Proc. Indian Acad. Sci. (Math. Sci.), Vol. 98, No. 1, June 1988, pp. 31–42.

Zeta functions of orders in quadratic fields

PROMODE KUMAR SAIKIA

Department of Mathematics, North Eastern Hill University, Laitumkhrah, Shillong 793 003, India

MS received 16 December 1987

Abstract. The zeta function of a lattice over an order in a semisimple Q-algebra introduced by L Solomon, is a generalization of Dedekind zeta function and has an Euler product in terms of local zeta functions. In this paper, the local zeta functions of localized orders in quadratic fields and hence the global ones are computed first by using Solomon's combinatorial method. Next the same formulae are obtained by considering orders in algebras (over p-adic fields) obtained by completing quadratic fields. In the complete case the methods of integration initiated by Bushnell and Reiner to study Solomon's zeta functions are used.

Keywords. Zeta functions; quadratic fields; Q-algebra.

1. Introduction

In this paper we determine Solomon's zeta functions for lattices in a quadratic number field over $\mathbb{Z}$-orders in the field. We begin by briefly discussing the concepts involved, and fix the notation. For detailed discussion see [9], [10]. For facts about orders we refer to [7].

Let A be a finite dimensional semisimple $\mathbb{Q}$-algebra and let Λ be an R-order in A where R is either $\mathbb{Z}$ or the localization $\mathbb{Z}_p$ of $\mathbb{Z}$ at a rational prime p. Let L be a (full) Λ-lattice in a finitely generated A-module V. Solomon's zeta function $\zeta_\Lambda(L;s)$ for the Λ-lattice L is defined as

$$\zeta_\Lambda(L;s) = \Sigma |L:N|^{-s}, \tag{1}$$

where the sum is over all Λ-sublattices N of L. Here $|L:N|$ denotes the index of N in L, and s is a complex variable. In the case $V = A$ where A is a number field and $L = \Lambda$ is the ring of integers in A, then Solomon's zeta function for L coincides with Dedekind's zeta function for A.

If Λ is a $\mathbb{Z}$-order in the $\mathbb{Q}$-algebra A, for a rational prime p, we form $\Lambda_p = \mathbb{Z}_p \otimes_Z \Lambda = \mathbb{Z}_P\Lambda$ and $L_p = \mathbb{Z}_p \otimes_Z L = \mathbb{Z}_p L$. Then Λ_p is a $\mathbb{Z}_p$-order in A and L_p is a Λ_p-lattice in V. So we may form local zeta function $\zeta_{\Lambda_p}(L_p;s)$ by using (1). It is shown in [10] that the Euler product formula

$$\zeta_\Lambda(L;s) = \Pi \zeta_{\Lambda_p}(L_p;s) \tag{2}$$

holds; here the product is over all rational primes p. Now fix a rational prime p, and let $\mathscr{P}$ be the partially ordered set of all Λ_p-lattices in V ordered by reverse inclusion.

$\mathscr{P}$ is locally finite and has a Mobius function μ. Fix an indeterminate t and for any pair of Λ_p-lattices $N \subseteq M$ write $[N:M] = t^n$ if the index $|M:N| = p^n$. It is known [7, p.228] that the number of Λ_p-isomorphism classes of Λ_p-lattices in V is finite; choose a complete set of representatives $L_1, L_2, \ldots, L_k$ of Λ_p-isomorphism classes in $\mathscr{P}$. Define two $k \times k$ matrices $\mathscr{A} = [a_{ij}]$ and $\mathscr{Z} = [Z_{ij}]$ as follows:

$$a_{ij} = \Sigma\, \mu(L_i, M)[M:L_i] \tag{3}$$
$$Z_{ij} = \Sigma [M:L_i],$$

where both the sums run through all Λ_p-sublattices M of L_i isomorphic to L_j. It can be shown that $\mathscr{A}$ depends only on the labelling of the classes and is independent of the choice of representative in each class. Furthermore, Solomon [9] proved that

$$\mathscr{A}^{-1} = \mathscr{Z}. \tag{4}$$

Now denote by $Z_i(t) = Z_i$ the sum of the ith row of $\mathscr{Z}$; it is clear that

$$Z_i(p^{-s}) = \zeta_{\Lambda_p}(L_i; s). \tag{5}$$

Thus the local zeta functions are known once we compute $\mathscr{A}$ and invert it.

