ENDOMORPHISM RINGS OF IDEALS

By

AMITA MITRA

DEPARTMENT OF MATHEMATICS
NORTH-EASTERN HILL UNIVERSITY

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CERTIFICATE

I certify that the dissertation entitled " ENDOMORPHISM RINGS OF IDEALS" submitted by Ms. Amita Mitra in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

Mangesh B. Rege

Mangesh B. Rege
Supervisor
Mathematics Department
North-Eastern Hill University
Shillong

Shillong
February 2, 1993.

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INTRODUCTION

The title of this dissertation derives from our original plan : a survey of some investigations carried out by Cox [C:73]* Clark [C:86], Wiegand [W:69], Alamelu, Vasconcelos and others in the area of endomorphism rings of ideals of commutative and non-commutative rings. However, while following this plan it became apparent that for such a survey to be fruitful it is necessary to include results concerning endomorphism rings of arbitrary modules over various rings. The necessity of including, for the sake of completeness and convenience of the reader, basic properties of the rings and modules under consideration was also felt.

The material surveyed (after this shift of emphasis) has been organised as follows. Results concerning directly finite modules and directly finite rings are collected in Chapter 1. Since hopfian modules and cohopfian modules (see text for definitions) are directly finite several results of independent interest concerning them are also recorded there. The main "endomorphism ring" results of this chapter

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*The bibliography is divided into two parts: books and monographs, research papers. In each reference to a research paper we have included a two-digit number, indicating the year of publication. This will give an approximate idea of when the research was carried out.

are lemma 1.9, Proposition 1.10, 1.12, 1.13 and Corollary 1.14.

Chapter 2 is devoted to a study of (von Neumann) regular, strongly regular, unit-regular and dependent rings. These results were originally obtained by, among others, Wiegand, Steinberg, Ehrlich and Henriksen. (Detailed mention of our debt is made in the bibliographical notes at the end of each chapter.) There are several "endomorphism ring" results in this chapter.

Chapter 3 is almost entirely devoted to our original theme. Sufficient conditions for the endomorphism ring of an ideal to be commutative are given. We end with an example of Clark that settles a question of Faith.

Since we have somewhat deviated from our original plan a change in the name of the dissertation would perhaps have been in the fitness of things. However, a change of title and synopsis would have required the approval of the University's academic authorities, caused administrative hassles and resulted in some delay. Because of these reasons no effort was made for a change of title. We seek the reader's indulgence for this.

We end this introduction with some remarks on notations, conventions etc. for undefined concepts and results we refer to the books and monographs mentioned in the bibliography. Standard notation is followed. The letter R always denotes a ring. Ideal means two-sided ideal. All our left-sided concepts and results have right-sided counterparts. A left R-module M is denoted by ${}_RM$ and a right R-module M by M_R. When ${}_RN$ is a submodule of ${}_RM$ we write $N \leq M$; the ring under consideration may not always be mentioned. Homomorphisms of modules are always written on the left except if otherwise mentioned. The letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of natural numbers, integers, rational numbers and real numbers respectively. For a ring S, $\overline{I}(S)$ denotes the set of all idempotents of S.

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CHAPTER 1

DIRECT FINITENESS

In this chapter we shall study the direct finiteness of modules. Proposition 1.10 connects direct finiteness of a module M with the direct finiteness of the ring $\mathrm{End}_R(M)$. Since hopfian and cohopfian modules (defined in § 1 below) are directly finite, § 3 is devoted to a study of these classes of modules.

§ 1. Basic definitions

In this section we collect some definitions needed by us.

1.1. Directly finite module. A module ${}_RM$ is *directly finite* if it is not isomorphic to a proper direct summand of M.

1.2. Directly finite ring. A ring R is *directly finite* if for elements a, b of R, ab = 1 implies ba = 1. The ring R is directly finite if and only if ${}_RR$ is a directly finite module (and if and only if R_R is a directly finite module). This can be seen directly; it can also be derived as a corollary of Proposition 1.10.

1.3. Hopfian module. A module ${}_RM$ is *hopfian* if every onto endomorphism of M is an automorphism.

1.4. Cohopfian module. A module ${}_RM$ is *cohopfian* if every one one endomorphism of M is an automorphism.

1.5. Noetherian module. A module ${}_RM$ is *noetherian* if for every ascending chain $N_1 \leq N_2 \leq \ldots..$ of submodules of M we have $N_r = N_{r+1}$ for some $r \in \mathbb{N}$.

1.6. Artinian module. A module ${}_RM$ is *artinian* if for every descending chain $N_1 \geq N_2 \geq \ldots..$ of submodules of M we have $N_r = N_{r+1}$ for some $r \in \mathbb{N}$.

1.7. Left duo ring. A ring R is a *left duo ring* if every left ideal of R is also a right ideal (and hence is an ideal).

1.8. Abelian ring. A ring R in which all the idempotent elements are central is an *abelian ring*.

§ 2. Basic results.

In this section we discuss a condition on a module M equivalent to the condition $\text{End}_R(M)$ is a directly finite ring.

1.9. Lemma. (Peterson) Let ${}_RM$ be a module. Then $\text{End}_R(M)$ is not a directly finite ring $\Leftrightarrow$ there exist submodules M', N of M with $M = M' \oplus N$ and $M' \cong M$ and $N \neq 0$.

Proof. Let $\alpha, \beta \in \text{End}_R(M)$ be such that $\alpha\beta = 1$ but $\beta\alpha \neq 1$.

Let $M' = \beta(M)$ and $N = \ker \alpha$. Now $\alpha\beta = 1$ implies that β is one-one. So $M \cong \beta(M) = M'$.
Let, if possible α be one-one. Also α is onto as $\alpha\beta = 1$. So α is an isomorphism. Now $\alpha\beta = 1 \Rightarrow \beta = \alpha^{-1}$. This implies that $\beta\alpha = 1$ which is a contradiction. Hence α cannot be one-one. So $N \neq 0$. Next we assert that
$N = (1- \beta\alpha)(M)$ .... (I).
Let $m \in M$. Since $\alpha\ ((1-\beta\alpha)m) = (\alpha - \alpha\beta\alpha)\ (m) = (\alpha - \alpha).(m) = 0$, it follows that $(1-\beta\alpha)\ (m) \in \ker \alpha = N$.
Next $m \in \ker \alpha \Rightarrow \alpha(m) = 0 \Rightarrow m = m - \beta\alpha(\ m\ ) = (1 - \beta\alpha)(m)$.
Thus $N \leq (1-\beta\alpha)(M)$. It follows that $N = (1 - \beta\alpha)(M)$. This proves assertion (I). If $m \in M$ then $m = m - \beta\alpha(m) + \beta\alpha(m) = \beta\alpha(m) + (1 - \beta\alpha)(m) \in M' + N$.

Now let $t \in M' \cap N$. Then $t = \beta(m)$ for some $m \in M$ and $\alpha(t) = 0$. Therefore $0 = \alpha(t) = \alpha\beta(m) = m$. Thus $t = \beta(m) = 0$. So $M = M' \oplus N$ with $M' \cong M$ and $N \neq 0$.

Conversely, suppose there exist M', N such that $M = M' \oplus N$, $N \neq 0$ and ϕ is an isomorphism of M' to M. We define a mapping α from M to M : $\alpha(m) = \phi(m')$ where $m = m'+n$, $m' \in M'$, $n \in N$. We define $\beta \in \mathrm{End}_R(M)$ as $\beta(x) = \phi^{-1}(x)$.
Then $\alpha\beta(m) = \alpha\phi^{-1}(m) = \phi\phi^{-1}(m) = m \quad \forall\ m \in M$. Thus $\alpha\beta = 1_M$.
However $\beta\alpha(M) \leq M' \neq M$. Thus $\beta\alpha \neq 1_M$. So $\mathrm{End}_R(M)$ is not a directly finite ring. ∎

1.10. <u>Proposition.</u> A module ${}_RM$ is a directly finite module if and only if $\mathrm{End}_R(M)$ is a directly finite ring.
Proof: We get this result by applying the above lemma. ∎

1.11. Example We show by an example that a submodule of a directly finite module need not be directly finite.

Let $R = K(+)V$, where V is an infinite dimensional vector space over a field K. We define addition and multiplication in R as follows:

$(x,a) + (y,b) = (x+a, y+b)$, $(x,a).(y,b) = (xy, xb + ya)$

where $x,y \in K$ and $a,b \in V$. With these definitions R becomes a ring. Now, $(x,a).(y,b) = (xy, xb + ya)$

$= (yx, ya + xb) = (y,b).(x,a)$.

Thus R is commutative. So ${}_RR$ is directly finite. Now consider $\tilde{V} = \{ (0,a) \mid a \in V \} \leq K(+)V$

$\tilde{V}$ is an R submodule of R and $\tilde{V} \cong V$, as K-vector spaces. It can be seen that the R-submodules of $\tilde{V}$ correspond to the K-subspaces of V. As V is an infinite dimensional vector space ${}_R\tilde{V}$ is not a directly finite module.

1.12. Proposition. The right R-module R^n is directly finite if and only if $M_n(R)$ is a directly finite ring.

Proof: R^n_R is a directly finite module

$\Leftrightarrow$ $\mathrm{End}(R^n{}_R)$ is a directly finite ring

$\Leftrightarrow$ $M_n(R)$ is a directly finite ring. $\left[\text{As } \mathrm{End}(R^n{}_R) \cong M_n(R). \right]$. ▣

1.13. Proposition. (Peterson) If M is a left R-module such that $\mathrm{End}_R(M)$ is not directly finite then there exists a proper ascending chain and a proper descending chain of submodules of M.

Proof: Since $\mathrm{End}_R(M)$ is not directly finite by Lemma 1.9

there exist submodules M_1 and N_1 of M such that $M = M_1 \oplus N_1$ with $M_1 \cong M$ and $N_1 \neq 0$. Let ϕ be an isomorphism of M_1 onto M.

Then there exists $\psi : M \longrightarrow M_1$ such that $\phi\psi = 1_M$ and $\psi\phi = 1_{M_1}$.

Let $M_2 = \psi(M_1)$ and $N_2 = \psi(N_1)$. Let $m_1 \in M_1$. Then $\phi(m_1) \in M$.

So $\phi(m_1) = m'_1 + n'_1$ for some $m'_1 \in M_1$, $n'_1 \in N_1$. This gives $\psi\phi(m_1) = \psi(m'_1) + \psi(n'_1) \Rightarrow m_1 = \psi(m'_1) + \psi(n'_1) \in M_2 + N_2$. Let $t \in M_2 \cap N_2$. Then $t = \psi(m) = \psi(n)$ for some $m \in M_1$, $n \in N_1$. Then $\phi(t) = m = n \in M_1 \cap N_1 = 0$. This implies that $t = 0$. Thus $M_1 = M_2 \oplus N_2$. Again, $M_2 \cong M_1$ and $N_2 \cong N_1$ under the mapping ϕ. So we have $M_1 = M_2 \oplus N_2$ with $M_2 \cong M$ and $N_2 \cong N_1$. Proceeding in this way, we can find sequences M_k and N_k of submodules of M such that $M_k = M_{k+1} \oplus N_{k+1}$ with $M_{k+1} \cong M$, $N_{k+1} \cong N_1$. Then the sequence $\left\{ M_k \right\}_{k\in\mathbb{N}}$ is a properly descending chain of submodules of M. Let $A_k = N_1 + \ldots + N_k$ Then $\left\{ A_k \right\}_{k\in\mathbb{N}}$ is a properly ascending chain of submodules of M. ▨

1.14. <u>Corollary</u>. If a module ${}_RM$ satisfies the ascending chain condition or the descending chain condition on its direct summands over R, then $\mathrm{End}_R(M)$ is directly finite.

1.15. <u>Corollary</u>. If R is a left noetherian ring then R is directly finite.

§ 3. Hopfian and cohopfian modules.

In this section we study hopfian and cohopfian modules (defined in § 1 above) and connect them with ascending chain condition, descending chain condition and direct finiteness.

1.16. Proposition. If ${}_RM$ is a hopfian module, then it is directly finite.

Proof: Let f, g $\in \mathrm{End}_R(M)$ be such that $fg = 1_M$. Then f is onto . But M is hopfian, so f must be one-one. So there exists $f' \in \mathrm{End}_R(M)$ such that $ff' = f'f = 1_M$. It follows that $f' = g$, so that $gf = 1_M$. ▩

1.17. Proposition. If M is a cohopfian module then it is directly finite.

Proof: Let f,g $\in \mathrm{End}_R(M)$ be such that $fg = 1_M$ which implies that g is one one. So g must be onto since M is cohopfian. Therefore there exists $g' \in \mathrm{End}_R(M)$ such that $gg' = g'g = 1_M$. So it follows that $f = g'$. Therefore, $gf = 1_M$.

1.18. Proposition. Let R be a ring. Then ${}_RR$ is hopfian

$\Leftrightarrow$ R is directly finite.

Proof: ($\Rightarrow$) We can prove this part by putting M = R in Proposition 1.16. But we can also prove it directly as shown below.

Let $xy = 1$ in R.

Consider the map $\theta_y : R \longrightarrow R$ defined by $\theta_y(z) = zy$

$\forall z \in R$. Since $1 = xy$, therefore $1 \in \operatorname{Im} \theta_y$. So θ_y is onto. Since ${}_RR$ is hopfian θ_y must be one one.

Now, $\theta_y(1-yx) = (1 - yx)y = y-yxy = y-y = 0$.

Hence $1 - yx = 0 \Rightarrow yx = 1$. So R is directly finite.

