n(X;G)$, the category of u, denoted cat u, is the least integer $k \geq 1$ with the property that X may be covered by k open subsets U_m such that $j_m^*(u) = 0$ where $j_m^*: H^n(X;G) \to H^n(U_m;G)$ is induced by the inclusion map $j_m \colon U_m \longrightarrow X$; if no such integer exists, set cat $u = \infty$.

Clearly cat $u \leq$ cat X.

Let $K(G,n)$ denote any Eilenberg-Maclane CW complex L such that $\Pi_n(L) \approx G$ and $\Pi_q(L) = 0$ for $q \neq n$; the homotopy type of L is uniquely determined, [W.1; p244, Thm 7.1] . The group $H^n(K(G,n); G)$ may be identified with the group Hom(G,G), and the identity map of G then corresponds to the fundamental class $t \in H^n(K(G,n);G)$, see [W.1; p236] . If X is a CW-complex, the set $\Pi(X,K(G,n))$ of homotopy classes of maps $X \to K(G,n)$ and the group $H^n(X;G)$ are in a one to one correspondence which is given by $f \mapsto f^*(t)$ where $f^* \colon H^n(K(G,n);G) \to H^n(X;G)$ is induced by $f \colon X \to K(G,n)$, [W.1;p244, cor.6.20] . For any $u \in H^n(X;G)$ we shall denote by f_u any of the homotopically equivalent maps $f \colon X \to K(G,n)$ such that $f^*(t) = u$.

<u>II.4.2. THEOREM.</u> If X is a CW-complex and if $u \in H^n(X;G)$, then cat u = cat f_u.

<u>PROOF.</u> Let V be an open subset of X with the inclusion map $j \colon V \longrightarrow X$. If $f_u \circ j \simeq 0$, then $j^*(u) = j^* \circ f_u^*(t) = 0$; so that cat $u \leq$ cat f_u. Conversely, if $j^*(u) = 0$ then for any CW-complex L and any map $h \colon L \to V$, one has $h^* \circ j^* \circ f_u^*(t) = 0$. So, by the one to one correspondence between $\Pi(L,K(G,n))$ and $H^n(L;G)$ as mentioned above, we have $f_u \circ j \circ h \simeq 0$ and

hence, by (II.2.1), we get cat $f_u \leq$ cat u. #

From theorem (II.4.2) and proposition (II.2.3) one gets

II.4.3 THEOREM. If X is a CW-complex of dimension r and if $u \in H^n(X;G)$, then cat $u \leq r/n + 1$. #

II.4.4 PROPOSITION. If X is a CW complex and if $u,v \in H^n(X;G)$, then cat$(-u)$=cat u and cat $(u+v) \leq$ cat u +cat v -1

PROOF. The first statement is clear from the definition (II.4.1). In order to derive the second, recall that $K(G,n)$ has an H-space structure which makes the set $\Pi(X,K(G,n))$ a group which is isomorphic to $H^n(X;G)$ under the correspondence displayed earlier. Therefore $f_{u+v} \simeq f_u f_v$ and the result now follows from (II.4.2) and (II.2.2). #

II.4.5 PROPOSITION. If X is any space and if $u_m \in H^{q_m}(X;R)$, $m = 1,2$; then cat $(u_1 \cup u_2) \leq \min \{$ cat u_1, cat $u_2\}$, where $\cup$ denotes the cup product, and R denotes a commutative ring.

PROOF. If V, with the inclusion map $i:V \to X$, is an open subset of X such that $i^*(u_1) = 0$, then $i^*(u_1 \cup u_2) = i^*(u_1) \cup i^*(u_2) = 0$ so that cat$(u_1 \cup u_2) \leq$ cat u_1. Similarly, cat$(u_1 \cup u_2) \leq$ cat u_2 and hence the proposition. #

Recall that, a cohomology operation T of type $(m,n;\ G,G')$ is a natural transformation from the functor

$H^m(\ ;G)$ to the functor $H^n(\ ;G')$, where $H^m(\ ;G)$ and $H^n(\ ;G')$ are contravariant singular cohomology functors defined on the category of topological pairs, m and n are fixed positive integers and G,G' are abelian groups.

II.4.6. PROPOSITION. If X is an arbitrary space and $v \in H^n(X;G')$ is the image of $u \in H^m(X;G)$ under a cohomology operation T, then cat $v \leq$ cat u.

PROOF. Let U with the inclusion map $j:U \to X$ be an open subset of X such that $j^*(u) = 0$. Then the following commutative diagram

$$\begin{tikzcd}
H^m(X;G) \arrow[r, "j^*"] \arrow[d, "T_X"] & H^m(U;G) \arrow[d, "T_u"] \\
H^n(X;G') \arrow[r, "j'^*"] & H^n(U;G')
\end{tikzcd}$$

gives $j'^*(v) = j'^* \circ T_X(u) = T_u \circ j^*(u) = 0$ which proves the proposition. #

II.4.7 PROPOSITION. Let E,X be any two spaces and $g:E \to X$ be a map. Let $u \in H^n(X;G)$, then cat $g^*(u) \leq$ cat u.

PROOF. If U with inclusion map $j:U \to X$ is an open subset of X such that $j^*(u) = 0$, then $i^* g^*(u) = d^* j^*(u) = 0$, where $i:N=g^{-1}(U) \longrightarrow E$ is the inclusion map and $d:N \to U$ is defined by $d(x) = g(x)$. Hence the proposition. #

As a relation between n-dimensional category and the category of a cohomology class we have the following result, which follows immediately from (II.4.2) and (II.3.8) with n replaced by $(n+1)$.

II.4.8 THEOREM. Let X be a connected CW-complex and $u \in H^n(X;G)$ then $\operatorname{cat} u \leq \operatorname{cat}_n X$. #

A lower bound for the category of a cohomology class is provided by the following theorem.

II.4.9 THEOREM. If X is a CW-complex and if the k-fold cup product $u \cup u \cup ---- \cup u$ of a cohomology class $u \in H^n(X;R)$ is nontrivial ($n \geq 1$), then $\operatorname{cat} u \geq k+1$, where R denotes commutative ring.

PROOF One has $f_u^* (t \cup t \cup --- \cup t) = u \cup u \cup --- \cup u \neq 0$. So by (II.4.2) and by part (vi) of (II.1.2) we have $\operatorname{cat} u = \operatorname{cat} f_u \geq k+1$. #

II.4.10 REMARK. Let $\mathbb{C}P^{n-1}$ be the complex projective space of real dimension $(2n-2)$. We have $\operatorname{cat} \mathbb{C}P^{n-1} = n$ (using the same argument as in (I.2.2.)). Also $H^{2q}(\mathbb{C}P^{n-1}; \mathbb{Z}) \approx \mathbb{Z}$ for $1 \leq q \leq (n-1)$. If $u \in H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$ is a generator, then $\operatorname{cat} u \leq \operatorname{cat} \mathbb{C}P^{n-1} = n$ and, since the $(n-1)$-fold cup product $u \cup u \cup --- \cup u$ generates $H^{2n-2}(\mathbb{C}P^{n-1}; \mathbb{Z})$, by the above theorem one has $\operatorname{cat} u \geq n$. Thus for every positive integer n there is a cohomology class of category n.

CHAPTER III

LUSTERNIK-SCHNIRELMANN THEORY ON CRITICAL POINTS.

About fifty years back, Lusternik and Schnirelmann showed that the number of critical points of a real valued function f on a closed manifold M is greater than or equal to the category of M. In this chapter we study a generalisation of this result for Finsler manifolds i.e. **for** paracompact Banach manifolds. Further, we introduce the notion of A-category leading towards the study of the topology of the set of critical points.

§ 1. Finsler manifolds and Pseudo-gradient vector fields — definition and elementary properties.

Let V and W be two Banach spaces and $\mathcal{O}$ be an open subset of V. Let $f:\mathcal{O} \longrightarrow W$ be a function of class C^{k-1}, k being a positive integer. We say that f is of class C^{k-}, if given any $p_0 \in \mathcal{O}$ there is a neighborhood U of p_0, $U \subseteq \mathcal{O}$, and a real number $r > 0$ such that

$$\|d^{k-1} f_p - d^{k-1} f_q\| \leq r \|p-q\| \qquad \text{for all}$$

$p, q \in U$.

III.1.1. THEOREM. If f is of class C^k then f is of class C^{k-}.

PROOF. Given $p_o \varepsilon \mathcal{O}$, we choose $r > 0$ such that $\| d^k f_{p_o} \| < r$. Then by continuity of $d^k f$, there exists some convex neighborhood $\mathcal{U}$ of p_o, $\mathcal{U} \subseteq \mathcal{O}$, such that $\| d^k f_x \| < r$ for all $x \varepsilon U$. If $p,q \varepsilon U$ then $\| d^{k-1} f_p - d^{k-1} f_q \| < r \| p-q \|$, by the mean value theorem. #

III.1.2. REMARK. It is elementary to see that the maps of class C^{k-} from $\mathcal{O}$ to W form a vector space and the composition of two maps of class c^{k-} is again of class C^{k-}.

By a C^k Banach manifold we mean a Hausdorff, second countable topological space M with an open cover $\{U_i\}_{i \varepsilon A}$ such that each U_i is homeomorphic to a Banach space under a homeomorphism h_i satisfying the condition that

$$h_j o h_i^{-1} : h_i(u_i \cap u_j) \to h_j(u_i \cap u_j)$$

is differentiable of class C^k.

Let M and N be two C^k Banach manifolds and $f: M \to N$ be any map. We say that f is of class C^{k-}, if given $p \varepsilon M$ there exists a chart $g: \mathcal{O} \to V$ for M at p and a chart $h: \mathcal{U} \to W$ for N at f(p) such that $h o f o g^{-1}$ is of class C^{k-}.

III.1.3 LEMMA. If M is a C^1 Banach manifold then for every sufficiently small open set $\mathcal{O}$ of M there is a

C^{1-} map $f:M \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in M$ and

$\mathcal{O} = \{ x \in M \mid f(x) > 0\}$.

PROOF. Let $\mathcal{O}$ be any subset of M such that $\overline{\mathcal{O}}$ is included in the domain U of a chart $g:U \to V$. Let $\tilde{\mathcal{O}} = g(\mathcal{O})$, then it is enough to show that there is a C^{1-} map $\tilde{f}:V \to \mathbb{R}$ with $\tilde{f} \geq 0$ and $\tilde{\mathcal{O}} = \{v \in V \mid \tilde{f}(v) > 0 \}$, as in that case we can define $f(x) = \tilde{f}(g(x)$ for $x \in U$ and $f(x) = 0$ for $x \notin \overline{\mathcal{O}}$. Define $\tilde{f}(v) = \operatorname{Inf} \{ \|v-w\| : w \notin \tilde{\mathcal{O}} \}$. Since $\tilde{\mathcal{O}}$ is open,. $\tilde{\mathcal{O}} = \{ v \in V \mid \tilde{f}(v) > 0 \}$. Let $v_1, v_2 \in V$ and $\varepsilon > 0$ then choose $w \notin \tilde{\mathcal{O}}$ such that $\|v_2-w\| < \tilde{f}(v_2) + \varepsilon$. Then,

$\tilde{f}(v_1) \leq \|v_1-w\| \leq \|v_1-v_2\| + \|v_2-w\| < \|v_1-v_2\| + \tilde{f}(v_2) + \varepsilon$;

and since ε is arbitrary $\tilde{f}(v_1) - \tilde{f}(v_2) \leq \|v_1 - v_2\|$ Interchanging v_1 and v_2 we have $|f(v_1)-f(v_2)| \leq \|v_1-v_2\|$, showing that $\tilde{f}$ is a C^{1-} map. #

III.1.4 THEOREM. If M is a paracompact C^1 Banach manifold and $\{U_a\}_{a \in A}$ is an open cover of M then there is a C^{1-} partition of unity $\{g_b\}_{b \in B}$ for M subordinate to $\{U_a\}_{a \in A}$.

PROOF. By lemma (III.1.3) and the fact that M is paracompact, we can find a locally finite open cover $\{\mathcal{O}_b\}_{b \in B}$ for M which refines $\{U_a\}_{a \in A}$ such that if V is an open subset of $\mathcal{O}_b$ for some $b \in B$ then there is a C^{1-} map $f:M \to \mathbb{R}$ with $f \geq 0$ and $V = \{ x \in M : f(x) > 0 \}$. Let $\{ V_b \}_{b \in B}$ be an open cover of M with $\overline{V}_b \subseteq \mathcal{O}_b$ (by shrinking lemma) and let $f_b : M \to R$ be a C^{1-} map with $f_b \geq 0$ and $V_b = \{ x \in M \mid f_b(x) > 0 \}$. Let $g_b = f_b / (\dot{\Sigma} f_b)$. Then $\{ g_b \}_{b \in B}$ is clearly a C^{1-} partition of unity for M and support $g_b \subseteq \overline{V}_b \subseteq O_b$ which is included in some U_a. #

III.1.5 DEFINITION. Let B be a topological space, V a Banach space and $E = B \times V$ the product Banach space bundle with fibre V over B. A function $\| \; \| : E \to \mathbb{R}$ is a Finsler structure for E if

(i) for each $b \in B$ the map $v \mapsto \|(b,v)\|$ is an admissible norm for V (call it $\| \; \|_b$), and

(ii) given $b_o \in B$ and a real number $r > 1$ there is a neighborhood U of b_o in B such that

$$\frac{1}{r} \| v \|_{b_o} \leq \| v \|_b \leq r \| v \|_{b_o}$$

for all $b \in U$ and all $v \in V$.

Suppose N is an admissible norm for V then the map $\| \; \| : E \to \mathbb{R}$ defined by $\|(b,v)\| = N(v)$ is called the flat Finster structure for E defined by N. (Here by an admissible norm on V, we mean a norm which induces same topology on V as the one induced by the original norm on the Banach space V).

III.1.6 DEFINITION. Let $p:E \to B$ be a C^0 Banach space bundle and let $\| \|:E \to \mathbb{R}$ be a function. We say $\| \|$ is a Finsler structure for E if given $b_0 \in B$ there is a bundle chart $g: \mathcal{O} \times V \approx E|\mathcal{O}$ for E with $\mathcal{O}$ a neighborhood of b_0 such that $\| \| \circ g$ is a Finsler structure for $\mathcal{O} \times V$.

III.1.7. DEFINITION. A Finsler manifold M is a C^1 Banach manifold together with a Finsler structure for $T(M)$, where $T(M) \longrightarrow M$ is the tengent bundle of M.

Every paracompact C^1 Banach manifold admits the structure of a Finsler manifold, [P.1] .

It can be easily checked that, if M is a connected C^k banach manifold and $p,q \in M$ then there is a C^k path $s: [a,b] \to M$ with $s(a) = p$ and $s(b) = q$.

Let M be a Finsler manifold and $s: [a,b] \to M$ be a C^1 path in M. We define $l(s)$, the length of s, by

$$l(s) = \int_a^b \|s'(t)\| \, dt \quad , \quad s'(t) \equiv \frac{d}{dt} s(t) \quad .$$

If p and q are in the same component of M, we define the distance $d(p,q)$ from p to q by

$$d(p,q) = \operatorname{Inf} \{ l(s) \mid s \text{ is a } C^1 \text{ path from } p \text{ to } q \}$$

This function d is a metric for each component of M and the topology given by this metric is same as the given topology of M, [P.1] . This metric d defined on each component of M

is called the Finsler metric for M. If each component of M is complete in the Finsler metric, then M is called a complete Finsler manifold.

The following proposition is immediate.

III.1.8 PROPOSITION. Every paracompact C^1 Banach manifold M is metrizable. Conversely ofcourse every metrizable Banach manifold is paracompact. #

In fact every paracompact C^1 Banach manifold admits a complete metric, [P.3] .

Let M be a C^1 Finsler manifold and $s:(a,b) \to M$ be a C^1 path in M. We define l(s), the length of s, by

$$l(s) = \lim_{\substack{\alpha \to a \\ \beta \to b}} \int_{\alpha}^{\beta} \| s'(t) \| \, dt \, .$$

Note that we may have $l(s) = \infty$.

Again, if χ is a C^{1-} vector field on a C^2 Finsler manifold M then by an integral curve for X we mean a C^1 map $s:(a,b) \longrightarrow M$ with the property that $s'(t) = X(s(t))$ for all $t \in (a,b)$. Moreover if $0 \in (a,b)$ then s(0) is called the initial condition of s.

III.1.9 PROPOSITION. Let X be a C^{1-} vector field on an open submanifold $\tilde{M}$ of a complete C^2 Finsler manifold M and let $s:(a,b) \to \tilde{M}$ be a maximal integral curve of χ i.e. every integral curve of X with the same initial condition as that of s is a restriction of s. If $b < \infty$ and

$\int_0^b \| \chi(s(t)) \| \, dt < \infty$ then $s(t)$ has a limit point in $M - \tilde{M}$ as $t \longrightarrow b$ i.e. there exists a sequence $\{ t_n \}$ in (a,b) converging to b such that $\{ s(t_n) \}$ converges to a point in $M - \tilde{M}$. Similarly if $a > -\infty$ and

$\int_a^0 \| \chi(s(t) \| \, dt < \infty$ then $s(t)$ has a limit point in $M - \tilde{M}$ as $t \longrightarrow a$.

PROOF. If $\int_0^b \| \cdot \chi(s(t) \| \, dt < \infty$ then, since $s'(t) = \chi(s(t)$, we have

$$\int_0^b \| s'(t) \| \, dt < \infty .$$

Given $\varepsilon > 0$ we choose a partition

$$0 = t_o < t_1 < t_2 < \cdots < t_n < b = t_{n+1}$$ of

$[o,b)$ so that, for $i = 0,1,2, ---, n$,

$$\int_{t_i}^{t_{i+1}} \| s'(t) \| \, dt < \varepsilon .$$

Then clearly $s([0,b))$ is included in the union of the ε- balls about the points $s(t_i)$, $i = 1,2,---, n$. That is $s([o,b))$ is a totally bounded subset of M and since M is complete, $s([o,b))$ has a compact closure. Hence $s(t)$ has a limit point q in M as $t \to b$. Again if

$b < \infty$ then by theorem (3) of [P.2; § 6] we have $q \notin \tilde{M}$. Hence $q \in M - \tilde{M}$. #

III.1.10 PROPOSITION. Let X be a C^{1-} vector field on a complete C^2 Finsler manifold M. If $\|X\|$ is bounded on M then X generates a global one parameter group i.e. X generates a one parameter family $\{g_t\}_{t \in \mathbb{R}}$ of C^{1-} diffeomorphisms of M such that the map $\mathbb{R} \to \mathrm{Diff}(M)$, $t \mapsto g_t$, is a continuous homomorphism with $g_t g_s = g_{t+s}$, where $\mathrm{Diff}(M)$ is the group of all diffeomorphisms on M. ($\|X\| = \| \| \circ X$, *where* $\| \|$ *is the Finsler structure for* $T(M)$).

PROOF. Let $s:(a,b) \to M$ be a maximal integral curve of X. It is enough to show that $a = -\infty$ and $b = \infty$. Suppose for example that $b < \infty$. Then since $\|X\|$ is bounded,

$$\int_0^b \| X(s(t)) \| \, dt < \infty \quad ; \text{ so by (III.1.9)}$$

$s(t)$ has a limit point in $M - M = \emptyset$ as $t \to b$, which is absurd. Now for $t \in \mathbb{R}$ define $g_t : M \to M$ as $g_t(x) = s_x(t)$, where s_x is the maximal integral curve of X with $s_x(0) = x$. Then $\{g_t\}_{t \in \mathbb{R}}$ forms the global one parameter group generated by X, [H.1; p 149-151]. #

If M is a Finsler manifold then there is a natural Finsler structure for the dual space $T(M)^*$ defined by $\|h\| = \sup \{ h(v) \mid \|v\|_b = 1 \}$ for $h \in T(M)^*_b$. In particular if $f: M \to \mathbb{R}$ is a C^1 map then $\|df_p\|$ is defined for each $p \in M$ and $\|df\| : M \to \mathbb{R}$ given by $p \mapsto \|df_p\|$ is well defined, nonnegative, continuous real valued function on M.

III.1.11 DEFINITION. Let M be a C^{k+1} Finsler manifold ($k \geq 0$) and let $f : M \to \mathbb{R}$ be a C^1 map. A vector $X \in T(M)_p$ is called a *pseudo-gradient vector* for f at $p \in M$ if

$$\text{(i)} \quad \| X \| \leq 2 \| df_p \|$$

and
$$\text{(ii)} \quad Xf = df_p(X) \geq \| df_p \|^2$$

If χ is a C^k vector field on a subset $S \subset M$, then χ is called a C^k *Pseudo-gradient vector field* for f on S if for each $p \in S$, $X_p = \chi(p) \in T(M)_p$ is a pseudo-gradient vector for f at p.

It is immediate from the above definition that the *set of pseudo-gradient vectors for f at $p \in M$ is a convex subset of* $T(M)_p$.

III.1.12 LEMMA. Let M be a C^{k+1} Finsler manifold ($k \geq 0$) and let $f : M \to \mathbb{R}$ be a C^1 map. Given $p \in M$, which is not a critical point of f, there is an open neighborhood $\mathcal{O}$ of p in M and a C^k pseudo-gradient vector field for f in $\mathcal{O}$.

PROOF. Given $p \in M$, which is not a critical point of f, we can find $Y_p \in T(M)_p$ with $\| Y_p \| = 1$ such that $df_p(Y_p)$ is as close to $\| df_p \|$ as we wish; say

$df_p(Y_p) > \frac{2}{3} \| df_p \|$. Consider $X_p = \frac{3}{2} \| df_p \| \, Y_p$

so that $\| X_p \| = \frac{3}{2} \| df_p \| < 2 \| df_p \|$ and

$X_p f = df_p(X_p) = \frac{3}{2} \| df_p \| \, df_p(Y_p) > \| df_p \|^2$. Now extend

X_p to a C^k vector field X in a neighborhood U of p in M (say by making it "constant" with respect to a chart at p) Then define

$$\mathcal{O} = \{ q \in U \mid \; X_q f \; > \; \| df_q \|^2 \text{ and } \| X_q \| \; < \; 2 \| df_q \| \}$$

where $X_q \equiv X(q)$ for all q. Since $X f = df \circ X$, $\| df \|$ and $\| X \|$ are all continuous on U, so $\mathcal{O}$ is open. Hence the lemma follows. #

III.1.13 THEOREM. Let M be a C^2 Finsler manifold and let $f: M \to \mathbb{R}$ be a C^1 map. Let M^* denote the open submanifold of M consisting of regular (i.e. noncritical) points of f. Then there is a C^{1-} pseudo-gradient vector field for f in M^*.

PROOF. For each $p \in M^*$, we choose by lemma (III.1.12) a neighborhood $\mathcal{O}_p$ of p in M^* and a C^1 pseudo-gradient vector field X^p for f in $\mathcal{O}_p$. Now, M^* is metrizable, being a Finsler manifold, and hence paracompact. So by theorem (III.1.4) there is a C^{1-} partition of unity $\{ g_b \}_{b \in B}$ for M^* such that for each $b \in B$ there is a point $p(b) \in M^*$ with support $g_b \subseteq \mathcal{O}_{p(b)}$. Then $X = \Sigma \, g_b X^{p(b)}$ is a C^{1-} vector field in M^* and hence is a C^{1-} pseudo-gradient vector field for f in M^*, as the set of pseudo-gradient vectors at any point forms a convex subset of the tangent space at that point.

#

§2. Existence of critical points.

Throughout and the later sections, M will denote a C^k- Banach manifold ($k \geqslant 1$), $K \subseteq M$ the set of critical points of f (where $f: M \to R$ is a map of class at least C^1), and $M^* = M - K$ the set of regular points of f. We denote the frontier of K i.e. $K \cap \overline{M^*}$, by $\dot{K}$. Note that K is a closed subset of M i.e. M^* is open in M.

III.2.1 THEOREM. Let M be a connected C^1 Banach manifold and $f: M \to \mathbb{R}$ be a nonconstant C^1 map. Then $f(K) = f(\dot{K})$.

PROOF. Let $p \in K$. We shall find a point $x \in \dot{K}$ such that $f(x) = f(p)$. Choose a point $q \in M$ such that $f(q) \neq f(p)$ and a C^1 path $s:I \to M$ such that $s(0) = p$ and $s(1) = q$ and let $g(t) = f(s(t))$. Then, $g'(t) = df_{s(t)}(s'(t))$. Since g is not constant, g' is not identically zero; so $s(I)$ is not included in K. Let

$$t_o = \mathrm{Inf}\ \{\ t \in I \mid s(t) \notin K\ \} = \mathrm{Inf}\ s^{-1}(M-K).$$

Then clearly $t_o \in I$, since I is closed and bounded, and also $s(t_o) \in \dot{K}$, because otherwise

$$t_o \in s^{-1}(M-\dot{K}) = s^{-1}(M-K) \cup s^{-1}(\mathrm{Int}\ K)$$

which contradicts the fact that $t_o = \mathrm{Inf}\ s^{-1}(M-K)$.
Set $x = s(t_o) \in \dot{K}$ and since $g'(t) = 0$ for $0 \leqslant t \leqslant t_o$, it follows that $f(x) = g(t_o) = g(0) = f(p)$. #

III.2.2. DEFINITION. Let M be a C^1 Finsler manifold and $f:M \to \mathbb{R}$ a C^1 map. We say that f satisfies condition (C) if, given any subset S of M on which $|f|$ is bounded but on which $\| df \|$ is not bounded away from zero (i.e. for each $\varepsilon > 0$ there exists some $x \in S$ such that $\| df_x \| < \varepsilon$), there is a critical point of f in $\overline{S}$.

III.2.3. THEOREM. Let M be a C^1 Finsler manifold and $f:M \to \mathbb{R}$ be a C^1 map satisfying condition (C). Then $f \mid \dot{K}$ is proper i.e. given $-\infty < a \leq b < \infty$, $\dot{K} \cap f^{-1} [a,b]$ is compact. In particular if $\mathrm{Int}(K) = \emptyset$ then $f \mid K$ is proper.

PROOF. Since M is metrisable, it is enough to prove that $\dot{K} \cap f^{-1} [a,b]$ is sequentially compact. Consider a sequence $\{p_n\}$ in $\dot{K} \cap f^{-1} [a,b]$. Since $p_n \in \dot{K}$, we can choose $q_n \notin K$ arbitrarily close to p_n. In particular since $\| d~~f~~ \|$, f are continuous and $\| df_{p_n} \| = 0$ we can choose q_n so close to p_n that

$$\| df_{q_n} \| < \frac{1}{n} , \quad a-1 < f(q_n) < b+1 \quad \text{and also} \quad d(q_n, p_n) < \frac{1}{n} ,$$

d being the Finsler metric. Then by condition (C) a subsequence of $\{ q_n \}$ will converge to a critical point p of f. Since

$$d(q_n, p_n) < \frac{1}{n} ,$$

the corresponding subsequence of $\{ p_n\}$ will also converge to p. Since $\dot{K}$ is closed, $p \in \dot{K} \cap f^{-1} [a,b]$. Thus

$\dot{K} \cap f^{-1}$ [a,b] is sequentially compact.

Particular case follows from the fact that if Int $K = \emptyset$ then $\dot{K} = K$. #

III.2.4 PROPOSITION. Let M be a complete C^2 Finsler manifold without boundary, $f: M \to \mathbb{R}$ be a C^1 function satisfying condition (C) and X be a C^{1-} pseudo-gradient vector field for f in M*. If $s:(a,b) \to M$ is a maximal integral curve for X then either $\lim\limits_{t\to b} f(s(t)) = \infty$ or s(t) has a critical point of f as a limit point as $t \to b$ (b may be ∞). Similarly either $\lim\limits_{t \to a} f(s(t) = -\infty$ or s(t) has a critical point of f as a limit point as $t \to a$ (a may be $-\infty$).

PROOF Let $g(t) = f(s(t))$. Then $g'(t) = df_{s(t)}(s'(~~t)~~)$ $= df_{s(t)}(X(s(t))) \geqslant \| df_{s(t)} \|^2 > 0$ (since $X(s(t))$ is a regular point for each $t \in (a,b)$). So, g is strictly monotonically increasing and hence has a limit B as $t \to b$. Suppoe $B < \infty$. Then

$$g(t) = g(o) + \int_o^t g'(x)dx \geqslant g(o) + \int_o^t \| df_{s(x)} \|^2 \, dx,$$

and it follows that

$$(*) \qquad \int_o^b \| df_{s(x)} \|^2 \, dx < \infty .$$

Suppose first that $b = \infty$. Then by $(*)$, $\| df_{s(x)} \|$ is not bounded away from zero for $x \in [0, \infty)$. Since $g(x) = f(s(x))$ is monotonically increasing

$$f(s(o)) \leqslant f(s(x)) \leqslant B \quad \text{for } x \in [0, \infty).$$

Hence by condition (C), s(t) has a critical point of f as limit point as $t \to \infty$.

Next suppose $b < \infty$. By Schwartz's inequality we have

$$\int_0^b \| df_{s(x)} \| \, dx \leqslant b^{\frac{1}{2}} \left[\int_0^b \| df_{s(x)} \|^2 \, dx \right]^{\frac{1}{2}}$$

and by $(*)$, $\int_0^b \| df_{s(x)} \|^2 \, dx < \infty$.

But since X is a pseudo-gradient vector field for f in M^*,

$$\| X(s(x)) \| \leqslant 2 \| df_{s(x)} \| ,$$

hence $\int_0^b \| X(s(x)) \| \, dx < \infty$.

Then by proposition (III.1.9), s(t) has a limit point in $M - M^* = K$ as $t \to b$.

The other conclusion can be drawn similarly. #

III.2.5 THEOREM. Let M be a complete C^2 Finsler manifold without boundary and $f: M \to \mathbb{R}$ be a C^1 map satisfying condition (C). If f is bounded below on a component M_o of M then $f \mid M_o$ assumes its greatest lower bound. If f is bounded below then either f assumes its greatest lower bound or else there is a sequence $\{M_k\}$ of components of M, on each of which f is constant, such that

$$f(M_k) \to \mathrm{Inf}\ \{ f(x) \mid x \in M \} .$$

PROOF Let $B_o = \mathrm{Inf}\ \{ f(x) \mid x \in M_o \}$. For every positive integer n, choose $x_n \in M_o$ with $f(x_n) < B_o + \frac{1}{n}$. We can assume $x_n \in K$ (otherwise by proposition (III.2.4) we can replace x_n by a limit point of s(t) as $t \to a$ where $s: (a,b) \to M^*$ is the maximal integral curve of a pseudo-gradient vector field for f on M^*; also note that $B_o > -\infty$ and f(s(t)) is monotonically increasing). In case f is constant on M_o, we are done. So we can assume that f is not constant on M_o. Then by Theorem (III.2.1), we can assume $x_n \in \dot{K}$ (In fact here Theorem (III.2.1) will look like $f(K \cap M_o) = f(K \dot{\cap} M_o)$, where $K \dot{\cap} M_o$ is the frontier of $K \cap M_o$ in M_o and we have $K \dot{\cap} M_o = \dot{K} \cap M_o$). Thus we have a sequence $\{x_n\}$ such that $x_n \in \dot{K}$ and

$$B_o \leqslant f(x_n) < B_o + \frac{1}{n} \leqslant B_o + 1$$

and hence by Theorem (III.2.3), there exists a subsequence of $\{x_n\}$ converging to some point $x_o \in \overline{M_o}$ (note that $x_n \in M_o$ for each n) and clearly $f(x_o) = B_o$. This proves the first assertion.

Now let $B = \mathrm{Inf}\ \{ f(x) \mid x \in M \}$.

For every positive integer n choose $p_n \in M$ such that

$B \leqslant f(p_n) < B + \frac{1}{n} \leqslant B+1$. By the first part of the theorem we can assume p_n to be the minimum of f on a component M_n of M. Then $p_n \in K$ and also by assuming that f is not constant on M_n for infinitely many n's, because otherwise $f(M_n) \to B$, we can use Theorem (III.2.1) to have $p_n \in \dot{K}$ for infinitely many n's. Hence by Theorem (III.2.3) there exists a subsequence of $\{p_n\}$ converging to some point p of M and clearly $f(p) = B$. #

III.2.6 THEOREM. Let M be a C^1 Finsler manifold, $f:M \to \mathbb{R}$ be a C^1 map satisfying condition (C) and c be a regular value of f. Then either c is in the interior of $f(M^*)$ or else there is a sequence M_k of components of M, on each of which f is constant, such that $f(M_k) \to c$.

PROOF Let $S = \{a \in \mathbb{R} \mid f$ has the constant value a on a component of $M\}$.