In the next section, we give a complete description of the matrix $\mathscr{A}$ for Λ_p-lattices in a quadratic field A where Λ_p is a $\mathbb{Z}_p$-order in A. As a consequence we obtain explicit formulae for the corresponding local zeta functions and hence global formulae also.

In contrast to Solomon's algebraic and combinatorial method, Bushnell and Reiner [2, 3] in a series of papers successfully adopted p-adic integration in computations involving Solomon's zeta functions. In § 3, we use their method to work out some local zeta functions.

2. Solomon's matrix $\mathscr{A}$ and zeta functions for quadratic orders

We begin by collecting some facts we need about $\mathbb{Z}_p$-orders in an algebraic number field A. Throughout this section p will be a fixed but arbitrary rational prime unless otherwise specified. For definitions and basic facts about order, refer to [7].

If O is the ring of algebraic integers in A, then $\mathbb{Z}_p O$ is the unique maximal $\mathbb{Z}_p$-order in A, denoted by Λ_0. It is well known that Λ_0 is a principal ideal domain and if $pO = \Pi P_i^{e_i}$ where P_i are distinct prime ideals of O, then $p\Lambda_0 = \Pi M_i^{e_i}$ where M_i are the distinct prime ideals of Λ_0. Now let Λ be a $\mathbb{Z}_p$-order in A. Since $p\mathbb{Z}_p$ is the unique maximal ideal in the local ring $\mathbb{Z}_p$, it follows that $p\Lambda \subseteq \operatorname{Rad}\Lambda$, where $\operatorname{Rad}\Lambda$ is the Jacobson radical of Λ. We can now prove

Lemma 1. Let L be a Λ-lattice in a vector space V over the number field A, where Λ is a $\mathbb{Z}_p$-order in A. Let μ denote the Mobius function of the poset of Λ-lattices in V. Then $\mu(L, M) = 0$ unless $M \supseteq pL$.

Proof. By a result due to Ph. Hall [8; Cor. to Prop. 2, § 5] $\mu(L, M) = 0$ unless M is an intersection of maximal Λ-sublattices of L. Since $p\Lambda \subseteq \operatorname{Rad}\Lambda$, the Λ-sublattice pL of L is contained in the intersection of all maximal Λ-sublattices of L. The lemma follows. □

Let Λ_1 and Λ_2 be two $\mathbb{Z}_p$-orders in A containing a $\mathbb{Z}_p$-order Λ in A such that Λ_1 and Λ_2 are isomorphic as Λ-lattices. If ϕ is the Λ-isomorphism, it can be extended uniquely to an A-module isomorphism of A (as $\mathbb{Q}\Lambda = \mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2 = A$). Such an isomorphism of A is given by the multiplication by a non-zero element of A. So $\Lambda_1 = \alpha\Lambda_2$ for some $\alpha \in A$. Comparing the 'order' we then see that $\Lambda_1 = \Lambda_2$. (By the order $O(L)$ of a Λ-lattice L in A we mean $O(L) = \{x \in A : xL \subseteq L\}$. $O(L)$ is actually a $\mathbb{Z}_p$-order in A containing Λ.) Thus we have

Lemma 2. Let Λ_1 and Λ_2 be $\mathbb{Z}_p$-orders in a number field A containing a $\mathbb{Z}_p$-order Λ in A. If Λ_1 and Λ_2 are isomorphic as Λ-lattices, then $\Lambda_1 = \Lambda_2$. □

From now onwards let A be a quadratic field. We choose $\Theta \in A$ such that $A = \mathbb{Q}(\Theta)$ and $O = \mathbb{Z} \oplus \mathbb{Z}\Theta$. Then

$$\Lambda_0 = \mathbb{Z}_p \oplus \mathbb{Z}_p \Theta$$

is the unique maximal $\mathbb{Z}_p$-order in A. For each integer $r \geqslant 0$, let

$$\Lambda_r = \mathbb{Z}_p \oplus \mathbb{Z}_p p^r \Theta \tag{6}$$

so that Λ_r is a $\mathbb{Z}_p$-order in A of index p^r in Λ_0. If Λ is any $\mathbb{Z}_p$-order of index p^r in Λ_0, then $p^r\Lambda_0 \subseteq \Lambda$ so that $\Lambda_r = \Lambda$. Thus we conclude that the collection of all $\mathbb{Z}_p$-orders in A forms a chain

$$\Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_r \supset \cdots \tag{7}$$

where Λ_r as given in (6) is the unique $\mathbb{Z}_p$-order of index p^r in Λ_0.

