A Note on Numerical Estimation of Sato's Two-Level CES Production Function

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I. Introduction: A production function is the technical relationship between the output and the inputs used for turning out the produce manufactured by an **efficient** firm. The simplest and most popular specification of the said technical relationship is the so-called Cobb-Douglas production function given as $Y = A(X_1^{\alpha} X_2^{\beta})$, where, Y is the output and $(X_1,\ X_2)$ are the inputs (often labour and capital, respectively) applied to raise the output. Given a sample of data (of $n \geq 4$ size, but better if larger) on Y and $(X_1,\ X_2)$, it is often required to fit the function $Y = A(X_1^{\alpha} X_2^{\beta})$ to the data as best as possible and to estimate the parameters ($A,\ \alpha$ and β) of the function. These parameters have a definite meaning in economics. While A is interpreted as the scale parameter, α and β are interpreted as the elasticities of produce (Y) with respect to labour (X_1) and capital (X_2) respectively. In turn, the elasticity of the produce, Y, with respect to any input (say X_i) is defined as $\xi_{YX_i} = (\partial Y/\partial X_i)/(\bar{Y}/\bar{X}_i)$ or the ratio of the marginal productivity to the average productivity of the input concerned. Estimation of these parameters is straightforward. A logarithmic transformation of the Cobb-Douglas function, $Y = A(X_1^{\alpha} X_2^{\beta})$, gives us $y = a + \alpha x_1 + \beta x_2$ where $y = \log(Y),\ a = \log(A),\ x_1 = \log(X_1)$ and $x_2 = \log(X_2)$. Linear (multiple) regression of y on $(x_1,\ x_2)$ readily gives us the estimated values of parameters if the sample data satisfy the required conditions of estimation.

Arrow et al. (1961) generalized the Cobb-Douglas production function. This generalized production function is known as the Constant Elasticity of Substitution (CES) production function. The formal specification of CES production function is $Y = A\left[\delta X_1^{-\beta} + (1-\delta) X_2^{-\beta}\right]^{-(\rho/\beta)}$. In this specification, $0 < \delta < 1$ is called the distribution parameter, $-1 \leq \beta$ is called the substitution parameter and $0 \leq \rho$ is called the returns (to scale) parameter. The elasticity of substitution $\sigma = 1/(1+\beta)$ is a constant, depending on the substitution parameter, β. The elasticity of substitution, σ, is, in particular, unity for the Cobb-Douglas production function, when $\beta = 0$. For the L-shaped Leontief production function, where there is no substitution between inputs, $\sigma = 0$ (while β is very large). For $-1 \leq \beta < 0$ the elasticity of substitution is larger than unity. Thus, the Cobb-Douglas, the Leontief and the linear production functions are only the special cases of the CES production function for $\beta = 0,\ \beta \to \infty$ and $\beta = -1$ respectively (Intriligator, 1978).

Since the CES production function is nonlinear and not amenable to any simple transformation so as to make the estimation of its parameters amenable to linear regression analysis, Kmenta approximated the original CES specification by Taylor's expansion (around $\beta = 0$), and linearizing it by dropping the terms involving powers of β

larger than unity (Kmenta, 1967, 1971). This approximation, known as Kmenta's approximation of the CES production function, is given as:

$$y = a + \rho\delta x_1 + \rho(1-\delta)x_2 - 0.5\rho\beta\delta(1-\delta)[x_1 - x_2]^2$$

where, y, a, x_1 and x_2 are $\log(Y)$, $\log(A)$, $\log(X_1)$ and $\log(X_2)$ respectively. The parameters of Kmenta's approximation are amenable to estimation by linear regression analysis. From these estimated parameters one may get back the estimated values of the parameters of the original CES specification. This is not to say that the original CES function cannot be estimated directly (by nonlinear regression). However, due to its simplicity (and some sort of general bias of economists in favour of assuming $\beta \cong 0$), Kmenta's approximation has received a wide acceptance.

II. Sato's Generalization of the CES Production Function: Kazuo Sato (1967) generalized the CES production function by nesting the CES at two levels and augmenting the list of inputs to the output. Sato's two-level CES production function may be specified as:

$$Y = A\left[\{\delta_1 X_1^{-\beta_1} + (1-\delta_1)X_2^{-\beta_1}\}^{\beta/\beta_1} + \{\delta_2 X_3^{-\beta_2} + (1-\delta_2)X_4^{-\beta_2}\}^{\beta/\beta_2}\right]^{-1/\beta}$$

Symbolically, $Y = A[CES_1 + CES_2]^{\gamma}$. In this specification, CES_1 may be close to the Leontief type (very little substitution between X_1 and X_2) function while CES_2 may be of the Cobb-Douglas type, etc. Then, at the higher level, they may be combined differently. Equally well, one may specify the models as $Y = A[\delta_3 CES_1^{-\beta_3} + (1-\delta_3)CES_2^{-\beta_3}]^{-1/\beta_3}$ and so on.

There are ample empirical evidences that suggest capital-skill complementarity (Griliches, 1969), or the wage differential between skilled and unskilled workers. It requires two types of labour (skilled and unskilled) to be separately dealt with in specifying the production function. To specify such models, the two-level CES production technology with capital, skilled labor and unskilled labor as inputs may be more suitable. Denoting X_1 as the skilled labour, X_2 as the unskilled labour, and X_3 as capital, we may define: $Y_1 = A_1\{\delta_1 X_1^{-\beta_1} + (1-\delta_1)X_3^{-\beta_1}\}^{-1/\beta_1}$. At the second level, Y_1 may be combined with the unskilled labour, X_2, to give $Y_2 = A_2\left[\delta_2 Y_1^{-\beta_2} + (1-\delta_2)X_2^{-\beta_2}\right]^{-1/\beta_2}$. By substituting Y_1 into the last equation we get (Papageorgiou and Saam, 2005),

$$Y_2 = A_2\left[\delta_2 A_1^{-\beta_2}[\delta_1 X_1^{-\beta_1} + (1-\delta_1)X_3^{-\beta_1}]^{\frac{\beta_2}{\beta_1}} + (1-\delta_2)X_2^{-\beta_2}\right]^{-1/\beta_2}$$

Such models cannot be linearized or approximated easily. Therefore, estimation of their parameters necessitates an application of nonlinear methods of optimization.

III. An Example of Sato's Two-Stage CES Production function and its Estimation: The data (table-1) on Y have been generated from (X_1, X_2, X_3, X_4) by the model given below. The sample size, $n = 50$.

$$Y = A\left[\{\delta_1 X_1^{-\beta_1} + (1-\delta_1)X_2^{-\beta_1}\}^{\beta/\beta_1} + \{\delta_2 X_3^{-\beta_2} + (1-\delta_2)X_4^{-\beta_2}\}^{\beta/\beta_2}\right]^{-1/\beta}$$

where, $A = 200$, $\delta_1 = 0.6$, $\beta_1 = 0.5$, $\delta_2 = 0.3$, $\beta_2 = -0.17$, $\beta = 0.6$. No errors of equation have been introduced. All figures in the table have been rounded off to three places after decimal.

