PRIME AND MAXIMAL IDEALS IN $C(X)$ AND HOMOMORPHISMS ON $C(X)$: A SURVEY

BY

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DECLARATION

I, Mr. Kshittiz Chettri, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

This dissertation is being submitted to the North - Eastern Hill University for the degree of Master of Philosophy in Mathematics.

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CERTIFICATE

I certify that the dissertation entitled 'PRIME AND MAXIMAL IDEALS IN C(X) AND HOMOMORPHISMS ON C(X): A SURVEY' submitted by Mr. Kshittiz Chettri in partial fulfillment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

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The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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dedicated to my parents...

PREFACE

This dissertation is a brief survey of some work done in the field of 'Prime and Maximal ideals in $C(X)$ and Homomorphisms on $C(X)$', where $C(X)$ is the ring of all real-valued continuous functions on an arbitrary topological space X. There is a fair amount of interplay between the topological properties of X and the algebraic properties of $C(X)$. The pioneering work in this field was done by A. Tychonoff in [36], where he showed that a completely regular space is a subspace of a compact space. However, the actual groundwork for the theory of rings of continuous functions was laid in three papers. The first was by M. H. Stone [35] in which the basic theory of $C^*(X)$, the subring of all bounded real-valued continuous functions on X, was studied. It is worth noting that he had considered the metric structure of $C^*(X)$. The second paper was by Gelfand and Kolmogoroff [16], where he generalized some of the Stone's result without considering the metric structure of $C^*(X)$ and did a similar study in $C^*(X)$. The third paper was by E. Hewitt [19], where he added to the knowledge of the ring $C(X)$ and set the direction for subsequent research. It was in this paper that the zero-sets were systematically exploited for the first time. It also contains information about zero-sets, pseudocompactness and relation between ideals, z-ideals, fixed ideals, free ideals etc.

Subsequent works by various generations of mathematicians led to further development of the subject. It is worth mentioning that a lot of work was done in Purdue University by Gillman, Jerison, C. W. Kohls and others.
In Chapter I we shall include some basic results on rings of continuous functions, without proof, which we shall need later on.

One of the major achievements of the study in rings of continuous functions has been the characterization of prime and maximal ideals in $C(X)$ and $C^*(X)$. However, the case for $C^*(X)$ is not as easy as $C(X)$. This gave rise to a question, 'Is it possible to address the distinct cases of $C(X)$ and $C^*(X)$ in the same setting? ' This question was answered by H.L.Byun and S. Watson in [7], where they considered the intermediate algebra $A(X)$, satisfying $C^*(X) \subseteq A(X) \subseteq C(X)$. In this setting $C^*(X)$ and $C(X)$ are just special cases of $A(X)$. Further works by Acharya, Chattopadhyay and Ghosh [1]; Domínguez, Gómez Pérez [11] among others led to further development of

the subject.
In Chapter II, we shall study the generalization of the results known for $C(X)$ and $C^*(X)$ to $A(X)$.

It is well known that a continuous map $\pi : X \to Y$ induces a homomorphism $g \circ \pi : C(Y) \to C(X)$, where X and Y are arbitrary topological spaces. J. M. Dominguez and M. A. Mulero have shown in [10] that the properties of the map $X \to Y$ determines the finiteness properties of the homomorphism $C(Y) \to C(Y)$, i.e., whether it is finite, integral, singly generated or finitely generated. In Chapter III, we shall study the finiteness properties of the homomorphisms $C(Y) \to C(X)$.

The rings of continuous funtions, $C(X)$ is an object in **CR**, the category of all commutative rings. If Y is a subspace of X then the restriction map from X to Y induces a restriction homomorphism $\rho : C(X) \to C(Y)$. Michael Barr, W. D. Burgess and R. Raphael in [4] have studied various conditions under which ρ is an epimorphism. In Chapter III, we shall briefly study some of the conditions under which 'ρ' is an epimorphism in **CR**.

Let A be a topological ring. If S is a multiplicatively closed subset of A, we define a natural topology on the localization A_S of A with respect to S. The ring A_S endowed with this topology is said to be the topological localization of A with respect to S. This topological localization can be studied in the topological ring $C(X)$ by considering $S = \{f \in C(X) \mid 0 \notin f(U)\}$, where U is a cozero-set in X as shown by B. Requezo, J.B. Sancho in [29]. In chapter IV, we shall briefly study this localization on $C(X)$.

If X is a Tychonoff space, a zero-set Z of X is $z-$complemented in X if there exists a zero-set $\hat{Z}$ of X such that $Z \cup \hat{Z} = X$ and $Z \cap \hat{Z}$ is nowhere dense in X. The notion of z-complemented zero-sets arises in determining $C(X)$ having thr property that the total ring of quotients $T(C(X))$ is von-Neumann regular. Ronnie Levy and Jay Shapiro studied in [21] the conditions on a space X under which every zero-set is z-complemented. In Section 2 of Chapter IV, some works done by Levy and Shapiro will be carried out.

Contents

Chapter 1

Preliminaries

In this chapter we recall some of the basic definitions, notations and conventions from the theory of rings of continuous functions, which will be used in the forth coming chapters.

1.1 Functions on a topological space

Definition and Notation 1.1.1 (1.1, [18]). Let X be a topological space and let $C(X)$ be the set of all continuous functions from X to $\mathbb{R}$. We define addition and multiplication on $C(X)$ by

$$(f+g)(x) = f(x) + g(x) \qquad \forall x \in X$$

and

$$(fg)(x) = f(x)g(x) \qquad \forall x \in X.$$

(i) With these definitions $C(X)$ is a commutative ring with the constant function 0 as the zero element and the constant function 1 as the unity element of $C(X)$.
The additive inverse of f is -f defined by $(-f)(x) = -(f(x)) \in \mathbb{R}$.
The multiplicative inverse f^{-1} (which exists when the function f does not vanish anywhere on X) is given by the formula

$$f^{-1}(x) = 1/f(x) \quad \forall x \in X.$$

(ii) We regard $C(X)$ as a partially ordered set under the definition:
for $f, g \in C(X)$, $f \leq g$ if $f(x) \leq g(x)$ $\forall x \in X$.

(iii) The cardinality of $C(X) \geq c$, where c is the cardinality of the linear continuum, when $X \neq \phi$. Indeed, we have uncountably many constant functions in $C(X)$ (when $X \neq \phi$). These are denoted by (for each $c \in \mathbb{R}$) $f_c : X \to \mathbb{R}$ where f_c is defined by $f_c(x) = c \quad \forall\, x \in X$.

$$\text{When } X = \{x\},\quad C(X) = \{f_c \mid c \in \mathbb{R}\}.$$

Notation 1.1.2 (1.4,[18]). We write $C^*(X)$ for the set of all bounded elements of $C(X)$. Clearly, $C^*(X)$ is a subring of $C(X)$.
If $C(X) = C^*(X)$, then we say X is *pseudocompact.*

Remark 1.1.3. Every compact space is *pseudocompact.*

Notation 1.1.4 (0.5, 1.3 [18]). In the partially ordered set $C(X)$ (resp. $C^*(X)$), the symbol
$f \vee g$ denotes $\sup\{f, g\}$, the smallest element $k \in C(X)$ satisfying $f \leq k$ and $g \leq k$. It is given by

$$k = 2^{-1}(f + g + |f - g|).$$

Likewise, $f \wedge g$ stands for $\inf\{f, g\}$, given by

$$f \wedge g = 2^{-1}(f + g - |f - g|).$$

We further have

$$|f| = f \vee -f, \quad \text{where } |f| \text{ satisfies } |f|(x) = |f(x)|, \text{ for all } x \in X.$$

Definition 1.1.5 (0.5, [18]). A partially ordered set R is a lattice if the inf and the sup of any two elements of R is in the set R. A subset S of R is said to be a sublattice of R provided that, for all $a, b \in S$, the elements $a \vee b$ and $a \wedge b$ of R belong to S.

Remark 1.1.6. It follows from Definition 1.1.5 and Notation 1.1.4 that the partially ordered set $C(X)$ is a lattice and $C^*(X)$ is a sublattice of $C(X)$.

1.1.1 Zero and cozero sets

Definition and Notation 1.1.7. Let X be a topological space. For $f \in C(X)$ and $r \in \mathbb{R}$, let

$$f^{\leftarrow}\{r\} = \{x \in X | f(x) = r\}.$$

The set $f^{\leftarrow}\{0\} = \{x \in X | \ f(x) = 0\}$ is called the *zero-set* of f. We shall denote this set by $Z(f)$. By $Z(X)$, we denote the set of all zero-sets in X.

Definition 1.1.8 (1.11, [18]). A complement in X of a zero-set is called a *cozero-set*.

We list some basic properties of zero-sets in X.

Properties 1.1.9 (1.10, [18]). (i) Zero-sets are closed in X.

(ii) $Z(f) = Z(|f|) = Z(f^n)$, for each positive integer n.

(iii) $Z(fg) = Z(f) \cup Z(g)$ and $Z(|f| + |g|) = Z(f) \cap Z(g) = Z(f^2 + g^2)$.

Remark 1.1.10 (1.13, [18]). The rings *C(X)* and $C^*(X)$ yields the same set of zero-sets.

Remark 1.1.11 (1.10, [18]). Every zero-set is a G_δ-set.

Remark 1.1.12. [1.14(a), [18]] Zero-sets are closed under countable intersections.

Definition 1.1.13 (1.15, [18]). Let Z be a zero-set in X and let $A \subset X$. Then Z is said to be a zero-set neighbourhood of A if Int $Z \supset A$.

Examples 1.1.14. Let X=$\mathbb{R}$, then

(i) for $f = 0$, $Z(f) = X$,

(ii) for $f = i$, where $i(r) = r \quad \forall\, r \in \mathbb{R}$, we have $Z(f) = \{0\}$,

(iii) for $f = c \neq 0$, $c \in \mathbb{R}$ we have $Z(f) = \phi$.

(iv) The zero-set X is a zero-set neighbourhood of every subset of X.

(v) X is a zero-set as well as a cozero-set.

1.1.2 Completely separated sets

Definition 1.1.15 (1.15, [18]). Let X be a topological space. Two subsets A and B of X are said to be *completely separated* if there exists a function $f \in C^*(X)$ such that

$$f[A] = \{0\}, f[B] = \{1\} \text{ and } 0 \leq f(x) \leq 1 \quad \forall x \in X.$$

Example 1.1.16. Let X be a normal spac. Then any two disjoint closed subsets in X are completely separated.

Remark 1.1.17 (1.15, [18]). To verify the assertion in Example [1.1.16] it is enough to find a function $g \in C(X)$ satisfying

$$g[A] \subseteq]-\infty, 0] \text{ and } g[B] \subseteq [1, \infty[,$$

then $h = (0 \vee g) \wedge 1$ completely separates A and B.

Theorem 1.1.18 (1.15, [18]). *Two sets are completely separated iff they are contained in disjoint zero-sets.*

1.1.3 C-embedding and C^*-embedding

Definition 1.1.19 (1.16, [18]). A subspace S of X is said to be *C-embedded* in X if every function f in $C(S)$ can be continuously extended to $C(X)$.

Example 1.1.20. If X is normal and S is closed subspace of X, then by Tietze's extension theorem S is C-embedded in X.

Similarly, we have

Definition 1.1.21 (1.16, [18]). A subspace S of X is said to be *C^*-embedded* in X if every function f in $C^*(S)$ can be continuously extended to $C^*(X)$.

Remark 1.1.22 (1.16, [18]). Every C-embedded subspace is C^*-embedded.

It follows from the above remark that S is C^*-embedded in X iff every function in $C(S)$ can be extended to a function in $C(X)$.

Theorem 1.1.23 (1.17, [18]). *A subspace S of X is C^*-embedded in X iff any two completely separated sets in S are completely separated in X.*

Definition 1.1.24 (1H, [18]). A space X is called *extremally disconnected* if every open set has an open closure; X is *basically disconnected* if every cozero-set has an open closure.

Example 1.1.25. A discrete space is extremally disconnected, hence basically disconnected.

Remark 1.1.26 (1H(6, [18]). X is extremally disconnected if and only if every open subspace is C^*-embedded.

1.2 Basic Topological results

Definition 1.2.1 (8.2, [18]). A topological space X is said to be *σ-compact* if it is expressible as a countable union of compact spaces.

Example 1.2.2. $\mathbb{R}$ with usual topology is σ-compact, as $\mathbb{R} = \bigcup\limits_{n \in \mathbb{N}} [-n, n]$.

Theorem 1.2.3 (0.13, [18]). *Every infinite Hausdorff space contains a copy of* $\mathbb{N}$.

Remark 1.2.4 (3L (3), [18]). Let X be an infinite completely regular space, then $C^*(X)$ contains a function with infinite range.

The next theorem would be used in chapter 3.

Theorem 1.2.5 (Theorem 57.3, [25]). *(Borsuk-Ulam theorem for S^2) Given a continuous map $f : S^2 \rightarrow \mathbb{R}^2$, there is a point x of S^2 such that $f(x) = f(-x)$.*

1.3 Some ring theoretic results

Definition 1.3.1 (0.19, [18]). Let A be a ring and $\geq$ be a partial ordering relation defined on the ring, then A is called a *partially ordered ring* if

(i)

$$a \geq b \text{ implies } a + x \geq b + x \text{ for all } x, \text{ and}$$

(ii)

$$a \geq 0 \text{ and } b \geq 0 \text{ implies } ab \geq 0.$$

Remark 1.3.2 (0.19, [18]). To show that a ring A is a partially ordered ring, it is enough to show that

(i)

$$a \geq 0 \text{ and } -a \geq 0 \text{ if and only if } a = 0$$

(ii)

$$a \geq 0 \text{ and } b \geq 0 \text{ implies } a + b \geq 0 \text{ and } ab \geq 0.$$

Definition 1.3.3 (5.1, [18]). An ideal I in a partially ordered ring is said to be *convex* if we have:

$$\text{whenever } 0 \leq x \leq y, \text{ and } y \in I, \text{ then } x \in I.$$

An ideal I in a lattice-ordered ring is said to be *absolutely convex* if, whenever

$$|x| \leq |y| \text{ and } y \in I, \text{ then } x \in I.$$

Theorem 1.3.4 (5.3, [18]). *The following conditions on a convex ideal I in a lattice-ordered ring A are equivalent.*

(i) *I is absolutely convex*

(ii) *$x \in I$ implies $|x| \in I$*

(iii) *$x, y \in I$ implies $x \vee y \in I$.*

(iv) *$I(a \vee b) = I(a) \vee I(b)-$ whence A/I is a lattice.*

(v) *$I(a) \geq 0$ if and only if $a \equiv |a|$ (mod I)*

Theorem 1.3.5 (0.16, [18]). *Let I be an ideal in A, and S a set that is closed under multiplication and disjoint from I. There exists an ideal P containing I, disjoint from S, and maximal with respect to this property. Such an ideal is necessarily prime.*

Corollary 1.3.6 (0.16, [18]). *Let I be an ideal If no power of a belongs to I, then there exists a prime ideal containing I but not a.*

Corollary 1.3.7 (0.16, [18]). *The intersection of all prime ideals containing a given ideal I is precisely the set of all elements of which some power belongs to I*

Theorem 1.3.8 (0.22, [18]). *The only non-zero homomorphism of $\mathbb{R}$ into itself is the identity*

Definition 1.3.9. Let R be a ring. If for each $a \in R$, there exists $b \in \mathbb{R}$ such that $aba = a$, then R is called *von Neumann regular* ring.

Example 1.3.10. Let X be a discrete space, then $C(X)$ is von Neumann regular.

Definition 1.3.11. Let R be a ring and let I be an ideal of R, then I is said to be a radical ideal if $a \in I$ whenever $a^n \in I$ for some $n > 0$, $a \in R$.

Example 1.3.12. Every prime ideal is a radical ideal.

1.4 Ideals and z-filters

Definition 1.4.1 (2.2, [18]). A non-empty sub-family $\mathcal{F}$ of $Z(X)$ is called a *z-filter* on X if

(i) $\phi \notin \mathcal{F}$;

(ii) $Z_1 \cap Z_2 \in \mathcal{F}$ whenever, $Z_1, Z_2 \in \mathcal{F}$;

(iii) If $Z \in \mathcal{F}$, and $Z' \in Z(X)$ such that $Z' \supset Z$ then $Z' \in \mathcal{F}$.

Examples 1.4.2. (i) Let $X = \{x\}$, then $\mathcal{F} = \{X\}$ is a z-filter.

(ii) Let $X = \mathbb{R}$, then $\mathcal{F} = \{F \in Z(X) | 0 \in Z\}$ is a z-filter.

(iii) If I is an ideal in $C(X)$, then $Z[I] = \{Z(f) : f \in I\}$ is a z-filter on X.

Remark 1.4.3 (2.2, [18]). Every family β of zero-sets having the finite intersection property is contained in a z-filter. The smallest such is the family $\mathcal{F}$ of all zero-sets containing the finite intersections of members of β. We say that β generates the z-filter $\mathcal{F}$. When β itself is closed under finite intersection it is called a base for β.

Theorem 1.4.4 (2.3, [18]). (i) *If I is an ideal in $C(X)$, then the family*

$$Z[I] = \{Z(f) : f \in I\}$$

is a z-filter on X.

(ii) *If $\mathcal{F}$ is a z-filter on X, then the family*

$$Z^{\leftarrow}[\mathcal{F}] = \{f \in C(X) \mid Z(f) \in \mathcal{F}\}.$$

is an ideal in $C(X)$.

Remark 1.4.5 (2.4, [18]). Let I be an ideal in $C(X)$, then

(i) $Z[Z^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$;

(ii) $Z^{\leftarrow}[Z[I]] \supset I$.

Example 1.4.6. [2.4, [18]] Let $X = \mathbb{R}$, with usual topology. $I = (i)$, the principal ideal generated by the identity function i, i.e. $I = \{f \in C(X) : f(x) = xg(x) \text{ for some } g \in C(X)\}$. Let $M_0 = Z^{\leftarrow}[Z[I]]$. Clearly, it consists of all functions in $C(\mathbb{R})$ that vanishes at 0. So, $M_0 \supseteq I$. Now consider $i^{\frac{1}{5}}$,then $i^{\frac{1}{5}} \in Z^{\leftarrow}[Z[I]]$ but $i^{\frac{1}{5}} \notin I$. For, if $i^{\frac{1}{5}} \in I$, then $i^{\frac{1}{5}} = gi$ for some $g \in C(\mathbb{R})$. So, $g(x) = x^{\frac{-4}{5}}$ for $x \neq 0$. Clearly, g must be discontinuous at 0. So, $M_0 = Z^{\leftarrow}[Z[I]]$ properly contains I. Furthermore, M_0 can be shown to be a maximal ideal.

Unless stated otherwise, M_0 will denote the above maximal ideal.

Definition 1.4.7 (2.5, [18]). A z-filter $\mathcal{F}$ on X is called a *z-ultrafilter* if it is not contained in any other z-filter.

Remark 1.4.8 (2.5, [18]). Every z-filter is contained in some z-ultrafilter.

Theorem 1.4.9 (2.5, [18]). (i) *If M is a maximal ideal in $C(X)$, then $Z[M]$ is a z-ultrafilter on X.*

(ii) *If $\mathcal{A}$ is a z-ultrafilter on X, then $Z^{\leftarrow}[\mathcal{A}]$ is a maximal ideal in $C(X)$.*

Example 1.4.10. Let $X = \mathbb{R}$, with usual topology and consider the maximal ideal $M_0 = \{f \in C(X) : f(0) = 0\}$. Then $Z[M_0]$ is a z-ultrafilter on X.

Theorem 1.4.11 (2.6 , [18]). (i) *Let M be a maximal ideal in $C(X)$; if $Z(f)$ meets every member of $Z[M]$, then $f \in M$.*

(ii) *Let $\mathcal{A}$ be a z-ultrafilter on X;if a zero-set Z intersects every member of $\mathcal{A}$, then $Z \in \mathcal{A}$.*

Definition 1.4.12 (2.7, [18]). An ideal I in $C(X)$ is said to be a z-ideal if $Z(f) \in Z[I]$ implies $f \in I$.

Example 1.4.13. Let M_0 be as in Example [1.4.6]. Suppose $Z(f) \in Z[M_0]$,then by Theorem 1.4.11 $f \in M_0$

Remarks 1.4.14 (2.7, [18]). (i) Every maximal ideal in $C(X)$ is a z-ideal.

(ii) Intersection of two z-ideals is a z-ideal

(iii) There is a one to one correspondence between z-ideals of $C(X)$ and the z-filters of X.

Theorem 1.4.15 (2.11, [18]). *Every prime ideal in C(X) is contained in a unique maximal ideal.*

Theorem 1.4.16 (5.5, [18]). *Every prime ideal P in C(X) [resp. $C^*(X)$] is absolutely convex, and the residue class ring C/P [resp. C^*/P] is totally ordered. Furthermore, the mapping $r \longrightarrow P(r)$ is an order-preserving isomorphism of the real field $\mathbb{R}$ into the residue class ring*

Remark 1.4.17 (14.3, [18]). Let P be a prime ideal in $C(X)$. Then the prime ideals in C/P form a chain.

Note 1.4.18. We know that every residue class field of $C(X)$ or $C^*(X)$ (modulo a maximal ideal M) contains a canonical copy of the real field $\mathbb{R}$. When the canonical copy of $\mathbb{R}$ is the entire field $C(X)/M$ [resp. $C^*(X)/M$], then we refer to M as *real ideal*. When M is not real, we call it is *hyper-real* ideal.
We say an element a in a totally ordered field is said to be *infinitely large* if $a > n$ for every $n \in \mathbb{N}$.

1.5 Completely regular spaces

Definition 1.5.1 (3.1, [18]). A space X is called *completely regular* if it is a Hausdorff space such that, whenever F is a non-empty closed set in X and x is a point in its complement, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f[F] = \{0\}$.

Example 1.5.2. Every normal space is completely separated.

Definition 1.5.3 (3.2, [18]). (i) A collection $\mathcal{B}$ of closed sets in a topological space is a *base* for the closed sets if every closed set in X is an intersection of members of $\mathcal{B}$

(ii) Equivalently, $\mathcal{B}$ is a base if whenever F is a closed set and x a point in its complement, there is a member of $\mathcal{B}$ containing F but not x.

Theorem 1.5.4 (3.2, [18]). *A Hausdorff space X is completely regular iff the family of all zero-sets is a base for the closed sets.*

Remarks 1.5.5 (3.2, [18]). (i) Every closed set F in a completely regular space is an intersection of zero-set neighbourhoods of F.

(ii) Every neighbourhood of a point in a completely regular space contains a zero-set neighbourhood of the point.

In the study of rings of continuous functions the topological space can be taken to be completely regular without any loss of generality. This can be done using the following theorem.

Theorem 1.5.6 (3.9, [18]). *For every topological space X, there exists a completely regular space Y and a continuous mapping τ of X onto Y such that the mapping $g \rightarrow g \circ \tau$ is an isomorphism of $C(Y)$ onto $C(X)$.*

Next we shall list some properties of completely regular spaces.

Properties 1.5.7 (3.11 (a),(b) and (c), [18]). (i) In a completely regular space, any two disjoint closed sets, one of which is compact, are completely separated.

(ii) In a completely regular space, every G_δ-set containing a compact set S contains a zero-set containing S.

(iii) Every compact set in a completely regular space is C-embedded.

In what follows, unless stated otherwise, the topological space X is a completely regular Hausdorff space (Tychonoff space).

1.5.1 Convergence of z-filters

Definition 1.5.8 (3.16, [18]). Let X be a completely regular space. A point $p \in X$ is said to be a *cluster point* of a z-filter $\mathcal{F}$ if every neighbourhood of p intersects every element of $\mathcal{F}$.

Example 1.5.9. Let $Z[M_0]$ be as in Example[1.4.6], then 0 is a cluster point of $Z[M_0]$ as $0 \in \bigcap_{f \in M_0} Z(f)$.

Remark 1.5.10 (3.16, [18]). Since the members of $\mathcal{F}$ are closed sets, p is a cluster point of $\mathcal{F}$ iff $p \in \bigcap \mathcal{F}$.

Definition 1.5.11 (3.16, [18]). A z-ultrafilter $\mathcal{A}$ is said to *converge* to the limit p if every neighbourhood of p contains a member of $\mathcal{A}$.

Example 1.5.12. $Z[M_0]$ converges to 0.

Result 1.5.13 (3.16, [18]). *If p is a cluster point of $\mathcal{F}$, then at least one z-ultrafilter containing $\mathcal{F}$ converges to p.*

Notation 1.5.14 (3.18, [18]). A_p denotes the family of all zero-sets containing a given point p. It is also a z-ultrafilter.

Results 1.5.15 (3.18 (a), (b) and (c), [18]). (i) *p is a cluster point of a z-filter $\mathcal{F}$ iff $\mathcal{F} \subset A_p$*

(ii) *A_p is the unique z-ultrafilter converging to p.*

(iii) *Distinct z-ultrafilters cannot have a common cluster point.*

1.5.2 Fixed and Free ideals

Definition 1.5.16 (4.1, [18]). An ideal I is *fixed* if $\bigcap Z[I] \neq \phi$ and *free* if $\bigcap Z[I] = \phi$.

Example 1.5.17. The zero ideal in $C(X)$ is fixed. In $C^*(\mathbb{N})$, the ideal $J = (j)$ generated by the function $j(n) = \frac{1}{n}$ is free.

Theorem 1.5.18 (4.6, [18]). (i) *The fixed maximal ideals in $C(X)$ are precisely the sets*

$$M_p = \{f \in C(X) : f(p) = 0\} \quad (p \in X).$$

The ideals M_p are distinct for distinct p For each p, $C(X)/M_p$ is isomorphic with the real field $\mathbb{R}$, in fact, the mapping $M_p(f) \rightarrow f(p)$ is the unique isomorphism of $C(X)/M_p$ onto $\mathbb{R}$

(ii) *The fixed maximal ideals in $C^*(X)$ are precisely the sets*

$$M^*{}_p = \{f \in C^*(X) : f(p) = 0\} \quad (p \in X).$$

The ideals $M^*{}_p$ are distinct for distinct p For each p, $C^*(X)/M^*{}_p$ is isomorphic with the real field $\mathbb{R}$; in fact, the mapping $M^*{}_p(f) \rightarrow f(p)$ is the unique isomorphism of $C^*(X)/M^*{}_p$ onto $\mathbb{R}$

1.5.3 Structure spaces

Let $\mathcal{M}$ be the collection of all maximal ideals in $C(X)$. Then we can topologize $\mathcal{M}$ by defining

$$\mathcal{M}(f) = \{M \in \mathcal{M} \cdot f \in M\} \quad .$$

as the base for closed sets. This topology on $\mathcal{M}$ is called hull-kernel topology or Stone topology and the resulting topological space $\mathcal{M}$ is known as structure space of $C(X)$.

Theorem 1.5.19 (4.9, [18]). *Two compact spaces X and Y are homeomorphic iff their rings $C(X)$ and $C(Y)$ are isomorphic*

1.6 Stone-Čech compactification

Among the major achievements in the study of rings of continuous functions is the characterization of maximal ideals. For a compact space X, we characterize X by associating to each maximal ideal M, the limit of the z-ultrafilter $Z[M]$. If X is non-pseudocompact, i e., $C(X) \neq C^*(X)$ then for studying $C^*(X)$ we look for a compactification T of X such that

$$C^*(X) \cong C(T)$$

In this case the maximal ideals of $C^*(X)$ corresponds to the maximal ideals of $C(T)$.

For studying $C(X)$ we associate with each z-ultrafilter of X some suitable space T in a natural way. This is done as follows

Theorem 1.6.1 (6.4, [18]). *Let X be a dense subspace of T. The following statements are equivalent.*

(i) *Every continuous mapping τ from X into any compact space Y has an extension to a continuous mapping from T into Y.*

(ii) *X is C^*-embedded in T*

(iii) *Any two disjoint zero-sets in X have disjoint closures in T*

(iv) *For any two zero-sets $Z_1 . Z_2$ in X,*

$$\mathrm{cl}_T(Z_1 \cap Z_2) = \mathrm{cl}_T Z_1 \cap \mathrm{cl}_T Z_2.$$

(v) *Every point of T is the limit of a unique z-ultrafilter on X.*

The following fundamental theorem is essentially due to Stone and Čech.

Theorem 1.6.2 (6.5, [18]). *Every (completely regular)space X has a compactification βX, with the following equivalent properties*

(i) *Every continuous mapping τ from X into any compact space Y has an extension $\bar{\tau}$ to a continuous mapping from βX into Y.*

(ii) *X is C^*-embedded in βX.*

(iii) *Any two disjoint zero-sets in X have disjoint closures in βX.*

(iv) *For any two zero-sets Z_1, Z_2 in X,*

$$\mathrm{cl}_{\beta X}(Z_1 \cap Z_2) = \mathrm{cl}_{\beta X} Z_1 \cap \mathrm{cl}_{\beta X} Z_2.$$

(v) *Distinct z-ultrafilters on X have distinct limits in βX.*

Furthermore, βX (Stone-Čech compactification of X) is unique, in the following sense: if a compactification T of X satisfies one of the listed conditions, then there exists a homeomorphism of βX onto T that leaves X pointwise fixed

Remark 1.6.3 (6.6(b), [18]). The mapping $f \rightarrow f^\beta$ is an isomorphism of $C^*(X)$ onto $C(\beta X)$

Theorem 1.6.4 (6.12, [18]). *Every compactification of X is a continuous image of βX. Moreover, if τ is any homeomorphism from X into a compact space Y, then its Stone extension $\bar{\tau}$ carries $\beta X \setminus X$ into $Y \setminus \tau[X]$.*

The following is a standard theorem in [18]

Theorem 1.6.5 (9.5, [18]). *Every non-empty zero-set in βX, if disjoint from X contains a copy of $\beta \mathbb{N}$ and so its cardinality is atleast 2^c*

Corollary 1.6.6 (9.6, [18]). *No point of $\beta X \setminus X$ is a G_δ in βX.*

1.6.1 Characterization of maximal ideals

Theorem 1.6.7 (7.2, [18]). *The maximal ideals in $C^*(X)$ are precisely the sets*

$$M^{*p} = \{f \in C^*(X) \cdot f^\beta(p) = 0\} \qquad (p \in \beta X)$$

and they are distinct for distinct p

Theorem 1.6.8 (7.3, [18]). *For the maximal ideals M^p in $C(X)$, we have*

$$M^p = \{f \in C(X) \,.\, p \in \mathrm{cl}_{\beta X} Z_X(f)\} \qquad (p \in \beta X).$$

Chapter 2

Prime and maximal ideals in intermediate subalgebras of $C(X)$

2.1 Introduction

This chapter is a survey of prime and maximal ideals in subrings of $C(X)$ and intersection of maximal ideals in intermediate algebras between $C*(X)$ and $C(X)$ which was studied extensively by H. L. Byun and S. Watson in 1990 [7]. Further works by Acharya, Chattopadhyay and Ghosh [1]; Domínguez, Gómez Pérez [11] among others led to further development of the subject. Many of the results which hold in $C(X)$ can be generalized to subrings $A(X)$ of $C(X)$ (also known as *intermediate algebra*) that contain $C^*(X)$. In this setting $C(X)$ and $C^*(X)$ are just special cases of $A(X)$. We shall record some results which hold in such $A(X)$.

Definition 2.1.1. An algebra $A(X)$ is called an *intermediate algebra* if $C^*(X) \subseteq A(X) \subseteq C(X)$.

2.2 Prime ideals and β -ideals

In this section we shall quote a map from $A(X)$ to $Z(X)$ that gives a correspondence between the algebraic properties of $A(X)$ and topological properties of X. We shall also quote some results on prime ideals and a class of prime ideals called β-ideals.

First we shall show that $A(X)$ is a lattice.

Theorem 2.2.1. *If $g \in A(X)$ then $|g| \in A(X)$.*

Proof. Let

$$F = \{x \in X \mid g(x) \geq 1\},$$
$$G = \{x \in X \mid g(x) \leq -1\}.$$

We shall show that F and G are completely separated.

Let

$$h(x) = (g(x) \vee -1) \wedge 1 \text{ clearly, } h \text{ is continuous.}$$

Now, if $x \in F$, then $h(x) = -1$ and if $x \in G$, then $h(x) = 1$, so $k(x) = \frac{h(x)+1}{2}$ completely separates F and G.
Now, let

$$h'(x) = h(x)g(x) - |g(x)|$$

Clearly, if $x \in F \cup G$, then $h'(x) = 0$ and if $x \notin F \cup G$, then

$$|h'(x)| = |h(x)g(x) - |g(x)|| \leq |h(x)g(x)| + |g(x)| = |g(x)|(|h(x)|+1) \leq 2|g(x)| < 2.$$

Further, h' is continuous as $C(X)$ is a lattice. Since, $-1 < |g| < 1$ on $X \setminus F \cap G$, implies $|h'| < 2$. So, $h' \in \mathbb{C}^*(X) \subseteq A(X)$, implies that $hg - |g| \in A(X)$, hence $|g| \in A(X)$.

□

Corollary 2.2.2. *$A(X)$ is a lattice.*

Proof. Let $f, g \in A(X)$, then

$$f \vee g = \frac{f + g + |f - g|}{2}$$

Since

$f \pm g \in A(X) \implies |f - g| \in A(X)$ [Theorem2.2.1], so, $f + g + |f - g| \in A(X)$.

Hence, $\frac{1}{2}(f + g + |f - g|) \in A(X)$ implies that $f \vee g \in A(X)$. Similarly, $f \wedge g \in A(X)$. Hence, $A(X)$ is a lattice.

□

Proposition 2.2.3. *If $A(X)$ is an intermediate algebra between $C^*(X)$ and C(X) then $A(X)$ is absolutely convex subalgebra of C(X) and hence a sublattice of C(X).*

Proof. Let $f, g \in C(X)$ such that $|f| \leq |g|$ and $g \in A(X)$.
Then $1 + g^2 \in A(X)$ as $A(X) \supset C^*(X) \ni 1$.
So, $\dfrac{1}{1+g^2} \in C^*(X) \subseteq A(X)$.
Further, $|f| \leq |g| \quad \Rightarrow -|g| \leq f \leq |g|$

$$\Rightarrow \frac{-|g|}{1+g^2} \leq \frac{f}{1+g^2} \leq \frac{|g|}{1+g^2} \Rightarrow -1 \leq \frac{-|g|}{1+g^2} \leq \frac{f}{1+g^2} \leq \frac{|g|}{1+g^2} \leq 1$$

$$\Rightarrow f(1+g^2)^{-1} \in C^*(X) \subseteq A(X)$$

Thus, $f(1+g^2)^{-1}(1+g^2) \in A(X) \Rightarrow f \in A(X)$
Hence, $A(X)$ is absolutely convex.

□

Unlike the case of $C(X)$ in Theorem [1.4.4], where I is an ideal in $C(X)$ imply that $Z[I]$ is a z-filter on X, the corresponding result for $A(X)$ need not be true.

Example 2.2.4. Consider the ring $C^*(\mathbb{N})$ where $\mathbb{N}$ is the set of natural numbers. Let I be the set of all sequences in $C^*(\mathbb{N})$ that converge to zero, then I is an ideal but $Z[I]$ is not a z-filter on X.

Proof. Let I be the set of al the sequences in $C^*(\mathbb{N})$ that converge to zero, i.e.,

$$I = \{(s_n)_{n\in\mathbb{N}} \in C^*(\mathbb{N}) \mid (s_n)_{n\in\mathbb{N}} \to 0 \text{ as } n \longrightarrow \infty\}$$

Since the constant sequence (0) converges to 0 and $(a_n)_{n\in\mathbb{N}} + (0) = (a_n)_{n\in\mathbb{N}}$, for all sequences$(a_n)_{n\in\mathbb{N}} \in I$ so,

$$\mathbf{0} = (0)_{n\in\mathbb{N}} \in I$$

Let $(s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}} \in I$
then,

$$(s_n)_{n\in\mathbb{N}} + (t_n)_{n\in\mathbb{N}} = (s_n + t_n)_{n\in\mathbb{N}} \longrightarrow 0 \quad as\ n \to \infty.$$

Hence, $(s_n)_{n\in\mathbb{N}} + (t_n)_{n\in\mathbb{N}} \in I \ \forall (s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}} \in I$.

Let $(a_n)_{n\in\mathbb{N}} \in C^*(\mathbb{N})$ & $(s_n)_{n\in\mathbb{N}} \in I$
then, there exists $k \in \mathbb{N}$ such that $|a_n| \leq k \;\; \forall n \in \mathbb{N}$, so

$$\lim_{n\to\infty} |(a_n)_{n\in\mathbb{N}} \cdot (s_n)_{n\in\mathbb{N}}| \leq \lim_{n\to\infty} |k||(s_n)_{n\in\mathbb{N}}| = |k|| \lim_{n\to\infty}(s_n)_{n\in\mathbb{N}}| = 0.$$

So $(a_n)_{n\in\mathbb{N}}(s_n)_{n\in\mathbb{N}} \in I$. Thus, I is an ideal.
Now we show that $Z[I]$ is not a z-filter. Let $f \in I$ be given by

$$f(n) = (\frac{1}{n}) \in I \quad \forall n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

but, $Z(f) = \phi$, so $\phi \in Z[I]$, hence $Z[I]$ is not a z-filter on X. □

This shows that the usual correspondence $I \longrightarrow Z[I]$ of ideals and z-filters may not be true. So there was a need to introduce a well defined correspondence between ideals in $A(X)$ and z-filters on X. This was resolved by Byun and Watson in [7].

Notation 2.2.5. For each $f \in A(X)$, we associate a collection of subsets of X, $\mathcal{Z}_A(f) = \{F \in Z(X) \mid \exists\, g \in A(X) \text{such that } fg\,|_{F^c} = 1\}$

Remark 2.2.6. $\mathcal{Z}_A(f)$ need not be a z-filter, for consider $C(\mathbb{N})$ and let $j(n) = \frac{1}{n}$, then there exists $i \in C(\mathbb{N})$, where $i(n) = n$ such that $ij\,|_{\phi^c} = 1$. This implies that $\phi \in \mathcal{Z}_C(j)$.

Notation 2.2.7. (i) For an ideal I of $A(X)$, we set

$$\mathcal{Z}_A[I] \;=\; \bigcup\{\mathcal{Z}_A(f) \mid f \in I\}.$$

(ii) For a z-filter $\mathcal{F}$ on X, we set $\mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}] \;= \{\, f \in A(X) \mid \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$.

We note a few basic facts about the maps $\mathcal{Z}_A$ and $\mathcal{Z}_A{}^{\leftarrow}$.

Remark 2.2.8.

$$\mathcal{Z}_A(f) = \mathcal{Z}_A|f|.$$

Lemma 2.2.9 (1.2, [7]; 1, [28]). *Let $g,h \in A(X)$*

(i) $\mathcal{Z}_A(gh) \subseteq \mathcal{Z}_A(g) \cap \mathcal{Z}_A(h)$

(ii) $\mathcal{Z}_A(g^2 + h^2) \supset \mathcal{Z}_A(g) \cup \mathcal{Z}_A(h)$

(iii) *If* $|g| \leq |h|$, *then* $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A(h)$

Proof. (i) Let $F \in \mathcal{Z}_A(gh)$ then $\exists\, k \in A(X)$, such that

$$(gh)k \mid_{F^c} = h(gk) \mid_{F^c} = g(hk) \mid_{F^c} = 1. \tag{2.1}$$

So, by equation (2.1) $F \in \mathcal{Z}_A(g)$. It also follows from the above equation that

$$(gh)k \mid_{F^c} = (hg)k \mid_{F^c} = h(gk) \mid_{F^c} = 1 \quad \Rightarrow F \in \mathcal{Z}_A(h).$$

Thus $F \in \mathcal{Z}_A(g) \cap \mathcal{Z}_A(h)$.

(ii) Let $F \in \mathcal{Z}_A(g) \cup \mathcal{Z}_A(h)$
Without loss of generality, let $F \in \mathcal{Z}_A(g)$. Then $\exists\, k \in A(X)$, $k \neq 0$ such that $gk \mid_{F^c} = 1$

$$\text{Since, } g, h \in A(X),\ g^2 + h^2 \in A(X).$$

Now on F^c,

$$g^2 + h^2 \geq g^2 > 0 \ \Rightarrow\ (g^2 + h^2)k^2 \geq g^2k^2 = (gk)^2 = 1$$

$$\Rightarrow 0 \ < \ \frac{1}{(g^2 + h^2)k^2} \leq 1$$

Let $l = \max\{(g^2 + h^2)k^2, 1\}$, then l is continuous and $l \geq 1$, this imply that $\frac{1}{l}$ is continuous. Let $l' = \frac{1}{l}$, then $l' \in C^*(X) \subseteq A(X)$. It is clear that $l = (g^2 + h^2)k^2$ on F^c, therefore,

$$(g^2 + h^2)k^2(l') \mid_{F^c} = (g^2 + h^2)k^2(\frac{1}{gf^2 + h^2}) \mid_{F^c} = 1 \Rightarrow F \in \mathcal{Z}_A(g^2 + h^2).$$

Similarly, $F \in \mathcal{Z}_A(h) \Rightarrow F \in \mathcal{Z}_A(g^2 + h^2)$. Hence the result follows.