Any $\mathbb{Z}$-lattice L in the quadratic field A is invertible. [1, p. 198]. Since $O(\mathbb{Z}_p L) = \mathbb{Z}_p(O(L))$ and $\mathbb{Z}_p L^{-1} = (\mathbb{Z}_p L)^{-1}$ [7, pp. 109, 192] it follows that any $\mathbb{Z}_p$-lattice in A is invertible. Consequently one has

Theorem 1. *Any $\mathbb{Z}_p$-lattice L in A is principal i.e. there is $\alpha \in L$ such that $L = O(L)\alpha$.* □

A proof can be found in [6, p. 98] where it is attributed to Ihara. It is a special case of more general results [4, Theorems 35.5, 35.14]. For us the importance of theorem lies in the following corollary,

COROLLARY 1

For a $\mathbb{Z}_p$-order Λ_r in the quadratic field A, two Λ_r-lattices in A are Λ_r-isomorphic if and only if their orders coincide. Thus for a fixed $r \geqslant 0$, $\{\Lambda_s; 0 \leqslant s \leqslant r\}$ is a complete set of representatives of Λ_r-isomorphism classes of Λ_r-lattices in A.

Proof. Because of Theorem 1, if two Λ_r-lattices have the same order, they must be isomorphic as Λ_r-lattices. Conversely, if they are isomorphic their orders will be isomorphic as Λ_r-lattices. Since these orders contain Λ_r, the proof is complete by lemma 2. □

Now fix a $\mathbb{Z}_p$-order Λ_r ($r \geqslant 0$). In view of the preceding results, in order to know the zeta functions of Λ_r-lattices in A, we need to compute only the zeta functions of the orders Λ_i $(0 \leqslant i \leqslant r)$ considering them as Λ_r-lattices. Let $\mathscr{A}(r) = \mathscr{A}$ and $\mathscr{Z}(r) = \mathscr{Z}$ be the matrices associated to the order Λ_r as in (3). Since $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_r\}$ is a full set of non-isomorphic Λ_r-lattices in the poset of Λ_r-lattices in A, both $\mathscr{A}(r)$ and $\mathscr{Z}(r)$ are $(r+1) \times (r+1)$ matrices. Let μ_r denote the Mobius function of this poset; then the entries of $\mathscr{A}(r)$ are given by

$$a_{ij}(r) = \Sigma \mu_r(\Lambda_i, M) \quad [M:\Lambda_i], \tag{8}$$

where the sum is over all Λ_r-sublattices M in Λ_i such that $p\Lambda_i \subseteq M$ and $M \cong \Lambda_j$ as Λ_r-lattices (see lemma 1). Also as mentioned in the introduction, $\mathscr{Z}(r) = \mathscr{A}(r)^{-1}$ and

$$\zeta_{\Lambda_r}(\Lambda_i; s) = Z_i(r), \tag{9}$$

where $Z_i(r)$ is the sum of the ith row of the matrix $\mathscr{Z}(r)$. Note that in both the matrices $\mathscr{A}(r)$ and $\mathscr{Z}(r)$, the subscripts i and j run from 0 to r. Our next result describes the nature of $\mathbb{Z}_p$-lattices between Λ_i and $p\Lambda_i$.