Table-1: Simulated Data of Sato's 2-Level CES Production Function Y=f(X_1, X_2, X_3, X_4)											
Sl	Y	X_1	X_2	X_3	X_4	Sl	Y	X_1	X_2	X_3	X_4
1	2568.875	26.276	91.892	14.870	62.191	26	4185.272	66.557	76.261	58.552	64.967
2	1769.669	39.317	76.181	3.234	33.776	27	5215.146	62.830	92.276	88.190	98.656
3	3032.155	73.555	37.344	62.064	36.294	28	745.067	98.222	1.433	24.597	34.681
4	3166.279	59.184	28.457	38.090	72.427	29	2022.232	86.719	8.184	68.628	38.115
5	2158.898	91.753	98.675	66.274	9.566	30	1295.326	11.033	23.167	39.700	29.700
6	3486.665	60.934	79.510	28.653	56.322	31	2274.001	20.897	58.079	60.404	39.712
7	3600.447	80.059	73.456	20.206	60.132	32	2186.084	94.634	10.394	10.659	70.407
8	2185.430	78.940	32.452	84.255	13.463	33	1947.648	26.514	25.420	13.892	54.569
9	1897.939	22.477	13.698	79.898	56.104	34	656.971	2.388	23.208	17.753	97.646
10	224.915	17.541	0.306	43.969	61.059	35	3103.832	26.102	77.795	31.857	90.985
11	3793.914	70.638	74.296	10.032	91.525	36	3908.014	59.194	36.302	85.062	83.656
12	1930.118	58.862	29.444	46.911	16.469	37	438.510	36.378	65.362	64.772	0.374
13	2720.323	76.614	11.462	79.242	72.301	38	1370.139	19.296	12.406	0.778	93.011
14	2576.268	20.622	73.028	52.563	58.133	39	2040.781	51.064	69.283	56.126	13.204
15	2350.199	75.726	31.162	5.433	52.141	40	3913.082	74.094	25.726	87.516	93.556
16	1783.949	63.951	14.440	11.079	36.513	41	147.777	73.695	0.165	37.737	24.932
17	1773.103	19.362	87.558	51.086	18.495	42	1057.765	9.961	35.379	22.626	17.345
18	2919.148	51.198	40.730	83.600	35.030	43	178.722	0.434	46.496	75.067	31.943
19	2565.703	57.786	31.442	68.578	28.491	44	1274.565	16.053	8.832	8.511	71.580
20	3362.018	53.743	66.042	12.563	81.003	45	2382.314	52.883	73.080	62.534	17.488
21	2586.172	72.951	8.674	95.490	94.870	46	4558.196	42.121	95.337	92.929	99.543
22	585.069	9.816	5.060	42.021	6.583	47	4536.696	60.895	69.479	68.821	87.620
23	1745.456	61.312	8.621	99.919	21.929	48	1420.395	6.325	49.374	39.318	91.542
24	3495.859	32.301	96.443	87.947	59.694	49	3645.612	95.388	48.448	32.194	57.132
25	4693.787	66.639	62.591	75.796	91.453	50	1352.639	53.045	4.084	47.099	45.842

We estimated the parameters of Sato's 2-level production function (given above) by a number of methods. The loss function, $(\sum_{i=1}^{n=50}(Y_i - \hat{Y}_i)^2)$, was minimized by five alternative methods, namely, (i) Hooke-Jeeves Pattern Moves (HJPM), (ii) Hooke-Jeeves-Quasi-Newton (HJQN), (iii) Rosenbrock-Quasi-Newton (RQN), (iv) Differential Evolution (DE), and (v) Repulsive Particle Swarm (RPS) methods of optimization. Of the five, the last two methods are population-based stochastic methods. Population-based stochastic methods are often successful at optimizing extremely nonlinear (often multi-modal) objective functions (Mishra, 2006-a and b). The parameters of Sato's 2-level production function so estimated by the said five methods are presented in Table-2.

Table-2: Estimated Parameters of Sato's 2-Level CES Production Function							
Method	A	δ_1	β_1	δ_2	β_2	β	Loss
HJPM	199.8385	0.599983	0.499798	0.300019	-0.170107	0.600422	0.2647
HJQN	199.8385	0.599981	0.499787	0.300020	-0.170106	0.600421	0.2653
RQN	200.0087	0.599993	0.499964	0.300001	-0.170012	0.599976	0.0928
DE	200.0000	0.600000	0.500000	0.300000	-0.170000	0.600000	0.4e-11
RPS	200.2410	0.600061	0.500979	0.299943	-0.169380	0.599396	1.0407
Optimization Methods: HJPM=Hooke-Jeeves Paatern Moves; HJQN=Hooke-Jeeves Quasi-Newton; RQN=Rosenbrock Quasi-Newton; DE=Differential Evolution; RPS=Repulsive Particle Swarm.							

IV. Introduction of Errors of Equation at the Second Level: For experiment we generated Y from the model specified in the earlier section, using (X_1, X_2, X_3, X_4) presented in Table-3. The parameters are:

$$A = 200,\ \delta_1 = 0.6,\ \beta_1 = 0.5,\ \delta_2 = 0.3,\ \beta_2 = -0.17,\ \beta = 0.6 .$$

We added normally distributed errors N(0,2) to Y (output at the highest level). All data pertaining to this experiment have been presented in table-3.

Table-3: Simulated Data of Sato's 2-Level CES Production Function Y=f(X_1, X_2, X_3, X_4)

Sl	Y	X_1	X_2	X_3	X_4	Sl	Y	X_1	X_2	X_3	X_4
1	1903.423	40.640	26.276	91.892	14.870	26	4470.175	91.453	66.557	76.261	58.552
2	1183.128	62.191	39.317	76.181	3.234	27	4726.950	64.967	62.830	92.276	88.190
3	3066.605	33.776	73.555	37.344	62.064	28	1585.638	98.656	98.222	1.433	24.597
4	2448.175	36.294	59.184	28.457	38.090	29	2713.325	34.681	86.719	8.184	68.628
5	4855.468	72.427	91.753	98.675	66.274	30	1658.498	38.115	11.033	23.167	39.700
6	1527.713	9.566	60.934	79.510	28.653	31	2333.070	29.700	20.897	58.079	60.404
7	2689.390	56.322	80.059	73.456	20.206	32	1242.910	39.712	94.634	10.394	10.659
8	4129.596	60.132	78.940	32.452	84.255	33	1613.757	70.407	26.514	25.420	13.892
9	1633.598	13.463	22.477	13.698	79.898	34	774.991	54.569	2.388	23.208	17.753
10	1309.395	56.104	17.541	0.306	43.969	35	2935.125	97.646	26.102	77.795	31.857
11	2025.191	61.059	70.638	74.296	10.032	36	4473.937	90.985	59.194	36.302	85.062
12	3415.342	91.525	58.862	29.444	46.911	37	3845.913	83.656	36.378	65.362	64.772
13	2185.764	16.469	76.614	11.462	79.242	38	79.187	0.374	19.296	12.406	0.778
14	2996.389	72.301	20.622	73.028	52.563	39	4110.150	93.011	51.064	69.283	56.126
15	1222.381	58.133	75.726	31.162	5.433	40	2173.410	13.204	74.094	25.726	87.516
16	1379.316	52.141	63.951	14.440	11.079	41	1497.740	93.556	73.695	0.165	37.737
17	2456.831	36.513	19.362	87.558	51.086	42	1281.619	24.932	9.961	35.379	22.626
18	2485.012	18.495	51.198	40.730	83.600	43	293.653	17.345	0.434	46.496	75.067
19	3016.364	35.030	57.786	31.442	68.578	44	827.824	31.943	16.053	8.832	8.511
20	1724.186	28.491	53.743	66.042	12.563	45	4048.398	71.580	52.883	73.080	62.534
21	3921.063	81.003	72.951	8.674	95.490	46	2591.249	17.488	42.121	95.337	92.929
22	1628.408	94.870	9.816	5.060	42.021	47	4690.477	99.543	60.895	69.479	68.821
23	1378.774	6.583	61.312	8.621	99.919	48	1757.240	87.620	6.325	49.374	39.318
24	2684.667	21.929	32.301	96.443	87.947	49	3442.878	91.542	95.388	48.448	32.194
25	4203.961	59.694	66.639	62.591	75.796	50	2240.162	57.132	53.045	4.084	47.099

Once again, we estimated the model by the said five methods of estimation. We observe that all the five methods perform more or less equally well. The estimated parameters have been presented in Table-4.