(iii) As $0 < |g| \leq |h|$, the result follows by imitating the proof of (ii) above. □

Lemma 2.2.10. (i) *If* $f \in A(X)$, *then* $\lim_{\mathcal{Z}_A(f)} fh = 0$ *for* $h \in A(X)$

(ii) *If* $\mathcal{F}$ *is a z-filter on* X *and if* $\lim_{\mathcal{F}} fh = 0\ \forall\, h \in A(X)$, *then* $\mathcal{Z}_A(f) \subseteq \mathcal{F}$

Proof. (i) We shall first show that $\lim_{\mathcal{Z}_A(f)} f = 0$

Let $W = (-\epsilon, \epsilon)$, $\epsilon > 0$, be a neighbourhood of 0 in $\mathbb{R}$ and consider $V = f^{\leftarrow}(W)$. So, $0 < \epsilon < |f|$ on the set V^c.

Let $l = \max\{\epsilon, |f|\}$, then $|f||_{V^c} = l\,|_{V^c}$ implies that

$$0 < l^{-1} \leq \epsilon^{-1} \Rightarrow l^{-1} \in C^*(X) \subseteq A(X).$$

Now

$$ll^{-1} = 1 \Rightarrow |f|l^{-1}\,|_{V^c} = 1 \;\Rightarrow V \in \mathcal{Z}_A(|f|) = \mathcal{Z}_A(f) \Rightarrow \lim_{\mathcal{Z}_A(f)} f = 0.$$

It is clear that $\mathcal{Z}_A(fg) \subseteq \mathcal{Z}_A(f)$ and the above argument implies that $\lim_{\mathcal{Z}_A(fg)} fg = 0$ i.e., preimage of ϵ- neighbourhood of 0 is in $\mathcal{Z}_A(fg) \subseteq \mathcal{Z}_A(f)$ $\forall\, \epsilon > 0$. Hence, $\lim_{\mathcal{Z}_A(f)} fg = 0$

(ii) Let $F \in \mathcal{Z}_A(f)$. We claim that $F \supset E$ for some $E \in \mathcal{F}$. On the contrary suppose there does not exists any such E in $\mathcal{Z}_A(f)$, then,

$$E \cap F^c \neq \phi \quad \forall E \in \mathcal{F}$$

Let $h \in A(X)$ such that $fh\,|_{F^c} = 1$ and consider $\mathcal{G} = \{fh(E) \mid E \in \mathcal{F}\}$

then $1 \in fh(E)$ $\forall E \in \mathcal{F}$ implies that 1 is a cluster point of $\mathcal{G}$, contradicting the fact that 0 is the unique limit of $\mathcal{G}$.

□

Theorem 2.2.11. *For X a normal space and $f \in C(X)$, the members of $\mathcal{Z}_C(f)$ are the zero-set neighbourhoods of $Z(f)$*

Proof. Let E be a zero-set neighbourhood of $Z(f)$. Then there exists an open set U such that $Z(f) \subseteq U \subseteq E$. Since U^c is closed and $\frac{1}{f}$ is defined on U^c, so by Tietze's extension theorem, it has a continuous extension $h \in C(X)$. Clearly, $fh|_{E^c} = 1$ implies that $E \in \mathcal{Z}_C(f)$. On the other hand, if $E \in \mathcal{Z}_C(f)$, then $E \supseteq E_\epsilon(f)$ for some $\epsilon > 0$ [Proposition 2.7.10], i.e., $E \supseteq E_\epsilon(f) \supseteq Z(f)$. Hence, E is a zero-set neighbourhood of $Z(f)$. □

Lemma 2.2.12. *Let $f \in A(X)$, I an ideal of $A(X)$, and $\mathcal{F}$ a z-filter on X.*

(i) *$\mathcal{Z}_A(f)$ is a z-filter on X iff f is not invertible in $A(X)$.*

(ii) *If I is an ideal in $A(X)$, then $\mathcal{Z}_A[I]$ is a z-filter on X.*

(iii) *If $\mathcal{F}$ is a z-filter on X, then $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ is an ideal in $A(X)$.*

Proof. (i) Let us suppose that $\mathcal{Z}_A(f)$ is a z-filter on X, then

$$\phi \notin \mathcal{Z}_A(f) \Rightarrow \nexists\ h \in A(X) \text{ such that } fh\mid_{\phi^c} = 1$$

Hence, f is not invertible on X.
On the other hand, if f is not invertible on $A(X)$, then

(a) $\phi \notin \mathcal{Z}_A(f)$

(b) Suppose $F, F' \in \mathcal{Z}_A(f)$, then there exists $g, g' \in A(X)$ such that

$$fg\mid_{F^c} = 1 \text{ and } fg'\mid_{F'^c} = 1.$$

It is clear that $F^c \supset (F \cup F')^c$ and $F'^c \supset (F \cup F')^c$.
further,

$$(F \cap F')^c = F^c \cup F'^c \tag{2.2}$$

Let $h = g + g' - fgg'$. Clearly, $h \in A(X)$.
We claim that f is locally invertible on $(F \cap F')^c$.
By equation (2.2), either $x \in F^c$, in which case

$$(hf)(x) = (gf + g'f - fgg'f)(x) = 1 + g'(x)f(x) - g'(x)f(x) = 1,$$

or $x \in F'^c$, in which case,

$$(hf)(x) = g(x)f(x) + 1 - g(x)f(x) = 1.$$

$$\text{Hence, } F \cap F' \in \mathcal{Z}_A(f).$$

(c) Let $F' \in Z[X]$ such that $F' \supset F$ for some $F \in \mathcal{Z}_A(f)$, then there exists $g \in A(X)$ such that $fg\mid_{F^c} = 1$
Since $F'^c \subseteq F^c$ so,

$$fg\mid_{F'^c} = 1 \quad \Rightarrow F' \in \mathcal{Z}_A(f).$$

Hence, $\mathcal{Z}_A(f)$ is a z-filter on X.

(ii) Let I be an ideal in $A(X)$, we know

$$\mathcal{Z}_A[I] = \bigcup\{\mathcal{Z}_A(f) \mid f \in I\}$$

(a) Since I is an ideal in $A(X)$, so each f in I is not invertible in $A(X)$. Consequently, by (i) above

$$\phi \notin \mathcal{Z}_A(f) \text{ for all } f \in I \Rightarrow \phi \notin \bigcup\{\mathcal{Z}_A(f) \mid f \in I\} \Rightarrow \phi \notin \mathcal{Z}_A[I].$$

(b) Let $F, F' \in \mathcal{Z}_A[I]$, then $F \in \mathcal{Z}_A(f)$ and $F' \in \mathcal{Z}_A(\mathrm{g})$ for some $f, g \in I$. So, there exists $f', g' \in A(X)$ such that $ff' \mid_{F^c}= 1$ and $gg' \mid_{F'^c}= 1$.
Now,

$$f, g \in I \Rightarrow f^2 + g^2 \in I \text{ and } f^2 + g^2 \geq f^2 > 0 \text{ on } F^c.$$

This implies that

$$(f^2 + g^2)f'^2 \geq f^2 f'^2 > 0 \Rightarrow (f^2 + g^2)f'^2 \geq 1 > 0.$$

Let

$l = \max\{(f^2+g^2)f'^2, 1\}$. Clearly, $l \geq 1$ so, $(f^2+g^2)f'^2(\frac{1}{l}) \mid_{F^c}= 1$.

Similarly,

$$(f^2 + g^2)g'^2(\frac{1}{k}) \mid_{F'^c}= 1 \text{ where } k = \max\{(f^2 + g^2)g'^2, 1\}.$$

Thus, $F \in \mathcal{Z}_A\ (f^2 + g^2)$ & $F' \in \mathcal{Z}_A\ (f^2 + g^2)$.
Since $(f^2+g^2) \in I$ implies (f^2+g^2) is not invertible, so $\mathcal{Z}_A(f^2 + g^2)$ is a z-filter. Hence,

$$F \cap F' \in \mathcal{Z}_A(f^2 + g^2) \subseteq \mathcal{Z}_A[I].$$

(c) Let $F \in Z[X]$ such that $F \supset F'$ with $F' \in \mathcal{Z}_A[I]$, then $\exists\ g \in A(X)$ such that $fg \mid_{F'^c}= 1$
Since, $F^c \subset F'^c$, so,

$$fg \mid_{F^c}= 1 \Rightarrow F \in \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[I].$$

Hence, $\mathcal{Z}_A[I]$ is a z-filter on X.

(iii) Clearly, $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] \neq \phi$ for,

$$X \in \mathcal{F} \;\&\; \mathcal{Z}_A(0) = \{X\} \subseteq \mathcal{F}, \text{so } 0 \in \mathcal{Z}_A^{\leftarrow}[\mathcal{F}].$$

Let $I = \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ and consider $f \in I$ & $g \in A(X)$,

$$\text{then } \mathcal{Z}_A(fg) \subseteq \mathcal{Z}_A(f) \subseteq \mathcal{F} \Rightarrow \mathcal{Z}_A(fg) \subseteq \mathcal{F} \Rightarrow fg \in I.$$

Now let $f, g \in I$.

$$\text{then, } \lim_{\mathcal{F}} fh = \lim_{\mathcal{F}} gh = 0 \ \Rightarrow \lim_{\mathcal{F}} (f+g)h = 0 \ \ \forall \ \ h \in A(X).$$

Thus, $\mathcal{Z}_A(f+g) \subseteq \mathcal{F}$ implies that $f+g \in I$. Hence, I is an ideal.

□

So, *we have seen that* $\mathcal{Z}_A$ *is a mapping from the set of ideals in* $A(X)$ into the set of z-filters on X. Also, $\mathcal{Z}_A{}^{\leftarrow}$ maps the set of z-filters on X into the set of ideals in X.

There arises a question whether $\mathcal{Z}_A$ maps maximal ideals of $A(X)$ to z-ultrafilters in X? The answer is not true and we shall record a counter as shown in [7]. However, $\mathcal{Z}_A$ and $\mathcal{Z}_A{}^{\leftarrow}$ satisfy some intersting properties as shown by the following lemma;

Lemma 2.2.13. *For an ideal* I *of* $A(X)$ *and a* z*-filter* $\mathcal{F}$ *on* X *we have:*

(i) $\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]] \supset I$

(ii) $\mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]]] = \mathcal{Z}_A[I]$

(iii) $\mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]] \subseteq \mathcal{F}$,

(iv) $\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]]] = \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$

Proof. (i) Let $f \in I$, then $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[I] \Rightarrow f \in \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]]$.
Hence proved.

(ii) Let $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[I]$, then $\exists\, g \in I$ such that $\mathcal{Z}_A(f) = \mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[I]$.
Using (i)

$$g \in \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]] \Rightarrow \mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]]].$$

Since $\mathcal{Z}_A(f) = \mathcal{Z}_A(g)$, so,

$$\mathcal{Z}_A[I] \subseteq \mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]]].$$

On the other hand, if follows from (i) that

$$\mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]]] \supset \mathcal{Z}_A[I], \text{ as } \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]] \supset I.$$

Hence, the equality follows.

(iii) Let $\mathcal{Z}_A(f) \subset \mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]]$. Then $\mathcal{Z}_A(f) = \mathcal{Z}_A(g)$ for some $g \in \mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]]$ and so,

$$\mathcal{Z}_A(g) \subseteq \mathcal{F} \Rightarrow \mathcal{Z}_A(f) \subseteq \mathcal{F}.$$

Hence proved.

(iv) Let $f \in \mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]$, then

$$\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]] \Rightarrow f \in \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]]] \Rightarrow \mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}] \subseteq \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]]].$$

On the other hand (iii) implies that

$$\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[\mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]]] \subseteq \mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}].$$

Hence, the equality follows.

□

We shall record examples to illustrate these cases, before that let us first look at the relationship between zero-sets $Z(f)$ and $\mathcal{Z}_A(f)$, $f \in A(X)$ as shown by Byun and Watson [7].

Remark 2.2.14. Let $f \in A(X)$, then

$$Z(f) = \bigcap \mathcal{Z}_A(f).$$

Proof. Let

$$E_\epsilon(f) = \{x \in X \mid |f(x)| \leq \epsilon\} \text{ where } f \in A(X).$$

We claim that $E_\epsilon(f) \subseteq \mathcal{Z}_A(f)$ for every $\epsilon > 0$. It is clear that $0 < \epsilon < |f(x)|$ on $E^c_\epsilon(f)$. By Theorem [2.2.1], $f \in A(X) \Rightarrow |f| \in A(X)$

Let $g = \max\{|f|, \epsilon\}$.Then $g \geq \epsilon$ on $E^c_\epsilon(f)$.

So,

$$g\mid_{E^c_\epsilon(f)} = |f|\mid_{E^c_\epsilon(f)} \text{ and } g \geq \epsilon > 0 \Rightarrow 0 < \frac{1}{g} \leq \frac{1}{\epsilon} \Rightarrow \frac{1}{g} \in C^*(X) \subseteq A(X).$$

Now,

$$|f|g^{-1}\mid_{E^c_\epsilon(f)} = |f|\frac{1}{|f|}\mid_{E^c_\epsilon(f)} = 1 \Rightarrow E_\epsilon(f) \in \mathcal{Z}_A(|f|) = \mathcal{Z}_A(f),$$

So, $E_\epsilon(f) \subseteq \mathcal{Z}_A(f)$ for every $\epsilon > 0$.
Next we claim that

$$Z(f) = \bigcap\{E_\epsilon(f) \mid \epsilon > 0\}$$

Let

$$x \in \bigcap\{E_\epsilon(f) \mid \epsilon > 0\} \text{ Then, } |f(x)| \leq \epsilon \; \forall \; \epsilon > 0 \Rightarrow -\epsilon \leq f(x) \leq \epsilon \; \forall \epsilon > 0.$$

Suppose $f(x) \neq 0$, let $a = |f(x)| > 0$. Let $\epsilon' > 0$ such that $0 < \epsilon' < a$, then $x \notin E_{\epsilon'}(f)$ which is not true.
Hence,

$$f(x) = 0 \Rightarrow x \in Z(f).$$

On the other hand,

$$x \in Z(f) \Rightarrow f(x) = 0 \Rightarrow x \in E_\epsilon(f) \; \forall \; \epsilon > 0 \Rightarrow x \in \bigcap\{E_\epsilon(f) \mid \epsilon > 0\}.$$

Hence, the equality follows.
Since $E_\epsilon(f) \subseteq E \quad \forall \; E \in \mathcal{Z}_A(f)$, so

$$\bigcap \mathcal{Z}_A(f) \supset \bigcap\{E_\epsilon(f) \mid \epsilon > 0\}$$

Now $E_\epsilon(f) \in \mathcal{Z}_A(f) \; \forall \epsilon > 0$. So the remark follows. □

Remark 2.2.15. In the study of *C(X)*, the zero set $Z(f)$ is a measure of where f is not invertible. The analogous role for that measure in arbitrary $A(X)$ is played by $\mathcal{Z}_A(f)$. In particular this is a measure of where f is not invertible in $A(X)$ even when $Z(f) = \phi$.

Example 2.2.16. Let $X = \mathbb{R}$ and consider the ring $A(X) = C(\mathbb{R})$. Let $M_0 = \{f \in C(\mathbb{R}) \mid f(0) = 0\}$ and consider the map $\phi : C(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by $\phi(f) = f(0)$. Let $f, g \in C(\mathbb{R})$, then

$$\phi(f+g) = (f+g)(0) = f(0) + g(0) = \phi(f) + \phi(g),$$
$$\text{and } \phi(fg) = fg(0) = f(0)\, g(0) = \phi(f)\phi(g).$$

Hence ϕ is a homomorphism.
Now, let $r \in \mathbb{R}$, then $\phi(r) = r(0) = r \Rightarrow \phi$ is onto. Further,

$$\operatorname{Ker}\phi = \{f \in C(\mathbb{R}) \mid f(0) = 0\} = M_0.$$

So, by the fundamental theorem of ring homomorphisms $\dfrac{C(\mathbb{R})}{\operatorname{Ker}\phi} \cong \mathbb{R}$. This implies that $\operatorname{Ker}\phi = M_0$ is a maximal ideal of $C(\mathbb{R})$. By Theorem [1.4.9] $Z[M_0]$ is a z-ultrafilter in $\mathbb{R}$. We now show that $\mathcal{Z}_A[M_0] \subsetneq Z[M_0]$.
Let $f \in M_0$ and consider $\mathcal{Z}_A(f)$. Then,

$$Z(f) = \bigcap_{E \in \mathcal{Z}_A(f)} E \Rightarrow Z(f) \subseteq E, \text{ hence, } E \in Z[M_0] \;\; \forall E \in \mathcal{Z}_A(f),$$

so,

$$\mathcal{Z}_A(f) \subseteq Z[M_0] \quad \forall f \in M_0$$

$$\Rightarrow \bigcup_{f \in M_0} \mathcal{Z}_A(f) \subseteq Z[M_0] \;\Rightarrow \mathcal{Z}_A[M_0] \subseteq Z[M_0] \Rightarrow \mathcal{Z}_A[M_0].$$

Now for $i \in M_0$, where $i(r) = r \;\; \forall r \in \mathbb{R}$,

$$Z(i) = \{0\} \;\in Z[M_0].$$

We claim that $\{0\} \notin \mathcal{Z}_A[M_0]$, for if $\{0\} \in \mathcal{Z}_A[M_0]$, then $\{0\} \in \mathcal{Z}_A(i)$, so $\exists\, k \in$ $C(X)$ defined by

$$k(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ a, & x = 0, \;\; a \in \mathbb{R}. \end{cases}$$

Clearly, k is not continuous at 0. Hence, $\mathcal{Z}_A[M_0] \subsetneq Z[M_0]$, so we conclude that $\mathcal{Z}_A[M_0]$ is not a z-ultrafilter on X.

Remark 2.2.17. The inclusion in Lemma [2.2.13 (iii)] may be proper.

Proof. Let M_0 be as in Example[2.2.16] and consider $\mathcal{F} = Z[M_0]$, then $\mathcal{Z}_C^{\leftarrow}[\mathcal{F}] = \mathcal{Z}_C^{\leftarrow}[Z[M_0]] \supset M_0$.
i.e.,

$$M_0 \subseteq \mathcal{Z}_C^{\leftarrow}[\mathcal{F}] \subseteq C(\mathbb{R}).$$

Since M_0 is maximal, so either $M_0 = \mathcal{Z}_C^{\leftarrow}[\mathcal{F}]$ or $\mathcal{Z}_C^{\leftarrow}[\mathcal{F}] = C(\mathbb{R})$.
Since $\mathcal{F}$ is a z-filter, so,

$$1 \notin \mathcal{Z}_C^{\leftarrow}[\mathcal{F}] \Rightarrow \mathcal{Z}_C^{\leftarrow}[\mathcal{F}] \nsubseteq C(\mathbb{R}).$$

Hence,

$$M_0 = \mathcal{Z}_C^{\leftarrow}[\mathcal{F}] \Rightarrow \mathcal{Z}_C[\mathcal{Z}_C^{\leftarrow}[\mathcal{F}]] = \mathcal{Z}_C[M_0] \subsetneq Z[M_0] = \mathcal{F},$$

$$i.e., \;\; \mathcal{Z}_C[\mathcal{Z}_C^{\leftarrow}[\mathcal{F}]] \subsetneq \mathcal{F}.$$

□

Remark 2.2.18. The inclusion in Lemma [2.2.13(i)] may be proper.

Proof. If we consider $I = (i)$, the ideal generated by the function $i(r) = r\ \forall r \in \mathbb{R}$, then $\mathcal{Z}_C(f) \subseteq \mathcal{Z}_C(i)\ \ \forall f \in M_0$.
For, let $E \in \mathcal{Z}_C(f)$, then $E \subseteq (-\epsilon, \epsilon)$ for some $\epsilon > 0$.
Define $k(r) = \begin{cases} \frac{1}{r} & r \notin (-\epsilon,\epsilon) \\ \frac{r}{\epsilon^2} & r \in (-\epsilon,\epsilon) \end{cases}$

This implies that,

$$\mathcal{Z}_C[M_0] \subseteq \mathcal{Z}_C[I]\ \&\ \mathcal{Z}_C[I] \subseteq \mathcal{Z}_C[M_0]\ \text{ as } I \subseteq M_0.$$

So,

$$\mathcal{Z}_C[M_0]\ =\ \mathcal{Z}_C[I].$$

Now,

$$M_0 = \mathcal{Z}_C{}^{\leftarrow}[\mathcal{Z}_C[M_0]]\ = \mathcal{Z}_C{}^{\leftarrow}[\mathcal{Z}_C[I]] \supset I.$$

As $M_0 \neq I$, we have $\mathcal{Z}_C{}^{\leftarrow}[\mathcal{Z}_C[I]]\ \neq I$. ☐

Definition 2.2.19. The ideals I of A(X) that satisfy $\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]]\ =\ I$ is known as β -ideals.

The following result leads us to a host of examples.

Proposition 2.2.20. *I is a β-ideal if and only if $\mathcal{Z}_A(f)\ \subseteq\ \mathcal{Z}_A[I]$ implies $f \in I$*

Proof. Let I be a β-ideal, then

$$\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]] = I.$$

Let

$$\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[I] \Rightarrow f \in \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]] = I.$$

Converse is obvious. $\mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]] \supset I$.

☐

Corollary 2.2.21. *Every maximal ideal is a β-ideal.*

Proof. Let M be a maximal ideal in *C(X)*. Then we have

$$M \subseteq \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[M]] \supset A(X).$$

$$\text{Since } \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[M]] \neq A(X),$$

So, by maximality of M, $M = \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[M]]$. Hence, M is a β-ideal. ☐

Result 2.2.22. *In $C(\mathbb{N})$ every ideal is a β-ideal*

Proof. Let I be an ideal in $C(\mathbb{N})$ and let $f \in C(\mathbb{N})$ such that $\mathcal{Z}_C(f) \subseteq \mathcal{Z}_C[I]$. For the function h defined as below

$$h(x) = \begin{cases} 0 & x \in Z(f) \\ \dfrac{1}{f} & x \in Z(f)^c, \end{cases}$$

$h \in C(\mathbb{N})$ and $hf\,|_{Z(f)^c} = 1 \;\Rightarrow Z(f) \subseteq \mathcal{Z}_C(f) \;\Rightarrow Z[I] \subseteq \mathcal{Z}_C[I]$. Hence $\mathcal{Z}_C[I] = Z[I]$. So, $Z(f) \in \mathcal{Z}_C[I] = Z[I]$ and $\exists\, g \in I$ such that $Z(f) = Z(g)$. Define a function k as

$$k(x) = \begin{cases} 0 & x \in Z(g), \\ \dfrac{f(x)}{g(x)} & x \in Z(g)^c. \end{cases}$$

Clearly,

$$k \in C(\mathbb{N}) \;\&\; f = gk \in I \Rightarrow f \in I.$$

□

Remark 2.2.23. The definition of a z-ideal applies to $C(X)$ only and is somewhat less restrictive than the notion of β -ideals in the sense that an ideal in $C(X)$ may be a z-ideal but fail to be a β-ideal.

Proof. Consider

$$O_0 = \{f \in C(\mathbb{R}) \mid Z(f) \text{ is a nbd. of } 0\}.$$

We claim that O_0 is a z-ideal.

$$\text{Let } Z(f) \in Z[O_0], \text{ then } Z(f) = Z(g) \text{ for some } g \in O_0.$$

Thus $Z(g)$ is a neighbourhood of 0 and hence $Z(f)$ is a neighbourhood of 0. This implies that $f \in O_0$. Now $O_0 \subseteq M_0$, where M_0 is as in Example [2.2.16] and $\mathcal{Z}_C[O_0] = \mathcal{Z}_C[M_0]$.

$$\text{But, } O_0 \subseteq \mathcal{Z}_C^{\leftarrow}[\mathcal{Z}_C[O_0]] = [\mathcal{Z}_C[M_0]] = M_0.$$

Consider the function i, given by $i(r) = r$ for all $r \in \mathbb{R}$.

$$\text{Clearly, } i \in M_0 \setminus O_0, \text{ implies } O_0 \neq M_0 \text{ thus, } O_0 \neq \mathcal{Z}_C^{\leftarrow}[\mathcal{Z}_C[O_0]].$$

Hence O_0 is not a β- ideal. □

Remark 2.2.24. Let I be an ideal in $C(X)$, then $\mathcal{Z}_A[I] \subseteq Z[I]$.

Proof. Let $E \in \mathcal{Z}_A[I]$, then $E \in \mathcal{Z}_A(f)$ for some $f \in I$. Consequently, $Z(f) \in Z[I]$.
By Remark [2.2.14], $\bigcap \mathcal{Z}_A(f) = Z(f) \Rightarrow Z(f) \subseteq E$. Hence, $E \in Z[I]$. So, $\mathcal{Z}_A[I] \subseteq Z[I]$. $\square$

Remark 2.2.25. In $C(X)$ every β-ideal is a z-ideal.

Proof. Let I be a β-ideal and let $Z(f) \in Z[I]$.
Then $f \in Z^{\leftarrow}[Z[I]] \supseteq Z^{\leftarrow}[\mathcal{Z}_A[I]] \supseteq \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[I]] = I$. Hence it is a z-ideal. $\square$

Next we shall record some properties of β -ideals.

Theorem 2.2.26. *Every β-ideal in $A(X)$ is an intersection of prime ideals.*

Proof. We first show that $\mathcal{Z}_A(f^n) = \mathcal{Z}_A(f)$ for every $n \in \mathbb{N}$.
Let

$$E \in \mathcal{Z}_A(f^n) \text{ then } \exists\, k \in A(X) \text{ such that } f^n k\,|_{E^c} = 1$$
$$\Rightarrow f(f^{n-1})\,|_{E^c} = 1 \quad \Rightarrow E \in \mathcal{Z}_A(f).$$

On the other hand, let

$$E \in \mathcal{Z}_A(f) \text{ then } \exists\, k \in A(X) \text{ such that } fk\,|_{E^c} = 1.$$

Since $A(X)$ is a subring of $C(X)$, so

$$f^n.k^n = (fk)^n \Rightarrow f^n k^n\,|_{E^c} = (fk)^n\,|_{E^c} = (fk\,|_{E^c})^n = 1.$$

Hence $E \in \mathcal{Z}_A(f^n)$. So the equality follows.

Let I be a β -ideal in $A(X)$, and let $f \in A(X)$ such that $f^n \in I$.

$$\text{Then } \mathcal{Z}_A(f^n) = \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[I] \text{ implies } f \in I.$$

By Corollary [1.3.7] I is the intersection of all prime ideals which contain it. Hence the result folows. $\square$

Theorem 2.2.27. *For any $\beta-$ideal I in $A(X)$, the following are equivalent.*

(i) *I is prime*

(ii) *I contains a prime ideal*

(iii) *For every $g, h \in A(X)$, if $gh = 0$, then $g \in I$ or $h \in I$.*

(iv) *For every* $f \in A(X)$ *and for every* $\epsilon > 0$ $\exists$ *a zero-set in* $\mathcal{Z}_A[I]$ *on which* $f \leq \epsilon$ *or* $f \geq -\epsilon$.

Proof. *(i)* $\Rightarrow$ *(ii)* Since I is prime so it holds trivially.

(ii) $\Rightarrow$ *(iii)* Trivial, as $I \supset P$ for some prime ideal P and $0 \in P$.

(iii) $\Rightarrow$ *(iv)* Let $f \in A(X)$, and consider $\epsilon > 0$, then $A(X)$ is a lattice, so, $f \vee 0, f \wedge 0 \in A(X)$. But,

$$(f \vee 0)(f \wedge 0) = 0 \Rightarrow \; f \vee 0 \in I \;\; or \;\; f \wedge 0 \in I.$$

So

$$\mathcal{Z}_A[f \vee 0] \subseteq \mathcal{Z}_A[I] \;\; or \;\; \mathcal{Z}_A[f \wedge 0] \subseteq \mathcal{Z}_A[I].$$

If $\mathcal{Z}_A(f \vee 0) \subseteq \mathcal{Z}_A[I]$, then consider

$$E_\epsilon(f \vee 0) = \{x \in X \mid |(f \vee 0)| \leq \epsilon\}.$$

Since,

$$E_\epsilon(f) \subseteq \mathcal{Z}_A(f) \;\text{ so, }\; E_\epsilon(f \vee 0) \in \mathcal{Z}_A(f \vee 0) \subseteq \mathcal{Z}_A[I].$$

Let $k = \begin{cases} (f \vee 0) + \epsilon & x \in E_\epsilon(f \vee 0)^c \\ 0 & x \in E_\epsilon(f \vee 0). \end{cases}$

then $k \in A(X)$ and $E_\epsilon(f \vee 0) = Z(k) \;\Rightarrow E_\epsilon(f \vee 0)$ is a zero-set. Further,

$$(f \vee 0) \leq \epsilon \text{ on } E_\epsilon(f \vee 0) \text{ implies } f \leq f \vee 0 \leq \epsilon \;\text{ on }\; E_\epsilon(f \vee 0).$$

Similarly, if $f \wedge 0 \in I$ then define

$k' = \begin{cases} (f \wedge 0) + \epsilon & x \in E_\epsilon(f \wedge 0)^c \\ 0 & x \in E_\epsilon(f \wedge 0). \end{cases}$

So, $K' \in A(X)$, and

$$E_\epsilon(f \wedge 0) \in \mathcal{Z}_A[I], \text{ then } E_\epsilon(f \wedge 0) \in \mathcal{Z}_A(f \wedge 0) \subseteq \mathcal{Z}_A[I].$$

then

$$E_\epsilon(f \wedge 0) = Z(k') \;\;\&\;\; f \geq (f \wedge 0) \geq -\epsilon \;\;\text{ on } E_\epsilon(f \wedge 0).$$

$(iv) \Rightarrow (i)$ Let $g \in A(X)$ such that $gh \in I$ and consider for $k \in A(X)$ the map $|gk| - |h|$. For $\epsilon > 0$, by (iv), there exists

$$E \in \mathcal{Z}_A[I], \text{ such that } |gk| - |h| \leq \epsilon \text{ on } E.$$

Thus for any $\delta > \epsilon$, $|gk| > \delta \Rightarrow \delta < |gk| \leq |h| + \epsilon \Rightarrow \delta - \epsilon < |h|$ on E.

Now let $E_{gk} = \{x \in X \mid |gk| \leq \delta\}$ & $E_h = \{x \in X \mid |h| \leq \delta - \epsilon\}$.

Then, we have $E_{gk} \in \mathcal{Z}_A(gk)$ and $E_h \in \mathcal{Z}_A(h)$.

i.e., there exists $g', h' \in A(X)$ such that $(gk)g' |_{E^c_{gk}} = 1$ and $hh' |_{E^c_h} = 1$.

$$\text{Thus } (gk)g'hh' |_{E^c_{gk} \cap E^c_h} = 1 \quad \text{and} \quad E^c_{gk} \cap E^c_h = (E_{gk} \cup E_h)^c$$

$$\text{Therefore, } ((gk)g'hh') |_{(E_{gk} \cup E_h)^c} = 1.$$

Hence,

$$E_{gk} \cup E_h \in \mathcal{Z}_A(gkh) \subseteq \mathcal{Z}_A[I]. \tag{2.3}$$

Now we claim that $E \cap E_{gk} \supset E \cap E_h$.

$$\text{Let } x \in E \cap E_h \Rightarrow |gk| - |h| \leq \epsilon \quad \& \ |h| \leq \delta - \epsilon$$

$$\Rightarrow |gk| \leq |h| + \epsilon \leq \delta - \epsilon + \epsilon = \delta \quad \Rightarrow x \in E_{gk}. \text{ Hence } x \in E_{gk} \cap E$$

$$\text{But, } \quad E_{gk} \supset E \cap E_{gk} = E \cap E_{gk} \cup (E \cap E_h) \quad \text{as } E \cap E_h \subseteq E.$$

Since $\mathcal{Z}_A[I]$ is a z-filter and

$$E_{gk} \supset E \cap E_{gk} \cup E_h \Rightarrow E_{gk} \in \mathcal{Z}_A[I] \ \Rightarrow \lim_{\mathcal{Z}_A[I]} gk = 0.$$

Since k is arbitrary, so the result holds true for all $k \in A(X)$.

So, by Theorem [2.2.10], $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[I]$ and I is a $\beta - ideal$ implies $g \in I$.

Hence I is prime.

□

Corollary 2.2.28. *Every prime ideal in $A(X)$ is contained in a unique maximal ideal.*

Proof. Let P be a prime ideal in $A(X)$. If possible let M, N be two distinct maximal ideals such that $P \subset M$ and $P \subset N$. We claim that $M \cap N$ is a $\beta - ideal$.

Let $Z_A(f) \subseteq Z_A[M \cap N]$. Without loss of generality, let $Z_A[M \cap N] \subseteq Z_A[M]$.

Since every maximal ideal is a $\beta - ideal$, so $f \in M$ and similarly, $f \in N$. This implies $f \in M \cap N$, so $M \cap N$ is a $\beta - ideal$ i.e., $M \cap N \supset P$.
So $M \cap N$ is prime by Theorem [2.2.27]. Since $M \neq N$, for $p \in M \setminus N$, $q \in N \setminus M$,

$$pq \in M \cap N, \quad \text{but } p \notin M \cap N \text{ and } q \notin M \cap N.$$

This contradicts the asumption that M and N are maximal. □

Here we shall record a generalization to an arbitrary $A(X)$ of a result known for $C^*(X)$ and $C(X)$.

Theorem 2.2.29. *If $A(X) \subseteq B(X)$, then P is a prime ideal in B(X) if and only if $P \cap A(X)$ is a prime ideal in A(X).*

Proof. Suppose P is a prime ideal in $B(X)$. Let $f, g \in A(X)$, such that $fg \in P \cap A(X)$.

$$\text{Then } f, g \in P \ \& \ fg \in A(X) \quad \Rightarrow f \in P \text{ or } g \in P$$

$$\Rightarrow f \in P \cap A(X) \text{ or } g \in P \cap A(X).$$

Hence $P \cap A(X)$ is prime.
Conversely, suppose that $P \cap A(X)$ is a prime ideal in $A(X)$. Let $f, g \in B(X)$ such that $fg \in P$.
Now for each $f \in B(X)$, there exists $u \in C^*(X)$ such that $uf = (f \vee -1) \wedge 1$ [1E , [18]]. Clearly, $u \in C^*(X) \subseteq A(X) \subseteq B(X)$ and $|uf| \leq 1 \Rightarrow uf \in C^*(X) \subseteq A(X)$.
Similarly, for $g \in A(X)$, there exists $v \in C^*(X)$ such that $vg = (g \vee -1) \wedge 1$.

So $ufvg \in C^*(X)$ and $fg \in P \Rightarrow ufvg = uvfg \in P \Rightarrow ufvg \in P \cap A(X)$

$$\Rightarrow uf \in P \cap A(X) \quad \text{or} \quad vg \in P \cap A(x).$$

Without loss of generality, let $uf \in P \cap A(X)$ this implies that $u(|f| \vee 1) = 1$. Since $|f| \vee 1 \in B(X)$, it is invertible in $B(X)$ and so is u. This implies that $f \in P$. Similar argument holds for g.
Hence P is a prime ideal in $B(X)$.

□

If $A(X)$ is any subring of $C(X)$, then Dominguez and Gomez Perez in [11] have shown an interesting result in terms of containment of a prime ideal of $A(X)$ in a prime ideal of $C(X)$.

Proposition 2.2.30. *Let $A(X)$ be a subring of $C(X)$. For every prime ideal Q of $A(X)$, there exists a prime ideal P of $C(X)$ such that $P \cap A(X) = Q$.*

Proof. Let Q be a prime ideal of $A(X)$ and let $S = A(X) \backslash Q$. We know that S is a multiplicatively closed subset of $A(X)$, and hence of $C(X)$. So by Theorem [1.3.5], there exists a prime ideal P in $C(X)$ such that $P \supset Q$ and $P \cap S = \phi$ and $0 \notin S$. This implies that

$$(P \cap A(X) \cap S) = \phi \Rightarrow (P \cap A(X)) \subseteq Q.$$

Hence the result follows.

$\square$

2.3 Maximal ideals and residue class rings.

In this section we study maximal ideals of $A(X)$ in terms of their associated residue class fields. The result that every prime ideal P in $A(X)$ is absolutely convex and the residue class ring $A(X)/P$ is totally ordered is recorded here.

In analogy with $C(X)$, we define

Definition 2.3.1. An ideal I of A(X) is *fixed* if $\bigcap \mathcal{Z}_A[I] \neq \phi$ and *free* if $\bigcap \mathcal{Z}_A[I] = \phi$

Example 2.3.2. Let $X = \mathbb{N}$, $C(X) = C(\mathbb{N})$, $A(X) = C^*(\mathbb{N})$. Let $f = 0$, and $I = \{0\}$, then $\bigcap \mathcal{Z}_A(f) = X$, hence fixed. On the other hand, if $f = j$, and $I = (j)$, where $j(n) = \frac{1}{n}$ then $\bigcap \mathcal{Z}_A(f) = \phi$, as $Z(j) = \phi$ hence I is free.

Remark 2.3.3. The definitions of fixed and free ideals with respect to $Z[I]$ and $\mathcal{Z}_A[I]$ are equivalent.

Proof. It is enough to show that $\bigcap \mathcal{Z}_A[I] = \bigcap Z[I]$. By Remark [2.2.14] we know that $Z(f) = \bigcap \mathcal{Z}_A(f)$ for each $f \in I$. So $\bigcap Z[I] = \bigcap_{f \in I} Z(f) = \bigcap_{f \in I} (\bigcap \mathcal{Z}_A(f)) = \bigcap \mathcal{Z}_A[I]$.

Hence the result follows.

$\square$

Notation 2.3.4. $I(f) = \{g \in A(X) \mid f \equiv g \ (mod\ I)\}$

Theorem 2.3.5. *The fixed ideals in $A(X)$ are precisely the sets*

$$M_A^P = \{f \in A(X) \mid f(p) = 0\} \qquad (p \in X).$$

The ideals M_A^p are distinct for distinct p and the mapping $M_A^p(f) \longrightarrow f(p)$ is the unique isomorphism of $A(X)/M_A^p$ onto $\mathbb{R}$. If $B(X)$ is another subring of $C(X)$ such that $A(X) \subseteq B(X)$ then there is a one to one correspondence between the fixed maximal ideals in $B(X)$ and those in $A(X)$ given by

$$M_B^p \longrightarrow M_A^p = M_B^p \cap A(X).$$

Proof. Fix $p \in X$ and let $\Phi_p : A(X) \longrightarrow \mathbb{R}$

such that $\Phi_p(f) = f(p)$

We claim that Φ_p is an onto homomorphism.

(i) **Φ_p is well defined.**
Let $f, g \in A(X)$ such that $f = g$ i.e. $f(x) = g(x) \ \ \forall x \in A(X)$. In particular, $f(p) = g(p)$, which implies that $\Phi_p(f) = \Phi_p(g)$.

(ii) **Φ is onto.**
Let $r \in \mathbb{R} \subseteq A(X)$ then define

$$r : X \to \mathbb{R}$$

given by $r(x) = r \ \ \forall x \in \mathbb{R}$. Then $\Phi_p(r) = r(p) = r$, hence Φ_p is onto.

(iii) **Φ_p is a homomorphism.**
Let $f, g \in A(X)$, then

$$\phi_p(f + g) = (f + g)(p) = f(p) + g(p) = \Phi_p(f) + \Phi_p(g)$$

and

$$\Phi_p(f.g) = (fg)(p) = f(p).g(p) = \Phi_p(f)\Phi_p(g)$$

Then by the fundamental theorem of ring homomorphisms

$$A(X)/\text{Ker}\,\Phi_p \cong \mathbb{R} \text{ where } \text{Ker}\,\Phi_p = \{f \in A(X) \mid f(p) = 0\} = M_A^p.$$

Hence M_A^p is a maximal ideal.
Let M be any fixed maximal ideal of $A(X)$, then

$$A(X)/M \cong \mathbb{R} \cong A(X)/\text{Ker}\,\Upsilon,$$

where Υ is the isomorphism between $A(X)/M$ and $\mathbb{R}$. So $M \cong \operatorname{Ker}\Upsilon = M_A^p$. Hence every maximal ideal is of the form M_A^p.

We claim that the ideals M_A^p are distinct for distinct p.
Let $p, q \in X$ such that $p \neq q$. Since X is a Tychonoff space, $\{p\}$ and $\{q\}$ are closed in X, there exists $f \in A(X)$ such that $f[p] = \{0\}$ and $f[q] = \{1\}$. So, $f \in M_A^p \setminus M_A^q$.

Similarly, there exists $g \in A(X)$ such that $g[q] = \{0\}$ and $g(p) = 1$. So, $g \in M_A^q \setminus M_A^p$. Hence $M_A^p \neq M_A^q$.
Now let

$$\Upsilon_p : A(X)/M_A^p \longrightarrow \mathbb{R}$$

such that $\Upsilon_p(M_A^p(f)) = f(p)$. Clearly, Υ_p is an isomorphism.

The uniqueness of this isomorphism follows from Theorem [1.3.8].
We now show that there is a one to one correspondence between the fixed maximal ideals in $B(X)$ and those in $A(X)$.

Let $\mathcal{M}_A = \{M_A^p \mid p \in X\}$, the set of all fixed maximal ideals in $A(X)$.

and $\mathcal{M}_B = \{M_B^q \mid q \in X\}$ the set of all fixed maximal ideals in $B(X)$.

For each M_B^p,

$$M_B^p \cap A(X) \quad = \{f \in A(X) \mid f(p) = 0\} \; = \; M_A^p \in \mathcal{M}_A$$

We define

$$k : \mathcal{M}_B \longrightarrow \mathcal{M}_A$$

as $k(M_B^p) = M_B^p \cap A(X)$.