Theorem 2. *Let M be a $\mathbb{Z}_p$-lattice lying between Λ_i and $p\Lambda_i$; $M \neq \Lambda_i$, $M \neq p\Lambda_i$. $(i \geqslant 0)$. Let $\varepsilon(p) = -1, 0$ or 1 depending on whether p is inert, ramifies or decomposes in A.*

(i) *If $i = 0$, then the order $O(M)$ of M is either Λ_0 or Λ_1. The number of M with $O(M) = \Lambda_0$ is $\varepsilon(p) + 1$, and with $O(M) = \Lambda_1$ is $p - \varepsilon(p)$.*
(ii) *If $i > 0$, then $O(M)$ is either Λ_{i-1} or Λ_{i+1}. The lattice $p\Lambda_{i-1}$ is the only one having the order Λ_{i-1}. The number of M with $O(M) = \Lambda_{i+1}$ is exactly p.*

Proof. We first observe that if for such a lattice $M, O(M) = \Lambda_s$ then $s \leqslant i+1$, for otherwise $\Lambda_s \subset \Lambda_i$. Also by Theorem 2, we may choose $\alpha \in M$ such that $M = \Lambda_s\alpha$. Now multiplying $p\Lambda_i \subset M = \Lambda_s\alpha \subset \Lambda_i$ by Λ_i we obtain $p\Lambda_i \subset \Lambda_s\alpha \subset \Lambda_i\alpha = \Lambda_i$ which implies $p^{s-i} = p$ contradicting $s > i+1$. This observation proves the first assertion in (i). To prove the corresponding assertion in (ii), assume $O(M) = \Lambda_s$ where $s \leqslant i$. Now multiply $p\Lambda_i \subset M \subset \Lambda_i$ by Λ_s to obtain $p\Lambda_s \subseteq M \subset \Lambda_s$ whence

$$p^2 \geqslant |\Lambda_s:M| = |\Lambda_s:\Lambda_i| \quad |\Lambda_i:M| = p|\Lambda_s:\Lambda_i|.$$

Thus $|\Lambda_s:\Lambda_i| \leqslant p$ so that either $\Lambda_s = \Lambda_{i-1}$ or $\Lambda_s = \Lambda_i$. Our first assertion in (ii) is proved once we exclude the second possibility. So let $O(M) = \Lambda_i$; we aim to obtain a contradiction. As before, we write $M = \Lambda_i\alpha$ for some $\alpha \in M$. In that case, multiplying $p\Lambda_i \subset M = \Lambda_i\alpha \subset \Lambda_i$ by Λ_0 we obtain $p\Lambda_0 \subseteq \Lambda_0\alpha \subseteq \Lambda_0$.

If $\Lambda_0\alpha = \Lambda_0$, then $\alpha \in U(\Lambda_0)$, the group of units in Λ_0. But $\alpha \in \Lambda_i$ and $U(\Lambda_0) \cap \Lambda_i = U(\Lambda_i)$ [7, Ex 25.4, p. 224] so $M = \Lambda_i\alpha = \Lambda_i$, a contradiction. If $\Lambda_0\alpha = p\Lambda_0$, then $\alpha \in p\Lambda_0 \cap \Lambda_i = p\Lambda_{i-1}$ and thus $M \subseteq p\Lambda_{i-1} \subset \Lambda_i$. As $|\Lambda_i:M| = p$, these inclusions imply that $M = p\Lambda_{i-1}$ which contradicts $O(M) = \Lambda_i$. Thus we can assume that $|\Lambda_0:\Lambda_0\alpha| = p$. In that case $\Lambda_0\alpha \cap \Lambda_1 = p\Lambda_0$ so $\alpha \in p\Lambda_0$ and $\Lambda_0\alpha \subseteq p\Lambda_0$, a contradiction. These show that $O(M) = \Lambda_s \neq \Lambda_i$.

To obtain the other assertions, note that if $s > i$, then $\Lambda_i/p\Lambda_i$ is a two-dimensional vector space over the field $\Lambda_s/p\Lambda_{s-1}$ of p elements and that the number of Λ_s-sublattices of Λ_i containing $p\Lambda_i$ and having index p in Λ_i is exactly $(p+1)$. On the other hand it is clear that the number of sublattices of Λ_0 having index p in Λ_0 with their order as Λ_0 is $\varepsilon(p) + 1$. Hence the second assertion in (i). If $i > 0$, we have already shown that the $\mathbb{Z}_p$-sublattices of index p in Λ_i are all Λ_{i+1}-sublattices so they number $(p+1)$ by the preceding observation. If, for such a lattice M, $O(M) = \Lambda_{i-1}$, then the inclusions

$p\Lambda_i \subset M \subset \Lambda_i$ yield $p\Lambda_{i-1} \subseteq M \subset \Lambda_i$ which forces $M = p\Lambda_{i-1}$. This completes the proof of (ii). □

