Projective Modules Over Certain Class of Rings

AND

Serre's Conjecture

BY

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I certify that the dissertation entitled "Projective modules over certain class of rings and Serre's conjecture" submitted by Miss Sipra Paul Choudhury in fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate. I certify that sources from which ideas have been borrowed are duly referred to.

This dissertation may be placed before the examiners for evaluation and necessary formalities.

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CONTENTS

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CHAPTER 0

Preliminary definitions

The purpose of this chapter is to explain briefly, some of the terms which are used in this dissertation. These terms will be used in the following sense, unless otherwise stated

1. **Ring** – A commutative ring with an identity element.
2. **Module** – A module means unitary module.
3. **Ring homomorphism** – A map respecting additive and multiplicative structures and carrying the multiplicative identity to multiplicative identity.
4. **Category** – A category C consists of

(a) A class of objects

(b) For every ordered pair of objects X and Y, a set $\mathrm{Hom}(X,Y)$ of "morphisms" with domain X and range Y; if $f \in \mathrm{Hom}(X,Y)$, we write $f : X \mapsto Y$.

(c) For every ordered triple of objects X, Y and Z, a function associating to a pair of morphism $f : X \mapsto Y$ and $g : Y \mapsto Z$ their composite $gf = g \circ f : X \mapsto Z$

These satisfy the following two axioms:

(i) Associativity – If $f : X \mapsto Y$, $g : Y \mapsto Z$ and $h : Z \mapsto W$ then $h(gf) = (hg)f$ where $(hg)f : X \mapsto W$.

(ii) Identity – For every object Y there is a morphism $1_Y : Y \mapsto Y$ such that if $f : X \mapsto Y$, then $1_Y f = f$ and if $h : Y \mapsto Z$, then $h1_Y = h$.

5. **Functor** - Let C and D be categories. A covariant functor (respectively contravariant functor) T from C to D consists of an object function which assigns to every object X of C an object T(X) of D and a morphism function which assigns to every morphism $f : X \longmapsto Y$ of C a morphism $T(f) : T(X) \longmapsto T(Y)$ (respectively $T(f) : T(Y) \longmapsto T(X)$) of D such that

   (a) $T(1_X) = 1_{T(X)}$

   (b) $T(gf) = T(g)\ T(f)$ respectively $T(gf) = T(f)\ T(g)$

6. **Complex** - A complex is a sequence of modules and homomorphisms :

$$\cdots \longmapsto S_n \xmapsto{d_n} S_{n-1} \xmapsto{d_{n-1}} S_{n-2} \longmapsto \cdots$$

   (indexed by all integers 'n') such that $d_{n-1}\, d_n = 0$.

7. **Exact sequence** - A sequence of R-modules and R-homomorphisms

$$\cdots \longmapsto M_n \xmapsto{d_n} M_{n-1} \xmapsto{d_{n-1}} M_{n-2} \longmapsto \cdots$$

   is exact if Ker "d_{n-1}" = Image "d_n" for all "n" in Z

8. **Local ring** - A ring with unique maximal ideal is called local ring.

9. **Unique Factorization Domain** (Abbreviation U.F.D) - An integral domain R is called an unique Factorization Domain if

   (a) Every element $r \in R - \{o\}$ can be written in the form

$$r = ua_1a_2 \ldots\ldots a_n.$$

where u is a unit of R, $n \geqslant o$ and the "a_i" are irreducible elements of R. By an irreducible element we mean an element "a" in R such that "a" is a non-unit and whenever "a" is expressed as a product of two elements "b" and "c" in R, one of "b" or "c" is an unit.

(b) If $ua_1a_2 \cdots a_m = u'b_1 \cdots b_n$ where "u" and "u'" are units of R and "a_i" and "b_j" are irreducible elements, then $n = m$ and $a_i = u_i b_{\pi(i)}$ where "u_i" is a unit and π is a permutation on $1, 2, \ldots, n$

10. Zerodivisor of a module - Suppose M is R-module. An element $r \in R$ is a zero divisor of M if there exist $m \in M$, $m \neq 0$ such that $rm = 0$.

11. Torsion free module - Suppose M is an R-module. M is a torsion free module if "λ" $\in$ R is a nonzero divisor of R implies "λ" is a nonzero divisor of M.

12. Free module - An R-module F is free if it is isomorphic to a direct sum of copies of R. If $Ra_k \cong R$, and $F = \bigoplus_{k \in K} Ra_k$, then $\{a_k : k \in K\}$ is called a basis of F.

13. Localisation - Let R be a ring. A multiplicatively closed subset of R is a subset "S" of R such that $1 \in$ "S" and "S" is closed under multiplication. Define a relation "$\sim$" on R x S as follows:

$(a,s) \sim (b,t)$ if and only if $(at - bs)u = o$ for some "u" in S. This is an equivalence relation on R x S. Let $S^{-1}R$ denote the set of equivalence classes. An element of $S^{-1}R$ is denoted by $\frac{a}{s}$ which is the equivalence class of (a,s). We put ring structure on $S^{-1}R$ by defining addition and multiplication of these "Fractions" $\frac{a}{s}$ as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

Let "P" be a prime ideal of R. Then $S = R - P$ is multiplicatively closed. We write R_P for $S^{-1}R$ in this case. The process of passing from R to R_P is called "localisation" at "P".

CHAPTER I

Projective, flat and faithfully flat modules

§1. Projective modules

The purpose of this section is to give definition, various equivalent conditions and examples of projective modules.

Definition: A R-module P is projective if $M \longmapsto N \longmapsto 0$ is an exact sequence of R-modules and if given a R-homomorphism from $P \longmapsto N$, there exists a R-homomorphism from $P \longmapsto M$ such that the following diagram

$$\begin{tikzcd}
P \arrow[d, dashed] \arrow[dr] & & \\
M \arrow[r, mapsto] & N \arrow[r, mapsto] & 0
\end{tikzcd}$$

is commutative

Example: Every free R-module is projective.

Proof: Let F be a free module on the basis $(e_i)_{i \in I}$

Let $M \overset{\theta}{\longmapsto} N \longmapsto 0$ be an exact sequence of R-modules and $\Phi : F \longmapsto N$ be a R-homomorphism. Consider the diagram:

$$\begin{tikzcd}
F \arrow[d, dashed, "\Psi"'] \arrow[dr, "\Phi"] & & \\
M \arrow[r, mapsto, "\theta"] & N \arrow[r, mapsto] & 0
\end{tikzcd}$$

Let $\Phi(e_i) = n_i$ for all 'i' in I. Since θ is surjective, there exists $m_i \in M$ such that $\theta(m_i) = n_i$ for all 'i' in I.

Define $\psi : F \longmapsto M$ such that $\psi(e_i) = m_i$ and extend it by linearity.

Then $\theta \cdot \psi = \Phi$ and hence F is projective.

Example of a non-free projective module

Consider the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. A nonzero free R-module with finitely many elements has 6^n elements for some integer $n \geqslant 1$.

Consider $a = \mathbb{Z}_2 \times (o)$, $b = (o) \times \mathbb{Z}_3$

Then "a" and "b" are ideals of R. Also $a \oplus b = R$. So "a" is a projective R-module but "a" is not free over R.

Proposition (1.1.1). For any R-module P, the following conditions are equivalent.

(i) P is projective.

(ii) P is a direct summond of a free R-module.

(iii) The functor $M \longmapsto \operatorname{Hom}(P,M)$ is an exact functor from the category of R-module to the category of abelian groups. i.e. if $0 \longmapsto M_1 \longmapsto M_2 \longmapsto M_3 \longmapsto 0$ is an exact sequence of R-modules, then

$0 \longmapsto \operatorname{Hom}(P,M_1) \longmapsto \operatorname{Hom}(P,M_2) \longmapsto \operatorname{Hom}(P,M_3) \longmapsto 0$ is exact.

(iv) There exist $x_i \in P$, $f_i \in \operatorname{Hom}(P,R)$ such that for each $x \in P$, $f_i(x) = 0$ for almost all 'i' in I and $x = \sum_{i \in I} x_i f_i(x)$.

Proof:

(i) ⇒ (ii) There exist a free module and a surjective R-homomorphism $\theta : F \longmapsto P$.

Consider the following diagram

$$\begin{tikzcd}
P \arrow[dr, "1_P"] & & \\
F \arrow[r, mapsto] & P \arrow[r, mapsto] & 0.
\end{tikzcd}$$

By (i), ∃ a R-homomorphism $\longmapsto F$ such that

Define $Q_0 = \text{Ker}\,\theta$, $P_0 = \bar{\Phi}(P)$.

$\theta \circ \Phi = 1_P \Rightarrow \Phi$ is an injection. Then $F = P_0 \oplus Q_0$ and $P \oplus Q_0 \cong F$ and (ii) follows.

(ii) $\Rightarrow$ (i) Consider the following diagram:

$$\begin{tikzcd} P \arrow[dr, "\Phi"] & & \\ M \arrow[r, mapsto] & N \arrow[r, mapsto] & 0 \end{tikzcd}$$

where M, N are R-module

θ and Φ are R-homomorphisms. and is surjective.

By (ii), there exists a free R-module F and a R-module Q s.t $P \oplus Q = F$.

$\exists$ a R-homomorphism $\alpha : P \oplus Q \longmapsto P$ such that $\alpha\big((p,q)\big) = P$ for all $(p,q) \in P \oplus Q$ and a R-homomorphism $\beta : P \longmapsto P \oplus Q$. Such that $\beta(P) = (p,o)$ for all $p \in P$

So consider:

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, dashed, "\Psi"'] & P \arrow[d, "\Phi"] & \\ M \arrow[r, mapsto] & N \arrow[r, mapsto] & 0 \end{tikzcd}$$

Since F is free, it is projective and hence there exists $\Psi : F \longmapsto M$ such that $\theta . \Psi = \Phi . \alpha$.

$\Psi \circ \beta : P \longmapsto M$ is such that $\theta . (\Psi . \beta) = \Phi$. Therefore P is projective.

(i) $\Leftrightarrow$ (iii) The four term sequence

$0 \longmapsto \text{Hom}(P, M_1) \longmapsto \text{Hom}(P, M_2) \longmapsto \text{Hom}(P, M_3)$ is always exact for any R-module P.

<u>Claim</u>: $\text{Hom}(P, M_2) \longmapsto \text{Hom}(P, M_3) \longmapsto 0$ is exact iff P is projective.

$\tilde{\beta}$ is defined as $\tilde{\beta}(\theta) = \beta . \theta$.

P is projective iff given $g : P \longmapsto M_3$, $\exists$ $f : P \longmapsto M_2$

such that $\beta \circ f = g$, i.e. $\tilde{\beta}(f) = g$, which is equivalent to $\tilde{\beta}$ is surjective.

(ii) $\Rightarrow$ (iv) By (ii) $\exists$ a free R-module F and a R-module Q such that $P \oplus Q = F$. Let $\{e_i\}_{i \in I}$ be a basis of F over R. Any $f \in F$ is of the form $f = \sum_{i \in I} a_i e_i$ where $a_i = 0$ for all but finitely many 'i'.

Define $p_i : F \mapsto R$ such that $p_i(f) = a_i$ and $f_i = p_i|_P : P \mapsto R$. Write $e_i = x_i + y_i$ where $x_i \in P$ and $y_i \in Q$.

Let $x \in P \subseteq F$. So

$$\begin{aligned} x &= \sum b_i e_i , \quad b_i \in R \\ &= \sum b_i (x_i + y_i) \\ &= \sum b_i x_i + \sum b_i y_i . \end{aligned}$$

$\sum b_i x_i \in P$ and $\sum b_i y_i \in Q$. Hence $\sum b_i y_i = 0$. Therefore $x = \sum b_i x_i$ where $f_i(x) = b_i$. That is $x = \sum_{i \in I} x_i f_i(x)$ and all but finitely many $f_i(x)$ are zero.

(iv) $\Rightarrow$ (ii) Let F be the free R-module $\bigoplus_{i \in I} Re_i$

Define $\theta : F \mapsto P$ s.t. $\theta(e_i) = x_i$.

Let $x \in P$. Then by assumption

$$\begin{aligned} x &= \sum_{i \in I} x_i f_i(x) \\ &= \theta\Big(\sum_{i \in I} e_i f_i(x)\Big) . \end{aligned}$$

So θ is surjective.

Define $\psi : P \longmapsto F$ such that $\psi(x) = \sum_{i \in I} e_i f_i(x)$.

Let $K = \operatorname{Ker} \theta$. Also $\theta \circ \psi = 1_P$. Therefore $0 \mapsto K \mapsto F \mapsto P \mapsto 0$ is split exact and $P \oplus K \cong F$.

§2. Direct sum, tensor product and scalar extension of projective modules.

Proposition (1.2.1) Let $(P_i)_{i \in I}$ be a family of R-modules indexed by the nonempty set I. Then $\oplus P_i$ is a projective R-module, if and only if each P_i is a projective R-module.

Proof: $\oplus P_i$ is projective $\Leftrightarrow$ $\oplus P_i$ is a direct summand of a free module $\Leftrightarrow$ each P_i is a direct summand of some free module $\Leftrightarrow$ each P_i is projective.

—

Proposition (1.2.2) Tensor product of projective modules is projective.

Proof: Let P and P^1 be two projective modules over R.

There exists free modules F, F^1 and modules Q, Q^1 such that $F = P \oplus Q$ and $F^1 = P^1 \oplus Q^1$.

Now $F \otimes_R F^1 = (P \oplus Q) \otimes_R (P^1 \oplus Q^1)$

$$= (P \otimes_R P^1) \oplus (Q \otimes_R P^1) \oplus (P \otimes_R Q^1) \oplus (Q \otimes_R Q^1)$$

$F \otimes F^1$ is a free module. So $P \otimes_R P^1$ is R-projective.

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Proposition (1.2.3). Scalar extension of projective module is projective.

Proof: Let f be a ring homomorphism from R to S where R and S are rings. Then S can be considered as a R-module. Suppose P is a R-projective module.

Claim: $P \otimes_R S$ is S-projective.

Since P is R-projective, there exist free R-module and a R-module Q such that $P \oplus Q = F$.

$F \otimes_R S = (P \oplus Q) \otimes_R S = (P \otimes_R S) \oplus (Q \otimes_R S)$

F x S is free S-module, therefore $P \otimes_R S$ is S-projective.

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§3. Flat and faithfully-flat modules.

In this section we will discuss flat and faithfully-flat modules which leads to a simple consequence like :

$P \otimes_R S$ is S-projective $\Leftrightarrow$ P is R-projective where S is a faithfully flat R-algebra and P is an R-module.

Definition: A R-module M (not necessarily finitely generated) is called "flat" if whenever

$0 \mapsto N_1 \mapsto N_2 \mapsto N_3 \mapsto 0$ is an exact sequence of R-modules,

$0 \mapsto N_1 \otimes_R M \longmapsto N_2 \otimes_R M \longmapsto N_3 \otimes_R M \longmapsto 0$ is exact.

Examples: (i) Free modules are flat.

(ii) Projective modules are flat.

Example of flat modules which are not projective:

(i) Q over $\mathbb{Z}$ is flat but not projective.

Proof: Q considered as a $\mathbb{Z}$-module is flat, being the localization of $\mathbb{Z}$ at the multiplicatively closed set of nonzero integers. But Q is not projective as a $\mathbb{Z}$-module, for otherwise Q will be a direct summand of a free $\mathbb{Z}$-module, say $\bigoplus_I \mathbb{Z}$, where I is some nonempty indexing set. Let $f_i : \bigoplus_I \mathbb{Z} \longmapsto \mathbb{Z}$ denote projection on the ith summand for $i \in I$.

For atleast one $i \in I$, $f_i|Q$ is a nonzero homomorphism. Call this homomorphism f. Let $f(1) = t$. Let p/q be any rational number, where p, q are integers and $q \neq 0$. Then

$$\begin{aligned} q\, f(p/q) &= f(\underbrace{p/q + p/q + \dots + p/q}_{q \text{ times}}) \\ &= f(q \tfrac{p}{q}) \\ &= f(p) \\ &= p\, f(1) \\ &= p\, t\,. \end{aligned}$$

Hence $f(p/q) = \frac{p}{q} t$. That is $tQ \subset Z$; but this is possible only when $t = 0$, in which case f is the zero map, a contradiction. Hence Q is not Z-projective.

(ii) $\mathbb{Z}/2\mathbb{Z}$ is not a $\mathbb{Z}$-flat module.

Proof: Consider the sequence

$0 \hookrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ where $f(x) = 2x \ \forall\ x \in \mathbb{Z}$ and g is the cannonical map, is exact. Tensoring with $\mathbb{Z}/2\mathbb{Z}$ we get $0 \hookrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$. But $Z \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$. The above sequence becomes $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\hat{f}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$, where $\hat{f}$ is multiplication by 2, i.e. $\hat{f}$ is the zero map. So the sequence is not exact. Hence $\mathbb{Z}/2\mathbb{Z}$ is not a flat Z-module.

Remark (1.3.1) Flat modules are torsion free.

Proof: Let M be a flat R-module. Suppose $\lambda \neq 0$, $\lambda \in R$ is a nonzero divisor of R. Then $0 \hookrightarrow R \xrightarrow{f} R$ where $f(x) = \lambda x$ for all $x \in R$ is an exact sequence.

Since M is flat R-module $0 \mapsto R \otimes_R M \mapsto R \otimes_R M$ is exact i.e. $0 \mapsto M \xmapsto{\hat{f}} M$ is exact where $\hat{f}(m) = \lambda m$ for all m in M. So M is torsion free R-module.

Definition: A module M over R is called faithfully flat R-module if $0 \mapsto N_1 \mapsto N_2 \mapsto N_3 \mapsto 0$ is a sequence of R-modules, then it is exact if and only if $0 \mapsto N_1 \otimes_R M \mapsto N_2 \otimes_R M \mapsto N_3 \otimes_R M \mapsto 0$ is exact.

Proposition (1.3.1) If M is flat and if $mM \neq M$ for every maximal ideal m of R, then M is faithfully flat.

Lemma: Let M be a flat module such that $m M \neq M$ for every maximal ideal m of R. Let A be a R-module such that $A \otimes_R M = 0$. Then $A = 0$.

Proof: Suppose $A \neq 0$; let $x \in A$ be a nonzero element. Then tensoring the exact sequence $0 \mapsto Rx \mapsto A \mapsto A/Rx \mapsto 0$, by M, we get the following exact sequence:
$0 \mapsto Rx \otimes_R M \mapsto A \otimes_R M \mapsto A/Rx \otimes_R M \mapsto 0$. Now, $A \otimes_R M = 0$ implies that $Rx \otimes_R M = 0$. Let m be any maximal ideal of R. Then $Rx \ x\ M \ x\ R/m = 0$, i.e. $Rx/m(Rx) \otimes_{R/m} M/mM = 0$. Since R/m is a field, $Rx/m(Rx)$ and M/mM are vector spaces over R/m; since $M/mM \neq 0$, by assumption, we get therefore $Rx/m(Rx) = 0$, i.e. $Rx = m(Rx)$. This being true for every maximal ideal m of R, we obtain $Rx = 0$, a contradiction. Hence $A = 0$.

Proof of Proposition (1.3.1) Let $0 \mapsto N_1 \mapsto N_2 \mapsto N_3 \mapsto 0$

be a sequence of R-modules. It is enough to show that if $0 \to N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$ .... (1) is exact, then $0 \to N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \to 0$ is exact.