(i) k **is well defined**

If $M_B^p = M_B^q \quad p, q \in X$ then

$$M_B^p \cap A(X) = M_B^q \cap A(X) \quad \Rightarrow k(M_B^p) = k(M_B^q).$$

(ii) k **is onto**

for each M_A^p, $\exists\, M_B^p$ such that $M_A^p = M_B^p \cap A(X)$ and so $k(M_B^p) = M_A^p$.

(iii) k **is one-one**

Let $k(M_B^p) = k(M_B^q)$ then,

$$M_B^p \cap A(X) = M_B^q \cap A(X) \quad \Rightarrow M_A^p = M_A^q.$$

Suppose $M_B^q \neq M_B^p$, then without loss of generality there exists $f \in M_B^q \backslash M_B^p$ such that $f(p) \neq 0$.
Let $g = (f \vee -1) \wedge 1$, then

$$g(p) = 0 \Rightarrow g \in M_A^p \subseteq C^*(X) \subseteq A(X) \text{ and } Z(f) = Z(g),$$

so,

$$g(p) \neq 0 \Rightarrow g \notin M_A^p \Rightarrow M_A^p \neq M_A^q$$

which is a contradiction.

□

Corollary 2.3.6. *Every fixed maximal ideal is real.*

Proof. Since $M = M_A^p$ for some $p \in X$ and $A(X)/M_A^p \cong \mathbb{R}$, so the result follows. □

Proposition 2.3.7. *$A(X)/I$ is a partially ordered ring iff I is convex.*

Proof. We first assume that I is convex. By Remark [1.3.2] we must verify that

(i) $I(f) \geq 0$ and $I(-f) \geq 0 \quad \Leftrightarrow I(f) = 0$, and

(ii) $I(f) \geq 0$ and $I(g) \geq 0 \Leftrightarrow I(f) + I(g) \geq 0$ and $I(f)I(g) \geq 0$.

Now $I(f) \geq 0$ & $I(-f) \geq 0$ implies $\exists\, h, k \in A(X)$ such that

$$h \geq 0, k \geq 0 \text{ and } f \equiv h(modI) \text{ and } \quad -f \equiv k(modI).$$

Hence $h + k \equiv 0(modI)$. But,

$$0 \leq h \leq h + k \quad \Rightarrow h \equiv 0,$$

by convexity of I which implies that $I(f) = I(h) = 0$.
Also $I(f) = 0 \Rightarrow I(f) \geq 0$ and $I(-f) \geq 0$, hence (i) holds.

Let $I(f) \geq 0$ and $I(g) \geq 0$ for $f, g \in I$. Then, there exists $h, k \in A(X)$ where $h \geq 0$, $k \geq 0$.So, $h + k \in A(X)$. Now

$$I(f) = I(h),\ I(g) = I(k) \quad \Rightarrow I(f) + I(g) \quad = I(h) + I(k) \quad = (h + k) + I.$$

Similarly, $\quad I(f)I(g) = hk + I.$
So, $I(f) + I(g) \geq 0$ & $I(f)I(g) \geq 0$, and hence (ii) holds.

Conversely, let $0 \leq h \leq k$, $\quad h, k \in A(X)$ with $k \in I$. But $0 \leq h \leq k \quad \Rightarrow 0 \leq k - h$; Also

$$I(h) \geq 0 \quad \& \ I(k-h) \geq 0 \text{ so } 0 \leq I(k - h) \ = I(k) - I(h) \ = -I(h) \Rightarrow I(h) \leq 0.$$

But $h \geq 0 \Rightarrow I(h) \geq 0$. Thus, $I(h) = 0 \Rightarrow h \in I$.

□

Proposition 2.3.8. *Every β-ideal in $A(X)$ is absolutely convex.*

Proof. Let I be a $\beta - ideal$ in $A(X)$. Let $|f| \leq |g|$ for $f, g \in A(X)$ and suppose $g \in I$, then $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[I]$
Now $|f| \leq |g| \quad \Rightarrow \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[I]$ by Lemma [2.2.9].
This implies that $f \in I$ as I is a β-ideal.
Hence, I is absolutely convex.

□

Corollary 2.3.9. *Every maximal ideal in $A(X)$ is absolutely convex.*

Proof. The result follows from the fact that every maximal ideal is a β-ideal. □

We know that if M is a maximal ideal in $C(X)$ then $M(f) \geq 0$ iff $f \geq 0$ on some member of $Z[M]$ [Remark page.68 of, [18]]. The corresponding conditions on $A(X)$ were studied by Byun and Watson in [7] and we shall record some of their important results.

Lemma 2.3.10. *For a $\beta - ideal$ I in $A(X)$ and $f \in A(X)$, $I(f) \geq 0$ iff $\lim_{\mathcal{Z}_A[I]} f|h| \geq 0$ for every $h \in A(X)$.*

Proof. Let I be a β-ideal, then I is absolutely convex by Proposition [2.3.8]. Suppose $I(f) \geq 0$ $\quad$ then $f - |f| \in I$ by Theorem [1.3.4] and for each

$$h \in A(X),\ |h| \in A(X) \ \& \ f|h| - |f||h| \in I$$

$$\Rightarrow \mathcal{Z}_A(f|h| - |f||h|) \subseteq \mathcal{Z}_A[I].$$

Now

$$Z(g) = \bigcap \mathcal{Z}_A(g) \quad = \bigcap \{E_\epsilon(g) \mid \epsilon > 0\} \quad \text{where } g = f|h| - |f||h|.$$

So, $E_\epsilon(g) \in \mathcal{Z}_A[I]$ for every $\epsilon > 0$ and thus $f|h| \geq -\epsilon$ on $E_\epsilon(f|h| - |f||h|)$, hence $\lim_{\mathcal{Z}_A[I]} f|h| \geq 0$ for every $h \in A(X)$.

Conversely, suppose $\lim_{\mathcal{Z}_A[I]} f|h| \geq 0$ for every $h \in A(X)$, i.e., for $\epsilon > 0$ and $h \in A(X)$, $f|h| \geq -\epsilon$ on some member E of $\mathcal{Z}_A[I]$.

Then $-2\epsilon \leq (f|h| - |f||h|) \leq 0$ on E and so $E \subset E_{2\epsilon}(f|h| - |f||h|)$.

Thus $E_{2\epsilon}(f|h| - |f||h|) \in \mathcal{Z}_A[I]$ $\forall \epsilon > 0$, since $\mathcal{Z}_A[I]$ is a z-ideal.

By Lemma [2.2.10] $\mathcal{Z}_A(f - |f|) \subseteq \mathcal{Z}_A[I]$ $\Rightarrow f - |f| \in I$ since I is a β-ideal so $I(f) = I(|f|) \geq 0$. □

Lemma 2.3.11. *Let I be a z-ideal of $A(X)$ and $f \in A(X)$. If there exists $h \in A(X)$ such that $\lim_{\mathcal{Z}_A[I]} f|h| > 0$ then $I(f) > 0$. If I is a maximal ideal the converse holds as well.*

Proof. Suppose that there exists $h \in A(X)$ and $\epsilon > 0$ such that $f|h| > \epsilon$ on $E \in \mathcal{Z}_A[I]$ then $f > 0$ on E and by Lemma [2.3.10] $I(f) \geq 0$. Also $f|h| > \epsilon$ on $E \in \mathcal{Z}_A[I]$. some member of $\mathcal{Z}_A(f|h|)$ does not meet E. Since $\mathcal{Z}_A(f|h|) \subseteq \mathcal{Z}_A(f)$ so some members of $\mathcal{Z}_A(f)$ does not meet E. So $f \notin I$ $\Rightarrow I(f) \neq 0$ thus $I(f) > 0$. Now suppose that I is a maximal ideal and $I(f) > 0$, then $(I, f) = A(X)$ and there exists $g \in I$ and $h \in A(X)$ s.t. $g + fh = 1$.
Fix $\epsilon > \frac{1}{2}$, then

$$E_\epsilon(fh) = \{x \in X \mid |fh(x)| \leq \epsilon\}$$

and

$$E_\epsilon(g) = \{x \in X \mid |g(x)| \leq \epsilon\} = \{x \in X \mid |(1-fh)(x)| \leq \epsilon\} = \{x \in X \mid \epsilon < (fh)(x) < 1 + \epsilon\}$$

$$\Rightarrow |fh[E_\epsilon(g)]| > \epsilon \quad \text{so } E_\epsilon(fh) \cap E_\epsilon(g) = \phi.$$

By Lemma [2.3.10] there exists $F \in \mathcal{Z}_A[I]$ on which $f|h| \geq -\epsilon$

$$\Rightarrow |fh| > \epsilon \;\&\; f|h| \geq -\epsilon \text{ on } E' \cap E_\epsilon(g).$$

Thus

$$f|h| > \epsilon \quad \text{on } E' \cap E_\epsilon(g) \in \mathcal{Z}_A[I] \quad \text{and hence } \lim_{\mathcal{Z}_A[I]} f|h| > 0.$$

□

Theorem 2.3.12. *Let M be a maximal ideal in $A(X)$ and $f \in A(X)$, then $M(f) > 0$ iff $\lim_{Z_A[I]} f|h| \geq 0$ for every $h \in A(X)$. Moreover $M(f) > 0$ iff there exists $h \in A(X)$ such that $\lim_{Z_A[M]} f|h| > 0$*

Proof. The theorem is a direct consequence of Lemma [2.3.10] and Lemma[2.3.11]. □

Theorem 2.3.13. *Every prime ideal P in $A(X)$ is absolutely convex and the residue class ring $A(X)/P$ is totally ordered. Moreover the mapping $r \longrightarrow P(r)$ is an order preserving isomorphism of the real field $\mathbb{R}$ onto $A(X)/P$.*

Proof. Let $f, g \in A(X)$ with $0 \leq |f| \leq |g|$ such that $g \in P$

$$\text{then} \quad 0 \leq f^2 \leq g^2 \quad \& \; g^2 \in P$$

Define $u(x) = \begin{cases} 1 & \text{if } x \in g^{2\leftarrow}[0,1) \\ \frac{1}{g^2(x)} & \text{if } x \in g^{2\leftarrow}[1,\infty) \end{cases}$
then u is continuous and $u \in C^*(X) \subseteq A(X)$, and

$$g^2 \wedge 1 = ug^2 \text{ implies that } \; 0 < u \leq 1.$$

Also $g^2 \in P$ implies that $ug^2 \in P$.

$$\text{Define } h(x) = \begin{cases} \dfrac{u(x)f^4(x)}{g^2(x)} & x \notin Z(g) \\ 0 & x \in Z(g) \end{cases}$$

then

$$h \in C^*(X) \subseteq A(X)$$

Now $h(x)g^2(x) = \dfrac{u(x)f^4(x)}{g^2(x)}g^2(x)$, i.e., $hg^2 = uf^4 \in P$
We claim that $u \notin P$, for if $u \in P$, then $u + ug^2 \in P$

$$\text{Also, if } g^2(x) < 1 \text{ then } u(x) = 1 \quad \Rightarrow u + ug^2 \geq 1$$

$$\text{and if } \; g^2(x) \geq 1 \text{ then } u(x) = \frac{1}{g^2(x)} \Rightarrow u + ug^2 \geq 1$$

and this implies that $u + ug^2$ is invertible in $A(X)$, and this is a contradiction.
So $u \notin P$ implies that $f^4 \in P$ and so $f \in P$.
Now we show that $A(X)/P$ is totally ordered.
Let $f \in A(X)$ and consider $P(f)$. Since

$$(f - |f|)(f + |f|) = 0 \in P \quad \text{so, either } \; f - |f| \in P \text{ or } f + |f| \in P$$

$$\Rightarrow f \equiv |f| (\mod P) \quad \text{or } f \equiv -|f| \; (\mod P)$$
$$\Rightarrow P(f) \geq 0 \quad \text{or} \quad P(f) \leq 0.$$

Hence $A(X)/P$ is totally ordered.
That the mapping $r \longrightarrow P(r)$ is an isomorphism follows from Theorem[1.4.16] ☐

Theorem 2.3.14. *If I is a $\beta -$ ideal in $A(X)$, then I is prime if and only if $A(X)/P$ is totally ordered.*

Proof. Neccesity follows from Theorem[2.3.13].

Conversely, if $A(X)/I$ is totally ordered then for each $f \in A(X)$, $I(f) \geq 0$ or $I(f) \leq 0$.
If $I(f) \geq 0$ then by Lemma [2.3.10] for every $\epsilon > 0$, $f \geq -\epsilon$ on some member of $\mathcal{Z}_A[I]$. Similarly if $I(f) \leq 0$ then for every $\epsilon > 0$, $f \leq \epsilon$ on some member of $\mathcal{Z}_A[I]$. Hence from Theorem [2.2.27] I is prime. ☐

Theorem 2.3.15. *Let $f \in A(X)$. For a given maximal ideal M in $A(X)$, the following conditions are equivalent.*

(i) *$|M(f)|$ is infinitely large.*

(ii) *f is unbounded on every member of $\mathcal{Z}_A[M]$.*

(iii) *For each $n \in N$, the zero set $Z_n = \{x \in X \| f(x)| \geq n\}$ belongs to $\mathcal{Z}_A[M]$.*

Proof. *(ii) $\Rightarrow$ (i)* Suppose $|M(f)|$ is not infinitely large i.e., there exists $n \in \mathbb{N}$ such that $|M(f)| \leq n$.

$$\text{Then} \quad \Rightarrow |M(f)| - n \leq 0 \quad \Rightarrow M(|f| - n) \leq 0.$$

By Theorem [2.3.12] $\exists \epsilon > 0$ such that

$$\lim_{\mathcal{Z}_A[M]} (|f| - n) \leq \epsilon \quad \Rightarrow (|f| - n) \leq \epsilon \text{ on some element } E \text{ of } \mathcal{Z}_A[M]$$

$$\Rightarrow -(|f| - n) \geq -\epsilon \quad \Rightarrow n - |f| \geq -\epsilon \quad \Rightarrow |f| \leq n + \epsilon \quad \textit{on } E.$$

This is a contradiction. Hence, $|M(f)|$ is infinitely large.

(i) $\Rightarrow$ (iii) $|M(f)|$ is infinitely large implies $|M(f)| > n \quad \forall n \in \mathbb{N}$. By Lemma [2.3.13] there exists $E_n \in \mathcal{Z}_A[M]$ for every $n \in \mathbb{N}$ on which $|f| - n > 0$. Then, $E_n \subseteq Z_n$ so, $Z_n \in \mathcal{Z}_A[M]$ since $\mathcal{Z}_A[M]$ is a z-filter.

$(iii) \Rightarrow (ii)$ Let

$$E \in \mathcal{Z}_A[M] \quad \Rightarrow E \cap Z_n \neq \phi \ \forall n \in \mathbb{N},$$

for each $n \in \mathbb{N}, \exists\, x \in E$ such that $|f(n)| > n$ hence f is unbounded on every member of $\mathcal{Z}_A[M]$. □

Theorem 2.3.16. *$|M(f)|$ is infinitely large for some maximal ideal M in $A(X)$ iff f is unbounded on X.*

Proof. Suppose $|M(f)|$ is infinitely large then by Theorem [2.3.15] f is unbounded on every member of $\mathcal{Z}_A[M]$. But $X \in \mathcal{Z}_A[M]$ as it is a z-filter. Hence the result follows.

Conversely, suppose f is unbounded on X, then by Theorem [2.3.15], there is a family of sets Z_n in $\mathcal{Z}_A[M]$. Then $\{Z_n\}_{n \in \mathbb{N}} \cup X$ has finite intersection property, so it is embeddable in some z-ultrafilter $\mathcal{F}$. Then $M = \mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]$ is a maximal ideal by Theorem [2.4.2] and $Z_n \in \mathcal{Z}_A[M]\ \forall n$. So $|M(f)|$ is infinitely large. □

Corollary 2.3.17. *Every maximal ideal in $A(X)$ is real when and only when $A(X) = C^*(X)$.*

Proof. Suppose every maximal ideal in $A(X)$ is real. Let $f \in A(X)$. Then $|M(f)| \leq n$ for some positive integer n and for all maximal ideals M, implies that

$$|f| \leq n \quad \Rightarrow f \in C^*(X).$$

Hence, $A(X) = C^*(X)$.

On the other hand, if $A(X) = C^*(X)$, for $f \in A(X)$, $|f| \leq n$ for some $n \in \mathbb{N}$, which implies that for each maximal ideal M in $A(X)$, $|M(f)| \leq n$ for some $n \in \mathbb{N}$. Hence, M is real.

□

2.4 Maximal ideals and z-ultrafilters of $A(X)$

The correspondence between ideals of $C(X)$ and z-ultrafilters on X is well known [18]. In this section we study the correspondence between the ideals of $A(X)$ and z-ultrafilters on X. L Redlin and S Watson in [28] have assumed that for a maximal ideal M, $\mathcal{Z}_A[M]$ is a z-ultrafilter but H.L.Byun and S.Watson in [7] showed that this may not be true. However, every $\mathcal{Z}_A[M]$ is contained in a unique z-ultrafilter, is recorded here.

Lemma 2.4.1. *Let $A(X)$ and $B(X)$ be subrings of C(X) such that $B(X) \subseteq A(X)$, then for any ideal I of A(X), $\mathcal{Z}_A[I] = \mathcal{Z}_B[I \cap B(X)]$*

Proof.

Let $\mathcal{Z}_{C^*}(f) \subseteq \mathcal{Z}_{C^*}[I \cap C^*(X)]$, then $\mathcal{Z}_{C^*}(f) = \mathcal{Z}_{C^*}(g)$ for some $g \in I \cap C^*(X)$.

$$\text{But } C^*(X) \subseteq B(X) \quad \Rightarrow g \in I \cap B(X) \quad \Rightarrow \mathcal{Z}_{C^*}(f) \subseteq \mathcal{Z}_B(f).$$

$$\text{So, } \mathcal{Z}_{C^*}(f) = \mathcal{Z}_{C^*}(g) \quad \subseteq \mathcal{Z}_B(g) \quad \subseteq \mathcal{Z}_B[I \cap B(X)]$$

$$\Rightarrow \mathcal{Z}_{C^*}[I \cap C^*(X)] \quad \subseteq \mathcal{Z}_B[I \cap B(X)]. \tag{2.4}$$

Now let $\mathcal{Z}_B(g) \subseteq \mathcal{Z}_B[I \cap B(X)]$, then $\mathcal{Z}_B(g) = \mathcal{Z}_B(f)$ for some $f \in I \cap B(X)$, then by [1E, [18]] $\exists \;\; u \in C^*(X)$ such that $fu = (f \vee 1) \wedge -1$. Then, $fu \in C^*(X) \quad \Rightarrow fu \in I \cap C^*(X)$.
Since u is a unit of $C^*(X)$,

$$\mathcal{Z}_B(f) = \mathcal{Z}_B(fu) \Rightarrow \mathcal{Z}_{C^*}(fu) \subseteq \mathcal{Z}_{C^*}[I \cap C^*(X)]$$

$$\text{Hence,} \qquad \mathcal{Z}_B[I \cap B(X)] \subseteq \mathcal{Z}_{C^*}[I \cap C^*(X)] \tag{2.5}$$

Thus from eqn[2.4] and [2.5] we have

$$\mathcal{Z}_{C^*}[I \cap C^*(X)] = \mathcal{Z}_B[I \cap B(X)].$$

Now we show that $\mathcal{Z}_A[I] = \mathcal{Z}_{C^*} I \cap C^*(X)]$.

Let $f \in I$ and consider $E \in \mathcal{Z}_A(f)$, then $\exists\; g \in A(X)$ such that $fg|_{E^c} = 1$. Let $h = fg$ and $u = \dfrac{1}{1+h^2}$. Clearly,

$$u \in C^*(X) \quad \text{and} \quad hu = \frac{fg}{1+(fg)^2} \in C^*(X) \quad \Rightarrow hu = fgu.$$

$$\text{Also } hu|_E = \frac{fg}{1+(fg)^2} \quad = \frac{1}{1+1} \quad = \frac{1}{2} > 0$$

Hence, $hu|_{E^c} = 1$ implies that $E \in \mathcal{Z}_{C^*}(hu)$. Therefore, $\mathcal{Z}_A[I] \subseteq \mathcal{Z}_{C^*} I \cap C^*(X)$.
On the other hand, if

$$\mathcal{Z}_{C^*}(f) \subseteq \mathcal{Z}_{C^*}[I \cap C^*(X)],$$

then

$$\mathcal{Z}_{C^*}(f) = \mathcal{Z}_{C^*}(g) \quad \text{for some } g \in I \cap C^*(X),$$

which implies that $g \in I$

$$\text{So, } \mathcal{Z}_{C^*}(g) \subseteq \mathcal{Z}_A[I] \quad \Rightarrow \mathcal{Z}_{C^*} I \cap C^*(X)] \subseteq \mathcal{Z}_A[I].$$

Hence $\mathcal{Z}_A[I] = \mathcal{Z}_{C^*} I \cap C^*(X)]$ □

Theorem 2.4.2. (i) *If M is a maximal ideal in $A(X)$, then $\mathcal{Z}_A[M]$ is contained in a unique z-ultrafilter.*

(ii) *If $\mathcal{F}$ is a z-ultrafilter on X, then $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ is a maximal ideal in $A(X)$. In particular, the map $\mathcal{Z}_A$ induces a bijection from the set of maximal ideals of $A(X)$ onto the set of z-ultrafilters on X.*

Proof. (i) Let M be a maximal ideal in $A(X)$, then $\mathcal{Z}_A[M]$ is a z-ultrafilter on X. Let $\mathcal{F}$ be the z-ultrafilter on X , such that $\mathcal{Z}_A[M] \subseteq \mathcal{F}$. Also, there exists a maximal ideal M in $C(X)$ such that $N = Z^{\leftarrow}[\mathcal{F}]$,then

$$Z[N] = Z[Z^{\leftarrow}[\mathcal{F}]] \quad = \mathcal{F}.$$

Since M is maximal, so $M = \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[M]] \subseteq \mathcal{Z}_A{}^{\leftarrow}[Z[N]]$.
So, $M \subset \mathcal{Z}_A{}^{\leftarrow}[Z[N]] \subseteq A(X)$ and by maximality of M,
either $M = \mathcal{Z}_A{}^{\leftarrow}[Z[N]]$ or $\mathcal{Z}_A{}^{\leftarrow}[Z[N]] = A(X)$.
But $1 \notin \mathcal{Z}_A{}^{\leftarrow}[Z[N]]$ otherwise $\phi \in Z[N]$ and this is not possible.
Thus, $M = \mathcal{Z}_A{}^{\leftarrow}[Z[N]]$. Now by Lemma [2.4.1],

$$\mathcal{Z}_A{}^{\leftarrow}[Z[N]] \supset \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_C[N]] \quad = \quad \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[N \cap A(X)]] \supset N \cap A(X).$$

Thus $N \cap A(X) \subseteq M$, which implies that

$$\mathcal{Z}_A[N] = \mathcal{Z}_A[N \cap A(X)] \subseteq \mathcal{Z}_A[M].$$

Now $\mathcal{F} = Z[N] \supset \mathcal{Z}_A[M]$ and $\mathcal{Z}_C[N] \subseteq \mathcal{Z}_A[M] \subseteq Z[N]$.
We claim that $Z[N]$ uniquely contains $\mathcal{Z}_A[M]$.
Suppose there exists another maximal ideal O of *C(X)* such that $\mathcal{Z}_A[M] \subseteq Z[O]$, then by repeating the above procedure we get

$$\mathcal{Z}_C[O] \subseteq \mathcal{Z}_A[M] \subseteq Z[O]$$

So any zero-set $E \in Z[O]$ intersects every member of $\mathcal{Z}_A[M]$ and hence also intersects every member of $\mathcal{Z}_C[N]$. Thus E intersects with every member of $Z[N]$ and $Z[N]$ is a z-ultrafilter implies that $E \in Z[N] \Rightarrow Z[O] \subseteq Z[N]$. Hence, the maximality of O and the fact that $Z[O]$ is a z-ultrafilter implies that $Z[O] = Z[N]$.

(ii) For each z-ultrafilter $\mathcal{F}$, $\mathcal{F} = Z[N]$ for some maximal ideal O in $C(X)$. So by Lemma [2.4.1]

$$\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] = \mathcal{Z}_A{}^{\leftarrow}[Z[N]] \supset \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_C[N]] = \mathcal{Z}_A{}^{\leftarrow}[\mathcal{Z}_A[N \cap A(X)]] \supset N \cap A(X)$$

Let M be a maximal ideal of $A(X)$ such that $N \cap A(X) \subseteq M$. Then $\mathcal{Z}_C[N] = \mathcal{Z}_A[N \cap A(X)] \subseteq \mathcal{Z}_A[M]$.
Now let O be a maximal ideal of $A(X)$ such that $\mathcal{Z}_A[M] \subseteq Z[O]$, where $Z[O] = \mathcal{F}$ is the unique z-ultrafilter containing $\mathcal{Z}_A[M]$. Then proceeding similarly as in (i), we get

$$\mathcal{Z}_C[O] \subseteq \mathcal{Z}_A[M] \text{ and } Z[O] = Z[N].$$

Since $M \subset \mathcal{Z}_A{}^{\leftarrow}[Z[N]]$, so by maximality of M, $M = \mathcal{Z}_A{}^{\leftarrow}[Z[N]] = \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$. Hence $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ is a maximal ideal in $A(X)$.

□

Note 2.4.3. It follows from Theorem [2.4.2] that the map $\mathcal{Z}_A$ induces a bijection from the set of maximal ideals of $A(X)$ onto the set of z-ultrafilters on X, with the following association.

$$M \longrightarrow \mathcal{F} \supset \mathcal{Z}_A[M] \quad and\, \mathcal{F} \longrightarrow \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$$

under $\mathcal{Z}_A$ with usual meanings of the symbols used.

Let $\mu(A)$ be the collection of all maximal ideals in $A(X)$,

$$i.e., \quad \mu(A) = \{M_A^p |\, p \in \beta X\}.$$

We topologize the collection with hull-kernel topology [7M,[18]] by considering the collection $M_f = \{M_A^p \in \mu(A) | f \in M_A^p\}$ as a base for the closed sets for each $f \in A(X)$.

Remark 2.4.4. The set of z-ultrafilters on X endowed with the Stone-Topology (hull-kernel)is βX.

Remark 2.4.5. For $p \in \beta X$, $\lim \mathcal{Z}_A[M_A^p] = p$

Remark 2.4.6. The set $\mu(A)$ of all maximal ideals of $A(X)$ endowed with Stone-topology is homeomorphic to βX.

Notation 2.4.7. For a filter $\mathcal{F}$ on X, we write $S[\mathcal{F}]$ for the set of cluster points of $\mathcal{F}$ in βX

$$i.e., \quad S[\mathcal{F}] = \bigcap \{cl_{\beta X} E | E \in \mathcal{F}\}$$

There is an analogue of the Gelfand-Kolmogoroff theorem [Theorem 1.6.8] in $A(X)$.

Theorem 2.4.8. *Let M_A^p be the maximal ideal of $A(X)$ corresponding to the point p of βX. Then $M_A^p = \{f \in A(X) | p \in S[\mathcal{Z}_A(f)]\}$.*

Proof. Since M_A^p is a β-ideal, so $f \in M_A^p$ iff $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M_A^p]$. Let $f \in M_A^p$. Then $\mathcal{Z}_A[M_A^p] \subseteq \mathcal{F}$, where $\mathcal{F}$ converges to p. So, $\mathcal{Z}_A[M_A^p]$ converges to p, which implies that $p \in S[\mathcal{Z}_A[M_A^p]]$ and $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M_A^p] \quad \Rightarrow p \in S[\mathcal{Z}_A(f)] \supset \mathcal{Z}_A(f)$.
On the other hand, $p \in S[\mathcal{Z}_A(f)] \ \Rightarrow \mathcal{Z}_A[M_A^p] \supset \mathcal{Z}_A(f) \Rightarrow f \in M_A^p$.
Hence $M_A^p = \{f \in A(X) | p \in S[\mathcal{Z}_A(f)]\}$. □

We shall record some more characterizations.

Proposition 2.4.9. $M_A^p = \{f \in A(X) | \lim_{\mathcal{Z}_A[M_A^p]} fh = 0 \quad \textit{for every } h \in A(X)\}$.

Proof. ($\Rightarrow$) Let $f \in M_A^p$. Then, $fh \in M_A^p$ for every $h \in A(X)$, which gives $\mathcal{Z}_A(fh) \subseteq \mathcal{Z}_A[M_A^p]$. Further, $\lim_{\mathcal{Z}_A(fh)} fh = 0 \quad \Rightarrow \lim_{\mathcal{Z}_A[M_A^p]} fh = 0$,

($\Leftarrow$) If $\lim_{\mathcal{Z}_A[M_A^p]} fh = 0$ for every $h \in A(X)$ then by Lemma [2.2.10]

$$\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M_A^p] \quad \Rightarrow f \in M_A^p \text{ as } M_A^p \text{is } \ a\,\beta - \textit{ideal}.$$

□

Let $A(X)$ and $B(X)$ be subrings of $C(X)$ that contain $C^*(X)$, then

Remark 2.4.10. $\mu(A)$ and $\mu(B)$ are in one to one correspondence with the points of βX.

Proof. Let $M \in \mu(A)$, then by Theorem [2.4.2] $\mathcal{Z}_A[M]$ is contained in a unique z-ultrafilter $\mathcal{F}$ which corresponds to a unique point p in βX to which it converges.
Conversely, for each p there exists a unique z-ultrafilter $\mathcal{F}$ converging to p and $M = \mathcal{Z}_A{}^{\leftarrow}[\mathcal{F}]$. Hence there is a one to one correspondence. The same argument holds for $\mu(B)$. □

Remark 2.4.11. $\mu(A)$ and $\mu(B)$ are in one to one correspondence.

Proof. Let $M_A^p \in \mu(A)$, $\ p \in \beta X$, then this p is identified with the maximal ideal M_B^p of $\mu(B)$ and vice-versa. Hence the identification $M_A^p \longleftrightarrow M_B^p$. □

Notation 2.4.12. $\overline{\mathcal{Z}_A[M]}$ denotes the unique z-ultrafilter containing $\mathcal{Z}_A[M]$.

Theorem 2.4.13. *Let $A(X)$ and $B(X)$ be subrings of $C(X)$ that contain $C^*(X)$ and let M be a maximal ideal in $A(X)$. The map $M \longrightarrow \mathcal{Z}_B^{\leftarrow}[\overline{\mathcal{Z}_A[M]}]$ gives a homomorphism between the spaces $\mu(A)$ and $\mu(B)$ with the hull-kernel topology. Moreover, under this map free ideals map to free ideals and fixed ideals map to fixed ideals*

Proof. By Theorem [2.4.10] $\mu(A)$ is homeomorphic to βX and βX is homeomorphic to $\mu(B)$, hence $\mu(A)$ is homeomorphic to $\mu(B)$. ☐

To study further the relationship between M_A^p and M_B^p for different subrings $A(X)$ and $B(X)$, Byun and Watson in [7] considered the extension of functions to βX. Every $f \in A(X)$ may be regarded as map into the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ and so has a Stone extension $f^* : \beta X \longrightarrow \mathbb{R}^*$.

Definition 2.4.14. Let $a \in K$, where K is a totally ordered field, then a is infinitely small (large) if $a < \frac{1}{n}$ ($> \frac{1}{n}$) $\forall n \in \mathbb{N}$.

Theorem 2.4.15. *Let $f \in A(X)$*

(i) $f^*(p) = \infty$ *iff* $|M_A^p(f)|$ *is infinitely large.*

(ii) $f^*(p) = r$ *iff* $|M_A^p(f) - r|$ *is either infinitely small or zero.*

Proof. (i) If $f^*(p) = \infty$, then for each $n \in \mathbb{N}$, p is the closure of the sets $E_n = \{x \in X : |f(x)| \geq n\}$, for if $p \notin \operatorname{cl} E_n$ for some $n \in \mathbb{N}$, then there exists some neighbourhood $N_r(p)$ of p such that $N_r(p) \cap E_n = \phi$ and $f^*(x) \leq n$ for all $x \in N_r(p)$. In particular $f^*(p) \leq n$, and this is a contradiction.
Now $E_n = Z(g_n)$ where $g_n = (|f(x)| - n) \wedge 0$.
Thus p is a cluster point of $\mathcal{Z}_A(g_n)$. By Theorem [2.4.8] $g_n \in M_A^p$ and so $\mathcal{Z}_A(g_n) \subseteq \mathcal{Z}_A[M_A^p]$. Clearly, each $E_n \in \mathcal{Z}_A(g_{n-1})$ for $n \geq 2$ and thus $E_n \in \mathcal{Z}_A[M_A^p]$ $\forall n \geq 2$. By Theorem [2.3.15] $|M_A^p(f)|$ is infinitely large.

On the other hand, if $|M_A^p|$ is infinitely large, then for each $n \in \mathbb{N}$ $\exists\, x \in X$ such that $f(x) > n$. So, $f^*(p) > n$ $\forall n$ $\Rightarrow f^*(p) = \infty$.

(ii) If $f^*(p) = r$, then $|M_A^p(f) - r| \leq \frac{1}{n}$ for each $n \in \mathbb{N}$, so that $|M_A^p(f) - r|$ is infinitely small or zero.
The converse follows from the fact that the posibilities considered are mutually exclusive and exhaustive.

☐

Corollary 2.4.16. *Let $B(X) \subseteq A(X)$. Then M_B^p is the set of all $f \in B(X)$ for which $|M_A^p(f)|$ is infinitely small or zero*

Proof. Since $B(X) \subseteq A(X)$, so $f \in M_B^p \Rightarrow f \in M_A^p \Rightarrow M_A^p(f) = 0$.
If $|M_A^p(f)|$ is infinitely small or zero then $f^*(p) = 0 \Rightarrow f \in M_B^p$ by Theorem [2.4.15]. □

Corollary 2.4.17. *Let $B(X) \subseteq A(X)$. Then M_A^p is hyper-real iff M_B^p contains a unit of $A(X)$.*

Proof. If M_A^p is hyper-real, then there exists $f \in A(X)$ such that $f \geq 1$ and $|M_A^p(f)|$ is infinitely large [2.3.15]. So,$g^{-1} \leq \frac{1}{n}$ implies that $|M_A^p(g^{-1})| \leq \frac{1}{n}$ $\forall n$. By Corollary [2.4.16] $f^{-1} \in M_B^p$
Conversely, let $u \in M_B^p$ such that u is a unit of $A(X)$, By Corollary [2.4.16] $|M_A^p(u)|$ is infinitely small or zero, then $|M_A^p u^{-1}|$ is infinitely large, hence M_A^p is hyper-real. □

Corollary 2.4.18. *Let $B(X) \subseteq A(X)$. Then $M_A^p \cap B(X) = M_B^p$ iff M_A^p is real.*

Proof. This can be equivalently expressed as $M_A^p \cap B(X) \neq M_B^p$ iff M_A^p is hyper-real. By Corollary [2.4.17] M_A^p is hyper-real iff M_B^p contains a unit of $A(X)$, so there exists $u \in M_B^p$ such that u is a unit of $A(X)$, which implies $u \notin M_A^p$, as M_A^p is a maximal ideal.
By Corollary [2.4.16], the converse holds as well. □

2.5 The Ideals O_A^p

In this section we survey the analogue in $A(X)$ of the ideals O^p defined in [18] for $C(X)$ as shown in [7], where they have shown that $\mathcal{Z}_A$ does not distinguish between prime ideals contained in a given maximal ideal in $A(X)$.

Definition 2.5.1. For each $p \in \beta X$, we define O_A^p as follows
$O_A^p = \{f \in A(X) | \, p \in \operatorname{Int} S[\mathcal{Z}_A(f)]\}$

Remark 2.5.2. O_A^p is an ideal.

Note 2.5.3. If f is bounded function on X, then f has a continuous extension f^β to βX.

Proof. This follows from Stone-Čech Compactification Theorem [1.6.2]. □

Thus from the continuity of f and the fact that $\lim \mathcal{Z}_A[M_A^p] = p$, we have $f^\beta(p) = \lim_{\mathcal{Z}_A[M_A^p]} f \quad (p \in \beta X)$.

Note 2.5.4. If $f \in M_A^p$, then $f^\beta(p) = 0$

Proof. If $f \in M_A^p$ then $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M_A^p]$. By Lemma [2.2.10]

$$\lim_{\mathcal{Z}_A(f)} f = 0, \Rightarrow \lim_{\mathcal{Z}_A[M_A^p]} f = 0 \quad \Rightarrow f^\beta(p) = 0.$$

□

Remark 2.5.5. $O_A^p \subseteq M_A^p$

Proof. Let $f \in O_A^p$. Then $p \in \operatorname{Int} S[\mathcal{Z}_A(f)] \subseteq S[\mathcal{Z}_A(f)] \quad \Rightarrow f \in M_A^p$. □

Theorem 2.5.6. *$\mathcal{Z}_A[O_A^p] = \mathcal{Z}_A[M_A^p]$ for all $p \in \beta X$*

Proof. Let $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[O_A^p]$. Then,

$$\mathcal{Z}_A(f) = \mathcal{Z}_A(g) \quad \text{for some } g \in O_A^p \subseteq M_A^p.$$

By Remark [2.5.5], $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[M_A^p]$ which implies that $\mathcal{Z}_A(f) = \mathcal{Z}_A(g) \subseteq \mathcal{Z}_A[M_A^p]$.
Again, let $\mathcal{Z}_A(f) \subset \mathcal{Z}_A[M_A^p]$. Then $f \in M_A^p$, without loss of generality, assume that f is bounded. By Note [2.5.3] it has a continuous extension f^β, with $f^\beta(p) = 0$.
Now let $E \in \mathcal{Z}_A(f)$, then there exists $g \in A(X)$ such that $fg|_{E^c} = 1$. Since $\{p\}$ and E^c are completely separated in βX by complete regularity of βX, there exists a zero-set neighbourhood V of p and $h_0 \in C(\beta X)$ such that $h_0(x) = 0 \quad$ for $x \in V$ and $h_0(E^c) = 1$.
Let $h = h|_X$. So, $Z(h) \supset V \cap X$,

$$\text{then } \operatorname{cl}_{\beta X} Z(h) \supset \operatorname{cl}_{\beta X}(V \cap X) = \operatorname{cl}_{\beta X} V \cap \operatorname{cl}_{\beta X} X = \operatorname{cl}_{\beta X} V \supset V.$$

$$\text{Also,} \quad E \in \mathcal{Z}_A(h) \quad \Rightarrow E \supset Z(h) \quad \Rightarrow \operatorname{cl}_{\beta X} E \supset \operatorname{cl}_{\beta X} Z(h) \supset V.$$

$$\text{So,} \quad V \subset \bigcap\{\operatorname{cl}_{\beta X} E | E \in \mathcal{Z}_A(h)\} = S[\mathcal{Z}_A(h)].$$

$$\text{Since} \quad p \in \operatorname{Int} V \subset S[\mathcal{Z}_A(h)] \quad \Rightarrow h \in O_A^p \ \text{ as } \ h|_{E^c} = 1$$

$$\text{it implies that} \quad E \in \mathcal{Z}_A(h), \text{ and } \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(h) \subseteq \mathcal{Z}_A[O_A^p].$$

Hence $\mathcal{Z}_A[O_A^p] = \mathcal{Z}_A[M_A^p]$. □

So the preceeding theorem yields that $\mathcal{Z}_A$ is not sensitive enough to differentiate between the ideals O_A^p and the maximal ideals containing it. The characterization of spaces X, which have the property that every ideal O_A^p is prime and also the spaces X which have the property that every ideal O_A^p is maximal, is recorded here.

Definition 2.5.7. A completely regular space X is called a P-space if every prime ideal in $C(X)$ is maximal.

We shall note an important result by D. Plank in [27].

Theorem 2.5.8. *The following are equivalent for any subring $A(X)$ of $C(X)$*

(i) $M_A^p = O_A^p$ *for* $p \in \beta X$.

(ii) *Every prime ideal in $A(X)$ is maximal.*

Proof. (*i*) $\Rightarrow$ (*ii*) Since every ideal of the form O_A^p is an intersection of prime ideals in $A(X)$ [18], so the result follows trivially.
(*ii*) $\Rightarrow$ (*i*) This follows since a prime ideal P in $A(X)$ is contained in M_A^p iff P contains O_A^p. $\square$

Lemma 2.5.9. $f^\beta(p) = 0$ *for every* $p \in S[\mathcal{Z}_A(f)]$

Proof. We know $f^\beta(p) = \lim_{\mathcal{Z}_A[M_A^p]} f$.

$$\text{Now let } p \in S[\mathcal{Z}_A(f)]. \text{ Then } f \in M_A^p \quad \Rightarrow \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M_A^p].$$

$$\text{So } 0 = \lim_{\mathcal{Z}_A(f)} f = \lim_{\mathcal{Z}_A[M_A^p]} f = f^\beta(p).$$

Hence, $f^\beta(p) = 0$. $\square$

Theorem 2.5.10. *If $M_A^p = O_A^p$ for every $p \in X$, then every zero-set of X is open.*

Proof. Let $Z(f)$ be a zero-set of X, where $f \in C(X)$.
We claim that $f(p) = 0 \quad \Rightarrow f(V) = 0$, for some neighborhood V of p. Now $f(p) = 0 \quad \Rightarrow \ p \in Z(f) \subseteq E \ \ \forall E \in \mathcal{Z}_A(f)$.

$$\text{So, } p \in \bigcap_{E \in \mathcal{Z}_A(f)} E \quad \Rightarrow p \in \operatorname{Int} S[\mathcal{Z}_A(f)] \quad \Rightarrow f \in O_A^p = M_A^p.$$

Let $V \in \operatorname{Int} S[\mathcal{Z}_A(f)]$ and let $q \in V$, then $f^\beta(q) = 0$. So, $p \in V \subset Z(f)$.
Hence $Z(f)$ is open in X. $\square$

Corollary 2.5.11. *X is a P-space iff every zero-set of X is open.*

Proof. Neccesity follows from Theorem [2.5.10].
Conversely, $O^p_A \subseteq M^p_A$ follows from Remark [2.5.5]. On the other hand, if $f \in M^p_A$, then $p \in S[\mathcal{Z}_A(f)]$. Since every zero-set of X is open, so $p \in \operatorname{Int} S[\mathcal{Z}_A(f)]$ implies that $f \in O^p_A$. Hence, $O^p_A = M^p_A$ and the result follows from Theorem [2.5.8]. □

Example 2.5.12. Every discrete space is a P-space.