($\Leftarrow$) Let R be directly finite. Let θ be an epimorphism from ${}_RR$ to ${}_RR$. Then θ must be of the form θ_y where $\theta_y(x) = xy$, for some $y \in R$.

Now θ_y is onto. So $1 = xy$, for some $x \in R$.

Since R is directly finite, $yx = 1$.

Now, let, $z \in \ker \theta_y$. Then $\theta_y(z) = 0 \Rightarrow zy = 0 \Rightarrow zyx = 0$ $\Rightarrow z = 0$. Thus $\theta = \theta_y$ is one-one. It follows that ${}_RR$ is hophian. ∎

1.19. <u>Proposition</u>. Let R be a ring. Then ${}_RR$ is cohophian $\Rightarrow R$ is directly finite.

Proof: This can be proved by putting $M = R$ in proposition

We note that the converse of the above statement is not true, e.g. ${}_{\mathbb{Z}}\mathbb{Z}$ is directly finite but not cohophian. ∎

1.20. <u>Lemma</u>. Let ${}_RM$ be a module. If θ is an onto endomorphism of M such that $\ker\theta = \ker\theta^2$, then θ is a monomorphism.

Proof: Let $\theta(m) = 0$. Since θ is an epimorphism, there exists $m' \in M$ such that $\theta(m') = m$. This yields $\theta^2(m') = \theta(m) = 0$.

Therefore $m' \in \ker\theta^2 = \ker\theta$. So $m = \theta(m') = 0$.

Hence θ is a monomorphism. ∎

1.21. <u>Lemma</u>. If θ is a one-one endomorphism of a module ${}_RM$ such that $\operatorname{Im}\theta = \operatorname{Im}\theta^2$ then θ is an epimorphism.

Proof: Let $z \in M$. Then $\theta(z) = \theta^2(y)$ for some $y \in M$. This implies that $\theta(z - \theta(y)) = 0 \Rightarrow z - \theta(y) = 0 \Rightarrow z = \theta(y)$ which shows that θ is an epimorphism. ▣

1.22. <u>Proposition</u>. If ${}_RM$ is a noetherian module then it is hopfian.

Proof: let f be an onto endomorphism of M. Since M is noetherian, the ascending chain

$\ker f \leq \ker f^2 \leq \ldots \leq \ker f^n \leq \ldots$ of submodules of M must terminate. Therefore $\ker f^n = \ker f^{2n}$ for some $n \in \mathbb{N}$. Since f^n is onto, f^n is a monomorphism by Lemma 1.20. Hence f is also one-one. Thus ${}_RM$ is a hopfian module. ▣

1.23. <u>Example</u>. Consider $R = \prod_{i=1}^{\infty} K_i$, K_i fields. Then R is commutative, so directly finite. So ${}_RR$ is hopfian. But ${}_RR$ is not noetherian.

1.24. <u>Proposition.</u> If ${}_RM$ is an artinian module then it is cohopfian.

Proof: Let f be a one-one endomorphism of M. Then the descending chain $\operatorname{Im} f \geq \operatorname{Im} f^2 \geq \ldots \geq \operatorname{Im} f^n \geq \ldots$ of submodules of M must terminate as M is artinian. Therefore $\operatorname{Im} f^n = \operatorname{Im} f^{2n}$ for some $n \in \mathbb{N}$. By Lemma 1.21 f^n is an epimorphism. Therefore f is an epimorphism. Thus M is cohopfian. ▣

1.25. **Quasi-injective module.** A module M is *quasi-injective* if given $K \leq M$ and any homomorphism $\phi : K \longrightarrow M$, there exists $g : M \longrightarrow M$ which extends ϕ.

1.26. **Proposition.** Let M be a quasi-injective directly finite module. Then M is cohopfian.

Proof: Let f be a one-one endomorphism of M. We apply definition 1.25 with $K = M$ and $1_M : M \longrightarrow M$ be the identity map. Then there exists $g : M \longrightarrow M$ such that $gf = 1_M$. Now $End_R(M)$ is a directly finite ring as M is a directly finite module. Therefore $fg = 1_M$ which shows that f is onto. Thus M is cohopfian. ▣

1.27. **Proposition.** If M is a *quasi-injective*, hopfian module then it is cohopfian.

Proof: As M is hopfian implies M is directly finite we get this result by the application of Proposition 1.23.

But we can also prove it directly as shown below.

Let f be a one-one endomorphism of M. Let $1_M : M \longrightarrow M$ be the identity map of M. Since M is quasi-injective, there exists $g : M \longrightarrow M$ such that $gf = 1_M$. Therefore g is onto. As M is hopfian g is one-one. Thus g is an automorphism. Therefore there exists $g' \in End_R(M)$ such that $gg' = 1 = g'g$. Also we know that $gf = 1$. Therefore $g' = f \Rightarrow fg = 1 \Rightarrow f$ is onto. Thus M is cohopfian. ▣

1.28. **Quasi-projective module.** A module M is *quasi-projective* if for any epimorphism $f : M \longrightarrow N$ and for each homomorphism $\psi : M \longrightarrow N$ there exists an R-homomorphism $g : M \longrightarrow N$ such that $fg = \psi$.

1.29. <u>Proposition.</u> If M is a quasi-projective directly finite module then M is hopfian.

Proof: We apply definition 1.28 with N = M. Let f be an onto endomorphism of M and $1_M : M \longrightarrow M$ be the identity map. Then there exists $g : M \longrightarrow M$ such that $fg = 1_M$. Hence $gf = 1_M$ by Proposition 1.10. It follows that f is a monomorphism. Hence M is hopfian. ▣

1.30. <u>Proposition.</u> If M is a quasi-projective cohopfian module then it is hopfian.

Proof: As M is cohopfian M is directly finite. So we get this result by applying Proposition 1.26. But we can also prove it directly as shown below.

Let f be an epimorphism of M. Let $1_M : M \longrightarrow M$ be the identity map. Then there exists $g : M \longrightarrow M$ such that $fg = 1_M$. Therefore g is one-one which implies that g is onto as M is cohopfian. Thus g is an automorphism. So there exists $g' \in \mathrm{End}_R(M)$ such that $gg' = 1 = g'g \Rightarrow g' = f$ as $fg = 1$.

Therefore $gf = 1 \Rightarrow f$ is one-one. Hence M is hopfian. ▣

1.31. <u>p-injective module.</u> A module ${}_RM$ is *p-injective* if for each principal left ideal I of R every R-homomorphism

$f : I \longrightarrow M$ can be extended to an R-homomorphism $g : R \longrightarrow M$.

1.32. p-injective ring. A ring R is left *p-injective* if the module ${}_R R$ is p-injective.{ Clearly, left self-injective rings are left p-injective. It is easily seen that regular rings (see Chapter 2) are left and right p-injective.}

1.33. Proposition. Let R be a directly finite left p-injective ring. Then ${}_R R$ is cohopfian.

Proof: Let $f : R \longrightarrow R$ be a monomorphism of left R-modules. As R is left p-injective there exists $g : R \longrightarrow R$ such that $gf = 1_R$, the identity map of R. Therefore $fg = 1_R$ as R is directly finite. Thus f is onto and hence ${}_R R$ is cohopfian. ∎

We now state two propositions without proof. Proposition 1.34 is due to Utumi and Proposition 1.35 is due to Vasconcelos.

1.34. Proposition. If R is left and right self-injective R is directly finite.

1.35. Proposition. If M is a finitely generated module over a commutative ring then M is hopfian.

§ 4. Bibliographical notes.

1. The defition of a directly finite module is in([6], p.165).

2. Lemma 1.9, Proposition 1.13, Corollary 1.14, Corollary 1.15 are due to Peterson ([P : 75], p.218).

3. For Proposition 1.22 we refer to Lambek ([L], p.23) and also to ([AF], p.138).

4. For Proposition 1.24 we refer to ([L], p.23).

5. Proposition 1.26 and Proposition 1.30 are due to Birkenmeier ([B : 76], Cor.2) and ([B : 76], Remark on page 102) respectively.

CHAPTER 2.

REGULARITY

The main objects of study in this chapter are unit-regular rings and one-sided unit-regular rings.

§ 1. Regular rings.

In this section we define a regular ring and discuss a few properties of a regular ring.

2.1. Regular element. An element a in a ring R is a *regular element* if there exists $b \in R$ such that $aba = a$.

2.2. Regular ring. A ring R is a *regular ring* in the sense of von Neumann if each element of R is regular.

2.3. Semi-prime ring. A ring R is *semi-prime* if it has no non-zero nilpotent ideal.

2.4. Semi-prime ideal. An ideal A in a ring R is *semi-prime* if R/A is a semi-prime ring.

2.5. Large left ideal. A left ideal A in a ring R is *large* if it has nonzero intersection with every nonzero ideal of R.

2.6. Left hereditary ring. A ring R is *left hereditary* if all left ideals of R are projective.

2.7. Left semi-hereditary ring. A ring R is *left semi-hereditary* if all finitely generated left ideals of R are projective.

2.8. Singular and non-singular modules. For a module $_RM$ we define $Z(M) = \left\{ m \in M \mid Lm = 0 \text{ for some large left ideal } L \text{ of } R \right\}$. A module $_RM$ is *singular* if $Z(M) = M$ and *non-singular* if $Z(M) = 0$.

2.9. Left non-singular ring. A ring R is *left non-singular* if $Z(_RR) = 0$.

2.10. Proposition. Let R be a regular ring. Then

(1) All one-sided ideals of R are idempotent.

(2) All two-sided ideals of R are semi-prime.

(3) The Jacobson radical of R is zero.

(4) R is left and right semihereditary.

(5) R is right and left non-singular.

Proof: (1) Let A be a left ideal of R. Then $A^2 \leq A$. Let $a \in A$. Then there exists $b \in R$ such that $a = aba = a(ba) \in A.A = A^2$, this shows that $A \leq A^2$. Hence $A = A^2$. ▣

(2) Let A be a left ideal of R satisfying $A^2 = 0$. Then $A = A^2 \Rightarrow A = 0$. Thus R regular $\Rightarrow$ R is a semiprime ring. Now as R is regular R/A is regular for every ideal A of R. So R/A is semi-prime for every ideal A of R. ▣

(3) Let $a \in \text{Rad } R$. There exists $x \in R$ such that $axa = a$. This implies that $(xa)^2 = xa \in \text{Rad } R$ as Rad R is an ideal of R. Since Rad R contains no non-zero idempotents, we get $xa = 0$. It follows that $a = axa = 0$. ▨

(4) Let A be a finitely generated left ideal of R. Then A is generated by an idempotent. Hence A is a direct summand of R and so A is projective over R. Thus R is left (similarly, right) semi-hereditary. ▨

(5) Let A be a large right ideal of R. Let $xA = 0$ for some $x \in R$. Now $Rx = Re$ for some $e = e^2 \in R$. Therefore $ReA = RxA = 0$. It follows that $A \leq (1 - e)R \Rightarrow A \cap eR = 0 \Rightarrow eR = 0$ as A is large. Hence $e = 0 \Rightarrow Re = 0 \Rightarrow Rx = 0 \Rightarrow x = 0$. Thus R_R is non singular. ▨

The following results are well-known ([S], p.41).

2.11. <u>Proposition.</u> Let ${}_R M$ be a semi-simple module. Then $\text{End}_R(M)$ is a regular ring.

2.12. <u>Corollary.</u> Let ${}_D V$ be a vector space over a division ring D. Then $\text{End}_D(V)$ is a regular ring.

In the rest of this section following Kaplansky we prove that if R is a regular ring then for each natural number n, $M_n(R) \cong \text{End}(R^n_R)$ is a regular ring.

First we prove the following lemma which is due to McCoy.

2.13. <u>McCoy's lemma.</u> Let a be an element in a ring R and b an element in R such that $aba-a$ is regular. Then is a regular element of R.

Proof: Since aba-a is regular, there exists $c \in R$ such that $(aba-a)c(aba-a) = aba-a \Rightarrow a(-bacab+bac+cab-c+b)a = a$ which shows that a is regular. ■

2.14. Proposition. If R is a regular ring then $M_n(R)$ is also a regular ring.

Proof: First we consider the case when $n = 2$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$. By regularity of R there exists $r \in R$ such that $crc = c$. So we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} arc-a & ard-b \\ 0 & crd-d \end{pmatrix}$$

So by McCoy's lemma, if matrices of the type $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ are regular then for each $a,b,c,d \in R$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is regular.

Again by regularity of R there exists $x,y \in R$ such that $axa = a$, $dyd = d$. Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & axb+byd-b \\ 0 & 0 \end{pmatrix}$$

Hence by McCoy's lemma, matrices of the type $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ will be regular if matrices of the type $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ are regular. Let $z \in R$ be such that $bzb = b$.

Then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & bzb \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$

showing $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ to be a regular element of $M_2(R)$ for each b. Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is regular for each a,b,c,d in R, that is. $M_2(R)$ is regular.

Now we consider the case when $n = 4$.

A 4×4 matrix can be written as a 2×2 matrix where each entry is a 2x2 matrix. So $M_4(R) = M_2(M_2(R))$. Since $M_2(R)$ is regular $M_4(R)$ is also regular.

If $n = 2^k$ then $M_{2^k}(R) = M_2(M_{2^{k-1}}(R))$. So by induction it can be easily seen that $M_n(R)$ is regular for $n = 2^k$.

Next we consider the case when $n \neq 2^k$. Choose k in such a way that $2^k > n$.

An element $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ of $M_n(R)$ can be identified

with the element $\left[\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & \\ \cdots & \cdots & \cdots & 0 \\ a_{n1} & \cdots & a_{nn} & \\ \hdashline & 0 & & 0 \end{array}\right]$ of $M_{2^k}(R)$.