Table-4: Estimated Parameters of Sato's 2-Level CES Production Function

Method	A	δ_1	β_1	δ_2	β_2	β	Loss
HJPM	200.6409	0.599964	0.501694	0.299676	-0.169270	0.598371	174.334
HJQN	200.7711	0.599987	0.501787	0.299662	-0.169173	0.598033	173.751
RQN	200.8801	0.600002	0.501856	0.299656	-0.169109	0.597749	173.751
DE	200.8802	0.600002	0.501856	0.299656	-0.169109	0.597749	173.751
RPS	200.9432	0.599936	0.501866	0.299690	-0.169138	0.597576	173.905

Optimization Methods: HJPM=Hooke-Jeeves Paatern Moves; HJQN=Hooke-Jeeves Quasi-Newton; RQN=Rosenbrock Quasi-Newton; DE=Differential Evolution; RPS=Repulsive Particle Swarm.

V. Introduction of Errors as well as Outliers: Once again we generated Y from the model specified before, using $(X_1,\ X_2,\ X_3,\ X_4)$ presented in Table-5. The parameters are: $A = 200,\ \delta_1 = 0.6,\ \beta_1 = 0.5,\ \delta_2 = 0.3,\ \beta_2 = -0.17,\ \beta = 0.6$. We added normally distributed errors N(0,2) to Y (output at the highest level). Additionally we generated five quantities within the rage (0, 500) randomly and added to randomly chosen observations on Y. All data pertaining to this experiment have been presented in table-5. It is well known that such perturbations amount to insertion of outliers in the dependent variable and cause a shift in the mean error. Further, such contamination affects the applicability and performance of the least squares method of estimation adversely.

Table-5: Simulated Data of Sato's 2-Level CES Production Function Y=f(X_1, X_2, X_3, X_4)

Sl	Y	X_1	X_2	X_3	X_4	Sl	Y	X_1	X_2	X_3	X_4
1	3126.284	29.387	65.769	30.135	88.887	26	1355.925	18.340	66.759	35.498	12.155
2	1916.264	62.111	72.682	77.093	8.425	27	1331.975	53.449	11.297	86.738	18.756
3	4250.869	56.711	64.441	76.247	77.518	28	2639.034	31.893	22.555	48.296	86.780
4	4594.667	78.884	75.638	43.873	82.494	29	779.031	86.015	35.068	36.275	2.034
5	479.087	0.111	58.215	48.293	65.823	30	3204.431	85.731	96.080	4.906	64.716
6	2530.694	60.301	69.401	73.697	17.572	31	3441.778	44.140	82.392	97.094	41.035
7	4296.466	42.160	94.811	89.427	83.269	32	3328.444	72.362	64.863	37.914	43.719
8	538.657	94.770	19.194	62.236	0.670	33	1798.501	21.090	33.067	8.593	53.949
9	3591.214	43.897	57.347	49.009	77.931	34	4182.461	46.359	92.613	38.449	97.179
10	1538.131	26.509	57.673	7.457	25.688	35	3371.976	37.037	47.606	52.306	85.382
11	4431.791	87.017	90.906	62.283	55.547	36	2296.028	41.537	80.784	10.873	38.184
12	3553.142	72.734	36.486	64.308	57.471	37	2933.948	31.242	43.795	81.596	95.419
13	4496.723	66.055	79.095	80.072	68.583	38	2877.132	38.151	70.138	77.470	33.580
14	3105.294	90.849	27.841	49.406	45.872	39	842.871	4.247	23.259	1.337	98.684
15	2927.776	30.577	70.609	48.464	55.300	40	4255.249	80.077	92.304	33.129	68.038
16	5620.657	95.627	76.062	95.729	89.821	41	1786.216	10.060	48.025	97.605	53.409
17	2778.937	77.364	55.795	38.494	28.808	42	3356.732	42.004	71.345	50.035	58.110
18	2909.113	26.404	99.156	57.296	51.378	43	99.680	98.346	67.079	17.367	0.486
19	2176.618	92.603	93.211	25.838	16.135	44	1957.756	46.615	75.314	32.355	16.300
20	2387.999	64.265	40.374	63.862	19.804	45	1527.971	6.607	92.937	98.162	52.540
21	2844.089	44.071	86.190	20.500	47.288	46	2088.333	31.781	17.826	20.926	65.119
22	2649.409	99.227	69.774	25.597	25.617	47	608.940	2.378	28.199	47.796	32.986
23	1799.172	23.327	9.406	46.494	94.314	48	547.702	67.748	9.612	47.942	1.149
24	1346.823	47.435	35.783	87.784	4.659	49	3957.006	75.409	42.118	74.026	65.091
25	3553.580	37.893	85.430	71.538	60.364	50	1343.291	24.318	60.084	41.644	9.109

Table-6: Estimated Parameters of Sato's 2-Level CES Production Function

Method	A	δ_1	β_1	δ_2	β_2	β	Loss
HJPM	184.4451	0.581338	0.588389	0.295359	-0.170136	0.643227	533239
HJQN	184.9474	0.581341	0.589104	0.295334	-0.169657	0.641591	533236
RQN	184.9301	0.581341	0.589081	0.295335	-0.169673	0.641647	533236
DE	184.9302	0.581341	0.589081	0.295335	-0.169673	0.641646	533236
RPS	184.9264	0.581395	0.589352	0.295306	-0.169454	0.641656	533236

Optimization Methods: HJPM=Hooke-Jeeves Paatern Moves; HJQN=Hooke-Jeeves Quasi-Newton; RQN=Rosenbrock Quasi-Newton; DE=Differential Evolution; RPS=Repulsive Particle Swarm.

We estimated the model by minimizing the least squares loss function, $\sum_{i=1}^{n=50}(Y_i-\hat{Y}_i)^2$. The results so obtained are presented in Table-6. The effects of presence of outliers in the data are clearly observable on the estimated values of A, δ_1, β_1 and β, being away from their true values. However, all the five methods are comparable at minimizing the loss.

Instead of minimizing the loss defined as $\sum_{i=1}^{n}(Y_i-\hat{Y}_i)^2$, we may minimize the sum of absolute deviations of the expected Y from the observed Y, that is, $\sum_{i=1}^{n}\left|Y_i-\hat{Y}_i\right|$. This is often referred as the Least Absolute Deviation (LAD) estimation. A large number of studies have indicated that the performance of LAD estimator is better than the least squares estimator in presence of outliers in the data (Dasgupta and Mishra, 2004).