Theorem 2.5.13. *Let $A(X)$ be any subring of $C(X)$. If every prime ideal in $A(X)$ is maximal, then X is a P-space.*

Proof. By Remark [2.5.5], if every prime ideal in $A(X)$ is maximal, then $M^p_A = O^p_A$, so by Theorem [2.5.10], every zero-set of X is open and finally by Corollary [2.5.11] X is a P-space. □

Remark 2.5.14. Converse of Theorem [2.5.13] is not true.

Proof. Clearly $\mathbb{N}$ is a P-space as it is discrete.
Let $A(X) = C^*(\mathbb{N})$. But $M^p_A \neq O^p_A$ for $p \in \beta\mathbb{N} \setminus \mathbb{N}$ [p.48, [27]]. □

2.6 Intersections of free maximal ideals

In this section we shall study the generalizations of the results on intersections of maximal ideals known for $C(X)$ and $C^*(X)$ to $A(X)$.

Notation 2.6.1. $A_F(X) = \underset{p\in\beta X\setminus X}{M^p_A}$,i.e., the intersection of free maximal ideals in $A(X)$, where F signifies that the maximal ideals concerned are all free.

Theorem 2.6.2. *If $B(X) \subseteq A(X)$, then $A_F(X) \subseteq B_F(X)$.*

Proof. Let $f \in A_F(X)$, then $f \in M^p_A$ forall $p \in \beta X \setminus X$.

$$\text{This implies that} \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M^p_A] \quad \forall p \in \beta X \setminus X.$$

Since every $\mathcal{Z}_A[M^p_A]$ is contained in a unique maximal z-ultrafilter, so let $\mathcal{F}^p$ be the free z-ultrafilter containing $\mathcal{Z}_A[M^p_A]$.
Further, $B(X) \subseteq A(X) \;\Rightarrow \mathcal{Z}_B(f) \subseteq \mathcal{Z}_A(f)$.

$$\text{So, } \mathcal{Z}_B(f) \subseteq \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M^p_A] \subseteq \mathcal{F}^p \quad \Rightarrow f \in \mathcal{Z}_B{}^{\leftarrow}[\mathcal{F}^p] \quad \forall\, p \in \beta X \setminus X.$$

By Theorem [2.4.13] $\mathcal{Z}_B{}^{\leftarrow}[\mathcal{F}^p] = M^p_B$, so $f \in B_F(X)$.
Thus $A_F(X) \subseteq B_F(X)$. □

Definition 2.6.3. Let $f \in C(X)$, then the support of the function $f \in C(X)$ is the set $\mathrm{cl}_X(X \setminus Z(f))$

Notation 2.6.4.

$$C_C(X) = \{f \in C(X) | \, \mathrm{cl}_X(X \setminus Z(f)) \text{ is compact}\}$$

We shall record an analogous result for $A(X)$ which is well known for the case of $C(X)$.

Lemma 2.6.5. *Let I be an ideal in $A(X)$. Then I is free iff for every compact set $K \subset X$, there exists $f \in I$ such that $Z(f) \cap K \neq \phi$.*

Proof. Let us suppose that I is free, i.e., $\underset{f\in I}{Z(f)} = \phi$. Suppose, K is a compact set in X such that $Z(f) \cap K \neq \phi$ for every $f \in I$.

$$\text{Since } Z(f) = \bigcap \mathcal{Z}_A(f), \text{ so } E \cap K \neq \phi, \text{ for every } E \in \mathcal{Z}_A[I]. \tag{2.6}$$

Now, $\mathcal{Z}_A[I]$ is a z-filter, so the collection $\rho = \{E \cap K | E \in \mathcal{Z}_A[I]\}$ is a collection of closed sets in K with finite intersection property by (2.6). Since K is compact and X is T_1, so every closed subset of K is compact and hence by the compactness of K, $\bigcap \rho \subseteq \bigcap \mathcal{Z}_A[I] = \bigcap Z[I]$.
Hence, the neccesity holds.
For sufficiency, suppose I is fixed and consider $\rho = \cap\{Z(f) | f \in I\}$, then $\rho \neq \phi$. Let $K \subseteq \rho$ such that K is compact. Such a K exists as finite sets are compact. Clearly, $K \cap Z(f) \neq \phi \quad \forall f \in I$. Hence the result. □

Lemma 2.6.6 (pg. 109, [18], 5.3, [7]).

$$C_C(X) = \{f \in C^*(X) | Z_{\beta X}(f^\beta) \text{ is a neighbourhood of } \beta X \setminus X\}.$$

Theorem 2.6.7. *The intersection of the free ideals in any $A(X)$ is $C_C(X)$*

Proof. Let I_F denote the intersection of all free ideals in A(X). Let $f \in I_F$. We first observe that O_A^p is free whenever $p \in \beta X \setminus X$,

$$\text{for } \bigcap_{f \in O_A^p} \mathcal{Z}_A(f) = \bigcap_{f \in O_A^p} Z(f) \subseteq \bigcap_{f \in O_A^p} \mathrm{cl}\, Z(f) = p$$

$$\textit{So, } \; p \in \beta X \setminus X \quad \Rightarrow p \in \mathrm{cl}\, Z(f) \setminus Z(f) \quad \Rightarrow p \notin Z(f) \;\; \forall\, f \in O_A^p.$$

$$\text{Hence, } \bigcap_{f \in O_A^p} \mathcal{Z}_A(f) = \phi.$$

Now, $f \in I_F \quad \Rightarrow f \in O^p_A \;\; \forall p \in \beta X \setminus X \quad \Rightarrow p \in \operatorname{Int} S[\mathcal{Z}_A(f)] \;\; \forall p \in \beta X \setminus X$

$$\Rightarrow \beta X \setminus X \subseteq \operatorname{Int} S[\mathcal{Z}_A(f)]$$

Thus, $f^\beta(p) = 0 \;\; \forall\, p \in \operatorname{Int} S[\mathcal{Z}_A(f)]$ by Lemma[2.5.9].

Also, $Z(f^\beta)$ is a compact set in βX such that $Z(f^\beta) \supset \operatorname{Int} S[\mathcal{Z}_A(f)] \supset \beta X \setminus X$.
So, by Lemma [2.6.6] $f \in C_C(X)$ implies that $I_F \subseteq C_C(X)$.
On the other hand, let $f \in C_C(X) \quad \Rightarrow \overline{(X \setminus Z(f))}^{X}$ is compact.
Let I be a free ideal in $A(X)$, then by Lemma [2.6.5] there exists $g \in I$ such that

$$Z(g) \cap \overline{X \setminus Z(f)} = \phi \quad \Rightarrow Z(g) \subseteq \overline{X \setminus Z(f)} \subseteq Z(f)$$

This implies that $Z(f)$ is a neighbourhood of $Z(g)$. Define h as

$$h(x) = \begin{cases} \frac{f(x)}{g(x)} & x \notin IntZ(f) \\ 0 & x \in Z(f) \end{cases}$$

The continuity of h follows from Pasting Lemma, for $\operatorname{Int} Z(f)$ and $Z(f)$ are closed in X, $\operatorname{Int} Z(f) \cup Z(f) = X$, $h|_{\operatorname{Int} Z(f)}, h|_{Z(f)}$ are continuous and whenever $x \in \operatorname{Int} Z(f) \cap Z(f)$, then $h(x) = 0$. So $h \in C^*(X)$ and

$$f = hg \quad \Rightarrow f \in I, \text{ so } f \in I_F.$$

□

We now consider the intersection of the free maximal ideals in $A(X)$.

Definition 2.6.8. A set $E \subseteq X$ is small if every zero set contained in E is compact.

Example 2.6.9. If X is a finite space, then every subset of X is compact.

Notation 2.6.10. Let $\mathcal{K} = \{E \subseteq Z[X] |\; E^c \text{ is small}\}$ and $A_K(X) = \{f \in A(X) | \mathcal{Z}_A(f) \subseteq \mathcal{K}\}$.

In this context we shall record a theorem originally due to Redlin and Watson [28].

Theorem 2.6.11.

$$A_K(X) = \bigcap \{M^p_A : p \in \beta X \setminus X\}.$$

Proof. Let $f \in A_K(X)$. If $\mathcal{F}$ is any z-ultrafilter on X such that $\mathcal{Z}_A(f) \nsubseteq \mathcal{F}$, then there exists $E \in \mathcal{Z}_A(f) \setminus \mathcal{F}$ and $F \in \mathcal{F} \setminus \mathcal{Z}_A(f)$, such that $E \cap F = \phi$ and hence $F \subset E^c$. By the above characterization of $A_K(X)$ Notation [2.6.10], F is compact. Since every element of $\mathcal{F}$ intersects with F so by compactness of F, $F \cap \mathcal{F} \neq \phi$ implies $\mathcal{F}$ is fixed . So E does not belong to any fixed z-ultrafilter and since E was chosen arbitrarily, so $\mathcal{Z}_A(f)$ is contained in every free z-ultrafilter.

$$i.e., \quad \mathcal{Z}_A(f) \subseteq \overline{\mathcal{Z}_A[M_A^p]} \quad \forall p \in \beta X \setminus X \quad \Rightarrow f \in M_A^p \ \forall p \in \beta X \setminus X$$

Conversely, if $f \in M_A^p$ $\forall p \in \beta X \setminus X$ then $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M_A^p] \subseteq \overline{\mathcal{Z}_A[M_A^p]}$, i.e., f belongs to every free z-ultrafilter (*)
We claim that $\mathcal{Z}_A(f) \subseteq \mathcal{K}$. On the contrarary suppose $E \in \mathcal{Z}_A(f)$ such that $E \notin \mathcal{K}$. Then E^c must contain a noncompact zero-set F. Since $E \cup F \supset E \in \mathcal{Z}_A(f)$ this implies that $E \cup F \in \mathcal{Z}_A(f)$. By (*), $E \cup F$ belongs to every free z-ultrafilter $\overline{\mathcal{Z}_A[M_A^p]}$, which implies that $F \notin \mathcal{Z}_A[M_A^p]$ for $E \cap F = \phi$. This is a contradiction for every non-compact set must belong to some free z-ultrafilter. Thus $E \in \mathcal{K}$ and hence $f \in A_K(X)$. □

Definition 2.6.12. A collection $\mathcal{F}$ of subsets of X is called A-stable if every $f \in A(X)$ is bounded on some member of $\mathcal{F}$.

Theorem 2.6.13. *The space X is A-compact iff every A-stable z-ultrafilter on X converges.*

Proof. Let us suppose that X is A-compact and if possible let $\mathcal{F}$ be a z-ultrafilter on X that do not converge. Then $\{\mathrm{cl}_{\beta X} F | F \in \mathcal{F}\}$ is a z-ultrafilter on βX, for X is dense in βX, $\mathrm{cl}(Z_1 \cap Z_2) = \mathrm{cl}\, Z_1 \cap \mathrm{cl}\, Z_2$ and $\mathrm{cl}_{\beta X} F$ is a zero-set in βX. So, there exists $p \in \beta X \setminus X$ which is a limit of $\mathcal{F}$. We know that $\overline{\mathcal{Z}_A[M_A^p]}$ also converges to p, so $\overline{\mathcal{Z}_A[M_A^p]} \subseteq \mathcal{F}$. Otherwise, if there exists $\overline{\mathcal{Z}_A[M_A^p]} \supset \mathcal{Z}_A[M_A^p]$, such that $\overline{\mathcal{Z}_A[M_A^p]} \neq \mathcal{F}$. Since $\mathcal{Z}_A[M_A^p]$ converges to p, so two distinct z-ultrafilters converge to p, which is a contradiction. Now, X is A-compact and the ideal M_A^p is free, so it is hyper-real. Therefore, there exists $f \in A(X)$ such that $|M_A^p(f)|$ is infinitely large, which yields $Z_n \in \mathcal{Z}_A[M_A^p]$ $\forall n \in \mathbb{N}$.
Let $F \in \mathcal{F}$, then $F \cap Z_n \neq \phi$ $\forall n \in \mathbb{N}$ implies $f[F] \geq n$ $\forall n \in \mathbb{N}$. Since F was arbitrarily chosen, so the result holds for all $F \in \mathcal{F}$. This contradicts that $\mathcal{F}$ is A-stable. Hence $\mathcal{F}$ must converge.

Conversely, suppose that X is not A-compact, then there exists a real free maximal ideal M_A^p, $p \in \beta X \setminus X$. So, $M_A^p(f)$ is finite for all $f \in A(X)$. By

Theorem [2.3.15] f is bounded on some member of $\mathcal{Z}_A[M_A^p]$. Since $\mathcal{Z}_A[M_A^p] \subseteq \overline{\mathcal{Z}_A[M_A^p]}$ and $\overline{\mathcal{Z}_A[M_A^p]}$ is A-stable such that it doesn't converge in X , hence the result follows. $\square$

Lemma 2.6.14. *If X is A-compact, then $f \in A(X)$ has compact support iff $\mathcal{Z}_A(f) \subseteq \mathcal{K}$*

Proof. Suppose $f \in A(X)$ has a compact support, then $\mathrm{cl}_X\,(X \setminus Z(f))$ is compact. Let $E \in \mathcal{Z}_A(f)$, then there exists $g \in A(X)$ such that $fg|_{E^c} = 1$. Since $E \supset Z(f)$, so $E^c \subseteq X \setminus Z(f) \subseteq \mathrm{cl}(X \setminus Z(f))$. Thus $\mathrm{cl}\,E^c$ is closed and hence compact. Let F be a zero-set in E^c, then $F \subseteq E^c \subseteq \mathrm{cl}\,F^c$. Since every zero-set is closed, hence it is compact. Thus $E \in \mathcal{K}$.

Conversely, if f has non-compact support, then $\mathrm{cl}_X(X \setminus Z(f))$ is non-compact, so it is not closed in βX

$$\text{Now } \mathrm{cl}_{\beta X}(X \setminus Z(f)) \setminus \mathrm{cl}_X(X \setminus Z(f)) \neq \phi \quad \text{,so } p \in \mathrm{cl}_{\beta X}(X \setminus Z(f)) \setminus X$$

So, M_A^p is free and by A-compactness of X, M_A^p is hyper-real. So, there exists $g \in A(X)$ *such that* $M_A^p(g)$ is infinitely large. Hence by Theorem [2.4.15] $g^*(p) = \infty$, i.e., g is unbounded on $X \setminus Z(f)$. It is well known [18] that $X \setminus Z(f)$ contains a non-compact set E such that E and $Z(f)$ are completely separated. As completely separated sets are contained in disjoint zero set neighbourhoods [Theorem 1.1.18], so $E \subset U$ and $Z(f) \subseteq V$ such that $U \cap V = \phi$. Since $\mathcal{Z}_A(f)$ is a z-ultrafilter (as f is not invertible), so $V \in \mathcal{Z}_A(f)$ but $V^c \supset E$ such that E is noncompact, so $V \notin \mathcal{K}$ implies that $\mathcal{Z}_A(f) \nsubseteq \mathcal{K}$. $\square$

Acharya, Chattopadhyay and Ghosh [1] have given a useful characterisation of A-compactness.

Theorem 2.6.15 (3.2, [1]). *A space X is A-compact iff for every p in $\beta X \setminus X$, there exists an f in $C^*(X)$ such that f is a unit of A and $f^\beta(p) = 0$ (or equivalently X is A-compact iff for every p in $\beta X \setminus X$, there exists a unit g of A such that $g^{-1} \in C^*(X)$ and $g^*(p) = \infty$).*

2.7 Intersections of maximal ideals in algebras between $C^*(X)$ and $C(X)$

In this section we include some more results about intersection of free maximal ideals in subalgebras. We end this section by quoting a counterexample to a thoerem [Theorem 5.8, [7]] of Byun and Watson. Most of the results recorded are due to Dominguez and Perez in [11, 1].

Notation 2.7.1. $C^*(X)[f] = \sum\limits_{i=0}^{n} g_i f_i : g_i \in C^*(X),\ n = 0, 1, 2, \ldots.$ It is also known as the singly generated algebra generated by 'f'.

The following remark also serves as an example of a proper intermediate algebra.

Remark 2.7.2. It is the smallest intermediate algebra containing f.

Proof. Let $A(X)$ be any intermediate algebra containing f. Consider $g \in C^*(X)[f]$ then $g = \sum\limits_{i=0}^{n} g_i f_i$, for some $n \in \mathbb{N}$, $g_i \in C^*(X)$ for $1 \leq i \leq n$.
Since $C^*(X) \subseteq A(X)$ and $f \in A(X)$, $g \in A(X)$. Hence, any intermediate algebra containing f contains $C^*(X)[f]$. □

Notation 2.7.3. $C(v_f X)$, denotes the largest subset of βX to which f can be continuously extended.

Remark 2.7.4. $C^*(X)[f] \subseteq C(v_f X)$ as $f \in C(v_f X)$.

We know that $A(X)$ is an absolutely convex subalgebra of *C(X)* by Proposition[2.2.3] and a sublattice of *C(X)*. Some interesting results have been shown to be true because of this property

Remark 2.7.5. Let c be a real number, $c > 1$. Every singly generated intermediate algebra on X is $C^*(X)[f]$ for some $f \geq c$.

Proof. Let $A(X)$ be a singly generated intermediate algebra. It is enough to show that $g \in A(X)$ iff $|g| + c \in A(X)$,

$$\text{for then } C^*(X)[g] = C^*(X)[|g| + c] = C^*(X)[f] \text{ where } f \geq c.$$

Now $g \in A(X)$ implies $-g \in A(X)$ and $|g| = g \vee (-g)$. Since $A(X)$ is a lattice by Corollary [2.2.2], so $|g| \in A(X)$ implies $|g| + c \in A(X)$ *for* $c \in C^*(X)$. The other implication is clear. □

Remark 2.7.6. If $f \geq c \geq 1$ for some $c \in \mathbb{R}$, then $C^*(X)[f] = \{g \in C(X) : |g| \leq f^k \ \textit{for some}\ k \in \mathbb{N}\}$

Proof. Let $g \in C(X)$ such that $|g| \leq f^k$ for some $k \in \mathbb{N}$. Then, $f^k \in C^*(X)[f]$ implies that $g \in C^*(X)[f]$ by absolute convexity of $C^*(X)[f]$. On the other hand, if $g \in C^*(X)[f]$, then there exists $g_0, g_1, \ldots, g_n \in C^*(X)$ such that $g = \sum\limits_{i=0} g_i f_i$ for some $n \in \mathbb{N}$.
For each i, $g_i \in C^*(X)[f]$ implies $|g_i| \leq m$ for some $m \in \mathbb{R}$, but $m \leq f^{l_i}$ for some $l_i \in \mathbb{N}$ as $f > 1$.
Let $k = n \times \max\{l_0, l_2, \ldots, l_n\}$, clearly, $|g| \leq f^k$. □

Remark 2.7.7. Every finitely generated intermediate algebra is singly generated. Explicitly, $C^*(X)[f_1, f_2, ..., f_n] = C^*(X)[|f_1| + |f_2| + ... + |f_n|]$, for any $n \in \mathbb{N}$.

Proof. We have $f_i \in C^*(X)[|f_1| + |f_2| + ... + |f_n|] \quad \Rightarrow |f_i| \in C^*(X)[|f_1| + |f_2| + ... + |f_n|]$ where $1 \leq i \leq n$.

$$\text{i.e., } |f_1| + |f_2| + ... + |f_n| \in C^*(X)[|f_1| + |f_2| + ... + |f_n|] \tag{2.7}$$

Also,

$$|f_i| \leq [|f_1| + |f_2| + ... + |f_n|] \quad \Rightarrow f_i \in C^*(X)[|f_1| + |f_2| + ... + |f_n|] \quad \forall \ 1 \leq i \leq n \tag{2.8}$$

Thus from Equations [2.7] and [2.8], we have

$$C^*(X)[f_1, f_2, .., f_n] = C^*(X)[|f_1| + |f_2| + ... + |f_n|]$$

□

Notation 2.7.8. $U(A)$ denotes the set of all units in $A(X)$

Remark 2.7.9. If $A(X) = C^*(X)[f]$ for some $f \geq c > 1$, then the multiplicatively closed subset S_A is given by $S_A = \{g \in C^*(X) : |g| \geq \frac{1}{f^n}$ for some $n \in \mathbb{N}\}$.

Proof. Now, $f \geq c > 1$ implies $Z(f) = Z(f^n) = \phi$. For, $g \in C^*(X)[f]$ if $|g| \geq \frac{1}{f^n}$ for some $n \in \mathbb{N}$, then $Z(g) = Z(\frac{1}{f^n}) = \phi \quad \Rightarrow g \in U(A) \cap C^*(X)$. On the other hand, if $g \in U(A) \cap C^*(X)$, then $Z(g) = \phi$ and let $m = |g| \neq 0$. Since $f \geq c > 1$, for some $n \in \mathbb{N}$ $f^n \geq \frac{1}{m}$ implies that $\frac{1}{f^n} \leq m = |g|$. Hence the equality follows. □

Proposition 2.7.10. *Let* $f \in C(X)$, $A(X) = C^*(X)[f]$ *and* $E \in Z(X)$, *then the following conditions are equivalent*

(i) $E \in \mathcal{Z}_A(f)$

(ii) $E \supset E_\epsilon(f)$ *for some* $\epsilon > 0$.

Proof. *(ii ⇒ i)* It follows from Remark [2.2.14] that, $E_\epsilon(f) \subseteq \mathcal{Z}_A(f)$ for every $\epsilon > 0$, so $f|_{E_\epsilon(f)^c}$ is locally invertible and

$E \supset E_\epsilon(f)$, implies that $E_\epsilon(f)^c \supset E^c$, implies $f|_{E^c}$ is locally invertible on $E_\epsilon(f)^c$

Hence, $E \in \mathcal{Z}_A(f)$.

$(i \Rightarrow ii)$ Let $E \in \mathcal{Z}_A(f)$ and let us supose that (ii) fails. So, for each $\epsilon > 0$, $f^{\leftarrow}(-\epsilon, \epsilon) \cap E^c \neq \phi$. For each n, we choose $x_n \in E^c$ such that $|f(x_n)| < \frac{1}{n}$. Consequently, for the sequence $(x_n)_{n \in \mathbb{N}}$ in $E^c, f(x_n) \in (\frac{-1}{n}, \frac{1}{n})$ $\forall n \in \mathbb{N}$ implies that $\lim\limits_{n \to \infty} f(x_n) = 0$.
Now, let $D = \{x_1, x_2, ...\}$ and $D_k = \{x_k, x_{k+1}, ...\}$, $k \in \mathbb{N}$.
But $D \subseteq E^c$ implies that $D \cap Z(f) = \phi$. So, $f(D)$ is non-compact. Otherwise $f(D)$ is compact implies that $f(D)$ is closed and bounded in $\mathbb{R}$. So 0 is a limit point of $f(D)$ implies that $0 \in f(D)$, a contradiction as $D \cap Z(f) = \phi$. So D must be non-compact as f is continuous. Hence, D is not closed in βX.

Let $p \in \text{cl}_{\beta X} D \setminus D$ then $p \in \text{cl}_{\beta X}\, D_k$, as every neighbourhood of p must intersect D at infinitely many points. Further, f is continuous implies that $f(\bar{F}) \subseteq \overline{f(F)}$ forall $F \subset X$.
So, $f[\overline{\text{cl}_{\beta X} D_k}] \subseteq \text{cl}_{\mathbb{R}^*} f(D_k)$ gives $f^*(p) \in \text{cl}_{\mathbb{R}^*} f(D_k)$, where $\mathbb{R}^*$ is the one-point compactification of $\mathbb{R}$. Since this is true for each $k \in \mathbb{N}$,

$$\text{so } f^*(p) \in \bigcap \text{cl}_{\mathbb{R}^*} f(D_k) = \underset{k \in \mathbb{N}}{\text{cl}_{\mathbb{R}^*}} (\frac{-1}{k}, \frac{1}{k}) = 0.$$

Since $E \in \mathcal{Z}_A(f)$, there exists $g \in F(X)$ such that $fg = 1$ on E^c and $f^*(p) = 0$. which gives $p \in v_f(X)$, the realcompactification of X for f.
So, by Remark [2.7.2] $f, g \in C(v_f X) \supset F$, which implies $(fg)^*(p) = f^*(p)g^*(p) = 0$. But $1.(fg) = 1$ on D, so $(fg)^*$ must be 1 on $\text{cl}_{\beta X}\, D$ as fg is continuous. Hence $(fg)^*(p) = 1$. This is a contradiction to the fact that $E \in \mathcal{Z}_A(f)$. ☐

Corollary 2.7.11. *$E \in \mathcal{Z}_A(f)$ implies that E is a zero set neighbourhood of $Z(f)$.*

Proof. This follows from the fact that $Z(f) \subseteq \{x \in X : f(x)g(x) \neq 1\} \subseteq E$. ☐

Remark 2.7.12. The converse need not be true.

Proof. Let $X = \mathbb{N}$ and define a function f as $f(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} & \text{if } n \neq 1. \end{cases}$

Since X is discrete so $Z(f)$ is a neighbourhood of itself. But $\{0\} \notin \mathcal{Z}_{C^*}(f)$, as the function f, given by $f(n) = n$ is not in $\mathcal{Z}_{C^*}(f)$. ☐

We now look at some more properties of intersections of free maximal ideals.

Proposition 2.7.13.

$$C_K(X) \subseteq C(X)_F \subseteq A_F \subseteq C^*(X)_F = C_\infty(X)$$

Proof. It is well known that $C_\infty(X) = C^*(X)_F$[18]. Let $f \in C_K(X)$ and $p \in \beta X \setminus X$.
By Theorem [1.6.8], $p \in \mathrm{cl}_{\beta X} Z(f) \quad \Rightarrow f \in M^p \quad \Rightarrow f \in C(X)_F$. So, $C_K(X) \subseteq C(X)_F$.
Let $B(X)$ be an intermediate algebra containing $A(X)$. Since $f \in M_A^p \quad \Rightarrow f^*(p) = 0$, for $f \in A_F$, $f^*(p) = 0 \quad \forall p \in \beta X \setminus X \quad \Rightarrow f \in C_\infty(X) \subseteq C^*(X)_F$. So $A_F \subseteq C^*(X)_F$. If $f \in B_F$ and $p \in \beta X \setminus X$ and also $f \in C^*(X)_F \subseteq A(X) \quad \Rightarrow f \in M_B^p \cap A(X) \subseteq M_A^p$. Thus $B_F \subseteq A_F$. Hence the result follows. ☐

We shall record few results taking into account ${S_A}^{-1}(C^*(X))$, where $S_A = U(A) \cap C^*(X)$.

Lemma 2.7.14. *Let $A(X)$ be an intermediate algebra on X and $S_A = U(A) \cap C^*(X)$. Then $A(X) = \{f \in C(X) : |f| \le |\frac{1}{g}| \text{for some } g \in S_A\}$*

Proof. Suppose $f \in C(X)$ such that $|f| \le |\frac{1}{g}|$ for some $g \in S_A$. Now $\frac{1}{g} \in S_A$. Since $A(X)$ is absolutely convex, $f \in A(X)$.

Conversely, let $f \in A(X)$, then $f^2 \in A(X)$ and $1+f^2 \in A(X)$ implies $\frac{1}{1+f^2} \in C^*(X) \cap U(A) = S_A$. Now $|f| \le 1 + f^2 = |\frac{1}{\frac{1}{1+f^2}}|$. Hence the equality holds. ☐

We shall state here a new description of the zero-sets in $\mathcal{Z}_A(f)$.

Notation 2.7.15.

$$E_g(f) = \{x \in X : |f(x)| \le |g(x)|\}$$

and

$$E^g(f) = \{x \in X : |f(x)| \ge |g(x)|\}$$

Remark 2.7.16. $E_g(f)$ and $E^g(f)$ are zero-sets, for $E_g(f) = Z((g - f) \wedge 0)$ and $E^g(f) = Z((f - g) \wedge 0)$

Proposition 2.7.17. *Let $A(X)$ be an intermediate algebra on X, $E \in Z(X)$ and $f \in A(X)$. The following conditions are equivalent.*

(i) $E \in \mathcal{Z}_A(f)$

(ii) $|f| \geq |g|$ *on* E^c, *for some* $g \in S_A$

(iii) $E \supset E_g(f)$ *for some* $g \in S_A$

Proof. ($i \Rightarrow ii$) Let $E \in \mathcal{Z}_A(f)$, then there exists $g \in A(X)$ such that $fg|_{E^c} = 1$. Now $g \in A(X)$ implies that $|g| \leq |\frac{1}{h}|$ for some $h \in S_A$. This implies that $|\frac{1}{g}| = \frac{1}{|g|} \geq |h|$ on E^c But $|f| = |\frac{1}{g}|$ on E^c implies that $|f| \geq |h|$, $h \in S_A$ on E^c

(ii) $\Rightarrow$ (i) Suppose $|f| \geq |g|$ on E^c for some $g \in S_A$, then $E \in \mathcal{Z}_A(g)$ as g is a unit of $A(X)$, so $|g| \leq |f|$ implies that $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A(f)$. Thus $E \in \mathcal{Z}_A(f)$.

(ii) $\Rightarrow$ (iii) Let $F = \{x \in X \quad |f(x)| < |g(x)|\}$, then, $F \subset E$. Let $g \in S_A$, so $Z(g) = \phi$ and let $G = \{x \in X : |f(x)| \leq \frac{1}{2}|g(x)|\}$, then $E \supset E_{\frac{g}{2}}(f)$ and $\frac{1}{2}g \in S_A$. Let $k = \frac{1}{2}g$, then $k \in S_A$ and $E \supset E_K(f)$.

(iii) $\Rightarrow$ (ii) Suppose $E \supset E_g(f)$ for some $g \in S_A$, then $F \subseteq E$. Let $x \in E^c$, then $E^c \cap F = \phi$, so that $|f(x)| \geq |g(x)|$. ☐

Definition 2.7.18. Let E be a subset of X. Then E is said to be small set if every zero-set contained in E is compact.

We can further characterize $\mathcal{K} = \{E \subset Z[X] \mid E^c \text{ is small}\}$ and $A_X(X)$ as shown by Dominguez and Perez in [11].

Remark 2.7.19.

$$\mathcal{Z}_A^{\leftarrow}(\mathcal{K}) = \{M_A^p \mid p \in \beta X \setminus X\}$$

The follwing result is due to Dominguez, Gomez Perez [10] is an immediate consequence of the properties of rings of fractions [1.3, [9]].

Theorem 2.7.20. *Let* $A(X)$ *be an intermediate algebra between* $C^*(X)$ *and* $C(X)$, *and* $S_A = U(A) \cap C^*(X)$

(i) *The map* $\imath_A$. spec $A(X) \longrightarrow$ spec $C^*(X)$, *that sends* $\bar{P}$ *in* spec $A(X)$ *to* $\bar{P} \cap C^*(X)$, *induces an homeomorphism between* spec $A(X)$ *and the subspace of* spec $C^*(X)$ *consisting of those prime ideals in* $C^*(X)$ *which do not cut* S_A.

(ii) *Every prime ideal* $\bar{P}$ *in* $A(X)$ *is absolutely convex, the quotient ring* $A(X)/\bar{P}$ *is totally ordered, and the canonical morphism* $A(X) \longrightarrow A(X)/\bar{P}$ *is order preserving*

(iii) *The prime ideals in $A(X)$ containing a given prime ideal form a chain. Hence, every prime ideal in $A(X)$ is contained in a unique maximal ideal.*

(iv) $\operatorname{Max} A(X)$ *is a compact Hausdorff space, and the map* $X \longrightarrow \operatorname{spec} A(X)$, *that sends* $x \in X$ *to* $M_A{}^x = \{f \in A(X) : f(x) = 0\}$, *establishes an homeomorphism between* X *and a dense subspace of* $\operatorname{Max} A(X)$.

Proof. (i) It follows trivially from the properties of rings of fractions.

(ii) Let $\bar{P}$ be a prime ideal in $A(X)$ and $P = \bar{P} \cap C^*(X)$. Then by the property of rings of fractions $\bar{P} = S_A{}^{-1}P$. We claim that $\bar{P}$ is absolutely convex.

Let f and g be two functions in $A(X)$ such that $|f| \leq |g|$ with $g \in \bar{P}$.

$$\text{Then } |f(1+g^2)^{-1}| \leq |g(1+g^2)^{-1}| \text{ and } g(1+g^2)^{-1} \in P$$

Since P is absolutely convex in $C^*(X)$ by Theorem [1.4.16], so $f(1+g^2)^{-1} \in P$.

Therefore, $f = (1+g^2)f(1+g^2)^{-1} \in \bar{P}$. Hence $\bar{P}$ is absolutely convex in $A(X)$. The other part of the proof follows directly from Theorem [1.4.16].

(iii) This follows from (i) and Theorem [1.4.17].

(iv) Since every prime ideal in $A(X)$ is contained in a unique maximal ideal, $\operatorname{Max} A(X)$ is a Hausdorff space.

Consider the following composition [Theorem 2.3.5];

$$\begin{aligned} X &\longrightarrow \operatorname{Max} A \longrightarrow \operatorname{spec} C^*(X) \\ \text{by } x &\longrightarrow M_X{}^x \longrightarrow M^{*x} \end{aligned}$$

where $M^{*x} = M_A{}^x \cap C^*(X)$. From (i), $\operatorname{Max} A \longrightarrow \operatorname{spec} C^*(X)$ is an immersion. So, $X \longrightarrow \operatorname{Max} C^*(X)$ is a canonical immersion. Hence, the result follows.

□

Theorem 2.7.21. *Let $A(X)$ be an intermediate algebra on X and let $S_A = U(A) \cap C^*(X)$. A function $f \in A(X)$ is in $\bigcap\{M_A^p : p \in \beta X \setminus X\}$ iff $E^g(f)$ is compact for every $g \in S_A$.*

Proof. Suppose $E^g(f)$ is non-compact for some $g \in S_A$. Since $E^g(f)$ is not closed in βX, there exists a $p \in \mathrm{cl}_{\beta X} E^g(f) \setminus X$.

$$\text{Consider } E_{\frac{g}{2}}(f) = \{x \in X \,|\, |f(x)| \leq |\frac{1}{2} g(x)|\},$$

then for each $x \in E_{\frac{g}{2}}$, $(f)^c \quad f(x) > \frac{g(x)}{2}$ implies $E_{\frac{g}{2}}(f) \in \mathcal{Z}_A(f)$ by Proposition[2.7.17].
We claim that $\mathcal{Z}_A(f) \not\subseteq \mu^p$. Since $E_{\frac{g}{2}}(f)$ and $E^g(f)$ are disjoint zero sets, so by Theorem [1.6.2], they have disjoint closures in βX.

$$\text{But } p \in \mathrm{cl}_{\beta X} E^g(f) \text{ implies that } p \notin \mathrm{cl}_{\beta X} E_{\frac{g}{2}}(f) \text{ so, } p \notin S[\mathcal{Z}_A(f)].$$

Thus $\mathcal{Z}_A(f) \not\subseteq \mu^p$, so, $f \notin M_A^p$, which is a contradiction.
Conversely, suppose $E^g(f)$ is compact for every $g \in S_A$, we shall show that $\mathcal{Z}_A(f) \subseteq \mathcal{K}$. Let $E \in \mathcal{Z}_A(f)$, then by Proposition [2.7.10] there exists $g \in S_A$ such that $|f| \geq |g|$ on E^c, where $E^c \subseteq E^g(f)$. Since $E^g(f)$ is compact, so any zero set $F \subseteq E^c \subseteq E^g(f)$ is compact, implies that E^c is small and hence $E \in \mathcal{K}$.

$$\text{Now } \mathcal{Z}_A(f) \subseteq \mathcal{K} \quad \Rightarrow \quad f \in A_K = \bigcap \{M^p : p \in \beta X \setminus X\}.$$

□

Remark 2.7.22. Let A(X) be an intermediate algebra on X and $f \in A(X)$. If h is a unit of A(X) and $E^g(f)$ is compact for every $g \in S_A$, then $E^h(f)$ is compact too.

Proof. Since h is a unit, $\dfrac{h^2}{1+h^2}$ is also a unit. Let $g = \dfrac{h^2}{1+h^2}$, then $|g| \leq 1$. So $g \in S_A$ and $|g| \leq |h|$.
Now $E^h(f) = \{x \in X \,|\, |f(x)| \geq |h(x)|\}$, since $|g(x)| \leq |h(x)|$,

$$\text{so } E^h(f) \subseteq \{x \in X \,|\, |f(x)| \geq |g(x)|\} \; = E^g(f).$$

Since $E^g(f)$ is compact so $E^h(f)$ is compact too. □

Corollary 2.7.23.

$$\bigcap \{M^{*p} : p \in \beta X \setminus X\} = C_\infty(X)$$

Corollary 2.7.24. *A function* $f \in C(X)$ *is in* $\bigcap \{M^p : p \in \beta X \setminus X\}$ *iff every zero set disjoint from* $Z(f)$ *is compact.*

Proof. Let $f \in C(X)$ and suppose there exists $h \in C(X)$ for which $Z(h)$ is non-compact such that $Z(h) \cap Z(f) = \phi$. Let $g = (|h| + |f|) \wedge 1$.

If $x \in Z(h)$ then $g(x) = |f(x)| \wedge 1 \Rightarrow |g(x)| \leq |f(x)| \Rightarrow Z(h) \subseteq E^g(f)$.

If $E^g(f)$ is compact then every closed subset of $E^g(f)$ is compact. As $Z(h)$ is non-compact so $E^g(f)$ is must be non-compact. By Theorem [2.7.21], $f \notin \bigcap\{M^p \in \beta X \setminus X\}$.
Conversely, suppose $f \notin \bigcap\{M^p : p \in \beta X \setminus X\}$ then $f \notin M^p$ for some $p \in \beta X \setminus X$ thus $p \notin \mathrm{cl}_{\beta X} Z(f)$. So, there is a neighbourhood V of p such that $V \cap Z(f) = \phi$. Since every neighbourhood of p contains a zero set neighbourhood F of p [1.5.5], then $F \cap Z(f) = \phi$. Since βX is compact Hausdorff, it is normal. So by Urysohn's Lemma there exists $h^\beta \in C(\beta X)$ such that $h^\beta[F] = \{0\}$ and $h^\beta[Z(f)] = 1$. This implies that $p \in F \subseteq Z(h^\beta)$. So, $Z(h^\beta)$ is a neighbourhood of p in βX implies that $Z(h) \cap Z(f) = \phi$, but $p \in \mathrm{cl}_{\beta X} Z(h) \setminus Z(h)$ which implies $Z(h)$ is not closed, hence non-compact. □

Corollary 2.7.25. *If X is a realcompact space, then* $\bigcap\{M^p : p \in \beta X \setminus X\} = C_K(X)$.

Proof. Let $f \in C_K(X)$, then f has a compact support. If $E \in \mathcal{Z}_C(f)$, then there exists $g \in C(X)$ such that $fg|_{E^c} = 1$. Now $E^c \subseteq \mathrm{cl}(X \setminus Z(f))$, so $\mathrm{cl}\, E^c$ is compact. Let F be any zero set in X such that $F \subset E^c$, then $F \subset \mathrm{cl}\, E^c$ implies F is compact. Hence $\bigcap\{M^p : p \in \beta X \setminus X\} \supset C_K(X)$.
On the other hand, suppose $\exists\ f \in C(X)$ such that $f \notin C_K(X)$, then $\exists\ p \in \mathrm{cl}_{\beta X}(X \setminus Z(f)) \setminus X$. Now M^p is free implies M^p is hypereal as X is realcompact, so $\exists\ h \in C(X)$ such that $h^*(p) = \infty \Rightarrow h$ is unbounded on $X \setminus Z(f)$. By [18] $X \setminus Z(f)$ contains a non-compact closed set S, that is C-embedded in X and if S and $Z(f)$ are completely separated, they have disjoint closures. Let $q \in \mathrm{cl}_{\beta X} S \setminus X$, then $q \in \mathrm{cl}_{\beta X} Z(f)$. So, $f \notin M^q$. Hence, the result follows. □

For a singly generated intermediate algebra, the characterization of the functions in all the free maximal ideals can be simplified further as shown in [11].