Thus $M_n(R)$ is a subring of $M_{2^k}(R)$.

Let $E = \left[\begin{array}{cccc:ccc} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \hdashline & & & & 0 & & \\ & & 0 & & & \ddots & \\ & & & & & & 0 \end{array}\right]$ where the number of one's in the diagonal is n and the number of zeroes in the diagonal is $2^k - n$. Then E is an idempotent in $M_{2^k}(R)$.

$\therefore E\, M_{2^k}(R)\, E$ is regular.

Let $2^k = t$ and $A = \left[a_{ij}\right]_{t \times t} \in M_t(R)$.

Then EAE is regular. Therefore

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & \cdot & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & 0 & & & & \cdot & \\ & & & & & & 0 \end{bmatrix} \begin{bmatrix} a_{11} & & a_{1t} \\ \cdots & \cdots & \cdots \\ a_{t1} & \cdots & a_{tt} \end{bmatrix} \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & & 1 & & & \\ & & & & 0 & & \\ & 0 & & & & \cdot & \\ & & & & & & 0 \end{bmatrix}$$

is regular. This implies that $\begin{bmatrix} a_{11} & \cdot & a_{1n} & & & \\ \cdots & \cdots & \cdots & & 0 & \\ a_{n1} & \cdots & a_{nn} & & & \\ & & & 0 & & \\ & 0 & & & \cdot & \\ & & & & & 0 \end{bmatrix}$

is regular, which implies that $\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{bmatrix}$

is regular. Therefore $M_n(R)$ is regular for every natural number n. ∎

As a consequence of the above theorem we have :

2.15. Corollary. If R is a regular ring, End(R^n_R) is a regular ring for each natural number n.

We also mention without proof the following result due to Ware ([W :71]).

2.16. Proposition. Let P_R be a finitely generated projective module over a regular ring R then $\mathrm{End}_R(P)$ is a regular ring.

§ 2. Endomorphism rings of injective modules.

In the main theorem of this section, we show that if E is an injective module and S = $\mathrm{End}_R(E)$ then S/Rad(S) is a regular ring. The results of this section are due to Faith and Utumi.

The concept of a large left ideal (2.5) has the following natural extension to modules.

2.17. Large submodule. A submodule L of a module M is *large or essential* in M if for a submodule K of M we have $K \cap L = 0 \Rightarrow K = 0$.

We write $L \triangleleft M$ for L is large in M.

2.18. Small submodule. A submodule K of M is *small or superfluous* in M, written as $K \ll M$ if for a submodule $L \leq M$, $K + L = M \Rightarrow L = M$.

2.19. Lemma. Let ${}_RM$ be a module and L a submodule of M. Then the following conditions are equivalent.

(1) L is large in M.

(2) For each $x \in M$, $x \neq 0$, $\exists\ r \in R$ such that $rx \in L$ and $rx \neq 0$.

Proof: (1) $\Rightarrow$ (2)

Let $x \in M$, $x \neq 0$. Then $Rx \leq M$, $Rx \neq 0$.

Therefore $L \cap Rx \neq 0$ as $L \triangleleft M$

Hence there exists $r \in R$ such that $rx \in L$, $rx \neq 0$.

(2) ⇒ (1) Let N be a nonzero submodule of M. Then there exists a nonzero element x in N. By hypothesis there exists an element $r \in R$ such that $rx \neq 0$, $rx \in L$. As $rx \in N$, clearly $L \cap N \neq 0$. Hence L is large in M. ▣

2.20. Lemma. Let R be any ring. Let $a \in R$.

Then $Ra \ll R \Rightarrow a \in \text{Rad}\ (R)$.

Proof. Let μ be a maximal left ideal in R.

Then $Ra + \mu \neq R$ since $Ra \ll R$. It follows that $Ra + \mu = \mu$ as μ is maximal left ideal.

Therefore $a \in \mu$ for each maximal left ideal μ of R.

Hence $a \in \text{Rad}\ R$. ▣

2.21. Proposition. Let ${}_RE$ be an injective module and $S = \text{End}_R(E)$. Let $a \in S$. Then $a \in \text{Rad}(S) \Leftrightarrow \ker a \triangleleft E$.

Proof: (⇒) Let $a \in \text{Rad}(S)$. Let M be a submodule of E such that $\ker a \cap M = 0$. It is required to prove that $M = 0$. Consider the mappings $j : M \longrightarrow E$ and $a : E \longrightarrow E$. Now a_j is a monomorphism : for $aj(m) = 0 \Rightarrow a(m) = 0 \Rightarrow m \in \ker a$; so $m \in \text{Ker}\ a \cap M = 0 \Rightarrow m = 0$.

Since E is injective there exists $\theta : E \longrightarrow E$ such that $\theta.aj = j \Rightarrow \theta.aj\ (m) = m\ \forall\ m \in M$

$\Rightarrow \theta a(m) = m \Rightarrow (\theta a - 1)m = 0\ \forall\ m \in M$.

As $a \in \text{Rad}(S)$, $\theta a - 1$ is left invertible. Hence $(\theta a - 1)m = 0 \Rightarrow m = 0$. Thus $M = 0$. Hence $\text{Ker}\ a \triangleleft E$.

($\Leftarrow$) Let Ker $a \lhd E$. To prove that $a \in \text{Rad}(S)$ it is enough to prove $Sa \ll_S S$ by Lemma 2.20.

Let $I \leq_S S$ and $I + Sa = S$. We assert that $I = S$.

Let $1_E = sa + b$ for $s \in S$, $b \in I$. We assert that b is one-one.

Let $z \in \text{Ker } b \cap \text{Ker } a$. Then $bz = 0$, $az = 0 \Rightarrow z = (sa + b)z = saz + bz = 0$. This shows that $\text{Ker } b \cap \text{Ker } a = 0$.

As Ker a E, it follows that Ker $b = 0$. Hence b is a monomorphism.

The exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "b"] & E \arrow[r] & E/bE \arrow[r] & 0 \end{tikzcd}$$

splits as E is injective. So there exists $c : E \longrightarrow E$ such that $cb = 1_E$. But $cb \in I$ as I is a left ideal.

Thus $1 \in I \Rightarrow I = S$. ■

2.22. Injective envelope. Let M be a left R-module. Then an injective left R-module E is an *injective envelope or injective hull* of M if there exists a monomorphism $i : M \longrightarrow E$ such that $i(M)$ is large in E.

2.23. Theorem. If E is an injective module and $S = \text{End}_R(E)$ then $S/\text{Rad}(S)$ is regular.

Proof : Let $a \in S$. Then $\text{Ker}(a) \leq E$. Let $\overline{\text{Ker}(a)}$ denote the injective hull of $\text{Ker}(a)$. As E is injective, $\overline{\text{Ker}(a)} \leq E$. As $\overline{\text{Ker}(a)}$ is injective it is a direct summand of E. So there exists E' such that $E = \overline{\text{Ker}(a)} \oplus E'$.

Now, $a_0 = a_{|E'} : E' \longrightarrow aE'$ is clearly onto. Further, we have: $a_o(e') = 0 \Rightarrow e' \in Ker(a) \Rightarrow e' \in Ker(a) \cap E' = 0 \Rightarrow e' = 0$

Thus a_o is an isomorphism.

Since E' is injective over R, therefore aE' is injective over R. Thus there exists E'' such that $E = aE' \oplus E''$.

Let $x : E = E'' \oplus aE' \longrightarrow \overline{Ker(a)} \oplus E' = E$ be defined by $x\,(y,z) = (0, a_o^{-1}z)$.

We now prove that $Ker(axa - a)$ is a large submodule of E.

Let $m \neq 0$, $m \in E$. Then we have to prove that there exists $r \in R$ such that $rm \neq 0$ and $rm \in Ker(axa - a)$.

Let $m = u + v$, $u \in \overline{Ker(a)}$, $v \in E'$.

Case 1. $u = 0$, $v \neq 0$.

Now $v \in E' \Rightarrow a(v) \in aE'$. Therefore $xav = a_o^{-1}av = v$. It follows that $(axa - a)v = axav - av = av - av = 0$. Thus with $r = 1$, the condition $rm \neq 0$ and $(axa - a)(rm) = 0$ are satisfied.

Case 2. $u \neq 0$. We have $u \in \overline{ker(a)}$. So there exists $r \in R$ such that $0 \neq ru \in ker(a)$. Hence $rm = r(u + v) = ru + rv \neq 0$. Also, $(axa - a)(rm) = (axa - a)(ru + rv) = (axa - a)(rv) = 0$.

[For $v \in E' \Rightarrow av \in aE' \Rightarrow xav = a_o^{-1}av = v$

Next $xav = v \Rightarrow axav = av \Rightarrow (axa - a)v = 0$]

Thus $ker(axa - a)$ is a large submodule of E. It follows that $axa - a \in Rad(S)$ by Propositon 2.16. Therefore $(\overline{axa - a}) = 0$ in $S/Rad(S)$ yielding $\bar{a}\,\bar{x}\,\bar{a} = \bar{a}$. This shows that $S/Rad(S)$ is regular. ∎

2.24. Proposition. If R is left or right self-injective then R/Rad(R) is regular.

Proof: Let R be left self-injective. Then $S = \mathrm{End}_R(R) \cong R$. Hence $S/\mathrm{Rad}(S) \cong R/\mathrm{Rad}(R)$. As ${}_RR$ is injective, S/Rad(S) is regular. Thus R/Rad(R) is regular. ◻

§ 3. Unit-regular rings.

In this section we shall define unit-regular rings, strongly regular rings, dependent rings and discuss their properties. The material included here is mainly due to Henriksen and Ehrlich.

2.25. Unit-regular ring. A ring R is called *unit-regular* if for every a in R there is a unit x in R such that axa = a.

2.26. Examples. Division rings and product of division rings are unit-regular.

2.27. Example. An example of a regular ring which is not unit-regular is $\mathrm{End}_D(V)$; V an infinite dimensional vector space over a division ring D. For proof see Proposition 2.55.

2.28. Remark. A ring R is unit-regular if and only if every element of R is a left (right) unit multiple of an idempotent.

Proof: Let R be a unit-regular ring and a be any element of R. Then there exists a unit $u \in R$ such that a = aua. Let

u' be the inverse of u in R. Then $a = (au)u'$ with au an idempotent, u' a unit. Conversely, let $a = tf$, t a unit. $f = f^2 \in R$. Let t' be the inverse of t in R. Then $at'a = tft'tf = tf^2 = tf = a$. Thus a is unit-regular. $\square$

2.29. __Proposition.__ Matrix rings over division rings are unit-regular.

Proof: Let A be a $n \times n$ matrix in $M_n(D)$, D being a division ring. By elementary matrix theory we know that there exists invertible matrices P and Q in $M_n(D)$ such that

$$PAQ = \left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$$

where the number of one's in the diagonal is equal to the rank of the matrix A.

Now, $PAQ\,PAQ = PAQ$. This gives

$P^{-1}PAQ\,PAQQ^{-1} = P^{-1}PAQQ^{-1} \Rightarrow AQPA = A$

Since, Q and P are both units, QP is also a unit. So A is unit-regular. Thus $M_n(D)$ is a unit-regular ring. $\square$

2.30. __Proposition.__ If R is a semi-simple ring then R is unit-regular.

Proof: If R is semi-simple then by Wedderburn's structure theorem

$$R = \prod_{i=1}^{t} M_{n_i}(D_i), \text{ where } D_i \text{ are division rings.}$$

But we have already proved that $M_{n_i}(D_i)$ is a unit regular ring for each i. So R being a direct product of unit regular rings is unit regular. ■

2.31. Lemma. Let R be a ring and a be an element in R. Then the following statements are equivalent:

(1) There is a unit u in R such that $aua = a$.

(2) There is a unit u in R such that au and ua are idempotents.

(3) There is a unit u in R such that au or ua is an idempotent.

(4) There are units p and q in R such that paq is an idempotent.

Proof. (1) ⇒ (2) and (2) ⇒ (3) are trivial.

(3) ⇒ (4) If au is an idempotent then we choose $p = 1$, $q = u$ so that paq is an idempotent.

(4) ⇒ (1) We have $paq\,paq = paq \Rightarrow p^{-1}paq\,paqq^{-1} = p^{-1}paqq^{-1}$ $\Rightarrow aqpa = a$ where qp is a unit. ■

2.32. Theorem. If R is unit regular and 2 is a unit in R then every element of R is equal to a sum of two units.

Proof: Let $a \in R$. Then by Remark 2.28 $a = et$ where e is an idempotent and t is a unit in R. Now $e = 2^{-1}(2e - 1) + 2^{-1}$.

Therefore, $a = 2^{-1}(2e - 1)t + 2^{-1}t$.

Now, $2^{-1}(2e - 1)t.t^{-1}(2e - 1)2 = 2^{-1}(2e2e - 2e - 2e + 1)2 = 1$.

Therefore $2^{-1}(2e - 1)t$ is a unit.

Also, $2^{-1}t.t^{-1}2 = 1 \Rightarrow 2^{-1}t$ is a unit.

This implies that a is a sum of two units. ■

2.33. Proposition. If R is a unit regular ring, then R is directly finite.

Proof: Let $a,b \in R$ and $ab = 1$. Then, by the unit-regularity of R, there exists a unit u in R such that $aua = a$. Let v be the inverse of u. Then, $1 = ab = auab = au.1 = au$. Therefore, $v = auv = a.1 = a$. Also, $vb = ab = 1 \Rightarrow uvb = u$. Again, $uvb = (uv)b = 1.b = b$. This gives $u = b$ Hence $uv = 1 \Rightarrow ba = 1$. So R is directly finite. ∎

2.34. Strongly regular ring. A ring R is *strongly regular* if for each a in R there exists an element x in R such that $a = a^2x$.