Now by Theorem (III.2.1), $f(K) = S \cup f(\dot{K})$ (because $K = \bigcup_i (K \cap M_i)$ where M_i's are components of M and if f is constant on some M_i then $f(K \cap M_i) \subset S$ and if f is not constant on some M_i then $f(K \cap M_i) = f(K \dot{\cap} M_i)$ by Theorem (III.2.1), where $K \dot{\cap} M_i$ is the frontier of $K \cap M_i$ in M_i and $K \dot{\cap} M_i \subset \dot{K}$.). Since c is given to be a regular value and also since a proper map is closed, so from Theorem (III.2.3) it follows that if c is not a limit point of S then c is not a limit of $f(K)$ also (because $\overline{f(K)} = \bar{S} \cup f(\dot{K})$, $f(\dot{K})$

being closed). That is c lies in the interior of $f(M^*)$. #

Let X be a C^{1-} vector field on an open submanifold $\tilde{M}$ of a C^2 Banach manifold M, M being without boundary. For each $p \in M$, let s_p be the maximal integral curve for X with initial condition p. Define two functions

$$t^+ : \tilde{M} \to (0, \infty) \text{ and } t^- : \tilde{M} \to (-\infty, 0)$$

such that the domain of s_p is $(t^-(p), t^+(p))$. Let

$$D = D(X) = \{(p,t) \in \tilde{M} \times \mathbb{R} \mid t^-(p) < t < t^+(p)\}$$

and for each $t \in \mathbb{R}$ let

$$D_t = D_t(X) = \{p \in \tilde{M} \mid (p,t) \in D\}.$$

Define $g : D \to \tilde{M}$ by $g(p,t) = s_p(t)$ and $g_t : D_t \to \tilde{M}$ by $g_t(p) = g(p,t) = s_p(t)$. The indexed set $\{g_t\}$ is called the <u>maximal local one parameter group generated by X</u>. Also (see [p.2]) D is open in $\tilde{M} \times \mathbb{R}$ and $g : D \to \tilde{M}$ is C^{1-}. For each $t \in \mathbb{R}$, D_t is open in $\tilde{M}$ and g_t is a C^{1-} isomorphism of D_t onto D_{-t} having g_{-t} as its inverse. If $p \in D_t$ and $g_t(p) \in D_s$ then $p \in D_{t+s}$ and $g_{t+s}(p) = g_s(g_t(p))$.

III.2.7. THEOREM. Let M be a complete C^2 Finsler manifold without boundary, $f : M \to \mathbb{R}$ be a C^{2-} function satisfying condition (C) and assume that the internal (a,b) contains no critical values of f, $-\infty \leq a < b \leq \infty$. If $c \in (a,b)$ then

$W = f^{-1}(c)$ is a closed C^2 submanifold of M and there is a C^{1-} homeomorphism F of $W \times (a,b)$ onto $\mathcal{O} = f^{-1}(a,b)$ such that for each $e \in (a,b)$ the map $w \to F(w,e)$ is a C^{1-} homeomorphism of W onto $f^{-1}(e)$ which for $e = c$ is the identity map.

PROOF. Since c is a regular value of f, $W = f^{-1}(c)$ is a C^2 submanifold of M. Let X be a C^{1-} pseudo-gradient vector field for f in M^*. Let $w \in W$ and $s:(\alpha,\beta) \to M^*$ be the maximal integral curve for X with $s(0) = w$. We claim that if $e \in (a,b)$ then there exists a unique $t_o \in (\alpha,\beta)$ such that $f(s(t_o)) = e$. Uniqueness is clear since $f \circ s$ is strictly monotonically increasing (see the proof of (III.2.4)). For existence suppose that there is no such t_o; also assume $c < e$, for definiteness. Now $c = f(s(0)) \leqslant f(s(t)) < e$ for all $t \in [0,\beta)$, by monotonicity and continuity of $f(s(t))$, so by Theorem (III.2.4) there is a critical point p of f and a sequence $\{t_n\} \to \beta$ such that the sequence $\{f(s(t_n))\} \to f(p)$. Also since $a < c \leqslant f(s(t_n)) < e < b$, it follows that $a < f(p) < b$; which is a contradiction, since $f(p)$ is a critical value. Now df, X are C^{1-}, so $Xf = df(X)$ is C^{1-}. Also $Xf \geqslant \|df\|^2 > 0$ on M^* so $1/(Xf)$ is C^{1-} on M^*. Hence $\tilde{X} = X/(Xf)$ is C^{1-} on M^*. Let $\{g_t\}$ be the maximal local one parameter group generated by $\tilde{X}$. So for $p \in M^*$, $t \to g_t(p)$ is a maximal integral curve for $\tilde{X}$ with initial condition p. Since $\tilde{X}$ is proportional to X with a nonzero proportionality factor, $t \to g_t(p)$ is nothing but the maximal integral curve for X with a reparametrisation such that $f(g_t(p)) = f(p)+t$, because $\frac{d}{dt} f(g_t(p)) = \tilde{X} f = 1$. Now note that $\mathcal{O} = f^{-1}(a,b) \subseteq M^*$. So for $p \in \mathcal{O}$, the above claim and the expression $f(g_t(p)) = f(p) + t$ imply that if $a - f(p) < t < b-f(p)$ then $g_t(p)$ is defined and the interval $(a-f(p), b-f(p))$ is in bijective correspondence with (a,b) under the map $t \to f(g_t(p))$. Define $F:W \times (a,b) \to M^*$ by $F(w,t) = g_{t-c}(w)$. Then F is of class C^{1-}. Note that $F(w,c) = w$ and $f(F(w,t)) = t$. So we have $F:W \times (a,b) \to \mathcal{O}$ and also W is mapped into $f^{-1}(e)$ under $w \to F(w,e)$. Also note that the inverse of F is given by $\tilde{F}:\mathcal{O} \to W \times (a,b)$ by $\tilde{F}(p) = (g_{c-f(p)}(p), f(p))$. Hence the theorem follows.

#

III.2.8 THEOREM. Let M be a complete C^2 Finsler manifold without boundary, $f: M \to \mathbb{R}$ be a C^{2-} map satisfying condition(C). Let $-\infty < a \leqslant b < \infty$ and assume that there are no critical values of f in $[a,b]$ (in $[a,\infty)$ if $b = \infty$) and that neither a nor b is a limit point of $S = \{ c \in \mathbb{R} \mid f$ has the constant value c on a component of $M \}$.

Then there is a C^{1-} map $H:M \times I \to M$ such that if we put $H_s(p) = H(p,s)$ then for some $\varepsilon > 0$:

(1) H_s is a C^{1-} homeomorphism of M into $\overline{\text{irself}}$ for all $s \in [0,1)$ (all $s \in [0,1]$ if $b < \infty$)

(2) $H_s(m) = m$ if $m \notin f^{-1}(a - 2\varepsilon, b+2\varepsilon)$

(3) H_0 = identity map of M

(4) $H_1(f^{-1}(-\infty, b+\varepsilon]) = f^{-1}(-\infty, a-\varepsilon]$, if $b<\infty$ and $H_1(M) = f^{-1}(-\infty, a-2\varepsilon]$, if $b=\infty$.

PROOF. By (III.2.6) there is an $\varepsilon > 0$ such that f has no critical values in $(a-3\varepsilon, b+3\varepsilon)$. By (III.2.7) there is a C^{1-} homeomorphism

$F: W \times (a-3\varepsilon, b+3\varepsilon) \approx f^{-1}(a-3\varepsilon, b+3\varepsilon)$ where $W = f^{-1}(a)$ and $F(W \times \{c\}) = f^{-1}(c)$ for all $c \in (a-3\varepsilon, b+3\varepsilon)$.

Case 1. $b=\infty$. Define $H_s(m) = m$ if $f(m) \leq a-2\varepsilon$. If $f(m) \geq a-2\varepsilon$, $m = F(w,t)$ with $t \geq a-2\varepsilon$, then

$$H_s(m) = F(w, t+s(a-2\varepsilon-t))$$

Case 2. $b<\infty$. Let $h: \mathbb{R} \to \mathbb{R}$ be the unique continuous function such that $h(t) = t$ for $t \leq a-2\varepsilon$, h is linear in $[a-2\varepsilon, b+\varepsilon]$ and $h(b+\varepsilon) = a-\varepsilon$, h is linear in $[b+\varepsilon, b+2\varepsilon]$, and $h(b+2\varepsilon) = b+2\varepsilon$, $h(t) = t$ for $t \geq b+2\varepsilon$ Define $H_s(m) = m$ if $m \in f^{-1}(a-2\varepsilon, b+2\varepsilon)$ and for $m \notin f^{-1}(a-3\varepsilon, b+3\varepsilon)$, say $m = F(w,t)$, define $H_s(m) = F(w, t+s(h(t)-t))$.

Hence the theorem follows. #

III.2.9. THEOREM. Let M be a complete C^2 Finsler manifold without boundary, $f: M \to \mathbb{R}$ a C^{2-} function satisfying condition (C) and given $c \in \mathbb{R}$ let $M_c = f^{-1}(-\infty, c]$ and let $K_c = K \cap f^{-1}(c)$ denote the set of critical points of f at the level c.

Then,

(1) If c is a regular value of f and is not a limit point of $\{a \in \mathbb{R} \mid f$ has the constant value a on some component of $M\}$, then for some $\varepsilon > 0$ there is an isotopy g_t of M with $g_1(M_{c+\varepsilon}) = M_{c-\varepsilon}$.

(2) If $c > \sup f(K)$ then there is a strong deformation retraction of M onto $M_{c-\varepsilon}$ for some $\varepsilon > 0$.

PROOF. Statement (1) follows from part (4) of (III.2.8) with $a = b = c$ and statement (2) follows from parts (3) and (4) of (III.2.8) with $a = c$ and $b = \infty$. #

Let X be a C^{1-} pseudo-gradient vector field for f on M^*, $f:M \to \mathbb{R}$ being a C^{2-} function satisfying condition (C). Let $\tilde{X} = X/(Xf)$. Since df is C^{1-} and X is C^{1-}, it follows that $Xf = df(X)$ is C^{1-}. Also since $Xf \geq \| df \|^2 > 0$ on M^* and $t \to 1/t$ is C^∞ for $t \neq 0$, $1/(Xf)$ is C^{1-} on M^*, hence $\tilde{X} = X/(Xf)$ is a C^{1-} vector field on M^*. Also $\tilde{X}f = 1$. Define

$V_k = \{x \in M : \| df_x \| < 1/k\}$ for each positive integer k, so V_k is an open neighborhood of K and $\bar{V}_{k+1} \subseteq V_k$. Let $L_k : M \to [0,1]$ be a C^{1-} function on M which is zero on $\bar{V}_{k+1}$ and 1 on $M-V_k$. Define a C^{1-} vector field $\tilde{X}^k$ on M by $\tilde{X}^k = L_k\tilde{X}$ in M^* and $\tilde{X}^k = 0$ in V_{k+1}.

III.2.10 LEMMA. For each positive integer k, $\tilde{X}^k$ is a bounded C^{1-} vector field on M with $\| \tilde{X}^k \| \leqslant 2(k+1)$; hence by proposition (III.1.10) $\tilde{X}^k$ generates a global one parameter group $\{ g_t^k \}$ of C^{1-} homeomorphisms of M. Moreover if $p \in M$ then $d(g_a^k (p), g_b^k(p)) \leqslant 2(k+1) | b-a |$ and in particular $d(p, g_t^k(p)) \leqslant 2(k+1) | t |$, (d is the Finsler metric.).

PROOF If $\tilde{X}^k(x) \neq 0$ then $x \notin V_{k+1}$ so $\| df_x \| \geqslant 1/k+1$; hence, since X is a pseudo-gradient vector field for f, we have

$$\| \tilde{X}(x) \| = \left\| \frac{X(x)}{X f(x)} \right\| < \frac{2 \| df_x \|}{\| df_x \|^2} = \frac{2}{\| df_x \|} \leqslant 2(k+1).$$

Hence $\| \tilde{X}^k(x) \| = L_k(x) \; \| \tilde{X}(x) \| \leqslant 2(k+1)$.

Now $g_t^k (p)$ with variable t and fixed point p is an integral curve for $\tilde{X}^k$ with the initial condition p and when $a \leqslant t \leqslant b$, $g_t^k(p)$ serves as a path from $g_a^k (p)$ to $g_b^k(p)$.

Therefore $d(g_a^k(p), g_b^k (p)) \leqslant \int_a^b \left\| \frac{d}{dt}(g_t^k(p)) \right\| dt$

$$= \int_a^b \| \tilde{X}^k(g_t^k (p)) \| \; dt \leqslant 2(k+1) \; | b-a |. \quad \#$$

III.2.11 LEMMA. $f(g_a^k (p)) + (b-a) \geqslant f(g_b^k (p)) \geqslant f(g_a^k (p))$.

In particular $f(g_t^k(p))$ is monotone nondecreasing in t.

Moreover if $g_t^k(p) \notin V_k$ for $a \leqslant t \leqslant b$ then $f(g_b^k(p)) - f(g_a^k(p)) = b - a$.

PROOF. $\frac{d}{dt} f(g_t^k(p)) = df_{g_t^k(p)} \cdot \frac{d}{dt}\big(g_t^k(p)\big)$

$$= df_{g_t^k(p)} \cdot \tilde{X}^k(g_t^k(p)) = \tilde{X}^k(g_t^k(p))\, f$$

$$= L_k(g_t^k(p))\, \tilde{X} f = L_k(g_t^k(p)),$$

since $\tilde{X} f = 1$. Hence

$$f(g_b^k(p)) - f(g_a^k(p)) = \int_a^b L_k(g_t^k(p))\, dt .$$

Since $0 \leqslant L_k \leqslant 1$ and since L_k is 1 outside V_k, the lemma follows.

III.2.12 LEMMA. If $\{f(p_n)\}$ is bounded as $n \to \infty$ and $\{t_n\}$ is bounded then $\{f(g_{t_k}^k(p_k))\}$ is bounded.

PROOF. Since $\{f(p_n)\}$ and $\{t_n\}$ are bounded, so there exists real numbers A and B such that $|f(p_n)| \leqslant A$ and $|t_n| \leqslant B$, for all n.

Now $|f(g_{t_k}^k(p_k))| \leqslant |f(g_{t_k}^k(p_k)) - f(g_o^k(p_k))|$

$$+ |f(g_o^k(p_k))| \leqslant |t_k| + |f(p_k)| \leqslant B + A,$$

where the second inequality follows from Lemma (III.2.11) and the ract that $g_o^k =$ identity. Hence $\{f(g_{t_k}^k(p_k))\}$ is bounded.

#

III.2.13 LEMMA. The set

$$U_k = \{ x \varepsilon M \mid | f(x) - c | < \frac{1}{k^2} \text{ and } g^k_t (x) \in V_k$$

for some $t \varepsilon [- \frac{2}{k^2}, 0] \}$ is an open neighborhood of K_c and each neighborhood U of K_c includes a U_k for some k.

PROOF. The first statement is clear. We can assume U to be a closed neighborhood of K_c, (by paracompactness). If no $U_k \subseteq U$, let $\{ p_k \}$ be a sequence with $p_k \varepsilon U_k$ and $p \notin U$. Since $|f(p_k)-c| < 1/k^2$, $f(p_k) \to c$ and since c is not a limit point of $f(\text{Int}(K-K_c))$ for large k, $p_k \notin \text{Int } K$, (note that $p_k \notin K_c$). Replacing p_k by a nearby point we can assume p_k is not a critical point of f (note that U is closed). Since $f(p_k) \to c$, $\{ f(p_k) \}$ is bounded. Choose $t_k \varepsilon [-2/k^2, 0]$ so that $g^k_{t_k} (p_k) \varepsilon V_k$. By the Lemma (III.2.12), $\{ f(g^k_{t_k} (p_k)) \}$ is bounded. By definition of V_k, the norm of df at $g^k_{t_k} (p_k)$ is less than $1/k$. Since for large k, $p_k \notin K$, it follows that for those k, $g^k_{t_k}(p_k) \notin K$ (since K is pointwise invariant under g^k_t). Hence by condition (C), a subsequence of $\{ g^k_{t_k} (p_k) \}$ converges to some point $q \varepsilon K$. Now by the Lemma (III.2.10),

$$d(p_k,q) \leq d(p_k, g^k_{t_k}(p_k)) + d(g^k_{t_k}(p_k), q)$$

$$\leq 2(k+1)|t_k| + d(g^k_{t_k}(p_k), q)$$

$$< 4\frac{(k+1)}{k^2} + d(g^k_{t_k}(p_k), q).$$

Hence the corresponding subsequence of $\{p_k\}$ will also converge to q. Since $f(q) = \lim_{k \to \infty} f(p_k) = c$, $q \in K_c$.

On the other hand since U is a neighborhood of K_c and $p_k \notin U$, no subsequence of $\{p_k\}$ can possibly converge to a point q of K_c. This contradiction proves the lemma.

<u>III.2.14 LEMMA.</u> If $0 < \varepsilon < 1/k^2$ then

$$g^k_{-1}(M_{c+\varepsilon} - U_k) \subset M_{c-\varepsilon}.$$

<u>PROOF.</u> If $x \in M_{c+\varepsilon} - U_k$ then $f(x) \leq c+\varepsilon<c+1/k^2$. Since by Lemma (III.2.11), $f(g^k_{-t}(x))$ is monotone non-decreasing in t, we can assume that $f(x) > c-1/k^2$ (otherwise $f(g^k_{-1}(x)) \leq f(x) \leq c-\frac{1}{k^2} < c-\varepsilon$ and so we are done). Hence $|f(x) - c| < \frac{1}{k^2}$ and so by definition of U_k, $g^k_t(x) \notin V_k$ for $-2/k^2 \leq t \leq 0$. Then by (III.2.11), with $a = -2/k^2$ and $b = 0$,

$$f(g^k_{-1}(x)) \leq f(g^k_{-2/k^2}(x)) = f(x) - \frac{2}{k^2}$$

$$< c+\frac{1}{k^2} - \frac{2}{k^2} = c - 1/k^2 < c - \varepsilon.$$

Hence the lemma follows.

III.2.15 THEOREM. Let M be a complete C^2 Finsler manifold without boun-dary, $f: M \to \mathbb{R}$ a C^{2-} function satisfying the condition (C). If $c \in R$ is not a limit point of $f(\mathrm{Int}(K-K_c))$, then there are arbitrarily small neighborhoods U of K_c and arbitrarily small $\varepsilon>0$ such that there exists an isotopy $\tilde{g}_t$ of M with $\tilde{g}_1(M_{c+\varepsilon} - U) \subset M_{c-\varepsilon}$.

PROOF. By Lemmas (III.2.13) and (III.2.14), we have arbitrarily small neighborhoods U_k of K_c and arbitrarily small $\varepsilon > 0$ such that the map $\tilde{g}: M \times I \to M$, given by $\tilde{g}(x,t) = \tilde{g}_t(x) = g^k_{-t}(x)$, is an isotopy of M with $\tilde{g}_1(M_{c+\varepsilon} - U_k) = g^k_{-1}(M_{c+\varepsilon} - U_k) \subset M_{c-\varepsilon}$. #

Let $\mathcal{F}$ be a family of subsets of a Banach manifold M. We shall say $\mathcal{F}$ is isotopy invariant if, given $F \in \mathcal{F}$ and an isotopy $\{g_t\}$ of M, $g_1(F) \in \mathcal{F}$. If $f: M \to \mathbb{R}$ is a function, we define minimax$(f,\mathcal{F})$ the minimax of f relative to $\mathcal{F}$, by

$$\operatorname{minimax}(f,\mathcal{F}) = \operatorname*{Inf}_{F \in \mathcal{F}} \operatorname{Sup}\{ f(x) \mid x \in F \}.$$

Equivalently, putting $M_a = f^{-1}(-\infty, a]$,

$$\operatorname{minimax}(f,\mathcal{F}) = \operatorname{Inf}\{ a \in R \mid \exists\ F \in \mathcal{F} \text{ with } F \subset M_a \}.$$

The following theorem gives critical values of f in terms of minimax $(f, \mathcal{F})$.

III.2.16 MINIMAX THEOREM. Let M be a complete C^2 Finsler manifold without boundary and $f: M \to \mathbb{R}$ a C^{2-} function satisfying condition (C). Let $\mathcal{F}$ be an isotopy invariant family of subsets of M such that $-\infty <$ minimax $(f, \mathcal{F}) < \infty$. Then either minimax $(f, \mathcal{F})$ is a critical value of f or else there is a sequence of distinct critical values $\{ c_k \}$ of f such that $c_k \to$ minimax $(f, \mathcal{F})$ and f assumes the constant value c_k on a component of M.

PROOF. Suppose that $c =$ minimax $(f, \mathcal{F})$ is neither a critical value of f nor a limit point of the set $\{a \in \mathbb{R} \mid f$ assumes the constant value a on a component of $M.\}$. Choose $\varepsilon > 0$ satisfying (1) of Theorem (III.2.9) and $F \in \mathcal{F}$ with $F \subseteq M_{c+\varepsilon}$. Then there is an isotopy g_t of M with $g_1(F) \subseteq M_{c-\varepsilon}$. Since $g_1(F) \in \mathcal{F}$ and $c =$ minimax $(f, \mathcal{F})$, so we have $c \leqslant c - \varepsilon$; which is absurd. Hence the theorem follows. #

§3. Main Theorem of Lusternik Schnirelmann Theory.

In this section we prove the main theorem of Lusternik and Schnirelmann regarding lower bound of the number of critical points of $f: M \to \mathbb{R}$.

III.3.1 THEOREM. Let M be a complete C^2 Finsler manifold without boundary and $f: M \to \mathbb{R}$ be a C^{2-} map satisfying condition (C). Assume for each $a \in R$ that a is not a limit point of $f(\text{Int}(K - K_a))$. For each positive integer $m \leqslant \text{cat } M$, define

$$c_m(f) = \text{Inf } \{ a \in R \mid \text{cat}_M(M_a) \geqslant m \}$$

Then,

(1) $c_1(f) = \text{Inf } \{ f(x) \mid x \in M \}$. In particular, if f is bounded below then $c_1(f) = \min \{ f(x) \mid x \in M \}$.

(2) $c_m(f) \leqslant c_{m+1}(f)$.

(3) If $-\infty < c_m(f) < \infty$ then $c_m(f)$ is a critical value of f.

(4) If some $c_m(f) = \infty$ then f is unbounded on K and hence K is infinite. In fact

$$c_m(f) \leqslant \text{Sup } \{ f(x) \mid x \in K \}$$

(5) If $0 < m < n \leqslant \text{cat } M$ and

$$-\infty < c = c_m(f) = c_n(f) < \infty ,$$

then $\text{cat}_M K_c \geqslant n-m+1$. In particular if M is connec ted then $\dim K_c \geqslant n-m$.

PROOF. Clearly $\text{cat}_M\ M_a \geqslant 1$ if $a > \text{Inf } \{ f(x) \mid x \in M \}$ and conversely if $\text{cat}_M\ M_a \geqslant 1$ then $a \geqslant \text{Inf } \{ f(x) \mid x \in M \}$.

Hence the statement (1) follows. The particular case in (1) follows from (III.2.5); the second conclusion of (III.2.5) is ruled out as $c_1(f)$ is not a limit point of $f(\mathrm{Int}(K-K_{c_1(f)}))$.

Statement (2) is immediate because if $\mathrm{cat}_M M_a \geqslant m+1$ then obviously $\mathrm{cat}\, M_a \geqslant m$ and so

$$\{a \in \mathbb{R} \mid \mathrm{cat}_M M_a \geqslant m+1\} \subset \{a \in \mathbb{R} \mid \mathrm{cat}_M M_a \geqslant m\}$$

and therefore

$$\mathrm{Inf}\, \{a \in \mathbb{R} \mid \mathrm{cat}_M M_a \geqslant m\} \leqslant \mathrm{Inf}\, \{a \in R \mid \mathrm{cat}_M M_a \geqslant m+1\}$$

To prove (3), let $\mathcal{F}_m = \{F \subseteq M \mid \mathrm{cat}_M F \geqslant m\}$.

Note, by monotonicity of cat, that $M_a \in \mathcal{F}_m$ iff there exists $F \in \mathcal{F}_m$ with $F \subseteq M_a$, hence

$$c_m(f) = \mathrm{Inf}\, \{a \in R \mid \exists\ F \in \mathcal{F}_m \text{ with } F \subseteq M_a\}$$

$$= \mathrm{minimax}\, (f, \mathcal{F}_m).$$

Now if $F \subseteq M$ and g_t is an isotopy of M then trivially $\mathrm{cat}_M g_1(F) = \mathrm{cat}_M F$, so $\mathcal{F}_m$ is isotopy in variant. Statement (3) now follows from (III.2.16); the second conclusion of (III.2.16) is ruled out as $c_m(f)$ is not a limit point of $f(\mathrm{Int}(K-K_{c_m(f)}))$.

To prove (4), notethat if $c > \sup\{f(x) \mid x \in K\}$ then by (2) of (III.2.9) and by (a) and (c) of (I.1.3) we have for some $\varepsilon > 0$, $\mathrm{cat}_M\, M_{c-\varepsilon} = \mathrm{cat}\, M$; so that if $m \leqslant \mathrm{cat}\, M$ then $c_m(f) \leqslant c-\varepsilon < c$. Hence

$$c_m(f) \leqslant \sup\{\, f(x) \mid x \in K \,\}\ .$$

In order to prove (5) we first show that, for some $\varepsilon > 0$, $\mathrm{cat}_M\, K_c \geqslant \mathrm{cat}_M\, M_{c+\varepsilon} - \mathrm{cat}_M\, M_{c-\varepsilon}$. Note that there is always a neighborhood U of K_c in M with $\mathrm{cat}_M\, U = \mathrm{cat}_M\, K_c$ (from definition of category of a space). Then by (III.2.15) there is an isotopy g_t of M with $g_1(M_{c+\varepsilon} - U) \subset M_{c-\varepsilon}$ for some $\varepsilon > 0$. Then $\mathrm{cat}_M\, M_{c-\varepsilon} \geqslant \mathrm{cat}_M\, g_1(M_{c+\varepsilon} - U) = \mathrm{cat}_M(M_{c+\varepsilon} - U)$, by monotonicity and invariance under homeomorphism of category of a space. Again by monotonicity and subadditivity of category we have

$$\mathrm{cat}_M\, M_{c+\varepsilon} \leqslant \mathrm{cat}_M\,(M_{c+\varepsilon} \cup U)$$

$$\leqslant \mathrm{cat}_M\,(M_{c+\varepsilon} - U) + \mathrm{cat}_M\, U \leqslant \mathrm{cat}_M\, M_{c-\varepsilon} + \mathrm{cat}_M\, K_c .$$

Therefore, $\mathrm{cat}\, K_c \geqslant \mathrm{cat}_M\, M_{c+\varepsilon} - \mathrm{cat}_M\, M_{c-\varepsilon}$. Since $c = c_n(f)$, so $\mathrm{cat}_M\, M_{c+\varepsilon} \geqslant n$ and since $c = c_m(f)$, so $\mathrm{cat}_M\, M_{c-\varepsilon} \leqslant m-1$. Hence, $\mathrm{cat}\, K_c \geqslant \mathrm{cat}_M\, M_{c+\varepsilon} - \mathrm{cat}_M\, M_{c-\varepsilon}$

$$\geqslant n-m+1.$$

The particular case follows from Theorem (I.2.4) since M is an ANR [p.3;Thm 5] . #

Now we are ready to prove the final result of Lusternik - Schnirelmann theory.

III.3.2. THEOREM. Let M be a complete C^2 Finsler manifold without boundary and $f : M \to \mathbb{R}$ a C^{2-} map satisfying condition (C). If f is bounded below (or, above) then f has at least cat M critical points.

PROOF. If some $c \in R$ is a limit point of $f(\mathrm{Int}(K-K_c))$ then f has infinitely many critical points so we can eliminate this case. For the same reason by (4) of (III.3.1) we can suppose $c_m(f) < \infty$ if $m \leq \mathrm{cat}\, M$. We can also assume f is bounded below (otherwise consider $-f$) so that by (1) of (III.3.1) $-\infty < c_1(f)$ and hence by (2) of (III.3.1) $-\infty < c_m(f) < \infty$ for all $m = 1,2,---, \mathrm{cat}\, M$. We shall prove by induction on m that there are at least m critical points of f in $M_{c_m(f)}$, $1 \leq m \leq \mathrm{cat}\, M$. For $m = 1$, this is immediate from (1) of (III.3.1), since an absolute minimum of f is certainly a critical point. Now suppose there are at least k critical points in $M_{c_k(f)}$ if $1 \leq k \leq n < m$. If $c_n(f) \neq c_{n+1}(f)$ then there is at least one critical point of f in $f^{-1}(c_{n+1}(f))$ by (3) of (III.3.1), and it is clearly different from those in $M_{c_n(f)}$. Hence there are at least $(n+1)$ critical points in $f^{-1}(c_{n+1}(f)) \cup M_{c_n(f)} \subseteq M_{c_{n+1}(f)}$. If $c = c_n(f) = c_{n+1}(f)$, let p be the least positive integer such that $c_p(f) = c_{n+1}(f)$. Then by (5) of (III.3.1) $\mathrm{cat}_M K_c \geq n+2-p$, so that

$$\mathrm{card}\,(K_c) \geq \mathrm{cat}_M K_c \geq n+2-p$$

i.e. there are at least n+2-p critical points on the level $c = c_{n+1}(f)$. If p = 1 we are done. If p> 1 then there are at least (p-1) critical points of f in $M_{c_{p-1}(f)}$, by induction hypothesis (note that $p-1 < n$ and also $c_{p-1}(f) \neq c_{n+1}(f)$). Hence there are at least (p-1) + (n+2-p) = n+1 critical points of f in

$$M_{c_{p-1}(f)} \cup f^{-1}(c_{n+1}(f)) \subseteq M_{c_{n+1}(f)} \quad .$$

Hence the theorem follows. #

III.3.3 REMARK. Let $\mu_M(f)$ be the number of critical points of a function $f:M \to \mathbb{R}$. Let $F(M) = \operatorname{Inf} \{ \mu_M(f) : f$ is constant maximal and regular on the boundary $\partial M \}$.

Lusternik and Schnirelmann [L.1] proved the inequality $F(M) \geq \operatorname{cat} M$ for closed manifolds M. However it is also known that the inequality $F(M) \geq \operatorname{cat} M$ holds even for the manifolds with boundary. Takens [T.2] considered the following problem:

For which manifolds M^n are $\operatorname{cat} M^n$ and $F(M^n)$ equal? Smale, [S.7] and [S.8] , found an answer for each of the following two special cases.

(i) Let M^n be a closed n-manifold. Does $\operatorname{cat} M^n = 2$ imply $F(M^n) = 2$? The answer is affermative for $n \geq 5$.

(ii) Let M^n be an n-dimensional compact manifold with boundary. Does cat $M^n = 1$ imply $F(M^n) = 1$? The answer is affermative if $\dim(M^n) \geqslant 6$ and $(M^n, \partial M^n)$ is 2-connected.

Takens found answers to the following cases:

For which manifolds M^n with boundary (resp. without boundary) with cat $M^n = 2$ (resp. 3) is $F(M^n)=2$ (resp. 3)?

§ 4. $\mathcal{A}$-category and the topology of critical sets.

We shall now discuss about a new generalisation namely $\mathcal{A}$-category, as introduced by Clapp and Puppe [c.1] , in order to have some informations about the topological structure of the set of critical points.

III.4.1 DEFINITION. Let $\mathcal{A}$ be a class of spaces which contains at least one nonempty space. We shall say that a subspace U of a space X is *deformable in X to $\mathcal{A}$* if the inclusion $i:U \hookrightarrow X$ factors through some space in $\mathcal{A}$ upto homotopy, i.e. if there exist $A \in \mathcal{A}$ and maps $a:U \to A$ and $b: A \to X$ such that $ba \simeq i$. A finite numerable covering $\{ U_1, U_2, ---, U_n\}$ of X, such that each U_t, $1 \leqslant t \leqslant n$, is deformable in X to $\mathcal{A}$, will be called an *$\mathcal{A}$-categorical covering* of X. We define the *$\mathcal{A}$-category ($\mathcal{A}$-cat(X))* of X to be the smallest cardinality n of such a covering. If no such covering exists, we say that $\mathcal{A}$-cat(X) $= \infty$. More generally : the *$\mathcal{A}$-category of a map* f: X → Y, denoted *$\mathcal{A}$-cat(f),* is the smallest cardinality n of a finite numerable covering $\{U_1, U_2, ---, U_n\}$ of X such that for each $t = 1,2,---, n$ the restriction $f \mid U_t : U_t \to Y$ factors through some space in $\mathcal{A}$ upto homotopy. Such a covering will be called an *$\mathcal{A}$-categorical covering associated to f.* Again if no such covering exists then $\mathcal{A}$-cat(f) $= \infty$.

III.4.2 EXAMPLES. (1) Let $\mathcal{P}$ be the class which consists only of the one point space. Then $\mathcal{P}$-cat(X) is nothing but the Lusternik - Schnirelmann category cat (X) of X , except that the coverings of X considered in the definition (I.1.1) of category of a space were open instead of numerable. If X is normal then $\mathcal{P}$-cat(X) is exactly equal to cat X.

(2) Let $\mathfrak{D}^q$ be the class of all CW-complexes of dimension $\leq q$. We denote by $\text{cat}^q(X)$ the $\mathfrak{D}^q$- category of X.

We now mention a few elementary properties of $\mathcal{A}$-category :

III.4.3. PROPOSITION.

(i) $\mathcal{A}\text{-cat}(1_X) = \mathcal{A}\text{-cat}(X)$, $1_X : X \to X$ is the identity map.