Notation 2.7.26. Let $f, l \in C(X)$ with $l \geq 0$ and $Z(l) = \phi$. For $n \in \mathbb{N}$, we shall write $F_n(f) = \{x \in X : |f(x)| \geq \frac{1}{l^n(x)}\}$

Corollary 2.7.27. *Let $A(X)$ be a singly generated intermediate algebra on X, $A(X) = C^*(X)[l]$ with $l \geq c > 1$. A function $f \in A(X)$ is in $\bigcap\{M_A^p : p \in \beta X \setminus X\}$ iff $F_n(f)$ is compact for all $n \in \mathbb{N}$.*

Proof. Let $f \in \bigcap\{M_A^p : p \in \beta X \setminus X\}$, then $l^n > 1$, so $0 < \frac{1}{l^n} < 1$ and

$$F_n(f) = \{x \in X : |f(x)| \geq \frac{1}{l^n(x)}\} = E^{\frac{1}{l^n}}(f).$$

By Theorem [2.7.21], $F_n(f) = E^{\frac{1}{l^n(f)}}$ is compact for all n.
Conversely, assume that $F_n(f)$ is compact for all $n \in \mathbb{N}$. Let $g \in S_A$, by Remark [2.7.9] $|g| \geq \frac{1}{l^n}$ for some $n \in \mathbb{N}$.

$$\text{Then } E^g(f) = \{x \in X : |f(x)| \geq |g(x)|\}.$$

Since $|g| \geq \frac{1}{l^n}$ for some $n \in \mathbb{N}$, so

$$E^g(f) \subseteq \{x \in X : |f(x)| \geq \frac{1}{l^n(x)}\} = F_n(f).$$

Since $F_n(f)$ is compact and $E^g(f)$ is closed so it is compact too. Hence by Theorem [2.7.21], $f \in \bigcap\{M_A^p : p \in \beta X \setminus X\}$. $\square$

Note 2.7.28 (3.4, [1]). If X is locally compact and $\sigma -$ *compact* but non-compact space, and if l is as in Corollary [2.7.27] such that l is a perfect mapping, i.e., $l^{-1}(K)$ is compact for each compact $K \subset \mathbb{R}$, then $\frac{1}{e^l}$ belongs to all the free maximal ideals of $C^*(X)[l]$, yet it does not belong to $C_K(X)$.

Dominguez and Perez [11] gave a counter example to a theorem by Byun and Watson [7]. We shall give an outline of that proof. However, we shall need some preliminaries.

Notation 2.7.29 ([27, 6, 11]).

Let $H = \{f \in C(\mathbb{N}) | \limsup\limits_{n \to \infty} \sqrt[n]{|f(x)|} \leq 1\}$, and $\bar{f}(m) = \sqrt[n]{|f(m)|}, m \in \mathbb{N}$

and $\bar{f}^\beta$ is the continuous extension of $\bar{f}$ to βX.

Clearly, $0 \geq \bar{f} \leq 1$.
We shall first show that H is an intermediate algebra over $\mathbb{R}$.

$$H \neq \phi \quad \text{for } \sqrt[n]{0} = 0 \quad \Rightarrow \limsup_{n \Rightarrow \infty} \sqrt[n]{0} = 0 \leq 1 \quad \Rightarrow 0 \in H.$$

Let $f, g \in H$, $r_1, r_2 \in \mathbb{R}$, then $r_1 f + r_2 g \in C(\mathbb{N})$ and $|r_1 f + r_2 g| \geq 0$.
So, we have the following cases.
Case (i) $|r_1 f + r_2 g| = 0$ which implies $\limsup\limits_{n\to\infty} \sqrt[n]{|r_1 f + r_2 g|} = 0 \leq 1$.
Case (ii) $|r_1 f + r_2 g| > 0$ which implies $\limsup\limits_{n\to\infty} \sqrt[n]{|r_1 f + r_2 g|} = 1$.
Thus $r_1 f + r_2 g \in H$. Hence it is a subspace of $C(\mathbb{N})$.

$$\text{Let } f, g \in H, \text{ then } \sqrt[n]{|fg|} = \sqrt[n]{|f|}\sqrt[n]{|g|} \leq 1 \Rightarrow fg \in H.$$

Further, $f \in C^*(\mathbb{N}) \quad \Rightarrow |f| \leq k$, for some $k \geq 0$
So, $\limsup\limits_{n\to\infty} \sqrt[n]{|f|} \leq 1 \quad \Rightarrow C^*(\mathbb{N}) \subseteq H$.

Finally, it is a proper intermediate algebra, for

$$f(n) = 2^n \in C(\mathbb{N}) \text{ but } \sqrt[n]{2^n} = 2 \text{ and } \underset{n\to\infty}{Lt.\ sup}\sqrt[n]{2^n} = 2 \quad \Rightarrow 2^n \notin H$$

Note 2.7.30. The above construction also serves as an example of proper intermediate algebra.

Proposition 2.7.31 (7.1, [27]). *$f \in H$ iff $\bar{f} \in C^*(X)(\mathbb{N})$ and $\bar{f}^\beta \leq 1$ on $\beta\mathbb{N} \setminus \mathbb{N}$.*

Proof. Let $f \in H$ then $\bar{f} = \underset{n\to\infty}{Lt.}\ \sup \sqrt[n]{|f|} \leq 1$. So, by continuity of f, $\bar{f}^\beta \leq 1$. This impies $\sup\{\bar{f}^\beta(p) : p \in \beta\mathbb{N} \setminus \mathbb{N}\} \leq 1$. Cearly, $f \in C^*(\mathbb{N})$.

Conversely, if $\bar{f} \in C^*(\mathbb{N})$ and $\bar{f}^\beta \leq 1 \quad \forall p \in \beta\mathbb{N} \setminus \mathbb{N}$, then $\sup\{\bar{f}^\beta\} \leq 1$. Further, f is bounded, so $\underset{n\to\infty}{Lt.}\ \sup \bar{f}(n) \leq 1$. Hence, $f \in H$. □

Proposition 2.7.32. *A function $f \in H$ is a unit of H iff $Z(f) = \phi$ and $\bar{f}^\beta = 1$ on $\beta\mathbb{N} \setminus \mathbb{N}$.*

Proof. Let $f \in H$ such that f is a unit, then there exists $g \in H$ such that $fg = 1$. Clearly

$$Z(f) = \phi \text{ and } f, g \in H \quad \Rightarrow \bar{f} \leq 1 \text{ and } \bar{g} \leq 1.$$

So by continuity of $\bar{f}^\beta$ and $\bar{g}^\beta$, we have

$$\bar{f}^\beta \leq 1 \text{ and } \bar{g}^\beta \leq 1. \tag{2.9}$$

$$\text{Since } fg = 1 \in C^*(X) \subseteq H, \quad (fg)^\beta \leq 1.$$

$$\text{But } \overline{(fg)}^\beta = \sqrt[n]{|(fg)^\beta|} = \sqrt[n]{|f^\beta g^\beta|} = \sqrt[n]{|f^\beta|}\sqrt[n]{|g^\beta|} = \bar{f}^\beta \bar{g}^\beta.$$

But $(\overline{fg})^\beta = 1$ by continuity, so

$$1 = (\overline{fg})^\beta = \bar{f}^\beta \bar{g}^\beta. \tag{2.10}$$

Since $\bar{f}^\beta \leq 1$ and $\bar{g}^\beta \leq 1$, so $\bar{f}^\beta = 1$ and $\bar{g}^\beta = 1$ on $\beta\mathbb{N} \setminus \mathbb{N}$.

Conversely, suppose that $Z(f) = \phi$ and $\bar{f}^\beta = 1$, this implies that

$$\frac{1}{\bar{f}^\beta} = 1 \Rightarrow \frac{\bar{1}^\beta}{\bar{f}^\beta} = (\frac{\bar{1}}{f})^\beta = 1 \Rightarrow \frac{1}{f} \in H,$$

which implies that f is a unit in H. □

Corollary 2.7.33 (2.3.1, [6]; 4.10, [11]). *A function $f \in H$ is in $\bigcap\{M_H^p : p \in \beta\mathbb{N} \setminus \mathbb{N}\}$ iff $\bar{f}^\beta < 1$ for every $p \in \beta\mathbb{N} \setminus \mathbb{N}$.*

Proof. Suppose there exists $p \in \beta\mathbb{N}\setminus\mathbb{N}$ such that $\bar{f}^\beta(p) = 1$. Since $\bar{f}^\beta(p) = 1$, 1 is a limit point of $\{\bar{f}(n)\}_{n\in\mathbb{N}}$. For each $k \in \mathbb{N}$, choose n_k in $\mathbb{N}$ such that $|\bar{f}(n_k) - 1| \leq \frac{1}{2^k}$, $n_k \neq n_j$ for $k \neq j$.

Let $D = \{n_k : k = 1, 2, ...\}$. Define

$$g(n) = \begin{cases} f(n) & n \in D \\ 1 & n \notin D. \end{cases}$$

Now $f(n) \neq 0$ for $n \in D$, for if not, then

$$|\bar{f}(n_k) - 1| = |0 - 1| = 1 \nleq \frac{1}{2^k} \quad \text{for all } k.$$

So, $Z(g) \neq \phi$ and $|g| \leq |f| \vee 1$.

Since every algebra is a lattice, $|f| \vee 1 \in H$.

$$\text{Now, } \underset{n\to\infty}{Lt.} \sup \bar{g}(n) \leq \underset{n\to\infty}{Lt.} \sup f \,\bar{\vee}\, 1 \leq 1, \quad \text{implies that } g \in H.$$

For each $k \in \mathbb{N}$, let $D_k = \{k, k+1, ...\}$. Let $q \in \mathrm{cl}_{\beta\mathbb{N}}(\mathbb{N} \setminus D)$, then $\bar{g}^\beta(q) = 1$, as $\bar{g}$ is continuous.

If $q \in \mathrm{cl}_{\beta\mathbb{N}} D$, then $\bar{g}^\beta(q) \in \mathrm{cl}_{\mathbb{R}}\, \bar{g}(D_k) = \mathrm{cl}_{\mathbb{R}}\, \bar{f}(D_k) \subseteq [1 - \frac{1}{2^k}, 1 + \frac{1}{2^k}]$ for all k

This implies $\bar{g}^\beta(q) = 1$. By Proposition [2.7.32], g is a unit.

Now, $E^g(f) = \{n \in \mathbb{N} : |f(x)| \geq |g(x)|\}$is non-compact for $D \subset E^g(f)$

Further, D is infinite but doesn't have a limit point in $E^g(f)$. Thus by Theorem [2.7.21] and Remark [2.7.22] $f \notin \bigcap\{M_H^p : p \in \beta\mathbb{N} \setminus \mathbb{N}\}$.

Conversely, suppose $f \in H$ such that $\bar{f}^{\beta} < 1$, $p \in \beta\mathbb{N} \setminus \mathbb{N}$. We claim that $f^{\beta}(p) = 0$, for, there exists $\delta < 1$ and a neighbourhood V of p in $\beta\mathbb{N}$ such that $\bar{f}(n) < \delta$, whenever $n \in V \cap \mathbb{N}$,

$$\text{i.e.,} \quad \sqrt[n]{|f(n)|} \leq \delta \quad \Rightarrow |f| \leq \delta^{n}.$$

Let U be a neighbourhood of p in $\beta\mathbb{N}$, then there exists $n \in U \cap V \cap \mathbb{N}$, such that n is arbitrarily large, i.e.,, $|f(n)|$ is arbitrarily small.
So, $f < 1$ and $\underset{n \to \infty}{Lt.} f(n) = 0$.
So, $f^{\beta}(p) = 0 \Rightarrow f \in M^{p}$ for all $p \in \beta\mathbb{N} \setminus \mathbb{N}$
$\Rightarrow f \in \bigcap\{M_{H}^{p} : p \in \beta\mathbb{N} \setminus \mathbb{N}\}$. □

We shall now end this section by recording the counter example given by Dominguez and Perez [11].

Counterexample 2.7.34. Let $H = \{f \in C(\mathbb{N}) | \underset{n \to \infty}{Lt.} \sup \sqrt[n]{|f(n)|} \leq 1\}$. Clearly, $\mathbb{N}$ is H-compact, for let $g \in C(\mathbb{N})$ be defined by $g(n) = n$, then $\underset{n \to \infty}{Lt.}\, sup \sqrt[n]{n} = 1 \Rightarrow g \in H$ and $\mathbb{N} = v_{g}\mathbb{N}$ by [18].
On the other hand, let $f(n) = \frac{n}{2^{n}}$, then

$$\underset{n \to \infty}{Lt.} \sup \sqrt[n]{|f(n)|} = \underset{n \to \infty}{Lt.} \sup \sqrt[n]{\frac{n}{2^{n}}} = \frac{1}{2} \underset{n \to \infty}{Lt.} \sup \sqrt[n]{n} = \frac{1}{2}.1 = \frac{1}{2} < 1$$

implies that $f \in \cap\{M_{H}^{p} : p \in \beta\mathbb{N} \setminus \mathbb{N}\}$. But $Z(f) = \phi$, so $\mathrm{cl}_{\mathbb{N}}(\mathbb{N} \setminus Z(f)) = \mathrm{cl}_{\mathbb{N}} \mathbb{N} = \mathbb{N}$, which is non-compact. Hence $f \notin C_{K}(\mathbb{N})$. This contradicts Theorem [2.6.14], in [7], where they have stated that
" X is A-compact iff $C_{K}(X) = \cap\{M_{A}^{p} : p \in \beta X \setminus X\}$".

Note 2.7.35. This error was also pointed out by Acharya, Chattopdhyay and Ghosh [3.4, [1]].

Chapter 3

A survey of homomorphisms

This chapter is a survey of finite homomorphisms and epimorphisms in the rings of continuous functions. The first section deals with results on finite homomorphisms due to Dominguez and Mulero [10], where they have shown that the study becomes simpler if it satisfies finiteness property. The remaining section deals with ring epimorphisms in the category of commutative rings (**CR**), where we shall record some results due to Barr, Burgess and Raphael in [4].

3.1 Finite homomorphisms on rings of continuous functions

It is well known that every continuous function $\pi : X \longrightarrow Y$ induces a ring homomorphism $C(Y) \longrightarrow C(X)$ by $f \longrightarrow f \circ \pi$ [18].

3.1.1 Finite spaces

In this subsection we study the case in which the space Y is finite. All our rings are commutative with unity.

Definition 3.1.1. Let $h : A \longrightarrow B$ be a ring homomorphism and consider B with the induced structure of A-module ($a \cdot b = h(a) \cdot b$). Then B is said to be an *A-algebra.*

Example 3.1.2. (i) Every ring is an algebra over itself.

(ii) Let $A = B = \mathbb{R}$ and $h = i$, the identity map, then $\mathbb{R}$ is an $\mathbb{R}$-algebra.

(iii) 0 is an algebra over any ring.

(iv) Let $i : \mathbb{Q} \hookrightarrow \mathbb{R}$, then $\mathbb{R}$ is an $\mathbb{Q}$-algebra.

(v) Every ring is a $\mathbb{Z}$-algebra.

Definition 3.1.3. An A-module B is said to be *finitely generated* if there exist $b_1, b_2, ..., b_n \in B$ such that every element b of B can be written as $b = \sum\limits_{i=1}^{n} a_i b_i$, $a_i \in A$.

Example 3.1.4. R-module 0 is finitely generated for any ring R.

Definition 3.1.5. Let $h : A \rightarrow B$ be any ring homomorphism. Then h is said to be *finite* if B if finitely generated as A-module. We say B is a finite *A-algebra.*

Definition 3.1.6. Let B be an A-algebra. An element $b \in B$ is *integral* over A, if there exists a monic polynomial $P(x) = x^n + a_{n-1}x^{n-1} + ... + a_0 \in A[x]$ such that $P(b) = 0$.

Example 3.1.7. (i) $\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ is integral over $\mathbb{Z}$.

(ii) 0 is integral over any A.

(iii) Every nilpotent element of B is integral over A.

Definition 3.1.8. A ring homomorphism $h : A \rightarrow B$ is *integral* and B is an *integral A-algebra*, if every element of B is integral over A.

Example 3.1.9. $B = 0$ is integral over any ring A.

Definition 3.1.10. The homomorphism h is of *finite type* and B is a *finitely generated A-algebra* if there exists a set of elements $b_1, b_2, ..., b_n \in B$ such that every element of B can be written as a polynomial in $b_1, b_2, ..., b_n$ with co-efficients in A, i.e., $B = A[b_1, b_2, ... b_n]$.

Remark 3.1.11. Every finite homomorphism is of finite type.

Definition 3.1.12. A homomorphism h is *singly generated* and B is a *singly generated A-algebra* if there exists $b \in B$ such that $B = A[b]$.

Example 3.1.13. The inclusion homomorphism $\iota : C^*(X) \hookrightarrow C^*(X)[f]$ is singly generated by $f \in C(X)$, where

$$C^*(X)[f] = \{\sum_{i=0}^{n} g_i f^i : g_i \in C^*(X),\ n = 0, 1, 2, ...\}$$

Remark 3.1.14. A ring homomorphism $\phi : A \longrightarrow B$ is finite iff it is of finite type and integral.

The following result relates the topological properties of X with the algebraic properties of $C(X)$.

Proposition 3.1.15. *The following conditions are equivalent for a completely regular (Hausdorff) space X.*

(i) *X is a finite space*

(ii) *C(X) is a finite* $\mathbb{R}$*-algebra.*

(iii) *C(X) is an integral* $\mathbb{R}$*-algebra*

(iv) *C(X) is a singly generated* $\mathbb{R}$*-algebra*

(v) *C(X) is a finitely generated* $\mathbb{R}$*-algebra*

Proof. $(i \Rightarrow ii)$ Let $X = \{x_1, x_2, ..., x_n\}$
We claim that

$$C(X) \cong C(x_1) \oplus C(x_2) \oplus ... \oplus C(x_n)$$

$$via\ \ f \xrightarrow{\Phi} (f(x_1), f(x_2), ..., f(x_n)).$$

Clearly, Φ is well defined. To show that it is a module-homomorphism, let $f, g \in C(X)$ and $r \in \mathbb{R}$,

$$\begin{aligned}\Phi(f+g) &= ((f+g)(x_1), (f+g)(x_2), ..., (f+g)(x_n)) \\ &= (f(x_1)+g(x_1), ..., f(x_n)+g(x_n)) \\ &= (f(x_1)+...+f(x_n)) + (g(x_1)+...+g(x_n)) \\ &= \Phi(f)+\Phi(g).\end{aligned}$$

Also, $\Phi(rf) = (rf(x_1), ..., rf(x_n)) = r(f(x_1), ..., f(x_n)) = r\Phi(f)$
and $\Phi(f) = 0 \Rightarrow ((f(x_1)+...+f(x_n))) = (0, ..., 0) \ \Rightarrow f(x_i) = 0 \ \forall i \ \Rightarrow f = 0.$

Hence Φ is an isomorphism, so

$$C(X) \cong C(x_1) \oplus C(x_2) \oplus ... \oplus C(x_n) \cong \mathbb{R} \oplus ... \oplus \mathbb{R} \cong \mathbb{R}^n.$$

But $\mathbb{R}^n$ is finitely generated $\mathbb{R}$-module with generators

$\{(0, 0, .., 1, .., 0) |$where 1 is in i^{th} place, for $1 \leq i \leq n$ and 0 elsewhere$\}$.

Hence *C(X)* is finite $\mathbb{R}$-algebra.

$(ii \Rightarrow iii)$ It follows from the fact that every finite ring homomorphism is integral.

$(iii \Rightarrow i)$ Suppose *C(X)* is integral, then $f \in C(X)$ implies

$$f_n + a_{n-1}f^{n-1} + \ldots + a_0 = 0 \text{ for some } n \in \mathbb{N},\ a_i \in \mathbb{R} \text{ for } 0 \leq i \leq n-1. \quad (3.1)$$

Since $f(x)$ satisfies (3.1) for each $x \in X$ and the equation in (3.1) has atmost n distinct roots, so $f[X]$ must be finite for all f. However, if X is infinite then by Remark [1.2.4] there exists a continuous function in $C(X)$ with infinite range, which is not possible, hence X must be finite.

$(i \Rightarrow iv)$ We claim that $C(X) = \mathbb{R}[f]$ for a function f that separates points in $X = \{x_1, x_2, \ldots, x_n\}$.
Let $\lambda_i = f(x_i)$ for $1 \leq i \leq n$, $f \in C(X)$.
Define

$$f_i = \frac{\prod\limits_{i \neq j} (f - \lambda_j)}{\prod\limits_{i \neq j} (\lambda_i - \lambda_j)}$$

then $f(x_i) = \lambda_i$ implies that

$$f_i(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i. \end{cases}$$

So, f_i separates x_i with $X \setminus \{x_i\}$, for $1 \leq i \leq n$, and $f_i \in \mathbb{R}[f]$, for f_i is a polynomial with real co-efficients. Let $g \in C(X)$. For $x \in X$, $x = x_k$ for some k, $1 \leq k \leq n$ and $g(x) = g(x_k)f_k(x_k)$

$$\text{So, } g = \sum_{i=1}^{n} g(x_k)f_k.$$

Hence, $g \in \mathbb{R}[f_i] \subseteq \mathbb{R}[f]$. So, $C(X) \subseteq \mathbb{R}[f]$ and $C(X) = \mathbb{R}[f]$.

$(iv \Rightarrow v)$ Trivial.

$(v \Rightarrow i)$ It is well known [3] that the number of minimal prime ideals of a finitely generated $\mathbb{R}$-algebra is finite. In *C(X)* each prime ideal is contained in a unique maximal ideal [Theorem 1.4.15]. Consider spec *C(X)* and order by inclusion. Let Σ be the collection of chains of prime ideals in spec *C(X)*. Since the number of minimal prime ideals is finite so there are only finitely many distinct chains. Hence every prime ideal in a particular chain is contained in the same maximal ideal. Thus, there are only finitely many maximal ideals. So, the number of fixed maximal ideals must be finite too. By Stone-Čech compactification[Theorem 1.6.2] we know that a fixed maximal ideal corresponds to a fixed z-ultrafilter, which in turn corresponds to a point of X, so X must be finite.

□

Corollary 3.1.16. *Let $\pi : X \longrightarrow Y$ be a continuous map between compact Hausdorff spaces. If the induced homomorphism $C(Y) \longrightarrow C(X)$ is finite (integral, singly generated or finitely generated), then each fibre $\pi^{-1}(y)$ is a finite set.*

Proof. Let $\pi : X \longrightarrow Y$ be a continuous map, then it induces a homomorphism $\Phi = f \circ \pi$ from $C(Y) \longrightarrow C(X)$. Suppose Φ is finite, then it would suffice to show that $C(\pi^{-1}(y))$ is a finitely generated algebra, then by Proposition [3.1.15] $\pi^{-1}(y)$ would be finite. We claim that $C(\pi^{-1}(y))$ is isomorphic to a quotient ring of $C(X)$. Define

$$\psi : C(X) \rightarrow C(\pi^{-1}(y))$$
$$\text{by } f \rightarrow f \circ i$$

where i is the inclusion map from $C(\pi^{-1}(y))$ to X. Clearly, ψ is a ring homomorphism. By Tietze's extension theorem every continuous function in $C(\pi^{-1}(y))$ has a continuous extension to X, hence ψ is onto. So by fundamental theorem of homomorphism $C(X)/I \cong C(\pi^{-1}(y))$, where I is the kernel of the homomorphism. Let Φ be the isomorphism. Now Φ is finite implies that $C(X)$ is finitely generated which implies that $C(X)/I$ is finitely generated and thus by Proposition [3.1.15] $C(\pi^{-1}(y))$ is a finitely generated algebra. Hence $\pi^{-1}(y)$ is a finite set. Similar argument holds if Φ is integral, singly generated or finitely generated. □

Corollary 3.1.17. *Let Y be a finite space and let $\pi : X \longrightarrow Y$ be a continuous map. The following condition are equivalent:*

(i) *X is a finite space.*

(ii) *$C(Y) \rightarrow C(X)$ is finite.*

(iii) *$C(Y) \rightarrow C(X)$ is integral.*

(iv) $C(Y) \to C(X)$ *is singly generated.*

(v) $C(Y) \to C(X)$ *is finitely generated*

Proof. If $C(Y) \to C(X)$ is finite, then by Corollary [3.1.16], each fibre $\pi^{-1}(y)$ is finite, but $X = \bigcup_{y \in Y} \pi^{-1}(y)$ which is a finite union of finite sets, so X must be finite so $(ii \Rightarrow i)$, Since a similar explanation holds for other cases ,so $(v \Rightarrow i)$, $(iv \Rightarrow i)$, $(iii \Rightarrow i)$.
Conversely, suppose that X is a finite space, then by Prop[3 1.15] $C(X)$ is a finitely generated $\mathbb{R}$-algebra. But $\mathbb{R} \subseteq C(Y)$ and $C(X)$ is finitely generated $\mathbb{R}$-algebra implies $C(X)$ is finitely generated as $C(Y)$-algebra.
So, $(i \Rightarrow v)$. Similarly, $(i \Rightarrow iv)$, $(i \Rightarrow iii)$ *and* $(i \Rightarrow ii)$. ☐

3.1.2 Extensions associated to compactifications

In this subsection we shall record more results on finiteness where spaces are taken to be compact, thereby recording some results originally proved by Faulkner in [14] to which Dominguez and Mulero in [10] gave simplified proofs.

Proposition 3.1.18. *Let $\pi : X \longrightarrow Y$ be continuous map between compact Hausdorff spaces If $F \subset Y$ is a closed subset such that $\pi : X \setminus \pi^{-1}(F) \to Y \setminus F$ is injective, then $C(X)$ is a finite (resp integral, singly generated, finitely generated) $C(Y)$-algebra if and only if $C(\pi^{-1}[F])$ is a finite (resp. integral, singly generated, finitely generated) $C(F)-$algebra.*

Proof Since X and Y are compact Hausdorff spaces so they are normal. Proceeding in a similar manner as in Corollary [3.1.16], it follows that

$$C(F) \cong C(Y)/I_F, \text{ where } I_F = \{f \in C(Y) | f[F] = \{0\}\}$$

and

$$C(\pi^{-1}[F]) \cong C(X)/I_{\pi^{-1}[F]} \text{ where } I_{\pi^{-1}[F]} = \{g \in C(X) | g[\pi^{-1}[F]] = \{0\}\}$$

We claim that g is constant on the fibre $\pi^{-1}(y)$ for each $y \in Y$ and for each $g \in I_{\pi^{-1}[F]}$.
Suppose $y \in F$, then $\pi^{-1}(y) \subseteq \pi^{-1}[F] \Rightarrow g[\pi^{-1}(y)] \subseteq g[\pi^{-1}[F]] = \{0\}$,
and if $y \in F^c$, then cardinality of $\pi^{-1}(y)$ is atmost one, so $g[\pi^{-1}(y)] = a$, for some $a \in \mathbb{R}$. Thus g is constant on the fibres so that

$$I_{\pi^{-1}[F]} \cong I_F \subseteq C(Y) \tag{3.2}$$

Now we claim that the homomorphisms $C(Y) \to C(X)$ and

$$C(F) = C(Y)/I_F \to C(\pi^{-1}[F]) = C(X)/I_F$$

have the same finiteness properties Suppose $h: C(Y) \to C(X)$ is finite, then $C(X)$ is finitely generated $C(Y)$ module We claim that $C(X)/I_{\pi^{-1}[F]}$ is finitely generated $C(Y)/I_F$ module
Let $\bar{g} \in C(X)/I_{\pi^{-1}[F]}$, then $\bar{g} = g + I_{\pi^{-1}[F]}$ for some $g \in C(X)$ Since $C(X)$ is finitely generated $C(Y)$ module so there exists $g_1, g_2, \cdots, g_n \in C(X)$ such that

$$g = f_1 g_1 + f_2 g_2 + \cdots + f_n g_n \quad \text{for} \quad \text{some } f_i \in C(Y) \ , 1 \leq i \leq n$$

so

$$\begin{aligned}
\bar{g} &= f_1 g_1 + f_2 g_2 + \cdots + f_n g_n + I_{\pi^{-1}[F]} \\
&= (f_1 g_1) + I_{\pi^{-1}[F]} + \cdots + (f_n g_n) + I_{\pi^{-1}[F]} \\
&= (f_1 + I_{\pi^{-1}[F]})(g_1 + I_{\pi^{-1}[F]}) + \cdots + (f_n + I_{\pi^{-1}[F]})(g_n + I_{\pi^{-1}[F]}) \\
\text{By equation(3 2) } \bar{g} &= (f_1 + I_F)(g_1 + I_{\pi^{-1}[F]}) + \cdots + (f_n + I_F)(g_n + I_{\pi^{-1}[F]}) \\
&= \bar{f}_1 \bar{g}_1 + \cdots + \bar{f}_n \bar{g}_n,
\end{aligned}$$

where $\bar{f}_i = f_i + I_F$ and $\bar{g}_i = g_i + I_{\pi^{-1}[F]}$ for $1 \leq i \leq n$ Since $\bar{g}$ was chosen arbitrarily so $\{g_1, g_2, \cdots, g_n\}$ generates $C(X)/I_{\pi^{-1}[F]}$ Hence, the result follows □

Corollary 3.1.19. *Let $\pi: X \to Y$ be a continuous map between compact Hausdorff spaces If the set $F = \{y \in Y | \pi^{-1}(y) > 1\}$ is finite, then the following conditions are equivalent*

(i) *$\pi^{-1}[F]$ is finite*

(ii) *$C(Y) \to C(X)$ is finite*

(iii) *$C(Y) \to C(X)$ is integral*

(iv) *$C(Y) \to C(X)$ is singly generated*

(v) *$C(Y) \to C(X)$ is finitely generated*

Proof Since F is finite, so it is closed in a compact Hausdorff space Y Let $\Psi = \pi|_{\pi^{-1}[F]}$, then Ψ is continuous map from $\pi^{-1}[F] \to F$ By Corollary [3 1 16] $C(F) \to C(\pi^{-1}[F])$ is finite (resp finitely generated, integral, singly generated) iff $\pi^{-1}[F]$ is finite By Proposition [3 1 18] $C(F) \to C(\pi^{-1}[F])$

have same finiteness properties if $\pi : X \setminus \pi^{-1}[F] \to Y \setminus F$ is injective.
We claim that π is injective.
Suppose

$$\pi(a) = \pi(b) \quad \text{for } a, b \in X \setminus \pi^{-1}[F].$$

So,

$$\pi(a), \pi(b) \notin F \text{ and } F^c = \{y \in Y : |\pi^{-1}(y)| \leq 1\}$$

implies that, $|\pi^{-1}(\pi(a))| \leq 1$ and $|\pi^{-1}(\pi(b))| \leq 1$.
Since $\pi(a) = \pi(b)$, so $a, b \in \pi^{-1}(\pi(a)) \cap \pi^{-1}(\pi(b))$.
But $|\Pi^{-1}\pi(a)) \cap \pi^{-1}(\pi(b))| \leq 1$ implies $a = b$. Hence, π is injective on $X \setminus \pi^{-1}[F]$ and the result follows by Corollary [3.1.17].

☐

In the following result we shall see that in order to study finiteness properties between the associated rings of continuous functions, the problem for compactification of a given locally compact Hausdorff space T is equivalent to the problem for arbitrary compact Hausdorff spaces. It is a well known result in [8] that

Theorem 3.1.20 ([8]). *If $\pi_{\gamma_\alpha} : \gamma T \to \alpha T$ is the projection map between two Hausdorff compactifications $\alpha T \leq \gamma T$ of T, then π_{γ_α} carries $\gamma T \setminus T$ onto $\alpha T \setminus T$, and so it induces an injective ring homomorphism $C(\alpha T \setminus T) \to C(\gamma T \setminus T)$.*

Corollary 3.1.21. *Let T be a locally compact, non-compact Hausdorff space and let $\alpha T \leq \gamma T$ be two Hausdorff compactifications of T. Then $C(\gamma T)$ is a finite (respectively integral, singly generated, finitely generated) $C(\alpha T)$ algebra if and only if $C(\gamma T \setminus T)$ is a finite (respectively, integral, singly generated,finitely generated) $C(\alpha T \setminus T)$-algebra*

Proof. Let $\pi_{\gamma_\alpha} : \gamma T \to \alpha T$, the projection map, then the result follows from Proposition [3.1.18] by replacing π with π_{γ_α}, X with γT, Y with αT and F with $\alpha T \setminus T$, then $\pi^{-1}[F] = \gamma T \setminus T$ ☐

We shall also quote a theorem due to Magill [7.2, [22]].

Theorem 3.1.22. *Suppose X is locally compact and K is Hausdorff. Then there exists a compactification $(\alpha X, h)$ of X such that $\alpha X \setminus h(X)$ is homeomorphic to K if and only if K is a continuous image of $\beta X \setminus X$.*

We shall adapt Theorem [3.1.22] and [2.1, [8]] to our needs.

Theorem 3.1.23. *Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. There exists a locally compact Hausdorff space T and Hausdorff compactifications $\alpha T \leq \gamma T$ such that $X = \gamma T \setminus T$, $Y = \alpha T \setminus T$ and π is the projection map between $\gamma T \setminus T$ and $\alpha T \setminus T$.*

Proof. From Theorem [4.17,[8]], it follows that there exists a locally compact Hausdorff space T and a homeomorphism $h_1 : \beta T \setminus T \to X$. So the composition $\pi \circ h_1$ is a surjective continuous map between $\beta T \setminus T$ and Y. By [[8],7.2]there exists a compactification αT of T and a homeomorphism $h_2 : \alpha T \setminus T \to Y$.

Further, it follows from [[8], 7.2] that the following diagram commutes.

$$\begin{tikzcd}
\beta T \setminus T \arrow[r, "h_1"] \arrow[d, "\pi_{\beta\alpha}"'] & X \arrow[d, "\pi"] \\
\alpha T \setminus T \arrow[r, "h_2"] & A_S
\end{tikzcd}$$

Hence, π is the projection map between $\gamma T \setminus T$ and $\alpha T \setminus T$. □

Remark 3.1.24. In order to study the finiteness properties of the homomorphism $C(Y) \to C(X)$ given by a continuous map $\pi : X \to Y$ between compact Hausdorff spaces, we can assume that π is a surjective map.

Proof. Let

$$\Psi : C(Y) \to C(\pi(X))$$
$$\text{given by } f \mapsto f|_{\pi(X)}$$

Clearly, Ψ is a homomorphism. Since X is compact, $\pi(X)$ is compact in Y, and hence closed. So, by Tietze's extension theorem every map $g \in C(\pi(X))$ has a continuous extension to Y. So, Ψ is surjective. □

Remark 3.1.25. It follows from the above remark that, $C(Y) \to C(X)$ is finite iff $C(\pi(X)) \to C(X)$ is finite.

One of the salient feature of the work done by Dominguez and Mulero [10] is that they have simplified proof of Faulkner's Theorems on extension of continuous function rings associated to compactifications [3,4 and 5, [14]] as we shall see.

Corollary 3.1.26. *Let αT have a finite remainder $\alpha T \setminus T$ and $\alpha T \leq \gamma T$. Then the following conditions are equivalent:*

(i) *γT has a finite remainder.*

(ii) *$C(\alpha T) \to C(\gamma T)$ is finite.*

(iii) *$C(\alpha T) \to C(\gamma T)$ is integral.*

(iv) *$C(\alpha T) \to C(\gamma T)$ is singly generated.*

(v) *$C(\alpha T) \to C(\gamma T)$ is finitely generated*

Proof. The result follows from Corollary [3.1.17] by taking $X = \gamma T \setminus T$, $Y = \alpha T \setminus T$ and the projecton map $\pi : \gamma T \setminus T \to \alpha T \setminus T$. ☐

Corollary 3.1.27. *Let $\pi_{\gamma_\alpha} : \gamma T \to \alpha T$ be the projection map between two Hausdorff compactifications $\alpha T \leq \gamma T$ of T. If the set $\bigcup\{{\pi_{\gamma_\alpha}}^{-1}(p) : |{\pi_{\gamma_\alpha}}^{-1}(p)| > 1\}$ is finite then there exists $f \in C(\gamma T)$ such that $C(\gamma T) = C(\alpha T)[f]$.*

Proof. Let $F = \{p \in \alpha T : |\pi^{-1}(p)| > 1\}$. We claim that F is finite.
Now $\pi_{\gamma_\alpha}^{-1}[F] = \bigcup\limits_{p\in F} \pi_{\gamma_\alpha}^{-1}(p) = \bigcup\limits_{p\in F} \{\pi_{\gamma_\alpha}^{-1}(p) : |{\pi_{\gamma_\alpha}}^{-1}(p)| > 1\}$ is finite by hypothesis. But, $\pi_{\gamma_\alpha}({\pi_{\gamma_\alpha}}^{-1}[F]) = F$, as π_{γ_α} is the projection map. So, F is image of a finite set, hence finite.
If we let $X = \gamma T$, $Y = \alpha T$ and $\pi_{\gamma_\alpha} : \gamma T \to \alpha T$, which is a continuous map then the result follows from Corollary [3.1.19] ☐

Corollary 3.1.28. *Let $\pi_{\gamma_\alpha} : \gamma T \to \alpha T$ be the projection map between two compact Hausdorff compactifications $\alpha T \leq \gamma T$ of T. If $C(\gamma T) = C(\alpha T)[f]$, then the fibres of π_{γ_α} are finite.*

Proof. If we let $X = \gamma T$, $Y = \alpha T$, $\pi = \pi_{\gamma_\alpha}$, then $C(\gamma T) = C(\alpha T)[f]$ implies that the homomorphism $C(Y) \to C(X)$ is finite, hence the result follows from Corollary [3.1.16]. ☐

In [14] Faulkner had asked whether complete converse of Corollary [3.1.27] holds? The answer is in negative as shown by Dwornik-Orzechowska and Wach [12]

Remark 3.1.29. The converse of Corollary [3.1.27] is not true.

Proof. Let $T = \mathbb{R} \times I$, where $I = [0, 1]$;
$\omega\mathbb{R} = \mathbb{R} \cup \{\infty\}$, the one point compactification of $\mathbb{R}$
and $\gamma\mathbb{R} = \mathbb{R} \cup \{+\infty, -\infty\}$, the two point compactification of $\mathbb{R}$.
Then $\alpha T = I \times \omega\mathbb{R}$ and $\gamma T = I \times \omega\mathbb{R}$ are the Hausdorff compactifications of T. Also $\alpha T \leq \gamma T$ for $(a, -\infty) \in \gamma T \setminus \alpha T$ $\forall a \in I$; $\alpha T \setminus T = I \times \{\infty\}$ and

$\gamma T \setminus T = I \times \{+\infty, -\infty\}$. So $C(\gamma T \setminus T) = C(\alpha T \setminus T) \oplus C(\alpha T \setminus T)$.
Since $\alpha T \setminus T$ is compact Hausdorff space, so it is normal and for disjoint closed sets $\{+\infty\} \times I$ and $\{-\infty\} \times I$, there exists $f \in C(\gamma T \setminus T)$ such that

$$f[\{+\infty\} \times I] = \{0\} \quad \text{and} \quad f[\{-\infty\} \times I] = \{1\}.$$

Then $C(\gamma T \setminus T) = C(\alpha T \setminus T)[f]$ and $C(\gamma T) = C(\alpha T)[f]$.
But $\bigcup\{{\pi_{\gamma_\alpha}}^{-1}(p) : |{\pi_{\gamma_\alpha}}^{-1}(p)| > 1\} = I \times \{+\infty, -\infty\}$ which is not finite. □

3.1.3 Finite homomorphism

In this section we shall record one of the main results of Dominguez and Mulero in [10]. However, one way of the theorem was proved independently by Mulero in 1998 [5.6, [24]] and we shall just record, without proof, the statement of that proposition.

Proposition 3.1.30 (5.6, [24]). *Let $\pi : X \to Y$ be a continuous map between realcompact spaces. If the homomorphism $C(Y) \to C(X)$ is finite, then the continuous extension of π to the Stone-Čech compactifications $\beta\pi : \beta X \to \beta Y$ is a locally injective map.*

Theorem 3.1.31. *Let $\pi : X \to Y$ be a continuous map between compact Hausdorff spaces. The homomorphism $C(Y) \to C(X)$ is finite iff the map $\pi : X \to Y$ is locally injective.*

Proof. Let $\pi : X \to Y$ be locally injective. Then every point $x \in X$ has a cozero neighbourhood U such that π is injective on $\bar{U}$, the closure of U. This implies that $C(\bar{U}) \cong C(\pi(\bar{U}))$. By Tietze's extension theorem, the restriction homomorphism

$$C(Y) \to C(\pi(\bar{U})) \text{ is onto.}$$

i.e., $C(Y) \to C(\pi(\bar{U})) \cong C(\bar{U})$ is surjective.

Let $f \in C(X)$ then $f|_{\bar{U}} \in C(\bar{U})$. Since $C(\bar{U}) \cong C(\pi(\bar{U}))$ and $C(\pi(\bar{U}))$ has continuous extension to $C(Y)$, $\exists\, g \in C(Y)$such that

$$f|_{\bar{U}} = g|_{\pi(\bar{U})}. \tag{3.3}$$

Let $\mathcal{F} = \{U_x | x \in X\}$, where U_x is a cozero-set neighbourhood of x, then $\mathcal{F}$ covers X. So by compactness of X, there exists a finite subcover $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$ such that
$X \subseteq \bigcup_{i=1}^{n} U_{x_i}$ and for each 'i', $\exists\, g_i \in C(X)$ such that $U_{x_i} = \text{Coz}(g_i)$ $1 \le i \le n$.