Let (1) be an exact sequence.

Let K = Kernel α. Then $0 \to K \to N_1 \to N_2$ is exact. By flatness of M, $0 \to K \otimes_R M \to N_1 \otimes_R M \to N_2 \otimes_R M$ is exact. But $N_1 \otimes_R M \to N_2 \otimes_R M$ is injective. Therefore $K \otimes_R M = 0$. Hence by lemma above, $K = 0$. That is $0 \to N_1 \to N_2$ is exact.

Let L = Cokernal β, then $N_2 \to N_3 \to L \to 0$ is exact. Therefore $N_2 \otimes_R M \to N_3 \otimes_R M \to L \otimes_R M \to 0$ is exact. But $N_3 \otimes_R M \to N_3 \otimes_R M$ is surjective. Therefore $L \otimes_R M = 0$ which implies $L = 0$. That is $N_2 \to N_3 \to 0$ is exact.

Let Ω = Image $(\beta \cdot \alpha)$. Then $0 \to \Omega \to N_3$ and $N_1 \to \Omega \to 0$ are exact sequences. Therefore $0 \to \Omega \otimes_R M \to N_3 \otimes_R M$ and $N_1 \otimes_R M \to \Omega \otimes_R M \to 0$ are exact. Composing them we get $N_1 \otimes_R M \to \Omega \otimes_R M \to N_3 \otimes_R M$. Image of this composite map is $\Omega \otimes_R M$. Since $0 \to N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$ is exact, $\Omega \otimes_R M = 0$ which implies $\Omega = 0$. Therefore $0 \to N_1 \to N_2 \to N_3 \to 0$ is a complex.

Next we show that $0 \to N_1 \to N_2 \to N_3 \to 0$ is exact. Let $A = \operatorname{Ker} \beta$, then $0 \to N_1 \to A$ is exact. Let T = Cokernel of $N_1 \to A$, so that $0 \to N_1 \to A \to T \to 0$ is exact. Tensoring with M, we get $0 \to N_1 \otimes_R M \to A \otimes_R M \to T \otimes_R M \to 0$ is exact. Also $0 \to A \to N_2 \to N_3 \to 0$ is exact. Therefore $0 \to A \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$ is exact. But

Kernel of $N_2 \otimes_R M \mapsto N_3 \otimes_R M$ is $N_1 \otimes_R M$. Therefore $A \otimes_R M = N_1 \otimes_R M$. From the sequence $0 \mapsto N_1 \otimes_R M \longmapsto A \otimes_R M \longmapsto T \otimes_R M \longmapsto 0$, $T \times M = 0$ which implies $T = 0$. That is Image α = Ker β. So $0 \mapsto N_1 \mapsto N_2 \mapsto N_3 \mapsto 0$ is exact. Hence M is faithfully flat.

Remark: (1.3.2) If M is faithfully flat, clearly M is a flat module. Moreover in this case it can be shown that $M \neq m M$ for any maximal ideal m of R.

Examples: (i) The $\mathbb{Z}$-module Q is flat but not faithfully flat, since $(p)Q = Q$, for any prime number p, where (p) denotes the ideal (maximal) generated by p.

(ii) Any non-zero free module is faithfully flat.

(iii) The polynomial ring $R[x]$ is a faithfully flat R-module.

(iv) Let R be a ring with a nonzero idempotent e. Then Re is a finitely generated flat module but not faithfully flat. For, Re is projective and hence flat. But $Re \otimes_R R(1-e) = 0$ where $R(1-e) \neq 0$, which implies Re is not faithfully flat.

(v) Suppose R is a ring and $f_1, f_2, \ldots, f_n$ are elements generating the unit ideal of R. Then $\prod_{i=1}^{n} R_{f_i}$ is a faithfully flat R-module.

Proof: Let $0 \mapsto N \mapsto M$ be an exact sequence of R-modules. Then $0 \mapsto N_{f_i} \longmapsto M_{f_i}$ is exact for every i. Hence $0 \mapsto \prod_{i=1}^{n} N_{f_i} \mapsto \prod_{i=1}^{n} M_{f_i}$ is exact, i.e. $0 \mapsto N \otimes_R \prod_{i=1}^{n} R_{f_i} \mapsto M \otimes_R \prod_{i=1}^{n} R_{f_i}$ is exact. That is $\prod_{i=1}^{n} R_{f_i}$ is R-flat.

Suppose m is a maximal ideal of R, then there exists at least one $f_i \notin m$. Then $m R_{(f_i)} \neq R_{(f_i)}$, (if $m R_{(f_i)} = R_{(f_i)}$ then $f_i \in m$.). Therefore

$$m \prod_{i=1}^{n} R_{(f_i)} \neq \prod_{i=1}^{n} R_{(f_i)} \text{ for all maximal ideal } m \text{ of } R.$$

Therefore $\prod_{i=1}^{n} R_{(f_i)}$ is a faithfully flat R-module.

<u>Proposition (1.3.2)</u> If S is an R-algebra which is faithfully flat as R-module and if P is an R-module, then P is R-projective if and only if $P \otimes_R S$ is S-projective.

<u>Lemma:</u> Suppose S is a faithfully flat R-algebra. If M is an R-module such that $M \otimes_R S$ is finitely generated as S-module then M is finitely generated as R-module.

<u>Proof:</u> Let $\{x_1 \otimes 1, x_2 \otimes 1, \ldots\ldots, x_n \otimes 1\}$ be a set of generators for $M \otimes_R S$ as S-module where $x_i \in M$. Let F be a free module on a basis $\{e_1, e_2, \ldots, e_n\}$.

Define a map $\theta : F \mapsto M$ such that $\theta(e_i) = x_i$, then the induced map, $\theta \otimes_R 1_S : F \otimes_R S \mapsto M \otimes_R S$ is surjective. By faithfull flatness, $F \mapsto M$ is surjective. So $\{x_1, x_2, \ldots, x_n\}$ generates M as R-module.

<u>Proof of the Proposition (1.3.2)</u> Since scalar extension of projective module is projective, P R-projective implies $P \otimes_R S$ is S-projective. Converse depends on the above lemma.

Suppose $P \otimes_R S$ is S-projective.

<u>Claim (i)</u> : $\mathrm{Hom}_R(P, M) \otimes_R S \cong \mathrm{Hom}_S(P \otimes_R S, M \otimes_R S)$ for

any R-module M.

Since P is finitely generated, there exists a finitely generated free module F_0 such that $F_0 \mapsto P \mapsto 0$ is exact. Let K = Kernel of $F_0 \mapsto P$. Therefore $0 \mapsto K \mapsto F_0 \mapsto P \mapsto 0$ is exact. Since S is flat, $0 \mapsto K \otimes_R S \mapsto F_0 \otimes_R S \mapsto P \otimes_R S \mapsto 0$ is exact. It splits as S-modules, since $P \otimes_R S$ is S-projective. Therefore $K \otimes_R S$ is finitely generated as S-module. By Lemma K is finitely generated as R-module. Therefore there exists a finitely generated free module F_1 such that $F_1 \mapsto K \mapsto 0$ is exact. Therefore $F_1 \mapsto F_0 \mapsto P \mapsto 0$ is exact. Applying the functor $\mathrm{Hom}_R(\ ,M)$ to this sequence, $0 \mapsto \mathrm{Hom}_R(P,M) \mapsto \mathrm{Hom}_R(F_0,M) \mapsto \mathrm{Hom}_R(F_1,M)$ is exact. Since S is R-flat,

$0 \mapsto \mathrm{Hom}_R(P,M) \otimes_R S \mapsto \mathrm{Hom}_R(F_0,M) \otimes_R S \mapsto \mathrm{Hom}_R(F_1,M) \otimes_R S$ is exact.

<u>Claim (ii)</u> For any module N, there exists a cannonical homomorphism of S-module: $\mathrm{Hom}_R(N,M) \otimes_R S \mapsto \mathrm{Hom}_S(N \otimes_R S, M \otimes_R S)$.

Given $f : N \mapsto M$, there exists $f \otimes_R 1_S : N \otimes_R S \mapsto M \otimes_R S$ as R-homomorphisms. Also $f \otimes_R 1_S$ is an S-homomorphism. Define $\eta : \mathrm{Hom}_R(N,M) \mapsto \mathrm{Hom}_S(N \otimes_R S, M \otimes_R S)$ by $\eta(f) = f \otimes_R 1_S$. Then η extends to an S-homomorphism

$\bar{\eta} : \mathrm{Hom}_R(N,M) \otimes_R S \mapsto \mathrm{Hom}_S(N \otimes_R S, M \otimes_R S)$ where $\bar{\eta}(\sum f_i \otimes s_i) = \sum s_i \, \eta(f_i)$. Hence Claim (ii) is proved. The above homomorphism is an isomorphism if N is finitely generated and free. For suppose $N \simeq R^n$ for some positive integer n. Then $\mathrm{Hom}_R(N,M) \cong \mathrm{Hom}_R(R^n,M) \cong \bigoplus_n M$ which implies $\mathrm{Hom}_R(N,M) \otimes_R S \cong (\bigoplus_n M) \otimes_R S \cong \bigoplus_n (M \otimes_R S)$.

Also $\mathrm{Hom}_S(N \otimes_R S, M \otimes_R S) \cong \mathrm{Hom}_S(R^n \otimes_R S, M \otimes_R S)$

$$\cong \mathrm{Hom}_S(S^n, M \otimes_R S)$$

$$\cong \bigoplus_n (M \otimes_R S).$$

Hence the assertion.

Since $F_1 \mapsto F_0 \mapsto P \mapsto 0$ is exact,

$F_1 \otimes_R S \mapsto F_0 \otimes_R S \mapsto P \otimes_R S \mapsto 0$ is exact. Applying functor $\mathrm{Hom}_S(\quad, M \times S)$,

$0 \mapsto \mathrm{Hom}_S(P \otimes_R S, M \otimes_R S) \mapsto \mathrm{Hom}_S(F_0 \otimes_R S, M \otimes_R S) \mapsto \mathrm{Hom}_S(F_1 \otimes_R S, M \otimes_R S$

is exact. From this sequence and the sequence

$0 \mapsto \mathrm{Hom}_R(P,M) \otimes_R S \mapsto \mathrm{Hom}_R(F_0,M) \otimes_R S \mapsto \mathrm{Hom}_R(F_1,M) \otimes_R S$ we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r, mapsto] & \mathrm{Hom}_R(P,M) \otimes_R S \arrow[r, mapsto] \arrow[d, "\theta_1"] & \mathrm{Hom}_R(F_0,M) \otimes_R S \arrow[r, mapsto] \arrow[d, "\theta_2"] & \mathrm{Hom}_R(F_1,M) \otimes_R S \arrow[d, "\theta_3"] \\
0 \arrow[r, mapsto] & \mathrm{Hom}_S(P \otimes_R S, M \otimes_R S) \arrow[r, mapsto] & \mathrm{Hom}_S(F_0 \otimes_R S, M \otimes_R S) \arrow[r, mapsto] & \mathrm{Hom}_R(F_1 \otimes_R S, M \otimes_R S)
\end{tikzcd}$$

Here $_2$ and $_3$ are isomorphisms. Therefore Ker   Ker That is, $\mathrm{Hom}_R(P,M) \otimes_R S \cong \mathrm{Hom}_S(P \otimes_R S, M \otimes_R S)$. Therefore Claim (i) is proved.

Let $0 \mapsto M_1 \mapsto M_2 \mapsto M_3 \mapsto 0$ be an exact sequence of R-modules. Then $0 \mapsto M_1 \otimes_R S \mapsto M_2 \otimes_R S \mapsto M_3 \otimes_R S \mapsto 0$ is exact. Since $P \otimes_R S$ is S-projective,

$0 \mapsto \mathrm{Hom}_S(P \otimes_R S, M_1 \otimes_R S) \mapsto \mathrm{Hom}_S(P \otimes_R S, M_2 \otimes_R S) \mapsto \mathrm{Hom}_S(P \otimes_R S, M_3 \otimes_R S)$

is exact. Also the sequence

$0 \mapsto \mathrm{Hom}_R(P,M_1) \mapsto \mathrm{Hom}_R(P,M_2) \mapsto \mathrm{Hom}_R(P,M_3)$ is exact.

Tensoring with S gives the exact sequence

$0 \mapsto \mathrm{Hom}_R(P,M_1) \otimes_R S \mapsto \mathrm{Hom}_R(P,M_2) \otimes_R S \mapsto \mathrm{Hom}_R(P,M_3) \otimes_R S$.

Clearly the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r, mapsto] & \mathrm{Hom}_S(P \otimes_R S, M_1 \otimes_R S) \arrow[r, mapsto] \arrow[d, "f"] & \mathrm{Hom}_S(P \otimes_R S, M_2 \otimes_R S) \arrow[r, mapsto] \arrow[d, "g"] & \mathrm{Hom}_S(P \otimes_R S, M_3 \otimes_R S) \arrow[r, mapsto] \arrow[d, "h"] & 0 \\
0 \arrow[r, mapsto] & \mathrm{Hom}_R(P,M_1) \otimes_R S \arrow[r, mapsto] & \mathrm{Hom}_R(P,M_2) \otimes_R S \arrow[r, mapsto] & \mathrm{Hom}_R(P,M_3) \otimes_R S &
\end{tikzcd}$$

Each of f, g, h are isomorphism. So

$0 \mapsto \mathrm{Hom}(P, M_1) \otimes_R S \mapsto \mathrm{Hom}_R(P, M_2) \otimes_R S \mapsto \mathrm{Hom}_R(P, M_3) \otimes_R S \mapsto 0$

is exact. By faithful flatness,

$0 \mapsto \mathrm{Hom}_R(P, M_1) \mapsto \mathrm{Hom}_R(P, M_2) \mapsto \mathrm{Hom}_R(P, M_3) \mapsto 0$ is exact.

Therefore P is R-projective.

CHAPTER 2

Projective modules - detailed study

Section 1 of this chapter is devoted to examine the structure of Projective modules over certain class of rings. Section 2 is dealt with the connection between projective modules and the prime spectrum of a ring. In the end we shall give an example of a finitely generated flat module which is not projective.

§1. Projective modules over certain class of rings:

Proposition (2.1.1) Finitely generated projective modules over local rings are free.

Proof: Let R be a local ring and let m be the maximal ideal of R. Let P be a f.g. projective module over R. Let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ be a basis of P/mP as a vector space over R/m; let $x_1, x_2, \ldots, x_n$ be representatives of $\bar{x}_1, \ldots, \bar{x}_n$ respectively. Let $\{e_1, e_2, \ldots, e_n\}$ be the cannonical basis for R^n. Define a R-homomorphisms $\phi : R^n \longmapsto P$ such that $\phi(e_i) = x_i$. Let $K = \text{Kernel } \phi$, $L = \text{Cokernel } \phi$. Then $0 \longmapsto K \longmapsto R^n \longmapsto P \longmapsto L \longmapsto 0$ is exact. Tensoring the exact sequence $R^n \longmapsto P \longmapsto L \longmapsto 0$ with R/m, we get the exact sequence $R^n/mR^n \longmapsto P/mP \longmapsto L/mL \longmapsto 0$. But $R^n/mR^n \longmapsto P/mP$ is surjective. So $L = mL$, therefore by Nakayama's Lemma $L = 0$.

Tensoring the split exact sequence, $0 \mapsto K \mapsto R^n \mapsto P \mapsto 0$ with R/m, we get that $0 \mapsto K/mK \mapsto R^n/mR^n \mapsto P/mP \mapsto 0$ is exact. Since $\dim\,(R^n/mR^n : R/m) = \dim\,(P/mP : R/m) = n$ we get, $R^n/mR^n \cong P/mP$. That is, $K = mK$, which implies $K = 0$ (K is finitely generated, since $K \oplus P = R^n$). So $R^n \cong P$, that is P is free R-module.

Proposition (2.1.2) Suppose R is a semi-local ring. P is a finitely generated projective R-module such that for all maximal ideal m of R, P_m is R_m-free of the same rank, then P is free.

Proof: Let J = Jacobson radical of R. Let $S = R/J$. Assume that the proposition is true for f.g. projective modules over the semi-local ring R/J.

Look at $P \otimes_R S$. Clearly $P \otimes_R S$ is S-projective and finitely generated as S-module. A maximal ideal of S is of the form $\bar{m} = m/J$ where m = maximal ideal of R.

Then $(P \otimes_R S)_{\bar{m}} = (P \otimes_R S) \otimes_S S_{\bar{m}}$

$= P \otimes_R (S \otimes_S S_{\bar{m}})$

$= P \otimes_R S_{\bar{m}}$

$= P \otimes_R R_m \otimes_{R_m} S_{\bar{m}}$. Since R_m is a local ring, $P \otimes_R P_m \cong R_m^n$ for some integer n.

Then $(P \otimes_R S)_{\bar{m}} \cong R_m^n \otimes_{R_m} S_{\bar{m}} \cong S_{\bar{m}}^n$. Since n is independent of m, it is clear from above that n is independent of $\bar{m}$. So by assumption, $P \otimes_R S$ is free over S.

Let $\{ x_1 \otimes 1, \; x_2 \otimes 1, \ldots, \; x_n \otimes 1 \}$ be a basis of P x S over S. Let F = free module on the basis $\{e_1, e_2, \ldots, e_n\}$ and let $\theta : F \longmapsto P$ be the homomorphism defined by $\theta(e_i) = x_i$. Let $K = \operatorname{Ker} \theta$, $L = \operatorname{Coker} \theta$, then $0 \mapsto K \mapsto F \mapsto P \longmapsto L \longmapsto 0$ is exact. Tensoring with $S = R/J$, $F/JF \longmapsto P/JP \mapsto L/JL \mapsto 0$ is exact. Since $F/JF \longmapsto P/JP$ is surjective, $L = JL$. Also L is finitely generated. So $L = 0$ by Nakayama's Lemma. Since $0 \mapsto K \mapsto F \longmapsto P \longmapsto 0$ is split exact. Therefore $0 \mapsto K/JK \mapsto F/JF \mapsto P/JP \longmapsto 0$ is exact. But F/JF, P/JP are free of same rank and $F/JF \longmapsto P/JP$ is surjective. So $F/JF \cong P/JP$. Therefore $K = JK$. Also $K + P = F$ implies K is finitely generated, hence $K = 0$. Therefore $F \cong P$, i.e. P is free.

So it's left to show that the result is true for $S = R/J$. Let Q be a finitely generated projective module over S such that Q_m is S_m-free of the same rank for all maximal ideal m of S. By Chinese Remainder Theorem, $S = R/J = \prod_{i=1}^{t} K_i$, K_i = fields and t = number of maximal ideals in R.

Now $m_i = \prod_{j \neq i} K_j = \{ (x_1, x_2, \ldots, x_t)$ such that $x_i = 0 \}$ are all the maximal ideals of S.

<u>Claim</u>: $S_{m_i} \cong K_i$.