(ii) If f:X → Y be any map then $\mathcal{A}$-cat(f) $\leq \min \{\mathcal{A}\text{-cat } X, \mathcal{A}\text{-cat } Y\}$ and $\mathcal{A}\text{-cat}(f) = 0$ iff $X = \emptyset$.

(iii) $\mathcal{A}$-cat$(X) = 1$ iff X is dominated by some space in $\mathcal{A}$ and $X \neq \emptyset$.

(iv) For any two maps $f: X \to Y$ and $g: Y \to Z$, $\mathcal{A}$-cat$(gf) \leq \min \{ \mathcal{A}\text{-cat}(f), \mathcal{A}\text{-cat}(g) \}$.

(v) If $\{ X_1, X_2 \}$ is a numerable covering of X then for any map $f: X \to Y$,
$\mathcal{A}$-cat$(f) \leq \mathcal{A}\text{-cat}(f|X_1) + \mathcal{A}\text{-cat}(f|X_2)$.

(vi) If $f \simeq g : X \to Y$ then
$\mathcal{A}$-cat$(f) = \mathcal{A}$-cat(g).

<u>PROOF.</u> (i), (ii), (iii), (iv) and (vi) are simple consequences of the definition.

For (V), simply note that

if $\{U_i, g_i\}_{i=1}^{m}$ and $\{V_j, h_j\}_{j=1}^{n}$ are numerable coverings of X_1 and X_2 respectively and if $\{k_1, k_2\}$ is a partition of unity subordinate to $\{X_1, X_2\}$ then $\{U_i\}_{i=1}^{m}$ $\{V_j\}_{j=1}^{n}$ forms a numerable covering for X, where a partition of unity $\{l_t\}_{t=1}^{m+n}$ subordinate to the new covering may be constructed as follows:

If $1 \leq t \leq m$, then define

$$l_t(x) = \begin{cases} k_1(x) g_t(x), & \text{if } x \in U_t \\ 0, & \text{if } x \notin \text{support } g_t. \end{cases}$$

and if $m+1 \leq t \leq m+n$, then define

$$l_t(x) = \begin{cases} k_2(x)\, h_{t-m}(x)\ , & \text{if } x \in V_{t-m} \\ 0\ , & \text{if } x \notin \text{support } h_{t-m} \end{cases}$$

Hence (v) follows. #

A formal consequence of (i), (ii), (iv) and (vi) of (III.4.3) is the following.

III.4.4. PROPOSITION. If X is dominated by Y then $\mathcal{A}$-cat$(X) \leq \mathcal{A}$-cat (Y). In particular $\mathcal{A}$-category is an invariant of the homotopy type.

Finally observe that if $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{B}$-cat(f) $\leq \mathcal{A}$-cat(f) for any map f. Applying this we have from (III.4.2),

$$\text{cat}^{q+1}(f) \leq \text{cat}^{q}(f) \leq \text{cat } f.$$

We now study the relationship between $\overline{\mathcal{A}}$-category and the set of critical points of differentiable functions.

Let M be a paracompact C^1- Banach manifold (possibly with boundary) and let $f: M \to \mathbb{R}$ be a C^1- function. Let K be the set of critical points of f. Our aim is to show, by extending the classical Lusternik-Schnirelmann method (Section 1 to 3 of chapter III), how $\mathcal{A}$-category may be used to obtain new informations about the topology of the sets K_a, $a \in \mathbb{R}$. For this purpose we need some assumptions on f and M. Consider the following deformation conditions:

(D_1) For any $a \in \mathrm{Int}(\mathbb{R}-f(K))$ there is an $\varepsilon > 0$ such that $M_{a+\varepsilon}$ is deformable into $M_{a-\varepsilon}$.

(D_2) For any isolated critical value a of f and any neighborhood V of K_a there is an $\varepsilon > 0$ such that $M_{a+\varepsilon} - V$ is deformable into $M_{a-\varepsilon}$.

(D_3) If $a > \sup f(K)$ then M is deformable into M_a.

We call (D_1), (D_2) and (D_3) together the generalised Palais-Smale conditions (GPS).

Now we are ready to state the new version of the main result of Lusternik and Schnirelmann (Theorem (III.3.1)). We shall write $\mathcal{A}\text{-cat}_M(X)$ for $\mathcal{A}\text{-cat}\,(X \quad M)$. Consider first the following lemma.

III.4.5. LEMMA. Let M be a paracompact C^1 - Banach manifold and $\mathcal{A}$ be a class of spaces having the homotopy type of CW-complexes. The set function defined as

$$M \supset X \to n(X) = \mathcal{A}\text{-cat}_M\,(X)$$

has the following properties.

(1) MONOTONICITY: If $X' \subset X \subset M$ then $n(X') \leq n(X)$.

(2) SUBADDITIVITY: If X_1, X_2 form a numerable covering $X \subset M$ then $n(X) \leq n(X_1) + n(X_2)$.

(3) DEFORMATION INVARIANCE: If $X \subset M$ is deformable in M into X' then $n(X) \leq n(X')$.

(4) CONTINUITY: If X is closed in M then there is a neighborhood U of X such that $n(U) = n(X)$.

PROOF. Properties (1), (2) and (3) can be easily obtained from parts (iv), (v) and (vi) of Proposition (III.4.3) with suitable changes.

(4) is proved using the fact that M is an ANR [p.3; Thm 5] and CW-complexes are ANE [M.6] where by an ANE (absolute neighborhood extensor) we mean a topological space E such that every map from any closed subspace A of a paracompact space Z into E can be extended to a neighborhood of A (in Z). To be precise, let $k = \mathcal{A}\text{-cat}_M(X)$ and $\{X_1, ---, X_k\}$ be an $\mathcal{A}$-categorical covering associated to $X \subset M$ i.e. a numerable covering of X such that there are $A_j \in \mathcal{A}$ and maps $a_j : X_j \to A_j$, $b_j : A_j \to M$ with $b_j a_j$ homotopic to the inclusion $X_j \to M$, $\alpha = 1,2,---k$. Choose open subsets U''_j of M such that $\overline{X_j} = U''_j \cap X$. Then X_j is closed in the paracompact space U''_j of M (note that M is metrisable). Since $A_j \in \mathcal{A}$, it has the homotopy type of a ANE. So it follows that upto homotopy a_j may be extended to a map $\overline{a}_j : U'_j \to A_j$, where U'_j is an open neighborhood of X_j in U''_j. Consider $b_j \overline{a}_j : U'_j \to M$. Since M is an ANR, there is an open neighborhood U_j of X_j in U'_j such that $b_j \overline{a}_j \mid U_j$ is

homotopic to the inclusion $U_j \hookrightarrow M$. Now $\{U_1, ---, U_k\}$ is an $\mathcal{A}$-categorical covering of its union U associated to the inclusion $U \hookrightarrow M$. Hence $\mathcal{A}\text{-cat}_M\, U \leq k$; and since $X \subset U$, by monotonicity (1) we have $k = \mathcal{A}\text{-cat}_M\, X \leq \mathcal{A}\text{-cat}_M\, U$. Hence (4) follows. #

<u>III.4.6 THEOREM.</u> Let M be a paracompact C^1- Banach manifold and $f: M \to \mathbb{R}$ a C^1 function satisfying the GPS condition. Consider the function

$$m: \mathbb{R} \to N \cup \{\infty\} \qquad \text{defined by}$$

$$m(a) = \mathcal{A}\text{-cat}_M\,(M_a)$$

where $\mathcal{A}$ is a class of spaces having the homotopy type of CW-complexes. Then

(i) The function m is monotonically increasing

(ii) The function m is locally constant in the interior of the set of regular values of f.

(iii) At any $a \in \mathbb{R}$, which is an isolated critical value of f, the function m jumps at most by $\mathcal{A}\text{-cat}_M\,(K_a)$.

(iv) $m(a) = \mathcal{A}\text{-cat}(M)$ for all $a > \operatorname{Sup} f(K)$.

<u>PROOF.</u> (i) follows directly from (1) of (III.4.5), (ii) from (1) and (3) of (III.4.5) using (D_1), (iv) also follows from (1) and (3) of (III.4.5) using (D_3). To prove (iii) take a neighborhood U of K_a such that $n(U) = n(K_a)$. Let V be a closed neighborhood of K_a in the interior of U and

choose $\varepsilon > 0$ as in (D_2). Then,

$$\begin{aligned} m(a+\varepsilon) &= n(M_{a+\varepsilon}) \\ &\leq n(M_{a+\varepsilon} - V) + n(U) \\ &\leq n(M_{a-\varepsilon}) + n(K_a) \\ &= m(a-\varepsilon) + \mathcal{A}\text{-cat}_M(K_a), \end{aligned}$$

using (2) and (3) of (III.4.5) together with (D_2). Hence the theorem follows. #

III.4.7. COROLLARY. If $f:M \to \mathbb{R}$ and $\mathcal{A}$ are as in the Theorem (III.4.6) and if in addition f is bounded below then

$$\mathcal{A}\text{-cat}(M) \leq \sum_{a \in \mathbb{R}} \mathcal{A}\text{-cat}_M(K_a)$$

PROOF. If f has infinitely many critical values then there is nothing to prove. Otherwise every $a \in \mathbb{R}$ is either a regular value or an isolated critical value of f. Choosing $c < \operatorname{Inf} f(M)$ and $b > \operatorname{Sup} f(K)$, we have $m(c) = 0$ and

$$\begin{aligned} \mathcal{A}\text{-cat}(M) &= m(b), \text{ by (iv) of (III.4.6)} \\ &= m(b) - m(c) \\ &\leq \sum_{a \in \mathbb{R}} \mathcal{A}\text{-cat}_M(K_a), \text{ by} \end{aligned}$$

(ii) and (iii) of (III.4.6), noting that (ii) of (III.4.6) implies that m is constant between any two consecutive critical values, (also note that, here $\operatorname{Inf} f(M) = \operatorname{Inf} f(K)$.). #

III.4.8 REMARK. The corollary (III.4.7) implies in particular that $\mathcal{A}$-cat (M) is a lower bound for the number of critical points of f; but since $\mathcal{A}$-cat(M) $\leq$ cat M, this is already known (III.3.2). But a useful consequence of (III.4.7) is that if $\mathcal{A}$ and f are as in (III.4.7) then either f has at least $\mathcal{A}$-cat M critical values or there is a critical value $a \in \mathbb{R}$ of f such that $\mathcal{A}\text{-cat}_M (K_a) > 1$. This means that the set K_a and hence the whole set K of critical points of f cannot be deformed in M to $\mathcal{A}$. In particular K_a and hence K is not dominated by any space in $\mathcal{A}$.

Following simple proposition explains what does the above remark mean in the examples given in (III.4.2).

III.4.9 PROPOSITION. Let M be a paracompact C^1 Banach manifold and $f: M \to \mathbb{R}$ be a C^1 function satisfying GPS. Then assuming f to be bounded below we have

(i) If f has less than cat M critical values then the set K of critical points of f is not contractible in M.

(ii) If f has less than $\text{cat}^q(M)$ critical values then the set K has covering dimension greater than q.

PROOF. (i) follows immediately from (III.4.8) taking $\mathcal{A}$ to be the class $\mathcal{P}$ consisting only of the one point space.

For (ii) we first note that by [p.3; Thm 14] there exists a simplicial complex S such that M is dominated by $|S|$, the associated CW-complex, i.e. there exist maps $h: M \to |S|$ and $\tilde{h} : |S| \to M$ with $\tilde{h} \circ h \simeq 1d_M$. Now take $\mathcal{A} = \mathfrak{D}^q$, the class of all CW-complexes of dimension $\leq q$. If possible, suppose that $\dim K \leq q$. Then by [S.3; Ex5, p152] there exists an open covering U of K, in which at the most $(q+1)$ members meet, and a simplicial map $g: N_U \to S$ such that $h \circ i \simeq |g| \circ t$, where N_U is the nerve of U, $|g| : |N_U| \to |S|$ is induced by $\underline{g}$, $i: K \to M$ is the inclusion and $t: K \to |N_U|$ is a canonical map. Clearly, the CW-complex $|N_U|$ is of dimension $\leq q$ and $i \simeq \tilde{h} \circ |g| \circ t$. That is i factors through a member of $\mathfrak{D}^q$, upto homotopy, which is a contradiction by (III.4.8). Hence $\dim K > q$. #

Throughout this chapter, unless otherwise mentioned, M will denote a paracompact, pathconnected C^1- Banach manifold on which a compact Lie group G acts.

<u>IV.1.1. DEFINITION.</u> If X is an invariant subspace of M, a homotopy $H:X \times I \to M$ is called <u>equivariant</u> if $H(gx,t) = g\, H(x,t)$, $x \in X$, $g \in G$. X is called <u>G-categorical</u> if there is an equivariant homotopy $H:X \times I \to M$ such that $H_0: X \to M$ is inclusion and $H_1: X \to M$ has image in a single orbit <u>orbx</u>, where orbx = $Gx = \{ gx: g \in G \}$ is the orbit of x.

<u>IV.1.2 DEFINITION.</u> If X is an invariant subset of M we set $G\text{-cat}_M X = n$ if X can be covered by n G-categorical open subsets of M and n is minimal with this propety. If X cannot be covered by a finite number of G-categorical subsets, we set $G\text{-cat}_M X = \infty$.

For simplicity we shall write G-cat M in place of $G\text{-cat}_M M$.

<u>IV.1.3 REMARK.</u> If G acts trivially on M, then G-category and (Lusternik - Schnirelmann) category coincide i.e.

$$G\text{-cat}_M X = \text{cat}_M X.$$

Moreover, for an arbitrary action of G, one can easily see that

$$G\text{-cat}_M X \geq \text{cat}_{\widetilde{M}} \widetilde{X} ,$$

where $\widetilde{X} = X/G$ and $\widetilde{M} = M/G$, by passing onto the quotients. In particular if the action is free then by the homotopy lifting property one can conclude that

$$G\text{-cat}_M X = \text{cat}_{\widetilde{M}} \widetilde{X}.$$

Thus using Krasnosielski's result [K.1] , that $\text{cat}\, S^n/G = n+1$ for all finite nontrivial group G acting freely on the n-sphere S^n , we immediately get $G\text{-cat}\, S^n = n+1$ if G is any finite nontrivial group acting freely on S^n.

The following proposition gives some elementary properties of G-category.

<u>IV.1.4. PROPOSITION.</u> Let S denote the set of invariant subsets of M and $h: S \to \mathbf{Z}^+$ the set function $h(X) = G\text{-cat}_M X$, Z^+ being the set of nonnegative integers. Then h has the following properties.

(i) <u>NORMALIZATION.</u> If $X \in S$ is a G-categorical open (or, closed) subset of M then $h(X) = 1$.

(ii) <u>MONOTONE.</u> If $X, Y \in S$ and $X \subset Y$, then $h(X) \leq h(Y)$.

(iii) <u>SUBADDITIVITY.</u> If $X, Y \in S$ then $h(X \cup Y) \leq h(X) + h(Y)$.

(iv) <u>INVARIANCE.</u> If $t: M \to M$ is an equivariant homeomorphism and $X \in S$, then $h(X) = h(t(X))$.

(v) CONTINUITY. If $X \in S$, then there is an open set $U \in S$, $U \supset X$ such that $h(U) = h(X)$.

(vi) ORBIT COUNT. If $h(X) = n$, $X \in S$, then X contains at least n orbits.

PROOF. Immediate from the definition (for the parenthetical part of (i) note that M is a G-ANR).

IV.1.5 OBSERVATION. Among the Z^+- valued functions on S satisfying properties (i), (ii) and (iii) above, G-category is maximal. For suppose $\overline{h}$ be any other such Z^+-valued function satisfying (i), (ii) and (iii) above. Then if $X \in S$ and $\text{G-cat}_M X = n < \infty$, we have $X \subset U_1 \cup U_2 \cup \ldots \cup U_n$, where U_i are G-categorical.

$$\text{Then } \overline{h}(X) \leqslant \overline{h}(U_1 \cup \ldots\ldots \cup U_n)$$

$$\leqslant \sum_{i=1}^{n} \overline{h}(U_i) = n = \text{G-cat}_M X.$$

§ 2. Role of G-category in the theory of critical points.

We shall discuss about the so called equivariant Lusternik - Schnirelmann method for invariant functionals, as illustrated by Fadell.

Let $f: M \to \mathbb{R}$ denote a C^1- functional which is invariant in the sense that $f(gx) = f(x)$, $x \in M$, $g \in G$.

An orbit <u>orbx is called critical</u> if the derivative of f is zero on orbx. As before, let K be the set of critical points of f and $K_c = K \cap f^{-1}(c)$, $M_c = f^{-1}(-\infty, c]$. Our aim is to study how the number of critical orbits in M is determined with the help of G-category. Because of the generality of the situation, namely M is a pathconnected paracompact C^1 - Banach G-manifold, Fadell made certain technical assumptions on f, which he refers as the <u>generalised Palais-Smale</u> condition (GPS).

<u>IV.2.1 DEFINITION.</u> A C^1 - functional $f: M \to \mathbb{R}$ is said to be <u>generalised Palais-Smale</u> (GPS) if $f \mid K$ is proper and for every $c \in \mathbb{R}$, $\bar{\varepsilon} > 0$ and neighborhood U of K_c, there exists an $\varepsilon > 0$, $\varepsilon < \bar{\varepsilon}$ and an equivariant homeomorphism $t: M \to M$ such that

(i) $t(x) = x$ if $|f(x) - c| \geqslant \bar{\varepsilon}$

(ii) $t(M_{c+\varepsilon} - U) \subset M_{c-\varepsilon}$, with the convention that $K_c = \phi$ implies $U = \phi$.

It may be worthwhile to mention that the GPS condition given in chapter III is slightly different from this GPS condition.

Let $\underset{\sim}{F}$ denote the set of closed invariant subsets of M and $\underset{\sim}{F}_j$ the subset of $\underset{\sim}{F}$ containing those $X \in \underset{\sim}{F}$ for which $\text{G-cat}_M X \geqslant j$. The Lusternik-Schnirelmann method begins by defining

$$c_j = \inf_{X \in \underset{\sim}{F}_j} \sup_{x \in X} f(x) \qquad , \quad 1 \leqslant j \leqslant n ,$$

where $n = \text{G-cat } M$. When $\text{G-cat } M = \infty$ it is understood that $1 \leqslant j < \infty$. Note that $c_j \leqslant c_{j+1}$. In order that the numbers c_j are well-defined, Fadell made the following two more assumptions, one on the G-manifold M and the other on the functional f.

(M) for each positive integer $N \leqslant n = \text{G-cat } M$ (or $N < \infty$ when $n = \infty$) there is a compact invariant set X_N with $\text{G-cat}_M\ X_N \geqslant N$.

(f) f is bounded below.

In the next step we will see that each value c_j is a critical value, that is the set $K_{c_j} \neq \emptyset$. Finally we will see that if

$$c_{j+1} = c_{j+2} = \text{------} = c_{j+k} = c \quad \text{then}$$

$\text{G-cat}_M\ K_c \geqslant k$. Then from this, the fact that f has at least $n = \text{G-cat } M$ critical orbits, follows immediately. These two steps are usually combined into one by observing that only the last step is required since $\text{G-cat}_M K_c \geqslant 1$ implies that $K_c \neq \emptyset$. Everything is illustrated in the following result.

<u>IV.2.2 THEOREM.</u> Let $f: M \to \mathbb{R}$ denote an invariant functional, M being a Banach G-manifold. Assume f is (GPS)

and the (M) and (f) as given above are satisfied. Then f possesses at least n = G-cat M critical orbits.

PROOF. As we have already observed that the numbers

$$c_j = \inf_{X \in \underset{\sim}{F}_j} \sup_{x \in X} f(x) \quad , \quad 1 \leqslant j \leqslant n$$

are well defined and it suffices to estimate the G-category of K_c, where

$$c = c_{j+1} = \text{-----} = c_{j+k} \ .$$

Suppose $\text{G-cat}_M K_c < k$. Choose an open invariant set U containing K_c so that

$$\text{G-cat}_M K_c = \text{G-cat}_M U \ ,$$

by part (V) of (IV.1.4). Let $\overline{\varepsilon} = 1$. Then by GPS condition there exists $0 < \varepsilon < 1$ and an equivariant homeomorphism $t: M \to M$ such that $t(M_{c+\varepsilon} - U) \subset M_{c-\varepsilon}$. Let $X \in \underset{\sim}{F}_{j+k}$ be such that $f(x) < c+\varepsilon$, $x \in X$. Then

$$\begin{aligned} j+k &\leqslant \text{G-cat}_M \ X \\ &\leqslant \text{G-cat}_M (X-U) + \text{G-cat}_M U \\ &\leqslant \text{G-cat}_M (X - U) + \text{G-cat}_M K_c \end{aligned}$$

by using parts (ii) and (iii) of (IV.1.4). This implies that $\text{G-cat}_M (X-U) \geqslant j+1$, as we have assumed that $\text{G-cat}_M K_c < k$. But this again implies that

$\text{G-cat}_M\, t(X-U) \geqslant j+1$, by (iv) of (IV.1.4).

Since $f(x) < c+\varepsilon$, for all $x \in X$, $t(X-U) \subset t(M_{c+\varepsilon} - U) \subset M_{c-\varepsilon}$ so that $\text{G-cat}_M\,(M_{c-\varepsilon}) \geqslant j+1$.

Therefore,

$$c_{j+1} = \inf_{Y \in \underset{\sim}{F}_{j+1}} \sup_{y \in Y} f(y) \leqslant c-\varepsilon = c_{j+1} - \varepsilon .$$

which is absurd. Thus

$$\text{G-cat}_M\, K_c \geqslant k.$$

Hence the theorem follows using part (vi) of (IV.1.4). #

IV.2.3. REMARK. It may be worthwhile to mention that Fadell's notion of G-category is an example of the $\mathcal{A}$-category (Chapter III) of Clapp and Puppe [c.1] ; taking $\mathcal{A}$ to be the class of G-$\mathcal{O}$ of all homogeneous spaces G/H, H being an isotropy subgroup of G. In fact the whole theory of $\mathcal{A}$-category works in an equivariant setting; in particular the Theorem (III.4.6) can be easily extended to the G-equivariant case with suitable obvious modifications. *For entire details see* [c.1] .

IV.2.4 REMARK. Recently W. Marzantowicz [M.3] has extended the theory of G-category to a topological space and has shown that this extension has all properties necessary for the minimax procedure. He has also proved results analogous to (I.2.4), for this extended notion of G-category.

§3. The use of G-cohomology in the Lusternik - Schnirelmann method.

G-cohomology (or equivariant cohomology) helps us in finding an alternative estimate for G-category and also it can be used to define a cohomological category theory which can serve as a substitute for the G-category in the equivariant Lusternik-Schnirelmann method. We shall first define what is this G-cohomology, G is our fixed compact Lie group. For our purpose we shall define the G-cohomology theory, denoted by H_G^*, in such a way that H_G^* becomes a continuous, multiplicative, equivariant cohomology theory.

Let $p: EG \to BG$ denote a Universal G-bundle i.e. EG is a paracompact, contractible, free G-space and BG is the corresponding orbit space. Then, if X is any paracompact G-space, $EG \times X$ becomes a free G-space with G acting coordinatewise. Let $EG \times_G X$ be the corresponding orbit space. Then the G-cohomology theory H_G^* is defined by $H_G^*(X, \underset{\sim}{K}) = H^*(EG \times_G X; \underset{\sim}{K})$, where $\underset{\sim}{K}$ denotes a commutative ring and H^* denotes a cohomology theory (say, Alexander-Spanier cohomology theory) with which H_G^* becomes a multiplicative, equivariant cohomology theory satisfying the following continuity property:

(P) Let X be a paracompact G-space and A be a closed G-subspace of X, then

$$H_G^*(A; \underset{\sim}{K}) = \varinjlim H_G^*(U; \underset{\sim}{K})$$

where the direct limit is taken over all G-neighborhoods U of A in X.

Similarly for a paracompact G-pair (X,A) the relative G-cohomology theory is defined by

$$H_G^*(X,A; \underset{\sim}{K}) = H^*(EG \times_G (X,A); \underset{\sim}{K}),$$

where $EG \times_G (X,A) = (EG \times_G X, EG \times_G A)$.

<u>IV .3.1 REMARK.</u> It may be noted that in general H_G^* does not satisfy the dimension axiom for a cohomology theory. If $X = pt$ is a single point then $H_G^*(X; \underset{\sim}{K})$ $= H_G^*(pt; \underset{\sim}{K}) = H^*(BG; \underset{\sim}{K})$, as $EG \times_G pt \approx BG$.

Analogous to the notion of cuplength in the nonequivariant sense (§2 of I), we have the notion of G-cuplength as given below.

<u>IV.3.2. DEFINITION.</u> The G-cuplength of a paracompact

G-space Y over $\underset{\sim}{K}$ is the largest positive integer k such that there exists k positive dimensional elements $a_1, a_2, \ldots, a_k$ in $H_G^*(Y; \underset{\sim}{K})$ with the product $a_1 \cdot a_2 \cdot \ldots \cdot a_k \neq 0$.

Convensionally we set G-cuplength (Y) over $\underset{\sim}{K} = \infty$, if there is no maximal k with the above property, and

G-cuplength (Y) ov er $\underset{\sim}{K} = 0$,

if $Y \neq \emptyset$ but $H_G^n(Y; \underset{\sim}{K}) = 0$ for $n \geqslant 1$.

In the nonequivariant case we have, (I.2.1),

Cuplength(Y) over $\underset{\sim}{K} < \text{cat}\ Y$.

But the G-analogue of this result requires some comment. In the non-equivariant case we have seen that if $U \subset Y$ is contractible to a point in Y then the inclusion map from U to Y induces zero homomorphism in the positive dimensional cohomology level. This due to the fact that $H^n(pt) = 0$ for $n \geqslant 1$. But in the equivariant case we do not have the similar situation in general; because H_G^n (orbx) may not be zero for $n \geqslant 1$, where orbx is a single orbit. In fact, as mentioned in (IV.3.1), if $x = \text{orb}x$ i.e. G operates trivially on x then

$$H_G^*(\text{orb}x; \underset{\sim}{K}) = H^*(BG; \underset{\sim}{K}),$$

the cohomology of the classifying space.

In order to avoid this difficulty we have the following definition.

<u>IV.3.3 DEFINITION.</u> We say that the action of G on M is <u>cohomologically free over $\mathbb{R}$</u> if H^n_G (orb x; $\underset{\sim}{K}$) = 0 for all $x \in M$ and $n \geqslant 1$.

Free actions are cohomologically free over any ring $\underset{\sim}{K}$, because in this case orb $x \approx G$ and $EG \times_G G \approx EG$ so that H^n_G (orb x; $\underset{\sim}{K}$) = $H^n(EG; \underset{\sim}{K}) = 0$ for all $n \geqslant 1$ and for all $x \in M$.

Also if the action has only finite isotropy groups then the action is cohomologically free over the field of rationals [F.4] .

The definition (IV.3.3) makes the following proposition obvious (using the method of proposition (I.2.1).

<u>IV.3.4 PROPOSITION.</u> If the action of G on M is cohomologically free over $\underset{\sim}{K}$, then

G-cuplength (M) over $\underset{\sim}{K}$ < G-cat M. #

In view of the above proposition we can say that under the hypothesis of theorem (IV.2.2) and the additional assumption that the G-action is cohomologically free over $\underset{\sim}{K}$, the number of critical orbits of f is greater than the G-cuplength of M over $\underset{\sim}{K}$.

We now try to replace the set function G-cat M defined on S (see (IV.1.4)), S being the set of invariant subsets of M, by one defined purely in terms of G-cuplength.

IV.3.5 DEFINITION. For $X \in S$, the G-cuplength of X in M over $\underset{\sim}{K}$, denoted by $L_G(X)$, is defined to be the largest positive integer k such that there are k positive dimensional elements $a_1, a_2, --- a_k$ in $H^*_G(M; \underset{\sim}{K})$ with $i^*_X(a_1 . a_2 . ----- a_k) \neq 0$ where $i_X : X \hookrightarrow M$ is the inclusion.

Clearly, when the action of G on M is cohomologically free then for $X \in S$,

$$L_G(X) < \text{G-cat}_M X ,$$

using (a) the G-cohomology exact sequence of the pair $(M, \bigcup_{i=1}^{n} U_i)$ where $\{U_i\}_{i=1}^{n}$ is a G-categorical open cover of X in M and $n = \text{G-cat}_M X$, and (b) the same method involved in (I.2.1).

Also in view of (IV.1.4) it is easy to see that, when the action of G on M is cohomologically free, $L_G(X)$ has the following properties:

(i) $L_G(X) = 0$ if $X \in S$ is a G-categorical open subset of M

(ii) $L_G(X) \leqslant L_G(Y)$ if X and $Y \in S$ are such that $X \subseteq Y$.

(iii) $L_G(X) = L_G(t(X))$ if $t: M \to M$ is an equivariant homeomorphism and $X \in S$.

(iv) If $L_G(X) = n$ then $X \in S$ has atleast $(n+1)$ orbits.

However in general, the following two properties are not valid.

(v) $L_G(X \cup Y) \leq L_G(X) + L_G(Y)$, $X, Y \in S$ and

(vi) given $X \in S$, $L_G(X) = L_G(U)$ for some open set $U \in S$, $U \supset X$.

Thus there are some difficulties in using the set function L_G directly to the critical point theorems, in place of G-cat_M.

To have a smooth sailing as far as the six properties mentioned in (IV.1.4) are concerned, one modifies the notion of G-cuplength and talks about what is known as the cohomological index theory. Here instead of fixing an ambient space M one works in the category of paracompact G-spaces. Fix an element $a \in H^*(BG; \underset{\sim}{K})$, $a \neq 0$, where G is the fixed compact Lie group and BG is the classifying space.

IV.3.6 DEFINITION. If X is a paracompact G-space and $P: EG \to BG$ is the universal G-bundle, then we have a a diagram

$$\begin{tikzcd}
EG \times X \arrow[r, "\Pi"] \arrow[d] & EG \arrow[d] \\
EG \times_G X \arrow[r, "q_X"] & BG
\end{tikzcd}$$

where the map q_X, induced by the projection Π, is unique upto homotopy and is called the classifying map for the G-bundle $EG \times X \longrightarrow EG \times_G X$. Define,

$$\text{Index}_a X = \min \{ k : q_X^*(a^k) = 0 \}$$

where $q_X^* : H^*(BG; \underset{\sim}{K}) \to H^*(EG \times_G X; \underset{\sim}{K})$ is induced by q_X. If $q_X^*(a^k) \neq 0$ for all k, then set $\text{Index}_a X = \infty$.

Before proceeding further let us consider, for any G-space Y, the map

$$\Pi_G : EG \times_G Y \to Y/G$$

with fibres $\Pi_G^{-1}(\text{orb } y) = EG \times_G \text{orb } y = EG/G_y = BG_y$, the classifying space for G_y; G_y is the isotropy subgroup of G at y. When the action of G on Y is cohomologically free, $H^*(\Pi_G^{-1}(\text{Orb } y)) = H_G^*(\text{Orb } y) = 0$ in positive dimensions for all $y \in Y$. We know that (by Vietoris mapping theorem [Q.1]) if a map $f:A \to B$ is closed and its fibres $f^{-1}(b)$, $b \in B$, are compact, relatively Hausdorff in A, and acyclic then

10243°
EASTERN HILL UNIVE
Dt.

$H^*(A) \approx H^*(B)$. Applying this to

$$\Pi_G : EG \times_G Y \to Y/G$$

we get $H^*_G(Y) \approx H^*(Y/G)$. That is when the action of G is cohomologically free, the G-cohomology of a G-space coincides with the cohomology of the orbit space.

We now return to our previous setting of an ambient paracompact pathconnected C^1 Banach G-manifold M and let C be the set of all invariant subsets of M. Then analogous to (IV.1.4) we havd

IV.3.7 PROPOSITION. With the notations same as above we have

(i) NORMALIZATION. If $X \in C$ is G-categorical in M and the action of G is cohomologically free over $\underset{\sim}{K}$ then

$$\mathrm{Index}_a\ X = 1$$

(ii) MONOTONE AND INVARIANCE. If $X, Y \in C$ and $t: X \to Y$ is an equivariant map, then

$$\mathrm{Index}_a\ X \leq \mathrm{Index}_a\ Y$$

In particular, if $t: X \to Y$ is an equivariant homeomorphism then

$$\mathrm{Index}_a\ X = \mathrm{Index}_a\ Y.$$

(iii) SUBADDITIVITY. If $X, Y \in C$ then

$$\text{Index}\ (X \cup Y) \leq \text{Index}_a X + \text{Index}_a Y.$$

(iv) CONTINUITY. If $X \in C$ is closed in M, then X has a neighborhood $U \in C$ such that

$$\text{Index}_a U = \text{Index}_a X.$$

(v) ORBIT COUNT. If $X \in C$ and $\text{Index}_a X > 0$, then $X \neq \emptyset$. If the action of G on M is cohomologically free over $\underset{\sim}{K}$ and $\text{Index}_a X > 1$, then X contains infinitely many orbits.