There are two well-known algorithms to carry out estimation by minimization of the least absolute deviations of the expected Y from the observed Y. They are: (i) the method of Linear Programming (Charnes et al., 1955, Taylor, 1974), and (ii) the Fair-Schlossmacher algorithm (Fair, 1974; Schlossmacher, 1973). Both of them assume a linear model. However, the Sato's model is extremely nonlinear and therefore, these methods are not applicable to its estimation.

We have used the five methods of optimization (listed before) to carry out the LAD estimation. The results are presented in Table-7. We observe that the Hooke-Jeeves method (hybridized with pattern move as well as Quasi-Newton) does not perform well. The estimated parameters are far away from the true ones. The Rosenbrock-Quasi-Newton method of optimization works quite well. On the other hand, the Differential Evolution and the Repulsive Particle Swarm methods work extremely well and the parameters estimated by them are very close the true ones.

Table-7: Estimated Parameters of Sato's 2-Level CES Production Function

Method	A	δ_1	β_1	δ_2	β_2	β	Loss
HJPM	133.1354	0.599898	0.336873	0.296548	-0.245451	0.925873	2776.879
HJQN	133.1354	0.599898	0.336873	0.296548	-0.245451	0.925873	2776.876
RQN	188.1826	0.599007	0.488200	0.299934	-0.178878	0.633467	1614.677
DE	199.8043	0.599550	0.503443	0.299929	-0.168905	0.600506	1513.135
RPS	199.8776	0.599548	0.504552	0.299900	-0.168845	0.600338	1513.520

Optimization Methods: HJPM=Hooke-Jeeves Paatern Moves; HJQN=Hooke-Jeeves Quasi-Newton; RQN=Rosenbrock Quasi-Newton; DE=Differential Evolution; RPS=Repulsive Particle Swarm.

V. Estimation of Service Production Function–An Exercise on Real Life Data: Lindenberger (2003) defines the output (Q) of German sector "Market-Determined Services" (for the years 1960-1989) in terms of three factors; capital (K), labour (L) and energy (E). He derives energy-dependent relations by specifying technological boundary conditions for the elasticities of production, and then obtains production functions by integration. His functions belong to the LINEX family [LINear EXponentials functions, derived by Kümmel (1982) and Kümmel et al. (1985)]. One of the (Lindenberger's) service production functions is defined as:

$Q = a_1 L \exp\left[a_2\{3 - 2(L/K) - (LE/K^2)\} + a_2 a_3^2 \{1 - (L/E)\}\right]$, such that the elasticities satisfy the restrictions : $\alpha = 2a_2(L/K)\{(E/K)+1\} \geq 0;\ \gamma = a_2\{a_3^2(L/E) - (LE/K^2)\} \geq 0;\ \beta = 1-\alpha-\gamma \geq 0$.

For the data given in Table-8, Lindenberger's production function (as specified above) has been estimated for two sub-periods separately (since Lindenberger observes a structural break in 1977-78). The estimation has been done by two methods of optimization, DE and RPS, and by each method Least Squares (LS) and Least Absolute Deviation (LAD) estimates of the parameters ($a_1,\ a_2$ *and* a_3), satisfying restrictions on the elasticity measure (for each year) have been obtained. Results are presented in Table-9. Figure-1 indicates that the structural break might have occurred sometime in 1975 or so.

Table-8: Output, Capital. Labour and Energy (Indices: Base=1960) Pertaining to German Sector "Market-Determined Services" - 1960-1989

Year	Output	Capital	Labour	Energy	Year	Output	Capital	Labour	Energy
1960	1000	1000	1000	1000	1975	1756	2795	843	2118
1961	1058	1082	1001	1061	1976	1840	2908	857	2279
1962	1108	1171	999	1279	1977	1930	3041	843	2244
1963	1149	1265	985	1505	1978	2015	3195	848	2400
1964	1250	1364	1004	1475	1979	2104	3373	853	2517
1965	1320	1478	992	1530	1980	2144	3575	865	2270
1966	1366	1599	988	1566	1981	2138	3778	857	2140
1967	1369	1720	953	1555	1982	2125	3963	856	1994
1968	1414	1824	940	1682	1983	2180	4127	843	2027
1969	1509	1934	926	1930	1984	2250	4308	846	2133
1970	1574	2057	921	1973	1985	2282	4486	834	2248
1971	1655	2195	932	2063	1986	2376	4659	838	2379
1972	1758	2342	925	2250	1987	2465	4837	840	2318
1973	1811	2505	912	2344	1988	2595	5026	861	2273
1974	1781	2675	882	2153	1989	2748	5256	878	2170

Source: Lindenberger, D. http://www.ewi.uni-koeln.de/ewi/content/e266/e283/e281/Ewiwp0302_ger.pdf , 2003. p.20.

Fig.-1: Graphical Presentation of Estimated Output of Lindenberger Model

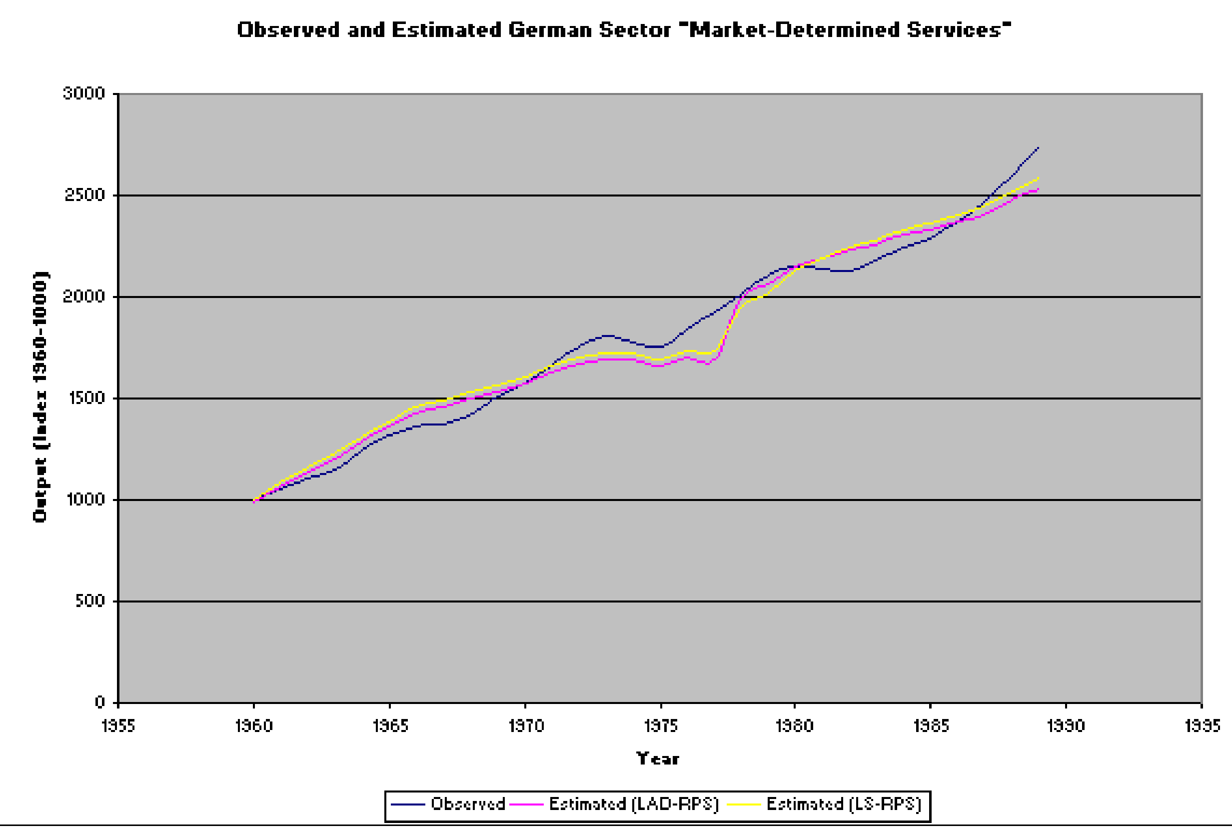

Table-9.1: Estimated Parameters of Lindenberger Production Function For Period-1 (1960-1977)