Since $Z(g_i{}^2) = Z(g_i)$, $\mathrm{Coz}(g_i{}^2) = \mathrm{Coz}(g_i)$. So without loss of generality, let $g_i \geq 0$.

Now let $h_i = \dfrac{g_i}{\sum_{i=1}^{n} g_i}$. We claim that $\{h_i\}i \overset{n}{=} 1$ generates $C(X)$ as $C(Y)$ module.

Let $f \in C(X)$ then by eqn (3.3), $\exists\ k_1, k_2, ...k_n \in C(Y)$ such that

$$\begin{aligned}
f|_{U_{n_i}} &= k_i|_{\pi(U_{x_i})} \\
\Rightarrow f &= k_i \quad in\ U_{x_i} \\
\Rightarrow g_i f &= g_i k_i \\
\Rightarrow \sum_{i=1}^{n} k_i g_i &= \sum_{i=1}^{n} f g_i = f \sum_{i=1}^{n} g_i \\
\Rightarrow f &= \frac{\sum_{i=1}^{n} k_i g_i}{\sum_{i=1}^{n} g_i} = \frac{k_1 g_1 + k_2 g_2 + ... + k_n g_n}{\sum_{i=1}^{n} g_i} \\
&= k_1 \frac{g_1}{\sum_{i=1}^{n} g_i} + k_2 \frac{g_2}{\sum_{i=1}^{n} g_i} + ... + k_n \frac{g_n}{\sum_{i=1}^{n} g_i} \\
&= k_1 h_1 + k_2 h_2 + ... + k_n h_n
\end{aligned}$$

Thus $C(X)$ is finitely generated $C(Y)$ module. Hence the homomorphism $C(Y) \rightarrow C(X)$ is finite.

Conversely, by Proposition [3.1.30], $\beta\pi : \beta X \rightarrow \beta Y$ is locally injective. Hence, $\beta\pi|_X = \pi$ is locally injective.

□

Proposition 3.1.32. *Let $\pi : X \rightarrow Y$ be a continuous map between realcompact spaces. If the homomorphism $C(Y) \rightarrow C(X)$ is finite, then π is a closed map and the space X can be covered by a finite number of cozero-sets, $X = \mathrm{Coz}(g_1) \cup ... \cup \mathrm{Coz}(g_n)$, such that π is injective on each closure $\overline{\mathrm{Coz}}(g_i)$. Consequently, $|\pi^{-1}(y)| \leq n$ for every $y \in Y$.*

Proof. Let $\Phi : C(Y) \rightarrow C(X)$ be the induced homomorphism such that Φ is finite. By Proposition [3.1.30], the map $\beta\pi : \beta X \rightarrow \beta Y$ is locally injective. So, by the same construction as in Theorem [3.1.31], βX can be covered by finite number of cozero-sets say $\{\mathrm{Coz}(g_1{}^\beta), ..., \mathrm{Coz}(g_n{}^\beta)\}$.

Let

$$U_i = \mathrm{Coz}(g_i{}^\beta) \cap X = \mathrm{Coz}(g_i) \quad \text{for } 1 \leq i \leq n$$

Clearly,

$$\begin{aligned} X &\subseteq \bigcup_{i=1}^{n} \mathrm{Coz}({g_i}^{\beta}) \\ &= \bigcup_{i=1}^{n} \mathrm{Coz}({g_i}^{\beta}) \cap X \subseteq X \\ &\Rightarrow X \subset \bigcup_{i=1}^{n} U_i \end{aligned}$$

Now $\beta\pi$ is injective on $\mathrm{cl}_{\beta X}\, U_i$ implies that $\beta\pi|_X = \pi$ is injective on $\mathrm{cl}_{\beta X}\, U_i \cap X = \mathrm{cl}_X\, U_i$.
Let $y \in Y$, if $\pi^{-1}(y) \neq \phi$, then there exists $x \in X$ such that $\pi(x) = y \Rightarrow x \in \pi^{-1}(y)$.
Now $s \in X \Rightarrow x \in U_i$ for some $'i'$, where $1 \leq i \leq n$. Since π is locally injective on $\mathrm{cl}_X\, U_i$, $|\pi^{-1}(y) \cap U_i|$ is atmost one. This is true for each i, so $\pi^{-1}(y)$ can have atmost n elements, so $|\pi^{-1}(y)| \leq n$.
Now we shall show that π is a closed map.
Since $\beta\pi : \beta X \longrightarrow \beta Y$ is a continuous map between compact Hausdorff spaces, so it is a closed map.
Next we claim that $\beta\pi$ transforms $\beta X \setminus X$ into $\beta Y \setminus Y$.

To achieve this we shall describe βX and βY in terms of prime ideals in *C(X)*. Consider spec *C(X)* and endow with hull-kernel topology [1.5.3]. Clearly, Max *C(X)* is a subspace of spec *C(X)*. It is well known that Max *C(X)* is same as βX. Also in *C(X)*, every prime ideal is contained in a unique maximal ideal [Theorem 1.4.15]. Define

$$\begin{aligned} \gamma_X &: \mathrm{spec}\, C(X) \longrightarrow \mathrm{Max}\, C(X) \\ &\text{such that } \gamma_X(P) = M_P, \end{aligned}$$

where M_P is the unique maximal ideal containing the prime ideal P. Clearly, γ_X is well defined and to see the continuity of γ_X it is enough to show that base elements of Max *C(X)* is taken to base element of spec *C(X)* under ${\gamma_X}^{-1}$
Let $\mathcal{M}$ be a base for closed sets in Max *C(X)* and let $M(f) \in \mathcal{M}$ for some $f \in C(X)$, where

$$M(f) = \{M \in \mathrm{Max}\, C(X) | f \in M\}$$

then

$$\gamma_X{}^{-1}[M(f)] = \{P \in \operatorname{spec} C(X) | P \subset M \text{ for each } M \in M(f)\} \ (**)$$

Since

$$\gamma_X[M] = M,$$

it is a continuous retraction. The map between the prime spectra

$$\Phi : \operatorname{spec} C(X) \to \operatorname{spec} C(Y)$$

that sends each prime ideal P in $C(X)$ to the prime ideal $\phi^{-1}(P) = \{f \in C(Y) : f \circ \pi \in P\}$ is also a continuous map. Clearly, the restriction of this map to βX, composed with the continuous retraction

$$\gamma_Y : \operatorname{spec} C(Y) \to \operatorname{Max} C(Y) = \beta Y,$$

is $\beta\pi : \beta X \to \beta Y$.

Given $p \in \beta X$, let M_p be the corresponding maximal ideal in $C(X)$[18]. Let $q = \beta\pi(p)$, then by the following identification the maximal ideal corresponding to q is $\gamma_Y(\phi^{-1}(M_p))$.

$$\begin{tikzcd}
\beta X \arrow[r] & \operatorname{Max} C(X) \arrow[r] & \operatorname{spec} C(X) \arrow[r] & \operatorname{spec} C(Y) \arrow[r] & \operatorname{Max} C(Y) \\
p \arrow[r] & M_p \arrow[r, hook] & M_p \arrow[r] & \phi^{-1}M_p \arrow[r] & \gamma_Y\phi^{-1}M_p = M_q
\end{tikzcd}$$

Let $M_q = \gamma_Y(\phi^{-1}(M_p))$. Since ϕ is finite, $\phi^{-1}(M_p)$ is maximal ideal in $C(Y)$ implies that $\phi^{-1}(M_p) = M_q$, i.e.,

$$M_q = \gamma_Y(\phi^{-1}(M_p)) = \phi^{-1}(M_p)$$

Let

$$\eta : C(Y)/M_q \to C(X)/M_p$$

given by

$$f + M_q \to \phi(f) + M_p.$$

By the same procedure as in Proposition [3.1.18] η is finite and $\phi^{-1}(M_p) = M_q$ implies that $\operatorname{Ker} \eta = \{0\}$, hence injective.
So, $C(X)/M_p$ is an algebraic extension of $C(Y)/M_q$. Further, $C(X)/M_p$ is totally ordered [Theorem 5.5, [18]] and $C(Y)/M_q$ is real closed [Theorem 13.4, [18]], i.e., it has no proper algebraic extension to an ordered field, so $C(Y)/M_q \cong C(X)/M_p$.

This is same as saying $C(Y)/M_q$ is real iff $C(X)/M_p$ is real
i.e., $C(Y)/M_q = \mathbb{R}$ iff $C(X)/M_p = \mathbb{R}$
i.e., $q \in Y$ iff $p \in X$. Hence $\beta\pi[\beta X \setminus X] = [\beta Y \setminus Y]$.

□

The converse of this theorem is true for normal spaces as we shall see.

Theorem 3.1.33. *Let $\pi : X \longrightarrow Y$ be a continuous map between realcompact spaces and suppose that Y is a normal space. The homomorphism $C(Y) \longrightarrow C(X)$ is finite iff π is a closed map and the space X can be covered by a finite number of cozero-sets, $X = \mathrm{Coz}(g_1) \cup ... \cup \mathrm{Coz}(g_n)$, such that π is locally injective on each closure $\overline{\mathrm{Coz}(g_i)}$.*

Proof. Neccesity follows from Theorem [3.1.32].
Converse follows by noting in the proof of Theorem [3.1.31] that compactness of the spaces was used to

(i) find a finite subcover of cozero-sets.

(ii) $C(\bar{U}) \cong C(\pi(\bar{U}))$ and $C(Y) \longrightarrow C(\pi(\bar{U}))$ is surjective.

Since X has a finite cover of cozero sets and Y is normal so every map $f \in C(\pi(\bar{U}))$ has a Tietze extension as Y is normal. So, the argument holds here too. □

Many questions were asked as whether the converse holds for Proposition[3.1.30].The following example shows that converse of Proposition [3.1.30] is not true and Theorem [3.1.31] does not hold for non-compact spaces.

Example 3.1.34. Let $\sum = \mathbb{N} \cup \{p\}$, where $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Then $\sum$ is a realcompact and normal space and $\mathbb{N}$ is dense and C^*−embedded in $\sum$ so $\beta\mathbb{N} = \beta\sum$. The homomorphism $C(\sum) \longrightarrow C(\mathbb{N})$ induced by the inclusion map $\mathbb{N} \xrightarrow{\pi} \sum$ is not finite for if it is finite then by Proposition [3.1.32] $\pi : \mathbb{N} \hookrightarrow \sum$ must be a closed map but $\mathbb{N}$ is not closed in $\sum$. A contradiction.

Example 3.1.35. A finite homomorphism $C(Y) \longrightarrow C(X)$ that is not singly generated. Let $\pi : S^2 \longrightarrow P^2$ be the natural projection map from the unit sphere onto the real projective plane satisfying

$$\pi(x_1, x_2, x_3) = \pi(-x_1, -x_2, -x_3).$$

Then π is locally injective, so the induced homomorphism $C(P^2) \longrightarrow C(S^2)$ is finite. However, it is not singly generated. Suppose there exists an $f \in$

$C(S^2)$ such that $C(S^2) = C(P^2)[f]$, then f must be injective on each fibre π, because $C(S^2)$ separates points in S^2. By Borsuk- Ulam's theorem [Theorem 1.2.5] for every $f \in C(S^2)$ $\exists p \in S^2$ such that $f(p) = f(-p)$. So, $C(S^2)$ is not singly generated over $C(P^2)$.

3.2 Ring epimorphisms and *C(X)*

In this section we study *C(X)* as an object in the category of all commutative rings (**CR**) and study the epimorphisms induced by function rings of subspaces of a topological space *X*. During 1960's a lot of mathematicians including Lazard, Mazet, Olivier and Storrer [20, 23, 26, 34] studied epimorphisms in various categories. The study was revived by N.Schwarz and Madden in [32] later in 1999. Here we shall record some of important works due to Barr, Burgess and Raphael [4] in the case of epimorphisms in **CR**.

We shall begin by recording the most fundamental theorem in this regard in [23] and [34].

Theorem 3.2.1 ([23], pg. 73 [34]). *A homomorphism $f : A \to B$ in **CR** is an epimorphism if and only if for each $b \in B$, there exists matrices C,D,E of sizes $1 \times n$, $n \times n$ and $n \times 1$ respectively, where*

(i) *C and E have entries in B,*

(ii) *D has entries in f(A)*

(iii) *the entries of CD and of DE are elements of f(A) and*

(iv) *b=CDE (such an expression is called an $n \times n$ zig-zag for b over A)*

Definition 3.2.2. A topological space X is said to be *perfectly normal* if every closed set is a zero set.

Example 3.2.3. Metric spaces and discrete spaces are perfectly normal spaces.

Remark 3.2.4. A perfectly normal space X is normal.

Proof. If A and B are disjoint closed sets in X then $A = Z(f)$ and $B = Z(g)$ for some $f, g \in C(X)$. Let $k = \frac{|f|}{|f|+|g|} \in C(X)$ such that $k[A] = \{0\}$, $k[B] = \{1\}$ and $0 \leq k \leq 1$. Clearly, $k^{-1}(\frac{-1}{2}, \frac{1}{2})$ and $k^{-1}(\frac{1}{2}, \frac{3}{4})$ are the disjoint open sets containing A and B respectively. ☐

Proposition 3.2.5. *Let Y be a subspace of a topological space X and $\rho : C(X) \to C(Y)$, the restriction homomorphism.*

(i) *If Y is C^*-embedded in X , then ρ is an epimorphism in **CR**.*

(ii) *If each $f \in C^*(Y)$ extends to a continuous function on a cozero-set of X, then ρ is an epmorphism in **CR**. This occurs in particular if Y is a cozero-set of X.*

(iii) *If for each $f \in C^*(Y)$, there are $g, h \in C(X)$ with $Y \subseteq \operatorname{Coz}(g)$ and $f\rho(g) = \rho(h)$ then ρ is an epimorphism.*

Proof. By Theorem [3.2.1] it is enough to construct a zig-zag over $C(X)$ for each $f \in C(Y)$.

(i) Let $f_1 = \frac{f}{1+f^2}$, $f_2 = \frac{1}{1+f^2}$. Clearly $f_1, f_2 \in C^*(Y)$ and $f_2 \neq 0$ implies ${f_2}^{-1}$ exists. Since Y is $C^*(X)$-embedded in X so there exists $g_1, g_2 \in C^*(X)$ such that $\rho(g_1) = f_1$, $\rho(g_2) = f_2$.
But

$$f = {f_2}^{-1} g_1 g_2 {f_2}^{-1} \text{ on Y}$$

for

$$\begin{aligned}({f_2}^{-1} g_1 g_2 {f_2}^{-1})(y) &= {f_2}^{-1}(y) g_1(y) g_2(y) {f_2}^{-1}(y) \\ &= (1 + f^2(y)) \frac{f(y)}{1+f^2(y)} 1 = f(y) \text{ for all } y \in Y.\end{aligned}$$

If we let

$$C = {f_2}^{-1},\ D = g_1 g_2,\ E = {f_2}^{-1},\ \text{then} f = CDE \qquad (1 \times 1\text{zig-zag})$$

which is the required 1 × 1 zig-zag.

(ii) If $f \in C^*(Y)$, then by Remark [1.1.22] there exists $g \in C^*(X)$ such that f extends continuously to a cozero-set of g, $\operatorname{Coz}(g)$ containing Y. Now $fg \in C^*(Y)$, so it extends to a continuous function $h \in C^*(X)$ on a cozero-set of X. On Y we have

$$g^{-1} g h g^{-1}(y) = g^{-1}(y) g(y) h(y) g^{-1}(y) = f(y)$$

$$\Rightarrow g^{-1} g h g^{-1} = f.$$

Now we put $f_1 = \frac{f}{1+f^2}$ and $f_2 = \frac{1}{1+f^2}$, then we have zig-zags $f_i = (g_i)^{-1} g_i h_i (g_i)^{-1}$ for $i = 1, 2$ on Y, where $h_i = f_i g_i$ on Y, and $h_2 = f_2 g_2 \neq 0$ on Y.

Thus

$$\begin{aligned} f &= ((g_1)^{-1} g_1 h_1 (g_1)^{-1})((g_2)^{-1} g_2 h_2 (g_2)^{-1})^{-1} \\ &= ({g_1}^{-1} g_1 h_1 {g_1}^{-1})(g_2 {g_2}^{-1} {h_2}^{-1} g_2) \\ &= ({g_1}^{-1} {h_2}^{-1})(g_1 g_2 h_1 h_2)({g_1}^{-1} {h_2}^{-1}) \quad \text{on } Y \end{aligned}$$

if we take $C = {g_1}^{-1} {h_2}^{-1}$, $D = g_1 g_2 h_1 h_2$, $E = {g_1}^{-1} {h_2}^{-1}$

then this is the required 1×1 zig-zag over $C(X)$.

(iii) Let $f \in C^*(Y)$. Now $Y \subseteq \mathrm{Coz}(g)$ and $\rho(g) = g|_Y$ implies that $\rho(g) \neq 0$ on Y. So $\rho(g)^{-1} \in C(Y)$ and

$$\rho(g)^{-1} \rho(g) \rho(h) \rho(g)^{-1} = \rho(g)^{-1} \rho(g) f \rho(g) \rho(g)^{-1} = f.$$

Hence the result follows by taking

$$C = \rho(g)^{-1}, \ D = \rho(g) f \rho(g), \ E = \rho(g)^{-1}.$$

□

Lemma 3.2.6. *Suppose that X is a topological space and Y a dense subspace. Let $f \in C(Y)$ and $x \in X \setminus Y$. Assume that $f = GAH$ is a zig-zag for f over $C(Y \cup \{x\})$ (as in Theorem[3.2.1]). Suppose, moreover, that for some neighbourhood U of x the entries of G and H are bounded in $U \cap Y$. Then f can be extended continuously to x.*

Proof. We have $f = GAH$ for each $f \in C(Y \cup \{x\})$. Let $\widehat{A} = A(x)$ denote the constant matrix and $\widehat{f} = G\widehat{A}H$. Let U be a neighbourhood of x such that the entries of G and H are bounded in $U \cap Y$. Let $\|G\| \leq M_1$ and $\|H\| \leq M_2$ for some $M_1, M_2 \in \mathbb{R}$, . Let $M = \max\{M_1, M_2\}$.
So in U

$$|f - \widehat{f}| = \|GAH - G\widehat{A}H\| \leq \|G\| \ \|A - \widehat{A}\| \ \|H\| \leq M^2 \|A - \widehat{A}\|$$

$$\text{Further, } \lim_{y \to x} A(y) = \widehat{A} \ \Rightarrow \lim_{y \to x} |f(y) - \widehat{f}(y)| = 0. \qquad (3.4)$$

Also in $U \cap Y$,

$$\|GA - G\widehat{A}\| \leq \|G\| \|A - \widehat{A}\| \leq M \|A - \widehat{A}\|,$$

this implies that

$$\lim_{y \to x} G(y)(A(y) - G(y))\widehat{A} = 0 \quad \Rightarrow \lim_{y \to x} G(y)\widehat{A} \ = \lim_{y \to x} G(y)A(y).$$

Similarly,

$$\lim_{y \longrightarrow x} \widehat{A}H(y) \quad = \lim_{y \longrightarrow x} A(y)H(y).$$

Since $\widehat{A}$ is constant there is a matrix B such that $\widehat{A}B\widehat{A}$ as the ring of matrices over $\mathbb{R}$ is regular, this implies that $\widehat{f} = G\widehat{A}B\widehat{A}H$ is defined at x, so is f. $\square$

The above lemma gives us an information about when $\rho^* : C^*(X) \longrightarrow C^*(Y)$, the restriction of ρ to $C^*(X)$ is an epimorphism.

Theorem 3.2.7. *Suppose that Y is a proper dense subset of X. Then $\rho^* : C^*(X) \longrightarrow C^*(Y)$ is an epimorphism in **CR** if and only if it is surjective, that is, if and only if Y is C^*-embedded in X. If, moreover, X is normal then the conclusion is valid even when Y is not dense.*

Proof. If ρ^* is surjective then it is an epimorphism trivially.

Conversely, Suppose ρ^* is an epimorphism. Let $f \in C^*(X)$, then there is a zig-zag $f = GAH$ over $C^*(X)$ as in Lemma [3.2.6]. Since G and H have entries in $C^*(Y)$, they are neccessarily bounded, so by Lemma [3.2.6], they can be extended to a function g over any point of $X \setminus Y$, without loss of generality, take $g \in C^*(X)$ [Remark 1.1.22]. Hence $\rho(g) = f$, thus ρ is a surjection.

If Y is C^*-embedded in X, then the result holds trivially. On the other hand, if X is normal and ρ^* is an epimorphism then for any $f \in C^*(X)$, we can construct a zig-zag for f over $C^*(X)$. If we allow the entries of zig-zag of f to be in $C^*(\mathrm{cl}(Y))$, then the zig-zag is over $C^*(\mathrm{cl}(Y))$. Thus every function f can be extended to $\widehat{f} \in C^*(\mathrm{cl}(Y))$. By Urysohn's extension theorem [Theorem 1.1.23], $\mathrm{cl}(Y)$ is C^*-embedded in X, as X is normal. Hence, ρ is surjective. $\square$

Corollary 3.2.8. *Suppose that Y is a pseudocompact proper dense subset of X. Then $\rho^* : C(X) \rightarrow C(Y)$ is an epimorphism in **CR** iff if it is surjective, i.e., iff Y is C-embedded. If, moreover, X is normal, then the conclusion is valid for pseudocompact Y even when not dense.*

Proof. If Y is pseudocompact then $C^*(Y) = C(Y)$, hence the result follows from Theorem [3.2.7]. $\square$

Definition 3.2.9. A space is called *almost compact* if $|\beta X \setminus X| \leq 1$.

Fine , Gillman and Lambek in [15] in the year 1965 proved an important result based on which a number of results on epimorphisms have been established.

Theorem 3.2.10. *Suppose that Y is a dense subset of X. Then for any $f \in C(Y)$, there is a largest subset of X to which f can be continuously extended.*

Remark 3.2.11. Since $\beta X = X$, every compact space is almost compact.

Notation 3.2.12. For any $f \in C(Y)$, we write dom(f) for the largest subspace of X to which f can be continuously extended.

Proposition 3.2.13. *Suppose that X is a topological space and Y is a dense subspace such that $\rho : C(X) \longrightarrow C(Y)$ is an epimorphism in* ***CR*** *then for $f \in C(Y)$,* dom(f) *contains an open set containing Y.*

Proof. It follows from Lemma [3.2.6] that there must exist G, A, H for which $f = GAH$. Suppose that $y \in X$. It follows from Lemma [3.2.6] that G and H are defined and continuous on Y. Also, there is an open neighbourhood U_y on which both G and H are bounded, such that for any $x \in U_y$ it is also a neighbourhood of x and by Lemma [3.2.6] f can be extended to x.
Let $U = \bigcup_{y \in Y} U_y$, then U is open and $Y \subset U \subset$ dom(f). Hence the result. □

Proposition 3.2.14. *In a space X, every first countable point is a zero-set.*

Proof. It follows from Remark [1.5.7] that in a completely regular space every G_δ-set U containing a compact set S contains a zero-set Z such that $S \subset Z \subset U$. In a first countable space every point is a G_δ set. Since finite sets are compact so there exists a zero-set Z' containing the compact set $\{x\}$ contained in $\{x\}$ implies that $Z' = \{x\}$. So every first countable point is a zero-set. □

Lemma 3.2.15. *Suppose that $t_1 > t_2 > t_3 > \dots$ is a sequence of numbers in the open unit interval that converges to 0. Then there is a continuous function $h : (0, \infty) \to [0, 1]$ such that $h(t) = 0$ for $t \geq t_1$, $h(t_n) = 0$ when n is odd and $h(t) = 1$ when n is even.*

Proof. The result follows by interpolating linearly between these values. □

Proposition 3.2.16. *The complement of a first countable non-isolated point is not C^*-embedded.*

Proof. Let x be a first countable non-isolated point. We claim that $Y = X - \{x\}$ is not C^*-embedded in X. On the contrary, suppose Y is C^*- embedded in X, then $\beta Y = \beta X$. By Theorem [1.6.5], and Corollary [1.6.6] $\{x\}$ is not a G_δ-set in βY, so it contradicts that x is a first countable point. □

Theorem 3.2.17. *If a dense subset Y of a first countable space X induces an epimorphism in* ***CR*** *then Y is open in X.*

Proof. We shall prove by contradiction. Suppose that $Y \subseteq X$ is not open, then $X \backslash Y$ is not closed, so there exists atleast one isolated point $y \in Y$. Since X is first countable there exists a sequence $\{x_n\}$ of points of $X \backslash Y$ converging to Y. By Proposition [3.2.14] each such point is a zero-set. So they can be regarded as distinct and forms a discrete subspace of X. By Proposition [3.2.16], we can choose for each $n \in \mathbb{N}$ a function $f_n : X \setminus \{x_n\} \longrightarrow [0,1]$ that cannot be extended to x_n. Then $f_n \in C(Y)$ for each n. Let $f = \sum 2^{-m} f_m$, clearly, the sequence converges uniformly so f is continuous implies that $f \in C(Y)$. However, f is not defined at each of the points x_n for all n. For each n, there is a neighbourhod V of x_n excluding the other points of the sequence. Clearly, $\sum 2^{-m} f_m$ extends continuously on V to x_n, for some $m \in \mathbb{N}$. We claim that f cannot be extended continuously, for suppose f can be extended continuously then $f - \sum_{m \neq n} 2^{-m} f_m$ can be extended too. But, this implies that f_n can be extended continuously to x_n, a contradiction to our choice of f_n.

Now let U be any open set containing Y, then U contains x_i for some i, but f is not defined at $x_\imath$, where i belongs to some index set J. This implies that f cannot have a continuous extension to any open set containing Y, hence Y is not C^*-embedded in X, so by Theorem [3.2.7] X cannot induce an epimorphism. This is a contradiction to the assumption that X induces an epimorphism. ☐

Definition 3.2.18. A subset U of X is said to be *locally closed* if it can be expressed as intersection of an open and a closed set of X.

Example 3.2.19. Every open (closed) subset of X is locally closed.

Corollary 3.2.20. *Let Y be a subspace of a first countable space X. If Y induces an epimorphism in* ***CR*** *then Y is locally closed in X.*

Proof. Let $\rho : C(\mathrm{cl}(Y)) \to C(Y)$ defined by $\rho(f) = f|_Y$, then ρ is clearly an onto morphism, hence it is an epimorphism. So by Theorem [3.2.17] Y is open in X, i.e., $Y = Y \cap X$, and X is open as well as closed implies that Y is locally closed in X. ☐

Corollary 3.2.21. *A subspace Y of a perfectly normal first countable space X induces an epimorphism in* ***CR*** *iff it is locally closed.*

Proof. Suppose Y is locally closed, then $Y = U \cap V$, where U is open and V is closed. So Y is perfectly normal implies that U^c is a zero-set, i.e., U is a cozero-set in X, hence also in V. Now define $\rho : C(V) \to C(Y)$ by $\rho(f) = f|_Y$. Hence ρ is an onto homomorphism by Tietze's extension theorem and thus an epimorphism. ☐

Remark 3.2.22. Let X be a topological space, then open sets can induce epimorphisms without being cozero-sets.

Proof. Let F be an extremally disconnected non-compact space and consider βF. Then βF is extremally disconnected compact Hausdorff space and hence normal. Since F is non-compact so there exists $a \in \beta F \backslash F$. Let $X = \beta F$ and $Y = \beta F \setminus \{a\}$. We know by Stone-Čech compactification [Theorem [1.6.2]] that a is not isolated in X. So $\text{cl}_X Y = X$. By [1.1.26] X is extremally disconnected iff every open subspace is C^*-embedded, so Y is C^* embedded in X.
Let $\rho : C(X) \to C(Y)$ be the restriction homomorphism. Clearly, ρ is onto hence an epimorphism. But Theorem [1.6.5] implies $\{a\}$ is not a zero-set in X, so Y is not a cozero-set. Hence the result follows. ☐

Proposition 3.2.23. (i) *Let Y be a subspace of a normal space X which has the following form : $Y = \bigcup_{n \in N} B_n$, where the B_n are closed sets of X and there is a family of pairwise disjoint open sets U_n from X with $B_n \subseteq U_n$, for each $n \in \mathbb{N}$. Then Y induces an epimorphism in **CR**.*

(ii) *If, under the hypothesis of (i), each B_n is compact then the conclusion of (i) holds for any X. This applies, in particular, if Y is homeomorphic to $\mathbb{N}$.*

(iii) *Let Y be a subspace of a space X which has the following form: $Y = \bigcup_{n \in N} B_n$ where there is a family of pairwise disjoint cozero-sets U_n, with $B_n \subseteq U_n$, for $n \in \mathbb{N}$. Suppose, moreover, that each B_n is C^*- embedded in U_n. Then $Y \subseteq X$ induces an epimorphism in **CR**.*

Proof. (i) Let $f \in C^*(Y)$ Let $M = \sup_Y |f(y)|$. For each $n \in \mathbb{N}$, $X \backslash U_n$ is closed and $B_n \bigcap X \backslash U_n = \phi$. By Urysohn's lemma there exists $h_n \in C(X)$ for each n, such that $h_n[B_n] = 1$ and $h_n[X \backslash U_n] = \{0\}$. Define $g \in C^*(Y)$ by $g(y) = (\frac{1}{n})f(y)$ for $y \in B_n$. Clearly, g is continuous. Let $g_n = \frac{g}{B_n}$, then X is normal and B_n is closed imply that by Tietze's extension theorem it has a continuous extension $l_n \in C(X)$.

Since $|f(y)| \leq M$, we may asume that $|f_n(x)| \leq \frac{M}{n} \quad \forall x \in X$. Now let
$a(x) = \begin{cases} h_n(x)l_n(x) & \text{if } x \in U_n \\ 0 & \text{if } x \in \bigcap_n (X \setminus U_n). \end{cases}$

and $b(x) = \begin{cases} (\frac{1}{n})h_n(x) & \text{if } x \in U_n \\ 0 & \text{if } x \in \bigcap_n (X \setminus U_n). \end{cases}$

then a is continuous , for let (c, d) be an open interval in $\mathbb{R}$,
so $a^{-1}(c, d) = U_N (h_n l_n)^{-1}(c, d)$ which is a countable union of open sets, hence open whenever $0 \notin (c, d)$.
If $0 \in [c, d]$, then the inverse image includes all but finitely many of the $U_n' s$. So its complement is a finite union of the sets of the form $(h_n l_n)^{-1}\{(-\infty, c] \bigcup [d, \infty)\}$ which is closed. Hence a is continuous and similarly b is continuous. Now $\rho(a) = a|_X$ and $\rho(b) = b|_X$, so $\rho(a)\rho(b)^{-1} = (h_n l_n)(n {h_n}^{-1})|_X = n l_n|_X = n.\frac{1}{n} f|_Y = f$. So the result follows by Proposition [3.2.5]

(ii) By Stone-Čech compactification [Theorem 1.6.2], X is C-embedded in βX. Since Y is a subspace of X so there exists open sets V_n for each n such that $U_n = X \cap V_n$ and $V_n' s$ are disjoint for each $n \neq m$. The compact sets B_n are closed in βX. However, βX is normal so by part (i), $C(\beta X) \to C(Y)$ is an epimorphism. Now the result is obtained by the following association.

$$C(\beta X) \longrightarrow C(X) \longrightarrow C(Y)$$

by

$$f^{\beta} \longrightarrow f^{\beta}|_X \longrightarrow f$$

(iii) Let $\varphi_n : X \longrightarrow [0, 1]$ and suppose that $U_n = \operatorname{Coz} \varphi_n$ for each n. If we let $g(y) = \frac{1}{n} f(y)\varphi_n(y)$, $y \in B_n$, then $f|_{B_n}$ extends to U_n so g extends to X. Also $g|_{U_n}$ extends to $l_n \in C(X)$ and if we take

$a(x) = \begin{cases} l_n(x) & x \in U_n \\ 0 & x \in \bigcap_n (X \setminus U_n). \end{cases}$

and $b(x) = \begin{cases} \frac{1}{n}\varphi_n(x) & x \in U_n \\ 0 & x \in \bigcap_n (X \setminus U_n). \end{cases}$

Then proceeding similarly as in (i), we get $f = \rho(a)g(b)^{-1}$. Hence the result follows.

□

Chapter 4

Localization in $C(X)$ and rings of quotients of $C(X)$

In this chapter we shall study localization in the ring of continuous functions and also study rings of quotients of rings of functions. Localization in rings of continuous functions has been studied by Blair and Hager in [5]. They gave a characterization of continuous functions in terms of localization. Later Requezo and Sancho in [29] carried further studies on this topic and settled the important question of obtaining compact open topology of $C(U)$ from the compact open topology of $C(X)$, whenever U is a cozero set of X. For the ring $C(X)$, its total ring of quotients, denoted by $T(C(X))$, is the localization of $C(X)$ at the multiplicative set of non-zero- divisors. Given a space X, one of the major problem is of determining, whether $T(C(X))$ is von-Neumann regular? It was shown by Levy and Shapiro in [21] that this question can be formulated topologically. In section 4.2.3, the general localization theory in the ring $C(X)$ is studied by defining Gabriel filter of ideals of $C(X)$ as certain filter of open sets of X. A generalization of [Theorem 2.6, [15]] has been recorded in which for certain Gabriel filters of ideals of $C(X)$, the ring of quotients is isomorphic to a ring of partial fractions on a subspace of X.

4.1 Localization in the rings of continuous functions

4.1.1 Preliminaries

Definition 4.1.1. A commutative ring A endowed with a topology τ is said to be a topological ring if the ring sum $+ : A \times A \longrightarrow A$ and the ring multiplication $\cdot : A \times A \longrightarrow A$ are continuous, and A has a continuous inverse, i.e., the map $a \mapsto a^{-1}$ is continuous. If A and B are topological rings, a ring morphism $A \to B$ is said to be a topological ring morphism if it is continuous. Finally, a topology τ on a ring A is said to be a ring topology if (A, τ) is a topological ring.

Example 4.1.2. The field $\mathbb{R}$ with usual topology is a topological ring.

Definition 4.1.3. Let X and Y be topological spaces. If C is a compact subspace of X and U is an open subset of Y, define

$$S(C,U) = \{f | f \in C(X,Y),\ \ f(C) \subseteq U\},$$

then the topology formed by taking $S(C,U)$ as a subbasis on $C(X,Y)$ is known as compact open topology.

Now we shall list two basic properties of topological rings.

Properties 4.1.4. (i) Let A be a ring and let $\{A_i : i \in I\}$ be a family of topological rings. If one has a ring morphism $f_i : A \to A_i$ for each index $i \in I$ then the coarsest topology on A such that the morphisms $\{f_i : i \in I\}$ are continuous is a ring topology.

(ii) If one has a ring morphism $f_i : A_i \to A$ for each index $i \in I$, then there exists on A the finest topology such that A is a topological ring and the morphisms $\{f_i : i \in I\}$ are continuous. This topology is said to be the final ring topology defined on A by the morphisms $\{f_i : i \in I\}$.

Example 4.1.5 (1.3, [29]). Let K be a compact topological space. Let $C(K)$ be the ring of all continuous functions from K to $\mathbb{K}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Consider the function $|| \ ||_K : C(K) \to \mathbb{R}$ defined by $||f||_K = \max\{|f(x)| : x \in K\}$, $f \in C(K)$, where $|\lambda|$ = absolute value of $\lambda \in \mathbb{K}$. It is well known that $C(\mathbb{K})$ endowed with the initial topology of the function $|| \ ||_{\mathbb{K}}$ is a topological ring [Theorem 1.4.8, [30]].

Now, let X be a topological space and let $\mathcal{K} = \{compact\ sets\ in\ X\}$ then consider the family of topological rings $\{(C(C), \|\ \|_C)\}_{C\in\mathcal{K}}$ and the family of restriction morphisms $\{r_C : C(X) \to C(C)\}_{C\in\mathcal{K}}$, the compact open topology in $C(X)$, denoted by τ_X is the coarsest topology on $C(X)$ such that the morphisms $\{r_C\}_{C\in\mathcal{K}}$ are continuous, i.e., τ_X is the initial topology of the functions $\{q_C = \|\ \|_C \circ r_C\}_{C\in\mathcal{K}}$.

Definition 4.1.6. Let A be a commutative ring. A *multiplicatively closed* subset S of A is a subset S of A such that $1 \in S$ and S is closed under multiplication: $s, t \in S \Rightarrow s.t \in S$. We define an equivalence relation $\sim$ *on* $A \times S$ as follows:

$$(a, s) \sim (b, t) \quad \Leftrightarrow \quad r.a.t = r.b.s \quad \textit{for some} \quad r \in S.$$

Let A_S denote the quotient set $A \times S/\sim$ and let a/s denote the equivalence class of (a, s). It is well known that A_S has a ring structure [3] by defining the addition and multiplication of fractions as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

and

$$\frac{a}{s}.\frac{b}{t} = \frac{ab}{st}.$$

Furthermore, the map

$$h : A \longrightarrow A_S$$

defined by $h(a) = \frac{a}{1}$ gives a ring homomorphism. The ring A_S is said to be the *localization* of A with respect to S and is characterized by a universal property.

Properties 4.1.7. If $f : A \longrightarrow B$ is a ring morphism such that $f(s)$ is invertible in B for all $s \in S$, then there exists a unique ring morphism $\bar{f} : A_S \longrightarrow B$ such that $f = \bar{f} \circ h$ and hence $\bar{f}(a/s) = f(a).f(s)^{-1}$.

Example 4.1.8. Let I be a prime ideal of a commutative ring R, then $R \setminus I$ is a multipliatively closed subset of R.

Let X be a topological space. For each open set U in X we denote by $C(X)_U$ the localization of $C(X)$ with respect to the multiplicatively closed subset $S_U = \{f \in C(X) : 0 \notin f(U)\}$ of $C(X)$. If for each $f \in C(X)$ we denote by f_U the restriction of f to U, then clearly f_U is invertible in $C(U)$ for all $f \in S_U$. Hence we have a natural ring morphism

$$\Phi : C(X)_U \to C(U),$$

$$\text{given by} \quad \Phi(\frac{f}{g}) = \frac{f_U}{g_U},$$

and

$$\Phi(\frac{f}{g} + \frac{k}{l}) = \Phi(\frac{fl+gk}{gl}) = \frac{(fl+gk)_U}{(gl)_U} = \frac{(fl)_U}{(gl)_U} + \frac{(gk)_U}{(gl)_U} = \Phi(\frac{f}{g}) + \Phi(\frac{k}{l}).$$

Similarly,

$$\Phi(\frac{f}{g}.\frac{k}{l}) = \Phi(\frac{fk}{gl}) \quad = \quad \frac{(fk)_U}{(gl)_U} \quad = \quad \frac{f_U k_U}{g_U l_U} \quad = \quad \Phi(\frac{f}{g}).\Phi(\frac{k}{l}).$$

The map is clearly one-one as

$$\text{Ker}\,\Phi = \quad \{\frac{f_U}{g_U} \in C(X)_U : \frac{f_U}{g_U} = \frac{0}{1}\}.$$

Since,

$$\frac{f_U}{g_U} = \frac{0}{1},$$

so there exists $k \in S_U$, such that

$$kf_U - g_U.0 = 0, \quad \Rightarrow k.f_U = 0.$$

Since $k(x) \neq 0 \ \forall x \in X$, so $f_U = 0$, hence, $\text{Ker}\,\Phi = 0$.
In the next lemma we shall prove that this morphism is an isomorphism whenever U is a cozero set which was first proved by Blair and Hager in [5] and later the argument was refined by Requezo and Sancho in [29].

Lemma 4.1.9. *If U is a cozero-set in X, then the natural ring homomorphism $C(X)_U \longrightarrow C(U)$ is an isomorphism.*

Proof. We have already shown that $\Phi : C(X)_U \to C(U)$ is an injective ring homomorphism. We claim that Φ is onto.
Let $g \in C(U)$ and let $d \in C(X)$ such that $d(x) \neq 0 \ \forall x \in X$. We define

$$\delta_2(g) = d.\min\{1, 1/|g|\} \quad \text{on } U \ , \ \delta_2(g) = 0 \quad \text{on } X - U,$$

$$\delta_1(g) = g.\delta_2(g) \ = \ (\text{sign } g) \ . \ d \ . \ \min\{1, |g|\} \quad \text{on } U, \quad \delta_1(g) = 0 \quad \text{on } X - U \ ,$$

where sign $g(x) = 1$ if $g(x) \geq 0$ and sign $g(x) = -1$ if $g(x) < 0$.

We shall first show that $\delta_2(g)$ is continuous. Let us take $A = X - U$ and $B = \bar{U}$, the closure of U in X.
Clearly, $A \cup B = (X \setminus U) \cup \bar{U} = X$ and the restriction of $\delta_2(g)$ is continuous on both A and B . Now let $h \in A \cap B$ then $d(h) = 0$, so $\delta_2(h) = 0$, hence they agree on the intersection. So, by Pasting lemma $\delta_2(g)$ is continuous on whole of X. By a similar argument $\delta_1(g)$ is continuous on X. As $\delta_2(g) \neq 0$ on U, so $\delta_2(g) \in S_U$. Now $\frac{\delta_1(g)_U}{\delta_2(g)_U} = g$, this implies that $g = \Phi(\frac{\delta_1(g)}{\delta_2(g)})$. Thus Φ is onto, and hence an isomorphism. $\square$

The above lemma raises the question, whether this result is true for all open sets? This was answered in [29].