2.35. Remark. Every commutative regular ring is strongly regular.

2.36. Example. Any division ring which is not a field is a non-commutative strongly regular ring.

2.37. Proposition. The following conditions are equivalent for a ring R.

(1) R is strongly regular.

(2) R is reduced and regular.

(3) R is abelian and regular.

Thus strong regularity is a left- right symmetric property.

2.38. Remark. Let R be a strongly regular ring with centre C. Let $a \in R$. Then there exists an element x in R

such that $axa = a$. Hence ax, xa are idempotents and therefore central. So we have

$ax = axax = axxa$ and $xa = xaxa = axxa$, as $xa, ax \in C$.

Therefore $ax = xa$.

2.39. Theorem. Every strongly regular ring is unit-regular.

Proof: Let $a \in R$. Then there exists $x \in R$ such that $a = a^2x$. Let $e = xa = ax \in$ Centre R. Then $e^2 = e$ and $ea = axa = a$, $ae = axa = a$.

We have $eax = ax = e$, $eae = ae = a$, $exe = e^2x = ex = xe$

We assert that $ex + 1 - e$ is the inverse of $ea + 1 - e$. A direct computation shows that $(ex + 1 - e)(ea + 1 - e)$

$= exea + ex - exe + ea + 1 - e - e^2a - e + e^2$

$= exa + ex - xe + a + 1 - e - a - e + e$

$= e^2 + ex - ex + 1 - e = 1$.

Similarly, $(ea + 1 - e)(ex + 1 - e)$

$= eaex + ea - eae + ex + 1 - e - e^2x - e + e^2$

$= ax + a - a + ex + 1 - e - ex - e + e = e + 1 - e = 1$.

Now, $a(ex + 1 - e)a = aexa + a^2 - aea = axa + a^2 - a^2$ $= axa = a$. Thus a is a unit regular element of R. So R is unit regular. ⊠

2.40. Left dependent ring. A ring R is called *left dependent* if for each $a, b \in R$, there exist $s, t \in R$, not both zero such that $sa + tb = 0$.

2.41. Remark. Note that Henriksen [H:73] has not distinguished between left dependent and right dependent rings. A natural question to ask is whether left dependent rings are right dependent and vice versa.

2.42. Theorem. Every unit regular ring is left (and right) dependent.

Proof: Let $a, b \in R$.

Case 1. Let a, b both have left inverses in R. Then $a^{-1}a + (-b^{-1})b = 0$ with $a^{-1} \neq 0$, $b^{-1} \neq 0$. So R is left dependent.

Case 2. Without loss of generality, suppose that a has no left inverse.

Let $a = xe$ with x a unit and e an idempotent. (See Remark 2.28). Since a is not a unit, e cannot be 1.

We take $s = (1-e)x^{-1}$, $t = 0$. Then $s \neq 0$. We have

$sa + tb = (1-e)x^{-1}.xe = 0 = 0$.

Therefore R is left dependent. (Similarly R can be shown to be right dependent.) □

2.43. Theorem. A regular ring R is left dependent if and only if whenever a, a', b, b' are elements of R such that $aa' = bb' = 1$ then $a(1-b'b)$ can not have a right inverse.

Proof: ($\Rightarrow$) Let R be left dependent. Let $a, b \in R$ be such that $aa' = bb' = 1$. Since R is left dependent, there exist $s, t \in R$ not both zero, such that

$$sa + tb = 0 \qquad \text{.............................} \qquad (1)$$

Therefore $sab' + t = 0 \rightarrow t = -sab'$ .... (2)

If $s = 0$ then $t = 0$. So s cannot be zero. From (2) we have $tb = -sab'b \Rightarrow -sa = -sab'b$ (using (1)) $\Rightarrow sa(1-b'b) = 0$

Thus $a(1-b'b)$ cannot have a right inverse. (Note that regularity of R is not required in the proof of this implication).

($\Leftarrow$) Assume that $aa' = bb' = 1 \Rightarrow a(1-b'b)$ fails to have a right inverse (where $a, a', b, b' \in R$).

Case 1. We assume that a does not have a right inverse. (The case when b does not have a right inverse can be treated similarly.) Then by the regularity of R, there exists $x \in R$ such that $a = axa$.

Therefore $(1-ax)a + 0.b = 0$ with $1-ax \neq 0$, which shows that R is left dependent.

Case 2. Suppose there exists $a', b' \in R$ such that $aa' = bb' = 1$. Let $u = a(1-b'b)$. As R is regular, there exists $y \in R$ such that $u = uyu$. Then we have:

$(1-uy)a - (1-uy)ab'b = a - uya - ab'b + uyab'b$

$= a - ab'b - (uya - uyab'b) = a(1-b'b) - uya(1-b'b)$

$= u - uyu = u-u = 0.$

Moreover $1-uy \neq 0$ by the hypothesis that u has no right inverse. □

2.44. Corollary. A directly finite and regular ring is left and right dependent.

2.45. Proposition. Let R be a regular ring. Consider the following conditions:

(1) R is unit regular.

(2) R is directly finite and regular.

(3) R is left and right dependent.

Then (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) but (3) does not imply (1).

Proof : We get (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) by Proposition 2.33 and Corollary 2.44. To show that (3) does not imply (1) we look at the following example which is due to Henriksen [H:73].

Take any regular ring R which is not directly finite, e.g., $\mathrm{End}_K(V)$ with $| V : K | = \infty$. Then K = Centre R. Let L be the ring of all sequences (a_n) of elements of R such that there exists a natural number N such that for all $n \geq N$ we have $a_n \in K$. The ring L is easily seen to be regular. Also L is not directly finite as R is not. Consider the elements (a_n), (b_n), (a'_n), (b'_n) of L such that $(a_n a'_n) = (1) = (b_n b'_n)$. Now, there exists a positive integer N such that b'_n, $b'_n \in K$ for each $n \geq N$. So $1 - b_n b'_n = 0 \ \forall\ n \geq N$. So $(a_n)\left\{(1) - (b'_n)(b_n)\right\}$ cannot have a right inverse. Hence by Theorem 2.43 L is dependent. ▫

In the following theorem the unit-regularity of the endomorphism ring of a left R-module is discussed.

2.46. Theorem. Let A be a left R-module such that the ring $T = \mathrm{End}_R(A)$ is regular. Then the following conditions are equivalent.

(a) T is unit-regular.

(b) If $A = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$ then $B_1 \cong B_2$.

(c) $\ker x \cong \operatorname{coker} x \ \forall\ x \in T$.

(d) If $e, f \in T$ are idempotents such that $eT \cong fT$ then $(1 - e)T \cong (1 - f)T$.

Proof: (a) $\Rightarrow$ (b). We define x in T in such a way that $xB_1 = 0$ and $x|_{A_1}$ is an isomorphism of A_1 onto A_2. Since T is unit-regular, there exists a unit u in T such that $xux = x$ $\Rightarrow$ $(ux)^2 = ux$. Since ux is an idempotent we have ,

$$A = uxA \oplus \ker ux = uxA \oplus \ker x = uA_2 \oplus B_1 \longrightarrow (1)$$

as $\ker x = \ker ux$ and $uxA = uA_2$.

Since u is an automorphism of A, we have

$$A = u(A) = u(A_2 \oplus B_2) = uA_2 \oplus uB_2 \longrightarrow (2).$$

From (1) and (2) we have $B_1 \cong uB_2 \cong B_2 \Rightarrow B_1 \cong B_2$.

(b) $\Rightarrow$ (c) Let $x \in T$. Since T is regular, there exists $y \in T$ such that $xyx = x \rightarrow (xy)^2 = xy$ and $(yx)^2 = yx$.

Hence $A = xyA \oplus \ker xy = xA \oplus (1 - xy)A$

Since $xyA \leq xA$ and $xA = xyxA \leq xyA$ we have $xA = xyA$

Also, $A = yxA \oplus \ker yx = yxA \oplus \ker x$

Since $xA = 0 \Rightarrow yxA = 0$ and $yxA = 0 \Rightarrow xyxA = 0 \Rightarrow xA = 0$, we have $\ker x = \ker yx$

Now, $x|_{yxA} : yxA \longrightarrow xA$ is an isomorphism.

Hence $\ker x \cong (1 - xy)A$.

Now, we define a map $\theta : A \longrightarrow (1 - xy)A$ by $\theta(a) = (1 - xy)a$. θ is clearly onto. Let $a \in \ker \theta$.

Then $\theta(a) = 0 \to (1 - xy)a = 0 \to a = xya \to a \in xyA \leq xA$.

Let $xa \in xA$. Then $\theta(xa) = (1 - xy)xa = xa - xyxa = 0$ implies that $xa \in \ker \theta$. Therefore $\ker \theta = xA$.

So we have, $A/xA \cong (1 - xy)A$, that is, $\operatorname{coker} x \cong (1 - xy)A$.

Thus $\ker x \cong \operatorname{coker} x$.

(c) $\Rightarrow$ (a) Let $x \in T$. Since T is regular, there exists $y \in T$ such that $x = xyx$. As we have already seen,

$$A = yxA \oplus \ker x = xA \oplus (1 - xy)A.$$

By hypothesis, $\ker x \cong \operatorname{coker} x$. Also $\operatorname{coker} x \cong (1 - xy)A$.

$x_{|yxA} : yxA \longrightarrow xA$ is an isomorphism. So there exists $x' : xA \longrightarrow yxA$ such that $xx' = 1_{xA}$, $x'x = 1_{yxA}$.

Let v be an isomorphism of $(1 - xy)A$ to $\ker x$. Then we define $u \in T$ as $u(a) = x'(a)$ for $a \in xA$

$$= v(a) \text{ for } a \in (1 - xy)A.$$

Then u is a unit in T and $xuxA = xu(xA) = xx'(xA) = 1(xA) = xA$

Thus $xux = x$ and hence T is unit regular.

(a) $\Rightarrow$ (d) Let $A = {}_TT$. Then $\operatorname{End}({}_TT) = T$. Let $e, f \in T$, $e = e^2$, $f = f^2$. Then $T = eT \oplus (1 - e)T$ and $T = fT \oplus (1 - f)T$.

Given $eT \cong fT$. Using the implication ((a) $\Rightarrow$ (b)), we get $(1 - e)T \cong (1 - f)T$.

(d) $\Rightarrow$ (a) We get this using the implication ((b) $\Rightarrow$ (a)). ▨

2.47. Example. Let V be a vector space over a division ring D. Then $\operatorname{End}_D(V)$ is unit regular $\Leftrightarrow$ V is finite dimensional.

Proof: Let ${}_DV$ be not finite dimensional. Then $\operatorname{End}_D(V)$ is not a unit regular ring as will be shown in Proposition 2.55.

Conversely, let ${}_DV$ be finite dimensional. Let $x \in \operatorname{End}_D(V)$.

Consider the exact sequence

$$0 \longrightarrow \ker x \longrightarrow V \longrightarrow xV \longrightarrow 0$$

$\dim V = \dim \ker x + \dim xV \Rightarrow \dim \ker x = \dim V - \dim xV$

Now $\dim \operatorname{coker} x = \dim (V/xV) = \dim V - \dim xV$.

Hence $\dim \ker x = \dim \operatorname{coker} x \Rightarrow \ker x \cong \operatorname{coker} x \ \forall\ x \in \operatorname{End}_D(V)$.

Hence $\operatorname{End}_D(V)$ is unit regular, by Theorem 2.38. ▨

Before proving the next theorem we shall define the inverse limit of a system of R-modules.

Let $\{M_\iota,\ i \in I\}$ be a set of modules over a ring R. Suppose there exist some pairs of indices $(k, j) \in K \times K$ and correspondingly there exist homomorphisms $\phi_{kj} : A_k \longrightarrow A_j$ such that $\phi_{ii} : A_i \longrightarrow A_i$ is the identity map for all $i \in I$ and $\phi_{ji}\phi_{kj} = \phi_{ki}$ whenever these maps are defined.

The inverse limit is a module M and a set of homomorphisms $a_k : M \longrightarrow A_k$ such that for any module X and any homomorphisms $f_k : X \longrightarrow A_k$ making the part of the following diagram shown by bold type arrows commute there exists a unique $\beta : X \longrightarrow M$ with all triangles commuting.

$$\begin{tikzcd}
X \arrow[rr, dashed, "\beta"] \arrow[dr, "f_k"'] \arrow[ddr, "f_j"'] & & \varprojlim A_k \arrow[dl, "a_k"] \arrow[ddl, "a_j"] \\
& A_k \arrow[d] & \\
& A_j &
\end{tikzcd}$$

2.48. <u>Theorem</u>. Let R be a strongly regular ring and J an ideal of R. Then $\operatorname{End}_R(J)$ is a strongly regular ring and $\operatorname{End}({}_R J) \cong \operatorname{End}(J_R)$.

Proof: We consider the set $X = \left\{ Rx \mid x \in J \right\}$. Then X is a family of (left) ideals of R and J is the union of its elements. Let $Rx, Ry \in X$ and $Rx \le Ry$. Since R is strongly regular, $Rx = Re$, $Ry = Rf$, for some central idempotents $e, f \in R$. Let $\alpha \in \mathrm{End}_R(Ry)$.

$\alpha(Rx) = \alpha(Re) = \alpha(eR) = \alpha(e^2R) = e\alpha(eR) \le eR = Re = Rx$.