We have the homomorphisms $K_i \overset{\Phi}{\longmapsto} S \overset{\Psi}{\longmapsto} S_{m_i}$, where

$\Phi(k_i) = (0, 0, \ldots 0, \underset{\text{ith place}}{k_i}, 0, \ldots, 0)$

and $\psi(x_1, x_2, \ldots, x_t) = \dfrac{(x_1, x_2, \ldots. x_t)}{1}$. It can be easily seen that $K_i \cong S_{m_i}$

Look at, $Q \otimes_S S_{m_i} = Q_{m_i} = S^n_{m_i}$ where n does not depend on m_i. Also $Q \otimes_S S_{m_i} = (Q \otimes_S K_i) \otimes_{K_i} S_{m_i}$. $Q \otimes_S K_i$ is finitely generated vector space over K_i. So $Q \otimes_S K_i = K_i^{n_i}$

for some +ve integer n_i. Therefore $Q \otimes_S S_{m_i} = K_i^{n_i} \otimes_{K_i} S_{m_i}$

$$= S_{m_i}^{n_i} .$$

Also $Q \otimes_S S_{m_i} = S^n_{m_i}$. Therefore $n_i = n$ for all $i=1,2,\dots,t$.

Therefore $Q \otimes_S K_i = K_i^n$ for $i=1,2,\dots t$, and

$$\begin{aligned} Q \otimes_S \Big(\prod_{i=1}^{t} K_i \Big) &= \prod_{i=1}^{t} (Q \otimes_S K_i) \\ &= \prod_{i=1}^{t} K_i^n \\ &= \Big(\prod_{i=1}^{t} K_i \Big)^n \\ &= S^n . \end{aligned}$$

Therefore $Q \cong S^n$, where $Q \otimes_S \Big(\prod_{i=1}^{t} K_i \Big) = Q \otimes_S S = Q$.

That is Q is S-free.

Next we look at projective modules over Dedekind rings.

Definition: A Dedekind ring is a noetherian domain satisfying the following properties:

i) integrally closed

ii) Any non-zero prime ideal is maximal.

It can be shown [10] that an integral domain is Dedekind if and only if the fractional ideals are invertible, i.e. projective [see also 1].

Definition: Let R be an integral domain, K its field of fractions. An R-submodules M of K is an invertible ideal if there exists a submodule N of K such that $MN = R$.

Remark: (2.1.1) The module N is unique and equal to $(R;M) = \{ x \in K$ such that $xM \subseteq R \}$.

Remark:(2.1.2) Every invertible ideal is finitely generated.

Definition: Let R be an integral domain, K its field of fractions. An R-submodule M of K is a fractional ideal of R if $xM \subseteq R$ for some $x \neq 0$ in R.

Proposition (2.1.3) For a fractional ideal M, the following are equivalent:

i) M is invertible

ii) M is finitely generated and, for each prime ideal p, M_p is invertible.

iii) M is finitely generated and, for each maximal ideal m, M_m is invertible.

Proof: (i) $\Rightarrow$ (ii)

M is invertible implies $R = M\,(R:M)$. Hence
$$R_p = (M(R:M))_p = M_p(R_p:M_p),$$
This holds because M is finitely generated by Remark (2,1.2).

(ii) $\Rightarrow$ (iii) is easy by noting that any maximal ideal is a prime ideal.

(iii) $\Rightarrow$ (i) Let $\mathfrak{a} = M(R:M)$ which is an ideal of R. For each maximal ideal m of R, $\mathfrak{a}_m = M_m(R_m: M_m)$. But since M_m is invertible $M_m(R_m: M_m) = R_m$ and hence $\mathfrak{a}_m = R_m$, i.e. $(R/\mathfrak{a})_m = 0$ for all maximal ideal m of R. Therefore $\mathfrak{a} = R$ by usual argument and hence M is invertible.

Proposition (2.1.4) Let R be a Dedekind ring. Then any finitely generated torsion free R-module is projective. Also the module is isomorphic to a finite direct sum of ideals of R.

Proof: Let P be a torsion free R-module. Let K = quotient field of R. Proof is by induction on the dimension of $P \otimes_R K$ over K. Suppose dimension of $P \text{ x } K$ over K is 1. Then $P \text{ x } K \quad K$. Since P is torsion free, the canonical map : $P \mapsto P \otimes_R K$ taking p to $p \otimes 1$ is injective, i.e. there exists injection : $P \longmapsto K$. Therefore P is isomorphic to a fractional ideal of R and hence projective. (because is invertible, see also 3.1.3 to follow

Let $x \in P$, $x \neq 0$. Look at $Kx \quad P$ in $P \otimes_R K$.

Claim: $Kx \cap P$ is isomorphic to a fractional ideal of R. Since P is finitely generated and R is noetherian, $Kx \cap P$ is a finitely generated submodule of P. There exists injection:

$$
\begin{aligned}
Kx \cap P \longmapsto (Kx \cap P) \otimes_R K &= (Kx \otimes_R K) \cap (P \otimes_R K) \\
&= Kx \cap (P \otimes_R K) \\
&= Kx \\
&\cong K.
\end{aligned}
$$

That is there exists injection: $Kx \cap P \longmapsto K$. Since $Kx \cap P$ is a finitely generated R-submodule of K, it is isomorphic to a fractional ideal of R, and hence projective.

Look at $P/(Kx \cap P)$.

Claim: $P/(Kx \cap P)$ is torsion free.

For if $y \in P$, and $0 \neq \lambda \in R$ is such that $\lambda y \in Kx \cap P$, then $\lambda y \in Kx$ which implies $y \in Kx$. Also $y \in P$. So $y \in Kx \cap P$. Hence the claim.

Also
$$P/(Kx \cap P) \otimes_R K \cong \frac{P \otimes_R K}{(Kx \cap P) \otimes_R K} = \frac{P \otimes_R K}{Kx}.$$

Therefore dimension of $P/(Kx \cap P) \otimes_R K$ over K is less than the dimension of $P \otimes_R K$ over K. So by induction $P/(Kx \cap P)$ is projective and isomorphic to finite direct sum of ideals of R. Look at the exact sequence

$0 \mapsto Kx \cap P \mapsto P \mapsto P/(Kx \cap P) \mapsto 0$. This is split exact, so $P \cong (Kx \cap P) \oplus P/(Kx \cap P)$. So P is projective being isomorphic to a direct sum of projective modules.

Now Kx P being isomorphic to a fractional ideal it is actually isomorphic to an ideal of R. Therefore P is isomorphic to a finite direct sum of ideals of R.

Corollary: If R is a Principal ideal domain, any finitely generated projective module over R is free.

Proof: Since R is a P.I.D., it is a Dedekind ring. Also projective modules are torsion free. Therefore by previous proposition any finitely generated projective module over R is isomorphic to a finite direct sum of ideals of R and hence free (since ideals are principal).

Proposition (2.1.5) Let R be a U.F.D. If P is a finitely generated projective module over R such that $P_p \cong R_p$ for all prime ideals p in R, then P is free.

Proof: Let K = field of fractions of R. We have $P_{(0)} \cong R_{(0)}$ i.e. $P \otimes_R K \cong K$. P is projective, therefore torsion free and hence there exists an injection: $P \mapsto P \otimes_R K$ and since $P \otimes_R K \cong K$, P is a finitely generated R-submodule of K and hence is isomorphic to a fractional ideal. Since $P_p \cong R_p$, P_p is an invertible ideal. By proposition (2.1.3), P is an invertible ideal. Therefore $P(R:P) = R$.

Let P be generated by $x_1, x_2, \ldots, x_n$.
$(R:P) = (R: \sum_{i=1}^{n} Rx_i) = \bigcap_{i=1}^{n}(R:Rx_i) = \bigcap_{i=1}^{n}(Rx_i^{-1})$.

Claim: In a U.F.D R, intersection of any two non zero principal ideals is principal.

Suppose $x, y \in R$, $x \neq 0$, $y \neq 0$. Write $x = \text{unit } x \prod p$ $y = \text{unit } x \prod p^{b_p}$ where a_p, b_p are integers. Clearly $Rx \cap Ry = Rz$ where $z = \prod_p p^{\max(a_p, b_p)}$. So $Rx \cap Ry$ is principal. Therefore finite intersection of principal ideals is principal.

Therefore $(R:P) = \bigcap_{i=1}^{n} Rx_i^{-1} = Rt$ for $t \neq 0$, $t \in R$.
Since $P(R:P) = R$

$P(Rt) = R$ which implies $Pt = R$ that is $P = Rt^{-1}$ where $t^{-1} \in K$. Therefore P is isomorphic to R. That is P is free.

§2. Spectrum of a ring

Definition: Let R be a ring. Underlying set of spectrum R is the set of prime ideals of R. For any subset E of R we define $V(E) = \{p \text{ prime ideal s.t } p \supseteq E\}$. It satisfies following properties:

(i) $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$, where $\mathfrak{a}$ is the ideal generated by E in R and $r(\mathfrak{a})$ = radical of $\mathfrak{a}$ in R.

(ii) If $\{E_i\}_{i \in I}$ is any family of subsets of R, then

$V(\bigcup_{i\in I} E_i) = \bigcap_{i\in I} V(E_i)$

(iii) $V(0)$ = set of all prime ideals of R and $V(R) = \phi$

(iv) $V(\mathfrak{a}\cap b) = V(\mathfrak{a}) \cup V(b)$ for ideals $\mathfrak{a}$ and b in R.

Above properties show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The topology formed by these closed sets is known as 'Zariski topology'. A base for open sets of this topology is given by the set $D(f) = \{ p$ prime ideal in R such that $p \not\ni f \}$ where $f \in R$. The set of prime ideals of R with Zariski topology is known as the prime spectrum of R and denoted by 'Spec R'. Set of the form $D(f)$ are known as special open set.

Proposition (2.2.1) Suppose P is a finitely generated projective module over R. There exists a covering of Spec R by special open sets say $D(f_1) \cup D(f_2) \cup \dots \cup D(f_n)$ = Spec R such that for every fixed i, $1 \leq i \leq n$, and whatever $p \in D(f_i)$, the R_p-module P_p is free of the same rank (rank depends on i).

Proof: Let p Spec R. We will construct $f \notin p$ such that for all q $D(f)$, P_q is free of the same rank over R_q. Thus we get a covering of spec R by special open sets $\{D(f_\alpha) : \alpha \in I\}$ with the property that for a fixed $D(f_\alpha)$, whatever $q \in D(f_\alpha)$ P_q is free of same rank.

Claim: Spec $R = \bigcup_{i=1}^{n} D(f_{\alpha_i})$ for $\alpha_1, \alpha_2, \dots \alpha_n \in I$.

The ideal generated by the f 's is the unit ideal by construction. Hence we can find indices $\alpha_1, \alpha_2, \dots, \alpha_n$, and elements $a_{\alpha_1}, a_{\alpha_2}, \dots, a_{\alpha_n}$ such that $1 = a_{\alpha_1} f_{\alpha_1} + a_{\alpha_2} f_{\alpha_2} + \dots + a_{\alpha_n} f_{\alpha_n}$ which implies, given any prime ideal at least one f_{α_i} for $1 \leq i \leq n$, is not contained in it, i.e Spec $R = \bigcup_{i=1}^{n} D(f_{\alpha_i})$.

So our problem reduces to finding a $f \in R$ for every $p \in$ Spec R such that $f \notin p$ and for all $q \in D(f)$, P_q is free of the same rank over R_q.

Given P is finitely generated R-projective, so $P_p = R_p \otimes_R P$ is finitely generated R_p-projective. Since finitely generated projective module over a local ring is free P_p is R_p-free. Let $\{x_1/s_1, x_2/s_2, \ldots, x_n/s_n\}$ be a basis of P_p over R_p where $x_i \in P$ and $s_i \in R-P$. Let F be the free R-module of rank n on the basis $\{e_1, e_2, \ldots, e_n\}$.

Construct a R-homomorphism $\theta : F \longmapsto P$, such that $\theta(e_i) = x_i$. Let K = Kernel θ, L = Cokernel θ. Then $0 \mapsto K \mapsto F \longmapsto P \longmapsto L \longmapsto 0$ is exact. Localising at p, $0 \mapsto K_p \longmapsto F_p \longmapsto P_p \longmapsto L_p \longmapsto 0$ is an exact sequence of R_p-modules. $F_p \longmapsto P_p$ is surjective and both have same rank over R_p. Hence $F_p \cong P_p$. Therefore $K_p = 0$, $L_p = 0$.

L is finitely generated. Let $\{y_1, y_2, \ldots, y_r\}$ be a set of generators for L. Since $L_p = 0$, $\exists\ s_i \notin p$ such that $s_i y_i = 0$ for all $i = 1, 2, \ldots, r$. Take $s = s_1 s_2 \ldots s_r$. Then $s y_i = 0$ for all i which implies $L_{(s)} = 0$.

Localising $0 \mapsto K \longmapsto F \longmapsto P \longmapsto L \longmapsto 0$ at the multiplicatively closed set generated by s, we get

$$0 \longmapsto K_{(s)} \longmapsto F_{(s)} \longmapsto P_{(s)} \longmapsto 0 \quad \text{———————} \quad (*)$$

is $R_{(s)}$ - exact. $P_{(s)}$ is $R_{(s)}$ - projective shows that above sequence splits and $F_{(s)} = K_{(s)} \oplus P_{(s)}$. Therefore $K_{(s)}$ is finitely generated $R_{(s)}$-module.

Note that $K_p = (K_s)_{pR_s}$ so that $(K_s)_{pR_s} = (0)$.

Since $K_{(s)}$ is finitely generated $R_{(s)}$-module, there exists $t \notin pR_{(s)}$ such that $(K_{(s)})_{(t)} = 0$. Without loss of generality we can take $t \notin p$. Then $(K_{(s)})_{(t)} = K_{(st)} = (0)$. Since $0 \mapsto K_{(st)} \mapsto F_{(st)} \mapsto P_{(st)} \mapsto 0$ is exact and $K_{(st)} = 0$, $F_{(st)} \cong P_{(st)}$.

Let $f = st$.

Claim: $\forall\ q \in D(f)$, rank if P is the same.

Let $q \in D(f)$, then $P_q = (P_{(f)})_{qR_f} \cong (F_f)_{qR_f} = F_q$.

Hence the proposition is proved.

Proposition (2.2.2) Suppose P is a finitely generated R-module such that, there exists a covering of Spec R by special open set, say Spec $R = \bigcup_{i \in I} D(f_i)$ such that $P_{(f_i)}$ is $R_{(f_i)}$ - free for every i, then P is projective.

Proof: Without loss of generality, we can assume that the set of elements f_i is finite say $f_1, f_2, f_3, \ldots, f_n$.

Look at $S = \prod_{i=1}^{n} R_{(f_i)}$. S is a faithfully flat R-algebra

Also $P \otimes_R S = P \otimes_R \prod_{i=1}^{n} R_{(f_i)} = \prod_{i=1}^{n} (P \otimes_R R_{(f_i)}) = \prod_{i=1}^{n} P_{(f_i)}$.

Given that for each i, $P_{(f_i)}$ is $R_{(f_i)}$ - free. So for every i, there exists a positive integer n_i such that $P_{(f_i)} \cong R^{n_i}_{(f_i)}$. Take N = maximum of $n_1, n_2, \ldots, n_n$. Then $\prod_{i=1}^{n} R^{N}_{(f_i)} = S^N$. Also $P \otimes_R S = \prod_{i=1}^{n} P_{(f_i)} = \prod_{i=1}^{n} R^{n_i}_{(f_i)}$.

Now $\prod_{i=1}^{n} R^{n_i}_{(f_i)} \oplus \prod_{i=1}^{n} R^{N-n_i}_{(f_i)} = \prod_{i=1}^{n} R^{N}_{(f_i)} = S^N$ and therefore $\prod_{i=1}^{n} R^{n_i}_{(f_i)} = P \otimes_R S$ is S-projective which implies P is R-projective by proposition (1.3.2).

Definition: A module M over R is said to be "finitely presented" if there exists an exact sequence $F_1 \mapsto F_0 \mapsto M \mapsto 0$ such that F_1 and F_0 are finitely generated free modules.

Examples:

(i) Any finitely generated module over a noetherian ring is finitely presented.

Proof: Let M be a finitely generated R-module. There exists F_0 finitely generated and free such that $F_0 \overset{\theta}{\mapsto} M \longmapsto 0$ is exact. Let $K = \text{Ker}\ \theta$. R is noetherian, K is a submodule of R_0, so K is finitely generated. Therefore there exists F_1 finitely generated free module such that $F_1 \mapsto K \mapsto 0$ is exact. Therefore $F_1 \mapsto F_0 \mapsto M \mapsto 0$ is exact.

(ii) Any finitely generated projective module is finitely presented.

Proof: Let P be a finitely generated projective R-module. There exist a finitely generated free module F_0 and a surjective homomorphism $\theta : F_0 \longmapsto P$. Let $K = \text{Ker}\ \theta$ the sequence $0 \mapsto K \mapsto F_0 \mapsto P \mapsto 0$ is split exact since P is projective. Hence $F_0 = P \oplus K$, hence K is finitely generated. So there exists a finitely generated free module F_1 and a surjective homomorphism: $F_1 \longmapsto K$, i.e $F_1 \mapsto F_0 \mapsto P \mapsto 0$ is exact.

Remark (2.2.1) If P is finitely presented R-module and there exists an exact sequence of R-modules $0 \mapsto K \mapsto F \mapsto P \mapsto ($ where F is finitely generated, then K is finitely generated.

Proof: There exists finitely generated free modules F_0, F_1 such that $F_1 \to F_0 \to P \to 0$ is exact. Therefore

$$\begin{tikzcd} & F_1 \arrow[r] & F_0 \arrow[r] & P \arrow[r] & 0 \\ {} \arrow[r] & K \arrow[r] & F \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

where α, β exists by the projective of F_1 and F_0, is commutative. By Snake Lemma, $\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker} 1_P \to \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker} 1_P$ is exact. Since $\operatorname{Ker} 1_P = 0$, $\operatorname{Coker} 1_P = 0$ we get, $0 \to \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to 0$ is exact, i.e. $0 \to K/\alpha(F_1) \to F/\beta(F_0) \to 0$ is exact. So $K/\alpha(F_1) \cong F/\beta(F_0)$ and hence $K/\alpha(F_1)$ is finitely generated. Also $\alpha(F_1)$ is finitely generated. Therefore K is finitely generated.

Remark (2.2.2) If P is a finitely presented module and is a prime ideal such that P_p is R_p-free, then there exists f p such that $P_{(f)}$ is $R_{(f)}$-free.

Proof: Let n = rank of P over R_p and $\{x_1/s_1, x_2/s_2, \ldots, x_n/s_n\}$ be a basis of P_p over R_p. Let F be a free R-module of rank n and $\{e_1, e_2, \ldots, e_n\}$ be a basis of F over R. Then there exists a R-homomorphism $\theta : F \to P$ taking e_i to x_i. Let $K = \operatorname{Ker}\theta$ and L = Coker then $0 \to K \to F \to P \to L \to 0$ is exact.

Now as in the proof of Proposition (2.2.1) we get an $s \notin p$ such that $0 \to K_{(s)} \to F_{(s)} \to P_{(s)} \to 0$ is an exact sequence of $R_{(s)}$-modules. Here P is a finitely presented R-module, so $P_{(s)}$ is a finitely presented $R_{(s)}$-module. By remark (2.2.1), $K_{(s)}$ is a finitely generated $R_{(s)}$-module and rest of the proof is similar to the said Proposition (2.2.1)

__Proposition:(2.2.3)__ If P is finitely presented and flat then P is projective.

__Proof:__ Case (i) Let R be a local ring. Let $x_1,x_2,\ldots,x_n \in P$ such that $\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n$ give a basis of the R/m - vector space P/mP where m is the maximal ideal of R. Let F be a free module of rank n with a basis $\{e_1,e_2,\ldots,e_n\}$. Then there exists a homomorphism $\theta : F \longmapsto P$ taking e_i to x_i. Let L = Cokernel θ, then $F \longmapsto P \longmapsto L \longmapsto 0$ is exact. Tensoring with R/m we get $F/mF \longmapsto P/mP \longmapsto L/mL \longmapsto 0$ is exact. But $F/mF \longmapsto P/mP$ is surjective, so $L/mL = 0$ which implies $L = 0$ by Nakayama's Lemma.