Note that a consequence of the last result is the following

COROLLARY 2

For $i > 0$, the $\mathbb{Z}_p$-order Λ_i is a local ring with $p\Lambda_{i-1}$ as the unique maximal ideal. □

Now that we know the number and the isomorphism classes of the required lattices, we just have to compute the values of the Mobius function to complete the description of $\mathscr{A}(r)$. Using the information of the preceding result and the recursive definition of μ_r we can compute these values easily (see [9] and [10]; for the sake of completeness we give the recursive definition of the Mobius function μ of a poset $\mathscr{P}$:

$$\mu(L, L) = 1, \quad \Sigma\mu(L, N) = 0 \quad \text{if} \quad L \supseteq M, \quad L \neq M,$$

where the sum is over all N in $\mathscr{P}$, such that

$$L \supseteq N \supseteq M$$

$$\mu(L, M) = 0 \quad \text{otherwise.})$$

We list these values

$$\begin{aligned}
&\mu_0(\Lambda_0, p\Lambda_0) = \varepsilon(p), \\
&\mu_r(\Lambda_r, p\Lambda_r) = 0 && \text{if} \quad r \geqslant 1, \\
&\mu_r(\mu_i, p\Lambda_i) = p && \text{if} \quad 0 \leqslant i \leqslant r, \\
&\mu_r(\Lambda_i, M) = -1 && \text{if} \quad 0 \leqslant i \leqslant r, \quad M \ \Lambda_r\text{-sublattice of } \Lambda_i \\
& && \text{of index } p.
\end{aligned}$$

Putting all these in (8), we can easily obtain the following description of the matrix $\mathscr{A}(r)$.

Theorem 3. *Let r be a non-negative integer and let $\mathscr{A}(r) = [a_{ij}]$ be the $(r+1) \times (r+1)$ matrix as defined in* (8).

If $r = 0$, then

$$a_{00} = 1 - (\varepsilon(p) + 1)t + \varepsilon(p)t^2.$$

If $r \geqslant 1$, then $a_{ij} = 0$ unless $|i - j| \leqslant 1$. The remaining entries are

$$\begin{aligned}
&a_{00} = 1 - (\varepsilon(p) + 1)t + pt^2; \quad a_{01} = -(p - \varepsilon(p))t, \\
&a_{rr} = 1, \\
&a_{ii} = 1 + pt^2; \quad 0 < i < r, \\
&a_{i,i+1} = -pt; \quad 0 < i < r, \\
&a_{i+1,i} = -t; \quad 0 \leqslant i < r.
\end{aligned}$$

The picture of $\mathscr{A}(r)$ for $r > 1$ obtained in Theorem 3 makes it clear that the suitable row and column operations allow us to link $\mathscr{A}(r)$ to $\mathscr{A}(r-1)$. To be precise, let E_{ij} denote the elementary $(r+1) \times (r+1)$ matrix with 1 at (i, j)th place and zeros everywhere else; and let I be the $(r+1) \times (r+1)$ identity matrix. Then using the picture of $\mathscr{A}(r)$ in Theorem 3 one easily verifies that

$$(I + ptE_{r-1,r})\mathscr{A}(r)(I + tE_{r,r-1}) = \left[\begin{array}{c|c} \mathscr{A}(r-1) & 0 \\ \hline 0 & 1 \end{array}\right], \tag{10}$$

$$(I + (p - \varepsilon(p))tE_{0,1})\mathscr{A}(1)(I + tE_{1,0}) = \left[\begin{array}{c|c} \mathscr{A}(0) & 0 \\ \hline 0 & 1 \end{array}\right]. \tag{11}$$

For simplicity, fix the following notation

$$\phi(t) = (1 - t)(1 - \varepsilon(p)t), \tag{12}$$

$$h(p) = p - \varepsilon(p).$$

The preceding relations then yield the following interesting result:

COROLLARY 3

For $r \geqslant 0$,

$$\det \mathscr{A}(r) = \phi(t),$$
$$\det \mathscr{Z}(r) = \phi(t)^{-1}.$$ □

Note that $\phi(t)^{-1} = Z_0(0)$ *is the local zeta function of the maximal order* Λ_0.