PROOF. (i) It is easy to see that $q^*_X : H^*(BG; \underset{\sim}{K}) \to H^*_G(X; \underset{\sim}{K})$ can be factored as $q^*_X = i^*_G \circ q^*_M$, where

$i^*_G : H^*_G(M; \underset{\sim}{K}) \to H^*_G(X; \underset{\sim}{K})$ is induced by inclusion $i: X \to M$. Also $i^*_G = 0$ as X is G-categorical in M and the action of G is cohomologically free. So (i) follows.

For (ii) simply note that $q^*_X = t^*_G \circ q^*_Y$, where $t^*_G : H^*_G(Y, \underset{\sim}{K}) \longrightarrow H^*_G(X; \underset{\sim}{K})$ is induced by $t: X \to Y$.

(iii) Let $\text{Index}_a X = m$ and $\text{Index}_a Y = n$. Note that

$$q^*_X = i^*_G \circ q^*_{X \cup Y} \quad \text{and} \quad q^*_Y = j^*_G \circ q^*_{X \cup Y},$$

where $i: X \hookrightarrow X \cup Y$ and $j: Y \hookrightarrow X \cup Y$.

Now $q^*_X(a^m) = 0$, so $q^*_{X \cup Y}(a^m) \in \ker i^*_G$ and therefore $q^*_{X \cup Y}(a^m)$ can be lifted back to an element $b_1 \varepsilon H^*_G(X \cup Y, X; \underset{\sim}{K})$, by exactness of the G-cohomology sequence of the pair $(X \cup Y, X)$. Similarly $q^*_{X \cup Y}(a^n)$ can be lifted back to an element $b_2 \varepsilon H^*_G(X \cup Y, Y; \underset{\sim}{K})$ and so the product $q^*_{X \cup Y}(a^m).q^*_{X \cup Y}(a^n) = q^*_{X \cup Y}(a^{m+n})$ can be lifted back to an element $b_1 b_2 \varepsilon H^*_G(X \cup Y, X \cup Y; \underset{\sim}{K})$, which is a trivial group. Hence $\text{Index}_a(X \cup Y) \leq m+n$.

(iv) Since X is closed in M and M is paracompact, so using the continuity property (P) of H^*_G we have

$$H^*_G(X; \underset{\sim}{K}) = \varinjlim H^*_G(V; \underset{\sim}{K}),$$

where the direct limit is taken over all possible G-neighborhoods V of X in M. Let $k = \text{Index}_a X$ then $q^*_X(a^k) = 0$. Hence it follows that there exists some G-neighborhood V_o of X in M such that $q^*_{V_o}(a^k) = 0$, since $q^*_X = i^*_G \circ q^*_V$ $(i: X \hookrightarrow V)$ for all G-neighborhoods V of X in M. Thus $\text{Index}_a V_o \leq k = \text{Index}_a X$. Choosing $U = V_o$ and using the monotonicity of Index_a, (iv) follows.

(v) First part is trivial. Second part follows from the following commutative diagram

$$\begin{tikzcd}
H^*(X/G) \arrow[r, phantom, "\approx"] & H^*_G(X) \arrow[r, "q_X^*"] & H^*(BG) \arrow[dl, "q_M^*"] \\
H^*(M/G) \arrow[u, "\bar{i}_G^*"'] \arrow[r, phantom, "\approx"] & H^*_G(M) \arrow[u, "i_G^*"'] &
\end{tikzcd}$$

where the homomorphism $\overline{i}_G^*$, induced by $i:X \hookrightarrow M$, is trivial as X/G is a finite subset of the pathconnected set M/G (hence $\overline{i} \simeq 0$). Note that $H_G^*(X) \approx H^*(X/G)$ by Vietoris mapping Theorem applied to the map $EG \times_G X \to X/G$. #

From the above proposition we see that when the action of G on M is cohomologically free, one can use Index_a in place of $G\text{-cat}_M$ in the Lusternik - Schnirelmann method of §2 to obtain an analogue of (IV.2.2). That is under the hypothesis of (IV.2.2), with property (M) modified suitably in terms of Index_a, f possesses at least $n = \mathrm{Index}_a\, M$ critical orbits.

The above method for an estimation of the number of critical orbits using Index_a, ofcourse when the action of G is cohomologically free, is quite useful because Index_a is more easily computable than $G\text{-cat}_M$; eventhough the conclusion, that $\mathrm{Index}_a\, M \leqslant$ no of critical orbits, is an immediate corollary to Theorem (IV.2.2), since for $X \in C$. $\mathrm{Index}_a\, X \leqslant G\text{-cat}_M\, X$ (by (i) and (iii) of (IV.3.7)).

IV.3.8 REMARK. The biggest drawback of the set function Index_a is that when the action of G is not cohomologically free then a single orbit may have infinite index. For example, if $X \in C$ is such that $gx = x$ for all $g \in G$ and $x \in X$, then we have a diagram

$$\begin{tikzcd}
& EG \times X \arrow[r] \arrow[d] & EG \arrow[d] \\
BG \times X \arrow[r, phantom, "="] & EG \times_G X \arrow[r, "q_X"] & BG
\end{tikzcd}$$

where q_X is projection; and so $q_X^*: H^*(BG) \to H_G^*(X)$ is a monomorphism. Hence if $a \in H^*(BG)$ is of infinite order in $H^*(BG)$, then $\mathrm{Index}_a\, X = \infty$.

§4. Relative cohomological Index theories and the Lusternik - Schnirelmann method.

In view of Remark (IV.3.8), when the action of G on M is not cohomologically free, one likes to have a new index theory, that is to fix a subspace A of M and for $X \in C$ define an index theory of X modulo A, that would be useful especially when the action is cohomologically free outside A. A detailed study on the relative cohomological index theories is contained in [F.3] , we shall expose here only the highlights appropriate for the Lusternik - Schnirelmann method.

The general theory can be described as follows: Consider a category whose objects are paracompact G-pairs (X,A) and morphisms are equivariant maps $f:(X,A) \to (Y,B)$. Let $p: EG \to BG$ be the universal G-bundle. Then for a paracompact G-pair (X,A) we have a classifying map for the

the G-bundle $EG \times X \to EG \times_G X$ *and a diagram*

$$\begin{tikzcd}
EG \times X \arrow[r] \arrow[d] & EG \arrow[d, "p"] \\
EG \times_G X \arrow[r, "q_X"] & BG
\end{tikzcd}$$

inducing a homomorphism

$$q_X^* : H^*(BG) \to H^*(EG \times_G X) = H_G^*(X)$$

where the coefficients are now taken in a fixed field $\underset{\sim}{K}$. Suppose now that $\Lambda \subset H^*(BG)$ is a subring (always with unity), then the cup product

$$H_G^*(X,A) \otimes H_G^*(X) \to H_G^*(X,A)$$

together with the homomorphism

$$q_X^* : H^*(BG) \to H_G^*(X)$$

endows $H_G^*(X,A)$ with a right Λ-module structure. Precisely,

$$x\lambda = x \cup q_X^*(\lambda), \quad \lambda \in \Lambda, \; x \in H_G^*(X,\Lambda).$$

<u>IV.4.1 REMARK</u>. An equivariant $f:(X,A) \to (Y,B)$ between paracompact G-pairs induces a map

$$1 \times_G f = f_G : EG \times_G (X,A) \to EG \times_G (Y,B)$$

and hence a homomorphism of Λ-modules

$$f^*_G : H^*_G(Y,B) \to H^*_G(X,A).$$

Thus, H^*_G is a cohomology functor from the category of paracompactG-pairs and equivariant maps to the category of Λ-modules, $\Lambda \subset H^*(BG)$.

IV.4.2. DEFINITION. Let $\Lambda \subset H^*(BG; \underset{\sim}{K})$ denote a subring and (X,A) a paracompact G-pair. Let $U \subset \Lambda$ denote the annihilator of $H^*_G(X,A; \underset{\sim}{K})$ i.e. $U = \{ \lambda \in \Lambda \mid x\lambda = 0 \text{ for all } x \in H^*_G(X,A) \}$. Then regarding Λ/U as a module over $\underset{\sim}{K}$ (vector space over K) one defines

$$\text{Index}_\Lambda (X,A) = \text{rank}_{\underset{\sim}{K}} \Lambda/U = \dim_{\underset{\sim}{K}} \Lambda/U.$$

If $A = \emptyset$ and $a \in H^*(BG)$ is a chosen element, let Λ denote the subring generated by 1 and a. Then, since we have a unity $1 \in H^0(X)$, the annihilator of $H^*_G(X)$ i.e. U is just the kernel of

$$q^*_X : \Lambda \to H^*_G(X).$$

Thus U is generated by $\{a^i\}_{i \geq k}$ for some k and hence

$$\text{Index}_\Lambda X = k = \text{Index}_a X.$$

When the subring $\Lambda \subset H^*(BG)$ is generated by 1 and a single element $a \in H^*(BG)$, it will be called monogenic.

<u>IV.4.3. REMARK.</u> Let $\Lambda \subset H^*(BG; \underset{\sim}{K})$ be monogenic i.e. $\Lambda = \underset{\sim}{K}\,[a]$ for some $a \in H^*(BG; \underset{\sim}{K})$. Let U be the annihilator of $H^*_G(X,A)$. Then any element $z \in U$ looks like $z = k_i\, a^i + k_{i+1} a^{i+1} + \!---$ where $k_i \neq 0$, $i \geqslant 0$ and $k_i \in \underset{\sim}{K}$.

Now $x.z = x \cup q^*_X(z) = 0$ for all $x \in H^*_G(X,A)$

$$\Rightarrow \quad x \cup (k_i\, q^*_X(a^i) + k_{i+1}\, q^*_X(a^{i+1}) + ---) = 0$$

$$\Rightarrow \quad x \cup k_i\, q^*_X(a^i) = 0$$

$$\Rightarrow \quad x \cup q^*_X(a^i) = 0 \quad \text{for all} \quad x \in H^*_G(X,A)$$

which in turn implies that $a^i \in U$.

Thus it follows that U is generated by $\{ a^i \}_{i \geqslant i_0}$, where $i_0 = \min \{i : a^i \in U\}$. Correspondingly, $\operatorname{rank}_{\underset{\sim}{K}} \Lambda / U = i_0$.

<u>IV.4.4 PROPOSITION.</u> (Monotone property for Index_Λ) Let $f:(X,A) \to (Y,B)$ denote a morphism in the category of paracompact G-pairs.

(a) If $f^*_G : H^*_G(Y,B) \to H^*_G(X,A)$ is surjective, then

$$\text{Index}_\Lambda (X,A) \leqslant \text{Index}_\Lambda (Y,B).$$

(b) If A and B are empty, then

$$\text{Index}_\Lambda X \leqslant \text{Index}_\Lambda Y$$

PROOF. (a) $f_G^* : H_G^*(Y,B) \to H_G^*(X,A)$ is a Λ-homomorphism. Let $U(X,A)$ and $U(Y,B)$ denote the annihilators of $H_G^*(X,A)$ and $H_G^*(Y,B)$ respectively.

If $x \in H_G^*(X,A)$, there exists some $y \in H_G^*(Y,B)$ such that $f_G^*(y) = x$.

Now let $\lambda \in U(Y,B)$ then

$$x.\lambda = f_G^*(y\lambda) = 0 \text{ and hence}$$

$$U(Y,B) \subset U(X,A).$$

Therefore, $\mathrm{Index}_\Lambda(X,A) \leqslant \mathrm{Index}_\Lambda(Y,B)$.

(b). Note that, $q_X^* = f_G^* \circ q_Y^*$.

Thus $\ker(q_Y^* \mid \Lambda) \subset \ker(q_X^* \mid \Lambda)$

Since $\ker(q_X^* \mid \Lambda) = U(X)$ and $\ker(q_Y^* \mid \Lambda) = U(Y)$, the result follows. #

IV.4.5 PROPOSITION. (Additive property for Index_Λ).

Let $X = X_1 \cup X_2$ and $A = A_1 \cup A_2$ with $A_1 \subset X_1$ and $A_2 \subset X_2$, where (X_1, A_1) and (X_2, A_2) are paracompact G-pairs. Let Λ be monogenic. Then

(a) $$\mathrm{Index}_\Lambda(X,A) \leqslant \mathrm{Index}_\Lambda(X_1,A_1) + \mathrm{Index}_\Lambda(X_2, A_2) + \mathrm{Index}_\Lambda(X_1 \cap X_2, A_1 \cap A_2).$$

(b) If $A_2 = \emptyset$,

$$\mathrm{Index}_\Lambda(X, A_1) \leqslant \mathrm{Index}_\Lambda(X_1, A_1) + \mathrm{Index}_\Lambda X_2.$$

(c) If $X_1 = X_2$,

$$\text{Index}_\Lambda\ (X, A_1 \cap A_2 \leq \text{Index}_\Lambda (X, A_1) + \text{Index}_\Lambda (A_1 \cup A_2,\ A_1)$$

PROOF (a) Let U, U_1, U_2, U_{12} denote the annihilators of $H^*_G\ (X,A)$, $H^*_G\ (X_1, A_1)$, $H^*_G\ (X_2,\ A_2)$, $H^*_G(X_1 \cap X_2,\ A_1 \cap A_2)$ respectively.

Consider the following Mayer-Vietoris sequence:

$$\to H^*_G(X_1 \cap X_2,\ A_1 \cap A_2) \xrightarrow{\delta} H^*_G(X_1 \cup X_2,\ A_1 \cup A_2\ ) \xrightarrow{j^*} H^*_G\ (X_1, A_1) \oplus H^*_G\ (X_2, A_2) \xrightarrow{i^*}$$

where i^*, j^*, δ have their usual meaning. Let $\lambda_1 \varepsilon\ U_1$, $\lambda_2\ \varepsilon\ U_2$ $\lambda_{12}\ \varepsilon\ U_{12}$ and x be any element of $H^*_G\ (X_1 \cup X_2,\ A_1 \cup A_2) = H^*_G\ (X,A)$. Consider the element $x.\ \lambda_1\ \ \lambda_2\ \varepsilon\ H^*_G\ (X,A)$. Clearly $j^*(x.\ \lambda_1\ \lambda_2) = 0$, so there is some $y\ \varepsilon\ H^*_G\ (X_1 \cap X_2,\ \ A_1 \cap A_2)$ such that

$$\delta(y) = x.\ \lambda_1\ \ \lambda_2\ .$$

Therefore $\delta(y.\ \lambda_3) = x.\ \lambda_1\ \ \lambda_2\ \ \lambda_3$. But $y\ \lambda_3 = 0$, hence $x.\ \lambda_1\ \lambda_2\ \ \lambda_3 = 0$; showing thereby $U_1 U_2 U_{12} \subset U$

So, in view Remark (IV.4.3),

$$\text{rank } \Lambda\ /\ U \leq \text{rank } \Lambda\ /\ U_1\ U_2\ U_{12}$$

$$= \text{rank } \Lambda/\ U_1 + \text{rank } \Lambda/\ U_2 + \text{rank } \Lambda/U_{12}$$

Hence (a) follows.

(b) Let $U_1 = \text{Ann } H^*_G(X_1, A_1)$

$U_2 = \text{Ann } H^*_G(X_2)$

$U = \text{Ann } H^*_G(X, A_1)$

Let $\lambda_1 \in U_1$ and $\lambda_2 \in U_2$. Consider the diagram,

$$\begin{tikzcd}
H^*_G(X_2) & \Lambda \arrow[l, "q^*_{X_2}"'] \arrow[d, "q^*_X"] \\
 & H^*_G(X) \arrow[ul, "i_2^*"] \\
 & H^*_G(X, X_2) \arrow[u, "j_2^*"']
\end{tikzcd}$$

where i_2, j_2 are inclusions and q's are classifying maps.

Since $U_2 = \ker (q^*_{X_2} \mid \Lambda)$, there exists some $\beta \in H^*_G(X, X_2)$ such that $j_2^*(\beta) = q^*_X(\lambda_2)$, by exactness.

Again let $x \in H^*_G(X, A_1)$ and consider the exact sequence,

$$\to H^*_G(X, X_1) \xrightarrow{k_1^*} H^*_G(X, A_1) \to H^*_G(X_1, A_1) \to$$

Then there exists some $\alpha \in H^*_G(X, X_1)$ such that $k_1^*(\alpha) = x.\ \lambda_1$, by exactness.

Thus $x.\ \lambda_1\ \lambda_2 = x \cup q^*_X(\lambda_1\ \lambda_2)$ (by definition)

$$= (x \cup q^*_X(\lambda_1)) \cup q^*_X(\lambda_2).$$

$$= k_1^*(\alpha) \,.\, j_2^*(\beta).$$

But $\alpha.\beta \in H^*_G(X,X) = 0$. Hence $\lambda_1 \lambda_2 \in U$, i.e. $U_1 U_2 \subset U$. So rank $\Lambda / U \leq$ rank Λ / U_1 + rank Λ / U_2 and hence (b) follows.

(c) Consider the exact sequence,

$$\to H^*_G(A_1 \cup A_2, A_1) \to H^*_G(X, A_1 \cup A_2) \to H^*_G(X, A_1) \to$$

Let $U = \text{Ann } H^*_G(X, A_1 \cup A_2)$

$U_1 = \text{Ann } H^*_G(X, A_1)$

$U_{12} = \text{Ann } H^*_G(A_1 \cup A_2, A_1)$

Using the same argument as in the proof of (a) and (b) above, we have

$$U_1 U_{12} \subset U$$

and therefore (c) follows. #

Similarly, there do exist analogues of other basic properties (see [F.3]). But what is to be noticed is that a stringent additional condition is required for the monotone property of Index_Λ ; namely the surjectivity of $f^*_G: H^*_G(Y,B) \to H^*_G(X,A)$, see (IV.4.4). Thus, the statement $X \subset Y$ and $A = B$, need not imply $\text{Index}_\Lambda(X,A) \leq \text{Index}_\Lambda(Y,A)$; for example consider $Y = D^2$, $X = S^1$ and $A = B = pt$. This makes Lusternik - Schnirelmann method difficult to apply.

However a slight modification of the definition (IV.4.2) does give the appropriate monotone property.

Sticking to the same notion as above, we have for a paracompact G-pair (X,A), $A \neq \emptyset$, the coboundary operator

$$H_G^{q-1}(A) \xrightarrow{\delta} H_G^q(X,A) \text{ for } q \geq 1 \text{ and the augmentation}$$

$$\underset{\sim}{K} \xrightarrow{\varepsilon} H_G^0(X,A)$$

and we let $M^q(X,A) = \text{Image } \delta$ for $q \geq 1$ and $M^0(X,A) = \text{Image } \varepsilon$. Also for completeness, we set $M^*(X,A) = H_G^*(X)$ when $A = \emptyset$. Then $M^0(X)$ contains the unity $1 \in H_G^0(X)$. In both the cases if $\Lambda \subset H^*(BG)$ is a subring, $M^*(X,A)$ is a Λ-submodule of $H_G^*(X,A)$.

IV.4.6. DEFINITION. Let U' denote the annihilator of the graded Λ-module $M^*(X,A)$. Define

$$\text{Index}_\Lambda^\delta (X,A) = \text{rank}_{\underset{\sim}{K}} \Lambda / U' .$$

We shall refer to the index theory given by the above definition as δ-index theory. Since $U' \supset U = \text{Ann } H_G^*(X,A)$, we have

$$\text{Index}_\Lambda^\delta (X,A) \leq \text{Index}_\Lambda (X,A)$$

and when $A = \emptyset$,

$$\text{Index}_\Lambda^\delta X = \text{Index}_\Lambda X.$$

In contrast to Index_Λ ; the monotone property for δ-index is adequate for the Lusternik - Schnirelmann method.

IV.4.7 PROPOSITION (Monotone property for δ-index) Let $f:(X,A) \to (Y,B)$ denote a morphism between paracompact

G-pairs (X,A), (Y,B). If $(f\mid A)^*_G : H^*_G(B) \to H^*_G(A)$ is surjective, then

$$\mathrm{Index}^{\delta}_{\Lambda}(X,A) \leq \mathrm{Index}^{\delta}_{\Lambda}(Y,B)$$

In particular, when $X \subset Y$ and $A = B$, we have

$$\mathrm{Index}^{\delta}_{\Lambda}(X,A) \leq \mathrm{Index}^{\delta}_{\Lambda}(Y,A).$$

PROOF One simply considers the following commutative diagram

$$\begin{tikzcd}
H^*_G(A) \arrow[r, "\delta"] & H^*_G(X,A) \\
H^*_G(B) \arrow[r, "\delta"] \arrow[u, "(f|A)^*_G"] & H^*_G(Y,B) \arrow[u, "f^*_G"']
\end{tikzcd}$$

and uses the same technique involved in the proof of (IV.4.4). #

IV.4.8. PROPOSITION (Additive property for δ-index). Let $X = X_1 \cup X_2$, $A = A_1 \cup A_2$ with $A_1 \subset X_1$, $A_2 \subset X_2$, where (X_1, A_1) and (X_2, A_2) are paracompact G-pairs. Let Λ be monogenic. Then

$$\text{(a)}\quad \mathrm{Index}^{\delta}_{\Lambda}(X,A) \leq \mathrm{Index}^{\delta}_{\Lambda}(X_1,A_1) + \mathrm{Index}^{\delta}_{\Lambda}(X_2,A_2) + \mathrm{Index}_{\Lambda}(X_1 \cap X_2, A_1 \cap A_2).$$

(b) If $A = \emptyset$,

$$\mathrm{Index}_{\Lambda}^{\delta} (X, A_1) \leq \mathrm{Index}_{\Lambda}^{\delta} (X_1, A_1) + \mathrm{Index}_{\Lambda} X_2$$

(c) If $X_1 = X_2$

$$\mathrm{Index}_{\Lambda}^{\delta} (X, A_1 \cup A_2) \leq \mathrm{Index}_{\Lambda}^{\delta} (X, A_1) + \mathrm{Index}_{\Lambda} (A_1 \cup A_2, A_1)$$

PROOF. Adopt the same technique involved in the proof of (IV.4.5) and use the naturality property for δ .

Like Index_{Λ} , the δ-index also satisfy the analogues of other basic properties, i.e. Invariance, Continuity etc., see [F.3] .

Conditions (Finiteness conditions)under which $\mathrm{Index}_{\Lambda} (X,A) < \infty$ (hence $\mathrm{Index}_{\Lambda}^{\delta} (X,A) < \infty$) are of interest. There are several criterions for $\mathrm{Index}_{\Lambda} (X,A)$ to be finite. For example:

(1) Suppose (X,A) is a compact G-pair and A is a G-neighborhood retract in X. Suppose Λ is Noetherian and $\mathrm{Index}_{\Lambda} (\mathrm{orb} x) < \infty$ for all $x \in X-A$. Then $\mathrm{Index}_{\Lambda}(X,A) < \infty$.

(2) Suppose (X,A) is a G-pair with finitely generated cohomology, Λ is Noetherian and $X-A$ has finitely many orbit types. Suppose further that $\mathrm{Index}_{\Lambda} (\mathrm{orb} x) < \infty$ for all $x \in X-A$. Then $\mathrm{Index}_{\Lambda} (X,A) < \infty$.

But the simplest condition for finiteness is the following.

IV.4.9 PROPOSITION. Suppose H_G^* (Orbx) is acyclic over $\underset{\sim}{K}$ for every $x \varepsilon X-A$. Suppose further that X is a finite dimensional and seperable metric space. Then, $\text{Index}_\Lambda (X,A) < \infty$.

PROOF. Consider the map

Π_G: EG x_G (X,A) $\rightarrow$ (X/G, A/G) with fibres

Π_G^{-1}(orbx) = EG x_G orbx = $EG/G_x = BG_x$, the classifying space for G_x. Thus $H^*(\Pi_G^{-1}$ (orbx)) = H_G^* (orbx), which is given to be acyclic over $\underset{\sim}{K}$ for every $x \varepsilon X-A$. Hence we may apply vietoris mapping theorem [Q.1] , to conclude that

$$H_G^* (X,A) \approx H^*(X/G, A/G).$$

Now $H^q(X/G, A/G)$ vanishes for large q, X being finite dimensional. So H_G^q (X,A) also vanishes for large q. Thus, $\text{Index}_\Lambda (X,\Lambda) < \infty$, since $\Lambda = \{\Lambda^q\}$ is of finite rank over $\underset{\sim}{K}$ in each dimension q.

We now illustrate the equivariant Lusternik-Schnirelmann method using the relative δ-index theory. We consider the following simple case (we shall use notations of §2).

Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$ and suppose that the compact Lie group G acts as a subgroup of the orthogonal group $O(n+1)$. We do not assume that G acts freely on S^n. However, we do assume that an invariant closed subset $A \subset S^n$ exists such that $\text{Index}^{\delta}_{\Lambda} \; (S^n, A) = k < \infty$, where Λ is assumeed to be monogenic. Let $f\colon S^n \to \mathbb{R}$ denote a C^1 functional which is GPS. We shall study the existence of the critical points of f in $S^n - A$.

Let E denote the family of invariant pairs (X,A), where $A \subset X \subset S^n$ and $E_j \subset E$ denote the subfamily consisting of pairs (X,A) such that $\text{Index}^{\delta}_{\Lambda} \; (X,A) \geqslant j$. Then following Lusternik - Schnirelmann method, set

$$c_j = \inf_{X \in E_j} \; \sup_{x \in X} \; f(x) ,$$

$$c_j = \sup_{X \in E_j} \; \inf_{x \in X} \; f(x) , \quad 0 \leqslant j \leqslant k,$$

where $\max f \,|\, A = c_0 \leqslant c_1 < \text{-------} \leqslant c_k$

$\min f \,|\, A = c'_0 \geqslant c'_1 \geqslant \text{------} \geqslant c_k .$

We confine our study to a situation where $c_0 \neq c_k$ or $c'_0 \neq c'_k$ i.e. there exist p and q (minimal) such that

(1) $c_0 = c_1 = \text{----} = c_p < c_{p+1} \leqslant \text{---} \leqslant c_k$

or (2) $c'_0 = c'_1 = \text{----} = c'_q > c'_{q+1} \geqslant \text{---} \geqslant c'_k$

V.4.10 PROPOSITION. Suppose we are in the case (1) (or case (2)) above. Then f has at least $k-p$ (or $k-q$) critical orbits in $S^n - A$.

PROOF. We shall confine ourselves to case (1). Case (2) is similar. Suppose

$$c_{j+1} = c_{j+2} = \text{----} = c_{j+s} = c, \; j > p, \; 1 \leqslant s \leqslant k-j.$$

It is enough to prove that

$$\text{Index}^{\delta}_{\Lambda} \; (K_c) \geqslant s \quad \text{i.e.} \quad \text{Index}_a \; (K_c) \geqslant s \; ,$$

K_c being the set of critical orbits on the level c. Since $c > c_0 = \max f \mid A$, A and K_c are disjoint invariant sets and so we may find an invariant open neighborhood U of K_c such that $U \subset S^n - A$ and $\text{Index}^{\delta}_{\Lambda} \; (U) = \text{Index}^{\delta}_{\Lambda} \; (K_c)$ (see (IV.3.7)). Let $\bar{\varepsilon} = \frac{1}{2}[c-c_0]$. Then since f is GPS, there exists $\varepsilon < \bar{\varepsilon}$ and $t:S^n \to S^n$ satisfying the conditions in GPS. There exists $X \in E_{j+s}$ such that $f(x) < c+\varepsilon$ for all $x \in X$. By part (b) of (IV.4.8), we have

$$\begin{aligned} \text{Index}^{\delta}_{\Lambda} \; (X,A) &\leqslant \text{Index}^{\delta}_{\Lambda} \; (X-U,A) \\ &\quad + \text{Index}_{\Lambda} \; (U \cap X) \\ &= \text{Index}^{\delta}_{\Lambda} \; (X-U,A) \\ &\qquad + \text{Index}_a \; (U \cap X) \end{aligned}$$

and by (IV.3.7)

$$\text{Index}_a (U \cap X) \leqslant \text{Index}_a \; (U) = \text{Index}_a \; (K_c \;)$$

Therefore, $\text{Index}_{\Lambda}^{\delta}\ (X-U,A) \geq \text{Index}_{\Lambda}^{\delta}\ (X,A)$

$$- \text{Index}_a\ (K_c)$$

If $\text{Index}_a\ (K_c) \leq (s-1)$, then

$$\text{Index}_{\Lambda}^{\delta}\ (X-U,A) \geq j+s-s+1 = j+1$$

But by the GPS condition,

$$t(X-U,A) \subset t(M_{c+\varepsilon} - U,A) \subset (M_{c-\varepsilon}, A),$$

(note that $t \mid A = 1_A$ as $\max f \mid A < c-\bar{\varepsilon}$)

Hence $M_{c-\varepsilon} \in E_{j+1}$, by (IV.4.7).

So, $c = c_{j+1} = \inf\limits_{X \in E_{j+1}} \sup\limits_{x \in X} f(x) \leq c-\varepsilon < c,$

which is absurd. Thus,

$$\text{Index}_{\Lambda}^{\delta}\ (K_c) = \text{Index}_a\ (K_c) \geq s. \qquad \#$$

§ 5. Relative G-category and its computations.

Let Y be a pathconnected, normal G-space, G being a compact Lie group, such that every orbit Gy of Y is a retract of some open invariant subset of Y containing Gy (e.g. Y is a G-manifold). Let $A \neq \emptyset$ be a fixed closed invariant subset of Y. Let V be any invariant subset of Y containing A. If there exists a G-map $r:(V,A) \to (A,A)$ such that $j \circ r$ is G-homotopic to i, where $j:(A,A) \to (Y,A)$ and $i:(V,A) \to (Y,A)$ are inclusion maps, then V is said to be G-categorical relative to A.

IV.5.1 DEFINITION. Let X be an invariant subset of Y which contains A. We say $G\text{-cat}_Y(X,A) = n$ if X can be covered by n open sets $\{U_1, U_2, ---, U_n\}$ such that each U_i is G-categorical relative to A, and n is minimal with this property.

IV.5.2. DEFINITION. Under the same hypothesis of (IV.5.1) we say $G\text{-cat}_Y^*(X,A) = n$ if n is the smallest positive integer with the property that X can be covered by n open sets $\{W, U_1, U_2, --., U_{n-1}\}$ such that W is G-categorical relative to A and each U_i is G-categorical in the sense of § 1.

IV.5.3. REMARK. (i) If the action of G on Y is considered to be trivial then one denotes $G\text{-cat}_Y(X,A)$ simply by $\text{cat}_Y(X,A)$ and $G\text{-cat}_Y^*(X,A)$ simply by $\text{cat}_Y^*(X,A)$.

(ii) If G acts freely on Y then $G\text{-cat}_Y(X,A)$ $= \text{cat}_{Y/G}(X/G, A/G)$, $G\text{-cat}_Y^*(X,A) = \text{cat}_{Y/G}^*(X/G, A/G)$.

(iii) If $A = \emptyset$ then we take G-categorical relative to A to be just G-categorical in the sense of §1, and

$$G\text{-cat}_Y(X, \emptyset) = G\text{-cat}_Y^*(X, \emptyset) = G\text{-cat}_Y X.$$

IV.5.4 PROPOSITION. If $A \subset Y$ contains minimal orbits;

i.e. for each $y \in Y$, there is a point $a \in A$ and path p_y from y to a such that $G_y \subset G_{p_y(t)}$ for all $t \in [0,1]$, where $G_y = \{g \in G;\ gy = y\}$ is the isotropy subgroup of G at y, then

$$G\text{-cat}_Y\,(X,A) \leq G\text{-cat}^*_Y\,(X,A)\ .$$

PROOF. Let $G\text{-cat}^*_Y\,(X,A) = n$ and $\{W, U_1, U_2, \text{--}, U_{n-1}\}$ be an open cover of X such that W is G-categorical relative to A and each U_i is G-categorical. Now by normality of Y there exists an open G-set $W' \subset Y$ such that $A \subset W' \subset \overline{W}' \subset W$. Let $U_i' = U_i \cap (Y - \overline{W}')$, $1 \leq i \leq n-1$. Then clearly $U_i' \cap W' = \emptyset$ and U_i' is G-categorical (see proposition (IV.1.4)). So we have for each i, $1 \leq i \leq n-1$, a G-homotopy commutative diagram

$$\begin{tikzcd}
U_i' \arrow[rr, hook] \arrow[dr] & & Y \\
& Gy \arrow[ur, hook] &
\end{tikzcd}$$

for some $y \in Y$. Define a G-homotopy $H: G \times I \to X$ by $H(g,t) = gp(t)$, where p is a path from y to some point $a \in A$ such that $G_y \subset G_{p(t)}$ for all $t \in [0,1]$. Let $\bar{y} : G \to Gy$ be the map $g \mapsto gy$, which is an open surjective map. Now define $\tilde{H}: Gy \times I \to X$ as $\tilde{H}(gy, t) = H(g,t) = gp(t)$. $\tilde{H}$ is well-defined since $G_y \subset G_{p(t)}$ for all t, and also it is continuous as $\tilde{H} \circ (\bar{y} \times 1d) = H$, where $\bar{y} \times 1d : G \times I \to Gy \times I$

is an open surjective map. Now $\tilde{H}(gy,1) = H(g,1)$ $= gp(1) = ga \in Ga \subset A$, so $\tilde{H}$ is a G-homotopy taking Gy to Ga. Thus we have the following G-homotopy commutative diagram

$$\begin{tikzcd} U_i' \arrow[rr, hook] \arrow[dr] & & Y \\ & Ga \arrow[ur, hook] & \end{tikzcd}$$

Hence $U_i' \cup W'$ is G-categorical relative to A. Thus $\{W, U_1' \cup W', U_2' \cup W', --- U_{n-1}' \cup W'\}$ is an open cover of X, all of whose members are G-categorical relative to A. Hence the proposition follows. #

In most of the cases G-cat and G-cat* are equal but there can be differences as illustrated in the following example due to Ramsay [R.1] :

Let $X = Y = T^2$ (the 2-torus) and let $A = S^1 \subset T^2$ be the inner meridian of X. Also let G be the trivial group (or the G-action on Y be trivial). Ramsay computed the values of cat (T^2, S^1) and cat*(T^2, S^1) and found that cat $(T^2, S^1) = 2$ (which is of course simple) and cat* $(T^2, S^1) = 3$.