Counterexample 4.1.10. Let X be the one-point compactification of an uncountable discrete space U. Let $S_U = \{f \in C(X) : f(x) \neq 0 \ \ \forall\, x \in U\}$, i.e., it is the collection of all invertible functions of $C(X)$. We claim that U is not a cozero-set in X. We know that any cozero-set of a compact Hausdorff space is $\sigma-$compact and in a discrete space the only compact subspaces are finite ones. So U is σ- compact is equivalent to the statement that U is a countable union of finite spaces and hence countable. a contradiction to our assumption that U is uncountable. Clearly, the canonical morphism $C(X) \longrightarrow C(X)_U$ given by $f \mapsto \frac{f}{1}$ is an isomorphism.
On the other hand, $C(X)$ is pseudocompact as X is compact. However, U being uncountable discrete space, there exists unbounded functions in $C(U)$. Since any unbounded function cannot be the restriction of a bounded function , the restriction morphism $C(X) \longrightarrow C(U)$ is not surjective. Hence it is not an isomorphism.

4.1.2 Topological Localization

Definition 4.1.11. Let S be a multiplicatively closed subset of a topological ring A. The topological localization of A with respect to S is the ring A_S endowed with the final ring topology τ_S of the morphism $h : A \rightarrow A_S$, given by $h(a) = \frac{a}{1}$.

As before the topological localization A_S is characterized by the following universal property.

Properties 4.1.12. If $f : A \longrightarrow B$ is a topological ring morphism such that $f(s)$ is invertible in B for all $s \in S$, then there exists a unique topological ring morphism $\bar{f} : A_S \longrightarrow B$ such that $\bar{f} \circ h = f$.

The following lemma is an immediate consequence of this universal property.

Lemma 4.1.13 (8 pg- 44, [3]; 2.2, [29]). *Let S and Z be two multiplicatively closed subsets of a topological ring A. If $S \subseteq Z$, then the natural morphism $A_S \rightarrow A_Z$ is a ring isomorphism, then it is also a homeomorphism.*

Notation 4.1.14. Let τ_c denote the quotient topology on A_S.

Lemma 4.1.15. *If (A_S, τ_c) is a topological ring, then $\tau_c = \tau_S$.*

Proof. Let S be a multiplicatively closed subset of a topological ring A. Clearly, the canonical map $\pi : A \times S \longrightarrow A_S$ given by $\pi(a, s) = \frac{a}{s}$ is continuous. Since the quotient topology on A_S is the largest topology on A_S such that π is continuous, so $\tau_S \leq \tau_c$. On the other hand, the morphism $h : A \longrightarrow A_S$ is also continuous whenever A_S is endowed with the quotient topology. Since (A_S, τ_c) is a topological ring , so by definition τ_S is the finest topology on A_S such that h is continuous. Hence, $\tau_c \leq \tau_S$. Combining, we have $\tau_c = \tau_S$. ☐

Definition 4.1.16. A multiplicatively closed subset S of a ring A is said to be saturated if $a.b \in S \Leftrightarrow a \in S$ and $b \in S$.

Example 4.1.17. Let $A = \mathbb{R}$, then let $S = \mathbb{R} \setminus \{0\}$, then S is saturated.

Result 4.1.18. *If S is a multiplicatively closed subset of A, then there exists a multiplicatively closed subset Z of A, such that $Z \supset S$ and $A_S \rightarrow A_Z$ is an isomorphism.*

Proof. Let us define

$$Z = \{a \in A | \frac{a}{1} \text{ is invertible in } A_S\}.$$

Let $s \in S$, then $\frac{s}{1}.\frac{1}{s} = \frac{1}{1}$, so $S \subset Z$.
If $ab \in Z$, then $\frac{ab}{1} = \frac{a}{1}.\frac{b}{1}$ is invertible in A_S, implies

$$(\frac{a}{1}.\frac{b}{1}).(\frac{c}{d}) = \frac{1}{1} \quad \Leftrightarrow \frac{a}{1}.(\frac{b}{1}.\frac{c}{d}) \quad = \frac{1}{1},$$

where $(\frac{b}{1}.\frac{c}{d}) \in Z$ thus $a \in Z$. Similarly, $b \in Z$. Hence, Z is multiplicatively closed subset of A.
Clearly, the natural morphism $A_S \longrightarrow A_Z$ is an isomorphism and hence the result.

□

Remark 4.1.19. By Lemma [4.1.13], we may assume without loss of generality that every multiplicatively closed subset considered in our discussion is saturated ; if S is any multiplicatively closed subset of A, then by above result there exists a saturated multiplicatively closed subset Z of A, such that $A_S \longrightarrow A_Z$ is an isomorphism , hence a homeomorphism.

Theorem 4.1.20. *Let S be a saturated multiplicatively closed subset of a topological ring A. If the canonical map $\pi : A \times S \longrightarrow A_S$ has a continuous section with respect to the quotient topology τ_c on A_S, then τ_c coincides with the localization topology τ_S*

Proof. It would be sufficient to show that (A_S, τ_c) is a topological ring, then the result follows from Lemma [4.1.15]. Let us define

$$\sigma : (A_S, \tau_c) \longrightarrow A \times S$$

given by $\sigma(\frac{a}{s}) = (a, s)$ in a natural way. We claim that σ is a section of π. Now

$$(\pi \circ \sigma)(\frac{a}{s}) \quad = \pi[\sigma(a, s)] \quad = \pi(a, s) \quad = \frac{a}{s}.$$

Similarly,

$$(\sigma \circ \pi)(a. s) \quad = \sigma[\pi(a. s)] \quad = \sigma(\frac{a}{s}) = (a, s).$$

Hence, it is a section. Our argument can be simplified by the following commutative diagrams

$$\begin{tikzcd}
(A \times S) \times (A \times S) \arrow[r, "Q"] \arrow[d, shift right, "\sigma \times \sigma"'] & A \times S \arrow[d, "\pi"] \\
A_S \times A_S \arrow[u, shift right, "\pi \times \pi"'] \arrow[r, "+"] & A_S
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times S) \times (A \times S) \arrow[r, "P"] \arrow[d, shift right, "\sigma \times \sigma"'] & A \times S \arrow[d, "\pi"] \\
A_S \times A_S \arrow[r, "\cdot"] \arrow[u, shift right, "\pi \times \pi"'] & A_S
\end{tikzcd}$$

where Q and P are continuous maps defined, respectively by

$$Q((a,s),(b,t)) = (a.t + b.s, s.t) \quad \text{and} \quad P((a.s),(b,t)) = (a.b, s.t)$$

$$\text{Let } \frac{a}{s}, \frac{b}{t} \in A_S.$$

$$\begin{aligned}
\text{Now } (\sigma \times \sigma)(\frac{a}{s}, \frac{b}{t}) &= ((a,s),(b,t)) \\
\text{then } Q((a,s),(b,t)) &= (at + bs, st), \\
\text{and } \pi(at + bs, st) &= \frac{at + bs}{st}. \\
\text{So, } \frac{a}{s} + \frac{b}{s} &= \pi\circ(Q\circ(\sigma\times\sigma)),
\end{aligned}$$

which is a composition of continuous functions and hence addition is continuous.

Similarly for $\frac{a}{s}, \frac{b}{t} \in A_S$,

$$\pi\circ(P\circ(\sigma \times \sigma))(\frac{a}{s}, \frac{b}{t}) = \pi\circ P((a,s).(b,t)) = \pi(a.b, s.t) = \frac{ab}{st} = \frac{a}{s}\frac{b}{t}.$$

Hence, multiplication being composition of continuous functions is continuous. Finally, for $\frac{a}{s} \in {A_S}^*$, where ${A_S}^*$ is the set of all invertible functions in A_S, we have the following commutative diagram

$$\begin{tikzcd}
S \times S \arrow[r, "T"] \arrow[d, shift left, "\pi"] & S \times S \arrow[d, "\pi"] \\
(A_S)^* \arrow[u, shift left, "\sigma"] \arrow[r, "inv"] & (A_S)^*
\end{tikzcd}$$

where $T(s, r) = (r, s)$ and

$$\pi \circ (T \circ \sigma(\frac{a}{s})) \; = \; \pi \circ T(a, s) \; = \; \pi(s, a) \; = \; \frac{s}{a}.$$

So, the inverse operation is also continuous. Hence, (A_S, τ_c) is a topological ring, so by Lemma [4.1.15], $\tau_c = \tau_S$.

□

Corollary 4.1.21. *Let S be a saturated multiplicatively closed subset of a topological ring A and let τbe a topology on A_S. If $\pi : A \times S \longrightarrow (A_S, \tau)$ is continuous and has a continuous section, then $\tau = \tau_c = \tau_S$.*

Proof. In the preceeding theorem, we have already shown that (A_S, τ_c) is a topological ring. So $\tau_c = \tau_S$ by Lemma [4.1.15]. We shall now show that $\tau = \tau_c$. Since π is continuous with respect to τ and τ_c is the finest topology such that π is continuous, so $\tau \leq \tau_c$. On the other hand, if π admits a continuous section with respect to τ, then for each $U \in (A_S, \tau_c)$ $\quad U \in (A_S, \tau)$, hence $\tau_c \leq \tau$, thus $\tau = \tau_c = \tau_S$. □

4.1.3 Localization of continuous functions

In this section we shall use the notations introduced in Example [4.1.5] and shall record the main result in [29].

Lemma 4.1.22. *For any $x, y \in \mathbb{K}$, where $\mathbb{K}$ is a real or complex field, we have:*

(i) $|\min(|x + y|, 1) - \min(|x|, 1)| \leq |y|$;

(ii) $|\min(1, \frac{1}{|x+y|}) - \min(1, \frac{1}{|x|})| \leq |y|$.

Proof. Let us denote $a = |\min(|x + y|, 1) - \min(|x|, 1)| \leq |y|$;

(i) We shall prove it case wise.
Case (a):

$$|x + y| \geq 1 \quad , \quad |x| \leq 1,$$

$$\text{then} \quad a = |1 - |x||.$$

$$\text{Since} \quad 1 \leq |x + y| \leq |x| + |y| \quad \Rightarrow 1 - |x| \leq |y| \quad \Rightarrow a \leq |y|.$$

Case (b):

$$|x + y| \geq 1 \quad , \quad |x| \geq 1.$$

$$\text{Here,} \quad a = 1 - 1 = 0 \leq |y|.$$

Case (c):

$$|x + y| \leq 1 \quad , \quad |x| \leq 1,$$

$$\text{then} \quad a = ||x + y| - |x|| \leq |y|$$

Case (d):

$$|x + y| \leq 1 \quad , \quad |x| \geq 1,$$

$$\text{then} \quad a = ||x + y| - 1|.$$

$$\text{then} |x + y| \leq 1 + |y|$$

$$\Rightarrow |x + y| - 1 \leq |y| \text{ and } 1 - |x + y| \leq |y|$$

$$\Rightarrow ||x + y| - 1 \leq |y||$$

(ii) The proof of (ii) follows by similar casewise argument.

□

Theorem 4.1.23. *If U is a cozero-set in a topological space X, then the topological ring $(C(U), \tau_U)$ is the topological localization of $(C(X), \tau_X)$ with respect to the multiplicatively closed subset $S_U = \{f \in C(X) : 0 \notin f(U)\}$.*

Proof. Since U is a cozero-set in X, then there exists $d \in C(X)$ such that $U = \{x \in X | d(x) \neq 0\}$, without loss of generality let $0 \leq d(x) \leq 1$.

$$\text{Define} \quad \delta : C(U) \longrightarrow C(X) \times S_U$$

$$\text{given by } \delta(g) = (\delta_1(g), \delta_2(g))$$

where δ is the map defined from d as in Lemma [4.1.9]. then the natural morphism $C(X)_U \rightarrow C(U)$ is an isomorphism by Lemma [4.1.9].

We claim that δ is a section of $\Phi : C(X) \times S_U \to C(X)_U = C(U)$. Let $g \in C(U)$, then

$$(\pi \circ \delta)(g) = \pi[(\delta_1(g), \delta_2(g))] \quad = \frac{\delta_1(g)_U}{\delta_2(g)_U} = g.$$

Hence, it is a section. Since S_U is saturated and π is continuous with respect to the topology τ_U, so by Corollary [4.1.21], it is suficient to show that δ is continous with respect to τ_U. Let $g_0 \in C(U)$, we shall show that δ is continuous at g_0. If K is a compact subset of X and $0 < \epsilon < 1$, then

$$W = \{(\iota, s) \in C(X) \times S_U : q_K(J - \delta_1(g_0)(g0)) < \epsilon, q_K(s - \delta_2(g_0)) < \epsilon\}$$

is clearly a basic neighbourhod of $\delta(g_0)$ in the product space $C(X) \times S_U$. Given $0 < \bar{\epsilon} < \frac{\epsilon}{2}$, $C = K \cap \{x \in X : d(x) \geq \bar{\epsilon}\}$ is a compact subset of U and $V = \{g \in C(U) : q_C(g - g_0) < \bar{\epsilon}\}$ is a basic neighbourhood of g_0 in $C(U)$. Let $g \in V$ so we must prove that $\delta(g) \in W$.

$$\text{Clearly,} \quad \delta_2(g) = d.\ \min\{1, \frac{1}{|g|} \leq d\},$$

$$\text{therefore,} \quad 0 \leq \delta_2(g)(x) \leq \bar{\epsilon} < \frac{\epsilon}{2} \quad \text{for all } x \in K - C.$$

Hence

$$|\delta_2(g)(x) - \delta_2(g_0)(x)| \leq |\delta_2(g)(x)| + |\delta_2(g_0)(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{4.1}$$

For each $x \in C$, we have by Lemma [4.1.22]

$$|\delta_2(g)(x) - \delta_2(g_0)(x)| \leq |\min\{1, \frac{1}{|g(x)|}|\} - \min\{1, \frac{1}{|g(x)|}\} \leq |g(x) - g_0(x)| < \bar{\epsilon} < \epsilon,$$

then

$$q_C(\delta_2(g) - \delta_2(g_0)) \quad = \|\delta_2(g) - \delta_2(g_0)\| \quad = \max|\delta_2(g)(x) - \delta_1(g_0)(x)| < \epsilon. \tag{4.2}$$

From equations (4.1) and (4.2) we have

$$q_K(\delta_2(g) - \delta_1(g_0)) < \epsilon.$$

Now $|\delta_1(g)| = d. \min\{|g|, 1\} \leq d$.

$$\text{Therefore,} \quad |\delta_1(g)(x)| \leq \bar{\epsilon} < \frac{\epsilon}{2} \quad \text{for all } x \in K - C.$$

$$\text{Hence } |\delta_1(g)(x) - \delta_1(g_0)(x)| < \epsilon \ \text{ if } x \in K - C. \tag{4.3}$$

Further, if $x \in C$ and $\mathrm{sign}(g(x)) = \mathrm{sign}(g_0(x))$, then by Lemma [4.1.22]

$$|\delta_1(g)(x) - \delta_1(g_0)(x)| \leq |\min\{|g(x)|, 1\}| - \min\{|g_0(x)|, 1\} \leq |g(x) - g_0(x)|$$
$$\leq \bar{\epsilon} < \epsilon$$

If $x \in C$and $\mathrm{sign}(g(x)) \neq \mathrm{sign}(g_0(x))$, then $|g(x) - g_0(x)| < \bar{\epsilon}$ implies that $|g(x)| < \bar{\epsilon}$ and $|g_0(x)| < \bar{\epsilon}$, therefore

$$|\delta_1(g)(x) - \delta_1(g_0)(x)| \leq |\min\{|g(x)|, 1\} + \min\{|g_0(x)|, 1\}| \leq |g(x)| + |g_0(x)|$$
$$\leq 2\bar{\epsilon} < \epsilon,$$

which implies that

$$q_C(\delta_1(g) - \delta_1(g_0)) = \|\delta_1(g) - \delta_1(g_0)\| = \max\{|\delta_1(g) - \delta_1(g_0)|\} < \epsilon. \tag{4.4}$$

Thus from the equations (4.3) and (4.4) we conclude that $q_K(\delta_1(g) - \delta_1(g_0)) < \epsilon$, which implies that $\delta(g) \in W$. This completes the proof. □

The above theorem has been used in [29] to give a characterization (of categorial type) of the compact-open topology τ_X in *C(X)*.

Notation 4.1.24. Let Γ denote the category of locally compact, σ-compact Hausdorff spaces and continuous maps.

Theorem 4.1.25. *Let us assume that for each $X \in \Gamma$ we have a ring topology $\tau(X)$ in C(X) such that:*

(i) *The morphism $(C(Y), \tau(Y)) \longrightarrow (C(X), \tau(X)), f \rightarrow f \circ h$, is continuous for each continuous map $h : X \rightarrow Y$*

(ii) *$\tau(K) = \tau_K$ for each compact Hausdorff space K.*
Then $\tau(X) = \tau_X$ for all $X \in \Gamma$.

Proof. Let $X \in \Gamma$, then for each compact subset C of X, the restriction morphism $(C(X), \tau(X)) \rightarrow (C(C), \tau_c)$ is continuous. But the compact open topology is the coarsest topology on *C(X)* such that all restriction morphisms on compact subsets C of X are continuous. Hence $\tau_X \leq \tau(X)$. On the other hand, if $S = \{g \in C(\beta X) : 0 \notin g(X)\}$, then S is multiplicatively closed and saturated. Now the restriction morphism $C(\beta X) \rightarrow C(X)$ induces a continuous morphism. Thus from Theorem [4.1.23] it follows that $(C(\beta X, \tau(\beta X)_S)) = (C(X), \tau_K)$ because X is a cozero-set in βX, thus $\tau(X) \leq \tau_X$. Hence the result follows. □

4.2 Rings of quotients of rings of functions

We have seen that the study of rings of continuous functions is heavily dependent on zero-sets. R.Levy and J Shapiro in [[21],2005] have added to this knowledge by characterizing topological spaces in terms of the properties of the zero-sets of X.

4.2.1 Preliminaries

Definition 4.2.1. Let X be a topological space. A subset E of X is nowhere dense if $\text{Int}(\text{cl}\, E) = \phi$.

Example 4.2.2. Let $X = \mathbb{R}$ with usual topology, then every finite point set is nowhere dense in X.

Definition 4.2.3. Suppose that X is a topological space and Z is a zero-set of X, then Z is *z-complemented* in X if there exists a zero-set $\widehat{Z}$ of X such that $Z \cup \widehat{Z} = X$ and $Z \cap \widehat{Z}$ is nowhere dense in X. The zero set $\widehat{Z}$ is called z-complement of Z. The space X is z-good if every zero-set of X is z-complemented in X ; otherwise X is z-bad.

Example 4.2.4. (i) Let $X = \mathbb{R}$, with usual topology. $Z = \{0\}$ and $\widehat{Z} = \mathbb{R}$, then $Z \cap \widehat{Z} = \{0\}$ implies that $\text{Int}(\text{cl}(Z \cap \widehat{Z})) = \phi$ and $Z \cup \widehat{Z} = \mathbb{R}$. So Z is z-complemented in X.
(ii) Every discrete space is z-good.

The above definitions can be reframed in terms of cozero-sets of X.

Remark 4.2.5. X is z-good (usually called cozero complemented) if given a cozero set C of X, there exists a cozero set $\widehat{C}$ such that $C \cup \widehat{C}$ is dense in X and $C \cap \widehat{C} = \phi$.

Proof. Let Z be a zero set in X. Consider $C = X \setminus Z$, then by hypothesis there is a co-zero set $\widehat{C}$ in X such that

(i) $\text{cl}(C \cup \widehat{C}) = X$.

(ii) $C \cap \widehat{C} = \phi$.

Let $\widehat{Z} = X \setminus \widehat{C}$, then

(i)
$$Z \cup \widehat{Z} = ((Z \cup \widehat{Z})^c)^c = (Z^c \cap \widehat{Z}^c)^c = (C \cap \widehat{C})^c = \phi^c = X.$$

(ii) We shall prove this by contradiction.
Suppose $\mathrm{Int}_X(Z \cap \widehat{Z}) \neq \phi$, then there exists $x \in \mathrm{Int}(Z \cap \widehat{Z})$ and a neighbourhood U of x, such that $x \in U \subseteq Z \cap \widehat{Z}$. This implies that $U \cap C = \phi$ and $U \cap \widehat{C} = \phi$, thus $U \cap (C \cup \widehat{C}) = \phi$. Since $(C \cup \widehat{C})$ is dense in X, so every neighbourhood of x must meet $(C \cup \widehat{C})$. This contradicts that $(C \cup \widehat{C})$ is dense in X. Hence $\mathrm{Int}(Z \cap \widehat{Z}) = \phi$.

□

Definition 4.2.6. Let X be a topological space and U be a subset of X, then U is z-embedded in Y if each zero set of U is the intersection with U of a zero set of X, i.e., $Z \in Z(U) \in Z(X) \Rightarrow Z = Z' \cap U$, where $Z' \in Z[X]$.

Example 4.2.7. X is z-embedded in βX for $Z = Z(f) \in Z(X) \Rightarrow Z \subset Z(f^\beta)$ and $Z(f^\beta) \cap X = Z(f)$.

Notation 4.2.8. $T(C(X))$ is the localization of $C(X)$ with respect to the multiplicative set of non-zero divisors.

Notation 4.2.9. Z_X denotes the zero-set in X and $\widehat{Z_X}$ denotes the z-complement of Z in X.

The following proposition establishes the relation between the topological properties of X in terms of z-goodness and the algebraic properties of $C(X)$ via its total ring of quotients $T(C(X))$.

Proposition 4.2.10 (1.3, [21]; 2.3, [2]). *Suppose X is a space. Then $T(C(X))$ is von Neumann regular if and only if X is z-good.*

Proposition 4.2.11. *Suppose X is dense subset of Y and that X is z-embedded in Y. Then X is z-good iff Y is z-good. In particular X is z-good iff βX is z-good.*

Proof. Let us assume that X is z-good. Suppose $Z_X \in Z(X)$, then $Z_X = Z_Y \cap X$ for some $Z_Y \in Z(Y)$. Now X is z-good implies that $Z_X \cup \widehat{Z_X} = X$ and $\mathrm{Int}_X(Z_X \cap \widehat{Z_X}) = \phi$, for some $\widehat{Z_X} \in Z(Y)$, where $\widehat{Z_X} = \widehat{Z_Y} \cap X$.
Now, Z_Y is closed in Y and $Z_X \subseteq Z_Y \Rightarrow \mathrm{cl}\, Z_X \subseteq \mathrm{cl}\, Z_Y$ and $Z_X \cup \widehat{Z_X} = X$.

$$\text{So, } \mathrm{cl}_Y(Z_X \cup \widehat{Z_X}) = \mathrm{cl}_Y X \quad \text{gives } \mathrm{cl}_Y Z_X \cup \mathrm{cl}_Y \widehat{Z_X} = Y.$$

$$\text{But } \mathrm{cl}_Y Z_X \cup \mathrm{cl}_Y \widehat{Z_X} \subseteq Z_Y \cup \widehat{Z_Y} \subseteq Y \quad \Rightarrow Z_Y \cup \widehat{Z_Y} = Y.$$

We know $\mathrm{Int}[(Z_X \cap \widehat{Z_X})] = \phi$.

$$\text{But } \mathrm{Int}[(Z_X \cap \widehat{Z_X})] = \mathrm{Int}[(Z_Y \cap X \cap \widehat{Z_Y} \cap X)] = \mathrm{Int}[[(Z_Y \cap \widehat{Z_Y}) \cap X]]$$
$$= \mathrm{Int}[(Z_Y \cap \widehat{Z_Y}) \cap \mathrm{cl}\, X] = \mathrm{Int}[(Z \cap \widehat{Z_Y}) \cap Y] = \mathrm{Int}[(Z_Y \cap \widehat{Z_Y})] = \phi.$$

Hence Y is z-good. On the other hand, suppose Y is z-good. Let Z be a zero-set in X, then X is z-embedded in Y implies that $Z = Z_Y \cap X$ for some $Z_Y \in Z(Y)$.
So, there exists $\widehat{Z_Y} \in Z(Y)$ such that $Z_Y \cup \widehat{Z_Y} = Y$ and $\mathrm{Int}_Y[(Z_Y \cap \widehat{Z_Y})] = \phi$. Let $\widehat{Z} = \widehat{Z} \cap X$, then $Z \cup \widehat{Z} = (Z_Y \cap X) \cup (Z_Y \cap X) = X$ and $\mathrm{Int}_X\,(Z \cap \widehat{Z}) \subseteq \mathrm{Int}_Y\,(Z_Y \cap \widehat{Z_Y}) = \phi$. Hence X is z-good. □

Definition 4.2.12. Suppose X is a space and $x \in X$. Then x is a P-point of X if every zero-set of X containing x is a neighbourhood of X. The point x is an almost P-point if every zero-set of X containing x has a non-empty interior in X. The space X is an (almost) P-space if every element of X is an (almost) P-point.

Example 4.2.13. (i) A discrete space is a P-space.
(ii) Every P-space is almost P-space.

Proposition 4.2.14. *Supose that X is a space with the property that for each cozero-set C of X, the closure $\mathrm{cl}_X\, C$ is a zero-set of X, then X is z-good.*

Proof. Let Z be a zero-set of X and consider $C = X \setminus Z$. So, $\mathrm{cl}_X C = \mathrm{cl}_X(X \setminus Z)$ is a zero-set in X. Now $Z \cup \mathrm{cl}_X C = X$ and $\mathrm{Int}(Z \cap \mathrm{cl}_X C) = \phi$, otherwise, there exists $p \in Z \cap X \setminus X$ which is a contradiction and hence X is z-good. □

Example 4.2.15. Every metric space is z-good. More generally, every perfectly normal space is z-good.

Proof. It follows from the Proposition [4.2.14] and the fact that every closed set in a discrete space is a zero-set. □

Example 4.2.16. Every P-space is z-good.

Example 4.2.17. If X is an almost P-space which is not a P-space or more generally, if X has an almost P-point which is not a P-point, then X is z-bad. In particular if D is an infinite discrete space, then $\beta D \setminus D$ is z-bad and if D is uncountable. the one-point compactification of D is z-bad.

Proof. Let Y be an almost P-point which is not a P-point, then there exists $Z \in Z(X)$ such that $x \notin \mathrm{Int}_X Z$. We claim that X is z-bad. On the contrary, suppose X is z-good, then there exists $\widehat{Z} \in Z(X)$ such that $Z \cup \widehat{Z} = X$ and $\mathrm{Int}_X(Z \cap \widehat{Z}) = \phi$ with $x \in Z \cap \widehat{Z}$. Clearly, $Z \cap \widehat{Z} \in Z(X)$, but $\mathrm{Int}_X(Z \cap \widehat{Z}) = \phi$, i.e., there exists a zero-set with empty interior, this contradicts that X is an almost P-space.

For the other part, suppose $D^* = D \cup \{\infty\}$, where ∞ is the limit point of D in D^*. We shall show that ∞ is an almost P-point but not a P-point. We assert that every zero-set containing ∞ must intersect D. We shall prove this by the method of contradiction. Suppose there is $f \in C(D^*)$, such that $Z(f) = \{\infty\}$.
Now, for each $n \in \mathbb{N}$, $f^{\leftarrow}(-1/n, 1/n)$ is open in D^*, hence

$$f^{\leftarrow}(-1/n, 1/n) = D^* \setminus C_n, \quad \text{where } C_n \quad \text{is} \quad \text{finite} \quad \text{subset} \quad \text{of} \quad D$$

Also,

$$D^* \setminus C_n \supseteq D^* \setminus C_{n+1} \supseteq C_{n+2} \supseteq \cdots$$

So,

$$C_n \subseteq C_{n+1} \subseteq C_{n+2} \subseteq \cdots$$

But, $f^{\leftarrow}(\bigcap\limits_{n\in\mathbb{N}}(-1/n, 1/n)) = \bigcap\limits_{n\in\mathbb{N}} D^* \setminus C_n = \bigcap\limits_{n\in\mathbb{N}} {C_n}^c = (\bigcup\limits_{n\in\mathbb{N}} C_n)^c$
Since $\bigcup\limits_{n\in\mathbb{N}} C_n$ is countable, so $(\bigcup\limits_{n\in\mathbb{N}} C_n)^c$ is uncountable. But, $(\bigcup\limits_{n\in\mathbb{N}} C_n)^c = \{\infty\}$.
This contradicts that $Z(f) = \{\infty\}$. Thus, ∞ is an almost P-point.
On the other hand, ∞ is not a P-point as any open set of D^* containing ∞ must intersect D. ☐

The following questions were not answered till recently,

(i) Is it true that all z-good spaces have the property of Proposition [4.2.14]?

(ii) Do z-bad spaces necessarily contain almost P-points which are not P-points? If not, what nice properties can z-bad spaces have?

The next few sections are devoted to record the answers for these questions.

4.2.2 Some z-bad spaces without almost P-points

In this section, we shall record one of the answers given by Levy and Shapiro in [21].

Theorem 4.2.18. *Suppose that X is an almost P-space and M is a compact metric space. Then $X \times M$ is z-good iff X is a P-space.*

Proof. Let us first assume that X is not a P-space. Then there is a point $p \in X$ such that $p \in \widehat{Z} = Z(g)$ for some $g \in C(X)$ but $p \notin \text{Int}_X \widehat{Z}$.

$$\text{Define } f : X \times M \longrightarrow \mathbb{R}, \text{ by}$$
$$f(x, y) = g(x).$$

So, the function is well defined and continuous by continuity of g. Let $Z = Z(f) = \widehat{Z} \times M$. We claim that Z is not z-complemented in $X \times M$. On the contrary, suppose Z is z-complemented in X, then there is a zero-set K in $(X \times M)$ where $K = F \times M$, $F \in Z(X)$ such that $Z \cup K = X \times M$ and $\text{Int}_{X \times M}(Z \cap K) = \phi$.Since M is a compact metric space so it contains a countable dense subset C. For each $c \in C$, let $K \cap (X \times \{c\}) = A_c \times \{c\}$, where A_c is a zero-set in X [Remark 1.1.12].
Clearly, $A \times \{c\} \subseteq K$ for each $c \in C$, implies that $A \times C \subseteq K$.
Also, $\text{cl}(A \times C) = A \times M$. So K is closed implies that $\text{cl}(A \times C) \subseteq \text{cl}\, K = K$.
Since K is closed in $X \times M$ and $A \times C$ is dense in $A \times M$,

$$A \times M \subseteq K \quad \text{and} \quad \text{Int}_{X \times M}\, Z \cap K = \phi, \tag{4.5}$$

$$\text{This implies that, } \text{Int}_{X \times M}(Z \cap (A \times M)) \subseteq \text{Int}(Z \cap K) = \phi. \tag{4.6}$$

So $\text{Int}_{X \times M}\, \widehat{Z} \times M \cap A \times M = \text{Int}_{X \times M}(\widehat{Z} \cap A \times M) = \text{Int}(\widehat{Z} \cap A) = \phi$.
Since $p \in \widehat{Z} \setminus \text{Int}_{X \times M}\, \widehat{Z}$, so $p \in A_c$ for each $c \in C$, $p \in \bigcap_{c \in C} A_c = A$ implies $p \in A \cap \widehat{Z} \in Z(X)$.
But X is an almost P-space, so every zero-set in X has non-empty interior, but $\text{Int}_{X \times M}(Z \cap (A \times M)) = \phi$, a contradiction. Thus $X \times M$ is z-god if X is a P-space.
Conversely, suppose that X is a P-space. Let $\mathcal{K}$ be the set of zero-sets of M. For $K \in \mathcal{K}$, let $K^* = \text{cl}_M(M \setminus K)$, then by Example [4.2.15] $K^* \in \mathcal{K}$.
Clearly, $K \cup K^* = M$ and $\text{Int}_M(K \cup K^*) = \phi$.
For $K \in \mathcal{K}$, let $F_K \in C(M)$ be a function such that $Z(F_K) = K$. Suppose $Z = Z(f)$ is a zero-set of $X \times M$. For each $x \in X$, define

$$f_x : M \mapsto \mathbb{R} \text{ by } f_x(m) = f(x, m).$$

Then $f_x \in C(M)$ by continuity of f, and let $K_x = Z(f_x)$.
Let $\hat{Z} = \bigcap\limits_{x \in X} \{x\} \times {K_x}^*$. So, our proof will be complete if we show that

(i) $\hat{Z}$ is a zero-set of $X \times M$, and

(ii) $\hat{Z}$ is a z-complement of Z in $X \times M$.

First, we shall assert the following:
(A): For each $x \in X$, there exists a neighbourhood U_x of x such that:

$$\text{for each } \ (y, m) \in U_x \times M, \quad f(y, m) = f(x, m).$$

Suppose $x \in X$. Since every compact metric space is separable, there exists countable dense subset C of M. By our hypothesis x is a P-point, so for each $c \in C$, there exists a neighbourhood $U_x(c)$ such that $f(y, c) = f(x, c)$ for all $y \in U_x(c)$. Let $U_x = \bigcap\limits_{c \in C} U_x(c)$. As C is countable and x is a P-point, U_x is a neighbourhood of x. If $y \in U_x$, then $f(y, c) = f(x, c)$ for all $c \in C$, and since C is dense in M and f is continuous, $f(y, m) = f(x, m)$ for all $m \in M$. Hence our assertion holds.

We define $h : X \times M \to \mathbb{R}$ by $h(x, m) = F_{Z(f_x)^*}$. If U_x is as in (A) and if $y \in U_x$, then $Z(f_y) = Z(f_x)$, hence $Z(f_y)^* = Z(f_x)^*$. So, if U_x is as in (A), then $h(y, m) = h(x, m)$ for all $y \in U_x$ and $F_{Z(f_x)^*}$ is continuous on M implies that h is continuous on $X \times M$. Clearly, $Z(h) = \hat{Z}$, hence (i) holds. Finally, (ii) follows from (A) and the fact that for any zero-set K of M, the zero-set K^* is the z-complement of K in M.

□

Example 4.2.19 (2.2, [21]). If follows from Theorem [4.2.18] that $(\beta\omega) \setminus \omega \times [0, 1]$, which has no almost P-points, is z-bad.

4.2.3 Good z-bad and bad z-good spaces

In this section we shall record few results and examples of z-bad spaces which have many nice topological properties. In addition, we shall also record an answer to the question of whether all z-good spaces have the property of Proposition [4.2.14]. The answer is in the negative as shown in [21].

Lemma 4.2.20. *Suppose X is a compact Hausdorff space and D is a dense uncountable set of isolated points of X, such that $X \setminus D$ is metrizable. Then there exists a countably infinite subset C of D such that* $\mathrm{cl}_X C \supset X \setminus D$.

Proof. Let $M = X \setminus D$, then M is closed. Since closed subsets of compact spaces are compact, M is a compact metric space. Every compact metric space has a countable base. Let β be a base of M.

For each $B \in \beta$, choose $p_B \in B$. We know that for each open set W in M, there exists an open set U_W in X such that $U_W \cap M = W$. So we have $U_B \cap M = B$ for each $B \in \beta$. Since X is compact it is locally compact and so there exists a compact neighbourhood V_B of p_B in X such that $V_B \subseteq U_B$. Let C_B be a countably infinite subset of $V_B \cap D$. Since every infinite subset of X has a limit point x in X, in particular C_B has a limit point in $V_B \subseteq U_B$. Since D is a set of isolated points of X, it must lie in $V_B \setminus D$, i.e., in B.
Let $C = \underset{B\in\beta}{\cup} C_B$, then C is countable and let $x \in M$ then x is a limit point of C_B, so $M \subseteq \text{cl}_X C$ □

Definition 4.2.21. Suppose that M is a meric space. Then M is called *weak* $0 -$ *extension* of M. If n is a positive integer, then a compact space X is called a *weak n-extension* of M if X has a dense set D of isolated points such that $X \setminus D$ is a weak $n - 1$ extension of M. If n is a positive integer, an *n-extension* of M is a weak n-extension whose set of isolated points is uncountable.

Remark 4.2.22. It follows from the definition that if X is an n-extension of the metric space M where $n > 0$, $n \in \mathbb{N}$, we can write $X = M \cup D_1 \cup \ldots \cup D_n$, where D_1 is the set of isolated points of X and for each $k > 1$, D_k is the set of isolated points of $X \setminus \overset{k-1}{\underset{j=1}{\cup}} D_j$.

Notation 4.2.23. For each $p \in X \setminus M$, let k_p be such that $p \in D_{k_p}$.

Lemma 4.2.24. *Suppose that m is a positive integer and Y is an m-extension of the metric space W such that for some countable subset C of* D_m, $W \subset \text{cl}_X C$. *Suppose further that for each* $p \in Y \setminus W$, *there is a clopen neighbourhood* V_p *of p such that* $V_p \cap D_{k_p} = \{p\}$ *and* V_p *contains only countably many isolated points of Y. Then Y is z-bad.*

Proof. Since C is countable, let us index the elements as

$$C = \{C_k : k \in W\},$$

where W is a countable set. Let V_{C_k} be the corresponding clopen neighbourhood of C_k.

Define $f : Y \longrightarrow \mathbb{R}$, by

$$f(x) = \begin{cases} 1 & \text{if } x \in V_{c_0}, \text{ where } c_0 \text{ is the first element of C} \\ \frac{1}{k+1} & \text{if} x \in V_{C_k} \setminus \bigcup_{j=1}^{k-1} V_{C_j}, k > 0. \\ 0 & \text{if } x \in X \setminus \bigcup_{k \in W} V_{C_k}. \end{cases}$$

We claim that f is continuous.

Since V_{C_k} is clopen for each $k \in W$, so V_{c_0} is closed, and

$$V_{C_k} \setminus \bigcup_{j=1}^{k-1} V_{C_j} = V_{C_k} \cap (\bigcup_{j=1}^{k-1} V_{C_j})^c = V_{C_k}{}^c \cap (\bigcap_{j=1}^{k-1} V_{C_j}{}^c)$$

hence closed. Also $\bigcup_{k \in W} V_{C_k}$ is open $\Rightarrow$ $X \setminus \bigcup_{k \in W} V_{C_k}$ is closed.

Further, $V_{C_k} \setminus \bigcup_{j=1}^{k-1} V_{C_k}$ are pairwise disjoint for $k \in W$ and f is continuous on each of these closed sets, so f is continuous and $Z(f) = X \setminus \bigcup_{k \in W} V_{C_k}$.

Since V_{C_k} contains only countably many isolated points of Y, Coz(f) = $\bigcup_{k \in W} V_{C_k}$ contains countably many isolated points of Y.

Claim : We claim that $Z = Z(f)$ is not z-complemented in Y.

We shall first show that if K is a compact subset of Y which does not interset W, then K contains only countably many isolated points of Y. Since $K \cap W = \phi$, $K \subseteq Y \setminus W$. Let $\{V_q : q \in K\}$ be an open cover of K, then by compactness it has a finite subcover, say $\{V_{q_0}, V_{q_1}, ..., V_{q_n}\}$. By hypothesis, V_{q_i} $0 \leq i \leq n$ contains only countably many isolated points, so does K. Also, every cozero-set is an F_δ-set, i.e., countable union of closed sets. Now each closed subset is compact and since it does not intersects W, by hypothesis it has only countably many isolated points of Y. So a F_δ set contains only countably many isolated points of Y. Now suppose that H is a zeroset of Y, such that $Z \cup H = Y$ and $Z \cap H$ is nowhere dense in Y, i.e., Int$(Z \cap H) = \phi$, then $Z \cap H$ has no isolated points of Y.

Furhter, $C \subset$ Coz(f) as $C \cap Z(f) = \phi$ and Coz$(f) \subseteq H$ as $Z \cup H = Y$.

$$\text{So,} \quad C \subseteq \text{Coz}(f) \subseteq H \quad \Rightarrow \text{cl}_X C \subset h. \tag{4.7}$$

But H is closed and $W \subset \text{cl}_X C$ by Lemma [4.2.24], so $W \subseteq \text{cl}_X C \subseteq H$.

So $Y \setminus H \subseteq Y \setminus W$ implies that it contains only countably many isolated points of Y as $Y \setminus H \cap W = \phi$. So, $H \cap Z$ contains an isolated point of Y. This is a contradicton to the fact that $Z \cap H$ is nowhere dense in Y. Hence, $Z(f)$ is not z-complemented in Y. Thus Y is z-bad.

□

Proposition 4.2.25. *Suppose n is a positive integer and X is an n-extension of the metric space M. Then X is z-bad.*

Proof. Since X is an n-extension of M, so $X \setminus M \neq \phi$. For each $p \in X \setminus M$, let U_p be a clopen neighbourhood of p such that $U_p \cap D_{k_p} = \{p\}$. Such a clopen neighbourhood exists because the space obtained by collapsing M to a single point is compact and scattered, and , therefore, zero-dimensional. The following cases arise:

Case (i): Suppose that for every $p \in X \setminus M$, the set U_p contains only countably many isolated points of M. Applying Lemma [4.2.20] to $M \cup D_n$, there exists a countable subset C of D_n such that $M \subseteq \mathrm{cl}_X C$. Now applying Lemma [4.2.24] to X with $m = n$, $Y = X$ $W = M$ and $V_p = U_p$ gives X is z- bad.