Now inverting the matrix

$$\mathscr{A}(1) = \begin{bmatrix} 1 - (\varepsilon(p) + 1)t + pt^2 & -h(p)t \\ -t & 1 \end{bmatrix}$$

where we use the fact that $\det \mathscr{A}(1) = \phi(t)$, we obtain

$$Z_0(1) = \phi(t)^{-1}[1 + h(p)t], \tag{13}$$

$$Z_1(1) = \phi(t)^{-1}[1 - \varepsilon(p)t + pt^2], \tag{14}$$

they being the sum of the first and the second rows of $\mathscr{Z}(1)$. For future reference we list two individual entries of $\mathscr{Z}(1)$ also:

$$Z_{0,1}(1) = \phi(t)^{-1}h(p)t, \tag{15}$$

$$Z_{1,1}(1) = \phi(t)^{-1}[1 - (1 + \varepsilon(p))t + pt^2].$$

To obtain the corresponding "partial" zeta functions for $r \geqslant 2$, we first take inverses in the matrix equation (10). This yields

$$\mathscr{Z}(r) = (I + tE_{r,r-1})\left[\begin{array}{c|c} \mathscr{Z}(r-1) & 0 \\ \hline 0 & 1 \end{array}\right](I + ptE_{r-1,r}).$$

Introducing the matrix

$$X = [x_{ij}] = \left[\begin{array}{c|c} Z(r-1) & 0 \\ \hline 0 & 1 \end{array}\right]$$

the last equation may be rewritten as

$$\mathscr{X}(r) = X + t\sum_{j=0}^{r} x_{r-1,j}E_{rj} + pt\sum_{i=0}^{r} X_{i,r-1}E_{ir} + pt^2 x_{r-1,r-1}E_{rr}.$$

Now, for $0 \leqslant i < r$, the sum of the ith row of X is $Z_i(r-1)$ whereas the sum of the last row i.e. the rth row is just 1. Therefore one can easily deduce the following from the preceding matrix equation:

$$\begin{aligned}
&\text{(i)}\ Z_i(r) = Z_i(r-1) + ptZ_{i,r-1}(r-1) && 0 \leqslant i < r,\\
&\text{(ii)}\ Z_{ir}(r) = ptZ_{i,r-1}(r-1) && 0 < i < r,\\
&\text{(iii)}\ Z_{rr}(r) = 1 + pt^2 Z_{r-1,r-1}(r-1) &&\\
&\text{(iv)}\ Z_r(r) = tZ_{r-1}(r-1) + Z_{rr}(r). &&
\end{aligned} \tag{16}$$

The main result shows that $\zeta_{\Lambda_r}(\Lambda_i; s)$ is a product of $\phi(p^{-s})^{-1}$ and certain polynomials in p^{-s}. We first define these polynomials: For any integer k, let the symbol $[k; t]$ denote the rational function in t given by

$$[k; t] = (1 - t^k)/(1 - t)$$

so that for $k \geqslant 1$, $[k; t]$ is a polynomial in t whereas $[0; t] = 0$. Now for integers i and r, $0 \leqslant i \leqslant r$, and a rational prime p let

$$f_p(r, i; t) = [r - i + 1; pt] + (pt)^{r-i+1} \cdot t[i - 1; pt^2].$$

It is clear that $f_p(r, i; t)$ is a polynomial in t if $1 \leqslant i \leqslant r$. One readily verifies that for integers i, r such that $1 \leqslant i \leqslant r$, we have

$$f_p(r, i; t) - tf_p(r, i - 1; t) = (1 - t)[r - i + 1; pt]. \tag{17}$$

Theorem 4. *Let p be a rational prime and let $\phi(t) = (1 - t)(1 - \varepsilon(p)t)$. Let i, r be integers such that $0 \leqslant i \leqslant r$, and let Λ_i and Λ_r be $\mathbb{Z}_p$-orders in the quadratic field A having index p^i and p^r respectively in the maximal $\mathbb{Z}_p$-order Λ_0 in A. Then*

$$\zeta_{\Lambda_r}(\Lambda_i; s) = \phi(p^{-s})^{-1}[(1 - \varepsilon(p)p^{-s})f_p(r, i; p^{-s}) + p^r p^{-s(r+i)}].$$