Least Absolute Deviation Estimation				Ordinary Least Squares Estimation				Coefficient & Elasticity	
Differential Evaluation		R Particle Swarm		Differential Evaluation		R Particle Swarm			
Coefficient	Elasticity	Coefficient	Elasticity	Coefficient	Elasticity	Coefficient	Elasticity		
0.98444153	0.097127	0.984526755	0.097127	1.00398064	0.088150	1.00447289	0.088150	a_1	α
0.24377507	0.507215	0.243766404	0.507216	0.24377507	0.489116	0.24372814	0.489116	a_2	β
1.04982963	0.395657	1.049883080	0.395657	1.04982963	0.422734	1.05016838	0.422734	a_3	γ
R^2 (DE) =0.939167; R^2(RPS) =0.939164				R^2 (DE) =0.939167; R^2(RPS) =0.939156					

Table-9.2: Estimated Parameters of Lindenberger Production Function For Period-2 (1978-1989)

Least Absolute Deviation Estimation				Ordinary Least Squares Estimation				Coefficient & Elasticity	
Differential Evaluation		R Particle Swarm		Differential Evaluation		R Particle Swarm			
Coefficient	Elasticity	Coefficient	Elasticity	Coefficient	Elasticity	Coefficient	Elasticity		
0.41632090	0.010616	0.41654885	0.010616	0.31333566	0.022155	0.31468959	0.022155	a_1	α
0.67647403	0.913503	0.67626526	0.913503	0.79809593	0.947287	0.79637622	0.947287	a_2	β
0.66626212	0.075880	0.66637632	0.075880	0.59236766	0.030558	0.59324793	0.030558	a_3	γ
R^2 (DE) =0.86724; R^2(RPS) =0.86727				R^2 (DE) =0.85059; R^2(RPS) =0.85079					

Table-10: Empirical and Estimated Output, Capital. Labour and Energy (Indices: Base=1960) Pertaining to German Sector "Market-Determined Services" - 1960-1989

Year	Output (Empirical)	Estimated Output (LAD)		Estimated Output (LS)	
		DE	RPS	DE	RPS
1960	1000.000	984.442	984.527	1003.981	1004.473
1961	1058.000	1061.452	1061.543	1082.519	1083.045
1962	1108.000	1139.257	1139.356	1161.868	1162.449
1963	1149.000	1206.711	1206.817	1230.662	1231.287
1964	1250.000	1287.267	1287.377	1312.816	1313.466
1965	1320.000	1357.279	1357.394	1384.218	1384.894
1966	1366.000	1424.714	1424.832	1452.991	1453.690
1967	1369.000	1461.218	1461.337	1490.220	1490.924
1968	1414.000	1499.578	1499.701	1529.342	1530.067
1969	1509.000	1535.126	1535.253	1565.595	1566.347
1970	1574.000	1574.000	1574.128	1605.241	1606.006
1971	1655.000	1629.414	1629.547	1661.755	1662.544
1972	1758.000	1666.690	1666.825	1699.770	1700.579
1973	1811.000	1694.042	1694.179	1727.666	1728.485
1974	1781.000	1687.491	1687.625	1720.985	1721.784
1975	1756.000	1655.044	1655.174	1687.894	1688.673
1976	1840.000	1697.206	1697.340	1730.892	1731.695
1977	1930.000	1694.822	1694.954	1728.460	1729.257
1978	2015.000	1990.675	1990.834	1948.893	1949.976
1979	2104.000	2060.285	2060.435	2025.229	2026.237
1980	2144.000	2144.000	2144.130	2123.978	2124.806
1981	2138.000	2192.444	2192.551	2187.508	2188.143
1982	2125.000	2232.817	2232.906	2240.367	2240.843
1983	2180.000	2255.816	2255.891	2271.604	2271.969
1984	2250.000	2306.220	2306.288	2326.788	2327.097
1985	2282.000	2330.152	2330.210	2356.872	2357.097
1986	2376.000	2376.000	2376.052	2406.254	2406.438
1987	2465.000	2408.461	2408.503	2445.877	2445.970
1988	2595.000	2473.966	2474.004	2516.432	2516.473
1989	2748.000	2532.341	2532.369	2582.498	2582.450

VI. Fitting of Sato's Two-Level Production Function to German Sector Merket-Determined Services: For various reasons, producers substitute a factor of production for others. However, substitutability of the one factor for the other is bound by technical considerations. Certain factors are complimentary to (rather than substitutes of) each other. In the present case of the German sector "Market-Determined Services", it may be interesting to investigate how the three factors of production (capital, labour and energy) combine with or substitute each other. For this purpose, we fit Sato's production function to the data given in Table-8. We have ignored any structural break pointed out before.

The crux of the problem is, however, to choose the schema of nesting. Nesting is basically an exercise in aggregation and therefore must satisfy the necessary conditions of aggregation so that the aggregate variable qualifies for being used to compute substitution elasticities (Leontief, 1947, Fisher, 1993; Felipe & Fisher, 2001). We observe that the coefficients of correlation r(K,L) = -0.87 , r(K,E) = 0.76 and r(L,E) = -0.88 in the data given in Table-8. The partial correlation coefficients are: $r_{KL.E}$ = -0.6565; $r_{KE.L}$ = -0.02440; $r_{LE.K}$ = -0.67812. Thus, capital and energy may be more suitably clubbed together.

However, we carry out nesting in three alternative ways: M[(K,E),L], in which K and E are aggregated in the manner of CES and makes a composite input. We will denote it by M[(1,3),2]. Similarly, M[(K,L),E] and M[(L,E),K] would be denoted by M[(1,2),3] and M[(2,3),1] respectively. Further, we will use two models: the one in which $\rho = 1$ and the other in which ρ is free to take on any non-negative value. Thus we have:

$$Q = A\left[\delta_2[\delta_1 X_a^{-\beta_1} + (1-\delta_1)X_b^{-\beta_1}]^{\frac{\beta_2}{\beta_1}} + (1-\delta_2)X_c^{-\beta_2}\right]^{-1/\beta_2} \text{; rho=1}$$

$$Q = A\left[\delta_2[\delta_1 X_a^{-\beta_1} + (1-\delta_1)X_b^{-\beta_1}]^{\frac{\beta_2}{\beta_1}} + (1-\delta_2)X_c^{-\beta_2}\right]^{-\rho/\beta_2} \text{; rho is free}$$

The symbolic X_a, X_b and X_c will be representing K, L or E as the schema of nesting suggests.