Therefore $\alpha(Rx) \le Rx \Rightarrow \alpha_{|Rx} \in \mathrm{End}_R(Rx)$. Thus there exists a map

$$\begin{aligned} \theta : \mathrm{End}_R(Ry) &\longrightarrow \mathrm{End}_R(Rx) \\ \alpha &\longmapsto \alpha_{|Rx} \end{aligned}$$

Now we determine the inverse limit of this system of endomorphism rings and restriction maps.

Consider $Rx = Re$, $Ry = Rf$ in X. Let $\alpha \in \mathrm{End}({}_RJ)$.

We write $\alpha_e = \alpha_{|Re} : Re \longrightarrow Re$

$$\begin{aligned} \alpha_{e,f} : Re &\longrightarrow Rf \\ re &\longmapsto rf \end{aligned}$$

We write $R_0 = \varprojlim \mathrm{End}_R(Rx) = \varprojlim \mathrm{End}_R(Re) = \varprojlim (Re)$

$= \left\{ (a_e) \mid (a_e)\alpha_{e,f} = (a_f) \right\}$

Note that R strongly regular $\Rightarrow$ Re strongly regular $\forall\, e$

$\Rightarrow \varprojlim (Re)$ strongly regular

$\Rightarrow \varprojlim \mathrm{End}_R(Re)$ strongly regular.

We consider the map

$$\begin{aligned} \phi : \mathrm{End}_R(J) &\longrightarrow \varprojlim \mathrm{End}_R(Re) \\ \alpha &\longmapsto (\alpha_e)_{e \in J \cap I(R)} \end{aligned}$$

Notice that $\alpha_{e|Rf} = \alpha_f$ whenever $Rf \le Re$. We shall show that ϕ is an isomorphism of rings.

We first show that ϕ is one-one.

Let $\phi(\alpha) = \phi(\beta)$. Then $\alpha_{|Re} = \beta_{|Re}$ $\forall$ $e \in J \cap \bar{I}(R)$. Let $j \in J$. Then $Rj = Re$ for some $e \in J \cap \bar{I}(R)$.

$\alpha_{|Rj} = \beta_{|Rj} \Rightarrow \alpha(j) = \beta(j)$ for all $j \in J$ $\Rightarrow \alpha = \beta$.

Now we will show that ϕ is onto.

Let $(\alpha_e)_{e \in J \cap \bar{I}(R)} \in \varprojlim \mathrm{End}_R(Re)$.

We define $\alpha : J \longrightarrow J$ as $\alpha(j) = j\alpha_e(e)$ where $Rj = Re$ for some unique $e \in J \cap \bar{I}(R)$. We will show that α is a R-linear endomorphism of J.

Let $j_1, j_2 \in J$. Let $Rj_1 = Re_1$, $Rj_2 = Re_2$,

$R(j_1 + j_2) = Re_3 \le Rj_1 + Rj_2 = Re_1 + Re_2 = Re_4$ (say).

We write $\alpha_{e_i} = \alpha_i$. Then,

$\alpha(j_1 + j_2) = (j_1 + j_2)\alpha_3(e_3) = (j_1 + j_2)\alpha_4(e_3)$

$= \alpha_4((j_1 + j_2)e_3) = \alpha_4(j_1 + j_2)$

Also, $\alpha(j_1) + \alpha(j_2) = j_1\alpha_1(e_1) + j_2\alpha_2(e_2) = j_1\alpha_4(e_1) + j_2\alpha_4(e_2)$

$= \alpha_4(j_1e_1) + \alpha_4(j_2e_2) = \alpha_4(j_1) + \alpha_4(j_2) = \alpha_4(j_1 + j_2)$

Thus $\alpha(j_1 + j_2) = \alpha(j_1 + \alpha(j_2)$

Let $j \in J$, $r \in R$. Let $Rr_j = Re$, $Rj = Re_1$, $e, e_1 \in J \cap \bar{I}(R)$.

Then $Re \le Re_1$.

$\alpha(rj) = rj\alpha_{e_1}(e_1) = \alpha_{e_1}(rje_1) = \alpha_{e_1}(rj)$

Thus $\alpha(rj) = r\alpha(j)$. Therefore $\alpha \in \mathrm{End}_R(J)$.

Now, $\alpha(e) = e\alpha_e(e) = \alpha_e(e^2) = \alpha_e(e)$

Thus $\alpha_{|eR} = \alpha_e$. Hence $\phi(\alpha) = (\alpha_{|eR})_{e \in J \cap \bar{I}(R)} = (\alpha_e)_{e \in J \cap \bar{I}(R)}$

This shows that ϕ is onto.

Now we will show that ϕ is a ring homomorphism, that is

$\phi(\alpha + \beta) = \phi(\alpha + \beta)$ and $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$, $\forall$ $\alpha, \beta \in \mathrm{End}_R(J)$

We have: $\phi(\alpha + \beta) = ((\alpha + \beta)_{|Re}) = (\alpha_{|Re} + \beta_{|Re})$

$= (\alpha|_{Re}) + (\beta|_{Re}) = \phi(\alpha) + \phi(\beta)$

$\phi(\alpha\beta) = ((\alpha\beta|_{Re}) = (\alpha|_{Re}\beta|_{Re}) = (\alpha|_{Re})(\beta|_{Re}) = \phi(\alpha)\phi(\beta)$

So we have proved that $\mathrm{End}_R({}_R J) \cong \varprojlim \mathrm{End}_R(Re) \cong \varprojlim (Re)$

$$= \varprojlim X$$

which shows that $\mathrm{End}_R({}_R J)$ is strongly regular.

Now, if we start with $\overline{X} = \{xR \mid x \in J\}$ then similarly we can see that each xR is strongly regular and $xR \cong \mathrm{End}_R(xR)$ and $\mathrm{End}_R(J_R) \cong \varprojlim \overline{X}$.

But, $\overline{X} = \{xR \mid x \in J\} = \{eR \mid e \in J \cap \overline{I}(R)\}$

$$= \{Re \mid e \in J \cap \overline{I}(R)\} = X$$

Therefore, $\mathrm{End}_R({}_R J) \cong \mathrm{End}_R(J_R)$ ■

2.49. Proposition. (Zelmanowitz) There exists a regular ring R and a left ideal J of R such that $\mathrm{End}_R(J)$ is not regular.

Proof: Let R be the ring of column finite countable matrices over a field. Let $e_{11}, e_{22}, e_{33}, \ldots$ be the usual matrix units of R. Then,

$$R = \left\{ [a_{ij}] \;\middle|\; \text{For each } j\text{, there exists } N_j \in \mathbb{N} \text{ such that } a_{ij} = 0 \text{ for each } i \geq N_j . \right\}$$

$$\text{Let } J = \prod_{j \text{ odd}} Re_{1j} \oplus \sum_{j \text{ even}} \oplus Re_{1j}$$

$$= \left\{ A = [a_{ij}] \;\middle|\; [a_{ij}] \in R \text{ and for each } a \; \exists \; N_A \text{ such that for all even values of } j \geq N_A,\ a_{ij} = 0 \ \forall\, i \right\}$$

Then J is a left ideal of R.

Let $\alpha \in \mathrm{End}_R(J)$ be defined as $\alpha\left(\prod_{j \text{ odd}} Re_{ij}\right) = 0$

$$\alpha(e_{1j}) = e_{1(j-1)} \quad \text{when} \quad j \quad \text{is even.}$$

Then $\alpha J = \left\{ [a_{ij}] \mid a_{ij} = 0 \text{ for } j \text{ even and } a_{ij} = a_{i(j-1)} \text{ for } j \text{ odd.} \right\}$

$$= \sum_{j \text{ odd}} \oplus Re_{1j}.$$

Now, if possible, let $\text{End}_R(J)$ be regular. Then αJ must be a direct summand of J and hence of $\prod_{j \text{ odd}} Re_{1j}$.

As $R = \prod_{j \text{ odd}} Re_{1j} \oplus \prod_{j \text{ even}} Re_{1j}$, αJ would be a direct summand of R and so it must be a principal left ideal of R. But $\alpha(J)$ is an infinite direct sum, it can not be generated by a single element. So we get a contradiction. Hence $\text{End}_R(J)$ is not regular. ▨

2.50. Lemma. (Zelmanowitz) Let M be a left R-module and let $\alpha \in$ centre of $\text{End}_R(M)$. Then there exists an element $\beta \in$ centre of $\text{End}_R(M)$ with $\alpha\beta\alpha = \alpha$ if and only if $M = \alpha(M) \oplus \ker \alpha$.

Proof: Let $\alpha \in$ centre of $\text{End}_R(M)$. Suppose there exists $\beta \in$ centre of $\text{End}_R(M)$ such that $\alpha\beta\alpha = \alpha$. Then $(\alpha\beta)^2 = \alpha\beta$ and $(\beta\alpha)^2 = \beta\alpha$. This implies that $\alpha\beta = \beta\alpha$ as $\alpha, \beta \in$ centre of $\text{End}_R(M)$.

Let $\alpha\beta = \beta\alpha = e$. Then $\alpha(M) = \alpha\beta\alpha(M) \leq \alpha\beta(M) \leq \alpha(M)$.

Hence $\alpha\beta(M) = \alpha(M) \Rightarrow e(M) = \alpha(M)$.

Let $m \in \ker \alpha$. Then $\alpha(m) = 0 \Rightarrow \beta\alpha(m) = 0 \Rightarrow m \in \ker \beta\alpha$.

If $m \in \ker \beta\alpha$, then $\beta\alpha(m) = 0 \Rightarrow \alpha\beta\alpha(m) = 0 \Rightarrow \alpha(m) = 0$.

$\Rightarrow m \in \ker \alpha$. Hence $\ker \alpha = \ker \beta\alpha = \ker e$.

Now, $M = e(M) \oplus (1 - e)(M) = \alpha(M) \oplus \ker e = \alpha(M) \oplus \ker \alpha$.

Conversely, let $\alpha \in$ centre of $\mathrm{End}(M)$ and $M = \alpha(M) \oplus \ker \alpha$.

Given $m \in M$ we write $m = \alpha(n) + k$ where $n \in M$ and $k \in \ker \alpha$.

Again, n can be written as $\alpha(n_1) + k_1$ with $n_1 \in M$, $k_1 \in \ker \alpha$. Then

$$\begin{aligned} \alpha(m) = \alpha\,(\alpha(n) + k) &= \alpha(\alpha(\alpha(n_1) + k_1)) \\ &= \alpha^2(\alpha(n_1)) \end{aligned}$$

Let $\alpha(n_1) = x_m \in \alpha(M)$. Then x_m is the unique element of $\alpha(M)$ such that $\alpha(m) = \alpha^2(x_m)$.For, let $y \in \alpha(M)$ with $\alpha(m) = \alpha^2(y)$. Then,

$$\begin{aligned} \alpha^2(y) = \alpha^2(x_m) &\Rightarrow \alpha^2(y - xm) = 0 \\ &\Rightarrow \alpha(y - x_m) \in \ker \alpha \cap \alpha(M) = 0 \end{aligned}$$

Therefore $y - x_m \in \ker \alpha \cap \alpha(M) = 0 \Rightarrow y = x_m$.

Now $\alpha^2(x_{rm}) = \alpha(rm) = r\alpha(m) = r\alpha^2(x_m) = \alpha^2(rx_m)$.

Hence $x_{rm} = rx_m$.

Also we have $\alpha^2(x_{m+n}) = \alpha(m + n) = \alpha(m) + \alpha(n)$

$= \alpha^2(x_m) + \alpha^2(x_n) = \alpha^2(x_m + x_n)$

This gives $x_{m+n} = x_m + x_n$

Let $\gamma \in \mathrm{End}_R(M)$. Then

$$\begin{aligned} \alpha^2(x_{\gamma m}) = \alpha(\gamma(m)) = \alpha\gamma(m) &= \gamma\alpha(m) \text{ as } \alpha \in \text{centre of } \mathrm{End}_R(M) \\ &= \gamma\alpha^2(x_m) = \alpha^2(\gamma x_m) \end{aligned}$$

Hence $x_{\gamma m} = \gamma x_m$

Now we define a map $\beta \in \mathrm{End}_R(M)$ by the rule $\beta(m) = x_m$

By what precedes $\beta(rm) = x_{rm} = rx_m = r\beta(m)$ and

$\beta(m + n) = x_{m+n} = x_m + x_n = \beta(m) + \beta(n)$

Therefore β is R-linear.

Also $\beta(\gamma(m) = x_{\gamma(m)} = \gamma\beta(m)$. Hence $\beta \in$ centre of $\mathrm{End}_R(M)$.

For any $m \in M$, $(\alpha\beta\alpha)(m) = \alpha(\beta\alpha(m)) = \alpha x_{\alpha(m)} = \alpha^2 x_m = \alpha(m)$

Therefore $\alpha\beta\alpha = \alpha$ with $\beta \in$ centre of $\mathrm{End}_R(M)$. ■

Before proving the next theorem we define a regular module.

2.51. Regular module. A left R-module M is *regular* if given any element $m \in M$ there exists $f \in \mathrm{Hom}_R(M, R)$ such that $(mf)m = m$.
In this definition and in the next theorem we write module homomorphisms on the right. By replacing M by R in the definition above we can see easily that ${}_R R$ is a regular module if and only if R is a regular ring.