G-cat$_Y$ and G-cat*$_Y$ satisfy the following elementary properties.

IV.5.5. PROPOSITION.

(i) If X is open G-categorical relative to A, then

$$\text{G-cat}_Y(X,A) = \text{G-cat}^*_Y(X,A) = 1.$$

(ii) If $A \subset X_1 \subset X_2 \subset Y$, then

$$\text{G-cat}_Y(X_1,A) \leq \text{G-cat}_Y(X_2,A),$$

$$\text{G-cat}^*_Y(X_1,A) \leq \text{G-cat}^*_Y(X_2,A).$$

(iii) If $t\colon Y \longrightarrow Y$ is an equivariant homeomorphism with $t \mid A = 1d_A$, then

$$\text{G-cat}_Y(X,A) = \text{G-cat}_Y(t(x),A),$$

$$\text{G-cat}^*_Y(X,A) = \text{G-cat}^*_Y(t(X),A).$$

(iv) For any closed invariant subset $X \subset Y$ containing A there are open invariant sets U,V containing X such that

$$\text{G-cat}_Y(\bar{U},A) = \text{G-cat}_Y(X,A),$$

$$\text{G-cat}^*_Y(\bar{V},A) = \text{G-cat}^*_Y(X,A)$$

(v) If $A \subset X_1$ and $A \subset X_2$ then

$$\text{G-cat}_Y(X_1 \cup X_2,A) \leq \text{G-cat}_Y(X_1,A) + \text{G-cat}_Y(X_2,A).$$

(vi) If $A \subset X_1$ and X_2 is any invariant subset of Y then

$$\text{G-cat}^*_Y(X_1 \cup X_2,A) \leq \text{G-cat}^*_Y(X_1,A) + \text{G-cat}_Y X_2.$$

PROOF. (i), (ii), (iii), (v) and (vi) are immediate from the definition.

For (iv), let $G\text{-cat}_Y(X,A) = n$ and $\{U_1, U_2, --- U_n\}$ be an open cover of X in Y such that each U_i is G-categorical relative to A. Now X being closed in $Y, \{U_1, U_2, --- U_n, Y-X\}$ is an open cover for Y. But Y is normal, so there is a refinement $\{V_1, V_2, --- V_{n+1}\}$ of this cover such that $V_i \subset \overline{V}_i \subset U_i$, $1 \leq i \leq n$, and $V_{n+1} \subset \overline{V}_{n+1} \subset Y-X$. Also $\bigcup_{i=1}^{n+1} V_i = y$. Hence $X \subset \bigcup_{i=1}^{n} V_i$. Again $A \subset U_i$ for $1 \leq i \leq n$, so there exists open sets W_i such that $A \subset W_i \subset \overline{W}_i \subset U_i$ (again by normality of Y). Let $O_i = V_i \cup W_i$, $1 \leq i \leq n$. Clearly $X \subset \bigcup_{i=1}^{n} O_i$ and $A \subset O_i \subset \overline{O}_i = \overline{V_i \cup W_i} = \overline{V}_i \cup \overline{W}_i \subset U_i$, $1 \leq i \leq n$. We choose $U = O_1 \cup O_2 \cup ------ \cup O_n$. Hence the first statement of (iv) follows. Second statement follows similarly. ≠

The following proposition is useful for the computations of relative G-category.

IV.5.6. PROPOSITION. Let A and B be G-spaces. If $X = A \circ B$ is the join of A and B, ($A \circ B = \{(ta, (1-t)b) : a \in A, b \in B, 0 \leq t \leq 1\}$). Consider the G-action on $A \circ B$ given by $g\,(ta, (1-t)b) = (tga, (1-t)\,gb)$.

Then

(a) $G\text{-cat}^*_X (X,A) \leq 1 + G\text{-cat } B$.

(b) $G\text{-cat}^*_{(X-B)} (X-B, A) = G\text{-cat}_{(X-B)} (X-B, A)$

$= 1.$

PROOF. For (a) note that if U is a G-categorical subset of B then clearly $\mathcal{O} = (A \circ U) - A = \{(ta, (1-t)b) : 0 \leq t \leq 1, b \in U, a \in A\}$ is a G-categorical subset of X. Thus if $G\text{-cat}_B\ B = k$ and $\{U_i\}_{i=1}^{k}$ is an open categorical covering of B then $\{(X-A), \mathcal{O}_1, ---, \mathcal{O}_k\}$, (where $(X-A)$ is clearly open and G-categorical relative to B, and $\mathcal{O}_i$'s are defined as above), gives a covering of X satisfying the conditions in definition (IV.5.2).

Proof of (b) is simple. #

A technique similar to that used in § 4 provides us with a lower bound for relative G-category. Let (X,A) be a paracompact G-pair, where $X \subset Y$. Consider $H_G^*(X,A)$ and regard it as a module over $H_G^*(Y)$ as follows. For

$\lambda \in H_G^*(Y)$, $x \in H_G^*(X,A)$ define $x.\lambda = x \cup j^*(\lambda)$, where j^* is the homomorphism induced by the inclusion $j : X \to Y$ and the cup product is $H_G^*(X,A) \otimes H_G^*(X) \longrightarrow H_G^*(X,A)$. (Throughout the section, the coefficients of G-cohomology are taken in a field $\underset{\sim}{K}$.)

IV.5.7 DEFINITION. We define

$$G\text{-Index}_Y (X,A) = k$$

if k is the largest positive integer with the property that there exists $(k-1)$ positive dimensional elements $y_1, y_2, ---, y_{k-1}$ in $H_G^*(Y)$ such that the product

$y_1 \cdot y_2 \cdot ----- \cdot y_k \notin \text{Ann}\ H_G^*(X,A)$, the Annihilator of

$H_G^*(X,A)$. If $H_G^*(X,A) = 0$, we set $G\text{-Index}_Y(X,A) = 0$. If $H_G^*(X,A) \neq 0$ but for any $\lambda \in H_G^p(Y)$, $p \geq 1$, λ lies in Ann $H_G^*(X,A)$ then set $G\text{-Index}_Y(X,A) = 1$.

<u>IV.5.8. PROPOSITION.</u> If the action of G on Y is cohomologically free over $\underset{\sim}{K}$ and if the inclusion $j_A\colon (X,A) \to (Y,A)$ induces a surjection $j_A^*\colon H_G^*(Y,A) \to H_G^*(X,A)$, then

$$G\text{-Index}_Y(X,A) < G\text{-cat}_Y^*(X,A).$$

<u>PROOF.</u> If $G\text{-cat}_Y^*(X,A) = 1$ the proposition is clear, for then $H_G^*(X,A) = 0$. Assuming $G\text{-cat}_Y^*(X,A) = n$, $n \geq 2$ and finite, let $\{W, U_1, ---, U_{n-1}\}$ be the corresponding cover of X. Also assume without any loss that $H_G^*(X,A) \neq 0$ and choose $x \in H_G^*(X,A)$. Let $y_1, y_2, ---, y_{n-1} \in H_G^*(Y)$ be any set of (n-1) positive dimensional elements. Now since each U_i is G-categorical and G-action is cohomologically free so $H_G^m(U_i) = 0$, $m \geq 1$. Therefore each y_i can be lifted back to some $v_i \in H_G^*(Y, U_i)$. In the same way, using the assumption that j_A^* is surjective, x can be lifted back to some $x' \in H_G^*(Y,A)$ and then to some $x'' \in H_G^*(Y,W)$, since W is G-categorical relative to A. Finally consider the following commutative diagram:

$$\begin{tikzcd}
H_G^*(Y,W)\otimes H_G^*(Y,U_1)\otimes\cdots\cdots\otimes H_G^*(Y,U_{n-1}) \arrow[dd, "\theta"'] \arrow[dr, "\beta^*\otimes\alpha_1^*\otimes\cdots\otimes\alpha_{n-1}^*"] & \\
 & H_G^*(X,X\cap W)\otimes H_G^*(X,X\cap U_1)\otimes\cdots\otimes H_G^*(X,X\cap U_{n-1}) \arrow[d, "p_1"] \\
H_G^*(Y,A)\otimes H_G^*(Y)\otimes\cdots\otimes H_G^*(Y) \arrow[d, "\psi"] & H_G^*(X,X)=0 \arrow[d, "j_A^*"] \\
H_G^*(X,A)\otimes H_G^*(X)\otimes\cdots\otimes H_G^*(X) \arrow[r, "p_2"] & H_G^*(X,A)
\end{tikzcd}$$

where p_1 and p_2 are cup products and other maps are induced by inclusions. We see that

$$x\cup j^*(y_1\cdot y_2\cdots\cdots y_{n-1})$$

$$= p_2(x\otimes j^*y_1\otimes\cdots\cdots\otimes j^*y_{n-1})$$

$$= p_2\psi(x'\otimes y_1\otimes\cdots\cdots\otimes y_{n-1})$$

$$= p_2\psi\theta(x''\otimes v_1\otimes\cdots\cdots\otimes v_{n-1})$$

$$= j_A^* p_1 \left(\beta^*(x'') \otimes \alpha_1^*(v_1) \otimes \cdots\cdots \otimes \alpha_{n-1}^*(v_{n-1}) \right)$$

$$= j_A^* \left(\beta^*(x'') \cdot \alpha_1^*(v_1) \cdots\cdots \alpha_{n-1}^*(v_{n-1}) \right)$$

$$= j_A^*(0) = 0$$

Hence the proposition follows. #

We now illustrate the computation of G-cat* with a number of examples. For simplicity we shall write G-cat* (X,A) for G-cat^*_X (X,A), also G-Index (X,A) for G-Index_X (X,A).

Example 1. Let $X = S^{2n+1} \subset \mathbb{C}^{n+1}$ and $G = S^1$. Let $A = S^{2k+1}$, $k < n$ and let G act freely on X by

$$g(z_0, z_1, \cdots, z_n) = (gz_0, gz_1, \cdots, gz_n)$$

Now since the action is free, so

$$H_G^*(S^{2n+1}, S^{2k+1}) \cong H^*(S^{2n+1}/G, S^{2k+1}/G)$$

$$\cong H^*(\mathbb{C}P^n, \mathbb{C}P^k)$$

From the exact cohomology sequence of the pair $(\mathbb{C}P^n, \mathbb{C}P^k)$ we find that $H^p(\mathbb{C}P^n, \mathbb{C}P^k) = 0$ for $p \leq 2k+1$, and $H^p(\mathbb{C}P^n, \mathbb{C}P^k) \cong H^p(\mathbb{C}P^n)$ for $p \geqslant 2k+2$.

Hence, G-Index $(S^{2n+1}, S^{2k+1}) = n-k$. Also S^{2n+1} can be viewed as $S^{2k+1} \circ S^{2(n-k-1)+1}$, so (IV.5.6) gives

$$\text{G-cat}^{*}\,(S^{2n+1}, S^{2k+1}) \leq 1+\text{G-cat}\,(S^{2(n-k-1)+1})$$

$$= 1+\text{cat}\,(\mathbb{C}P^{(n-k-1)}) = 1+(n-k-1)+1 = n-k+1.$$

Hence by (IV.5.8),

$$\text{G-cat}^{*}\,(S^{2n+1}, S^{2k+1}) = n-k+1. \qquad \#$$

EXAMPLE.2. Let $G = S^1$ act on $X = S^{2n+1} \subset \mathbb{C}^{n+1}$ by

$$g(z_0, z_1, \cdots z_n) = (g^{m_0} z_0, \cdots, g^{m_n} z_n), \quad m_i \neq 0 \text{ for } i=0,1,\cdots n.$$

Also let $A = \emptyset$. Clearly all the isotropy groups will be finite subgroups of S^1.

(a) Let $m_o = m_1 = \ldots = m_n = 1$. Clearly G-cat* $(S^{2n+1}, \emptyset)$ = G-cat (S^{2n+1}) = cat (CP^n) = n+1, because in that case the G-action is free.

(b) Let $m = m_o = m_1 = \cdots = m_n \neq 1$. Then $G_x = Z_m$ for all $x \in S^{2n+1}$. Also $S^{2n+1}/G \approx CP^n$, so that G-Index $(S^{2n+1}) = n$ (since all isotropy groups are finite, so the G-action is cohomologically free over the field of rationals) and thus G-cat $(S^{2n+1}) \geq n+1$, by (IV.5.8). Now since $S^{2n+1}/G \approx CP^n$, cat $(S^{2n+1}/G) = n+1$ and also since we have the single isotropy type only so by Palais covering homotopy theorem, [p.4; p51] , the homotopies on the categorical sets in the orbit space can be lifted to yield a G-categorical cover for S^{2n+1} . Hence G-cat $(S^{2n+1}) = n+1$.

(c) For the general case we may assume $m_o \geq m_1 \geq -- \geq m_n \neq 0$ and we may view S^{2n+1} as the join $S^{2n_1-1} \circ \text{-----} \circ S^{2n_k-1}$, $n = (\sum_{i=1}^{k} n_i) - 1$, where the action on each S^{2n_i-1} has only one isotropy type. In fact k is the number m's which have distinct values, say $c_1, c_2, ---c_k$, and n_i is the number of m's which have the constant value c_i. For example if all m's are different then $k = n+1$ and $n_i = 1$ for all $i = 1,2,---,k=n+1$, and if all m's are equal then $k = 1$ and $n_1 = n+1$.

Now clearly,

$$\text{G-cat } (S^{2n+1}) \leq \text{G-cat } (S^{2n_1-1}) \quad + \ldots + \text{G-cat} (S^{2n_k-1})$$

$$= \sum n_i = n+1$$

Also by [F.4], one has G-Index $(S^{2n+1}) = n$, so G-cat $(S^{2n+1}) = n+1$. #

Before giving the next example let us discuss something about <u>spectral sequence.</u>

Let us consider modules over a fixed P.I.D. R (for our purpose we take $R = \mathbb{Q}$, the field of rationals). A <u>bigraded module</u> E (over R) is an indexed collection of R-modules $E^{s,t}$ for every pair of integers s and t.

A differential $d: E \to E$ of <u>bidegree</u> $(r, 1-r)$ is a collection of homomorphisms $d: E^{s,t} \to E^{s+r,t+1-r}$ for all s and t, such that $d^2 = d \circ d = 0$. The <u>cohomology module</u> $H(E)$ is the bigraded module defined by

$$H^{s,t}(E) = \frac{\ker [d : E^{s,t} \longrightarrow E^{s+r,t+1-r}]}{d(E^{s-r,t-1+r})}$$

An <u>E_k spectral sequence</u> E (k stands for the starting level) is a sequence $\{E_r, d_r\}$ for $r \geq k$ such that

(1) E_r is a bigraded module and d_r is a differential of bidegree $(r, 1-r)$ on E_r.

(2) For $r \geq k$ there is given an isomorphism $H(E_r) \cong E_{r+1}$.

To define the <u>limit term</u> E_∞ of a spectral sequence, for $r \geq k$, we identify E_{r+1} with $H(E_r)$. Let Z_k be the bigraded module $Z_k^{s,t} = \ker [d_k : E_k^{s,t} \longrightarrow E_k^{s+k,t+1-k}]$ and let B_k be the bigraded module $B_k^{s,t} = d_k (E_k^{s-k,t-1+k})$. Then $B_k \subset Z_k$ and $E_{k+1} = \frac{Z_k}{B_k}$. Let $Z(E_{k+1})$ be the bigraded module $Z(E_{k+1})^{s,t} = \ker [d_{k+1} : E_{k+1}^{s,t} \to E_{k+1}^{s+k+1,t-k}]$ and $B(E_{k+1})$ be the bigraded module $B(E_{k+1})^{s,t}$ $= d_{k+1} (E_{k+1}^{s-k-1,t+k})$. Then there exists bigraded submodules Z_{k+1} and B_{k+1} of Z_k containing B_k such that $Z(E_{k+1})^{s,t} \approx Z_{k+1}^{s,t} / B_k^{s,t}$ and $B(E_{k+1})^{s,t} \approx B_{k+1}^{s,t} / B_k^{s,t}$

for all s and t. Thus we have $B_k \subset B_{k+1} \subset Z_{k+1} \subset Z_k$. Continuing by induction we get for $r \geq k$

$$B_k \subset B_{k+1} \subset \cdots \subset B_r \subset \cdots \subset Z_r \subset \cdots \subset Z_{k+1} \subset Z_k ,$$

such that $E_{r+1} = Z_r/B_r$. We define bigraded modules $Z_\infty = \bigcap_r Z_r$ and $B_\infty = \bigcup_r B_r$ and $E_\infty = Z_\infty / B_\infty$. E_∞ is called the limit of the spectral sequence E.

The spectral sequence E is said to <u>converge</u> if for every s and t there exists an integer $r(s,t) \geq k$ such that for $r \geq r(s,t)$, $d_r : E_r^{s,t} \to E_r^{s+r,t+1-r}$ is trivial. Then $E_{r+1}^{s,t}$ is isomorphic to a quotient of $E_r^{s,t}$ and $E_\infty^{s,t}$ is isomorphic to the direct limit of the sequence

$$E_{r(s,t)}^{s,t} \longrightarrow E_{r(s,t)+1}^{s,t} \longrightarrow \cdots\cdots\cdots$$

The spectral sequence E is said to <u>converge in the strong sense</u> if for given s and t there exists $r(s,t)$ such that $E_r^{s,t} \approx E_\infty^{s,t}$ for $r \geq r(s,t)$.

The spectral sequence E is said to be a <u>first-quadrant spectral sequence</u> if there exists some $r \geq k$ such that $E_r^{s,t} = 0$ whenever $s < 0$ or $t < 0$. A first-quadrant spectral sequence is always convergent in the strong sense and for any q there are only a finite number of nontrivial modules $E_\infty^{s,t}$ with $s+t = q$.

We now state the following main result, which will be used for calculation of G-cat*.

Theorem [S.3; Thm 6, p495] : Let $p:E \to B$ be an orientable fibration over a path connected base and let $F = p^{-1}(b_o)$. Then there is a convergent E_2 cohomology spectral sequence, with $E_2^{s,t} \approx H^s(B;H^t(F;\mathbb{Q}))$ and E_∞ is the bigraded module *associated to some filtration of* $H^*(E;Q)$ *i.e. for each* s and t there exists a sequence of submodules of $H^{s+t}(E;Q)$ given by

$$H^{s+t}(E,\mathbb{Q}) = F^{0,s+t} \supset F^{1,s+t-1} \supset \cdots \supset F^{s,t} \supset F^{s+1,t-1} \supset \cdots$$

$$\cdots \supset F^{s+t,0} \supset \{0\},$$

such that $E_\infty^{s,t} = F^{s,t} / F^{s+1,t-1}$. Moreover this spectral sequence is a first-quadrant spectral sequence. #

We are now in a position to give the third example to illustrate the computation of G-cat*.

EXAMPLE 3. Let $G = S^1$ act on $X = S^{2n+1} \subset \mathbb{C}^{n+1}$, by $g(z_o, z_1, \cdots, z_n) = (g^{m_0}z_0, g^{m_1}z_1, \cdots, g^{m_n}z_n)$ with $m_o \geq m_1 \geq \cdots \geq m_n \neq 0$ so that for all $x \in X$, G_x is finite. Also let $A = S^{2k+1}$ represent the first $(k+1)$ coordinates of S^{2n+1}. If we take the coefficients in the rational field $\mathbb{Q}$, the action becomes cohomologically free and so we have $H_G^*(S^{2n+1}) \cong H^*(S^{2n+1}/G)$, by victoris mapping theorem applied to the map $EG \times_G X \longrightarrow X/G$. Thus

H^p_G (S^{2n+1}) vanishes for p large, in fact for $p > 2n+1$. Now consider the fibration

$$S^{2n+1} \hookrightarrow EG \times_G S^{2n+1} \xrightarrow{q} BG$$

where $BG \approx BS^1 \approx \mathbb{C}P^\infty$, [H.2; p54, EX.11.4] .

By the above theorem there is a convergent E_2 cohomology spectral sequence, with

$$E_2^{s,t} = H^s(\mathbb{C}P^\infty ; H^t(S^{2n+1}))$$

$$= \begin{cases} H^s(\mathbb{C}P^\infty ; \mathbb{Q}) & , \text{ if } t = 0 \text{ or } 2n+1. \\ 0, \text{ otherwise} & . \end{cases}$$

Thus it follows that whenever $t \neq 0$ and $t \neq 2n+1$ or s is negative or s is odd then

$$0 = E_2^{s,t} = E_3^{s,t} = \text{-----} = E_\infty^{s,t} .$$

Now, $$E_3^{s,o} = \frac{\ker[d_2 : E_2^{s,o} \longrightarrow E_2^{s+2,-1}]}{d_2(E_2^{s-2,1})}$$

$$= E_2^{s,o} \quad , \quad s \geq 0.$$

In this way we have, for all $s \geq 0$,

$$E_2^{s,o} = E_3^{s,o} = \text{----} = E_{2n+2}^{s,o} .$$

Again, $E_{2n+3}^{s,o} = \dfrac{\ker\left[d_{2n+2} \cdot E_{2n+2}^{s,o} \longrightarrow E_{2n+2}^{s+2n+2,-2n-1}\right]}{d_{2n+2}\left(E_{2n+2}^{s-2n-2,\,2n+1}\right)}$

$$= E_{2n+2}^{s,o}\ ,\ \text{if}\quad 0 \leq s \leq 2n+1.$$

Thus it follows that, for $0 \leq s \leq 2n+1$,

$$E_{2n+2}^{s,o} = E_{2n+3}^{s,o} = \text{-----} = E_{\infty}^{s,o}\ .$$

Therefore, for $0 \leq s \leq 2n+1$, we have

$$E_{\infty}^{s,o} = E_{2}^{s,0} = \begin{cases} \mathbb{Q}, & \text{if } s \text{ is even} \\ 0, & \text{if } s \text{ is odd} \end{cases}$$

Similarly, $E_2^{s,2n+1} = E_3^{s,2n+1} = \text{-----} = E_{2n+2}^{s,2n+1}$

$$= \begin{cases} \mathbb{Q} & \text{if } s \text{ is even} \\ 0 & \text{if } s \text{ is odd.} \end{cases}$$

Again by the above theorem, E_∞ term is given by a filtration of $H^*(EG \times_G S^{2n+1})$ i.e. for each s and t there is a sequence given by

$$H^{s+t}(EG \times_G S^{2n+1}) = F^{o,s+t} \supset F^{1,s+t-1} \supset \text{-----}\ F^{s,t} \supset \cdot$$
$$F^{s+1,t-1} \supset \text{------} \supset F^{s+t,o} \supset \{o\}\ ,$$

such that $E_{\infty}^{s,t} = F^{s,t} / F^{s+1,t-1}$.

Therefore, for $0 \leq s+t \leq 2n+1$.

$$F^{s+1,0} = E_{\infty}^{s+t,0} = \begin{cases} \mathbb{Q}, & (s+t) \text{ is even} \\ 0, & (s+t) \text{ is odd.} \end{cases}$$

We first assume that $(s+t)$ is even and $0 \leq s+t < 2n+1$. Then $F^{s+t,0} = \mathbb{Q}$.

Also $\dfrac{F^{s+t-1,1}}{F^{s+t,0}} = E_{\infty}^{s+t-1,1} = 0$

so that $F^{s+t-1,1} = \mathbb{Q}$.

Proceeding in the similar manner we get

$$F^{s+t-1,1} = F^{s+t-2,2} = ---- = F^{0,s+t} = \mathbb{Q}.$$

Therefore, $H_G^{s+t}(S^{2n+1}) = F^{0,s+t} = \mathbb{Q}$, when $(s+t)$ is even and $0 \leq s+t < 2n+1$.

Similarly one can see that $H_G^{s+t}(S^{2n+1}) = 0$, for $0 \leq s+t < 2n+1$ and $(s+t)$ odd. Also we have already seen that $H_G^{s+t}(S^{2n+1}) = 0$, whenever $s+t > 2n+1$. Thus it remains to compute $H_G^{2n+1}(S^{2n+1})$ only. For that we need the following Gysin sequence for a sphere bundle

$$S^{2n+1} \hookrightarrow E \longrightarrow B ,$$

$$\begin{tikzcd}
{} \arrow[r] & H^{2n+1}(\bar{E}) \arrow[r, "i^*"] & H^{2n+1}(E) \arrow[r, "\delta"] & H^{2n+2}(\bar{E},E) \arrow[r, "j^*"] \arrow[d, "\Phi"', "\cong"] & H^{2n+2}(\bar{E}) \arrow[r, "i^*"] & H^{2n+2}(E) \arrow[r] & {} \\
{} & H^{2n+1}(B) \arrow[u, "\cong"', "\bar{q}^*"] & & H^{o}(B) \arrow[r, dashed, "\Psi"] & H^{2n+2}(B) \arrow[u, "\cong"', "\bar{q}^*"] & &
\end{tikzcd}$$

where $\bar{E}$ is the associated disc bundle and $\bar{q}$ is the projection map of $\bar{E}$, Φ is the Thom isomorphism, i and j are inclusions and δ is the connecting homomorphism.

In our situation the sphere bundle is

$$S^{2n+1} \hookrightarrow ES^1 \times_{S^1} S^{2n+1} \xrightarrow{q} BS^1 = \mathbb{C}P^\infty .$$

So, $H^o(B) = H^o(BS^1) \approx H^o(\mathbb{C}P^\infty) = \mathbb{Q}$.

and $H^{2n+2}(B) = H^{2n+2}(BS^1) = H^{2n+2}(\mathbb{C}P^\infty) = \mathbb{Q}$.

Moreover we can identify $H^o(B)$ with $E_{2n+2}^{o,2n+1}$ and $H^{2n+2}(B)$ with $E_{2n+2}^{2n+2,0}$, and under this identification the map $\Psi : H^o(B) \longrightarrow H^{2n+2}(B)$, given by $\Psi = (\bar{q}^*)^{-1} \circ j^* \circ \Phi^{-1}$, can be identified with d_{2n+2}.

Now, $H^{2n+2}(E) = H^{2n+2}(ES^1 \times_{S^1} S^{2n+1})$

$\approx H^{2n+2}(S^{2n+1}/S^1) = 0$

Also we have $H^{2n+1}(\bar{E}) \approx H^{2n+1}(B)$ $\approx H^{2n+1}(\mathbb{C}P^{\infty}) = 0.$

Thus we get the following exact sequence

$$0 \longrightarrow H^{2n+1}(E) \longrightarrow \mathbb{Q} \xrightarrow{d_{2n+2}} \mathbb{Q} \longrightarrow 0$$

Therefore d_{2n+2} is a surjection and hence a module isomorphism from $\mathbb{Q}$ to $\mathbb{Q}$. Hence it follows that

$$H^{2n+1}(E) = H^{2n+1}(ES^1 \times_{S^1} S^{2n+1}) = 0$$

i.e. $H_G^{2n+1}(S^{2n+1}) = 0.$

Therefore we ultimately have

$$H_G^*(S^{2n+1}) \approx H^*(\mathbb{C}P^n), \text{ for all } n \geq 1.$$

Hence by Five lemma,

$$H_G^*(S^{2n+1}, S^{2k+1}) \approx H^*(\mathbb{C}P^n, \mathbb{C}P^k).$$

Thus, using the same argument as in Example (1), we have

$$\text{G-Index}(S^{2n+1}, S^{2k+1}) = n-k.$$

Now we view $S^{2n+1} = S^{2k+1} \circ S^{2(n-k-1)+1}$

so that, by (IV.5.6) and by Example (2),

$$\begin{aligned} &\text{G-cat}^*(S^{2n+1}, S^{2k+1}) \\ &\leq 1+\text{G-cat}\,(S^{2(n-k-1)+1}) \\ &\leq 1+ (n-k-1)+1 = n-k+1. \end{aligned}$$

Hence, $\text{G-cat}^* (S^{2n+1}, S^{2k+1}) = n-k+1$, by (IV.5.8). #

CHAPTER V

COBORDISM CLASSIFICATION OF MANIFOLDS WITH SMALL CATEGORY.

Throughout this chapter, unless mentioned otherwise, by a manifold we shall always mean a closed, smooth and compact manifold. Further we shall consider only mod 2 cohomologies i.e. the coefficients will always be in Z_2 .

In [M.2] , Mielke has shown that an n-dimensional manifold M^n with cat $M^n \leq 3$ and with $n \equiv 3 \mod 4$ is a boundary. In this chapter we study the complete classification of manifolds M^n with cobcat $(M^n) \leq 3$, as done by H. Singh [S.4]. The notion of cobcat is given in §2. We have also studied the cobordism classification for M^n with cobcat $(M^n) \leq 4$ for some cases, which is again due to H.Singh, [S.5] . Finally we close this section giving a couple of conjectures made by H.Singh, [S.5] , towards the cobordism classification of manifolds M^n with cobcat$(M^n) \leq 4$.

§1. Preliminaries.

In this section we give a very brief description of some basic materials that will be used at various places throughout this chapter. In most part of the description we have considered only the particular cases of various notions which are appropriate for the forthcoming sections.

(a) Stiefel-Whitney classes.

Let ξ be an n-dimensional vector bundle with base space $B(\xi)$. Then there exists a unique sequence of cohomology classes

$$w_i(\xi) \in H^i(B(\xi)) \quad , \quad i = 0, 1, 2, \cdots .$$

satisfying the following four axioms.

AXIOM 1. $w_0(\xi) = 1 \in H^0(B(\xi))$.

and $w_i(\xi) = 0$ if $i > n$.

AXIOM 2. If $f: B(\xi) \longrightarrow B(\eta)$ is covered by a bundle map from ξ to η, then

$$w_i(\xi) = f^* w_i(\eta)$$

AXIOM 3. If ξ and η are vector bundles over the same base space then

$$w_k(\xi \oplus \eta) = \sum_{j=0}^{k} w_i(\xi)\, W_{k-i}(\eta)$$

where $\oplus$ denotes the Whitney sum and the product on the right hand side is the cup product.

AXIOM 4. For the canonical line bundle γ_1^1 over the projective space $\mathbb{R}P^1$, the class $W_1(\gamma_1^1) \neq 0$.

V.1.1 DEFINITION. The cohomology classes

$W_i(\xi)$, $i = 0, 1, 2, \ldots\ldots$, satisfying the four axioms mentioned above, are known as Stiefel-Whitney classes of ξ. Also $W(\xi) = 1 + W_1(\xi) + \cdots + W_n(\xi)$ is called the total Stiefel-Whitney class of the vector bundle ξ.

Note that the AXIOM 3 above can also be stated as

$$W(\xi \oplus \eta) = W(\xi) . W(\eta).$$

(b) Universal vector bundle.

Consider $BO_n = G_n(\mathbb{R}^\infty)$ = the set of all n-dimensional linear subspaces of $\mathbb{R}^\infty$ and

$$EO_n = \{ (X, x) : X \in G_n(\mathbb{R}^\infty) , x \in X \} .$$

Define $P_n : EO_n \longrightarrow BO_n$ by $P_n(X, x) = X$

Then $\gamma^n : EO_n \xrightarrow{P_n} BO_n$ is a Universal vector bundle of dimension n in the sense that if ξ is a vector bundle of dimension n over a paracompact base $B(\xi)$ then there exists a map f, unique upto homotopy, from $B(\xi)$ to BO_n such that $\xi \cong f^*(\gamma^n)$. BO_n is called the classifying space for the n-dimensional vector bundles and f is called the classifying map for ξ .

$$\text{Define } BO = \operatorname*{dir\,lim}_{n \to \infty} BO_n$$

$$EO = \operatorname*{dir\,lim}_{n \to \infty} EO_n$$

$$\text{and} \quad \gamma' = \operatorname*{dir\,lim}_{n \to \infty} \gamma'^n ,$$

then $\gamma' : EO \to BO$ is a Universal vector bundle which classifies vector bundles of all dimensions over paracompact base i.e. if ξ is any vector bundle over a paracompact base then there exists a map $f : B(\xi) \to BO$ unique upto homotopy such that $\xi \cong f^*(\gamma')$ . .

Also the cohomology ring

$$H^*(BO) = \operatorname*{inv\,lim}_{n \to \infty} H^*(BO_n) .$$

It is well known, [M.4], that $H^*(BO_n)$ is a polynomial algebra over $\mathbf{Z}_2$ freely generated by the Stiefel-Whitney classes $W_1(\gamma^n), ----, W_n(\gamma^n)$ and $H^*(BO)$ is a polynomial algebra over $\mathbf{Z}_2$ freely generated by the Stiefel-Whitney classes $W_i(\gamma)$, $i = 1, 2, ---$.