Let K = Kernel $F \longmapsto P$. Therefore $0 \mapsto K \mapsto F \mapsto P \mapsto 0$ is exact. Tensoring with m, $K \otimes_R m \longmapsto F \otimes_R m \longmapsto P \otimes_R m \longmapsto 0$ is exact. Also, tensoring $0 \longmapsto m \longmapsto R$ with K, F and P successively, we get homomorphism

$\alpha : m \otimes_R K \longmapsto K$ (since $R \otimes_R K = K$)

$\beta : m \otimes_R F \longmapsto F$

$\gamma : m \otimes_R P \longmapsto P$.

By flatness $0 \longmapsto m \otimes_R P \longmapsto P$ is exact. We get the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & m \otimes_R K \arrow[r] \arrow[d, "\alpha"] & m \otimes_R F \arrow[r] \arrow[d, "\beta"] & m \otimes_R P \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & K \arrow[r] & F \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

So by Snake lemma, we get an exact sequence:

$\operatorname{Ker}\alpha \longmapsto \operatorname{Ker}\beta \longmapsto \operatorname{Ker}\gamma \longmapsto \operatorname{Coker}\alpha \longmapsto \operatorname{Coker}\beta \longmapsto \operatorname{Coker}\gamma$. Here $\operatorname{Ker}\gamma = 0$, so it gives the following exact sequence:

$0 \longmapsto \operatorname{Coker}\alpha \longmapsto \operatorname{Coker}\beta \longmapsto \operatorname{Coker}\gamma$. But $\operatorname{Coker}\alpha = K/mK$, $\operatorname{Coker}\beta = F/mF$ and $\operatorname{Coker}\gamma = P/mP$.

But $F/mF \cong P/mP$, therefore $K/mK = 0$ that is $K = mK$. But by the Remark (2.2.1), K must be finitely generated. Hence by Nakayama's Lemma $K = 0$, that is $F \cong P$ and hence P is projective.

Case (ii) R is any commutative ring.

Let p Spec R. Since P is flat, P_p is flat R_p-module. Also if $F_1 \mapsto F_0 \mapsto P \mapsto 0$ is a finite presentation for P, then $(F_1)_p \mapsto (F_0)_p \mapsto P_p \mapsto 0$ is a presentation of P_p as R_p-module. By Case (i) P_p is free over R_p. So by the Remark (2.2.2) there exists $f \notin p$ such that $P_{(f)}$ is $R_{(f)}$-free. Call this f as f_p to show its dependance on p. Then the ideal $(f_p)_{p \in \mathrm{Spec}\, R}$ is not contained in any maximal ideal. (If $(f_p)_{p \in \mathrm{Spec}\, R} \subseteq m$, m maximal ideal, then $f_m \in m$). Hence $R =$ ideal $(f_p)_{p \in \mathrm{Spec}\, R}$ which implies Spec $R = \bigcup D(f_p)$, where $P_{(f_p)}$ is $R_{(f_p)}$-free so by the Proposition (2.2.2) P is R-projective.

Next we shall give example to show that a finitely generated flat module need not be projective. This we do by constructing an example of a flat module which is finitely generated but not finitely presented.

As a general remark let us take R to be any non noetherian ring. Let $\mathfrak{a}$ be any non finitely generated ideal of R. Then by remark (2.2.1) it is clear that $R/\mathfrak{a}$ is not finitely presented (although $R/\mathfrak{a}$ is finitely generated), however we can also prove this in the following manner.

finitely generated.

Claim: $R/\mathfrak{a}$ is not finitely presented.

Suppose $R/\mathfrak{a}$ is finitely presented, then we have an exact sequence of the form $F_1 \mapsto F_0 \mapsto R/\mathfrak{a} \longmapsto 0$ where F_1 and F_2 are finitely generated free modules. Also $0 \mapsto \mathfrak{a} \mapsto R \mapsto R/\mathfrak{a} \mapsto 0$ is exact. Then we have the following diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathfrak{a} \arrow[d] & \\
 & & & R \arrow[d] & \\
0 \arrow[r, mapsto] & \theta(F_1) \arrow[r, mapsto] & F_0 \arrow[r, mapsto] & R/\mathfrak{a} \arrow[r, mapsto] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

Here R is a R-projective module, F_0 is a projective module. So by Schanuel's Lemma in Homological Algebra, $\mathfrak{a} \oplus F_0 \cong \theta(F_1) \oplus R$ which implies $\mathfrak{a}$ is a finitely generated R-module. This is a contradiction. So $R/\mathfrak{a}$ is not finitely presented.

Therefore $R/\mathfrak{a}$ is an example of a finitely generated module which is not finitely presented.

Remark (2.2.3) Suppose M is a finitely generated projective R-module. We define the rank map associated with M as the map r_M: Spec $R \mapsto \mathbb{Z}$ as follows:

If $p \mapsto$ Spec R, then M_p is R_p-free and finitely generated. Let $\text{rank}_{R_p} M_p = n_p$. Define $r_M(p) = n_p$. Take discrete topology on $\mathbb{Z}$, then r_M is continuous with respect to the Zariski topology in Spec R. It is enough to show that r_M is locally constant.

Suppose p Spec R and $\text{rank}_{R_p} P_p = n_p$ then we proved earlier that, there exists f p such that for all q $D(f)$, $\text{rank}_{R_q} P_q = \text{rank}_{R_p} P_p$. Here $D(f)$ is a neighbourhood of p. So the map r_M is locally construct and hence r_M is continuous.

<u>**Criteria for projectivity:**</u> Suppose P is a finitely generated R-module. Suppose for all $p \in \text{Spec } R$, P_p is R_p-free of rank n_p, and moreover the map r_P: Spec $R \mapsto \mathbb{Z}$ taking p to n_p is locally constant (i.e. continuous), then P is projective.

<u>**Proof:**</u> Let p Spec R. Let $\{p_i/1\}_{1 \leq i \leq n}$ be a basis of P_p as R_p-module where $p_i \in P$. Let F be a free module of rank n with the basis $\{e_1, e_2, \ldots, e_n\}$. Then there exists a homomorphism $u : F \mapsto P$ taking e_i to p_i. Then the map $u_p : F_p \mapsto P_p$ is surjective. Hence there exists $f \notin p$ such that $u_f : F_f \mapsto P_f$ is surjective. Also since Spec $R \mapsto \mathbb{Z}$ is locally constant, there exists a neighbourhood $D(g)$ of p such that for all $q \in D(g)$ $\text{rank}_{R_q} P_q = n$. Take $h = fg$, here $h \notin p$. Then $u_h : F_h \mapsto P_h$ is still surjective.

<u>**Claim:**</u> u_h is an isomorphism of R_h-modules.
We have only to show that $(u_h)_{q^1} : (F_h)_{q^1} \mapsto (P_h)_{q^1}$ is an isomorphism for all q^1, prime ideal of R_h.

The prime ideal q^1 corresponds to a prime ideal q of R which does not contain h. Clearly $q \not\ni g$, that is $q \in D(g)$.

Note that $(F_h)_{q^1} = F_q$ and $(P_h)_{q^1} = P_q$.

Here $(F_h)_{q^1}$ and $(P_h)_{q^1}$ are both free $(R_h)_{q^1} = R_q$-modules. Since for all $q \in D(g)$, rank $P_q = \text{rank}_{R_q} F_q$, So $(F_h)_{q^1}$, $(P_h)_{q^1}$ are free $(R_h)_{q^1}$-module of same rank n. Also $u_h : F_h \mapsto P_h$ is surjective. So $(u_h)_{q^1} : (F_h)_{q^1} \mapsto (P_h)_{q^1}$ is surjective. Hence $(F_h)_{q^1} \mapsto (P_h)_{q^1}$ is isomorphisms of $(R_h)_{q^1}$-module. Therefore $u_h : F_h \mapsto P_h$ is an isomorphisms of R_h-modules.

<u>Conclusion:</u> For all $p \in \text{Spec } R$, there exists $h \notin p$, such that P_h is R_h-free. Thus we get a covering of Spec R. Then by the proposition (2.2.2) P is a R-projective module.

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§3. <u>Example of a finitely generated flat module which is not projective</u>

<u>Lemma:</u> Suppose M is an R-module such that M_p is R_p-free for every prime ideal p of R, then M is R-flat.

<u>Proof:</u> Let $0 \mapsto N_1 \mapsto N_2$ be any exact sequence of R-modules. Let L = Kernel of $(N_1 \otimes_R M \mapsto N_2 \otimes_R M)$. Let p be a prime ideal of R, then $0 \mapsto (N_1)_p \mapsto (N_2)_p$ is exact sequence of R_p-modules. Now M_p is R_p-free, in particular R_p-flat. So $0 \mapsto (N_1)_p \otimes_{R_p} M_p \mapsto (N_2)_p \otimes_{R_p} M_p$ ———————— (*) is exact. Also from the exact sequence

$0 \mapsto L \mapsto N_1 \otimes_R M \mapsto N_2 \otimes_R M$ we get the exact sequence

$0 \mapsto L_p \mapsto (N_1 \otimes_R M)_p \mapsto (N_2 \otimes_R M)_p$ ———————— $(*^1)$

But $(N_1 \underset{R}{\otimes} M)_p = (N_1)_p \underset{R_p}{\otimes} M_p$ and $(N_2 \underset{R}{\otimes} M)_p = (N_2)_p \underset{R_p}{\otimes} M_p$. From $(*)$ and $(*^1)$ we get $L_p = 0$ for all p, prime ideals of R which implies $L = 0$. So $0 \mapsto N_1 \underset{R}{\otimes} M \mapsto N_2 \underset{R}{\otimes} M$ is exact. So M is R-flat.

__Definition:__ Let R be a ring (not necessarily commutative) is called V.N. Regular if for every $a \in R$, there exists R such that $aba = a$.

__Example of a V.N. Regular ring__ : $R = \prod_{\alpha \in \Lambda} K_\alpha$ (infinite product of fields) is V.N. Regular.

Let $a \in R$, then $a = (a_\alpha)_{\alpha \in \Lambda}$. If $a_\alpha \neq 0$, let $b_\alpha = a_\alpha^{-1}$, if $a_\alpha = 0$, let $b_\alpha = 0$, then $a_\alpha b_\alpha a_\alpha = a_\alpha$ in both the cases. Take $b = (b_\alpha)_{\alpha \in \Lambda}$, then $aba = (a_\alpha b_\alpha a_\alpha)_{\alpha \in \Lambda} = (a_\alpha)_{\alpha \in \Lambda}$
$= a$.

__Remark (2.3.1)__ If R is V.N.Regular, then $S^{-1}R$ is V.N.Regular for any multiplicative closed set S of R.

Let $x \in S^{-1}R$ then $x = a/s$ for $a \in R$, $s \in S$. So there exists b such that $aba = a$. Therefore $a/s(\frac{sb}{1})\ a/s = (\frac{ab}{1})\ a/s = \frac{aba}{s} = a/s$. Hence $S^{-1}R$ is V.N.Regular. So R_p is V.N.Regular for any prime ideal p in R.

__Remark (2.3.2)__ If R = local, commutative, V.N.Regular, then R is a field.

If $a \in R$, then there exists $b \in R$ such that $aba = a$ which implies $abab = ab$, that is ab is an idempotent of R. Also if $a \neq 0$, $ab \neq 0$ (since $aba = a$); ab being an idempotent and R local, show that $ab=1$. Therefore R is a field.

Example: Let R = infinite product of fields, then R is V.N.Regular, non noetherian. Let $\mathfrak{a}$ = non finitely generated ideal of R. Let $M = R/\mathfrak{a}$. Then $M_p = R_p/\mathfrak{a}_p$ for p prime ideal of R. R V.N.Regular implies R_p is V.N.Regular. Also R_p is local and commutative. So R_p is a field by Remarks (2.3.1) and (2.3.2). Here $\mathfrak{a}_p$ is an ideal of R_p, so either $\mathfrak{a}_p = 0$ or $\mathfrak{a}_p = R_p$, that is $M_p = R_p$ or 0. Therefore M_p is a free R_p-module for all prime ideal p of R. Hence by Lemma, M is R-flat. Also M is a finitely generated R-module.

Claim: M is not R-projective.

If M is R-projective then $0 \mapsto \mathfrak{a} \mapsto R \mapsto M \mapsto 0$ is split exact and hence $R \cong \mathfrak{a} \oplus M$. Therefore $\mathfrak{a}$ is a quotient of R and hence $\mathfrak{a}$ is a finitely generated R-module which is a contradiction. So M is not projective.

Therefore M is a finitely generated flat R-module which is not projective.

ABSTRACT

PROJECTIVE MODULES OVER CERTAIN CLASS OF RINGS AND SERRE'S CONJECTURE.

The first chapter of this dissertation aims at studying the definition and various equivalent conditions for a module to be projective. We also relate flat and faithfully flat modules to projective modules.

The second chapter is devoted to examine the structure of Projective modules over certain class of rings & also deals with the connection between projective modules and the prime spectrum of a ring.

In the thrid chapter, Picard functor & Grothendieck group of a ring are studied which are useful tools in examining projective modules over certain class of rings.

The fourth chapter is devoted to proving a general version of the classical theorem of Hilbert on syzygies. In this regard a category of modules called stably free modules enter into the discussion.

The fifth chapter, that is the last chapter, is devoted to the proof of Serre's conjecture on projective modules. The proof we give follows the presentation of Vasherstein and Suslin.

* * *

Certified that this is an abstract of the dissertation written by Miss. Sipra Paul Chowdhury, which is a product of her study of the subject.

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CHAPTER 3

Functors associated with Projective modules

The purpose of this chapter is to study some functors which will be useful in examining projective modules over certain class of rings.

§1. Picard functor

Picard group of a ring R: Let X be the set of isomorphism classes of finitely generated projective modules of constant rank 1 over R. We make X into a group through the following operations. If $[P], [Q] \in X$, define $[P][Q] = [P \otimes_R Q]$. $P \otimes_R Q$ is projective. If p spec R, then $(P \otimes_R Q)_p \cong P_p \otimes_{R_p} Q_p \cong R_p \otimes_{R_p} R_p \cong R_p$, $(P_p, Q_p$ are free of rank 1 which implies $P \times Q$ is projective of rank 1. Also $P \otimes_R Q$ is finitely generated, i.e. $[P \otimes_R Q] \in X$. Clearly this binary operation is well defined.

1. Associativity - It follows from the associativity of tensor product.
2. Identity - $[R]$ is the identity, since $[P][R] = [P \otimes_R R] = [P] \ \forall [P] \in X$.
3. Inverse - Inverse of $[P]$ will be $[P^*]$ where $P^* = \mathrm{Hom}_R(P,$

Claim: P^* is finitely generated projective of rank 1.

P is finitely generated projective implies there exists a finitely generated free module F and a module P^1 such that $P \oplus P^1 = F$. Then $\mathrm{Hom}_R(F,R) \cong \mathrm{Hom}_R(P \oplus P^1, R)$

$$\cong \mathrm{Hom}_R(P,R) \oplus \mathrm{Hom}_R(P^1,R).$$

Suppose $F \cong R^n$ for some positive integer n, then $\mathrm{Hom}_R(F,R) \cong \oplus R^n$. Therefore $\mathrm{Hom}_R(P,R) \oplus \mathrm{Hom}_R(P^1,R) \cong R^n$. So P^* is finitely generated projective.

Claim: Rank of P^* is 1.

Since P is projective, it is finitely presented, therefore $\exists$ F_0, F_1 finitely generated free modules such that $F_0 \mapsto F_1 \mapsto P \mapsto 0$ is exact. We have for all prime ideal p,

$$[\mathrm{Hom}_R(F_1,R)]_p \cong \mathrm{Hom}_{R_p}[(F_1)_p, R_p]$$

$$[\mathrm{Hom}_R(F_0,R)]_p \cong \mathrm{Hom}_{R_p}[(F_0)_p, R_p].$$

From the above exact sequence we get

$$0 \mapsto \mathrm{Hom}_R(P,R) \mapsto \mathrm{Hom}_R(F_1, R) \mapsto \mathrm{Hom}_R(F_0,R)$$ is exact.

Localising at a prime ideal p,

$$0 \mapsto [\mathrm{Hom}_R(P,R)]_p \mapsto [\mathrm{Hom}_R(F_1,R)]_p \mapsto [\mathrm{Hom}_R(F_0,R)]_p$$ is exact. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r, mapsto] & {[\mathrm{Hom}_R(P,R)]_p} \arrow[r, mapsto] \arrow[d] & {[\mathrm{Hom}_R(F_1,R)]_p} \arrow[r, mapsto] \arrow[d] & {[\mathrm{Hom}_R(F_0,R)]_p} \arrow[d] \\
0 \arrow[r, mapsto] & \mathrm{Hom}_{R_p}(P_p,R_p) \arrow[r, mapsto] & \mathrm{Hom}_{R_p}[(F_1)_p,R_p] \arrow[r, mapsto] & \mathrm{Hom}_{R_p}[(F_0)_p,R_p].
\end{tikzcd}$$

The map : $\mathrm{Hom}_R(P,R)_p \mapsto \mathrm{Hom}_{R_p}(P_p,R_p)$ is natural, namely suppose $f: P \mapsto R$, is a, R-homomorphism and, define $(f/s)(r/t) = \frac{f(r)}{st}$ for $r \in P$, $t \notin p$. The diagram is commutative, so $[\mathrm{Hom}_R(P,R)]_p \cong \mathrm{Hom}_{R_p}(P_p,R_p)$. Since P is of rank 1, $P_p \cong R_p$ Hence $\mathrm{Hom}_{R_p}(P_p,R_p) \cong R_p$, i.e. $\mathrm{Hom}_R(P,R)_p \cong R_p$. P^* is of rank 1, therefore $[P^*] \in X$, since P is finitely generated projective P^* is finitely generated.

Claim: $[P^*]$ is the inverse of $[P]$.

a natural homomorphism, $\Phi : P^* \otimes_R P \mapsto R$, where

$\phi(f \otimes a) = f(a)$ for $f \in P^*$, $a \in P$ and extend by additivity. To show ϕ is isomorphism it is enough to show that ϕ is an isomorphism when localised at any prime ideal. If $p \in$ spec R, then $(P^* \otimes_R P)_p \cong P^*_p \otimes_{R_p} P_p \cong R_p \otimes_{R_p} R_p \cong R_p$. Hence $\phi_p : (P^* \otimes_R P)_p \mapsto R_p$ is reduced to an isomorphism. Therefore ϕ is an isomorphism. We have $[P][P^*] = [P \otimes_R P^*] = [R]$. Therefore X with this binary operation is a group known as Picard group of the ring R and is denoted by Pic R.

Definition: Nonzero fractional ideals of R form a group under multiplication. The group of fractional ideals modulo principal ideals is known as ideal class group.

Theorem:(3.1.1) Let R be a Dedekind ring. We shall show that Pic R $\cong$ ideal class groupp

We know from Proposition (2.1.4) that torsion free modules over Dedekind rings are isomorphic to a finite direct sum of ideals of R. Since projective modules are torsion free the result quoted now implies that a projective module of rank 1 is isomorphic to a fractional ideal of R.