2.52. Theorem. (Zelmanowitz) If ${}_R M$ is a regular module then the centre of $\mathrm{End}_R(M)$ is a regular ring.
Proof: Let $\alpha \in \mathrm{End}_R(M)$ and $m \in M$. Then $m\alpha \in M$. By the regularity of ${}_R M$ there exists $f \in \mathrm{Hom}_R(M, R)$ such that $((m\alpha)f)m\alpha = m\alpha = ((mf)m)\alpha^2$.
Now $m = ((mf)m)\alpha + [m - ((mf)m)\alpha]$ with $((mf)m)\alpha \in M\alpha$ and $m - ((mf)m)\alpha \in \ker \alpha$. So $M = M\alpha + \ker \alpha$.
Let $m\alpha \in M\alpha \cap \ker \alpha$. Then $m\alpha = ((mf)m)\alpha^2 = (mf)(m\alpha^2) = 0$. Hence $M\alpha \cap \ker \alpha = 0$. Thus $M = M\alpha \oplus \ker \alpha$. Now applying Lemma 2.50 we see that there exists an element β in $\mathrm{End}_R(M)$ such that $\alpha\beta\alpha = \alpha$. Hence the centre of $\mathrm{End}_R(M)$ is a regular ring. ■

§ 4. Endomorphism rings of infinite dimensional vector spaces.

In this section some properties of the endomorphism ring of an infinite dimensional vector space are discussed. While these results are "folklore" material for ring theorists, we do not know a convenient reference where they have been recorded.

First we define a V-ring and a quasi-simple ring.

2.53. Left V-ring. A ring R such that every simple left R-module is injective is called a *left V-ring*.

2.54. Quasi-simple ring. A ring is called a *quasi-simple ring* if it has only two ideals.

2.55. Proposition. Let V be an infinite dimensional vector space over a field K. Let R = $\mathrm{End}_K(V)$. Then

(1) R is regular.

(2) R is not directly finite.

(3) R is not unit-regular.

(4) R is not a left V-ring.

(5) R is not left/right noetherian.

(6) R is not left/right artinian.

(7) R is not quasi-simple.

Proof: For convenience, we shall prove some of these results for the case $\dim_K(V)$ is countable; the extension in the general case is easy.

(1) Since V is a vector space over a field K, it is a semi-simple module and it is well known that the endomorphism ring of a semi-simple module is regular. (See 2.11)

(2) Let $\{e_1, e_2, e_3, \ldots\}$be a basis of ${}_K V$. We define f, g in $\mathrm{End}_K(V)$ as follows.

$f(e_i) = e_{i+1}$, $i = 1, 2,$

$g(e_1) = e_1$, $g(e_i) = e_{i-1}$, $i = 2, 3,$

Then, $fg(e_1) = f(e_1) = e_2$ and $gf(e_i) = g(e_{i+1}) = e_i$

Therefore $gf = 1_V$ but $fg \neq 1_V$ which shows that R is not directly finite.

(3) We know that if a ring is unit-regular then it is directly finite. But as R is not directly finite it is not unit-regular.

(4) [Here we follow Sarath and Varadarajan [SV:74].]

We consider the case when V have a countably infinite base $\{v_n\}_{n\geq 1}$ over K. Then V is an R-module under the defition $\theta.x = \theta(x)$, $\theta \in R$, $x \in V$. It is well known that V is a simple R-module. We will prove that V is not injective over A.

Let, if possible, V be injective over A. Let $W_n = V$ for each interger $n \geq 1$ and $g_n : W_n \longrightarrow W$ be the identity map. The maps g_n give rise to a unique map

$$g : \bigoplus_{n\geq 1} W_n \longrightarrow V$$
$$(a_1, a_2, \ldots) \longmapsto a_1 + a_2 + \ldots$$

If V is injective, there will be some extension

$h : \prod_{n\geq 1} W_n \longrightarrow W$ of g. Let $q : R \longrightarrow \prod_{n\geq 1} W_n$ be defined by $q(\theta) = (\theta(v_n))$. Then q is a homomorphism of R-modules.

Let $w = f(1)$ where $f : R \longrightarrow V$ is the map hq. Let $\theta \in R$.

Then $f(\theta) = f(\theta.1) = \theta.f(1) = \theta.w = \theta(w)$ .... (1)

For any integer $n \geq 1$, let

F_{n+1} = vector subspace of V spanned by v_k for $k \geq n + 1$

$I_n = \{ \theta \in A \mid \theta(v_k) = 0 \text{ for } k \geq n + 1 \}$

Let $\theta \in I_n$. Then $\theta(v_k) = 0$ for $k \geq n + 1$.

Hence $\theta(v_k) \in W_1 + W_2 + \quad + W_n$

So $q(\theta) = (\theta(v_1), \theta(v_2), \ldots, \theta(v_n)) \in W_1 + W_2 + \ldots + W_n \leq \prod_{n\geq 1} W_n$

Since $h|_{\oplus_{n\geq 1} W_n} = g$, therefore, $f(\theta) = hq(\theta) = g(q(\theta))$

$= g(\theta(v_1), \ldots, \theta(v_n), 0, 0, \ldots) = \theta(v_1) + \ldots + \theta(v_n)$

$= \theta(v_1 + v_2 + \ldots + v_n)$ for each $\theta \in I_n$.

Here $v_1 + \ldots + v_n$ denotes the sum in W.

Let $\psi_n : W \longrightarrow W$ be the K-linear map determined by

$\psi_n(v_i) = v_i$ for $1 \leq i \leq n$ and $\psi_n(v_k) = 0$ for $k \geq n + 1$.

Then $\psi_n \in I_n$. Therefore, $f(\psi_n) = \psi_n(v_1 + v_2 + \quad + v_n)$

But $f(\psi_n) = \psi_n(w)$ from (1).

Hence $\psi_n(w) = \psi_n(v_1 + \ldots + v_n) \Rightarrow \psi_n(w - \sum_{i=1}^{n} v_i) = 0$

Since $\ker \psi_n = F_{n+1}$, we get $w - \sum_{i=1}^{n} v_i \in F_{n+1}$ .... (2)

Now (2) should be valid for every $n \geq 1$. Since $\{v_n\}_{n\geq 1}$ is a base for W and F_{n+1} is generated by v_k for $k \geq n + 1$, it is clear that there is no element $w \in V$ satisfying (2) for all $n \geq 1$. This contradition proves that V is not an injective left R-module. This proves that R is not a left V-ring. ■

Before proving (4), (5) and (6) we fix some notation and recall some facts.

Let V be any vector space over a field K. Let $W \leq V$.

Let $A_W = \{f \mid \operatorname{Im} f \leq W \}$, $B_W = \{f \mid W \leq \ker f \}$

Then A_W and B_W are clearly additive subgroups of $\text{End}_K(V)$. Let $f \in A_W$, $g \in \text{End}_K(V)$. Then $fg(V) = f(g(V)) \leq f(V) \leq W$. Therefore $fg \in A_W$. Thus A_W is a right ideal of $\text{End}_K(V)$. Let $f \in B_W$, $g \in \text{End}_K(V)$. Then $f(W) = 0 \Rightarrow gf(W) = 0$ $\Rightarrow W \leq \ker gf$. Therefore $gf \in B_W$. Thus B_W is a left ideal in $\text{End}_K(V)$.

Let $W_1 \leq W_2$. Recall that $A_{W_i} = \{f \mid \text{Im } f \leq W_i\}$, $i = 1, 2$

Now $f \in A_{W_1} \Rightarrow \text{Im } f \leq W_1 \leq W_2 \Rightarrow f \in A_{W_2}$ which shows that $A_{W_1} \leq A_{W_2}$.

Also $f \in B_{W_2} \Rightarrow f(W_2) = 0 \Rightarrow f(W_1) = 0 \Rightarrow f \in B_{W_2}$ which shows that $B_{W_2} \leq B_{W_1}$.

Again, let $W_1 \lneqq W_2$. Then there exists $x \in W_2$ such that $x \notin W_1$. Let β be a basis of V. We define a map f in $\text{End}_K(V)$ such that $f(e) = x$ for all e in β. Then $f(V) \leq W_2$ but $f(V) \leq W_1$. Therefore $A_{W_1} \lneqq A_{W_2}$.

Now we shall show that $W_1 \lneqq W_2 \Rightarrow B_{W_2} \lneqq B_{W_1}$

Let $z \in W_2$, $z \notin W_1$ such that $\tilde{\beta} \cup \{z\}$ is a linearly independent set where $\tilde{\beta}$ is a basis of W_1.

We consider the map $g : V \longrightarrow V$ where $g(\tilde{\beta}) = 0$ and $g(z) = z$. Then $W_1 \leq \ker g$ but $W_2 \leq \ker g$

Therefore $g \in B_{W_1}$, $g \in B_{W_2}$ that is $B_{W_2} \lneqq B_{W_1}$.

From the above results it is clear that whenever we have infinite chain $W_1 \lneqq W_2 \lneqq \ldots$ of subspaces of V then we have infinite chains $A_{W_1} \lneqq A_{W_2} \lneqq \ldots$ of right ideals of

$\mathrm{End}_K(V)$ and $B_{W_1} \gneq B_{W_2} \gneq \dots$ of left ideals of $\mathrm{End}_K(V)$.

Let ${}_K\bar{V}$ be a countably infinite dimensional vector space and β be a basis of ${}_K\bar{V}$. Let $\bar{R} = \mathrm{End}_K(\bar{V})$.

Let $A = \{f \mid \mathrm{Im}\ f \text{ is finite dimensional}\}$ and let $f, g \in A$.

Then, $(-f)(\bar{V}) = \{(-f)(x) \mid x \in \bar{V}\} = \{-f(x) \mid x \in \bar{V}\}$ $= \{f(-x) \mid x \in \bar{V}\} = f(\bar{V})$ which shows that $(-f) \in A$

Again, $(f + g)(\bar{V}) \leq f(\bar{V}) + g(\bar{V})$ shows that $f + g \in A$.

Let $h \in \mathrm{End}_K(\bar{V})$. Clearly $\dim hf(\bar{V}) \leq \dim f(\bar{V}) < \infty$. Therefore $hf \in A$.

Also $fh(\bar{V}) \leq f(\bar{V})$ implies that $\dim fh(\bar{V}) \leq \dim f(\bar{V}) < \infty$. Therefore $fh \in A$. Thus A is a two-sided ideal of $\mathrm{End}_K(\bar{V})$ which proves that $\mathrm{End}_K(\bar{V})$ is not a quasi-simple ring. ∎

Now we will show that A can not be written as B_W or A_W for some $W \leq \bar{V}$.

Let, if possible, $A = B_W$ for some $W \leq \bar{V}$. Then $W \leq \ker f$ $\forall f \in A$. For $f = 0$, $f \in A$ and $\ker f = 0$.

So we have $W \leq 0 \Rightarrow W = 0 \Rightarrow B_W = \{f \mid 0 \leq \ker f\} \Rightarrow B_W = \mathrm{End}_K(\bar{V})$.

Therefore $A = \mathrm{End}_K(\bar{V})$. But $1 \notin A$ as $1(V) = V$ and V is not finite dimensional. This contradiction shows that A cannot be written as B_W for any $W \leq \bar{V}$.

Again, let $A = A_W$ for some $W \leq \bar{V}$. Then $\mathrm{Im}\ f \leq W$ for each $f \in A$. Therefore $\bar{V} = W$. Then $A_W = \mathrm{End}_K(\bar{V}) = A$. But $1 \notin A$. From this contradiction it is clear that A cannot be written as A_W for any $W \leq \bar{V}$.

Now, let $\beta = \{e_1, e_2, \dots\}$ be a basis of $\bar{V}$. Then we can have infinite chain of subspaces

$W_1 \gneq W_2 \gneq \dots$ (1) where A_ι is the subspace of $\bar{V}$ generated by $\{e_1, e_2, \dots\}$.

This gives rise to an infinite chain of right ideals $A_{W_1} \gneq A_{W_2} \gneq \dots$ which shows that $\mathrm{End}_K(\bar{V})$ is not right artinian and $B_{W_1} \gneq B_{W_2} \gneq \dots$ which shows that $\mathrm{End}_K(\bar{V})$ is not left artinian.

Again we can have an infinite chain of right ideals $W_1 \gneq W_2 \gneq \dots$ This gives rise to an infinite chain of endomorphism rings $A_{W_1} \gneq A_{W_2} \gneq \dots$ which shows that $\mathrm{End}_K(\bar{V})$ is not right artinian and $B_{W_1} \gneq B_{W_2} \gneq \dots$ which shows that $\mathrm{End}_K(\bar{V})$ is not left noetherian.

So we have proved (5) and (6). ∎

§ 5. One-sided unit-regularity in rings.

In this section we shall discuss one-sided unit-regularity in rings. Everything that is discussed here is due to Ehrlich.

2.56. Right/left unit-regular element. An element a in a ring R is *right/left unit-regular* if there is a right/left invertible element $x \in R$ such that $axa = a$.

2.57. Right/left unit-regular ring. A ring R is *right/ left unit-regular* if each of its elements is right/left unit-regular.

2.58. Some notations.

(1) In a ring R, $(Ra)^r$ will denote the right annihilator of the principal left ideal generated by the element a of R and $(aR)^l$ will denote the left annihilator of the principal right ideal generated by the element a of R.

(2) Let ${}_RM$ be a module and X,Y are submodules of M. Then $X \overset{\sim}{\subset} Y$ will indicate that X is isomorphic to a submodule of Y.

2.59. Remarks. Let ${}_RM$ be a module such that S = $\mathrm{End}_R(M)$ is regular. Let $x \in S$. Then

(i) x is right invertible in S if and only if it is an onto endomorphism of M.

(ii) x is left invertible if and only if it is a one-one endomorphism of M.

Proof: (i) Let x be right invertible and y be a right inverse of x. Then $xy = 1 \Rightarrow xy(m) = m \ \forall \ m \in M$ which shows that x is an epimorphism. Conversely, let x be an epimorphism. Since S is regular, there exists $y \in S$ such that $xyx = x$. Now, let $m \in M$. Then $m = x(m')$ for some $m' \in M$. As $xyx = x$, we have, $xyx(m') = x(m') \Rightarrow xy(m) = m$ which shows that $xy = 1_M$. Thus x is right invertible.