(c) Steenrod Squaring Operations, Wu's formula and Adem's relations.

For each pair $X \supset Y$ of spaces and each pair n, i of nonnegative integers, there exists a unique additive homomorphism

$$Sq^i : H^n(X,Y) \longrightarrow H^{n+i}(X,Y)$$

such that

(1) If $f: (X,Y) \longrightarrow (X', Y')$ is a map of pairs then the following diagram commutes:

$$\begin{tikzcd}
H^n(X,Y) \arrow[r, "Sq^i"] & H^{n+i}(X,Y) \\
H^n(X', Y') \arrow[u, "f^*"'] \arrow[r, "Sq^i"] & H^{n+i}(X', Y') \arrow[u, "f^*"']
\end{tikzcd}$$

(2) If $a \in H^n(X,Y)$, then $Sq^0(a) = 0$ $Sq^n(a) = a \cup a$ and $Sq^i(a) = 0$ for $i > n$.

(3) (The Cartan formula). The identity

$$Sq^k(a \cup b) = \sum_{i+j=k} Sq^i(a) \cup Sq^j(b)$$

is valid whenever $a \cup b$ is defined.

V.1.2. DEFINITION. The homomorphisms Sq^i, $i = 0,1,2,---$, are known as Steenrod squaring operations.

From property (2) above we see that the most interesting squaring operations are those for which $0 < i < n$. For many purposes it is convenient to introduce the total squaring operations; namely

$$Sq(a) = a+Sq^1(a)+Sq^2(a) +----+Sq^n (a)$$

for $a \in H^n(X,Y)$. Also, note that the cartan formula above can now be expressed as

$$sq(a \cup b) = Sq(a) \cup Sq(b).$$

WU'S FORMULA. Let ξ be a vector bundle and let $W_i = W_i(\xi)$ be its Stiefel-Whitney classes. Then

$$Sq^k (W_m) = \sum_{t=0}^{k} \binom{m-k+t-1}{t} W_{k-t} W_{m+t} ,$$

where $0 \leq k < m$ and $\binom{m-k+t-1}{t}$ denote the Binomial coefficients, reduced modulo 2. Note that this formula seems to differ from that given in [M.4; p94] . But they are actually same, the difference is infact in the meanings associated to the symbol ( ).

ADEM'S RELATIONS. The Steenrod squares generate an algebra, over $\mathbb{Z}_2$, of cohomology operations called the modulo 2 Steenrod Algebra [S.3; p276]. In this algebra the following Adem's relations hold:

$$Sq^i\, Sq^j = \sum_{0 \leq k \leq [i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k}\, Sq^k ,$$

where $0 \leq i < 2j$, $[i/2]$ denotes as usual the largest integer $\leq i/2$ and $\binom{j-k-1}{i-2k}$ denote the Binomial coefficients, reduced modulo 2.

(d) Wu class and Wu's theorem [M.4]

Let M be an n-dimensional manifold. For any integer $k \geq 0$, consider the homomorphism

$$x \longmapsto \langle Sq^k(x), [M] \rangle$$

from $H^{n-k}(M)$ to $\mathbb{Z}_2$. Using the Duality theorem [M.4;p128], there exists one and only one cohomology class $v_k \in H^k(M)$ which satisfies the identity

$$\langle v_k \cup x, [M] \rangle = \langle Sq^k(x), [M] \rangle$$

for every $x \in H^{n-k}(M)$. In fact, if one considers M as the disjoint union of its connected components, then v_k

satisfies the sharper condition

$$v_k \cup x = Sq^k(x) \in H^n(M)$$

for every $x \in H^{n-k}(M)$. Ofcourse $v_k = 0$ when $k > n-k$. The class v_k is called the kth <u>Wu class.</u> We define the total Wu class v to be the formal sum

$$v = 1+v_1+\text{---}+v_n.$$

v is characterised by the identity

$$\langle v \cup x, [M] \rangle = \langle Sq(x), [M] \rangle$$

which holds for every cohomology class $x \in H^*(M)$. Here Sq denotes the total suqaring operation

$$Sq^0 + Sq^1+\text{---}.$$

<u>WU'S THEOREM [M.4; p132, Thm 11.14]</u>.

The total Stiefel-Whitney class W(M) of the tangent bundle of M, is equal to Sq(V). In other-words

$$W_k(M) = \sum_{i+j=k} Sq^i(v_j) \quad , k \geq 0 .$$

(e) <u>Unoriented Cobordism algebra $\mathfrak{N}_*$ and Dold manifolds.</u>

For any integer $n \geq 0$ let N_n be the set of all closed unoriented n-dimensional manifolds. Define a relation $\sim$ in N as $M_1^n \sim M_2^n$ if and only if M_1^n is bordic to M_2^n i.e. iff there exists an (n+1)-dimensional compact smooth manifold W^{n+1} with boundary $\partial W^{n+1} = M_1^n \sqcup M_2^n$. It is easy to see that $\sim$ is an equivalence relation in N. The quotient set $N_n/\sim$, denoted by $\mathfrak{N}_n$, is an abelian group with respect to the operation "disjoint union", i.e.

$$[M_1^n] + [M_2^n] = [M_1^n \sqcup M_2^n] .$$

$\mathfrak{N}_n$ is called the n-dimensional unoriented bordism group with the class of bounding manifolds as the identity element. More precisely $\mathfrak{N}_n$ is a vector space over $\mathbb{Z}_2$. The topological product of manifolds induces a bilinear map

$$\mathfrak{N}_p \times \mathfrak{N}_q \longrightarrow \mathfrak{N}_{p+q}$$

by means of which the direct sum

$$\mathfrak{N}_* = \bigoplus_{q=0}^{\infty} \mathfrak{N}_q$$

becomes a graded commutative algebra over $\mathbb{Z}_2$. Thom has shown that $\mathfrak{N}_*$ is a free commutative algebra generated by indeterminates x_q of degree q for all q not of the form $2^k - 1$; for q even he showed that x_q can be chosen as the class of the projective space $\mathbb{R}P^q$. For q odd, $q \neq 2^k - 1$, the class x_q was given by Dold [D.2] as follows: write $q+1 = 2^r(2s+1)$. Then x_q can be taken as the class of $P(2^r-1, 2^r s)$, where in general $P(m,n)$ is the manifold obtained from $S^m \times \mathbb{C}P^n$ by identifying $(x, [z])$ with $(-x, [\bar{z}])$, where $x \in S^m$, $[z] \in \mathbb{C}P^n$.

P(m,n) is called the Dold manifold. Clearly dimension of P(m,n) is (m+2n). Dold also gave the ring structure of $H^*(P(m,n); \mathbb{Z}_2)$, which can be described as

$$H^*(P(m,n);\mathbb{Z}_2) \cong \left[\frac{\mathbb{Z}_2[c]}{c^{m+1}=0} \right] \otimes \left[\frac{\mathbb{Z}_2[d]}{d^{n+1}=0} \right]$$

Here c is the nozero element of $H^1(P(m,n); \mathbb{Z}_2) \approx \mathbb{Z}_2$ with $c^{m+1} = 0$ and d is a suitable nonzero element of $H^2(P(m,n); \mathbb{Z}_2)$ with $d^{n+1} = 0$. Note that $H^2(P(m,n); \mathbb{Z}_2) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 = (0, c^2, d, c^2 + d)$. The total Stiefel-Whitney class of P(m,n) is given by

$$W(P(m,n)) = (1+c+d)^{n+1} (1+c)^m .$$

(f) Dual submanifolds

Consider a special case of [S.3;p428, Theorem 10] i.e. by taking $n = 1$, $\pi = \mathbb{Z}_2$. The the map

$$\psi : [M^n, \mathbb{R}P^\infty] \longrightarrow H^1(M^n, \mathbb{Z}_2)$$

defined as $\psi(f) = f^*(W_1(\gamma^1))$ is a bijection, where $W_1(\gamma^1)$ is the first Stiefel-whitney class of the canonical 1-dimensional universal vector bundle $E(\gamma^1) \longrightarrow \mathbb{R}P^\infty$. Therefore for a given $c \in H^1(M^n; \mathbb{Z}_2)$ there exists a map $f: M^n \longrightarrow \mathbb{R}P^\infty$ unique upto homotopy such that $\psi(f) = f^*(W_1(\gamma^1)) = c$. Now f induces a line bundle $\xi : E(\xi) \longrightarrow M^n$, namely $\xi = f^*(\gamma^1)$. Therefore $c = f^*(W_1(\gamma^1)) = W_1(\xi)$. Moreover, since M^n is a compact finite dimensional manifold, $f(M^n)$ will be embedded inside some $\mathbb{R}P^k$ for sufficiently large k. Now $\mathbb{R}P^{k-1}$ is embedded inside $\mathbb{R}P^k$ as a submanifold of codimension 1. By transversality homotopy theorem [G.1; p70] there exists a map $\tilde{f}$ with $f \simeq \tilde{f}$ such that $\tilde{f}$ is transversal to $\mathbb{R}P^{k-1}$. Thus without any loss we can take f itself to be transversal to $\mathbb{R}P^{k-1}$ Then $f^{-1}(\mathbb{R}P^{k-1})$ is a submanifold of M^n of codimension 1 i.e. of dimension (n-1), and is known as the submanifold of M^n dual to $c \in H^1(M^n)$.

(g) Complex cobordism groups in a topological space and Smith homomorphisms.

Consider a family of n-dimensional complex closed manifolds M^n together with continuous maps $f: M^n \longrightarrow X$, X being a fixed topological space, $n \geqslant 0$. Define a relation $\sim$ by saying $(M_1^n, f_1) \sim (M_2^n, f_2)$ if and only if there exists a complex compact manifold W^{n+1} with a map $F: W^{n+1} \longrightarrow X$ such that $\partial W^{n+1} = M_1^n \sqcup M_2^n$ and $F|M_i^n = f_i$, i = 1,2. This is an equivalence relation in the above family and the set $\Omega_n^U(X)$ of bordism classes $[M^n, f]$ forms a group with respect to "disjoint union". $\Omega_n^U(X)$ is called the n-dimensional complex bordism group in X. As usual the direct sum

$$\Omega_*^U(X) = \bigoplus_{n=0}^{\infty} \Omega_n^U(X) .$$

is an algebra over $\mathbb{Z}_2$, called the complex bordism algebra over $\mathbb{Z}_2$ in X. For our purpose we take $X = \mathbb{C}P^\infty$. Then any element $[M^n, f] \in \Omega^U_{\wedge}(\mathbb{C}P^\infty)$ can also be written as $[M^n, \lambda]$ where λ is a 1-dimensional complex vector bundle over M^n. The idnetification is done in the following manner:

Given any $f: M^n \longrightarrow \mathbb{C}P^\infty$, we may take $\lambda = f^*(\gamma^1)$ where γ^1 is the canonical 1-dimensional universal complex vector bundle over $\mathbb{C}P^\infty$. Conversely given any 1-dimensional complex vector bundle λ over M^n, we may take $f: M^n \longrightarrow \mathbb{C}P^\infty$ to be a classifying map (which is unique upto homotopy) of λ.

Further, $\Omega^U_*(\mathbb{C}P^\infty)$ is a Ω^U_*-module where Ω^U_* may be viewed as $\Omega^U_*(X)$ with X being a one-point space. The module structure is given by

$$\Omega^U_*(\mathbb{C}P^\infty) \times \Omega^U_* \longrightarrow \Omega^U_*(\mathbb{C}P^\infty)$$

defined as $[M^n, f] \times [N^r] \longmapsto [M^n \times N^r, \bar{f}]$, where $\bar{f}: M^n \times N^r \longrightarrow \mathbb{C}P^\infty$ is given by $\bar{f}(x,y) = f(x)$, $x \in M^n$, $y \in N^r$.

Also there is defined an <u>augmentation homomorphism</u>

$$\varepsilon : \Omega^U_*(\mathbb{C}P^\infty) \longrightarrow \Omega^U_*$$

sending $[M^n, f]$ to $[M^n]$

Analogous to dual submanifolds in the real case, there does also exist a complex submanifold N^{n-1}, of dimension (n-1), of the complex manifold M^n dual to a given cohomology class $d \in H^2(M^n, Z_2)$. Using this, one defines a homomorphism

$$\Delta : \Omega^U_n(\mathbb{C}P^\infty) \longrightarrow \Omega^U_{n-1}(\mathbb{C}P^\infty)$$

given by $\Delta([M^n, f]) = [N^{n-1}, f | N^{n-1}]$

This homomorphism is known as the Smith homomorphism of of degree - 2. If we take $M^n = \mathbb{C}P^n$ and if γ^1_n is the canonical complex line bundle over $\mathbb{C}P^n$ then it can be easily seen that

$$\Delta([\mathbb{C}P^n, \gamma^1_n]) = [\mathbb{C}P^{n-1}, \gamma^1_{n-1}] .$$

§2. Poincare algebra and cobcat (M).

Let M^n be an n-dimensional manifold. Consider the tangent bundle $\tau(M^n)$. Then there exists a map $f : M^n \longrightarrow BO$, unique upto homotopy (f is the classifying map for $\tau(M^n)$), such that $W_i(M^n) = f^*(W_i(\gamma))$, where $\gamma : EO \longrightarrow BO$ is the universal vector bundle. Associated to M^n, there is a homomorphism

$$t : H^n(BO; Z_2) \longrightarrow Z_2$$

given by

$$
\begin{aligned}
& t\left(W_{i_1}(\gamma)\cdot W_{i_2}(\gamma)\cdots\cdots W_{i_p}(\gamma)\right) \\
&= \left\langle f^*(W_{i_1}(\gamma))\cdot f^*(W_{i_2}(\gamma))\cdots f^*(W_{i_p}(\gamma)), [M^n] \right\rangle \\
&= \left\langle W_{i_1}(M)\cdot W_{i_2}(M)\text{----}W_{i_p}(M), [M^n] \right\rangle
\end{aligned}
$$

where $i_1+i_2+\text{----}+i_p = n$ is a partition of n and $\left\langle W_i(M)\cdot\text{---} W_{i_p}(M), [M^n]\right\rangle$ is the corresponding Stiefel-Whitney number for M^n, which we shall denote by $\left\langle W_{i_1}\text{--------}W_{i_p}, [M^n]\right\rangle$ also.

It may be worthwhile to mention that Wu classes $v_i \in H^i(BO)$, $i \geq 0$, are defined inductively by the relation

$$W_i(\gamma) = \sum_{t=0}^{i} Sq^t(v_{i-t})$$

Thus for $x \in H^{n-i}(BO)$, one has $t(Sq^i(x) + v_i\cdot x) = 0$

i.e. $(Sq^i(x) + v_i\cdot x) \in \operatorname{Ker} t$.

Let, $J = \{x \in H^*(BO)$: either $\dim x > n$ or for all $y \in H^{n-\dim x}(BO)$, $\langle f^*(x)\cdot f^*(y), [M^n]\rangle = 0\}$.

It is easy to see that J is an ideal of the graded algebra $H^*(BO)$. For, if $x_1, x_2 \in J$, such that $\dim x_1 = \dim x_2 = m \leq n$, then we have for all $y \in H^{n-m}(BO)$

$$\langle f^*(x_1+x_2)\, f^*(y), [M^n] \rangle$$

$$= \langle f^*(x_1)\, f^*(y), [M^n] \rangle + \langle f^*(x_2)\cdot f^*(y), [M^n] \rangle$$

$$= 0, \text{ i.e., } x_1+x_2 \in J.$$

Also if $x \in J$ and $x' \in H^*(BO)$ such that $\dim(x.x') \leq n$ then for all $y \in H^{n-\dim(x.x')}(BO)$ we have

$$\langle f^*(x\, x')\cdot f^*(y), [M^n] \rangle$$

$$= \langle f^*(x). f^*(x'y), [M^n] \rangle$$

$$= 0, \text{since } x'y \in H^{n-\dim x}(BO).$$

So, $x\, x' \in J$.

Let $P^* = \dfrac{H^*(BO)}{J}$, the quotient algebra.

Let $q: H^*(BO) \longrightarrow P^*$ be the quotient map. Clearly $P^* = 0$ if and only if all Stiefel-Whitney numbers of M^n are zero i.e. $P^* = 0$ if and only if M is a boundary. If $x \in H^*(BO)$, then by '$x = 0$ in P^*', we will mean $x \in J$.

V.2.1 PROPOSITION. If M^n is not a boundary then

(a) P^* is an n-dimensional graded algebra with Poincaré duality [M.4; p.128].

(b) The Steenrod algebra acts on P^* with the action given by

$$Sq^i(q(x)) = q(Sq^i(x)).$$

(c) $q : H^n(BO) \longrightarrow P^n \approx \mathbb{Z}_2$ is the homomorphism t i.e. if $x \in H^n(BO)$, then $q(x) = 0$ if and only if

$$t(x) = \langle f^*(x), [M^n] \rangle = 0$$

PROOF. From the construction of P^*, it is clear that $\dim P^* \leq n$. Now $P^n = \dfrac{H^n(BO)}{J^n}$, where

$$J^n = \{x \in H^n(BO) : \text{ for all } y \in H^0(BO) \approx \mathbb{Z}_2, \langle f^*(x).f^*(y), [M^n] \rangle = 0\}$$

$$= \{x \in H^n(BO) : \langle f^*(x), [M^n] \rangle = 0\}$$

Thus if $x \in H^n(BO)$ then $x \in J^n$ if and only if $t(x) = \langle f^*(x), [M^n] \rangle = 0$

Hence $J^n = \text{Ker } t$. Again

$t : H^n(BO) \longrightarrow \mathbb{Z}_2$ is an epimorphism, since M is not a boundary. Therefore

$$\mathbb{Z}_2 \approx \frac{H^n(BO)}{\text{Ker } t} = \frac{H^n(BO)}{J^n} = P^n$$

. Thus the dimension of P^* is n. Note here that we have eventually completed the proof of (c), since for $x \in H^n(BO)$, $q(x) = 0$ iff $x \in J^n$.

Now clearly $P^i \otimes P^{n-i}$ maps into $P^n \approx \mathbb{Z}_2$ by multiplication. If $\overline{x} \in P^i$ is the image of $x \in H^i(BO)$ under q and $\overline{x}.\overline{y} = 0$ for all $\overline{y}$ P^{n-i} then $q(x.y) = q(x).q(y)$ $= \overline{x}.\overline{y} = 0$. Therefore $x.y \in J$ for all $y \in H^{n-i}$ (BO). Hence from the definition of J it follows that $x \in J$, since dim (xy) = n. Thus $\overline{x} = 0$ in P*. So the multiplication provides a dual pairing, noting that for any base $\{\overline{x}_1, \overline{x}_2, \text{------------}, \overline{x}_n\}$ of P*, dim $\overline{x}_i \neq$ dim $\overline{x}_j$ if $i \neq j$. This completes the proof of (a).

For (b) it is enough to see that if $x \in J$ then $Sq^i x \in J$. Now $Sq^0(x) = x \in J$. Inductively suppose that $Sq^i\ x \in J$ for $0 \leq j < i$. Let $y \in H^{n-\dim x-i}(BO)$, (if $i+\dim x > n$, we are done). Then

$$t(Sq^i(x).y) = \langle f^*(Sq^i x)\, f^*(y),\ [M^n] \rangle$$

$$= \langle f^*(Sq^i(x)\cdot y),\ [M^n] \rangle$$

$$= \langle f^*(Sq^i(x\cdot y) + \sum_{\substack{j+k=i \\ k\neq 0}} Sq^j(x)\, Sq^k(y)),\ [M^n] \rangle$$

$$= \langle f^*(Sq^i(x\cdot y)),\ [M^n] \rangle$$

$$+ \sum_{\substack{j+k=i \\ k\neq 0}} \langle f^*(Sq^j(x))\, f^*(Sq^k(y)),\ [M^n] \rangle$$

$= \langle f^*(Sq^i(x \cdot y)), [M^n] \rangle$, since $Sq^j x \in J$ for $0 \leq j < i$,

$= \langle f^*(v_i \cdot x \cdot y), [M^n] \rangle$, where $v_i \in H^i(BO)$ are the Wu classes,

$= \langle f^*(x \cdot v_i \cdot y), [M^n] \rangle$, since the coefficients are all in $\mathbb{Z}_2$,

$= \langle f^*(x) \cdot f^*(v_i y), [M^n] \rangle$

$= 0$, since $x \in J$.

Therefore $Sq^i(x) \in J$. #

P^* is called the <u>Poincaré algebra</u> associated to M. In view of the above proposition we can say that the usual relations between the Stiefel-Whitney classes, the Wu classes, and the Steenrod squares still hold in P*.

V.2.2 DEFINITION. Let A^* be any graded algebra. An element $x \in A^*$ will be called k-decomposable if it is zero or it is the sum of products $y_1 y_2 \cdots y_p$ where dim $y_i > 0$ for all i, and $p \geqslant k$.

V.2.3 DEFINITION. Let M be an n-dimensional manifold. The cobordism caregory of M, denoted by cobcat (M), is the smallest positive integer k such that $\langle w_{i_1} \cdots w_{i_p}, [M^n] \rangle = 0$ for all partitions $i_1 + i_2 + \cdots + i_p = n$ with $k \leq p \leq n$. If no such k exists then define cobcat(M) = n+1.

Clearly cobcat (M) is a cobordism invariant, since two manifold are cobordant if and only if each Stiefel-Whitney number of one equals the corresponding Stiefel-Whitney number of the other. Also cobcat (M) $\leqslant$ nil $\tilde{H}^*(M) \leq$ cat M.

Further, if M_1^n and M_2^n be two n-dimensional manifolds then cobcat $(M_1^n \sqcup M_2^n)$ $\leqslant$ max (cobcat (M_1^n), cobcat (M_2^n)) because any Stiefel-Whitney number of $M_1^n \sqcup M_2^n$ equals the sum of the corresponding Stiefel-Whitney numbers of M_1^n and M_2^n.

V.2.4 PROPOSITION. If M is an n-dimensional manifold with cobcat(M) $\leqslant$ k, and P* is the associated Poincaré algebra, then

(1) If $x \in P^*$ is k-decomposable, then x is zero

(2) If $x \in P^*$ is (k-1)-decomposable and $\dim x < n$, then x is zero.

PROOF. (1) Consider the quotient map

$$q: H^*(BO) \longrightarrow P^*$$

Clearly, every k-decomposable element x in P^* can be obtained from a k-decomposable element y in $H^*(BO, \mathbb{Z}_2)$ i.e. $x = q(y)$. Since $H^*(BO)$ is generated by the Stiefel-Whitney classes $W_1(\gamma)$, $i = 1, 2, ---$, so y can be written as sum of products $W_{i_1}(\gamma) ----- W_{i_p}(\gamma)$, $p \geqslant k$.

Let $z \in H^{n-\dim y}(BO)$ then we have

$$\langle f^*(W_{i_1}(\gamma)) \cdots f^*(W_{i_p}(\gamma)) f^*(z), [M] \rangle$$

$$= \langle W_{i_1}(M) \cdots W_{i_p}(M) f^*(z), [M] \rangle$$

$= 0$, since $\operatorname{cobcat}(M) \leqslant k$.

Thus $W_{i_1}(\gamma) \cdots\cdots W_{i_p}(\gamma)$ and hence y lies in J. So $q(y) = x$ is zero in P^*.

(2) Note that if $x \in P^*$ is (k-1) decomposable and $\dim x < n$ then there exists some $y \in H^{\dim x}(BO)$ which is also (k-1) - decomposable, such that $q(y) = x$ and for all $z \in H^{n-\dim x}(BO)$, $y.z$ is k-decomposable. Hence

by (1) $y.z \in J$, for all $z \in H^{n-\dim x}(BO)$, and therefore by definition of J, $y \in J$, since $\dim(y.z) = n$. So, $q(y) = x$ is zero in P^*.

V.2.5. PROPOSITION. Let M be an n-dimensional manifold and let $c \in H^1(M)$. Let N be the submanifold of M dual to c, (see § 1). If $\langle c^2x, [M] \rangle = 0$ for all x in the subalgebra of $H^*(M)$ generated by c, $w_1(M)$, ---, $W_n(M)$ with $\dim x = n-2$ then

$$\langle W_{i_1}(N). \cdots\cdots W_{i_p}(N), [N] \rangle$$

$$= \langle c W_{i_1}(M) \cdots\cdots W_{i_p}(M), [M] \rangle$$

for any partition $i_1 + \cdots + i_p = n-1$.

PROOF. Let $\nu(N)$ be the normal bundle of N' in M , then $W(\nu) = 1+j^*(c)$ where $j:N \to M$ is the inclusion. Now

$$\nu(N) \oplus \tau(N) = j^*(\tau(M)), \qquad [M.4; p.30]$$

So, $W(\nu) \cdot W(N) = j^*(W(M))$.

Therefore $W(N) = j^*(W(M)) / 1+j^*(c)$

Also one has $\langle j^*(x), [N] \rangle = \langle c x, [M] \rangle$ for all $x \in H^{n-1}(M)$, see [S.6; p.79].

Now,

$$W(N) = \frac{j^*(W(M))}{1 + j^*(c)}$$

$$= (j^*(1 + W_1(M) + \cdots + W_n(M)))(1 + j^*(c))^{-1}$$

$$= j^*[(1 + W_1(M) + \cdots\cdots + W_n(M))(1+c)^{-1}]$$

$$= j^*[(1 + W_1(M) + \cdots\cdots + W_n(M))(1 + c + c^2 + \cdots\cdots)]$$

So, $W_i(N) = j^*[W_i(M) + c\,W_{i-1}(M) + \cdots\cdots + c^i]$

Thus for any partition $i_1 + i_2 + \cdots\cdots + i_p = n - 1$

we have

$$\langle W_{i_1} \cdot W_{i_2} \cdots\cdots W_{i_p}, [N] \rangle$$

$$= \langle j^* \prod_{k=1}^{p} (W_{i_k}(M) + c\,W_{i_k - 1}(M) + \cdots\cdots + c^{i_k}), [N] \rangle$$

$$= \langle c \prod_{k=1}^{p} (W_{i_k}(M) + c\,W_{i_k - 1}(M) + \cdots\cdots + c^{i_k}), [M] \rangle$$

$$= \langle c\,W_{i_1} \cdots\cdots W_{i_p}, [M] \rangle$$

since $\langle c^2 x, [M] \rangle = 0$, for all x in the subalgebra of $H^*(M)$ generated by c, $W_i(M)$, $i = 1, 2, \ldots, n$. #

§ 3. Lemmas for the cobordism classification of manifolds with cobordism category ≤ 3.

Throughout this section M^n will denote an n-dimensional manifold with cobcat $(M^n) \leq 3$ and with associated Poincaré algebra P^*.

V.3.1 LEMMA. (a) If $x\, W_i(\gamma) = 0$ in P^*, where $x \in H^{n-i}(BO)$ and $0 < i < n$ then $x = 0$ in P^*.

(b) For $j > 0$

$$W_{2j+1}(\gamma) = \begin{cases} Sq^1(W_{2j}(\gamma)), & \text{if } 2j+1 < n \\ 0, & \text{if } 2j+1 = n \end{cases}$$

in P^*.

PROOF. (a) Let $y \in H^i(BO)$. If $y = W_i(\gamma)$ then by hypothesis $x.y = 0$ in P^*. If $y \neq w_i(\gamma)$ then y is decomposable and so xy is 3-decomposable. Hence $x.y = 0$ in P^*, since cobcat$(M^n) \leq 3$, see (V.2.4). Thus $x \in J$, as $\dim(x.y) = n$. So (a) follows.

(b) By Wu's formula, we have

$$Sq^1(W_{2j}(\gamma)) = W_1(\gamma)\, W_{2j}(\gamma) + \binom{2j-1}{1} W_{2j+1}(\gamma)$$

$$= W_1(\gamma)\, W_{2j}(\gamma) + W_{2j+1}(\gamma).$$

Now assume $2j+1 < n$, then $W_1(\gamma) . W_{2j}(\gamma) = 0$ in P^* by (V.2.4). Therefore,

$$Sq^1(W_{2j}(\gamma)) = W_{2j+1}(\gamma) \text{ in } P^*,$$

if $2j+1 < n$. On the other hand if $2j+1 = n$ then

$$Sq^1(W_{2j}(\gamma)) = v_1 \cdot W_{2j}(\gamma) \text{ in } P^*$$

where $v_1 \in H^1(BO)$ is the first Wu class. Also by Wu's theorem, [§ 1],

$$W_1(\gamma) = Sq^0(v_1) + Sq^1(v_0) = v_1.$$

So, $Sq^1(W_{2j}(\gamma)) = W_1(\gamma) \cdot W_{2j}(\gamma)$ in P^*. Therefore

we get $W_{2j+1}(\gamma) = 0$ in P^*. #

V.3.2. LEMMA. If n is even and $n > 2$, then $W_i(\gamma) = 0$ in P^* for all odd i, i.e. all Stiefel-Whitney numbers of M divisible by $W_i(M)$ for odd i are zero.

PROOF. Firstly we take $i = 1$. By the last lemma, since n is even and $n > 2$

$$Sq^1(W_{n-2}(\gamma)) = W_{n-1}(\gamma) \text{ in } P^*$$

$$\begin{aligned} \text{Then } W_1(\gamma).W_{n-1}(\gamma) &= v_1 \cdot W_{n-1}(\gamma) \\ &= Sq^1(W_{n-1}(\gamma)) \\ &= Sq^1 Sq^1(W_{n-2}(\gamma)) \end{aligned}$$

Now by Adem's relations, [§ 1], we get $Sq^1 Sq^1 = 0$. Thus

$$W_1(\gamma)\, W_{n-1}(\gamma) = 0 \text{ in } P^*.$$

Next suppose that $i = 2j+1$, $j > 0$, and $n-i = 2k+1$, $k > 0$. Then by lemma (V.3.1) we have, in P^*,

$$W_i(\gamma) \cdot W_{n-i}(\gamma) = Sq^1(W_{2j}(\gamma))\, Sq^1(W_{2k}(\gamma))$$

$$= Sq^1(W_{2j}(\gamma)\ Sq^1(W_{2k}(\gamma)))$$, since $Sq^1\ Sq^1 = 0$. Therefore,

$$W_i(\gamma)\ W_{n-i}(\gamma) = v_1\, W_{2j}(\gamma)\, Sq^1 W_{2k}(\gamma)$$

$$= 0 \text{ in } P^*,$$ as it is 3-decomposable.

Thus by lemma (V.3.1), we see that $W_i(\gamma) = 0$ in P* for all odd i. #

<u>V.3.3. PROPOSITION.</u> If M is a nonbounding n-dimensional manifold with cobcat (M) ≤ 3, where n is even and $n > 2$, then M is cobordant to N x N, where N is also nonbounding and cobcat (N) ≤ 3.

<u>PROOF.</u> By lemma (V.3.2), all Stiefel-Whitney numbers of M divisible by $W_i(M)$ for odd i are zero and so, by [M.5], there exists a manifold N of dimension n/2 such that M is cobordant to N x N. From [M.5], we also know that

$$< w_{i_1} \cdots\cdots W_{i_p}, [N] >$$

$$= < w_{2i_1} \cdots\cdots W_{2i_p}, [M] >$$

for any partition $i_1 + i_2 + \cdots\cdots + i_p = n/2$. Thus N is nonbounding, because if N were a boundary then

$< w_{i_1} \cdots\cdots w_{i_p}, [N] > \; = 0$, for all

partition $i_1 + \cdots + i_p = n/2$

So, $< W_{2i_1} \cdots\cdots W_{2i_p}, [M] > \; = 0$ for all partitions of n of the form $2i_1+\cdots+2i_p = n$ and we already know that all Stiefel-Whitney numbers of M divisible by $W_i(M)$, for odd i, are zero. Thus all Stiefel-Whitney numbers of M becomes zero, showing that M becomes a boundary; whic h is a contradiction. Also clearly cobcat (N) = Cobcat (M) ≤ 3. #

V.3.4. COROLLARY. Let M be an n-dimensional manifold with cobcat (M) ≤ 3. Let $n = 2^t.m$ where either m is odd and $m \geq 3$, or $m = 2$. Then either M is a boundary or else M is cobordant to $(N)^{2^t}$, where N is a nonbounding m-dimensional manifold with cobcat (N) ≤ 3.

PROOF. For $t = 0$, there is nothing to prove. So assume $t > 0$. Then by the above proposition, if M is nonbounding, M is cobordant to $N_1 \times N_1 = (N_1)^2$ where the dimension of N_1 is 2^{t-1} m and N_1 is nonbounding with cobcat$(N_1) \leq 3$. Thus the corollary follows by the process of induction.

V.3.5 COROLLARY. Let M be an n-dimensional manifold with cobcat(M) ≤ 3.

(1) If $n = 2^s$, $s > 0$ then either M is a boundary or else M is cobordant to $(\mathbb{R}P^2)^{2^{s-1}}$.

(2) If $n = s^s.3$, $s \geq 0$ then M is a boundary.