Suppose P and Q are projective modules of rank 1 over R. Say $P \cong \mathfrak{a}$, $Q \cong \mathfrak{b}$ where $\mathfrak{a}, \mathfrak{b}$ are fractional ideals of R. We shall show that $P \otimes_R Q \cong \mathfrak{a}\mathfrak{b}$. Let K be the quotient field of R. Then $P \otimes_R K \cong K$ and the injection : $P \mapsto P \otimes_R K$ identifies P with a fractional ideal of R. This is how we obtained the isomorphism $P \cong \mathfrak{a}$, $Q \cong \mathfrak{b}$. Note that $P \otimes_R Q \mapsto P \otimes_R Q \otimes_R K$ is still injective since $P \otimes_R Q$ is torsion free, being projective. Also $P \otimes_R Q \otimes_R K \cong (P \otimes_R K) \otimes_R (Q \otimes_R K)$ and the homomorphism $P \otimes_R Q \mapsto P \otimes_R Q \otimes_R K$ can be identified with the tensor product of the homomorphisms $P \mapsto P \otimes_R K$, $Q \mapsto Q \otimes_R K$, i.e with the homomorphism $P \otimes_R Q \mapsto (P \otimes_R K) \otimes_R (Q \otimes_R K)$. If we identify $P \otimes_R K$ with K and $Q \otimes_R K$ also with K, then noting that $K \otimes_R K \cong K$, we get an injection $P \otimes_R Q \mapsto K$. Since the map $K \otimes_R K \mapsto K$ is the product homomorphism we get that image of $P \otimes_R Q$ can be identified with $\mathfrak{a}\mathfrak{b}$.

Let $\bar{\mathfrak{a}}$ denote the image of $\mathfrak{a}$ in the ideal class group of R. Define a map : Pic R $\xmapsto{\theta}$ Ideal class group of R as follows:

$\theta([P]) = \bar{\mathfrak{a}}$ if $P \cong \mathfrak{a}$, $\mathfrak{a}$ fractional ideal of R. Clearly this map is well defined. By what precedes we find that $\theta([P][Q]) = \theta([P \otimes_R Q]) = \overline{\mathfrak{a}\cdot\mathfrak{b}} = \bar{\mathfrak{a}}\cdot\bar{\mathfrak{b}}$, where $\mathfrak{a}$, $\mathfrak{b}$ are fractional ideals and $P \cong \mathfrak{a}$, $Q = \mathfrak{b}$. Also $\theta([P])\cdot\theta([Q]) = \bar{\mathfrak{a}}\,\bar{\mathfrak{b}}$. Hence $\theta([P][Q]) = \theta([P])\,\theta([Q])$, i.e θ is a homomorphism of groups. This homomorphism is surjective, for let $\mathfrak{a}$ be any nonzero fractional ideal of R. Then $\mathfrak{a}.(R:\mathfrak{a}) = R$. Notice that $(R:\mathfrak{a}) \cong \mathrm{Hom}_R(\mathfrak{a}, R)$. There exist $a_i \in \mathfrak{a}$ and $b_i \in (R:\mathfrak{a})$ such that $1 = \sum a_i b_i$. Let $b_i \longleftrightarrow f_i$ in $\mathrm{Hom}_R(\mathfrak{a}, R)$, then for any $x \in \mathfrak{a}$, $f_i(x) = b_i x$ so that $x = \sum a_i(b_i x) = \sum a_i f_i(x)$. This implies by projectivity criterion that $\mathfrak{a}$ is projective.

For any prime ideal $\mathfrak{p}$ of R, $\mathfrak{a}_{\mathfrak{p}} \otimes \mathrm{Hom}_R(\mathfrak{a}_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong R_{\mathfrak{p}}$ hence $\mathfrak{a}_{\mathfrak{p}}$ is free of rank 1.

θ is injective since $\theta([P]) = 1$ implies $\mathfrak{a}$ is principal where $P \cong \mathfrak{a}$, but then P is free; hence $[P]$ will be the unity element of Pic R. Hence the theorem.

Situation: Let R be a ring. In the rest of this section S denotes the multiplicative set of non-zero divisors of R. Let $\widehat{R} = R_S$.

Remark (3.1.1) The canonical homomorphism from $R \mapsto \widehat{R} = R_S$ is injective.

Definition: Let M be any R-submodule of R. Then M is said to be non-degenerate if $\widetilde{R} M = \widetilde{R}$.

Example: If R happens to be a domain in which case $\widehat{R}$ = quotient field of R, then non-degenerate $\Leftrightarrow$ non-zero.

Proposition (3.1.1) Let M be any R-submodule of R. The following are equivalent.

(i) M is non-degenerate.

(ii) $M \cap S \neq \phi$.

(iii) If $M \overset{f}{\hookrightarrow} \widetilde{R}$ is the canonical injection, then $S^{-1}M \mapsto \widetilde{R}$ is an isomorphism.

Proof: (i) $\Rightarrow$ (ii) given $\widetilde{R} M = \widetilde{R}$. So $1 = \sum_{\text{finite}} \frac{a_i}{s} m_i$ where $a_i \in R$, $m_i \in M$. Then $1 = \frac{1}{s} \sum a_i m_i = \frac{1}{s} a$, where $a = \sum a_i m_i \in M$. So $a = s \in M \cap S$. That is $M \cap S \neq \phi$.

(ii) $\Rightarrow$ (iii) Let $s \in M \cap S$. Then $1 = s/s \in S^{-1}M$. So image of s/s in R is 1. So $S^{-1}M \mapsto \widetilde{R}$ is surjective. Also it is injective. Therefore $S^{-1}M \cong R$.

(iii) $\Rightarrow$ (i) $\widetilde{R}$ = Image of $S^{-1}M$. Let $S^{-1}M \xrightarrow{S^{-1}f} S^{-1}\widetilde{R}$ be the induced homomorphism from $M \xrightarrow{f} \widetilde{R}$. Then we have Image of $S^{-1}M = \widetilde{R} M$. Let $x \in$ Image of $S^{-1}M$. So $x = (S^{-1}f)(\theta)$ where $\theta \in S^{-1}M$. So $\theta = \frac{m}{s}$ for $s \in S$, $m \in M$. That is $x = (S^{-1}f)(\frac{m}{s}) = \frac{f(m)}{s}$ (by definition of $S^{-1}f$) $= \frac{m}{s} \in \widetilde{R} M$.

If $y \in \tilde{R}M$, then $y = \sum_{\text{finite}} \frac{a_i}{s} m_i$, where $a_i \in R$, $s \in S$, $m_i \in M$. Since $\sum a_i m_i \in M$, $y = \frac{1}{s} \sum a_i m_i \in$ Image of $(S^{-1}M)$. Therefore $\tilde{R}$ = Image of $S^{-1}M = \tilde{R}M$. M is non degenerate.

Definition: Let M be any R-submodule of $\tilde{R}$. Then M is said to be invertible if there exists another R-submodule N of $\tilde{R}$ such that $MN = R$.

Remark (3.1.2)(a) If M is invertible, then it is non-degenerate.

Since M is invertible so there exists N, R-submodule of $\tilde{R}$ such that $MN = R$. For $\tilde{R}M = \tilde{R}(\tilde{R}M) \supseteq \tilde{R}(NM) = \tilde{R}R = \tilde{R}$. Also $\tilde{R}M \subseteq \tilde{R}$. Therefore $\tilde{R}M = \tilde{R}$ and hence M is non-degenerate

Proposition (3.1.2) Let M be non degenerate. Let N be any R-submodule of $\tilde{R}$. Define $^{£}(N:M) = \{ a \in \tilde{R}$ such that $aM \subseteq N \}$. Then there exists a canonical R-isomorphism from $(N:M) \xrightarrow{\theta} \mathrm{Hom}_R(M,N)$.

Proof: Let $a \in (N:M)$. Define $\theta(a)$ as follows:

$\theta(a)(m) = am$ for $m \in M$.

θ is injective: Let $s \in S \cap M$ (possible since M is non-degenerate). Suppose $\theta(a) = 0$, then $\theta(a)(s) = 0$ that is $as = 0$. Since s is a unit in $\tilde{R}$, $a = 0$.

θ is surjective: Let $f \in \mathrm{Hom}_R(M,N)$ be given. Let $s \in S \cap M$. Define $a = \frac{f(s)}{s}$. If $x \in M$, then there exists $t \in S$ such that $tx \in R$. Look at $f(stx) = stf(x)$. $s, t \in R$. Also $f(stx) = txf(s)$, $tx \in R$, $s \in M$. So $f(x) = \frac{tx}{st} f(s) = \frac{x}{s} f(s)$ (since $t \in S$). Therefore $f(x) = x \frac{f(s)}{s} = xa$. So $a \in (N:M)$ since $f(x) \in N$. Also $\theta(a) = f$. Hence θ is surjective. Therefore θ is an isomorphism.

Remark(3.1.2)(b): If M, N are R-submodules of $\tilde{R}$ such that $M.N = R$, then it is easily seen that $N = (R:M)$.

Proposition (3.1.3) Suppose M is a non-degenerate R-submodule of $\tilde{R}$. Then the following are equivalent.

(i) M is invertible

(ii) M is a finitely generated projective R-module of rank 1.

Proof: (i) $\Rightarrow$ (ii): $\exists$ N, R-submodule of $\tilde{R}$ such that $MN = R$, that is $\exists$ $n_i \in N$, $m_i \in M$ such that $\sum_{\text{finite}} m_i n_i = 1$. Define $f_i \in \mathrm{Hom}_R(M,R)$ as follows $f_i(x) = n_i x$ for $x \in M$. f_i is a R-homomorphism. Also $x = m_i n_i x = m_i f_i(x)$ $\forall$ $x \in M$. This shows that M is a finitely generated projective R-module. Suppose p is a prime ideal of R. Let $r_p = \mathrm{rank}_{R_p}$ We have to show that $r_p = 1$ $\forall$ $p \in \mathrm{spec}\, R$. Let S^1 = Image of S in R_p, T = Image of $R-p$ in $S^{-1}R = \tilde{R}$. Then clearly $T^{-1}(S^{-1}R) = S^{1^{-1}}(R_p)$ and $T^{-1}(S^{-1}M) = S^{1^{-1}}(M_p)$. Therefore $T^{-1}(S^{1^{-1}}R_p) = T^{-1}(T^{-1}(S^{-1}R)) = T^{-1}(S^{-1}R)$ and $T^{-1}(S^{1^{-1}}M_p)$ equals to $T^{-1}(T^{-1}(S^{-1}M)) = T^{-1}(S^{-1}M)$. Since M is non-degenerate, by the proposition (3.1.1) $S^{-1}M = \tilde{R} = S^{-1}R$. So $T^{-1}(S^{1^{-1}}M_p)$ is free of rank 1 over $T^{-1}(S^{1^{-1}}R_p)$.* Hence $r_p = 1$. Therefore M is a finitely generated projective module of rank 1.

(ii) $\Rightarrow$ (i): $\exists$ $f_i \in \mathrm{Hom}_R(M,R)$, $m_i \in M$ such that $\forall$ $x \in M$, we have $x = \sum m_i f_i(x)$, where all but finitely many $f_i(x) = 0$.** Since M is non-degenerate, by the proposition (3.1.2) $\theta : (R:M) \hookrightarrow \mathrm{Hom}_R(M,R)$ is an isomorphism. So $\exists$ $a_i \in (R:M)$ such that $\theta(a_i) = f_i$, that is $f_i(x) = a_i x$. Hence $x = \sum a_i m_i x$. That is $[\sum(a_i m_i) - 1]x = 0$. Choose $x \in S \cap M$ (possible since M is non degenerate). In this case, $\sum a_i m_i = 1$. Let N = R-submodule of $\tilde{R}$ generated by the

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*This is because $T^{-1}(S^{1^{-1}}M_p) = T^{-1}(S^{-1}M) = T^{-1}(S^{-1}R) = T^{-1}(S^{1^{-1}}R_p)$

** in fact we can choose f_i's to be finite in number.

a_i's. Then $MN = R$, that is M is invertible.

Note: In the previous proposition it is clear that the part (ii) $\Rightarrow$ (i) follows only from the fact that M is finitely generated projective.

Proposition (3.1.4) Suppose M is an invertible R-submodule of $\tilde{R}$. Let N be any other R-submodule of R. Then the canonical map $M \otimes_R N \mapsto MN$ is an R-isomorphism.

Proof: Tensoring the canonical injection $N \mapsto \tilde{R}$ with M over R we get an injection from $N \otimes_R M \mapsto \tilde{R} \otimes_R M$ (Since M is projective, M is flat). Since M is non-degenerate $\tilde{R} \otimes_R M = S^{-1}M \cong S^{-1}R$. So $N \otimes_R M \mapsto S^{-1}R$ is injective. Image is NM, therefore $M \otimes_R N \mapsto MN$ is an isomorphism.

Proposition (3.1.5) Suppose A and B are rings. Let $f: A \mapsto B$ be a ring homomorphism. If P is a finitely generated projective A-module of rank 1, then $P \otimes_A B$ is a projective, finitely generated B-module of rank 1.

Proof: $P \otimes_A B$ is B projective and finitely generated. Let q be any prime ideal of B. Let p be its contraction to A. Then p is a prime ideal in A. Look at $(P \otimes_A B) \otimes_B B_q = P \otimes_A B_q = P \otimes_A A_p \otimes_{A_p} B_q$. $P \otimes_A A_p$ is a free A_p-module of rank 1. Hence $(P \otimes_A A_p) \otimes_{A_p} B_q$ is a free B_q-module of rank 1. That is $(P \otimes_A B) \otimes_B B_q$ is a free B_q-module of rank 1. That is $P \otimes_A B$ is a projective B-module of rank 1.

Remark (3.1.8) Let A,B be of as before. We have a group homomorphism from Pic A $\longmapsto$ Pic B taking $[P]$ to $[P \otimes_A B]$. Since

$$[P \otimes_A Q \otimes_A B] = [P \otimes_A B \otimes_A Q \otimes_A B] = [P \otimes_A B][Q \otimes_A B]$$

Situation: R, S, $\widetilde{R}$ are as before. In the rest of this section R^* will denote the group of invertible elements of R. The invertible R-submodules of $\widetilde{R}$ form a group under multiplication with the identity being R. Let us denote this by K.

Theorem (3.1.2) group homomorphisms making the following sequence exact, $1 \mapsto R^* \overset{\theta}{\mapsto} \widetilde{R}^* \overset{\phi}{\mapsto} K \overset{\psi}{\mapsto} \text{Pic } R \overset{\eta}{\to} \text{Pic } \widetilde{R}$.

Proof: Now $\theta : R^* \mapsto \widetilde{R}^*$ is induced from $R \mapsto \widetilde{R}$. The mapping ϕ is defined as follows: Let $a \in \widetilde{R}^*$, then $\phi(a) = Ra$. Ra is invertible since $Ra\,Ra^{-1} = R$. ψ is defined as follows: Suppose M is an invertible R-submodule of R. Then M is finitely generated projective module of rank 1. So $\psi(M) = [M]$. $\psi(MN) = [MN]$. But we have an isomorphism MN to $M \otimes_R N$. Therefore $\psi(MN) = [M \times N] = [M][N] = \psi(M)\,\psi(N)$. And η : Pic R $\mapsto$ Pic $\widetilde{R}$, is the canonical homomorphism. Clearly $1 \mapsto R^* \longmapsto \widetilde{R}^*$ is exact.

Exactness at R^*: $(R^* \longmapsto \widetilde{R}^* \longmapsto K)$.

$(\phi \cdot \theta)(a) = \phi(\theta(a)) = \phi(a) = Ra = R$. Also let $a \in \widetilde{R}^*$ such that $\phi(a) = 0$, that is $Ra = R$. This implies $\exists\ b \in R$

such that $ba = 1$ and $a = 1a \in R$. Therefore a is a unit in R. That is $a \in R^*$.

Exactness at K: $(\widetilde{R}^* \mapsto K \mapsto \text{Pic } R)$

Let $b \in \widetilde{R}^*$, $(\psi\cdot\phi)(b) = \psi(\phi(b)) = \psi(Rb) = [Rb]$. Since $b \in \widetilde{R}^*$, $R \mapsto Rb$ $(r \mapsto rb)$ is an R-isomorphism. So $(\psi\circ\phi)(b) = [R]$. Let M be invertible R-submodule of R which is isomorphic to R. Let the isomorphism be $R \mapsto M$ $(1 \mapsto b)$. Then $M = Rb$ and Rb is invertible module. Hence Rb is non degenerate. That is $\widetilde{R}\cdot Rb = \widetilde{R}$ which implies $\widetilde{R}b = \widetilde{R}$, that is there exists $c \in \widetilde{R}$ such that $cb = 1$. Therefore $b \in \widetilde{R}^*$.

Exactness at Pic R: $(K \xrightarrow{\psi} \text{Pic } R \xrightarrow{\eta} \text{Pic } \widetilde{R}.)$

Take M invertible R-submodule of $\widetilde{R}$.

$(\eta\cdot\psi)(M) = \eta[M] = M \otimes_R \widetilde{R} = M\widetilde{R} = \widetilde{R}$ because of the non degeneracy of M.

Let $[P] \in \text{Pic } R$ and $P \otimes_R \widetilde{R} \cong \widetilde{R}$. $\exists$ R-injection from $R \mapsto \widetilde{R}$. Since P is projective, it is flat. Therefore $P \cong R \otimes_R P \mapsto \widetilde{R} \otimes_R P$ is injection. That is $P \mapsto \widetilde{R} \otimes_R P$ is an injection. That is $\exists$ injection $P \mapsto \widetilde{R}$. So P can be identified with R-submodule of $\widetilde{R}$.

Claim: This submodule is non-degenerate.

Consider the canonical injection from $P \mapsto \widetilde{R}$. Then $S^{-1}P = P \otimes_R \widetilde{R} \mapsto \widetilde{R}$ is injective and image is $\widetilde{R}P$. But $P \otimes_R \widetilde{R} \cong \widetilde{R}$ and so $P\cdot\widetilde{R} = \widetilde{R}$ and P is non-degenerate. Since P is finitely generated projective of rank 1 and non-degenerate, by the proposition (3.1.3), it is invertible.

Consequences of the previous Theorem:

Definition: The co-kernel of the homomorphism $\widetilde{R}^{*} \longmapsto K$, is called the group of classes of invertible R-submodules of $\widetilde{R}$. In case R is a domain, it is called 'ideal class group'. If Pic $\widetilde{R} = 0$, then Pic R $\simeq$ group of classes of invertible R-submodule of $\widetilde{R}$.

Remark (3.1.4): If R is a domain, then Pic $\widetilde{R} = 0$. Clearly in this case $\widetilde{R}$ = quotient field of R and so Pic $\widetilde{R} = 0$.

Remark (3.1.5): If R is a noetherian ring, then Pic $\widetilde{R} = 0$.

Proof: In this case R will be semi-local.

Let $(0) = \bigcap_{i=1}^{n} q_i$ be a reduced primary decomposition of (0). Let p_i be the associated prime ideals of q_i $\forall\, i = 1,2,\dots,n$. Then the set of zero divisors of R is precisely $\bigcup_{i=1}^{n} p_i$ by the theory of primary decomposition. Hence $S = (R - \bigcup_{i=1}^{n} p_i)$ is the set of non-zero divisors of R.