Table-11: Estimated Parameters of Sato Production Functions with Different Nesting For German Sector "Market-Determined Services" - 1960-1989 (RPS Method)

Model	A	δ_1	β_1	δ_2	β_2	ρ	R^2
M[(1,3), 2]	1.00001036	0.568035245	-0.507113532	0.610262062	-0.82042005	-	0.99065
M[(1,2), 3]	0.9998600	0.47067414	-0.75217065	0.72896991	-0.9983358	-	0.99160
M[(2,3), 1]	1.0001654	0.62247131	-0.99580462	0.63284852	-0.5961938	-	0.99074
M[(1,3), 2]	0.0071629	0.65312869	0.201809927	0.39424369	-0.4885967	1.713434	0.99506
M[(1,2), 3]	0.0113824	0.31307504	-0.38522739	0.86763721	-0.9973683	1.646839	0.99566
M[(2,3), 1]	0.0057898	0.86192909	-0.99083213	0.71398938	-0.1976769	1.744525	0.99582
Note: The Parameters δ *and* β may not be comparable across rows as they relate to different variables							

In the manner explained above, we have estimated the parameter of the Two-level Sato functions for Market-Determined Services. All estimations have been done by the LAD procedure and the sum of absolute deviations has been minimized by the RPS and the DE methods. The parameters so estimated are presented in Table-11 (RPS method) and Table-13 (DE method). The estimated output values for different models have been presented in Table-12 and Table-14. Graphical presentations [Fig.-2, Fig.-3 (RPS) and Fig.-4, Fig.-5 (DE)] also have been made. The DE performs slightly better than the RPS.

VII. Conclusion: In real life we do not know as to the nature and magnitude of contamination of data originating from our survey (or collected from secondary sources) and whether outliers are present in the data or not. Nor can we have a clear idea on a correct nesting schema of different factors of production. Our experiments suggest that in any case the Least Absolute Deviation estimation based on population-based global optimization methods such as the Differential Evolution (Storn and Price, 1995) or the Particle Swarm (Eberhart and Kennedy, 1995) may work better than the more popularly used methods of nonlinear regression. Therefore, estimation of two-level CES production function may preferably be carried out by minimization of the sum of absolute deviations and such minimization should be attempted by the methods of global optimization.

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The FORTRAN Codes of the Program based on Differential Evolution and the Repulsive Particle Swarm methods to estimate the Sato's Production function may be downloaded from http://www1.webng.com/economics/satoprog.txt

Table-12: Estimated Output of Sato Production Functions with Different Nesting For German Sector "Market-Determined Services" - 1960-1989 [Estimated by R Particle Swarm]

Year	Empirical	M[(1,3), 2]	M[(1,2), 3]	M[(2,3), 1]	M[(1,3), 2]	M[(1,2), 3]	M[(2,3), 1]
1960	1000.000	1000.010	999.860	1000.165	989.444	992.546	991.374
1961	1058.000	1044.785	1044.765	1045.094	1040.363	1043.029	1043.248
1962	1108.000	1131.181	1133.124	1129.031	1126.527	1128.748	1123.689
1963	1149.000	1214.982	1220.240	1211.901	1200.120	1206.535	1195.342
1964	1250.000	1249.718	1252.507	1247.627	1261.993	1263.460	1260.306
1965	1320.000	1297.818	1300.020	1296.291	1310.126	1311.184	1309.763
1966	1366.000	1346.246	1347.503	1345.362	1365.930	1365.563	1367.360
1967	1369.000	1369.029	1369.004	1369.028	1367.936	1368.619	1370.398
1968	1414.000	1430.892	1431.131	1430.544	1423.436	1424.452	1424.624
1969	1509.000	1524.821	1527.211	1524.364	1506.645	1511.459	1507.885
1970	1574.000	1574.226	1575.395	1574.004	1560.429	1563.271	1562.142
1971	1655.000	1647.154	1647.672	1647.116	1655.387	1655.406	1658.529
1972	1758.000	1739.773	1740.971	1740.378	1749.988	1751.671	1755.952
1973	1811.000	1811.010	1811.019	1811.998	1819.076	1819.596	1826.125
1974	1781.000	1802.297	1797.920	1802.475	1788.850	1783.580	1790.070
1975	1756.000	1812.960	1807.172	1812.984	1758.701	1754.947	1757.169
1976	1840.000	1897.467	1891.426	1897.675	1867.766	1861.265	1868.386
1977	1930.000	1922.835	1915.594	1922.707	1882.671	1874.891	1881.219
1978	2015.000	2014.559	2006.476	2014.593	1991.581	1980.704	1991.914
1979	2104.000	2102.999	2093.846	2103.070	2099.948	2085.202	2101.618
1980	2144.000	2101.322	2092.930	2100.486	2121.600	2105.534	2119.838
1981	2138.000	2119.968	2114.246	2119.363	2132.014	2122.605	2131.382
1982	2125.000	2129.228	2129.008	2129.914	2136.613	2140.506	2142.381
1983	2180.000	2179.929	2180.021	2179.987	2175.303	2180.020	2178.757
1984	2250.000	2265.007	2263.512	2264.048	2279.094	2277.445	2279.520
1985	2282.000	2344.825	2340.959	2342.542	2355.124	2346.962	2350.315
1986	2376.000	2435.379	2429.116	2432.192	2469.445	2452.644	2462.059
1987	2465.000	2468.075	2465.603	2465.019	2506.480	2497.063	2501.169
1988	2595.000	2516.903	2519.203	2514.600	2593.010	2591.066	2594.006
1989	2748.000	2555.749	2566.970	2555.255	2655.236	2672.763	2668.247

Table-13: Estimated Parameters of Sato Production Functions with Different Nesting For German Sector "Market-Determined Services" - 1960-1989 [Estimated by Differential Evolution]							
Model	A	δ_1	β_1	δ_2	β_2	ρ	R^2
M[(1,3), 2]	1.0000000	0.5698514	-0.4866956	0.6098272	-0.8235084	-	0.99060
M[(1,2), 3]	1.0000000	0.4701859	-0.7540159	0.7289937	-1.0000000	-	0.99161
M[(2,3), 1]	1.0000000	0.6230672	-1.0000000	0.6320454	-0.5917516	-	0.99072
M[(1,3), 2]	0.0022075	0.7212857	0.9360296	0.3901159	-0.3497840	1.8806986	0.99503
M[(1,2), 3]	0.0030934	0.3340066	-0.1669982	0.8911314	-1.0000000	1.8313481	0.99629
M[(2,3), 1]	0.0033309	0.8808258	-1.000000	0.6842986	-0.0296768	1.8217535	0.99619
Note: The Parameters δ *and* β may not be comparable across rows as they relate to different variables							