(2) If x is left invertible in S then there exists y in S such that $yx = 1$. Now, $x(r) = 0 \rightarrow yx(r) = 0 \rightarrow r = 0$. So x

is a monomorphism. Conversely, let x be a monomorphism. Since S is regular there exists y in R such that $x = xyx$. Therefore, $x(yx - 1) = 0 \Rightarrow yx - 1 = 0 \Rightarrow yx = 1$. So x is left invertible.

For the proof of the following lemma we refer to [E:76].

2.60. Lemma. Let M_R be a module such that $S = \mathrm{End}_R(M)$ is a regular ring. Let $\mathcal{L}_P(S)$ be the lattice of all principal right ideals of S and let $\mathcal{L}_C(M)$ be the set of all complimented submodules of M. Then

(1) $\mathcal{L}_C(M)$ is a sublattice of the lattice of all submodules of M.

(2) $\psi : aS \longrightarrow aM$ where $a \in S$ is an isomorphism of the latice $\mathcal{L}_P(S)$ onto the lattice $\mathcal{L}_C(M)$.

(3) If $a \in S$ then $\ker a = \psi\ (Sa)^r$.

2.61. Theorem. Let M be a right R-module and $S = \mathrm{End}_R(M)$ be a regular ring. Let $a \in S$. Then the following conditions are equivalent:

(1) a is right unit regular.

(2) There is a monomorphism $u : M \longrightarrow M$ such that

$\mathrm{Im}\ a \cap u(\ker a) = 0$.

(3) $\ker a \overset{\sim}{\subset} \mathrm{coker}\ a$.

Proof: (1) $\Rightarrow$ (2). There exists a right invertible element $x \in S$ such that $axa = a$. Let u be the right inverse of x. Then $xu = 1$ implies that $u : M \longrightarrow M$ is monomorphism. Let $m \in \mathrm{Im}\ a \cap u(\ker a)$. Then $m = a(m') = u(k)$ for some

$m' \in M$, $k \in \ker a$. Therefore, $m = a(m') = axa(m') = axu(k) = a(k) = 0$. Therefore, $\operatorname{Im} a \cap u(\ker a) = 0$.

(2) $\Rightarrow$ (3). Let u be a monomorphism from M to M such that $\operatorname{Im} a \cap u(\ker a) = 0$. Since $\operatorname{End}_R(M)$ is regular $\operatorname{Im} a$ and $\ker a$ are direct summands of M. Let P be a submodule of M such that $\operatorname{Ker} a \oplus P = M$. Therefore, $u(\ker a) \oplus u(P) = u(M)$ as u is a monomorphism. Again, $u(M)$ is a direct summand of M as $\operatorname{End}_R(M)$ is regular. Thus $u(\ker a)$ is a direct summand of M, and so $\operatorname{Im} a \oplus u(\ker a)$ is a direct summand of M.

Let $Q \leq M$ be such that $M = (\operatorname{Im} a \oplus u(\ker a)) \oplus Q$. Then $\operatorname{coker} a = \dfrac{M}{\operatorname{Im} a} \cong u(\ker a) \oplus Q$. Therefore,

$\ker a$ m$\cong$ n $u(\ker a) \overset{\sim}{\subset} u(\ker a) \oplus Q \cong \operatorname{coker} a$.

(3) $\Rightarrow$ (1). Let $\ker a \overset{\sim}{\subset} \operatorname{coker} a$. Let $\alpha : \ker a \longrightarrow \operatorname{coker} a$ be a monomorphism. We define $y : M \longrightarrow M$ by

$y_{|T} = a_{|T}$, where T is a complement of $\ker a$ in M.

$y_{|\ker a} = \alpha$.

Now, $y_{|T}$ is a monomorphism, for $y(t) = 0 \Rightarrow a(t) = 0 \Rightarrow t \in \ker a$. Also, $t \in T$. But $T \cap \ker a = 0$. So, $t = 0$. Thus, $y_{|T}$ and $y_{|\ker a}$ are both monomorphisms, hence left invertible in R. So y is left invertible in R. Let $x \in R$ be a left inverse of y. Then for $t \in T$, $axat = axyt = at$. For $k \in \ker a$, $axak = 0 = ak$. Therefore $axa = a$ with x right invertible.

2.62. Theorem. Let M be a right R module such that $S = \operatorname{End}_R(M)$ is a regular ring. Let $a \in S$. Then the following conditions are equivalent.

(1) a is left unit-regular.

(2) There is an epimorphism $u : M \longrightarrow M$ such that $\operatorname{Im} a + u(\ker a) = M$.

(3) There is an epimorphism $u : M \longrightarrow M$ such that $\operatorname{Im} a \oplus u(\ker a) = M$.

(4) $\operatorname{coker} a \overset{\sim}{\subset} \ker a$.

Proof: (1) $\Rightarrow$ (2). Since a is left unit-regular, there exists a left invertible element $x \in S$ such that $axa = a$. Let $u \in S$ be a left inverse of x. Then u is an onto endomorphism of M. As $axa = a$, $xa \in \bar{I}(S)$. So, we have,

$M = xa(M) \oplus \ker xa = xaM \oplus \ker a$. This gives

$uM = uxaM + u(\ker a) \Rightarrow M = aM + u(\ker a)$.

(2) $\Rightarrow$ (3). Let $M = \operatorname{Im} a + u(\ker a)$ where $u : M \longrightarrow M$ is an epimorphism. Since $u(\ker a) = \psi(u(Ra)^r$, $\operatorname{Im} a \cap u(\ker a)$ is a complemented submodule of M, hence of $u(\ker a)$. Let Q be a submodule of M such that $(\operatorname{Im} a \cap u(\ker a)) \oplus Q = u(\ker a)$. Then Q is a homomorphic image of $u(\ker a)$. We have,

$M = \operatorname{Im} a + u(\ker a)$. This implies that

$M = \operatorname{Im} a + (\operatorname{Im} a \cap u(\ker a)) \oplus Q = \operatorname{Im} a \oplus Q$.

Hence there is an epimorphism $\bar{u} : M \longrightarrow M$ such that $\bar{u}|_T = a|_T$ where T is a complement of $\ker a$ in M and $\bar{u}|_{\ker a}$ is an epimorphism from $\ker a$ to Q. Thus $M = \operatorname{Im} a \oplus \bar{u}(\ker a)$ with $\bar{u} : M \longrightarrow M$ an epimorphism.

(3) $\Rightarrow$ (4). Let $u : M \longrightarrow M$ be an epimorphism such that $\operatorname{Im} a \oplus u \ker a = M$. Let L be a submodule of M such that $(\ker a \cap \ker u) \oplus L = \ker a$. Then

$$\text{coker } a = \frac{M}{\text{Im } a} \cong u(\ker a) \cong \frac{\ker a}{\ker a \cap \ker u} \cong L \subset \ker a.$$

Therefore coker a $\overset{\sim}{\subset}$ ker a.

(4) ⇒ (1). Let coker a $\overset{\sim}{\subset}$ ker a. Let T be a complement of ker a in M and let S be a complement of Im a in M.

Let $\beta : \text{Im } a \longrightarrow T$ be the isomorphism which is inverse to $a_{|T}$ and let $\alpha : S \longrightarrow \ker a$ be a monomorphism. Then there is a monomorphism $x : M \longrightarrow M$ such that $x_{|\text{Im } a} = \beta$ and $x_{|S} = \alpha$. If $t \in T$ then $axa(t) = a\beta a(t) = a(t)$. If $k \in \ker a$ then $axa(k) = 0 = a(k)$. Therefore, $axa = a$ with x left invertible.

2.63. Corollary. Let V be a vector space over a division ring D. Let $R = \text{End}_D(V)$.

(1) If $a \in R$, then a is right unit regular if and only if dim (ker a) ≤ dim (coker a); a is left unit regular if and only if dim (coker a) ≤ dim (ker a).

(2) R is one sided unit-regular.

Proof: These are easy consequences of theorem 2.61 and 2.62.

2.64. Theorem. Let M be a right R-module such that $S = \text{End}(M_R)$ is a regular ring. Then S is one sided unit-regular if and only if whenever $M = P_1 \oplus Q_1 = P_2 \oplus Q_2$ with $P_1 \cong P_2$ then it implies that $Q_1 \overset{\sim}{\subset} Q_2$ or $Q_2 \overset{\sim}{\subset} Q_1$.

Proof: Let $M = P_1 \oplus Q_1 = P_2 \oplus Q_2$ and a be an isomorphism of P_1 onto P_2 such that $a_{|Q_1} = 0$. Then Im a = P_2 and ker a =

Q_1. Coker $a = M/\mathrm{Im}\ a \cong Q_2$. Therefore by Theorem 2.61 and 2.62 respectively, we have, S is right unit-regular if and only if $Q_1 \overset{\sim}{\subset} Q_2$ and S is left unit-regular if and only if $Q_2 \overset{\sim}{\subset} Q_1$.

2.65. Theorem. Let R be a regular ring and let $a \in R$. Then

(1) a is right unit-regular if and only if there is a left invertible element $u \in R$ such that $aR \cap u(Ra)^r = 0$.

(2) a is left unit-regular if and only if there is a right invertible element $u \in R$ such that $aR + u(Ra)^r = R$.

Proof: If $a \in R$, then let a_L is an element of $\mathrm{End}_R(R)$ where a_L is the left multiplication by a. First we note that if $a \in R$, (i) a is right invertible if and only if a_L is an epimorphism; (ii) a is left invertible if and only if a_L is a monomorphism.

Obviously, $\mathrm{Im}\ a_L = aR$. Let $k \in \ker\ a_L$. This implies that $ak = 0 \Rightarrow rak = 0\ \forall\ r \in R$. Thus $k \in (Ra)^r$.

Again if $k \in (Ra)^r$ then $Rak = 0$ which implies that $ak = 0$ $\Rightarrow k \in \ker\ a_L$. Therefore $\ker\ a_L = (Ra)^r$. Then we get this result directly by applying theorem 2.61 and 2.62.

§ 6. Bibliographical notes.

1. For Proposition 2.10 see ([G] : p.2).
2. For Proposition 2.11 and Proposition 2.37 see ([S], pp.40–41)
3. For Proposition 2.14 see ([K], p.114).

4. Proposition 2.16 is due to Ware [W :71].
5. For Section 2 we have followed Lambek ([L]: pp.102–104).
6. Remark 2.28, Proposition 2.30, Theorem 2.32, 2.39 and 2.42 are due to Ehrlich [E : 68].
7. Lemma 2.31, Proposition 2.33 and Theorem 2.43 are due to Henriksen [H : 73].
8. For Theorem 2.46 see [G].
9. Theorem 2.48 was first proved for commutative regular rings by Wiegand [W : 69], and then in general by Steinberg [S :73]. Here we have amplified Goodearl's treatment ([G], Chapter 3).
10. Proposition 2.49, 2.50 and Theorem 2.52 are all due to Zelmanowitz ([Z : 72], pp. 349–350).
11. Section 6 is completely due to Ehrlich [E : 76].
12. The definition of inverse limit is taken from Rotman ([R], pp 28–29).

CHAPTER 3
COMMUTATIVITY

In this chapter we shall mainly discuss the commutativity of endomorphism rings of ideals. Also we define the total quotient ring and the complete ring of quotients of a commutative ring. It follows from the proof of Theorem 2.48 that if R is a commutative regular ring then $\mathrm{End}_R(I)$ is also a commutative regular ring for every ideal I of R. In this chapter we prove the commutativity part of this theorem in a different manner. The main results of this chapter are as follows :

(1) If R is commutative and reduced then $\mathrm{End}_R(A)$ is commutative and reduced for every ideal A of R.

(2) If R is a quasi-regular ring then $\mathrm{End}_R(A)$ is commutative for every ideal A of R.

§ 1. Basic definitions.

In this section we recall the defition of $S^{-1}R$ where R is a commutative ring and S is a multiplicatively closed subset of R. We also note down some basic results.

3.1. Let R be a commutative ring and S a multiplicatively closed subset of R. Then $S^{-1}R$ consists of elements of type r/s where $r \in R$, $s \in S$, $s \neq 0$ and for two elements r_1/s_1 and r_2/s_2 of $S^{-1}R$ $r_1/s_1 = r_2/s_2$ if and only if there exists some $u \in S$ such that $(r_1s_2 - r_2s_1)u = 0$. With addition and multiplication defined by

$r_1/s_1 + r_2/s_2 = (r_1s_2 + r_2s_1)/s_1s_2$,

$(r_1/s_1).(r_2/s_2) = (r_1r_2)/(s_1s_2)$ $S^{-1}R$ becomes a ring.

3.2. Remark. $S^{-1}R$ is the zero ring if and only if $0 \in S$.

3.3. Remark. The set S_o of non-zero-divisors in a commutative ring R is a multiplicatively closed set.

3.4. Total quotient ring. Let R be a ring and S_o be the set of non-zero-divisors of R. Then $S_o^{-1}R$ is its *total quotient ring*, (T.Q.R.). It is also known as the *classical ring of quotients*.

3.5. Remark. Let R be a commutative ring and A be an ideal in R.Let S be a multiplicatively closed subset of R. Then $S^{-1}A = \left\{ {}^{a}/s \mid a \in A,\ s \in S \right\}$ is an ideal of $S^{-1}R$, for $(r/s).(a_1/s_1) = ra_1/ss_1$ where $a_1 \in A$, $s_1, s \in S$, $r \in R$.