Proof. It suffices to prove that

$$Z_i(r) = \phi(t)^{-1}[(1 - \varepsilon(p)t)f_p(r, i; t) + p^r t^{r+i}]. \tag{18}$$

Case I: $i = 0$. In this case if $r = 0$, then (18) is $Z_0(0) = \phi(t)^{-1}$ which we have verified already. So we assume $r > 0$. Writing out the right hand side of (18) in terms of $h(p)$ where $h(p) = p - \varepsilon(p)$, we can put (18) in the form

$$Z_0(r) = \phi(t)^{-1}\left[1 + h(p)t\sum_{j=0}^{r-1}(pt)^j\right]. \tag{19}$$

This we prove by induction on r. If $r = 1$, (19) is just the relation (13). So assume $r > 1$. In this case note that the second relation in (16) together with (15) implies

$$Z_{0,r-1}(r - 1) = \phi(t)^{-1}h(p)t(pt)^{r-2}.$$

Using this in the first relation of (16) and applying the induction hypothesis we can obtain (19) easily.

Case II: $i > 0$. We first note that the following relation

$$Z_i(r) = Z_i(i) + pt[r-1; pt]Z_{ii}(i); \quad 1 \leqslant i \leqslant r \tag{20}$$

can be obtained from (i) and (ii) of (16). Now coming back to the proof of (18) in the present case of $i > 0$, we use induction on i. The case of $i = 1$ is a straightforward substitution of values of $Z_1(1)$ and $Z_{1,1}(1)$ from (14) and (15) in (20). So we may assume $i > 1$. First we put (20) in the form

$$Z_i(r) = tZ_{i-1}(i-1) + [r-i+1; pt]\{1 + pt^2 Z_{i-1,i-1}(i-1)\}$$

using the values of $Z_i(i)$ and $Z_{ii}(i)$ from (16).

On the other hand, since $i > 1$, we may replace $i-1$ by i in (20) to obtain

$$Z_{i-1}(r) = Z_{i-1}(i-1) + pt[r-i+1; pt]Z_{i-1,i-1}(i-1).$$

These two relations imply that

$$Z_i(r) = tZ_{i-1}(r) + [r-i+1; pt],$$

whence (18) follows by the induction hypothesis and (17). □

In the special case of $i = r$, since $f_p(r, r; t) = [r; pt^2]$ a simple calculation shows that

$$(1-\varepsilon(p)t)f_p(r, r; t) + p^r t^{2r} = \frac{(1-\varepsilon(p)t)(1-(pt^2)^r) + (1-pt^2)(pt^2)^r}{1-pt^2}.$$

Thus, in the special case of $i = r$, the formula for the corresponding zeta function in the preceding theorem reduces to the one given by Galkin [5]. Galkin's formula is for odd primes only, and is obtained by solving congruences modulo powers of primes.

The deduction of the formulae for the global case is now fairly routine. Let Γ be the maximal $\mathbb{Z}$-order in the quadratic field A, Λ an arbitrary $\mathbb{Z}$-order and L a Λ-lattice in A; the subscript p as before denotes the localization at a rational prime p. We are interested in an explicit formula for $\zeta_\Lambda(L; s)$. Now Γ_p being the maximal $\mathbb{Z}_p$-order in A, if $\zeta_A(s)$ denotes the Dedekind zeta function for the quadratic field A, then

$$\zeta_A(s) = \Pi \zeta_{\Gamma_p}(\Gamma_p; s)$$

by (2), where the product is over all rational primes p. Since theorem 4 shows that

$$\zeta_{\Gamma_p}(\Gamma_p; s) = \phi_p(p^{-s})^{-1}$$

for any rational prime p (where $\phi_p(t)$ is the polynomial $(1-t)(1-\varepsilon(p)t)$ it follows that

$$\zeta_A(s) = \Pi \phi_p(p^{-s})^{-1}. \tag{21}$$

The product being over all rational primes p. Now given a $\mathbb{Z}$-order Λ in A, there is a set $E(\Lambda)$ of exceptional primes p, (the set of primes that divide the index $|\Gamma:\Lambda|$) such that $\Gamma_p = \Lambda_p$ if and only if $p \notin E(\Lambda)$. Consequently for primes $p \notin E(\Lambda)$, $\zeta_{\Lambda_p}(L_p; s) = \zeta_{\Gamma_p}(\Gamma_p; s)$. On the other hand, if $p \in E(\Lambda)$ and if the $\mathbb{Z}_p$-order isomorphic to L_p has index p^i in Γ_p, then our last theorem implies that