Now $\exists$ a one to one correspondence between the prime ideals of $\widetilde{R}$ with the prime ideals of R, which do not meet S and maximal ideals of $\widetilde{R}$ correspond to prime ideals of R which are maximal with respect to the property of not meeting S and the latter are found among the prime ideals $p_1, p_2, \dots, p_n$. Hence $\widetilde{R}$ is semi local. Since finitely generated projective modules of constant rank over semilocal rings are free, we get Pic $\widetilde{R} = 0$.

Remark (3.1.6): From the Proposition (2.1.5) we know that Pic R = 0 for R any U.F.D. We can observe that this means the ideal class group of an U.F.D is trivial.

Example: Let R be the ring of algebraic integers in the number field $Q(\sqrt{2})$. Then one can show that R is a U.F.D. Hence the class number of K i.e. the order of the ideal class group of R is 1.

S2. Grothendieck group of a ring R:

Definition : Let C = category of all finitely generated projective R-modules. Let X = free abelian group on the generators (P) where (P) denotes the isomorphism class of the finitely generated projective R-module P. Let Y = subgroup of X with generators $(P \oplus Q) - (P) - (Q)$ where $P \in C$, $Q \in C$. Then the group $\frac{X}{Y}$ is known as grothendieck group of R and is denoted by $K_0(R)$. We let $[P]$ to denote the coset of (P) in $\frac{X}{Y}$ where $P \in C$.

Remark (3.2.1): Any element of $K_0(R)$ can be written as $[P] - [Q]$ or as $[P] - [R^n]$ where $P, Q \in C$ and n positive integer.

Proof: Any element of X has the form $\sum n_i(P_i)$ where $n_i \in \mathbb{Z}$. In $K_0(R)$, the image of this element will look like $\sum n_i[P_i]$. Note that if $P, Q \in C$, then $[P \oplus Q] = [P]$ $[Q]$. If n is a positive integer, then $n[P] = [P \oplus P \oplus \dots \oplus P] =$ [Some projective module]. If $n = -m$, m positive integer, then $n[P] = -m[P] = -$ [Projective module]. Define $0[P] = 0$. Using these $\sum n_i[P_i]$ can be written as

$[Q_1] + [Q_2] + \dots + [Q_t] - [L_1] + [L_2] + \dots + [L_s]$ where $Q_i, L_j \in C$. Since $\sum_{i=1}^{t} [Q_i] = [\bigoplus_{i=1}^{t} Q_i]$ and $\sum_{j=1}^{s} [L_j] = [\bigoplus_{j=1}^{s} L_j]$, we get $\sum n_i [P_i] = [\lambda] - [\Omega]$ where $\lambda, \Omega \in C$ which proves the first part of the remark.

__2nd part:__ Choose a finitely generated projective module Ω' such that $\Omega \oplus \Omega'$ is free, say $\Omega \oplus \Omega' = R^n$, n some positive integer. Then $[\lambda] - [\Omega] = [\lambda] + [\Omega'] - [\Omega] - [\Omega']$

$= [\lambda \oplus \Omega'] - [\Omega \oplus \Omega']$.

$= [T] - [R^n]$ where $T \in C$.

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§3. __Appendix:__

__Definition:__ A ring R is called graded ring if it is a direct sum of additive subgroups R_q of R satisfying the relation $R_q R_{q'} \subseteq R_{q+q^1}$, q, q^1 ranging over the set $\mathbb{Z}$. An element of R is said to be homogenous of degree q if it belongs to R_q and is different from 0. So in a graded ring R we have a direct decomposition $R = \sum_{q=-\infty}^{\infty} R_q$.

If S is a subring of R we say that S is graded subring of R if S is the direct sum of its subgroups $S_q = S \cap R_q$, that is if $S = \sum S_q$.

If R and R^1 are two graded rings $R = \sum_q R_q$, $R^1 = \sum_q R^1_q$, a homomorphism ϕ of R into R^1 is said to be homogenous of degree s if $\phi(R_q) \subseteq R^1_{q+s}$, for all q.

__Definition:__ Let R be a graded ring, say $R = \sum_{q=-\infty}^{\infty} R_q$, where the sum is direct. A graded module M over R is a module M, together with a direct sum decomposition $M = \sum_{q=-\infty}^{\infty} M_q$ of the additive group of M, such that, for every pair of integers (q,r), we have $R_r M_q \subseteq M_{q+r}$. Nonzero elements of M_q are said to be homogenous of degree q. Given any element $x \in M$, we can write in a unique way $x = \sum_{-\infty}^{\infty} x_q$ where $x_q \in M_q$ and where all but finitely many x_i's are 0.

A submodule N of M is said to be homogenous if the relation $x \in N$ implies that all the homogenous components of x belong to N.

Given two graded R-modules $M = \sum_{q} M_q$, $M^1 = \sum_{q_1} M^1_q$ and an integer d, a homomorphism θ of M into M^1 is said to be homogenous of degree d if $\theta(M_q) \subseteq M^1_{d+q}$ for every q.

__Example:__ Let R be a ring. The polynomial ring $R[X_1, X_2, \ldots, X_n]$ in n in determinates is a graded ring. Every element F in $R[X_1, X_2, \ldots, X_n]$ can be written as $F = F_0 + F_1 + \ldots + F_j + \ldots$ where F_j is either 0 or a form of degree j.

__Lemma:__ (Nakayama's lemma for Graded modules). Let M be any graded module over $R = \sum_{i=0}^{\infty} R_i$. Let $I = \sum_{i \geq 0} R_i$, then $M = IM \Rightarrow M = 0$.

__Proof:__ We have $M = IM$ implies $M = IM = I^2M = \ldots\ldots$, that is $M = \bigcap_{j \geq 0} I^j M$ (defining $I^0 = R$). Note that $I^j M \subseteq M_j + M_{j+1} + \ldots$ where $M = \bigoplus_{i=0}^{\infty} M_i$ is the gradation of M. Hence $M = \bigcap_{j \geq 0} I^j M \subseteq \bigcap_{j \geq 0} (M_j + M_{j+1} + \ldots)$. Then it will turn out that $\bigcap_{j \geq 0} (M_j + M_{j+1} + \ldots) = (0)$. If it is not (0), then $\exists\ x \neq 0$, x belonging to the intersection.

$x = x_{i_1} + x_{i_2} + \ldots + x_{i_m}$ which implies $x \notin M_{i_m+1} + M_{i_m+2} + \ldots$

and which is a contradiction. Hence $M = 0$.

<u>Remark (3.3.1)</u>: R as in lemma. Let M and N be any graded R-module and N graded R-submodule of M, then if $N+IM = M$ we get $N = M$.

<u>Proof</u>: Now $I\frac{M}{N} = \frac{IM + N}{N} = \frac{M}{N}$ which implies $\frac{M}{M} = 0$ by Lemma. That is $M = N$.

<u>Definition</u>: R is a graded ring (gradation by non-negative integers). A graded module over R is said to be graded free if it has a basis of homogenous elements.

<u>Definition</u>: A graded module P is said to be graded projective if $\exists$ another graded module Q such that $P \oplus Q$ (with obvious gradation) is graded free.

<u>Proposition (3.3.1)</u> A graded module P over R is graded projective iff it is projective in the usual sense.

<u>Proof</u>: By the definition of graded projective module if it is clear that graded projective $\Rightarrow$ projective. Conversely, suppose P is projective in the usual sense. Choose a set $\{x_i\}_{i\in I}$ of homogenous generators for P. Let F be a free module on a basis $(e_i)_{i \in I}$, that is $F = \sum_{i\in I} Re_i$. We grade F as follows $F_n = \sum_{m+\deg x_i = n} R_m e_i$ (R_m = mth homogenous components of R). Then $F = \bigoplus_{n\geq 0} F_n$. Define $f: F \mapsto P$ by $f(e_i) = x_i$. This is a graded homomorphism of degree 0. Since P is projective, $\exists$ $g: P \mapsto F$ such that $f\circ h = I_P$.*/ We define h on graded component of P and extend it by additivity. ....

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*We define a graded homomorphism $h: P \mapsto F$ to satisfy the condition $f\circ h = 1_P$.

Let $p \in P_n$, then define $h(p)$ = nth component of $g(p)$.

R-linearity of h: For $m \in R_m$, $p \in P_n$ $h(r_m p) = (m+n)$th component of $g(r_m p)$. Therefore $h(r_m p) = (m+n)$th component of $r_m g(p) = r_m$(nth component of $g(p)) = r_m h(p)$. It can be easily checked that $foh = I_p$. Enough to check it on the generators. Suppose $\deg x_i = n_i$. Then $(foh)(x_i) = f(h(x_i))$ $= f(n_i$th component of $g(x_i)) = n_i$th component of $f(g(x_i)$ (since f is of degree 0) $= n_i$th component of x_i (since $fog = I_p$). $= x_i$ (since x_i is homogenous of degree n_i). This shows that $foh = I_p$.

Proposition (3.3.2) Let R be a graded ring (gradation by non-negative integers). Assume that $R_0 = K$, a field. (For example $R = K[X \ldots., X_n]$, a polynomial ring over a field K). Let P be any finitely generated graded projective module over R. Then P is actually graded free.

Proof: Since P is finitely generated graded projective, $\exists$ another finitely generated graded projective module Q such that $P \oplus Q = F$, F is graded free. Let $I = R_1 + R_2 + R_3 + \ldots.$ of R. Then I is an ideal of R. Tensoring $P + Q$ with $\frac{R}{I}$ we get $\frac{P}{IP} \oplus \frac{Q}{IQ} = \frac{F}{IF}$. But $\frac{P}{IP}$, $\frac{Q}{IQ}$, $\frac{F}{IF}$ are graded modules over the ring $\frac{R}{I} = R_0 = K$. That is $\frac{P}{IP}$, $\frac{Q}{IQ}$, $\frac{F}{IF}$ are graded vector spaces over k. (Here K has trivial gradation $K_0 = K$, $K_i = 0$ for $i \neq 0$). Choose elements $x_i \in P$, $y_j \in Q$, $1 \leq i \leq n$, $1 \leq j \leq m$ homogenous such that the elements $\{\overline{x_i}\}_{1 \leq i \leq n}$ form a homogenous K-basis of $\frac{P}{IP}$

and such that elements $\{\bar{y}_j\}_{1 \leq j \leq m}$ form a homogenous K-basis of Q/IQ. Then clearly $\{\bar{x}_i\} \cup \{\bar{y}_j\}$ is a homogenous K-basis of $\frac{F}{IF}$. Let P^1 be the R-graded submodule of P generated by $x_1, x_2, \ldots, x_n$ and Q^1 be the submodule of Q generated by $y_1, y_2, \ldots, y_m$. Clearly $P^1+IP = P$, $Q^1+IQ = Q$. So by the Remark (3.3.1) $P^1 = P$, $Q^1 = Q$. That is $\{x_1, x_2, \ldots, x_n\}$ is a homogenous system of generators for P. Similarly $\{y_1, y_2, \ldots, y_n\}$ is a homogenous system of generators for Q. We shall show that $x_1, x_2, \ldots, x_n$ are linearly independent over R. Actually one can see that $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ is a basis for the free R-module F. Note that $\text{rank}_R F = \dim_K\left(\frac{F}{IF}\right) = n+m$. Let $\{e_1, e_2, \ldots, e_{n+m}\}$ be a homogenous R-basis of F. Clearly $x_1, x_2, \ldots, x_n, y_1, \ldots, y_m$ generates F as an R-module. Define a Map : $F \hookrightarrow F$ taking e_i to x_i for $1 \leq i \leq n$ and e_{n+j} to y_j for $1 \leq j \leq m$. This is an R-homomorphism. Also it is surjective. Hence this must be an isomorphism. That is $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ is also an R-basis of F. So in particular $\{x_1, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ are linearly independent. Therefore P is R-graded free.

CHAPTER 4

Stably free modules, regular rings and Hilbert's Syzygy Theorem

§1. In this Chapter which is comprised of one section only we shall investigate a category of modules called stably free modules and apply it to prove a general version of the classical theorem of Hilbert on syzygies.

Definition: A R-module P is said to be stably free if $\exists$ positive integers r, s such that $P \oplus R^r \cong R^s$.

Example: All free modules are stably free.

Remark (4.1.1) All projective modules are not stably free.

Proof: Suppose P is a projective module which is stably free, that is $\exists$ r, s $\mathbb{N}$ such that $P \oplus R^r \cong R^s$ which implies rank $P = (s-r)$. That is, P has constant rank. This shows that any projective modules of nonconstant rank will serve as an example.

Consider $R = K_1 \times K_2$ (K_i = fields). Then $P = K_1 \times (0)$ is projective, but not stably free because P does not have constant rank.

Proposition (4.1.1) Let P be a finitely generated projective R-module. Then P is stably free iff $[P] \in [R]\mathbb{Z}$, where $[R]\mathbb{Z}$ means the subgroup of K_0R generated by $[R]$.

Proof: We use the notations of §2 of chapter 3.

Suppose P is stably free. Then there exists r and positive integers such that $P \oplus R^r \cong R^s$. Hence $[P \oplus R^r] = [R^s]$. Also $[P \oplus R^r] = [P] \oplus [R^r]$. So

$[P] + [R^r] = [R^s]$ which implies $[P] + r[R] = s[R]$, that is $[P] = (s-r)\ [R] \in [R]Z$.

Next suppose $[P] \in [R]Z$. Write $[P] = n[R]$, that is $(P) - n(R) \in Y$. Write $(P) - n(R) = \sum n_i \{(P_i + Q_i) - (P_i) - (Q_i)\}$. Without loss of generality we can take $n_i = \pm 1$. Then by renaming, $(P) - n(R) = \sum_{i=1}^{r}\{(P_i + Q_i) - (P_i) - (Q_i)\} - \sum_{j=r+1}^{s}\{(P_j + Q_j) - (P_j) - (Q_j)\}$ Therefore, $(P) + \sum_{j=r+1}^{s}(P_j + Q_j) + \sum_{i=1}^{r}(P_i) + \sum_{i=1}^{r}(Q_i)$ $= n(R) + \sum_{i=1}^{r}(P_i + Q_i) + \sum_{j=r+1}^{s}(P_j) + \sum_{j=r+1}^{s}(Q_j)$. Since X is free abelian group on generators (P), we get if $\sum(P_i) = \sum(Q_i)$, then $\oplus P_i \cong \oplus Q_i$. So we get $P \oplus \sum_{j=r+1}^{s} P_j \oplus \sum_{j=r+1}^{s} Q_j \oplus \sum_{i=1}^{r} P_i \oplus \sum_{i=1}^{r} Q_i$ $= R^n \oplus \sum_{i=1}^{r} P_i \oplus \sum_{i=1}^{r} Q_i \oplus \sum_{j=r+1}^{s} P_j \oplus \sum_{j=r+1}^{s} Q_j$. Write $\sum_{i=1}^{s} P_i = L$, $\sum_{i=1}^{s} Q_i = M$, we get $P \oplus L \oplus M = R^n \oplus L \oplus M$. $L \oplus M$ is projective $\Rightarrow \exists$ finitely generated projective module T such that $L \oplus M \oplus T = R^k$, $k \geqslant 0$. So $P \oplus R^k = R^n \oplus R^k$ which implies $P \oplus R^k = R^{n+k}$. That is P is stably free.

<u>Proposition (4.1.2)</u> Suppose $P \in C$, admits a resolution by finitely generated free modules, then P is stably free.

<u>Proof:</u> Suppose the resolution be

$0 \mapsto F_n \mapsto F_{n-1} \mapsto \ldots\ldots \mapsto F_1 \mapsto F_0 \mapsto P \mapsto 0$. We get short exact sequences: $0 \mapsto L_1 \mapsto F_0 \mapsto P \mapsto 0$,

$0 \mapsto L_2 \mapsto F_1 \mapsto L_1 \mapsto 0, \ldots\ldots\ldots\ 0 \mapsto 0 \mapsto F_n \mapsto L_n \mapsto 0$.

Now P projective $\Rightarrow L_1$ projective, $\Longrightarrow L_2$ projective etc.

So $F_0 \cong L_1 \oplus P$, $F_1 \cong L_2 \oplus L_1, \ldots\ldots$ $F_{n-1} \cong L_n \oplus L_{n-1}$, $F_n \cong L_n$. Therefore $[F_0] = [P] \oplus [L_1]$, $[F_1] = [L_1] \oplus [L_2]$ ......, $[F_n] = [L_n]$. Taking alternating sum we get $[F_0] - [F_1] + [F_2] - \ldots + (-1)^{n-1}[F_n] = [P]$. This shows that $[P] = n[R]$ for some $n \in \mathbb{Z}$, that is P is stably free.

Definition: Suppose R is a ring. Then R is regular if

(i) R is noetherian.

(ii) Every finitely generated modules M over R admits a resolution of the type $0 \mapsto P_n \longmapsto P_{n-1} \mapsto \ldots \mapsto P_0 \mapsto M \mapsto 0$, where P_i are finitely generated projective.

Example: EvEry field is regular.

Theorem (4.1.1): If R is regular, then $R[X_1, X_2, \ldots, X_n]$ is regular for every $n \in \mathbb{N}$.

Lemma (i): Suppose R is a noetherian ring. X is an indeterminate over R. Let N_0 be a finitely generated R-module and M be a $R[X]$- submodule of $R[X] \otimes_R N_0 = N$. Then $\exists$ finitely generated $R[X]$- modules A and B of the type $A = R[X] \otimes_R A_0$ and $B = R[X] \otimes_R B_0$ where A_0 , B_0 are finitely generated R-modules and homomorphisms $B \mapsto A$ and $A \mapsto M$ such that $0 \mapsto B \mapsto A \mapsto M \mapsto 0$ is exact.

Proof: Considering N_0 as a R-submodule of M, define $N_t = \sum_{i=0}^{t} X^i N_0$. Any element $a \in N_t$ is of the form $a = a_0 + Xa_1 + X^2 a_2 + \ldots + X^t a_t$ where $a_i \in N_0$.

Also $N_0 \subseteq N_1 \subseteq N_2 \subseteq \dots\dots$ Also it can be seen that $N = \bigcup_{t \geq 0} N_t$. Take $x \in N$, then $x = 1 \otimes a_0 + X \otimes a_1 + \dots + X^t \otimes a_t$ where $a_i \in N_0 \ \forall\ i$. Then $x \in N_t$ by identification of $X^i \otimes a_i$ with $X^i a_i$. Also $N_t \subseteq N \ \forall\ t$. Therefore $N = \bigcup_{t \geq 0} N_t$. Define $M_t = M \cap N_t$.

$M = M \cap N = M \cap (\bigcup_{t \geq 0} N_t) = \bigcup_{t \geq 0} (M \cap N_t) = \bigcup_{t \geq 0} M_t$ and $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots\dots$ Since N is a finitely generated $R[X]$- module and $R[X]$ is noetherian, M is a finitely generated $R[X]$- submodule. Let $m_1, m_2, \dots, m_k$ generate M as $R[X]$- module. Take n large enough so that $\{m_1, m_2, \dots, m_k\} \subseteq M_{n+1}$. Define $A = R[X] \otimes_R M_{n+1}$, $B = R[X] \otimes_R M_n$.