Table-14: Estimated Output of Sato Production Functions with Different Nesting For German Sector "Market-Determined Services" - 1960-1989 [Estimated by Differential Evolution]							
Year	Empirical	M[(1,3), 2]	M[(1,2), 3]	M[(2,3), 1]	M[(1,3), 2]	M[(1,2), 3]	M[(2,3), 1]
1960	1000.000	1000.000	1000.000	1000.000	968.265	964.899	972.352
1961	1058.000	1044.767	1044.883	1044.945	1023.365	1020.604	1027.818
1962	1108.000	1130.958	1133.227	1128.811	1108.000	1108.000	1108.000
1963	1149.000	1214.527	1220.333	1211.637	1177.361	1185.956	1178.909
1964	1250.000	1249.445	1252.577	1247.429	1251.237	1253.185	1250.646
1965	1320.000	1297.599	1300.067	1296.136	1303.877	1305.539	1303.931
1966	1366.000	1346.107	1347.527	1345.254	1366.000	1366.000	1366.000
1967	1369.000	1369.000	1369.000	1368.980	1369.000	1369.000	1371.505
1968	1414.000	1430.836	1431.120	1430.505	1423.627	1424.557	1425.467
1969	1509.000	1524.632	1527.208	1524.325	1501.033	1508.321	1505.575
1970	1574.000	1574.118	1575.380	1574.000	1558.615	1563.031	1561.983
1971	1655.000	1647.089	1647.651	1647.138	1660.183	1661.868	1661.569
1972	1758.000	1739.684	1740.955	1740.435	1753.458	1758.000	1758.000
1973	1811.000	1811.000	1811.000	1812.100	1822.986	1825.224	1828.005
1974	1781.000	1802.536	1797.870	1802.609	1795.216	1786.400	1793.562
1975	1756.000	1813.281	1807.119	1813.139	1756.000	1746.479	1756.000
1976	1840.000	1897.798	1891.382	1897.856	1869.023	1856.794	1867.900
1977	1930.000	1923.234	1915.545	1922.894	1882.108	1867.218	1879.671
1978	2015.000	2015.000	2006.437	2014.815	1992.802	1973.958	1989.586
1979	2104.000	2103.497	2093.815	2103.320	2104.000	2080.286	2099.186
1980	2144.000	2101.791	2092.879	2100.660	2133.804	2111.730	2124.435
1981	2138.000	2120.319	2114.200	2119.478	2138.000	2127.949	2135.963
1982	2125.000	2129.330	2128.972	2129.949	2134.079	2147.973	2148.333
1983	2180.000	2180.000	2180.000	2180.000	2164.923	2180.000	2180.000
1984	2250.000	2265.136	2263.503	2264.066	2270.295	2276.746	2278.873
1985	2282.000	2345.044	2340.967	2342.573	2341.242	2336.648	2343.211
1986	2376.000	2435.699	2429.138	2432.246	2458.306	2441.327	2452.704
1987	2465.000	2468.203	2465.641	2465.000	2488.623	2486.283	2491.272
1988	2595.000	2516.803	2519.251	2514.508	2576.768	2592.489	2589.127
1989	2748.000	2555.221	2567.040	2555.042	2630.481	2684.674	2667.436

Fig.-2: Estimated Output of Sato Production Functions with Different Nesting For German Sector "Market-Determined Services" - 1960-1989 [Rho=1] [R Particle Swarm]

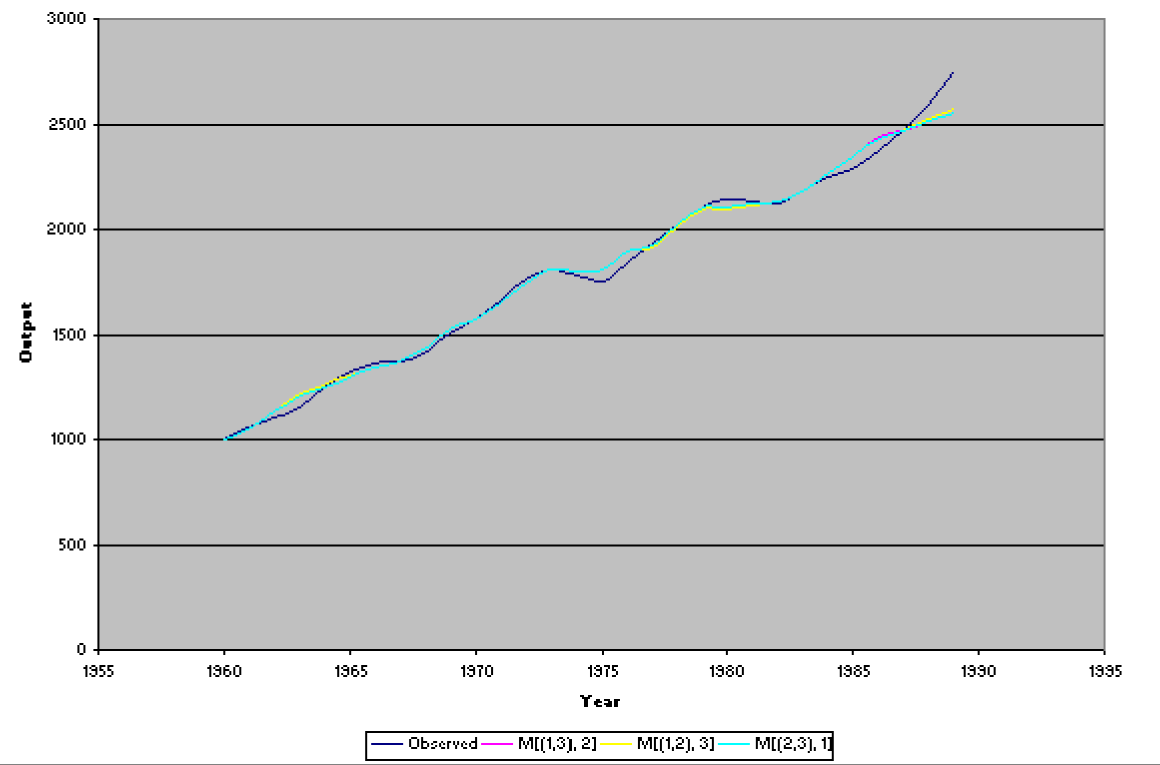


Fig.-3: Estimated Output of Sato Production Functions with Different Nesting For German Sector "Market-Determined Services" - 1960-1989 [Rho=Free] [R Particle Swarm]