PROOF. By the corollary (V.3.4), it is enough to look at the group $\mathfrak{N}_2$ and $\mathfrak{N}_3$ of cobordism classes of manifolds of dimension 2 and 3 respectively. We know that $\mathfrak{N}_2 \cong \mathbb{Z}_2$ with generator $[\mathbb{R}P^2]$ and $\mathfrak{N}_3 = 0$; also cobcat $(\mathbb{R}P^2) \leq$ cat $(\mathbb{R}P^2) = 3$. Hence the corollary follows. #

V.3.6 LEMMA. Let $n = 2^r + m$, where $1 \leq m < 2^r$.

Then

(1) $W_{2^s+k}(\gamma) = Sq^k(W_{2^s}(\gamma))$ in P^* if $0 \leq k < 2^s$ and $2^s + k < n$.

(2) $W_j(\gamma) = 0$ in P^* if $0 < j \leq m$.

PROOF. (1) By Wu's formula we have

$$Sq^k(W_{2^s}(\gamma)) = W_{2^s+k}(\gamma) + \sum_{t=0}^{k-1} \binom{2^s-k+t-1}{t} W_{k-t}(\gamma)\, W_{2^s+t}(\gamma).$$

Since by (V.2.4) decomposable elements of dimension less than n are zero in P^*, so (1) follows.

(2) Let $0 < j \leq m$. Then $n-j = 2^r+k$, where $k = m-j$ with $0 \leq k < m < 2^r$ and $2^r+k < 2^r+m = n$. Thus by (1), $W_{n-j}(\gamma) = Sq^k(W_{2^r}(\gamma))$. It is well-known that the Wu class v_j

is decomposable, if j is not a power of 2, and hence zero in P^*. By Wu's theorem

$$W_{2^r}(\gamma) = v_{2^r} + \sum_{i=1}^{2^r} Sq^i(v_{2^r-i})$$

$$= v_{2^r} + \sum_{i=1}^{2^{r-1}} Sq^i(v_{2^r-i}) ,$$

since $Sq^i(x) = 0$ if $i > \dim x$. Now v_{2^r-i} is not a power of 2 if $1 \leq i < 2^{r-1}$ and so zero in P^*. If $i = 2^{r-1}$ then $2^r - i = 2^{r-1}$, and $Sq^{2^{r-1}}(v_{2^{r-1}}) = v_{2^{r-1}} \cdot v_{2^{r-1}}$ (since $Sq^k(a) = a^2$ if $\dim a = k$) with total dimension $= 2^r < n$ and so zero in P^*. Thus it follows that

$$W_{2^r}(\gamma) = v_{2^r} \quad \text{in } P^* .$$

Also since $2^r > n/2$ $(n = 2^r + m < 2^r + 2^r = 2^{r+1})$,

so $W_{2^r}(\gamma) = v_{2^r} = 0$ in P^*.

Thus $W_{n-j}(\gamma) = 0$ in P^*.

So by lemma (V.3.1), $W_j(\gamma) = 0$ in P^*. #

V.3.7 PROPOSITION. Let $n = 2^r + m$, $1 \leq m < 2^r$. If $v_1 = v_2 = 0$ in P^* and if $n > 3$ is odd, then M is a boundary.

PROOF. We shall show that all positive Wu classes are zero in P^*. In that case by Wu's theorem all positive Stiefel-Whitney classes will be zero in P^* and hence all Stiefel-Whitney numbers of M will be zero. Now we know that $v_j = 0$ in P* if j is not a power of 2, as in that case v_j is decomposable. Also by hypothesis $v_1 = v_2 = 0$ in P*. Further $v_{2^r} = 0$ in P* as $2^r > n/2$. So it is enough to show that $v_{2^t} = 0$ in P* if $2 < 2^t \leq 2^{r-1}$. Now by lemma (V.3.1) we have

$$v_{2^t} W_{n-2^t}(\gamma') = Sq^{2^t} Sq^1 \left(W_{n-2^t-1}(\gamma')\right)$$

and by Adem's relation, § 1,

$$Sq^2 Sq^{2^t-1} = \binom{2^t-1-0-1}{2} Sq^{2^t+1} + \binom{2^t-1-1-1}{2-2} Sq^{2^t} Sq^1$$

$$= Sq^{2^t+1} + Sq^{2^t} Sq^1 .$$

By Adem's relation, we also have

$$Sq^1 Sq^{2^t} = \binom{2^t-0-1}{1-0} Sq^{2^t+1} = Sq^{2^t+1} .$$

So, $Sq^2 Sq^{2^t-1} = Sq^1 Sq^{2^t} + Sq^{2^t} Sq^1$

or, $Sq^{2^t} Sq^1 = Sq^2 Sq^{2^t-1} + Sq^1 Sq^{2^t}$.

Thus,

$$v_{2^t} W_{n-2^t}(\gamma') = Sq^2 Sq^{2^t-1} \left(W_{n-2^t-1}(\gamma')\right) + Sq^1 Sq^{2^t} \left(W_{n-2^t-1}(\gamma')\right).$$

$$= v_2 \cdot Sq^{2^t-1} \left(W_{n-2^t-1}(\gamma')\right) + v_1 Sq^{2^t} \left(W_{n-2^t-1}(\gamma')\right).$$

= 0 in P*, by hypothesis.

Thus, $v_{2^t} = 0$ in P*.

Hence the proposition follows. #

Let $\alpha(n)$ denote the number of terms in the diadic expansion of n.

V.3.8. COROLLARY. If M is an n-dimensional manifold with $\alpha(n) \geq 3$ and with cobcat $(M) \leq 3$, then M is a boundary.

PROOF. Suppose n is even then, since $\alpha(n) \geq 3$, $n = 2^t.m$ where m is odd and $m > 3$. Then by corollary (V.3.4) either M is a boundary or M is cobordant to $(N)^{2^t}$, where N is a nonbounding m dimensional manifold with cobcat $(N) \leq 3$. Moreover $\alpha(m) \geq 3$, since $\alpha(n) \geq 3$. Thus it is enough to assume that n is odd and we write $n = 2^r+k$ where $3 \leq k < 2^r$ and k is odd. By lemma (V.3.6), $W_1(\gamma) = W_2(\gamma) = 0$ in P^*. So, $v_1 = W_1(\gamma) = 0$ in P^* and therefore

$$\begin{aligned} W_2(\gamma) &= Sq^2(v_0) + Sq^1(v_1) + Sq^0(v_2) \\ &= 0 + 0 + v_2 = v_2 \end{aligned}$$

Hence, v_2 is also zero in P^*. So by proposition (V.3.7), M is a boundary. #

V.3.9 COROLLARY. Let M be an n-dimensional manifold with cobcat $(M) \leq 3$. If $n = 2^s+1$, $s \geq 2$, and $\langle W_2 W_{n-2}, [M] \rangle = 0$, then M is a boundary.

PROOF. By lemma (V.3.6) $W_1(\gamma) = 0$ in P^*.

If $\langle W_2 \cdot W_{n-2}, [M] \rangle = 0$

i.e. if $\langle f^*(W_2(\gamma)) \cdot f^*(W_{n-2}(\gamma), [M] \rangle = 0$

then $W_2(\gamma)\cdot W_{n-2}(\gamma) = 0$ in P^*. So by (V.3.1) $W_2(\gamma) = 0$ in P^*. Therefore $v_1 = v_2 = 0$ in P^* (see the proof of (V.3.8)), where $v_i \in H^i(BO)$, $i = 1,2$, are Wu classes. Hence by (V.3.7), M is a boundary. #

V.3.10 PROPOSITION. Let $n = 2^s+1$, where $s \geq 2$. If there exist two nonbounding n-dimensional manifolds M_1 and M_2 with cobcat $(M_1) \leq 3$ and cobcat$(M_2) \leq 3$, then they are cobordant.

PROOF By (V.3.9),

$$\langle W_2\cdot W_{n-2}, [M_1]\rangle = \langle W_2\cdot W_{n-2}, [M_2]\rangle = 1$$

Now

$$\langle W_{i_1}\cdots W_{i_p}, [M_1 \sqcup M_2]\rangle = \langle W_{i_1}\cdots W_{i_p}, [M_1]\rangle + \langle W_{i_1}\cdots W_{i_p}, [M_2]\rangle$$

for any partition $i_1 + i_2 + \cdots\cdots i_p = n$, where $\sqcup$ is the disjoint union. Thus clearly

$$\langle W_2\cdot W_{n-2}, [M_1 \sqcup M_2]\rangle = 1+1 = 0$$

and also cobcat $(M_1 \sqcup M_2) \leq 3$.

Hence by (V.3.9), $M_1 \sqcup M_2$ is a boundary. Thus the proposition follows. #

§ 4 Complete classification of manifolds with cobordism category ≤ 3.

A manifold M^n is called decomposable if

$$M^n = \bigsqcup \left(N_1^{i_1} \times N_2^{i_2} \times \cdots \times N_r^{i_r} \right),$$

where $i_1 + i_2 + \cdots + i_r = n$, $0 < i_1, i_2, \ldots, i_r < n$ and each $N_k^{i_k}$ is a i_k - dimensional manifold ($1 \leq k \leq r$).

Now for a manifold M^n, consider the total Stiefel-Whitney class

$$W(M) = 1 + W_1(M) + \cdots + W_n(M)$$

$$= (1+t_1)(1+t_2)\cdots(1+t_n) \qquad \text{(Say)}$$

where $t_i \in H^1(M^n)$, $1 \leq i \leq n$.

Then

$$W_1(M) = \sum_i t_i$$

$$W_2(M) = \sum_{i<j} t_i t_j$$

$$\vdots$$

$$W_n(M) = t_1 \cdot t_2 \cdots t_n .$$

The expressions in t's, associated to each W_i above, are called elementary symmetric polynomials. We know that any symmetric polynomial can be generated by elementary symmetric polynomials. Therefore it follows that $t_1^n + t_2^n + \cdots + t_n^n$ is a polymial in $W_1(M)$, $W_2(M)$, $\cdots W_n(M)$. Let us denote this polynomial by S_n.

Thom [T.3] has proved that, M^n is indecomposable iff $\langle S_n, [M^n] \rangle \neq 0$.

Further it is well-known that for a given odd integer $d > 0$, not of the form $2^k - 1$, there exists a unique *(upto bordism) indecomposable Dold manifold P(m,n) of* dimension d. In fact one can take $m = 2^r - 1$ and $n = 2^r.s$ where $(d+1) = 2^r(2s+1)$. In particular we can see that the Dold manifold $P(1,2^{s-1})$ is indecomposable. Therefore $\langle s_{2^s+1}, [P(1,2^{s-1})]\rangle = 1$, by Thom's result mentioned as above.

<u>V.4.1 THEOREM.</u> Let $n = 2^s+1$, $s \geqslant 2$, and

$$M^n = \coprod_{j=0}^{2^{s-2}} P(1, 2^{s-1}-2j) \times N_j \quad ,$$

where $P(1,2^{s-1}-2j)$ is a Dold manifold, N_j is the 4j degree term in the formal inverse

$$(1 + \mathbb{C}P^2 + \mathbb{C}P^4 + \mathbb{C}P^6 + \text{-----})^{-1}.$$

Then M is nonbounding, and cobcat (M) $\leqslant 3$.

<u>PROOF.</u> Let P_j denote $P(1,2^{s-1}-2j)$.

$$\text{Now} \quad \langle s_n, [M^n]\rangle = \sum_{j=0}^{2^{s-2}} \langle s_{2^s+1}, [P_j \times N_j]\rangle$$

$$= \langle s_{2^s+1}, [P(1,2^{s-1})]\rangle = 1 \quad ,$$

using Thom's result mentioned above. Hence M^n is not a boundary. Now it remains to prove that cobcat $(M^n) \leqslant 3$.

Let $i_1 + i_2 + \text{-----} + i_p = 2^s+1$ be a partition.

Then,

$$\langle W_{i_1} \text{---} W_{i_p}, [M]\rangle = \sum_{j=0}^{2^{s-2}} \langle W_{i_1} \text{---} W_{i_p}, [P_j \times N_j]\rangle$$

Since $P(1,0) \approx S^1$ is a boundary, we may assume that $j < 2^{S-2}$. It is known (§ 1) that the total Stiefel-Whitney class

$$W(P_j) = (1+c)(1+c+d)^{2^{S-1}-2j+1}$$

where $c \in H^1(P_j)$ and $d \in H^2(P_j)$ such that $c^2 = 0$ and $d^{2^{S-1}-2j+1} = 0$. Thus

$$\begin{aligned} W(P_j) &= (1+c)(1+c+d)(1+c+d)^{2^{S-1}-2j} \\ &= (1+c)(1+c+d)(1+d^2)^{2^{S-2}-j} \\ &= (1+d+cd)(1+d)^{2^{S-1}-2j} \end{aligned}$$

Also clearly $W_{odd}(N_j) = 0$ and so by using the formula for the Stiefel-Whitney classes of the cartesian product we have for $i > 1$, $W_{2i+1}(P_j \times N_j) = \bar{c}\,\bar{d}\,y$, where y is the (2i-2) degree term in

$$(1+d)^{2^{S-1}-2j} \otimes W(N_j), \qquad \bar{c} = c \otimes 1 \text{ and } \bar{d} = d \otimes 1$$

(1 is the unity of $H^0(N_j)$.).

Clearly $\bar{c}^2 = 0$.

Thus $W_{odd}(P_j \times N_j) \cdot W_{odd}(P_j \times N_j) = 0$

Therefore the only nonzero Stiefel-Whitney numbers of M are of the form

$$\begin{aligned} &\left\langle W_{2i+1} W_{2i_2} \cdots W_{2i_p}, [M] \right\rangle \\ &= \sum_{j=0}^{2^{S-2}} \left\langle \bar{c}\,\bar{d}\,y\, W_{2i_2} \cdots W_{2i_p}, [P_j \times N_j] \right\rangle \end{aligned}$$

Now since $c^2 = 0$, by proposition (V.2.5)

$$\langle W_{i_1} \cdots W_{i_q}, [\tilde{P}_j] \rangle = \langle c W_{i_1} \cdots W_{i_q}, [P_j] \rangle$$

for any partition $i_1 + i_2 + \cdots + i_q = 2^s$, where $\tilde{P}_j$ is the submanifold of P_j dual to c. We claim that $\tilde{P}_j$ is cobordant to $\mathbb{C}P^{2^{s-1}-2j}$. We will show that for all partition $i_1 + i_2 + \cdots + i_p = 2^s$

$$\langle W_{i_1} \cdots W_{i_p}, [\tilde{P}_j] \rangle = \langle W_{i_1} \cdots W_{i_p}, [\mathbb{C}P^{2^{s-1}-2j}] \rangle$$

i.e. $\langle c W_{i_1} \cdots W_{i_p}, [P_j] \rangle = \langle W_{i_1} \cdots W_{i_p}, [\mathbb{C}P^{2^{s-1}-2j}] \rangle$

But $W(\mathbb{C}P^{2^{s-1}-2j}) = (1+d)^{2^{s-1}-2j+1}$

and $W(P_j) = (1+c)(1+c+d)^{2^{s-1}-2j+1}$

Since one c is already present in $\langle c W_{i_1} \cdots W_{i_p}, [P_j] \rangle$ and $c^2 = 0$, the only term from $(1+c)(1+c+d)^{2^{s-1}-2j+1}$ which may give nonzero Stiefel-Whitney numbers $\langle c W_{i_1} \cdots W_{i_p}, [P_j] \rangle$ will actually come from $(1+d)^{2^{s-1}-2j+1} = W(\mathbb{C}P^{2^{s-1}-2j})$. Therefore, $\tilde{P}_j$ is bordic to $\mathbb{C}P^{2^{s-1}-2j}$. Hence, the repeated use of product formula for Stiefel-Whitney classes yields,

$$\langle \bar{c}\bar{d} y W_{2i_2} \cdots W_{2i_p}, [P_j \times N_j] \rangle$$

$$= \langle \bar{d} y W_{2i_2} \cdots W_{2i_p}, [\tilde{P}_j \times N_j] \rangle$$

$$= \langle \bar{d} y W_{2i_2} \cdots W_{2i_p}, [\mathbb{C}P^{2^{s-1}-2j} \times N_j] \rangle$$

Here y is the $(2i-2)$ degree term in

$$(1+d)^{2^{s-1}-2j} \otimes W(N_j) = \frac{W(\mathbb{C}P^{2^{s-1}-2j})}{(1+d)} \otimes W(N_j)$$

$$= (1+\bar{d}+\bar{d}^2+\cdots)\, W(\mathbb{C}P^{2^{s-1}-2j} \times N_j)$$

Thus, $y = \displaystyle\sum_{k=0}^{i-1} \bar{d}^k\, W_{2i-2k-2}(\mathbb{C}P^{2^{s-1}-2j} \times N_j)$

and so $\langle \bar{c}\bar{d}\, y\, W_{2i_2}\cdots W_{2i_p}, [P_j \times N_j] \rangle$

$$= \sum_{k=0}^{i-1} \langle \bar{d}^{k+1} W_{2i-2k-2} W_{2i_2}\cdots W_{2i_p}, [\mathbb{C}P^{2^{s-1}-2j} \times N_j] \rangle$$

Hence, $\langle W_{2i+1} W_{2i_2}\cdots W_{2i_p}, [M] \rangle$

$$= \sum_{j=0}^{2^{s-2}} \sum_{k=0}^{i-1} \langle \bar{d}^{k+1} W_{2i-2k-2} W_{2i_2}\cdots W_{2i_p}, [\mathbb{C}P^{2^{s-1}-2j} \times N_j] \rangle$$

$$= \sum_{k=0}^{i-1} \langle \tilde{d}^{k+1} W_{2i-2k-2} W_{2i_2}\cdots W_{2i_p}, [Y_{2^s}] \rangle$$

where $Y_{2^s} = \displaystyle\coprod_{j=0}^{2^{s-2}} (\mathbb{C}P^{2^{s-1}-2j} \times N_j)$ and

$$\tilde{d} = \oplus \bar{d} \in H^2(Y_{2^s}) = \bigoplus_{j=0}^{2^{s-2}} H^2(\mathbb{C}P^{2^{s-1}-2j} \times N_j).$$

For generality we write

$$Y_{4k} = \coprod_{j=0}^{k} (\mathbb{C}P^{2k-2j} \times N_j).$$

The pair $[\mathbb{C}P^{2k-2j}, \gamma^1_{2k-2j}]$ can be regarded as a class in the bordism group $\Omega^U_{2k-2j}(\mathbb{C}P^\infty)$ (see §1), where γ^1_{2k-2j} is the canonical complex line bundle over $\mathbb{C}P^{2k-2j}$. Also we can regard $[Y_{4k}, \bar{\gamma}^1_{2k-2j}]$ as a class in $\Omega^U_*(\mathbb{C}P^\infty)$, as $\Omega^U_*(\mathbb{C}P^\infty)$ has the structure of a Ω^U_*- module ($\bar{\gamma}^1_{2k-2j}$ is the 1-dimensional complex vector bundle over Y_{4k} induced by γ^1_{2k-2j}). Conner and Floyed [C.2] proved that there is a unique (upto bordism) basis $\{ X_{2i} \mid i = 0,1,---- \}$ of $\Omega^U_*(\mathbb{C}P^\infty)$ as an Ω^U_*-module such that

(i) $X_0 = 1$

(ii) $\Delta X_{2i} = X_{2i-2}$ for $i > 0$

(iii) $\varepsilon X_{2i} = 0$ for $i > 0$.

where ε and Δ are augmentation and Smith homomorphisms respectively, § 1.

For $k > 0$, put $\tilde{Y}_{2i} = \begin{cases} Y_{4k} & \text{if } i = 2k \\ \Delta Y_{4k} & \text{if } i = 2k-1 \end{cases}$.

One can see that the class $\tilde{Y}_{2i}$ defined above satisfy the properties (i), (ii), (iii) mentioned above and hence forms a basis for $\Omega^U_*(\mathbb{C}P^\infty)$.

There is a direct construction of the basis elements of $\Omega^U_*(\mathbb{C}P^\infty)$, due to Stong. Let γ^1_1 be the canonical complex line bundle over $\mathbb{C}P^1$. Let ξ be the complex k-plane bundle over $\mathbb{C}P^1$ defined as $\xi = \gamma^1_1 \oplus (k-1)\mathbb{C}$ i.e. Whitney sum of γ^1_1 and (k-1) dimensional trivial complex bundle over $\mathbb{C}P^1$. Then $\mathbb{C}P(\xi)$ is the space of complex lines in the fibres of ξ. Let λ be the complex line bundle over $\mathbb{C}P(\xi)$ whose total space consists of the pairs (α, x) where $\alpha \in \mathbb{C}P(\xi)$ and x is a point on α.

Choose

$$Z_{2k} = [\mathbb{C}P(\xi), \lambda] = [\mathbb{C}P(\gamma_1^1 \oplus (k-1)\mathbb{C}), \lambda]$$

Then $\{ Z_{2k} \}_{k=0}^{\infty}$ forms a basis of $\Omega_*^U(\mathbb{C}P^\infty)$. By the uniqueness of the basis of $\Omega_*^U(\mathbb{C}P^\infty)$ upto bordism, we have that $\tilde{Y}_{4k}$ is bordic to Z_{4k} for all k. That is Y_{2s} is bordic to $\mathbb{C}P(\gamma_1^1 \oplus (2^{s-1}-1)\mathbb{C})$. Thus we have

$$\langle W_{2i+1} W_{2i_2} --- W_{2i_p}, [M] \rangle$$

$$= \sum_{k=0}^{i-1} \langle d^{k+1} W_{2i-2k-2} W_{2i_2} --- W_{2i_p}, [\mathbb{C}P(\gamma_1^1 \oplus (2^{s-1}-1)\mathbb{C}] \rangle$$

Where d is now the mod 2 reduction of the first Chern class $c_1(\lambda)$ of λ. Also, the total Stiefel-Whitney class of $\mathbb{C}P(\gamma_1^1 \oplus (2^{s-1}-1)\mathbb{C})$ is given by, [C.3; P36],

$$W = (1+d+a)(1+d)^{2^{s-1}-1},$$

where a is the mod 2 reduction of the first Chern class of $\mathbb{C}P^1$ (so $a^2 = 0$). Thus, $W = (1+d)^{2^{s-1}} + a(1+d)^{2^{s-1}-1}$

and so, for $0 < 2j < 2^s$,

$$W_{2j}(\mathbb{C}P(\gamma_1^1 \oplus (2^{s-1}-1)\mathbb{C})) = a\, d^{j-1}.$$

Hence it follows that if $p \geq 3$, then

$$\langle W_{2i+1} W_{2i_2} ----- W_{2i_p} ; [M] \rangle = 0,$$

as $a^2 = 0$. Thus cobcat $(M) \leq 3$. #

Thus summing up we have

V.4.2 THEOREM. If M is an n-dimensional manifold with cobcat(M) ≤ 3, then M is either a boundary or else one of the following is true:

(1) $n = 2^s$, $s \geq 1$ and M is cobordant to $(\mathbb{R}P^2)^{2^{s-1}}$.

(2) $n = 2^r(2^s+1)$, $r \geq 0$, $s \geq 2$ and M is cobordant to $(\tilde{M})^{2^r}$ with

$$\tilde{M} = \coprod_{j=0}^{2^{s-2}} P(1, 2^{s-1} - 2j) \times N_j ,$$

N_j being the $4j$ degree term in the formal inverse $(1 + \mathbb{C}P^2 + \mathbb{C}P^4 + \cdots)^{-1}$.

PROOF. Suppose M is not a boundary. Then by (V.3.8) we may assume $\alpha(n) \leq 2$. If $\alpha(n) = 1$ i.e. $n = 2^s$ then by (V.3.5) we get (1). Let $\alpha(n) = 2$ i.e. $n = 2^r(2^s+1)$ where $r \geq 0$, $s \geq 1$. By (V.3.5) we may assume $s \geq 2$. Then by (V.3.4), M is bordic to $(\tilde{M})^{2^r}$, where $\tilde{M}$ is nonbounding $2^s + 1$ dimensional manifold with cobcat$(\tilde{M}) \leq 3$. Now by (V.3.10) and (V.4.1), $\tilde{M}$ is bordic to

$$\coprod_{j=0}^{2^{s-2}} P(1, 2^{s-1} - 2j) \times N_j .$$

#

§ 5 Cobordism classification of manifolds with Cobordism category ≤ 4.

In this section we shall try to describe the cobordism class of an n-dimensional manifold M with cobcat (M) ≤ 4. Let P_2^* denote the ideal of the Poincare' Algebra P^*, consisting of the decomposable elements of P^*.

To begin with we consider only the even dimensional manifolds and we shall see how the classification problem can be reduced to one of looking at odd-dimensional manifolds. Analogous to what we have done for the case cobcat (M) ≤ 3, all we need to do here is to examine the product $W_i(\gamma)\, W_j(\gamma)\, W_k(\gamma)$ where $i, j, k \geq 0$ and $i+j+k = n$; because here cobcat (M) ≤ 4 and so 4-decomposable elements are all zero in P^*, (V.2.4).

From Wu's formula we have

(1) $W_{2^s+k}(\gamma) \equiv Sq^k(W_{2^s}(\gamma)) \bmod P_2^*$,
where $2^s + k \leq n$ and $0 \leq k < 2^s$, and from Wu's theorem we have

(2) $W_{2^s}(\gamma) \equiv V_{2^s} \bmod P_2^*$, (see the proof of (V.3.6))

V.5.1. LEMMA. $W_1^2(\gamma) = 0$ in P^* if $n \geq 3$

PROOF. We have

$$Sq^1(w_1^2(\gamma)) = Sq^1(w_1(\gamma))\, w_1(\gamma) + w_1(\gamma) Sq^1(w_1(\gamma))$$

$$= 2\, w_1^3(\gamma) = 0 \quad \text{in} \quad P^* ,$$

and similarly, $Sq^2\,(W_1^2\,(\gamma)) = W_1^4(\gamma) = 0$ in P^*.

Thus clearly $sq^j(W_1^2\,(\gamma)) = 0$ in P^* if $j > 0$. Let $n-2 = 2^s+k$, where $0 \leq k < 2^s$.

If $k > 0$, $W_1^2(\gamma)\, W_{n-2}(\gamma)$

$= W_1^2(\gamma)\ Sq^k\ (W_{2^s}(\gamma))$

$= Sq^k\big(w_1^2(\gamma)\ w_{2^s}(\gamma)\big)$

$= v_k\ w_1^2(\gamma)\ w_{2^s}(\gamma)$

$= 0$ in P^*

If $k = 0$, $\quad w_1^2(\gamma)\, w_{n-2}(\gamma) = w_1^2(\gamma)\, w_{2^s}(\gamma)$

$= w_1^2(\gamma)\, v_{2^s} = Sq^{2^s}(w_1^2(\gamma)) = 0$ in P^*.

Hence it follows that $W_1^2(\gamma) = 0$ in P^*. #

<u>V.5.2 LEMMA.</u> If $n \geqslant 3$ then $W_1(\gamma) = 0$ in P^* unless $n = 2^r(\,2^s + 1) + 1$, where $r \geqslant 0$ and $s \geqslant 2$.

<u>PROOF.</u> Let N be the (n-1)-dimensional submanifold of M dual to $W_1(M)$. Then, by (v.2.5) and (v.5.1) we have

$$\langle W_{i_1} \text{----} W_{i_p}\, , [N] \rangle = \langle W_1 W_{i_1} \text{----} W_{i_p}\, , [M] \rangle$$

for any partition $i_1 + i_2 + \cdots + i_p = n-1$.

Thus cobcat $(N) \leq 3$ and $W_1(M) = 0$ in P^* iff N is a boundary. By (V.4.2), N is a boundary unless $(n-1) = 2^t$ or $2^r(2^s+1)$ where $t \geq 1$, $r \geq 0$ and $s \geq 2$. Further if $n-1 = 2^t$ and N is not a boundary then by (V.3.5), N is cobordant to $(\mathbb{R}P^2)^{2^{t-1}}$ and

$$\langle W_{2^t-1}^2, [N]\rangle = \langle W_1^2, [\mathbb{R}P^2]\rangle \neq 0$$

(see the proof of (V.3.3)). However,

$$\langle W_{2^t-1}^2, [N]\rangle = \langle W_1 W_{2^t-1}^2, [M]\rangle$$

and in P^*, $W_1(\gamma)\, W_{2^t-1}^2(\gamma)$

$$= v_1 W_{2^t-1}^2(\gamma) = Sq^1(W_{2^t-1}^2(\gamma))$$

$$= 2\, W_{2^t-1}(\gamma) \cdot Sq^1(W_{2^t-1}(\gamma)) = 0.$$

Thus N is a boundary even when $n-1 = 2^t$. Hence the lemma follows. #

V.5.3. LEMMA. If $W_1(\gamma) = 0$ in P^* then,

(1) $W_{2j+1}(\gamma) = Sq^1(W_{2j}(\gamma))$ in P^*

(2) $Sq^1(x) \cdot Sq^1(y) = 0$ in P^*, if

$\dim x + \dim y + 2 = n$.

PROOF. For (1) note that

$$Sq^1(W_{2j}(\gamma)) = W_1(\gamma)\, W_{2j}(\gamma) + W_{2j+1}(\gamma)$$

For (2) we have

$$Sq^1(x) \cdot Sq^1(y) = Sq^1(x \cdot Sq^1(y)) = W_1(\gamma) \cdot x \cdot Sq^1(y).$$

#

V.5.4. LEMMA. If n is even, $n > 2$ and $w_1(\gamma) = 0$ in P^* then

(1) $$W_i(\gamma)\, W_{n-i}(\gamma) = 0 \text{ in } P^* \text{ for all odd } i$$

(2) $$W_2(\gamma)\, Sq^1(x) = 0 \text{ in } P^* \text{ if } \dim x = n-3$$

PROOF. (1). Since $w_1(\gamma) = 0$ we may assume that $1 < i < n-1$. Then

$$W_i(\gamma)\, W_{n-i}(\gamma) = Sq^1(W_{i-1}(\gamma))\, Sq^1(W_{n-i-1}(\gamma))$$

$= 0$ in P^*, by (V.5.3).

(2). Firstly we note that if x is 3-decomposable then there is nothing to prove. Now, n-3 is odd; so by Wu's formula

$$Sq^1(W_{n-3}(\gamma)) = W_1(\gamma) \cdot W_{n-3}(\gamma)$$

in P^*. Thus, $w_2(\gamma).Sq^1(w_{n-3}(\gamma)) = 0$ in P^*. Now let $n-3 = j+k$ and assume that j is odd. Then in P^*

$$\begin{aligned}
& w_2(\gamma)\; Sq^1(w_j(\gamma)\; w_k(\gamma)) \\
&= W_2(\gamma)\, Sq^1(w_j(\gamma))\, w_k(\gamma) + w_2(\gamma)\, w_j(\gamma)\, Sq^1(w_k(\gamma)) \\
&= W_2(\gamma)\; w_j(\gamma)\; w_{k+1}(\gamma) = v_2\; w_j(\gamma)\; w_{k+1}(\gamma) \\
&= Sq^2(w_j(\gamma)\; w_{k+1}(\gamma)) \\
&= Sq^2(w_j(\gamma))\, w_{k+1}(\gamma) + w_j(\gamma)\, Sq^2(w_{k+1}(\gamma)) \\
&\quad + Sq^1(w_j(\gamma))\, Sq^1(w_{k+1}(\gamma))
\end{aligned}$$

$$= Sq^2(w_j(\gamma))\, w_{k+1}(\gamma) + w_j(\gamma)\cdot Sq^2(w_{k+1}(\gamma)),$$

by (V.5.3). Also by Wu's formula

$$Sq^2(w_i(\gamma)) = w_2(\gamma)\, w_i(\gamma) + \binom{i-1}{2} w_{i+2}(\gamma)$$

in P* for all $i \geq 2$.

Thus, $w_2(\gamma)\, Sq^1(w_j(\gamma)\, w_k(\gamma))$

$$= \binom{j-1}{2} w_{j+2}(\gamma)\, w_{k+1}(\gamma) + \binom{k}{2} w_{k+3}(\gamma)\, w_j(\gamma)$$

$= 0$ in P*, by (1)

Therefore, (2) follows. #

<u>V.5.5 LEMMA.</u> Let $i+j+k < n$. Then

$$Sq^i(w_j(\gamma)\, w_k(\gamma)) = \sum_{\beta=0}^{i} \binom{j-1}{\beta}\binom{k-1}{i-\beta} w_{j+\beta}(\gamma)\, w_{k+i-\beta}(\gamma)$$

in P*.

<u>PROOF.</u> We have by the Cartan formula, § 1,

$$Sq^i(w_j(\gamma)\, w_k(\gamma)) = \sum_{\beta=0}^{i} Sq^\beta(w_j(\gamma))\, Sq^{i-\beta}(w_k(\gamma))$$

The result now follows using Wu's formula and the fact that 4-decompasable elements and the 3-decomposable elements not in the top dimension are zero in P*. #

<u>V.5.6 PROPOSITION.</u> If n is even, $n > 2$ and $w_1(\gamma) = 0$ in P* then $w_j(\gamma) = 0$ in P* for all odd j.