Define $\theta : A \mapsto M$ taking $X^t \otimes m$ to $X^t m$ and extend by linearity. Suppose $a \in M$, then $a = \sum_{i=1}^{k} f_i m_i$ where f_i's are polynomials in X. Also $\sum_{i=1}^{k} f_i m_i = \theta(\sum_{i=1}^{k} f_i \otimes m_i)$ which implies θ is surjective. Define $\varphi : B \mapsto A$ as $\varphi(X^t \ x\ m) = X^{t+1} \otimes m - X^t \otimes Xm$ and extend by linearity. $m \in M_n \subseteq M_{n+1}$ and $m \in M_n$ implies $Xm \in M_{n+1}$. So φ is well-defined. φ and θ are $R[X]$- homomorphisms. We need to show that $0 \mapsto B \longmapsto A \mapsto M \mapsto 0$ is exact. Now $(\theta \cdot \varphi)(X^t \ x\ m) = \theta(X^{t+1} \otimes m - X^t \otimes Xm) = X^{t+1}m - X^t xm = 0$. Therefore $\theta \cdot \varphi = 0$.

φ is injective: Suppose $\varphi(X^t \otimes a_t + X^{t-1} \otimes a_{t-1} + \dots + 1 \otimes a_0) = 0$ where $a_i \in M_n$ and $a_t \neq 0$. This shows that

$$(X^{t+1} \otimes a_t - X^t \otimes xa_t) + (X^t \otimes a_{t-1} - X^{t-1} \otimes xa_{t-1}) + \dots\dots + (X \otimes a_0 - 1 \otimes xa_0) = 0.$$

That is $X^{t+1} \otimes a_t + X^t \otimes (a_{t-1} - Xa_t) + \ldots\ldots + = 0$. Which implies $a_t = 0$, $a_{t-1} - Xa_t = 0$ e.t.c. Therefore $a_t = 0$ is a contradiction. So ϕ is injective.

Exactness at A: Since $\theta \phi = 0$. So image $\phi \subseteq$ Ker θ . Let $Z \in A$ such that $\theta(z) = 0$.
$Z = X^t \otimes a_t + X^{t-1} \otimes a_{t-1} + \ldots\ldots + X \otimes a_1 + 1 \otimes a_0$ where $a_i \in M_{n+1}$. $a_i \in M_{n+1}$ implies $a_i = \lambda_{i,n+1} X^{n+1} + \lambda_{i,n}X^n + \ldots\ldots + \lambda_{i,0}$ where $\lambda_{i,n+1}$ e.t.c belong to N_0. Then $\theta(Z) = \lambda_{t,n+1} X^{n+1+t}$ + lower degree terms = 0 . So $\lambda_{t,n+1} = 0$. Therefore $a_t = \lambda_{t,n}X^n + \lambda_{t,n-1} X^{n-1} + \ldots\ldots + \lambda_{t,0} \in M_n$.
Call $Y = Z - \phi(X^{t-1} x\, a_t)$. $\theta(Y) = \theta(Z) - 0 = 0$.
Therefore $Y = x^t \otimes a_t + X^{t-1} \otimes a_{t-1} + \ldots\ldots + 1 \otimes a_0 - X^t \otimes a_t + X^{t-1} \otimes Xa_t = X^{t-1} \otimes a_{t-1} + \ldots\ldots + 1 \otimes a_0 + X^{t-1} \otimes Xa_t$.
We prove Ker $\theta \subseteq$ Image ϕ by induction on t. If $t = 0$, then $Z = 1 \otimes a_0$. $\theta(Z) = 0$ implies $a_0 = 0$ and hence $Z=0$. So $Z = \phi(0)$. Let us assume the result for all $n < t$. Then by induction $Y = \phi(\tau)$, $\tau \in B$. $Z = Y + \phi(X^{t-1} \otimes a_t) = \phi(\tau) + \phi(X^{t-1} \otimes a_t) = \phi(\tau + X^{t-1} \otimes a_t)$. That is $Z \in$ Image ϕ . So $0 \mapsto B \mapsto A \mapsto M \mapsto 0$ is exact.

Lemma (ii) Let R be a regular ring. Let M be any finitely generated $R[X]$- module. Then $\exists$ an exact sequence of the following type:
$0 \mapsto P_n \mapsto P_{n-1} \mapsto \ldots\ldots \mapsto P_1 \mapsto P_0 \mapsto M \mapsto 0$ where P_i's are finitely generated projective $R[X]$- modules of the

form $P_i = R[X] \otimes_R Q_i$ where Q_i's are finitely generated projective R-modules.

Proof: M is a finitely generated $R[X]$-module implies $\exists$ n belonging to N such that $R[X]^n \mapsto M \mapsto 0$ is exact. Let $\tilde{M}$ be the kernel of $R[X]^n \longmapsto M$. Then $0 \mapsto \tilde{M} \longmapsto R[X]^n \longmapsto M \mapsto 0$ is exact. $\tilde{M}$ is a finitely generated $R[X]$-submodule of $R[X] \otimes_R R^m$. So applying previous lemma we get $\exists$ $A = R[X] \otimes_R A_0$, $B = R[X] \otimes_R B_0$ where A_0, B_0 finitely generated R-modules such that $0 \mapsto B \mapsto A \mapsto M \mapsto 0$ is exact. Since R is regular, A_0, B_0 finitely generated R-modules, we get exact sequences of the form $0 \mapsto X_s \mapsto X_{s-1} \mapsto \dots\dots \mapsto X_0 \mapsto A_0 \mapsto 0$

$0 \mapsto Y_k \mapsto Y_{k-1} \mapsto \dots\dots \mapsto Y_0 \mapsto B_0 \mapsto 0$ where X_i and Y_j are finitely generated projective R-modules. $R[X]$ is a flat R-module so we get exact sequences of the form

$0 \mapsto X_s \otimes_R R[X] \mapsto X_{s-1} \otimes_R R[X] \mapsto \dots\dots \mapsto A \mapsto 0$

and $0 \mapsto Y_k \otimes_R R[X] \mapsto Y_{k-1} \otimes_R R[X] \mapsto \dots\dots \mapsto B \mapsto 0$

write $C_i = X_i \otimes_R R[X]$, $D_i = Y_i \otimes_R R[X]$, that is the above sequences look like

$$\begin{tikzcd}
0 \arrow[r, mapsto] & C_s \arrow[r, mapsto] & C_{s-1} \arrow[r, mapsto] & \dots\dots \arrow[r, mapsto] & C_1 \arrow[r, mapsto] & C_0 \arrow[r, mapsto] & A \arrow[r, mapsto] & 0 & \\
 & & & & & & & & \text{\_\_\_\_\_(*)} \\
0 \arrow[r, mapsto] & D_k \arrow[r, mapsto] & D_{k-1} \arrow[r, mapsto] & \dots\dots \arrow[r, mapsto] & D_1 \arrow[r, mapsto] & D_0 \arrow[r, mapsto] & B \arrow[r, mapsto] \arrow[uu, "\varphi"'] & 0. &
\end{tikzcd}$$

Here φ is the map $B \mapsto A$ given by the lemma. Since D_0 is projective, $\exists$ a map : $D_0 \longmapsto C_0$. Also we have a commutative diagram

$$\begin{tikzcd}
D_1 \arrow[r, mapsto] & \text{Image } D_1 \arrow[d] & \\
C_1 \arrow[r, mapsto] & \text{Image } C_1 \arrow[r, mapsto] & 0
\end{tikzcd}$$

So $\exists$ a map : $D_1 \mapsto C_1$. By this way, we get mappings $f : D_i \mapsto C_i \ \forall\ i$ and the above diagram (*) is commutative. Define $Z_k = D_{k-1} \oplus C_k$ for $k \geq 1$. $Z_0 = C_0$. Define $\eta : Z_k \mapsto Z_{k-1}$ by $\eta(d,c) = (-\partial(d), \partial(c) + f(d))$. Therefore η is $R[X]$-homomorphism.

We need to show that

$$\cdots\cdots \mapsto Z_{k+1} \mapsto Z_k \mapsto Z_{k-1} \mapsto \cdots\cdots \mapsto Z_1 \mapsto Z_0 \mapsto 0$$

is exact. It's easy to see that $\eta^2 = 0$. Take $(d,c) \in Z_k$ such that $\eta(d,c) = (0,0)$, that is

$\partial(c) + f(d) = 0$, $\partial(d) = 0$. $\partial(d) = 0$ implies $\exists$ $d^1 \in D_k$ such that $d = \partial(d^1)$. Therefore

$\partial(c) + f(d) = \partial(c) + f(\partial(d^1)) = \partial(c) + \partial(f(d^1))$

$= \partial(c + f(d^1))$. $c + f(d^1) \in C_k$, so $c + f(d^1) = \partial(c^1)$ for $c^1 \in C_{k+1}$. Consider $(-d^1, c^1)$.

$\eta(-d^1, c^1) = (-\partial(-d^1), \partial(c^1) + f(-d^1))$

$= (\partial(d^1), \partial(c^1) - f(d^1)) = (d, c)$. So the sequence is exact at each Z_i.

Next we will show that Coker $Z_1 \mapsto Z_0 (= C_0)$ = Coker $(B \mapsto A)$

$$\text{Coker } Z_1 \mapsto Z_0 = \frac{Z_0}{\text{Image } (Z_1 \mapsto Z_0)} = \frac{C_0}{\partial(C_1) + f(D_0)}$$

$$\text{Coker } B \mapsto A = \frac{A}{\text{Image } (B \mapsto A)} = \frac{A}{\text{Image } (D_0 \mapsto B \mapsto A)}$$

$$\text{Also } A = \frac{C_0}{\text{Image } (C_1 \mapsto C_0)}$$

$$\text{Coker } B \mapsto A = \frac{C_0}{\text{Image } (C_1 \mapsto C_0) + \text{Image } (D_0 \mapsto C_0 \mapsto A)}$$

$= \dfrac{C_0}{\partial(C_1) + f(D_0)}$. Therefore Cokernel of $Z_1 \mapsto Z_0$ = Cokernel of $B \mapsto A$. Since Coker $(B \mapsto A) = \widetilde{M}$ ($\because\ 0 \mapsto B \mapsto A \mapsto \widetilde{M} \mapsto 0$ is exact).

We get $\ldots\ldots \mapsto Z_n \mapsto Z_{n-1} \mapsto \ldots\ldots \mapsto Z_1 \mapsto Z_0 \mapsto \tilde{M} \mapsto 0$ is exact. We have $Z_k = D_{k-1} \oplus C_k = (R[X] \otimes_R Y_{k-1})$ $\oplus (R[X] \otimes_R X_k) = R[X] \otimes_R (Y_{k-1} \oplus Y_k)$. $Z_0 = C_0 = R[X] \otimes_R X_0$. Here X_0 and $Y_{k-1} \oplus Y_k$ are finitely generated projective R-module for every k. Therefore Lemma is proved.

Proof of the Theorem (4.1.1): The above Lemma shows that R regular $\Rightarrow R[X]$ regular $\Rightarrow R[X_1, X_2, \ldots, X_n]$ regular for every n.

Remark (4.1.2) If $R \mapsto S$ is a homomorphism of commutative rings, then $\exists$ a induced group homomorphism from $K_0(R) \mapsto K_0(S)$ taking $[P]$ to $[S \otimes_R P]$. Also if $R \mapsto S \mapsto T$ are homomorphism of rings, then the composite map $K_0(R) \mapsto K_0(S) \mapsto K_0(T)$ is induced from the composite : $R \mapsto T$ (Actually $K_0(R)$ is a functor from the categories of rings to the category of groups).

Corollary (i) Suppose R is regular, let $X_1, X_2, \ldots, X_n$ be indeterminates over R, then the homomorphism $K_0(R) \mapsto K_0(R[X_1, \ldots, X_n])$ given by $[P] \mapsto [R[X_1, \ldots, X_n] \otimes_R P]$ is an isomorphism of groups [illegible] projective R-module.

Proof: Enough to consider the case of one variable. For suppose $K_0(R) \mapsto K_0(R[X_1])$ is an isomorphism, then working with the regular ring $R[X_1]$ we get

$K_0(R[X_1]) \mapsto K_0(R[X_1,X_2])$ is an isomorphism and by the remark (4.1.2), $K_0(R) \mapsto K_0(R[X_1,X_2])$ is an isomorphism and so the result is true for n variables. So we will prove for one variable. $\exists$ a ring homomorphism: $R[X] \mapsto R$ given by $R \mapsto R$ and $X \mapsto 0$. The composite $R \overset{i^*}{\mapsto} R[X] \mapsto R$ is identity on R. Hence the composite $K_0(R) \mapsto K_0(R[X]) \mapsto K_0(R)$ is identity. That is $K_0(R) \mapsto K_0(R[X_1])$ is injective. Next we will show that $K_0(R) \mapsto K_0(R[X_1])$ is surjective. Let $[P]$ be any generator for $K_0(R[X_1])$ where P is a finitely generated projective $R[X]$- module. We will show that $[P]$ is the image of some element in $K_0(R)$. By Lemma (ii), $\exists$ an exact sequence of the following type

$0 \mapsto P_n \mapsto P_{n-1} \mapsto \cdots\cdots \mapsto P_0 \mapsto P \mapsto 0$ where P_i are finitely generated projective $R[X]$- module of the type $P_i = R[X] \otimes_R Q_i$ where Q_i's are finitely generated R-module. Then $[P] - [P_0] + [P_1] + \cdots\cdots + (-1)^{n-1}P_n = 0$ which implies $[P] = [P_0] - [P_1] + \cdots\cdots + (-1)^n[P_n]$
$= [R[X] \otimes_R Q_0] - [R[X] \otimes_R Q_1] + \cdots\cdots + (-1)^n[R[X] \otimes_R Q_n]$.
So $[P] \in$ Image of $K_0(R) \mapsto K_0(R[T])$. Hence $K_0(R) \mapsto K_0(R[X])$ is an isomorphism.

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*here i means inclusion map.

<u>**Corollary (ii)**</u> Suppose R is a regular local ring. Let $X_1, X_2, \ldots, X_n$ be indeterminate over R. Then any finitely generated projective $R[X_1, X_2, \ldots, X_n]$- module is stably free.

<u>Proof:</u> By Corollary (i) $K_0(R) \longmapsto K_0(R[X_1,\ldots,X_n])$ is an isomorphism. Now R local implies, any finitely generated projective R-module is actually free. That is, $K_0(R) \cong [R] Z$. Hence $K_0(R[X_1,X_2,\ldots,X_n])$ is also infinite cyclic. So by Proposition (4.1.1), any finitely generated projective $R[X_1,\ldots,X_n]$- module is stably free.

<u>Remark (4.1.3):</u> Suppose K is a field ; $X_1,X_2,\ldots,X_n$ are n indeterminates. Then K is regular and local. by Corollary (ii) , any finitely generated projective $K[X_1,\ldots,X_n]$- module is stably free.

CHAPTER 5

Serre's Conjecture

This chapter is devoted to the proof of Serre's Conjecture [6] on projective modules. In the language of algebraic geometry the conjecture states that any algebraic vector bundle over an affine space is trivial. In terms of projective modules the conjecture can be stated as follows :

Conjecture : Any finitely generated projective module over $K[X_1,X_2,\ldots,X_n]$, K being a field, is free. This is trivial for $n = 1$; for $n = 2$, it was first proved by Seshadri [8]. Later, generalisations of Seshadri's theorem were obtained by several authors [3], [1]. Murthy proved the conjecture for the case $n = 3$ and K algebraically closed [4]. Finally the conjecture was settled in the affirmative by Quillen [5] and Suslin [9] simultaneously. The proof we give follows the presentation of Vasherstein and Suslin.

Theorem (5.1.1) (Quillen-Suslin) : Let $K[X_1,X_2,\ldots,X_n]$ be the polynomial ring in n variables over a field K. Then any finitely generated projective $K[X_1,\ldots,X_n]$-module P is free.

For the proof of the above theorem we need the following definition.

Definition : Let R be a ring. A sequence $(a_1,a_2,\ldots,a_n)$ of elements a_i in R is said to be unimodular if

$\exists\ (b_1,b_2,\dots,b_n) \in R^n$ such that $\sum_{i=1}^{n} a_i b_i = 1$.

Remark (5.1.1) Let $x = (a_1,\dots,a_n) \in R^n$. Then Rx is a direct summond of R^n iff x is unimodular.

Proof of Theorem (5.1.1) : It is enough to prove the following.

Claim : Suppose $(f_1,f_2,\dots,f_r)$ is a sequence of unimodular elements of $K[X_1,X_2,\dots,X_n]$. Then there exists an invertible $r \times r$ matrix whose entries are in $K[X_1,X_2,\dots,X_n]$ and whose first row is $(f_1,f_2,\dots,f_r)$.

We shall first assume that the claim is correct and prove Theorem (5.1.1).

By earlier theory we know that there exists a finitely generated free mosule F such that $P \oplus F$ is free, say $P \oplus F \cong R^t$, for some positive integer t. Using induction on the rank of F we shall prove that P is free. Suppose to start with that F is free of rank 1. Then the isomorphism $P \oplus F \cong R^t$, gives a split exact sequence $0 \to R \xrightarrow{\alpha} R^t \xrightarrow{\beta} P \to 0$. Let $\tau : R^t \to R$ be a splitting of $R \xrightarrow{\alpha} R^t$. Let $e_1,e_2,\dots,e_t$ be the canonical basis of R^t. Let $\tau(e_i) = b_i$, for $1 \le i \le t$. Also suppose that $\alpha(1) = (a_1,a_2,\dots,a_t)$. Then clearly $\sum_{i=1}^{t} a_i b_i = 1$, i.e. the sequence $(a_1,a_2,\dots,a_t)$ is unimodular. By the assumed validity of the claim, there exists a $t \times t$ invertible matrix H with entries in $K[X_1,X_2,\dots,X_n]$ and whose first row is $(a_1,a_2,\dots,a_t)$. The matrix H yields an R-linear automorphism $R^t \to R^t$ such that if we let $a = (a_1,a_2,\dots,a_t)$, then $\gamma(a) = e_1$. The following diagram is clearly commutative :

$$\begin{tikzcd}
0 \arrow[r, mapsto] & R \arrow[r, mapsto, "\alpha"] \arrow[d, "\wr"] & R^t \arrow[d, "\gamma"] \\
0 \arrow[r, mapsto] & Re_1 \arrow[r, mapsto] & R^t
\end{tikzcd}$$

where the isomorphism $R \mapsto Re_1$ is defined by mapping 1 onto e_1 and the map $Re_1 \mapsto R^t$ is the natural inclusion. Passing to quotients we get the isomorphism $P \simeq R^t/Re_1 \simeq R^{t-1}$, i.e. P is free. Next suppose F is free of rank greater than 1, say $F \simeq R^s$ where $s > t$. Then $P \oplus F = (P \oplus R^{s-1}) \oplus R \simeq R^t$; by what precedes we get that $P \oplus R^{s-1}$ is free. By induction we then obtain the result that P itself is free.

In order to prove the claim we need the following Lemmas.

Lemma 1 : Let R be any commutative ring. T an indeterminate over R. Let $f,g \subset R[T]$. Assume f is monic of degree 'd'. Let degree $g < d$. Then every non-zero coefficient of 'g' will be the leading coefficient of a polynomial of degree $(d-1)$ lying in the ideal (f,g).

Proof : We agree to write polynomials of degree $< d$ in the form $C_{d-1}T^{d-1} + C_{d-2}T^{d-2} + \dots + C_1T + C_0$, by allowing C_{d-1}, C_{d-2}, etc. to be zero if necessary. Consider the following statement $A(k)$, where k is any integer such that $0 \leq k \leq d-1$.