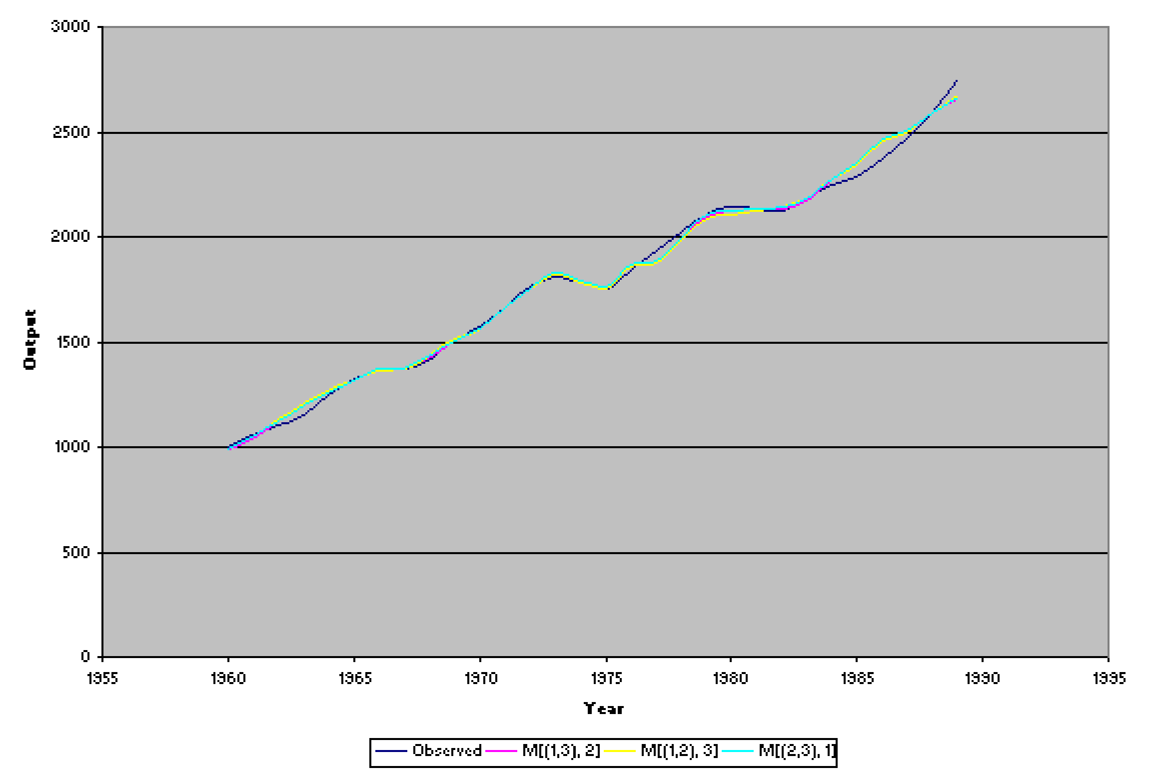


Fig.-4: Estimated Output of Sato Production Functions with Different Nesting For German Sector "Market-Determined Services" - 1960-1989 [Rho=1] [Differential Evolution]

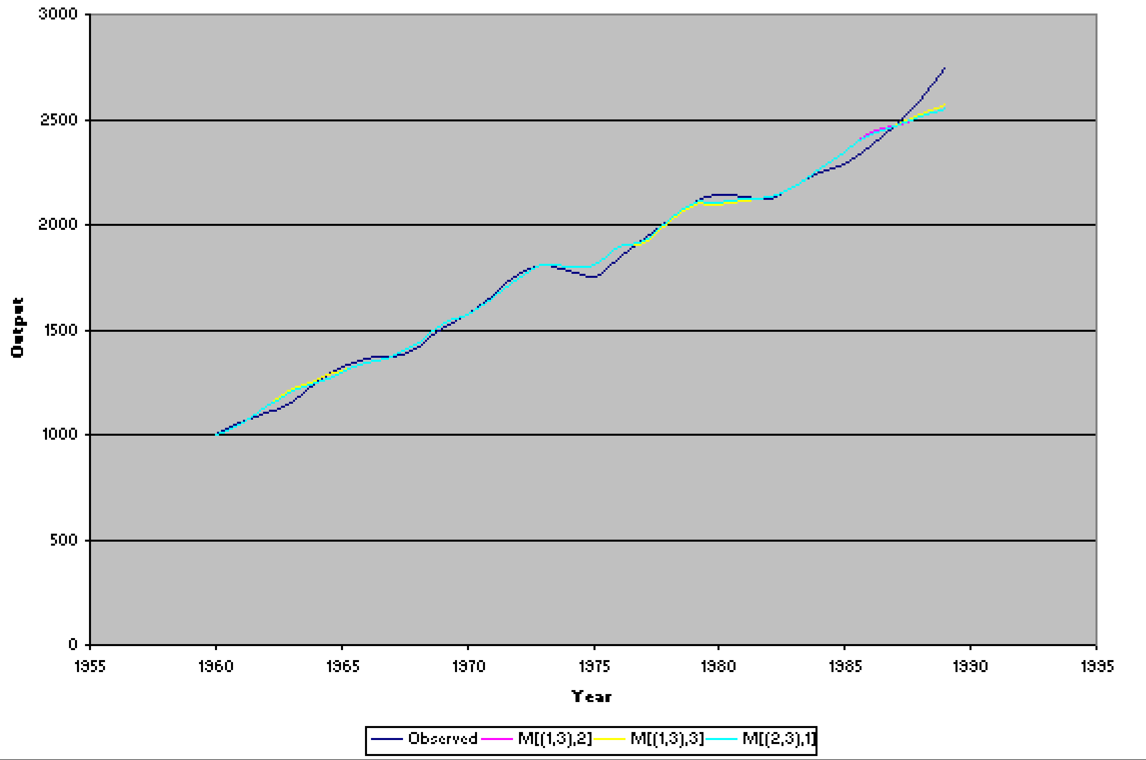
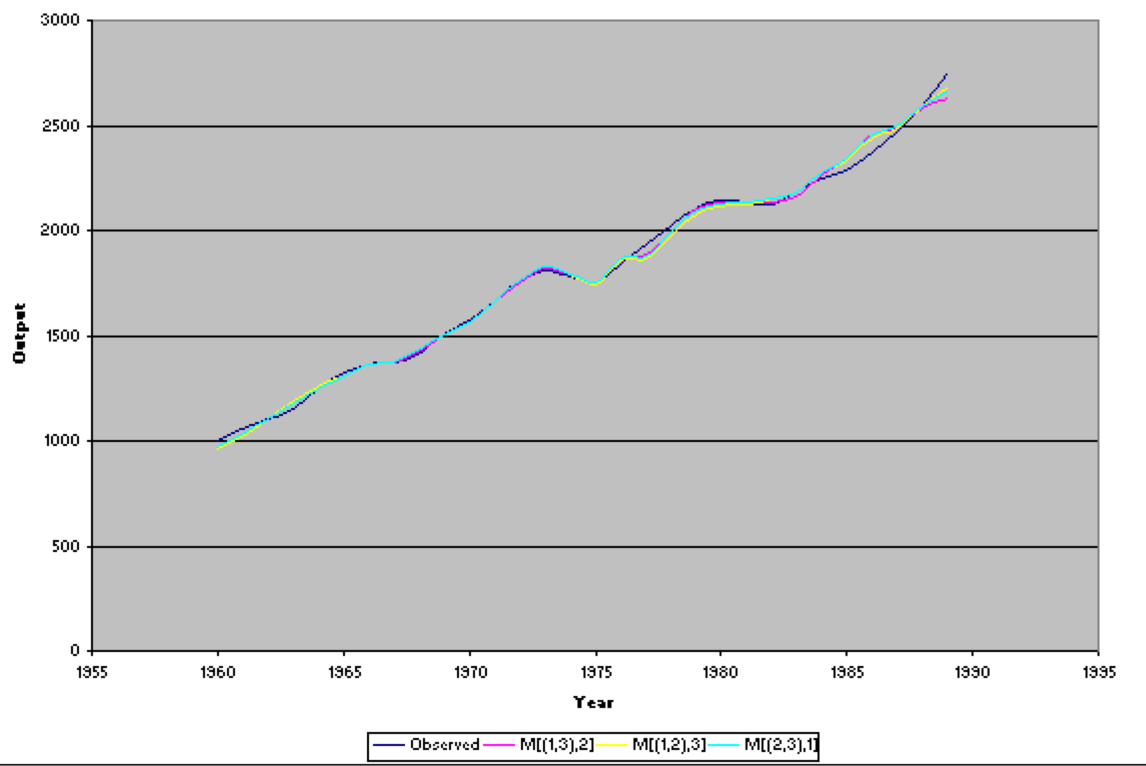


Fig.5: Estimated Output of Sato Production Functions with Different Nesting For German Sector "Market-Determined Services" - 1960-1989 [Rho=Free] [Differential Evolution]



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