<u>PROOF.</u> Since $w_1(\gamma) = 0$ we may assume that $j > 1$. By (V .5.4), $w_j(\gamma)\, w_{n-j}(\gamma) = 0$ and clearly $w_j(\gamma).x = 0$ in P* if x

is 3-decomposable. Thus all we need to show is that $w_j(\gamma).x = 0$ in P* if x is decomposable and $\dim(x) = n-j$.

(1) Let $K = n-j-2$. Thus k is odd and so by (V.5.3) and using Adem's relation, we have

$$w_j(\gamma)\, w_2(\gamma)\, w_k(\gamma) = w_2(\gamma)\, Sq^1 (w_{j-1}(\gamma)\, w_k(\gamma))$$

$= 0$ in P*, by (V.5.4).

(2) Let $k = n-j-2^a$ where $a \geqslant 2$ and $k > 0$. Thus k is odd and so

$$w_j(\gamma)\, w_{2^a}(\gamma)\, w_k(\gamma) = w_{2^a}(\gamma)\, Sq^1 (w_{j-1}(\gamma)\, w_k(\gamma))$$

$$= v_{2^a}\, Sq^1 (w_{j-1}(\gamma)\, w_k(\gamma))$$

$$= Sq^{2^a}\, Sq^1 (w_{j-1}(\gamma)\, w_k(\gamma))$$

By using Adem's relation we have

$$\begin{aligned} w_j(\gamma)\, w_{2^a}(\gamma)\, w_k(\gamma) &= Sq^2\, Sq^{2^a-1} (w_{j-1}(\gamma)\, w_k(\gamma)) \\ &\quad + Sq^{2^a+1} (w_{j-1}(\gamma)\, w_k(\gamma)). \end{aligned}$$

$$\begin{aligned} &= w_2(\gamma)\, Sq^1\, Sq^{2^a-2} (w_{j-1}(\gamma)\, w_k(\gamma)) \\ &\quad + Sq^1\, Sq^{2^a} (w_{j-1}(\gamma)\, w_k(\gamma)). \end{aligned}$$

$$\begin{aligned} &= w_2(\gamma)\, Sq^1\, Sq^{2^a-2} (w_{j-1}(\gamma)\, w_k(\gamma)) \\ &\quad + w_1(\gamma)\, Sq^{2^a} (w_{j-1}(\gamma)\, w_k(\gamma)). \end{aligned}$$

$= 0$ in P* by (V.5.4).

Thus from (1) and (2) it follows that $w_j(\gamma)\, w_{2^a}(\gamma)\, w_k(\gamma) = 0$ in P* for any odd j and any $a \geqslant 0$, where $k = n-j-2^a$.

Assume inductively that,

$$W_j(\gamma)\, W_{2^a+t}(\gamma)\, W_{k'}(\gamma) = 0 \quad \text{for any odd } j$$

and any $t < b$, where $k' = n-j-2^a-t$, $a \geq 1$, $0 < b < 2^a$.

Let $k = n-j-2^a-b$ and $k > 0$. Then in P^*,

$$W_j(\gamma)\, W_{2^a+b}(\gamma)\, W_k(\gamma) = Sq^b(W_{2^a}(\gamma))\, W_j(\gamma)\, W_k(\gamma)$$

$$= \sum_{\alpha=0}^{b-1} Sq^{\alpha}(W_{2^a}(\gamma))\, Sq^{b-\alpha}(W_j(\gamma)\, W_k(\gamma))$$

(by Cartan formula, since $Sq^b(W_{2^a}(\gamma)\, W_j(\gamma)\, W_k(\gamma))$

$$= V_b\, W_{2^a}(\gamma)\, W_j(\gamma)\, W_k(\gamma) = 0 \quad \text{in } P^*.)$$

$$= \sum_{\alpha=0}^{b-1} \sum_{\beta=0}^{b-\infty} \binom{j-1}{\beta}\binom{k-1}{b-\alpha-\beta} W_{2^a+\alpha}(\gamma)\, W_{j+\beta}(\gamma)\, W_{k+b-\alpha-\beta}(\gamma)$$

by (V.5.5). If β is odd $\binom{j-1}{\beta} = 0$, and if β is even then $j+\beta$ is odd, and so by the induction hypothesis

$$W_{2^a+\alpha}(\gamma)\, W_{j+\beta}(\gamma)\, W_{k+b-\alpha-\beta}(\gamma) = 0$$

in P^*. Thus,

$$W_j(\gamma)\, W_{2^a+b}(\gamma)\, W_k(\gamma) = 0 \quad \text{in } P^*.$$

Hence the proposition follows. #

V.5.7 PROPOSITION. Let M be a non-bounding n-dimensional manifold with cobcat(M) $\leq$ 4. If n is even, $n > 2$ and $W_1(\gamma) = 0$ in P^*, then M is cobordant to N x N where N is also non-bounding and cobcat (N) $\leq$ 4.

PROOF. By (V.5.6) $W_j(\gamma) = 0$ in P^* for all odd j. Thus M is cobordant to a square $N \times N$ (see [M.5]) with

$$\langle w_{i_1} w_{i_2} --- w_{i_p}, [N] \rangle = \langle w_{2i_1} w_{2i_2} --- w_{2i_p}, [M] \rangle$$

for any partition $i_1 + i_2 + ---- + i_p = n/2$.

Hence the proposition follows. #

V.5.8 COROLLARY. Let M be a non-bounding n-dimensional manifold with cobcat(M) ≤ 4, and let $n = 2^t m$ where $t \geq 1$ and $m > 1$.

(1) If $m = 2^s + 1$ for any $s \geq 1$, then M is cobordant to $(N)^{2^t}$ where N is a non-bounding, m-dimensional manifold with cobcat (N) ≤ 4.

(2) If $m = 2^s + 1$ for some $s \geq 1$, then M is cobordant to $(N)^{2^{t-1}}$ where N is a non-bounding, 2m-dimensional manifold with cobcat(N) ≤ 4.

PROOF. By (V.5.2) $W_1(M) = 0$ in P^* if dim(M) = 2r, where $r \neq 2^s + 1$ for any $s \geq 1$. The result now follows from (V.5.7) by induction. #

V.5.9 COROLLARY. Let M be a 2^t- dimensional manifold with cobcat(M) ≤ 4 and $t \geq 1$. Then M is either a boundary, or M is cobordant to $(\mathbb{R}P^2)^{2^{t-1}}$.

PROOF. If M is not a boundary, then by (V.5.8), M is cobordant to $(N)^{2^{t-1}}$ where N is a non-bounding 2-dimensional manifold with cobcat(N) ≤ 4. Clearly N is cobordant to $\mathbb{R}P^2$. #

V.5.10. PROPOSITION. If $n = 2^s + 2$ where $s \geq 2$, and if

$\langle w_1 w_2 w_{2^s-1}, [M] \rangle = 0$, then M is either a boundary or else M is cobordant to $N \times N$ where N is a nonbounding $(2^{s-1} + 1)$- dimensional manifold with $\mathrm{cobcat}(N) \leq 4$.

PROOF. Suppose M is not a boundary, and let M' be the $(2^s + 1)$-dimensional submanifold of M dual to $w_1(M)$. Then as in (V.5.2), $\mathrm{cobcat}(M') \leq 3$ and

$$\langle w_2 w_{2^s-1}, [M'] \rangle = \langle w_1 w_2 w_{2^s-1}, [M] \rangle$$

$$= 0.$$

Thus by (V.3.9) M' is a boundary, and so

$$\langle w_1 w_j w_{n-1-j}, [M] \rangle$$

$$= \langle w_j w_{n-1-j}, [M'] \rangle = 0$$

for all $0 \leq j \leq n-1$. Therefore $w_1(\gamma) = 0$ in P^* and hence the result follows by (V.5.7). #

V.5.11 COROLLARY. Let M_1 and M_2 be two non-bounding $(2^s + 2)$-dimensional manifolds, where $s \geq 2$, $\mathrm{cobcat}(M_1) \leq 4$ and $\mathrm{cobcat}(M_2) \leq 4$. If neither M_1 nor M_2 is a square, then M_1 is cobordant to $M_2 \sqcup (N \times N)$ for some manifold N with $\mathrm{cobcat}(N) \leq 4$.

PROOF. By (V.5.10),

$$\langle w_1 w_2 w_{2^s-1}, [M_1] \rangle = \langle w_1 w_2 w_{2^s-1}, [M_2] \rangle = 1$$

and so $\left\langle W_1 W_2 W_{2^s-1}, [M_1 \sqcup M_2] \right\rangle = 0$

Also clearly cobcat $(M_1 \sqcup M_2) \leq 4$.

The result now follows from (V.5.10). #

There exists classes $S_k \in H^k(BO)$ defined inductively by Newton's formula:

$$S_k + w_1(\gamma) S_{k-1} + \cdots + w_{k-1}(\gamma) S_1 + [k]_2 w_k(\gamma) = 0$$

for $k > 0$, where $[k]_2$ is the mod 2 reduction of k, and $S_0 = 1$. Further it can be shown that

$$Sq^j(S_k) = \binom{k}{j} S_{k+j} \quad \text{for all } k \geq j > 0$$

Also if M is an n-dimensional manifold then $S_n \in H^n(BO)$ corresponds to the symmetric polynomial $S_n \in H^n(M)$ under the homomorphism $f^*: H^*(BO) \to H^*(M)$ where $f: M \to BO$ is the classifying map of $\tau(M)$, and as we have mentioned in § 4, M is decomposable iff $\langle f^*(S_n), [M] \rangle$ $= \langle S_n, [M] \rangle = 0$.

V.5.12 COROLLARY. Let $n = 2^r + 2$, where $r \geq 2$, and let M be a non-bounding n-dimensional manifold which is not cobordant to a square. Then M is indecomposable.

PROOF. $S_{2^r+2} = Sq^3(S_{2^r-1}) = V_3 \cdot S_{2^r-1}$

Moreover V_3 is decomposable and by Newton's formula

$$S_{2^r-1} \equiv W_{2^r-1}(\gamma) \mod P_2^* .$$

It follows that,

$$\begin{aligned}
s_{2^r+2} &= v_3 W_{2^r-1}(\gamma) \\
&= (W_3(\gamma) + Sq^1(W_2(\gamma)))\, W_{2^r-1}(\gamma) \\
&= (W_3(\gamma) + W_1(\gamma) W_2(\gamma) + W_3(\gamma))\, W_{2^r-1}(\gamma) ,
\end{aligned}$$

by Wu's formula.

$$= W_1(\gamma)\, W_2(\gamma)\, W_{2^r-1}(\gamma) \qquad \text{in } P^*$$

By hypothesis and (V.5.10), we have

$$W_1(\gamma)\, W_2(\gamma)\, W_{2^r-1}(\gamma) \neq 0 \text{ in } P^* \text{ and so}$$

$$\langle S_{2^r+2}, [M] \rangle \neq 0 \text{ i.e. } M \text{ is indecomposable.} \qquad \#$$

The results we have obtained so far in this section have reduced the problem to one of looking at the odd dimensional manifolds M with cobcat(M) 4

V.5.13 LEMMA. If n is odd, then

(1) $x^2 = 0$ in P^* for any x, where $\dim(x) > 0$

(2) $W_{\text{odd}}(\gamma)\, W_{\text{odd}}(\gamma) = 0$ in P^*.

PROOF (1) Let $0 < 2i < n$, $n-2i$ is odd, and so

$$w_{n-2i}(\gamma) = Sq^1(w_{n-2i-1}(\gamma)) + w_1(\gamma)\, w_{n-2i-1}(\gamma),$$

by Wu's formula.

Hence, $w_i^2(\gamma)\, w_{n-2i}(\gamma) = w_i^2(\gamma)\, Sq^1(w_{n-2i-1}(\gamma))$

$$= Sq^1(w_i^2(\gamma)\, w_{n-2i-1}(\gamma))$$

$$= w_1(\gamma)\, w_i^2(\gamma)\, w_{n-2i-1}(\gamma) = 0 \text{ in } P^*$$

and so, $w_i^2(\gamma) = 0$ in P^* for all $i \geq 1$. It follows that $x^2 = 0$ in P^* for all x with $\dim(x) > 0$.

(2) Let i, j be odd and let $0 < i,\ j < n$. Then by Wu's formula,

$$w_i(\gamma)\, w_j(\gamma)\, w_{n-i-j}(\gamma)$$

$$= w_i(\gamma)\, w_j(\gamma)\, Sq^1(w_{n-i-j-1}(\gamma))$$

$$= Sq^1(w_i(\gamma)\, w_j(\gamma)\, w_{n-i-j-1}(\gamma))$$

$$= v_1\, w_i(\gamma)\, w_j(\gamma)\, w_{n-i-j-1}(\gamma)$$

$$= 0 \text{ in } P^*$$

and so $w_i(\gamma)\, w_j(\gamma) = 0$ in P^*. #

V.5.14 PROPOSITION. Let $n = 2^r + m$, where m is odd and

$0 < m < 2^r$, and let $\alpha(n) \geq 4$, $\alpha(n)$ being the number of nonzero terms in the diadic expansion of n. Then $W_{2^r+k}(\gamma) = 0$ in P^* for any $0 \leq k \leq m$.

PROOF
$$W_{2^r+k}(\gamma) \equiv Sq^k(W_{2^r}(\gamma)) \bmod P_2^* .$$
$$\equiv Sq^k(v_{2^r}) \bmod P_2^*$$

Moreover, $v_{2^r} = 0$ as $2^r > n/2$, and so, $W_{2^r+k}(\gamma)$ is decomposable for each $0 \leq k \leq m$. Hence, $W_{2^r+k}(\gamma) \cdot x = 0$ in P^* if x is decomposable.

It is easy to see, by Wu's theorem, that
$$W_n(\gamma) = 0 \quad \text{and} \quad W_1(\gamma) W_{n-1}(\gamma) = 0$$
in P^*. Moreover, since $W_{n-2}(\gamma)$ is decomposable
$$W_2(\gamma) W_{n-2}(\gamma) = v_2 W_{n-2}(\gamma)$$
$$= Sq^2(W_{n-2}(\gamma)) = Sq^2 Sq^{\frac{n-3}{2}}\left(v_{\frac{n-1}{2}}\right),$$
by Wu's formula
$$= Sq^{\frac{n-1}{2}} Sq^1\left(v_{\frac{n-1}{2}}\right) \quad \text{, by Adem's relation}$$
$$= v_{\frac{n-1}{2}} Sq^1\left(v_{\frac{n-1}{2}}\right).$$

Since $\alpha(n) \geq 4$, $\frac{n-1}{2}$ is not a power of 2, and so $v_{(n-1)/2}$

is decomposable. Thus $W_2(\gamma)\,W_{n-2}(\gamma) = 0$ in P^*. Assume inductively that $W_i(\gamma)\,W_{n-i}(\gamma) = 0$ in P^* for all $0 \leq i < j$, where $2 < j \leq m$. Then by Wu's formula,

$$W_j(\gamma)\,W_{n-j}(\gamma) = Sq^j(W_{n-j}(\gamma))$$
$$+\sum_{t=1}^{j}\binom{n-j-j+t-1}{t} W_{j-t}(\gamma)\,W_{n-j+t}(\gamma)$$

and so by the induction hypothesis,

$$W_j(\gamma)\,W_{n-j}(\gamma) = Sq^j(W_{n-j}(\gamma))$$
$$= v_j\,W_{n-j}(\gamma) \text{ in } P^*.$$

Now $W_{n-j}(\gamma) = W_{2^r+m-j}(\gamma)$ is decomposable and so is V_j if j is not a power of 2. Therefore in this case $W_j(\gamma)\,W_{n-j}(\gamma) = 0$ in P^*. If $j = 2^s$, for some $s \geq 2$, then

$$W_j(\gamma)\,W_{n-j}(\gamma)$$

$= V_{2^s} W_{n-2^s}(\gamma)$, since $W_{n-2^s}(\gamma)$ is decomposable

$= Sq^{2^s}(W_{n-2^s}(\gamma))$

$= Sq^{2^s} Sq^1(W_{n-2^s-1}(\gamma))$, since $w_1(\gamma) = 0$ in P^* by (V.5 1).

$= Sq^1\,Sq^{2^s}(W_{n-2^s-1}(\gamma)) + Sq^2 Sq^{2^s-1}(W_{n-2^s-1}(\gamma))$,

by Adem's relations.

$$= W_1(\gamma)\, Sq^{2^s}(w_{n-2^s-1}(\gamma)) + v_2\, Sq^{2^s-1}(w_{n-2^s-1}(\gamma))$$

$$= 0 + \sum_{t=0}^{2^s-1} \binom{n-2^s-1-2^s+1+t-1}{t} w_2(\gamma)\, w_{2^s-1-t}(\gamma)\, w_{n-2^s-1+t}(\gamma)$$

$$= \sum_{t=0}^{2^s-1} \binom{n-2^{s+1}+t-1}{t} w_2(\gamma)\, w_{2^s-1-t}(\gamma)\, w_{n-2^s-1+t}(\gamma)$$

Since $\alpha(n) \geq 4$ and $2^s = j \leq m$, so $2^s + 1 \leq m$. Hence for all $0 \leq t \leq 2^s - 1$ we have $n-2^s - 1 + t = 2^r + k$ for some $0 \leq k \leq m$. That is $W_{n-2^s-1+t}(\gamma)$ is decomposable for all $0 \leq t \leq 2^s - 1$. Thus it follows that $W_j(\gamma)\, W_{n-j}(\gamma) = 0$ in P^* if $j = 2^s$, for some $s \geq 2$. So we have proved that $W_j(\gamma)\, W_{n-j}(\gamma) = 0$ in P^* for all $o \leq j \leq m$. Hence the result follows. #

V.5.15 THEOREM. Let n be odd and let $W_j(\gamma)\, W_{n-j}(\gamma) = 0$ for all $0 \leq j \leq n$. Then M is a boundary.

PROOF. Let $i = \min(i,j,k)$, where $i,j,k \geq 0$ and $i+j+k = n$. By hypothesis, $w_i(\gamma)\, w_j(\gamma)\, W_k(\gamma) = 0$ in P^* if $i = 0$. Let $0 < i < n$ and assume inductively that

$W_p(\gamma)\, W_q(\gamma)\, W_r(\gamma) = 0$ in P^* if $\min(p,q,r) < i$, where

$p,q,r \geq 0$ and $p+q+r = n$. By lemma (V.5.13) we may assume that $i < j < k$. If $i = 2^a$, then

$$W_{2^a}(\gamma) \cdot W_j(\gamma) . W_k(\gamma) = v_{2^a}\, w_j(\gamma)\, w_k(\gamma)$$

$$= Sq^{2^a}(W_j(\gamma)\, W_k(\gamma)) = \sum_{\alpha=0}^{2^a} Sq^{\alpha}(w_j(\gamma))\, Sq^{2^a-\alpha}(W_k(\gamma))$$

$$= \sum_{\alpha=0}^{2^a} \sum_{t=0}^{\alpha} \binom{j-\alpha+t-1}{t} W_{\alpha-t}(\gamma)\, W_{j+t}(\gamma)\, Sq^{2^a-\alpha}(W_k(\gamma))$$

By the induction hypothesis,

$$W_{\alpha-t}(\gamma)\, W_{j+t}(\gamma)\, Sq^{2^a-\alpha}(W_k(\gamma)) = 0 \text{ in } P^*,$$

if $0 < \alpha - t < 2^a$.

So,

$$W_{2^a}(\gamma)\; W_j(\gamma)\; W_k(\gamma)$$

$$= W_{2^a}(\gamma)\, W_j(\gamma)\, W_k(\gamma) + \sum_{\alpha=0}^{2^a} \binom{j-1}{\alpha} W_{j+\alpha}(\gamma)\, Sq^{2^a-\alpha}(W_k(\gamma))$$

$$= W_{2^a}(\gamma)\; W_j(\gamma)\; W_k(\gamma) + \sum_{\alpha=0}^{2^a} \binom{j-1}{\alpha} W_{j+\alpha}(\gamma) \sum_{\beta=0}^{2^a-\alpha} \binom{k-2^a-\alpha+\beta-1}{\beta} W_{2^a-\alpha-\beta}(\gamma)\, W_{k+\beta}(\gamma),$$

by Wu's formula.

$$= W_{2^a}(\gamma)\, W_j(\gamma)\, W_k(\gamma) + W_j(\gamma)\, W_{2^a}(\gamma)\, W_k(\gamma) + \sum_{\alpha=0}^{2^a} \binom{j-1}{\alpha} \binom{k-1}{2^a-\alpha} W_{j+\alpha}(\gamma)\, W_{k+2^a-\alpha}(\gamma)$$

by induction hypothesis.

$= 0$ in P^*, since $W_h(\gamma)\, W_{n-h}(\gamma) = 0$ in P^* for all $0 \leq h \leq n$ by hypothesis.

If $i = 2^a + b$, $0 < b < 2^a$ then,

$$W_{2^a+b}(\gamma)\, W_j(\gamma)\, W_k(\gamma) = Sq^b(W_{2^a}(\gamma))\, W_j(\gamma)\, W_k(\gamma)$$

$$= \sum_{\alpha=0}^{b-1} Sq^{\alpha}(w_{2^a}(\gamma))\, Sq^{b-\alpha}(w_j(\gamma)\, w_k(\gamma)),$$

since $Sq^{b}(w_{2^a}(\gamma)\, w_j(\gamma)\, w_k(\gamma)) = 0$ in P^*.

$$= \sum_{\alpha=0}^{b-1} \sum_{\beta=0}^{b-\alpha} \binom{j-1}{\beta}\binom{k-1}{b-\alpha-\beta} w_{2^a+\alpha}(\gamma)\, w_{j+\beta}(\gamma)\, w_{k+b-\alpha-\beta}(\gamma),$$

by (V.5.5).

= 0 in P*, by the induction hypothesis.

Hence the theorem follows. #

V.5.16 COROLLARY. If $n = 2^{r+1} - 1$, $r \geq 3$, then M is a boundary.

PROOF. Since $n = 2^{r+1} - 1 = 2^r + (2^r - 1)$; for any $0 \leq k \leq n$, either $k \geq 2^r$ or $n-k \geq 2^r$. Thus by (V.5.14), $w_k(\gamma)\, w_{n-k}(\gamma) = 0$ in P* for all $0 \leq k \leq n$, and so by (V.5.15), M is a boundary. #

V.5.17 LEMMA. If n is odd, $w_1(\gamma) = 0$ in P*, and M is decomposable, then the Wu class $v_j = 0$ in P* whenever j is not a power of 2.

PROOF. Suppose that j is not a power of 2. Therefore v_j is decomposable and so $v_j \cdot x = 0$ in P* for any $x \in P_2^*$.

If j is odd, then

$$v_j\, w_{n-j}(\gamma) = Sq^j\, w_{n-j}(\gamma)$$

$$= Sq^1 Sq^{j-1}(w_{n-j}(\gamma)),$$ by Adem's relations.

$$= w_1(\gamma)\, Sq^{j-1}(w_{n-j}(\gamma)) = 0 \text{ in } P^*.$$

If j is even, then $n-j$ is odd and so by Newton's formula,

$$w_{n-j}(\gamma) \equiv s_{n-j} \mod P_2^* .$$

Hence, $v_j w_{n-j}(\gamma) = v_j s_{n-j}$

$$= Sq^j(s_{n-j}) = \binom{n-j}{j} s_n = 0 \quad \text{in} \quad P^*$$

since M is decomposable.

Hence the lemma follows. #

V.5.18 LEMMA. If $v_k w_{n-k}(\gamma) = 0$ in P^* for all $0 \leq k \leq n$, then $w_k(\gamma) w_{n-k}(\gamma) = 0$ in P^* for all $0 \leq k \leq n$.

PROOF. $w_1(\gamma) w_{n-1}(\gamma) = v_1 w_{n-1}(\gamma) = 0$ in P^*.

Let $1 < k < n$, and assume inductively that $w_i(\gamma) w_{n-i}(\gamma) = 0$ for all $0 \leq i < k$.

Then $w_k(\gamma) w_{n-k}(\gamma) = Sq^k(w_{n-k}(\gamma))$, by the induction hypothesis, and so $w_k(\gamma) w_{n-k}(\gamma)$

$= v_k w_{n-k}(\gamma) = 0$ in P^*. #

V.5.19 PROPOSITION. Let $n = 2^a(p) + (2^t-1)$, where p is odd, $1 \leq t < a$, and $\alpha(p) \geq 3$. Then M is decomposable.

PROOF. We can write $n = 2^r + 2^a(m) + (2^t-1)$ where $2^a(m) < 2^r$, m is odd, and $\alpha(m) \geq 2$. Thus,

$$Sq^{2^a+2^t}\left(s_{2^r+2^a(m-1)-1}\right) = \binom{2^r+2^a(m-1)-1}{2^a+2^t} s_n$$

Moreover, since m is odd and $m \geq 3$,

$$2^r+2^a(m-1)-1 = 2^r + 2^a(m-3) + (2^{a+1}-1)$$

$$= 2^r+2^a(m-3) + (1+2+----+2^a),$$

and so the binomial coefficient above is nonzero, since $t < a$.

Thus, $S_n = v_{2^a+2^t} S_{2^r+2^a(m-1)-1}$

Now $v_{2^a+2^t}$ is decomposable,

and $S_{2^r+2^a(m-1)-1} \equiv W_{2^r+2^a(m-1)-1}(\gamma) \bmod P_2^*$

and $W_{2^r+2^a(m-1)-1}(\gamma) = 0$ in P^*, by (V.5.14).

Hence $S_n = 0$ in P^* and so the proposition follows. #

<u>V.5.20 THEOREM.</u> Let $n = 2^r + 2^{r-1} + m$, where m is odd, $m < 2^{r-1}$, $m+1$ is not a power of 2 and $\alpha(m) \geq 2$. Then M is a boundary.

<u>PROOF.</u> $v_{2^r} = 0$ in P^*. If $2^a < 2^r$ then $n - 2^a > 2^r$, and so by (V.5.14) $v_{2^a} W_{n-2^a}(\gamma) = 0$ in P^*. Moreover, by (V.5.19) M is decomposable. It follows then from (V.5.12), (V.5.17) and (V.5.18), that $W_k(\gamma) W_{n-k}(\gamma) = 0$ in P^* for all $0 \leq k \leq n$, and so by (V.5.15) M is a boundary.

#

We conclude this chapter and also the dissertation by giving the following conjectures for an n-dimensional manifold M with cobcat (M) ≤ 4, as suggested by H.Singh [S.5] and some problems:

Conjecture 1. If $\alpha(n) \geq 4$ then M is a boundary.

Conjecture 2. If $n = 2^s + 1$ then cobcat(M)≤ 3, and so M is cobordant to the manifold given in (V.4.1),

In reference to the conjecture 1, following result due to F.P. Paterson [P.5] is available:

Result. Let M^n be an n-dimensional manifold. Assume that all products of Stiefel-Whitney classes of the normal bundle of M of length greater than d vanish. Then $\alpha(n) \leq d$ or M^n is a boundary. #

We remark that the condition on Stiefel-Whitney classes is satisfied if M^n has L.S. Category at most d.

Problem 1. Bordism classification of G-manifold with small G-category.

Problem 2. For a fixed topological space X to study the bordism classification of singular manifolds $f: M^n \longrightarrow X$ with small cat f .

Problem 3. Bordism classification of G-manifolds using different index theories as given in Chapter IV.

Problem 4. In view of the Paterson's result mentioned above, to study the bordism classification of manifolds M^n with $\alpha(n) \leq \text{cat}\,(M^n)$.

BIBLIOGRAPHY

[B.1] I. BERSTEIN and T. GANEA: The category of a map and of a cohomology class. Fundam. Math. 50 (1962) 265-279.

[C.1] M. CLAPP and D.PUPPE: Invariants of Lusternik - Schnirelmann type and the topology of the critical sets. Trans. Amer. Math. Soc.298(2) (1986) 603-620.

[C.2] P.E. CONNER and E.E. FLOYED: Torsion in SU-bordism. Mem. Amer. Math. Soc. No.60(1966).

[C.3] P.E. CONNER and E.E. FLOYED: Fibering within a cobordism class. Michigan Math. J.12 (1965), 33-47.

[D.1] A. DOLD: Lectures on Algebraic Topology, Springer-Verlag (1972).

[D.2] A. DOLD: Erzeugende der Thomschen Algebra N , Math.Z.65 (1956), 25-35.

[F.1] R.H. FOX: On the Lusternik-Schnirelmann category, Ann. Math. 42 (1941), 333-370.

[F.2] E. FADELL: The equivariant Lusternik-Schnirelmann method for invariant functionals and relative cohomological theories, A Granas, ed. Me'thodes Topologiques on Analyse Non linear (Univ. of Montreal Press, Montreal, 1985) 41-71.

[F.3] E. FADELL and S. HUSSEINI: Relative cohomological index theories, Adv. Math. 64, 1-31 (1987).

[F.4] E. FADELL and P. RABINOWITZ: Generalised cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems, Invent. Math. 45 (1978) 139-174.

[G.1] V. *GUILLEMIN* and *A. POLLACK*: Differential Topology, Prentice-Hall, Inc. Englewood Cliffs, N.J., 1974.

[H.1] M.W. HIRSCH: Differential Topology, Springer-Verlag, N.Y., Heidelberg, Berlin, 1076.

[H.2] D. HUSEMOLLER: Fibre Bundles, McGraw-Hill, N.Y., 1967.

[J.1] I.M. JAMES: On category, in the sense of Lusternik-Schnirelmann. Top. 17 (1978) 331-348.

[J.2] I.M. JAMES: General Topology and Homotopy Theory, Springer-Verlag. N.Y., Berlin Heidelberg Tokyo (1984).

[K.1] M.A. KRASNOSIELSKI: On special coverings of a finite-dimensional sphere (Russian), Dokl. Akad. Nauk.SSSR 103(1955), 961-964.

[L.1] L. LUSTERNIK and L.SCHNIRELMANN: Me'thodes topologiques dans les problemes variationnels. Paris: Hermann 1934.

[M.I] J.R. MUNKRES: Topology - A first course, Prentice Hall, New Delhi, 1978.

[M.2] M.V. MIELKE: Cobordism properties of manifolds of small category, Proc. Amer. Math. Soc. 21(1969) 332-334.

[M.3] W. MARZANTOWICZ: A G-Lusternik-Schnirelmann category of a space with an action of a compact Lie group. Topology

28 (1989) 403-412.

[M.4] J.W. MILNOR and J.D. STASHEFF: Characteristic classes, Ann. of Math. Studies. No. 76, Princeton Univ. Press, Princeton, N.J., 1974.

[M.5] J. MILNOR: On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds. Topology, 3 (1965) 223-230.

[M.6] M. MURAYAMA: On G-ANR's and their G-homotopy types, Osaka J. Math 20 (1983), 479-512.

[P.1] R.S. PALAIS: Lusternik-Schnirelmann theory on Banach manifolds. Topology 5 (1966) 115-132.

[P.2] R.S. PALAIS: Morse theory on Hilbert manifolds. Topology 2 (1963), 299-340.

[P.3] R.S. PALAIS: Homotopy theory on infinite dimensional manifolds. Topology 5 (1966), 1-16.

[P.4] R.S. PALAIS: The classification of G-spaces, Memoirs of the Amer. Math. Soc. 36 (1960).

[P.5] F.P. PETERSON: A-generator for certain polynomial algebras, Math. Proc. Cambridge Philos. Soc.105 (1989). 311-312 .

[Q.1] D. QUILLEN: The spectrum of an equivariant cohomology ring I, Ann. of Math. 94 (1971), 549-572.

[R.1] J.R. RAMSAY: Extensions of Lusternik-Schnirelmann category theory to relative and equivariant theories with an application to an equivariant critical point theorem. Topology and its Application 32 (1989) 49-60.

NEHU Library
Acc. No 102430
Acc. by
24.II.92
Checking by
Date by
Prescribed by

[S.1] R.M. SWITZER: Algebraic Topology - Homotopy and Homology. Springer-Verlag 1975.

[S.2] W. SINGHOF: Minimal co vering of manifolds with balls. Manuscripta math. 29, 385-415 (1979).

[S.3] E. SPANIER: Algebraic Topology, McGraw-Hill (1966).

[S.4] H. SINGH: Lusternik-Schnirelmann category and cobordism. Proc. Amer. Math. Soc. 102 (1988) 183-120.

[S.5] H. SINGH: Cobordism classes of manifolds with category four. Trans. Amer. Math. Soc. 310 (1988) 619-628.

[S.6] R.E. STONG: Notes on cobordism theory. Math. Notes, Princeton Univ. Press. Princeton, N.J., 1968.

[S.7] S. SMALE: On the structure of manifolds. Amer. J. Math 84, 387-399 (1962).

[S.8] S. Smale: On the structure of 5-manifolds. Ann. of Math 75, 38-46 (1962).

[T.1] F. TAKENS: The Lusternik-Schnirelmann categories of a product space. Composito math. 22 (1970), 175-180.

[T.2] F. TAKENS: The minimal number of critical points of a function on a compact manifold and the L.S. category. Invent. Math. 6 (1968), 197-244.

[T.3] R. THOM: Quelques proprie'te's globales des varie'te's differentiables, comment.Math.Helv. 28 (1954), 17-86.

[W.1] G.W. WHITEHEAD: Elements of Homotopy Theory. Springer-Verlag, N.Y., 1978.