3.6. Remark. Let R be a commutative ring and A is an ideal in R. If f is an R-linear endomorphism of A then f gives rise to an $S^{-1}R$-linear endomorphism $S^{-1}f$ of $S^{-1}A$ defined by $S^{-1}f(a/s) = f(a)/s$.

3.7. Quasi-regular ring. A commutative ring R is *quasi-regular* if the T.Q.R. of R is regular.

The following proposition can be verified easily.

3.8. Proposition. Let R be a commutative ring and S be a multiplicatively closed subset of R. Let M, N be $S^{-1}R$ modules. Then $\mathrm{Hom}_{S^{-1}R}(M, N) \cong \mathrm{Hom}_R(M, N)$.

3.9. Corollary. Let R be a commutative domain and K be its field of fractions. Let M, N be K-vector spaces. Then $\mathrm{Hom}_K(M, N) \cong \mathrm{Hom}_R(M, N)$.

In particular, if $R = \mathbb{Z}$ then $K = \mathbb{Q}$ and for $M = N = \mathbb{Q}$, we have $\mathrm{End}_{\mathbb{Q}}(\mathbb{Z}) = \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q}$.

§ 2. Complete ring of quotients.

In this section we give a brief introduction to the complete ring of quotients.

The process of forming the field of fractions of an integral domain is well known. In the same way we can construct theclassical ring of quotients of any commutative ring. (See 3.4. above).But the complete ring of quotients is formed in a different way which is discussed below.

3.10. Dense ideal. An ideal D in a commutative ring R is *dense* if for all $r \in R$, $rD = 0 \Rightarrow r = 0$. Thus for all these ideals $\mathrm{Ann}(D) = 0$.

3.11. Fraction. If D is a dense ideal in a commutative ring R, then any element f of $\mathrm{Hom}_R(D, R)$ is a *fraction*.

The fractions form an additive abelian semi-group with zero and a multiplicative abelian semi-group with 1. We define a relation θ on the set F of all fractions as $f_1 \ \theta \ f_2$ whenever f_1 and f_2 are equal on the intersection of their domains. Then θ is a congruence relation. Now θ decomposes F into equivalence classes. This set of equivalence classes is also a commutative ring which is denoted by Q(R) and is called the complete ring of quotients of R.

§ 3. Commutativity of endomorphism rings.

In this section we shall discuss different conditions on a ring R under which the endomorphism rings of ideals of that ring are reduced or are domains or are commutative.

3.12. Proposition. If R is a commutative domain then $\mathrm{End}_R(A)$ is commutative for every ideal A in R.

Proof: Let $a \in A$ and $f, g \in \mathrm{End}_R(A)$.

Case 1. Let $a = 0$. Then $fg(a) = 0 = gf(a)$.

Case 2. Let $a \neq 0$. and let $z = fg(a) - gf(a)$. Then

$za = \left[fg(a) - gf(a)\right] a = fg(a)a - gf(a)a = fg(a^2) - gf(a^2)$

$= f(a)g(a) - g(a)f(a) = 0$ as R is commutative.

Since R is a domain, $za = 0 \Rightarrow z = 0$.

Therefore $fg(a) = gf(a)$.

From case 1 and case 2, we see that $fg(a) = gf(a)$ for all

$a \in A$ which implies that $fg = gf$. Therefore $\mathrm{End}_R(A)$ is commutative. ▨

3.13. Proposition. If R is a commutative regular ring, then $\mathrm{End}_R(A)$ is a commutative ring for every ideal A of R.

Proof: Let $a \in A$. Since R is regular, there exists $b \in A$ such that $a = aba$. Let $f, g \in \mathrm{End}_R(A)$. Then

$fg(a) - gf(a) = fg(aba) - gf(aba) = f\left[ag(ba)\right] - g\left[f(a)ba\right]$

$= f(a)g(b)a - g(b)f(a)a = 0$. So $fg(a) = gf(a)$ $\forall$ $a \in A$. This implies that $fg = gf$. (Note that, by Theorem 2.48 $\mathrm{End}_R(A)$ is necessarily regular).

3.14. Proposition. If R is a commutative domain and A an ideal of R then $\mathrm{End}_R(A)$ is a domain.

Proof: Let f and g be two non-zero endomorphisms of A. Then there exist $a, b \in A$ such that $f(a) \neq 0$ and $g(b) \neq 0$. Now, $fg(ab) = f\left[ag(b)\right] = f(a)g(b)$. As R is a domain, $f(a)g(b) \neq 0 \Rightarrow fg(ab) \neq 0 \Rightarrow fg \neq 0$. Therefore $\mathrm{End}_R(A)$ is a domain. ▨

3.15. Proposition. (Cox) If R is a commutative ring and A an ideal of R, then $\mathrm{End}_R(A)$ is commutative if $A \cap \mathrm{Ann}(A) = 0$.

Proof: Let f, g be two non-zero endomorphisms of A. Then fg, gf and $fg - gf$ are all R-linear maps. Let $a, b \in A$. Then $a(fg - gf)b = (fg - gf)(ab) = fg(ab) - gf(ab)$

$= f(a)g(b) - g(b)f(a) = 0$. Therefore, $A(fg - gf)A = 0$

$\Rightarrow (fg - gf)A \subset A \cap \mathrm{Ann}(A) = 0$. So, $(fg - gf)A = 0 \Rightarrow$

$fg - gf = 0 \Rightarrow fg = gf$. Thus $\mathrm{End}_R(A)$ is a commutative ring. ∎

3.16. Corollary. If R is commutative and reduced then then $\mathrm{End}_R(A)$ is commutative and reduced for each ideal A in R.

Proof: First we assert that $A \cap \mathrm{Ann}(A)$ consists of nilpotents. For, if $a \in A \cap \mathrm{Ann}(A)$ then $aA = 0 \Rightarrow a.a = 0$ $\Rightarrow a^2 = 0 \Rightarrow a$ is nilpotent. As R is reduced, $A \cap \mathrm{Ann}(A) = 0$. So by Proposition 3.16 $\mathrm{End}_R(A)$ is commutative. (Note that this conclusion is an extension of 3.13).

If $f \in \mathrm{End}_R(A)$ be such that $f^n = 0$ then $\left[f(a)\right]^n = f^n(a^n) = 0 \ \forall \ a \in A$. As R is reduced, this implies that $f(a) = 0 \ \forall \ a \in A$. Thus $f = 0$ and hence $\mathrm{End}_R(A)$ is reduced. ∎

Proposition 3.16 has the following non-communicative extension.

3.17. Proposition. If R is a left duo reduced ring and A is a left ideal of R, then $\mathrm{End}_R(A)$ is reduced.

Proof: Let $f \in \mathrm{End}_R(A)$ be such that $f^2 = 0$. Let $I = \mathrm{Im}\ f$ and $K = \ker f$. Now $f^2 = 0 \Rightarrow f^2(A) = 0 \Rightarrow f\ f(A) = 0$ $\Rightarrow f(A) \leq \ker(f)$, that is $I \leq K$. This gives $I^2 \leq KI = Kf(A)$ $= f(KA) = f(K)A = 0$. As R is reduced, $I^2 = 0$ implies $I = 0$. Thus $f = 0$ and hence $\mathrm{End}_R(A)$ is reduced. ∎

The following result extends 3.13.

3.18. Proposition. If R is a commutative p-injective ring then $\mathrm{End}_R(A)$ is commutative for every ideal A in R.

Proof: Let $f, g \in \mathrm{End}_R(A)$ and let $a \in A$. Then $Ra \leq R$. Consider the map $\bar{f} = f_{|Ra} : Ra \longrightarrow R$. Since R is p-injective, there exists a map $\psi : R \longrightarrow R$ which extends $\bar{f}$. So $f(a) = \bar{f}(a) = \psi(a) = \psi(1.a) = a\psi(1) = at_1$ for some $t_1 \in R$. Similarly, $g(a) = at_2$ for some $t_2 \in R$.

Now $fg(a) - gf(a) = f(at_2) - g(at_1) = t_2f(a) - t_1g(a) = t_2at_1 - t_1at_2 = 0$. This gives $fg = gf$. Thus $\mathrm{End}_R(A)$ is commutative. ∎

3.19. Proposition. If R is a quasi-regular ring then $\mathrm{End}_R(A)$ is commutative for every ideal A in R.

Proof: Let S_o be the set of all non-zero-divisors of R. Let $f, g \in \mathrm{End}_R(A)$. Then $S_o^{-1}f, S_o^{-1}g \in \mathrm{End}_{S_o^{-1}R}(S_o^{-1}A)$. As $S_o^{-1}R$ is regular, $\mathrm{End}_{S_o^{-1}R}(S_o^{-1}A)$ is commutative by Proposition 3.13. So we have, for $a \in A$,

$$\left(S_o^{-1}f.S_o^{-1}g\right)(a/1) = \left(S_o^{-1}g.S_o^{-1}f\right)(a/1)$$

$$\Rightarrow S_o^{-1}f(g(a)/1) = S_o^{-1}g(f(a)/1) \Rightarrow fg(a)/1 = gf(a)/1$$

$\Rightarrow (fg(a) - gf(a))/1 = 0$. Therefore there exists $t \in S_o$ such that $t\left[fg(a) - gf(a)\right] = 0 \Rightarrow fg(a) - gf(a) = 0$ as t is a non-zero-divisor. This gives $fg = gf$. Thus $\mathrm{End}_R(A)$ is commutative. ∎

3.20. Remark. By analogy with the term "quasi-regular" we may call a commutative ring R "quasi-p-injective" if the T.Q.R. of R is p-injective. We have p-injective $\Rightarrow$

quasi-p-injective and quasi-regular ⇒ quasi-p-injective. (See Chapter 2). An application of the Proposition 3.18 yields, by an argument similar to that in 3.19, the following extension of 3.19 : If R is a commutative quasi-p-injective ring and A an ideal of R, then $\mathrm{End}_R(A)$ is commutative. A study of quasi-p-injective rings seems to be a topic of independent interest.

§ 4. A question of Faith.

A consequence of a result from Stenström [S] (see 3.21) is that if Q(R) is a self-injective ring then $\mathrm{End}_R(I)$ is commutative for every ideal I of R. (Proposition 3.22 below.) The proof of this result is similar to that of Proposition 3.15 above due to Cox. Carl Faith [F] had asked whether the converse of Proposition 3.22 holds. This section is devoted to the proof of Proposition 3.22. We also record without proof a counter-example due to Clark which shows that the question of Faith has a negative answer.

We record the following result from [S], p.279] without proof and then prove Proposition 3.22 with its help.

3.21. Theorem. Let Q(R) denote the complete ring of quotients of a commutative ring R. (See § 2.) Then Q(R) is a self-injective ring if and only if for every ideal I of R and every R-homomorphism $\phi : I \longrightarrow R$ there exists a dense ideal $J \supseteq I$ and an R-homomorphism $\psi : J \longrightarrow R$ such that ψ extends ϕ.

3.22. Proposition. If $Q(R)$ is a self-injective ring then $\mathrm{End}_R(I)$ is commutative for every ideal I in R.

Proof: Let $f, g \in \mathrm{End}_R(I)$. Since $Q(R)$ is self-injective, by Theorem 3.21 there exists an ideal $J_1 \supseteq I$ such that $\mathrm{Ann}(J_1) = 0$ and an R-homomorphism $h_1 : J_1 \longrightarrow R$ which extends f. Similarly there exists an ideal $J_2 \supseteq I$ such that $\mathrm{Ann}(J_2) = 0$ and an R-homomorphism $h_2 : J_2 \longrightarrow R$ extending g. Let $i \in I$, $j \in J_1 \cap J_2$. First we note that $ij \in I$, $f(ij) \in I$, $g(ij) \in I$, $I \subset J_1 \cap J_2$. Next we have, $fg(ij) = f(g(ij)) = h_1(g(ij)) = h_1h_2(ij) = h_1(h_2(i)j) = h_2(i)h_1(j)$.

Again, $gf(ij) = g(f(ij)) = h_2f(ij) = h_2(h_1(ij)) = h_2(ih_1(j)) = h_2(i)h_1(j)$.

Therefore $fg(ij) = gf(ij) \Rightarrow \left[fg(i) - gf(i)\right] j = 0 \ \forall \ j \in J_1 \cap J_2$.

But $\mathrm{Ann}(J_1 \cap J_2) = 0$. Hence $fg(i) - gf(i) = 0 \ \forall \ i \in I$.

This gives $fg = gf$. Thus $\mathrm{End}_R(I)$ is commutative. ◻

We conclude with an example (which is due to John Clark [C:86]) of a commutative ring R which is not self-injective but such that for every ideal I of R $\mathrm{End}_R(I)$ is commutative. The proof is not given here. Interested readers may refer to ([C:86]) for the proof.

3.23. Example. Let A be a countable discrete valuation ring with field of fractions K. Let M be the A-module K/A. Let $R = A \, (+) \, M$. Then R becomes a ring on defining addition and multiplication as

$(a_1, m_1) + (a_2, m_2) = (a_1 + a_2, m_1 + m_2)$

$(a_1, m_1).(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)\ \forall\ a_1, a_2 \in A, m_1, m_2 \in M.$

Then $\mathrm{End}_R(I)$ is commutative for every ideal I of R but $Q(R) = R$ is not self-injective.

§ 5. Bibliographical notes.

1. For section 1 we have followed Atiyah and McDonald [AM].
2. For section 2 we have followed Lambek ([L], pp. 36-38).
3. Proposition 3.15 is due to Cox [C : 73].
4. Theorem 3.21 is in Stenstrom ([S],p.279).
5. Example 3.23 is due to Clark [C : 86].

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