$A(k)$: "For any polynomial $g(T) = C_{d-1}T^{d-1} + \dots + C_1T + C_0$ of degree $< d$, and any integer 't' such that $k \leq t \leq d-1$, there exists a polynomial in the ideal (f,g), of the form C_tT^{d-1} + terms of degree less than $(d-1)$".

The lemma then asserts that $A(0)$ is true. We shall prove the truth of $A(k)$ for any k by descending induction on k. Trivially $A(d-1)$ is true, since in this case the polynomial itself will do. Assuming the truth of $A(k)$, we shall prove the truth of $A(k-1)$.

Write, $g(T) = C_{d-1}T^{d-1} + C_{d-2}T^{d-2} + \dots + C_0$

$$f(T) = T^d + a_{d-1}T^{d-1} + \dots + a_0 \ .$$

Consider $Tg(T) - C_{d-1}f(T)$. It is of degree $< d$. Coefficient of T^k is $C_{k-1} - C_{d-1}a_k$. So by applying the validity of statement $A(k)$ to the polynomial $Tg(T) - C_{d-1}f(T)$, we find that there exists some polynomial $\lambda(T)$ belonging to $(f, Tg - C_{d-1}f)$ such that $(T) = (C_{k-1} - C_{d-1}a_k)T^{d-1} + \dots$ Consider $\lambda(T) + a_k g(T) = C_{k-1}T^{d-1}$ + lower degree terms. Since $\lambda(T) + a_k g(T) \in (f,g)$, we find that $A(k-1)$ is true. Hence the lemma.

In what follows, for a ring R we shall denote by $GL_r(R)$, the group of invertible R-linear transformations of the free module R^r. If $\sigma \in GL_r(R)$ and $\underline{t} = (t_1, t_2, \dots, t_r) \in R^r$, then $\sigma(\underline{t})$ will be the result of applying σ to $\underline{t}$. Occasionally we shall confuse $GL_r(R)$ with the group of $r \times r$ invertible matrices over R. The context will make this clear.

<u>Lemma 2</u> : Let R be local with maximal ideal η. Let $\underline{f} = (f_1, f_2, \dots, f_r)$ be unimodular sequence of elements in $R[T]$, assume that one of $f_1, f_2, \dots, f_r$ is monic. Then $\exists\ \sigma \in GL_r(R[T])$ such that $\sigma \cdot \underline{f} = (1, 0, \dots, 0)$.

<u>Proof</u> : If $r = 1$, then $\exists\ g_1 \in R[T]$ such that $f_1 \cdot g_1 = 1$. Here g_1 is invertible. Take $\sigma = g_1$. If $r = 2$, we know $g_1, g_2 \in R[T]$ such that $f_1g_1 + f_2g_2 = 1$. Take $\sigma = \begin{pmatrix} g_1 & g_2 \\ f_2 & -f_1 \end{pmatrix}$

belonging to $GL_2(R[Y])$. Then $\begin{pmatrix} g_1 & g_2 \\ f_2 & -f_1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

that is $\lambda \cdot \underline{f} = \begin{matrix} 1 \\ 0 \end{matrix}$. So we are through in this case.

Take $r \geqslant 3$. By elementary operations on a row $(g_1,g_2,\ldots,g_r)$ we shall mean the following operations or combinations of them.

(1) $(g_1,g_2,\ldots,g_r) \longmapsto (g_{i_1},g_{i_2},\ldots,g_{i_r})$ where $i_1,i_2,\ldots,i_r$ is a permutation of $1,2,3,\ldots,r$.

(2) $(g_1,g_2,\ldots,g_r) \longmapsto (g_1,\ldots,g_{i-1},g_i + hg_j,g_{i+1},\ldots,g_j,\ldots,g_r)$ where $i \neq j$.

It is easy to see that these operations correspond to the effects of applications of invertible linear transformations to $(g_1,g_2,\ldots,g_r)$. Moreover if $g_1,g_2,\ldots,g_r$ is unimodular, so are the sequences obtained through these operations.

Now if all f_i's are constant then one of them must be unit. Otherwise the ideal $\langle f_1,f_2,\ldots,f_r\rangle$ will be contained in η, contradicting the unimodularity of $\underline{f}$. So in this case by elementary operations $\underline{f}$ changes into the row $(1,0,\ldots,0)$ and we are through.

Next assume that not all f_i's are constants. Let us choose a monic polynomial of least degree among $f_1,f_2,\ldots,f_r$. After an elementary operation let f_1 be the chosen monic polynomial of least degree.

Now we will apply Euclidean algorithm which asserts that if R is a commutative ring, and $f,g \in R[X]$ (polynomial ring in one variable) of degree $\geqslant 0$ and g is a monic polynomial, then $\exists$ unique polynomials q and r belonging to $R[X]$ such that $f = gq + r$ and $\deg r < \deg g$ or $r = 0$. Applying Euclidean algorithm to f_1 and f_i for $i \geq 2$ we get a sequence $(f_1, g_1', g_2', \ldots, g_r')$ where either $g_i' = 0$ or $\deg g_i' < \deg f_1$. Now we choose a monic polynomial of least degree among $f_1, g_1', \ldots, g_r'$ and repeat the process. Now this process stops when we reach a sequence where one of them is monic and others are either zero or non-monic of degrees strictly less than the degree of the monic polynomial. Hence after an elementary operation we can assume that the given sequence $(f_1, f_2, \ldots, f_r)$ is such that f_1 is monic and the others are non-monic of degrees less than that of f_1. Let $\deg f_1 = 0$, then $(f_1, \ldots, f_r) = (f_1, 0, 0, \ldots, 0)$ and we are through. Let $\deg f_1 = d \geq 1$. By unimodularity $\exists$ $g_1, g_2, \ldots, g_r \in R[T]$ such that $\sum_{i=1}^{r} f_i g_i = 1$. Reducing modulo $\mathfrak{m}$, we get $\sum_{i=1}^{r} \bar{f}_i \bar{g}_i = 1$ where $\bar{f}_i, \bar{g}_i \in \frac{R}{\mathfrak{m}}[T]$.

<u>Claim</u> : not all $\bar{f}_2, \bar{f}_3, \ldots, \bar{f}_r$ are zero. For otherwise $\bar{f}_1 \bar{g}_1 = 1 \Rightarrow \bar{f}_1$ is a unit in $\frac{R}{\mathfrak{m}}[T]$, that is $\bar{f}_1$ is a unit of $\frac{R}{\mathfrak{m}}$. Let $f_1 = T^d + \lambda_1 T^{d-1} + \cdots + \lambda_d$. Now, $\bar{f}_1$ is a unit of $\frac{R}{\mathfrak{m}} \Rightarrow d = 0$, a contradiction. Hence claim is proved. Assume without loss of generality that $\bar{f}_2 \neq 0$ which implies f_2 has a coefficient which does not belong to $\mathfrak{m}$, that is, it is a unit 'u' in R. We have $\deg f_2 < \deg f_1 = d$ and f_1 is monic. So by Lemma 1, $\exists$ a polynomial g of degree $(d-1)$ lying in the ideal (f_1, f_2) and such that its leading coefficient is 'u'.

Look at $f_3 + g$. Without loss of generality we can assume that the leading coefficient of f_3 is a non-unit. So the leading coefficient of $f_3 + g$ is unit. Consider the sequence

$(f_1, f_2, f_3+g, f_4, \ldots, f_r)$. Since $g \in (f_1, f_2)$, the above sequence is till unimodular. To prove the Lemma we apply induction on $\deg f_1 = d$. Earlier we disposed of the case when $d = 0$. If $d = 1$, then $\deg f_1 = 1$ and $f_2, f_3, \ldots, f_r$ are constants and since $\bar{f}_2, \bar{f}_3, \ldots, \bar{f}_r$ are not all 0, one of them is unit. So by elementary transformations $(f_1, f_2, \ldots, f_r) \longmapsto (1, 0, \ldots, 0)$ and we are through. Let us assume the result for all degrees $\leq d-1$. We arrived at the step that the sequence $(f_1, f_2, f_3+g, f_4, \ldots, f_r)$ is unimodular and that the polynomial $(f_3 + g)$ is monic of degree $d-1$. Now from $(f_1, f_2, f_3+g, \ldots, f_r)$ we can get a unimodular sequence (by elementary transformation) $(h_1, h_2, \ldots, h_r)$ where h_1 is monic of degree $\leq (d-1)$ and $h_2, \ldots, h_r$ are either 0 or non-monic of degree $< \deg h_1$. Now applying induction to this sequence we get $(h_1, h_2, \ldots, h_r) \longmapsto (1, 0, \ldots, 0)$, by elementary transformations. But $(h_1, h_2, \ldots, h_r)$ has been got from $(f_1, f_2, \ldots, f_r)$ by elementary transformation. So we get a $\varepsilon \in GL_r(R[T])$ such that $\varepsilon \, \underline{f} = (1, 0, \ldots, 0)$.

Therefore the lemma is proved.

__Lemma 3__ : Let R be any ring, let $(f_1, f_2, \ldots, f_r)$ be a unimodular sequence in $R[T]$, one of $f_1, f_2, \ldots, f_r$ being monic. Then $\exists$ $\varepsilon \in GL_r(R[T])$ such that $\varepsilon \, (f_1, f_2, \ldots, f_r) = (f_1(0), f_2(0), \ldots, f_r(0))$.

__Proof__ : Let $\underline{f}$ denote the row $(f_1, f_2, \ldots, f_r)$. Let X be any indeterminate over the ring $R[T]$. Let I be the set of elements $x \in R$ such that $\exists\ \delta(T,X) \in GL_r([T,X])$ such $\delta(T,X)\underline{f}(T + xX) = \underline{f}(T)$.

Claim : I is an ideal of R.

Let $x,y \in I$. $\exists\ \delta(T,X)$ and $\mu(T,X) \in GL_r(R[T,X])$ such that $\delta(T,X)\underline{f}(T + xX) = \underline{f}(T)$, $\mu(T,X)\cdot\underline{f}(T + yX) = \underline{f}(T)$. Now $\mu(T + xX,X)\underline{f}(T + xX + yX) = \underline{f}(T + xX)$. So $\delta(T,X)\mu(T + xX,X)\underline{f}(T + xX + yX) = \delta(T,X)\underline{f}(T + xX) = \underline{f}(T)$. So we get $(x + y) \in I$. Let $\lambda \in R$. Then $\delta(T + \lambda X)\underline{f}(T + \lambda xX) = \underline{f}(T)$ which implies $\lambda x \in I$. Therefore I is an ideal of R. It is enough to show that $1 \in I$, for in this case $\exists\ \delta(T,X) \in GL_r(R[T,X])$ such that $\delta(T,X)\underline{f}(T + X) = \underline{f}(T)$. Take $T = 0$, then $\delta(0,X)\underline{f}(X) = \underline{f}(0)$. Change X to T to get the result. Next we claim that $1 \in I$. Suppose $1 \notin I$. Choose a maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} \supseteq I$. Let us look at our situation over $R_{\mathfrak{m}}[T]$. Now $\underline{f}$ is a unimodular sequence in $R_{\mathfrak{m}}[T]$. So applying previous lemma to it, we get a $\delta(T) \in GL_r(R_{\mathfrak{m}}[T])$ such that $\delta\underline{f} = (1,0,0,\ldots,0)$. Define $\tau(T,X) = \delta(T)^{-1}\delta(T + X)$ which belongs to $GL_r(R[T,X])$ and $\tau(T,0) = \delta(T)^{-1}\delta(T)$ which is the identity. This shows that no entry of $\tau(T,X)$ involves monomials in T alone with positive degrees. So $\exists\ s \notin \mathfrak{m}$ such that $\tau(T,sX)$ belongs to $GL_r(R[T,X])$. $\tau(T,X)\cdot\underline{f}(T + X) = \delta(T)^{-1}\delta(T + X)\underline{f}(T + X) = \delta(T)^{-1}(1,0,\ldots,0) = \underline{f}(T)$ in $GL_r(R_{\mathfrak{m}}[T,X])$. Look at $\tau(T,sX)\underline{f}(T + sX) - \underline{f}(T)$. This is '0' when we localise at $\mathfrak{m}$, that is $\exists\ s' \notin \mathfrak{m}$ such that s' annihilates $\tau(T,sX)\underline{f}(T + sX) - \underline{f}(T)$. Let the (i,j)th entry of $\tau(T,sX)$ be denoted by $A_{i,j}$. Since $\tau(T,0)$ is $r \times r$ identity matrix, we can write $A_{i,i} = 1 + \sum a^{(i)}_{\alpha_1\alpha_2} T^{\alpha_1}(sX)^{\alpha_2}$ and $a^{(i)}_{\alpha_1,\alpha_2} \in R_\eta$; also we can write $A_{ij} = \sum a^{(i,j)}_{\beta_1\beta_2} T^{\beta_1}(sX)^{\beta_2}$ if $i \neq j$, where summation is extended over certain indices β_1, β_2 with $\beta_2 > 0$. The kth entry of the vector $\tau(T,sX)\underline{f}(T + sX) - \underline{f}(T)$ is equal to $A_{k1}f_1(T + sX) + A_{k2}f_2(T + sX) + \ldots + A_{kr}f_r(T + sX) - f_k(T)$. Clearly if $\ell \neq k$, then $A_{k,\ell}f_\ell(T + sX)$ can be written as sX times a polynomial in $R_\eta[T,sX]$. Also $A_{kk}f_k(T + sX) - f_k(T)$ $= \{f_k(T + sX) - f_k(T)\} + \sum a^{(k)}_{\alpha_1,\alpha_2} T^{\alpha_1}(sX)^{\alpha_2}f_k(T + sX)$.

Hence $A_{kk}f_k(T + sX) - f_k(T)$ can also be written as sX times a polynomial in $R_\eta[T,sX]$, i.e., any entry of $\tau(T,sX)\overset{f(T+sX)}{-}\underline{f}(T)$ can be written as sX times a polynomial in $R_\eta[T,sX]$; let us write $\tau(T,sX)\underline{f}(T + sX) - \underline{f}(T) = sX\underline{h}(T,sX)$ ....(*) then $\tau(T,s'sX)\underline{f}(T + s'sX) - \underline{f}(T) = s'sX\underline{h}(T,s'sX)$. Since s' annihilates $\tau(T,sX)\underline{f}(T + sX) - \underline{f}(T)$ we find that $s'sX\underline{h}(T,sX) = 0$. Note that $sX\underline{h}(T,sX) \in R[T,X]$, since the L.H.S. of (*) belongs to $R[T,X]$. Then clearly $s'sX\underline{h}(T,s'sX) = 0$ in $R[T,X]$. That is $\tau(T,s'sX)\underline{f}(T + s'sX) = \underline{f}(T)$. This implies $s' \in I$. Since , $s' \notin \eta$ and $I \subset \eta$, we obtain a contradiction. Hence $1 \in I$ and we are through.

Now we give the proof of the claim which states that if $(f_1,\ldots,f_n)$ is a sequence of unimodular elements in $K[X_1,X_2,\ldots,X_n]$, then $\exists$ $r \times r$ invertible matrix over $K[X_1,\ldots,X_n]$ such that its first row is $(f_1,f_2,\ldots,f_r)$.

Proof : Write $f_1 = \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} a_{\alpha_1,\alpha_2,\ldots,\alpha_n} X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, where

$a_{\alpha_1,\alpha_2,\ldots,\alpha_n} \neq 0$ and summation is taken over certain sequences of positive integers $(\alpha_1,\alpha_2,\ldots,\alpha_n)$. We change the variables by putting $X_1 = T_1$, $X_2 = T_2 + T_1^{\lambda_2}, \ldots, X_n = T_n + T_1^{\lambda_n}$ where λ_i's are positive integers so chosen that $\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n$ are all distinct for all sequences $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ occuring in the sum above. We have only to choose $\lambda_2 = k$, $\lambda_3 = k^2, \ldots, \lambda_n = k^{n-1}$ where k is any integer larger than any of $\alpha_1,\ldots,\alpha_n$ for all sequences $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ occuring in the sum above. Let $\tilde{f_1}(T_1,T_2,\ldots,T_n)$ be the polynomial $f_1(X_1,X_2,\ldots,X_n)$ after the change of variables. Let $a \in K^*$ be the coefficient of the monomial $X_1^{\alpha'_1}X_2^{\alpha'_2}\cdots X_n^{\alpha'_n}$. Multiply f_1 by a^{-1} to get $a^{-1}f_1$. Suppose C is an invertible matrix over $K[X_1,X_2,\ldots,X_n]$ such that $(a^{-1}f_1,f_2,\ldots,f_r)C = (1,0,\ldots,0)$. Then clearly C^{-1} is an invertible matrix whose first row is $(a^{-1}f_1,f_2,\ldots,f_r)$.

Let $$L = C^{-1}\begin{pmatrix} a & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}.$$

Then the first row of L is $(f_1,f_2,\ldots,f_r)$.

Here $\begin{pmatrix} a & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$ is invertible. So L is also invertible.

By the above discussion it is clear that existence of C ensures an invertible matrix L whose first row is $(f_1, f_2, \ldots, f_r)$. So W.L.O.G. we can assume that the coefficient of the monomial $X_1^{\alpha_1'} X_2^{\alpha_2'} \ldots X_n^{\alpha_n'}$ is unity. Consider $\tilde{f}_1$ as a polynomial over $R = K[T_2, T_3, \ldots, T_n]$, then $\tilde{f}_1(T_1)$ is monic in the variable T_1. Let $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_r)$. Clearly $\tilde{f}$ is unimodular over $R = K[T_2, \ldots, T_n]$. Now by Lemma 3, $\exists$ $\sigma \in GL_r(R[T_1])$ such that $\sigma(f_1, f_2, \ldots, f_r) = (\tilde{f}_1(0, T_2, \ldots, T_n), \tilde{f}_2(0, T_2, \ldots, T_n), \ldots, \tilde{f}_n(0, T_2, \ldots, T_n))$. Also $(\tilde{f}_1(0, T_2, \ldots, T_n), \ldots, \tilde{f}_r(0, T_2, \ldots, T_n))$ is an unimodular sequence in $K[T_2, T_3, \ldots, T_n]$. Changing to original variables X_i we get a $\sigma \in GL_r(K[X_1, \ldots, X_n])$ such that $\sigma(f_1, f_2, \ldots, f_r) = (f_1(0, X_2, \ldots, X_n), f_2(0, X_2, \ldots, X_n), \ldots, f_n(0, X_2, \ldots, X_n))$. Since this sequence is still unimodular over $K[X_1, \ldots, X_n]$, we will repeat the process until we get an invertible transformation over $K[X_1, X_2, \ldots, X_n]$ which takes $(f_1, f_2, \ldots, f_r)$ to $(f_1(0, \ldots, 0), \ldots, f_r(0, \ldots, 0))$. The resulting sequence $(f_1(0, \ldots, 0), \ldots, f_r(0, 0, \ldots, 0))$ is unimodular over K, so one of them is a non-zero constant in K. So by an elementary transformation

$(f_1(0, \ldots, 0), \ldots, f_r(0, \ldots, 0)) \longmapsto (1, 0, \ldots, 0)$. Hence an invertible transformation $T \in GL_r(K[X_1, \ldots, X_n])$ which takes $(1, 0, \ldots, 0)$ to $(f_1, \ldots, f_r)$ of T. So the $r \times r$ matrix has first row $(f_1, f_2, \ldots, f_r)$.

The claim is proved and so also the Theorem (5.1.1).

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