Case (ii): Suppose there exists a $p \in X \setminus M$ such that U_p contains uncountably many points of X. Then there exists a p such that k_p is as small as possible. Further, p is not an isolated point of X. For if p is an isolated point of X, then $p \in D_1$ and $k_p = 1$. By our assumption $U_p \cap D_{k_p} = \{p\}$, so it implies that U_p contains only one isolated point of X. This is a contradiction to our assumption. Then, $k_p \geq 2$. The clopen subspace $Y = U_p$ of X is an m-extension of the metric space $W = \{p\}$ for $m = k_p - 1$. For $q \in U_p$, if we take $V_p = U_p \cap U_q$. Then by minimality of k_p, each of the sets V_q contain only countably many isolated points of X, i.e., only countably many points of U_p. As p is the only limit point of $D_m \cap U_p$, p is a limit point of every countably infinite subset of $D_m \cap U_p$. So, applying Lemma [4.2.24] with $Y = U_p$ and $W = \{p\}$, we get U_p is z-bad. Since U_p is clopen in X, X is also z-bad.

□

Remark 4.2.26. z-bad spaces need not have subspaces which have almost P-points which are not P-points.

Let D be an uncountable discrete space. Let βD be its Stone-Čech compactification and D^* be its one-point compactification. By Examples [4.2.15 and 4.2.17] βD and D^* are z-good and z-bad respectively. Clearly, βD maps onto D^*, which in turn maps onto the one point space $\{p\}$ (say) which is z-good. By compactness of $\beta D, D^*$ and $\{p\}$ each of the maps are perfect. So we have the following remark:

Remark 4.2.27. Neither z-goodness nor z-badness is preserved by either perfect maps or inverse images of perfect maps.

Proposition 4.2.28. *Let $S = D \cup \{\infty\}$ be the one point compactification of an uncountable discrete space. Let (M, ρ) be a metric space. Suppose $p \in M$.*

Let $X = (D \times M) \cup \{(\infty, p)\} \subseteq S \times M$. Then every zero-set in X has a z-complement if and only if p is not an isolated point of M.

Proof. Suppose P is an isolated point of M, then $\{p\}$ is clopen in the metric space M. So, $S \times \{p\}$ is clopen subset of $S \times M$. Clearly, $S \times \{p\}$ is homeomorphic to S. By Example [4.2.17] ∞ is an almost P-point of S which is not a P-point. By similar argument (∞, p) is an almost P-point of $S \times \{\infty\}$ and hence of X. So, by Example [4.2.17] X has a zero-set which is not z-complemented.

Conversely, suppose that p is not isolated in M. Let Z be a zero-set of X. For each $d \in D$, $Z_d = Z \cap (\{d\} \times M)$ is a zero-set in the space $\{d\} \times M$. Being z-good it has a z-complement, $\hat{Z}_d$ in $\{d\} \times M$. Let $\hat{Z}_d = Z(f_d)$, where $f_d \in C(\{d\} \times M)$.
Let $g_d = (0 \vee f_d) \wedge 1$. Then $0 \leq g_d(d, y) \leq 1$ $\forall y \in M$ with $Z(g_d) = \hat{Z}_d$. Define

$$g : X \to \mathbb{R} \text{ by } g(d, y) = g_d(y, y)\rho(p, y) \text{ and } g((\infty, p)) = 0$$

Clearly, $g \in C(X)$ and $\hat{Z}$ is a z-complement of Z in X. □

Example 4.2.29. If in Proposition 4.2.51 we let $M = [0, 1]$ and p be any element of M, then the space X has a cozero-set whose closure is not a zero-set.

Proof. Let C be a countable subset of D, then C is a cozero-set in D such that $C = D \setminus Z(f)$ for some $f \in C(D)$. Let $U = C \times M$.
Then U is a cozero-set in X, but its closure is $U \cup \{(\infty, p)\}$ which is not a zero-set of X because every zero-set of X which contains (∞, p) intersects $\{d\} \times M$ for a co-countable set of d's. □

4.2.4 Gabriel filters

In this subsection the conception of a Gabriel filter of ideals and its various properties are discussed [21]. Among those, the localization of the ring at $\mathcal{F}$ is of special interest and we shall record some of the important results. We shall first state the following definitions.

Definition and Notation 4.2.30. A collection of ideals of a commutative ring R is called a filter, if it is closed under the operation of taking finite intersections and such that any ideal containing an element of the collection is in the collection. A filter of ideals $\mathcal{F}$ of a commutative ring R is called a Gabriel filter if it satisfies the additional property that if I is an arbitrary ideal and J is an element of $\mathcal{F}$ such that for all $a \in J$, $a^{-1}I \in \mathcal{F}$, then $I \in \mathcal{F}$, where

$$a^{-1}I = \{b \in R : ba \in I\}.$$

More generally, a filter is called multiplicative if it satisfies the weaker condition that the ideal $IJ \in \mathcal{F}$ whenever $I, J \in \mathcal{F}$.

Associated to a multiplicative filter $\mathcal{F}$ is a left exact functor $q_{\mathcal{F}}$ on the category of R-modules (i.e., for M an R-module), defined by

$$q_{\mathcal{F}}(M) = \bigcup_{I \in \mathcal{F}} \mathrm{Hom}(I, M). \tag{4.8}$$

Let $f, g \in q_{\mathcal{F}}(M)$ then there exists $I, J \in \mathcal{F}$ such that

$$f : I \overset{hom}{\rightarrow} M \quad \text{and} \quad g : J \overset{hom}{\rightarrow} M.$$

We identify $f = g$, if $f[L] = g[L]$, where $L \subseteq I \cap J$ and $L \in \mathcal{F}$.
We define addition as $f + g = (f + g)[I \cap J]$.
If $M = \mathbb{R}$, then we can define multipication as $f.g = f \circ g[IJ] \quad = f[g[IJ]]$.
Since the functions 0 and 1 are in $q_{\mathcal{F}}(\mathbb{R})$, it has a ring structure.

It has been shown by N.Schwartz in [31] that the map from R to $q_{\mathcal{F}}(R)$ given by r goes to multiplication by r, is a ring homomorphism. Further, it follows from the same that the localization of the ring at $\mathcal{F}$ is given by $q_{\mathcal{F}}(q_{\mathcal{F}}(R))$. We denote this functor by $Q_{\mathcal{F}}$. For a multiplicative filter $\mathcal{F}$ one defines the torsion submodule $\tau_{\mathcal{F}}(M)$ of an arbitrary module via

$$\tau_{\mathcal{F}}(M) = \{m \in M : mI = 0 \text{ for some } I \in \mathcal{F}\},$$

for let $m, n \in \tau_{\mathcal{F}}(M)$, then $mI = 0$ and $nJ = 0$ for some $I, J \in \mathcal{F}$,
then $(m + n)(IJ) = m(IJ) + n(IJ) = 0 \quad \Rightarrow m + n \in \tau_{\mathcal{F}}(M)$.
Further, let $r \in R$, $m \in \tau_{\mathcal{F}}(M)$, then $rm\ (I) = r.(mI) = r.0 = 0 \quad \Rightarrow rm \in \tau_{\mathcal{F}}(M)$. Hence it is a submodule and it follows from B.Stenstrom [33] that

$$Q_{\mathcal{F}}(M) = \bigcup_{I \in \mathcal{F}} Hom(I, M/\tau_{\mathcal{F}}(M)) = q_{\mathcal{F}}(M/\tau_{\mathcal{F}}(M)), \tag{4.9}$$

whenever $\mathcal{F}$ is a Gabriel filter.

Example 4.2.31. Let R be a commutative ring, and let $\mathcal{F}$ be the set of dense ideals, i.e., $I \in \mathcal{F}$ if $xI = 0 \quad \Rightarrow x = 0$ for any $x \in R$. Then $\mathcal{F}$ is a Gabriel filter in R.

Notation 4.2.32. The ring of quotients of R with respect to the filter sF in [4.2.31] is called the complete ring of quotients of R and is denoted by $Q(R)$.

Let X be a topological space and F be a filter of open sets in X, i.e., a collection of non-empty open subsets of X which is closed under finite intersections and such that an open superset of an element of the collection is again an element of the colection.

Notation 4.2.33. We denote the ring of partial fractions $C_F(X)$ by

$$C_F(X) = \bigcup_{U \in \mathcal{F}} C(U).$$

Here we identify two elements f_1, f_2, if $f_1[F] = f_2[F]$ for some $F \in \mathcal{F}$, such that $F \in \operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$. It has a natural ring structure and a ring homomorphism from $C(X)$ to $C_F(X)$, i.e., inclusion as $X \in F$.

Notation 4.2.34. For an ideal I of $C(X)$, let $\operatorname{Coz}(I) = \bigcup_{g \in I} \operatorname{Coz}(g)$.

Remark 4.2.35. I is dense iff $\operatorname{Coz}(I)$ is a dense subset of X.

In [21], the result of Fine, Lambek and Gilman [15], where they had shown that $Q(\mathbb{R})$ and $C_F(X)$ are isomorphic if F is the filter of open dense sets, have been generalized. We shall first record the generalization to the case of abitrary filter of open sets in X and their associated multiplicative filter of ideals of $C(X)$. The second is for certain filter of open sets and the associated Gabriel filter of ideals of $C(X)$.

Remark 4.2.36. Let F be a filter of open sets of X. Let $\mathcal{F}$ be a collection of ideals of $C(X)$ by declaring $I \in \mathcal{F}$ if $\operatorname{Coz}(I) \in \mathcal{F}$. Then $\mathcal{F}$ is a multiplicative filter of ideals.

Proof. $\mathcal{F} \neq \phi$, for consider zero ideal,then $\operatorname{Coz}(0) = X \in \mathcal{F} \quad \Rightarrow 0 \in \mathcal{F}$.
Let $I, J \in \mathcal{F}$ so $\operatorname{Coz}(I), \operatorname{Coz}(J) \in \mathcal{F} \quad \Rightarrow \operatorname{Coz}(I) \cap \operatorname{Coz}(J) \in \mathcal{F}$.
Consider $I \cap J$ then $\operatorname{Coz}(I \cap J) = \operatorname{Coz}(I) \cap (J) \in \mathcal{F}$,
for, let

$$x \in \operatorname{Coz}(I \cap J) \quad \Rightarrow \exists g \in I \cap J \text{ such that } g(x) \neq 0.$$

$$\text{Now}, g \in I \text{ and } g \in J$$

with $g(x) = 0$ implies that $x \in \mathrm{Coz}(I) \cap \mathrm{Coz}(J)$.

On the other hand, let $x \in \mathrm{Coz}(I) \cap \mathrm{Coz}(J)$, then there exists $g \in I$, $h \in J$ such that

$$g(x) \neq 0 \text{ and } h(x) \neq 0.$$

But, $gh \in I \cap J$ and $gh(x) = g(x)h(x) \neq 0$, implies

$$x \in \mathrm{Coz}(I) \cap \mathrm{Coz}(J).$$

Hence, the equality follows.
Finally, let $I, J \in \mathcal{F}$ and consider IJ,

$$\text{then} \quad \mathrm{Coz}(I \cap J) = \mathrm{Coz}(I) \cap \mathrm{Coz}(J).$$

Let $x \in \mathrm{Coz}(IJ)$, then there exists $g \in IJ$ such that $g(x) \neq 0$.

$$\text{Since} \quad IJ \subseteq I \cap J \text{ so } g \in I \text{ and } g \in J \Rightarrow x \in \mathrm{Coz}(I) \cap \mathrm{Coz}(J).$$

Now, let $x \in \mathrm{Coz}(I) \cap \mathrm{Coz}(J)$, then there exists $g \in I$ and $h \in J$ such that $h(x) \neq 0$ and $g(x) \neq 0$.
But $gh \in IJ$ and $gh(x) = g(x)h(x) \neq 0$, implies that $x \in \mathrm{Coz}(IJ) \Rightarrow IJ \in \mathcal{F}$.
Hence $\mathcal{F}$ is a multiplicative filter. □

Theorem 4.2.37. *Let F be a filter of open subspaces of X and $\mathcal{F}$ be its associated filter of ideals of $C(X)$. Then the rings $C_F(X)$ and $q_{\mathcal{F}}(C(X))$ are isomorphic.*

Proof. We first define a map $\Phi : q_{\mathcal{F}}(C(X)) \longrightarrow C(X)$.
Let $f \in \mathrm{Hom}(I, C(X))$, where $I \in \mathcal{F}$. Then $\Phi(f)$ must be a real valued continuous function on $\mathrm{Coz}(I)$. If $x \in \mathrm{Coz}(I)$ then there exists $g \in I$ such that $g(x) \neq 0$. So define,

$$\Phi(f) : \mathrm{Coz}(I) \longrightarrow \mathbb{R}$$

$$\text{by } \Phi(f)(x) = \frac{f(g)(x)}{g(x)}.$$

Showing that $\Phi(f)$ is well-defined is equivalent to showing that the definition is independent of our choice of g. Suppose that there exists another function $h \in I$ such that $h(x) \neq 0$, then we claim that

$$\frac{f(g)(x)}{h(x)} = \frac{f(h)(x)}{h(x)}.$$

So, the condition reduces to

$$f(g)(x)h(x) = f(h)(x)g(x).$$

Since f is a *C(X)*- module homomorphism, we have

$$\begin{aligned} f(g)(x)h(x) &= [f(g)h](x) \\ &= [f(gh)](x) \qquad \text{since } C(X) \text{ is commutative.} \\ &= [f(h)g](x) \\ &= f(h)(x)g(x). \end{aligned}$$

Thus for each $x \in \mathrm{Coz}(I)$, $\Phi(f)$ is defined and continuous on a neighourhood of x. Further, $\Phi(f)$ is continuous on its domain $\mathrm{Coz}(I)$ as they agree on any overlap between neighbourhoods. Φ is a ring homomorphism. for let $f_1, f_2 \in q_{\mathcal{F}}(C(X))$ then $\exists\ I. J \in \mathcal{F}$ and since $f_1 + f_2$ is defined on $I \cap J$, so

$$\begin{aligned} \Phi(f_1 + f_2) &= \frac{(f_1 + f_2)(g)(x)}{g(x)} \\ &= \frac{f_1(g)(x) + f_2(g)(x)}{g(x)} \\ &= \frac{f_1(g)(x)}{g(x)} + \frac{f_2(g)(x)}{g(x)} \\ &= \Phi(f_1) + \Phi(f_2). \end{aligned}$$

Also, $f_1 f_2$ is defined on $IJ \in \mathcal{F}$, so

$$\begin{aligned} \Phi(f_1 f_2) &= \frac{(f_1 f_2)(g)(x)}{g(x)} \\ &= \frac{f_1 f_2(g)(x)}{g(x)} \times \frac{g(x)}{g(x)} \qquad \text{since, } g(x) \neq 0. \\ &= \frac{f_1(g)(x) f_2(g)(x)}{g(x)g(x)} \\ &= \frac{f_1(g)(x)}{(g)(x)} \times \frac{f_2(g)(x)}{g(x)} \\ &= \Phi(f_1)\Phi(f_2). \end{aligned}$$

Next. we construct an inverse for Φ. Let $U \in \mathcal{F}$ and

$$I_U{}^* = \{g \in C(X) : \mathrm{cl}(\mathrm{Coz}(g)) \subseteq U\}.$$

We first show that ${I_U}^*$ is an ideal in $C(X)$.
Let $f, g \in {I_U}^*$ and consider Coz$(f - g)$, then

$$\begin{aligned} \mathrm{Coz}(f-g) &\subseteq \mathrm{Coz}(f) \cap \mathrm{Coz}(g) \subseteq U \\ &\Rightarrow f - g \in U \quad \text{and for any} h \in C(X) \\ \mathrm{Coz}(fh) &= \mathrm{Coz}(f) \cap \mathrm{Coz}(h) \subseteq U \subseteq \mathrm{Coz}(f) \subseteq U. \\ &\Rightarrow fh \in {I_U}^*. \end{aligned}$$

Hence ${I_U}^*$ is an ideal. It is well known that in a Tychonof space every open set is the union of the closure of Coz g_i, for some collection $\{g_i\} \subseteq C(X)$ $1 \leq i \leq n$. Next we claim that Coz ${I_U}^* = U$.
Let $x \in \mathrm{Coz}({I_U}^*)$. Then $\exists\, g \in {I_U}^*$ such that $x \in \mathrm{cl}\,\mathrm{Coz}(g) \subseteq U$
$\Rightarrow \mathrm{Coz}({I_U}^*) \subseteq U$.
On the other hand ,let $x \in U$, where $U = \bigcup\limits_{g_i \in A} \mathrm{cl}(\mathrm{Coz}(g_i))$, $A \subset X$
Then $x \in \mathrm{cl}(\mathrm{Coz}(g_i))$ for some i.
Since, $\mathrm{cl}(\mathrm{Coz}(g_i)) \subseteq U$, so $g_i \in {I_U}^*$.
Thus, $x \in \mathrm{Coz}({I_U}^*)$ and ${I_U}^* \in \mathcal{F}$.
We now define

$$\Psi : C_F(X) \to q_{\mathcal{F}}(C(X)).$$

Let $g \in C(U)$ for some $U \in \mathcal{F}$.

$$\begin{aligned} &\text{Define} \quad \Psi(g) : {I_U}^* \to C(X) \\ &\text{given by} \quad \Psi(g)(h) = \widehat{gh} \qquad \text{where} \quad \text{for} \quad \text{any} \quad h \in {I_U}^*. \end{aligned}$$

$$\widehat{gh}(x) = \begin{cases} g(x)h(x) & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\widehat{gh}$ is continuous on the closed sets $X \setminus \mathrm{Coz}(g)$ and $\mathrm{cl}\,\mathrm{Coz}(g)$. Further, $(X \setminus \mathrm{Coz}(g)) \cup \mathrm{cl}(\mathrm{Coz}(g)) = X$, so by Pasting lemma $\widehat{gh}$ is continuous. Finally,

$$\begin{aligned} \Psi[\Phi(f)] &= f, \quad \text{for each} f \in q_{\mathcal{F}}(C(X)) \\ \text{and} \Phi[\Psi(g)] &= g \quad \text{for each} g \in C_F(X). \end{aligned}$$

Thus, the rings are isomorphic. □

We call a filter F of open subsets of X a Gabriel filter if the associated filter of ideals of $C(X)$ is Gabriel. The Gabriel sub-filters of the the filter of dense sets satisfy the following corollary.

Corollary 4.2.38. *Let F be a Gabriel sub-filter of the filter of open dense sets in X and let $\mathcal{F}$ be its associated filter of ideals of $C(X)$. Then the rings $C_F(X)$ and $Q_{\mathcal{F}}(C(X))$ are isomorphic.*

Proof. We know

$$\begin{aligned} \tau_{\mathcal{F}}(C(X)) &= \{f \in C(X) : fI = 0 \text{ for some } I \in \mathcal{F}\} \\ &= \{0\} \\ \text{Clearly,, } \quad C(X)/\tau_{\mathcal{F}}(C(X)) &= C(X). \end{aligned}$$

Thus by equation (4.9), $Q_{\mathcal{F}}(C(X)) = q_{\mathcal{F}}(C(X))$.

So,the result follows by Theorem[4.2.37]. □

Now we shall record some criterion for a filter of open subsets to be Gabriel.

Notation 4.2.39. $I_U = \{g . \operatorname{Coz}(g) \subseteq U\}$

Remark 4.2.40. $\operatorname{Coz}(I_U) = U$

Proof. Let $x \in \operatorname{Coz}(I_U)$, then $\exists\, g \in I_U$ such that $x \in \operatorname{Coz}(g)$.

$$\text{But,} \quad \operatorname{Coz}(g) \subseteq U \text{ implies that } \operatorname{Coz}(I_U) \subseteq U$$

On the other hand, for a Tychonoff space it is well known [13] that $U = \bigcup_{g_i \in A} \operatorname{cl}\operatorname{Coz}(g_i)$, where $A \subset C(X)$.

So, $\operatorname{cl}\operatorname{Coz}(g_i) \subseteq U \Rightarrow g_i \in I_U$. Hence the result follows. □

Lemma 4.2.41. *Let U be an open set in X and let $g \in C(X)$. Then* $\operatorname{Coz}(g^{-1}I_U) = \operatorname{Int}(U \cup Z(g))$.

Proof. Let $y \in \operatorname{Int}(U \cup Z(g))$, then there exists an open neighbourhood V of x, such that $y \in V \subseteq \operatorname{Int}(U \cup Z(g))$

Now $X \setminus V$ is a closed set in X not containing Y, so by complete regularity of X, there exists a function $h \in X$ such that $h[X \setminus V] = \{0\}$ and $h(y) = 1$,

$$\begin{aligned} \text{i.e., } Z(h) = X \setminus V &\supset X \setminus \operatorname{Int}(U \cup Z(g)) \\ &\Rightarrow X \setminus Z(h) \subseteq \operatorname{Int}(U \cup Z(g)) \\ &\Rightarrow \operatorname{Coz}(h) \subseteq \operatorname{Int}(U \cup Z(g)). \end{aligned}$$

We claim that $g \in g^{-1}I_U$. It suffices to show that $\operatorname{Coz}(hg) \subseteq U$, for then $hg \in I_U \quad \Rightarrow h \in g^{-1}I_U$.
If $x \in \operatorname{Coz}(hg)$, then $hg(x) = h(x)g(x) \neq 0$.
But $\operatorname{Coz}(h) \subseteq \operatorname{Int}(U \cup Z(g)) \subseteq U \cup Z(g)$, so, $g(x) \neq 0 \Rightarrow x \in U$.
Thus $y \in \operatorname{Coz}(h) \subseteq \operatorname{Coz}(g^{-1}I_U)$ and $\operatorname{Int}(U \cup Z(g)) \subseteq \operatorname{Coz}(g^{-1}I_U)$.
On the other hand, let $y \in \operatorname{Coz}(g^{-1}I_U)$. Then $y \in \operatorname{Coz}(f)$ for some $f \in g^{-1}I_U$, implies that

$$fg \in I_U \Rightarrow \operatorname{Coz}(fg) \subseteq U.$$

But $\operatorname{Coz}(fg) = \operatorname{Coz}(f) \cap \operatorname{Coz}(g) \subseteq U$.
Thus if $y \in \operatorname{Coz}(g)$, then $y \in \operatorname{Coz}(f) \cap \operatorname{Coz}(g) \subseteq U$. If not, then $y \in Z(g)$. In any case $y \in U \cup Z(g)$. Since $\operatorname{Coz}(g^{-1}I_U)$ is an open set such that $y \in \operatorname{Coz}(g^{-1}I_U) \subseteq U \cap Z(g)$, so $y \in \operatorname{Int}(U \cup Z(g))$. Hence the equality holds. ☐

Theorem 4.2.42. *Let $\mathbb{F}$ be a filter of open sets on the space X. Then $\mathbb{F}$ is Gabriel iff it satisfies the following : Let $\{g_k\}_{k\in K}$ be a collection of elements of C(X) such that $\bigcup \operatorname{Coz}(g_k) \in \mathbb{F}$. Let V be any open set such that for each $k \in K$, $V \cup Z(g_k)$ contains an element of $\mathbb{F}$, then $V \in \mathbb{F}$.*

Proof. Let us suppose that $\mathbb{F}$ is Gabriel. Let $\{g_k\}_{k\in K}$ be a family of functions in *C(X)* such that $\bigcup \operatorname{Coz}(g_k) \in \mathbb{F}$. Let V be an open set such that $V \cup Z(g_k)$ contains an element of $\mathbb{F}$ for each $k \in K$.
We claim that $V \in \mathcal{F}$ It is enough to show that $I_V \in \mathcal{F}$, since $\operatorname{Coz}(I_V) = V$.
Let J be the ideal generated by the set $\{g_k\}_{k\in K}$.
Now,

$$\begin{aligned}
\operatorname{Coz}(J) &= \underset{g_k \in \mathcal{F}}{\operatorname{Coz}(g_k)} \in \mathbb{F} \\
&\Rightarrow J \in \mathcal{F}. \\
\text{By Lemma[4.2.41]}\ \operatorname{Coz}({g_k}^{-1}I_V) &= \operatorname{Int}(V \cup Z(g_k)) \\
\text{Since } V \cup Z(g_k) &\supset L \ \text{ for some } L \in \mathbb{F} \\
\text{so, } \operatorname{Int}(V \cup Z(g_k)) &\supset L \\
&\Rightarrow \operatorname{Int}(V \cup Z(g_k)) \in \mathcal{F} \\
&\Rightarrow {g_k}^{-1}I_V \in \mathcal{F} \forall k \in K \\
\text{for } \operatorname{Coz}(h^{-1}I_V) &= \operatorname{Int}(v \cup \operatorname{Coz}(h)), \quad \text{where } \operatorname{Coz}(h) \subseteq .
\end{aligned}$$

Hence $h^{-1}I_V \in \mathcal{F}, \quad \forall h \in I_V$. Since $\mathcal{F}$ is a Gabriel filter, so $I_V \in \mathcal{F}$.
Conversely, Suppose that $\mathbb{F}$ satisfies the given conditions. Let I and J be ideals of *C(X)* such that $J \in \mathcal{F}$ satisfying $h^{-1}I \in \mathcal{F} \quad \forall h \in J$. If we let $U = \operatorname{Coz}(J)$ and $V = \operatorname{Coz}(I)$, then $\operatorname{Coz}(h^{-1} \in \mathcal{F}) \ \forall h \in J$.

By Lemma [4.2.41] $\mathrm{Coz}(h^{-1}I) = \mathrm{Int}(V \cup Z(h)) \in \mathbb{F}$. Thus for all $h \in J$, we have $V \cup Z(h) \supset \mathrm{Int}(V \cup Z(h))$. So, by the given condition $V \in \mathbb{F} \Rightarrow I \in \mathcal{F}$. Hence $\mathcal{F}$ is a Gabriel filter and consequently its associated filter $\mathbb{F}$ is Gabriel filter too. ☐

Remark 4.2.43. If $\mathcal{F}$ is a Gabriel filter of the ring *C(X)* which comes from a filter of open sets of X, then $\mathcal{F}$ satisfies the property that whenever I is an arbitrary ideal and $J \in \mathcal{F}$ such that $\mathrm{Coz}(J) \subseteq \mathrm{Coz}(I)$, then $I \in \mathcal{F}$. A filter with this property is called a z-Gabriel filter. If $\mathcal{F}$ is a z-Gabriel filter of *C(X)*, then the family $\{\mathrm{Coz}(J) : J \in \mathcal{F}\}$ is a Gabriel filter of X as its associated filter of ideals is $\mathcal{F}$. Thus there is a bijection between Gabriel filters of open sets of X and z-Gabriel filter of *C(X)*.

Now we shall give more examples of Gabriel filters of a space X.

Example 4.2.44. Let X be any non-empty space and let Y be a non-empty subspace. Let F be the set of open subsets U of X such that $U \cap Y$ is dense in Y. Then F is a Gabriel filter.

Proof. We shall first show that $\mathbb{F}$ is a filter,
let $U_1, U_2 \in \mathbb{F}$ such that $\mathrm{cl}(U_1 \cap Y) = Y$ and $\mathrm{cl}(U_2 \cap Y) = Y$, then $\mathrm{cl}[(U_1 \cap U_2) \cap Y] = Y$
Also if $U \supset V$, where $V \in \mathbb{F}$, then $\mathrm{cl}(V \cap Y) = Y$, so $\mathrm{cl}(U \cap Y) \supset \mathrm{cl}(V \cap Y) = Y$. Hence it is a filter.
To show that it is a Gabriel-filter, let $U = \bigcup \mathrm{Coz}(g_k) \in \mathbb{F}$ for some collection $\{g_k\} \subseteq \mathbb{F}$. We will show that for some g_k, $V \cup Z(g_k)$ contains no element of $\mathbb{F}$.
Now, $V \notin \mathbb{F} \Rightarrow V \cap Y$ is not dense in Y, we can choose an open set W of the space Y with $W \subseteq Y \setminus V$. This is possible for $\exists\, y \in Y$ such that for some neighbourhood of Y in Y, say, Nbd.(Y) Nbd. (y) $\cap U = \phi$.
Since $U \in \mathbb{F}$ there exists $g_k = g$ such that $\mathrm{Coz}(g) \cap W \neq \phi$. Thus,

$$\begin{aligned}(V \cup Z(g)) \cap (\mathrm{Coz}(g) \cap W) &= (V \cap \mathrm{Coz}(g) \cap W) \cup (Z(g) \cap (\mathrm{Coz}(g) \cap W)) \\ &= \phi\end{aligned}$$

So, $V \cup Z(g)$ cannot contain an element of $\mathbb{F}$, otherwise it contradicts that $\mathbb{F}$ is a filter. So by Theorem [4.2.42] $\mathbb{F}$ is a Gabriel filter.

☐

Example 4.2.45. Let X be any space and let Y be a subset of X. Let $\mathbb{F}$ be the filter of all open sets that contain Y. Then $\mathbb{F}$ is a Gabriel filter.

Proof. Clearly $\mathbb{F}$ is a filter for any two elements of $\mathbb{F}$ containing Y, their intersection is open and contains Y. Similarly, if $U \supset V$, where $V \in \mathbb{F}$ and U is an open set of $\mathbb{F}$, then U also contains Y, hence it is in $\mathbb{F}$. Similarly, as done above let $U = \bigcup \text{Coz}(g_k) \in \mathbb{F}$ and let V be an open set such that V is not in $\mathbb{F}$. Then there exists $y \in Y \setminus V$. Since, $U \in \mathbb{F}$, there is a $g = g_k$ such that $y \in \text{Coz}(g)$. Thus $y \notin V \cup Z(g)$ and so $V \cup Z(g) \notin \mathbb{F}$ as $Y \not\subseteq V \cup Z(g)$. Therefore, by Theorem [4.2.42], $\mathbb{F}$ is a Gabriel filter. □

Remark 4.2.46. Intersections of two Gabriel filters is again a Gabriel filter.

The above two examples were of similar category, now we shall give an example that is not in this category.

Example 4.2.47. Let X be the closed unit interval and let $\mathbb{F}$ be the collection of open sets of X which are co-countable. Then $\mathbb{F}$ is a Gabriel filter.

Proof. To show that is a filter, let $U, V \in \mathbb{F}$ such that $X \setminus U$ and $X \setminus V$ are countable,

then $X \setminus U \cap V = U^c \cup V^c$, which is countable. Now let $U \supset V$ such that $V \in \mathbb{F}$ where U is an open set in X, then $X \setminus U \subseteq X \setminus V$ which is countable, hence $X \setminus U$ is countable too, so $U \in \mathbb{F}$.

Now let V be an open subset of X and let $\{g_i\}$ be a collection of functions in $C(X)$ such that $U = \bigcup \text{Coz}(g_i) \in \mathbb{F}$. Also, suppose that $V \cup Z(g_i)$ contains an element of $\mathbb{F}$ for each function g_i and on the contrary $V \notin \mathbb{F}$. This implies that $X \setminus V$ is uncountable set which means that it must contain a copy of the Cantor set, C. Since $V \cup Z(g_i)$ contains an element of $\mathbb{F}$, each $Z(g_i)$ must contain all but finitely many elements of C. However, since $Z(g_i)$ is closed, it must therefore contain all of C. Hence U cannot contain C, which is a contradiction as U is in $\mathbb{F}$. Hence F is a Gabriel filter.

□

We shall now give an example of a filter of open sets which is not a Gabriel filter, even though it is similar to the previous example.

Example 4.2.48. Let X be the closed inteval and let $\mathbb{F}$ be the filter of all co-finite open subsets of X. Then $\mathbb{F}$ is not a Gabriel filter.

Proof. The proof that it is a filter is similar to that of previous example. Now let $\{a_n\}_{n \in \mathbb{N}}$ be a convergent sequence in X with limit point a. Let $V = X \setminus A$, where $A = \{a_n\} \cup \{a\}$. Since $X \setminus V = A$, V is not in $\mathbb{F}$. Let g_n be an element of $C(X)$ with zero-set $A \setminus \{a_n\}$.

Then $U = \bigcup \text{Coz}(g_n) = X \in \mathbb{F}$. On the other hand, $V \cup Z(g_n)$ is clearly in $\mathbb{F}$ for each n. Thus by Theorem [4.2.42], $\mathbb{F}$ is not a Gabriel filter. □

Remark 4.2.49. Let $\mathbb{F}$ be a Gabriel filter of open sets in X and let $\mathcal{F}$ be its associated filter of ideals of $C(X)$, then

$$\tau_{\mathcal{F}}(C(X) = \{f \in C(X) | fI = 0 \text{ for some } I \in \mathcal{F}\}.$$

Let $Y_{\mathbb{F}}$ denote the intersection of the closures of the elements of $\mathbb{F}$.
Clearly, $f \in \tau_{\mathcal{F}}(C(X)), \Rightarrow Y_{\mathbb{F}} \subseteq Z(f)$, in particular the kernel of the canonical map $\Psi : C(X) \longrightarrow C(Y_{\mathbb{F}})$ contains $\tau_{\mathcal{F}}(C(X)$. Moreover, if $\text{Int}(Y_{\mathbb{F}}) \in \mathbb{F}$, then the two ideals are equal.

Note 4.2.50. Suppose that X is a normal space then the restriction map

$$\phi : C(X) \to C(Y)$$

is onto whenever Y is a closed subset of X by Tietze's extension theorem. In particular if $\mathbb{F}$ is a Gabriel filter of X, then the natural map

$$\varphi : C(X) \longrightarrow C(Y)$$

where $Y = Y_F$ is onto as $Y_{\mathbb{F}}$ is closed. Using the filter $\mathbb{F}$ we define a filter F_Y of open sets of Y by declaring $U \in F_Y$ if $U = W \cap Y$, where $W \in \mathbb{F}$. We shall verify that F_Y is indeed a filter.

Let $U_1, U_2 \in F_Y$ Then there exists $W_1, W_2 \in \mathbb{F}$ such that $U_1 = W_1 \cap Y$, $U_2 = W_2 \cap Y$and $U_1 \cap U_2 = (W_1 \cap W_2)$ as $W_1 \cap W_2 \in \mathbb{F}$.

Also, if U is an open set of Y such that $U \supset V$, where $V \in F_Y$ and $V = W \cap Y$, then $U = W' \cap Y$, where W' is open in X.So,

$$U \supset V \quad \Rightarrow W' \supset W \quad \Rightarrow W' \in \mathbb{F}.$$

Hence $U \in \mathbb{F}$. In addition, if we suppose that $\text{Int}(Y) \in \mathbb{F}$, then we claim that F_Y is also Gabriel. We observe that every element U of F_Y contains an element of $\mathbb{F}$, viz. $W \cap \text{Int}(Y)$, where $W \in \mathbb{F}$ and $U = W \cap Y$. Now suppose that V is an open set in Y and there is a family of functions $\{g_i\}_{i \in \mathbb{N}}$, defined on Y such that $\bigcup \text{Coz}(g_i) \in F_Y$ and $V \cup Z(g_i)$ contains an element of F_Y for each g_i. Since $\bigcup \text{Coz}(g_i) \in F_Y$ it contains an element of $\mathbb{F}$ as $\bigcup \text{Coz}(g_i)$ contains an element of $\mathbb{F}$. Furthermore, if W is any open set of X, such that $W \cap Y = V$, it follows that $W \cup Z(g_i') \supset V \cup Z(g_i)$ which cantains an element of $\mathbb{F}$. Since $\mathbb{F}$ is a Gabriel filter, $W \in \mathbb{F}$ and so $V \in F_Y$. Hence, F_Y is Gabriel.

So, the stage is set for describing the localization of $C(X)$ at the associated filter of ideals in terms of partial functions on a certain topological spaces

Proposition 4.2.51. *Let* $\mathbb{F}$ *be a Gabriel filter on* X, *where* X *is normal and suppose that* $\mathrm{Int}(F_Y)$ *is in* $\mathbb{F}$. *If* $\mathcal{F}$ *is the associated filter of ideals of* $C(X)$, *then* $Q_{\mathcal{F}}(C(X)) \cong C_{F_Y}(Y_{\mathbb{F}})$.

Proof. Let $Y = Y_{\mathbb{F}}$ and let $\mathcal{F}_Y$ denote the filter of ideals of $C(Y)$ associated to $\mathbb{F}_Y$. Then $\mathcal{F}_Y$ is a Gabriel filter of the filter of dense ideals. Thus $q_{\mathcal{F}_Y}(C(Y)) = C_{\mathbb{F}}(Y)$ by Theorem [4.2.37]. Suppose the ideal I of $C(Y)$ is in $\mathbb{F}_Y$, then its inverse image $J = \Phi^{-1}(I)$ in $C(X)$ is in $\mathcal{F}$, for

$$\mathrm{Coz}(I) = \bigcup_{g \in I} \mathrm{Coz}(g) \subseteq \bigcup_{h \in J} \mathrm{Coz}(h) = \mathrm{Coz}(J).$$

We know from above note that if $\mathrm{Coz}(I) \in \mathbb{F}_Y$, then it contains an element of $\mathbb{F}$. This implies that $\mathrm{Coz}(J) \in \mathbb{F}$ and hence $J \in \mathcal{F}$
Conversely, if $J \in \mathcal{F}$, then $(\bigcup_{h \in J} \mathrm{Coz}(h)) \cap Y$ is contained in $\bigcup_{g \in \Phi(J)} \mathrm{Coz}(g)$ i.e.,

$$\begin{aligned} \mathrm{Coz}(J) \cap Y &\subseteq \mathrm{Coz}(\Phi(J)) \\ &\Rightarrow \mathrm{Coz}(\Phi(J)) \in \mathbb{F}_Y \\ &\Rightarrow \Phi(J) \in \mathcal{F}_Y \end{aligned}$$

Furthermore, if J is any ideal of $C(X)$, then

$$\mathrm{Hom}_{C(X)}(J, C(Y)) \cong \mathrm{Hom}_{C(Y)}(\Phi(J), C(Y)) \text{ as } C(Y) \text{ modules.}$$

Since $C(Y) = C(X)/\tau_{\mathcal{F}}(C(X))$,
Therefore, $Q_{\mathcal{F}}(C(X)) = q_{\mathcal{F}_Y}(C(Y)) = C_{\mathbb{F}}(Y)$. Hence proved. □

Proposition 4.2.52. *Let* $\mathbb{F}$ *be a Gabriel filter of open sets in* X. *Suppose that* $\bigcap \mathbb{F} \subseteq \mathrm{Int}(Y_{\mathbb{F}})$ *and* $Bd(Y_{\mathbb{F}})$ *contains only finitely many accumulation points, then* $\mathrm{Int}(Y_{\mathbb{F}}) \in \mathbb{F}$.

Proof. Let us denote $Y_{\mathbb{F}}$ by Y and let $V = X \setminus \mathrm{Bd}(Y)$ We shall first show that $V \in \mathbb{F}$. Since $\cap \mathbb{F} \subseteq \mathrm{Int}(Y_{\mathbb{F}})$, for each $b \in \mathrm{Bd}(Y)$ there exists $I \in \mathbb{F}$. Let A be any finite subset of $\mathrm{Bd}(Y)$,then $X \setminus A$ is in $\mathbb{F}$, for let $A = \{a_1, a_2, \ , a_n\}$, then there exists $I_1, I_2, .. , I_n$ such that $a_i \notin I_i$, $1 \leq i \leq n$.

$$\begin{aligned} \text{Now} \quad X\setminus A &\supset I_1\cap \ldots \cap I_n = I' \ (say) \\ &\Rightarrow X\setminus A \supset I' \\ &\text{and } I' \in \mathbb{F} \quad \Rightarrow X\setminus A \in \mathbb{F}. \end{aligned}$$

For each $b \in \mathrm{Bd}(Y)$, b not an accumulation point of $\mathrm{Bd}(Y)$, the set $\mathrm{Bd}(Y)\setminus\{b\}$ is a closed set. Hence, there exists a family $\{{g_i}^b\}$ of functions such that the intersection of the zero-sets of these functions is $\mathrm{Bd}(Y)\setminus\{b\}$. The union of the cozero-sets of these families of functions over all $b \in \mathrm{Bd}(Y)$, b not an accumlation point is the set $X\setminus\{\text{acumulation points of } \mathrm{Bd}(Y)\}$. Since, there are only finitely many acumulation points and $\cap\mathbb{F} \subseteq \mathrm{Int}(Y_{\mathbb{F}})$, this set is in $\mathbb{F}$.

On the other hand, for each function g_i^b, $V \cup Z(g_i^b) \supset X\setminus\{b\} \in \mathbb{F}$. Hence by Theorem [4.2.42] $V \in \mathbb{F}$. Now for each $U \in \mathbb{F}$, let C_U be the closure of $U\setminus\mathrm{Int}(Y)$ and let $\{{h_i}^U\}$ be a family of functions such that $\cap Z({h_i}^U) = C_U$. Then for each $U \in \mathbb{F}$ and for each ${h_i}^U$, we have $\mathrm{Int}(Y)\cup Z({h_i}^U)$ contains U. Furthermore, since the intersection of the closures of the elements of $\mathbb{F}$ is Y, we have $\cup\,\mathrm{Coz}({h_i}^U)$ contains $X\setminus\mathrm{Bd}(Y) \in \mathbb{F}$. Hence, using Theorem[4.2.42] $\mathrm{Int}(Y) \in \mathbb{F}$. ☐

We shall conclude this section by giving an example of a Gabriel filter $\mathbb{F}$ such that $\mathrm{Int}(Y_{\mathbb{F}})$ is not empty and not in $\mathbb{F}$

Example 4.2.53 (4.11, [21]). Let X be the unit square and let Y be a closed disk contained in X. Let $\mathbb{F}$ be the collection of all open sets U in X, such that $U\cap\mathrm{Bd}(X)$ is a co-countable subset of $\mathrm{Bd}(Y)$ and $\mathrm{Int}(Y)\subseteq U$. Then clearly the intersection of the closures of the elements of $\mathbb{F}$ is the set Y and $\mathrm{Int}(Y)$ is not in $\mathbb{F}$.

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