$$\zeta_{\Lambda_p}(L_p; s)/\zeta_{\Gamma_p}(\Gamma_p; s) = (1 - \varepsilon(p)p^{-s}) f_p(r, i; p^{-s}) + p^{r(1-s)-si},$$

where r is determined by the relation $p^r = |\Gamma_p:\Lambda_p|$. $(r > 0)$. The right hand side of the preceding equation is a polynomial in p^{-s}; denote it by $g_p(p^{-s})$. Therefore, the Euler product expression for $\zeta_\Lambda(L; S)$ together with (21) yield the following global formula:

$$\zeta_\Lambda(L; s) = \zeta_A(s) \Pi g_p(p^{-s}), \tag{22}$$

where the product is over all the primes p in the finite set $E(\Lambda)$.

3. Computations using the method of Bushnell and Reiner

Shortly after the appearance of Solomon's [9, 10] papers, Bushnell and Reiner [2, 3] gave an alternative treatment of Solomon's zeta functions by adopting the idea of "zeta integrals on the idele group". Their treatment requires working with lattices over orders in semisimple algebras over the p-adic field $\hat{\mathbb{Q}}_p$, so that suitable topology can be introduced and a Haar measure can be chosen on certain entities related to these algebras. These lead to the formulation of the zeta functions in terms of Haar integrals. We do not go into the details, for our aim is limited to use these integrals in our special case of orders in quadratic fields. In our case, this new machinery may not appear to simplify the matters, but there can be little doubt that it is more powerful, at least for theoretical purposes, than the combinatorial methods of Solomon. For facts about orders with respect to complete base fields, we refer to Reiner [7].

Let $\hat{\mathbb{Q}}_p$ and $\hat{\mathbb{Z}}_p$ denote the p-adic field and its ring of integers ($\mathbb{Z}_p$ still denotes the localization of $\mathbb{Z}$ at p). If Λ is a $\mathbb{Z}_p$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra A, and L a Λ-lattice in a finitely generated A-module V, then by setting

$$\hat{A} = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} A, \quad \hat{\Lambda} = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} \Lambda, \quad \hat{V} = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} V, \quad \hat{L} = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} L.$$

We see that $\hat{\Lambda}$ is a $\hat{\mathbb{Z}}_p$-order in $\hat{A}$, $\hat{V}$ a finitely generated $\hat{A}$-module and $\hat{L}$ a $\hat{\Lambda}$-lattice in $\hat{V}$. Now, in a manner completely analogous to the definition (1), we can define the zeta function $\zeta_{\hat{\Lambda}}(\hat{L}; s)$. The natural map between Λ-lattices in V and $\hat{\Lambda}$-lattices in $\hat{V}$ allow us to conclude that

$$\zeta_\Lambda(L; s) = \zeta_{\hat{\Lambda}}(\hat{L}; s). \tag{23}$$

See [lemma 9; 10]. This relation shows that even for global formulae, we may as well work with completions.

As in earlier sections, let $A = \mathbb{Q} \dot{\oplus} \mathbb{Q}\Theta$ be a quadratic field. Since we shall be working with a fixed prime p, we shall drop the subscript p henceforth while discussing p-adic field or its integers. Then $\hat{A} = \hat{\mathbb{Q}} \oplus \hat{\mathbb{Q}}\Theta$. We note that, depending on the nature of the minimal polynomial of a primitive element of A over $\mathbb{Q}, \hat{A}$ is either a quadratic extension of $\hat{\mathbb{Q}}$ or $\hat{A}$ is isomorphic to the sum of two copies of $\hat{\mathbb{Q}}$ as a $\hat{\mathbb{Q}}$-algebra. [§ 5c; 7]. We set

$$\hat{\Lambda}_r = \hat{\mathbb{Z}} \oplus p^r \hat{\mathbb{Z}}\theta, \quad r = 0, 1, 2, \ldots.$$

Then from what we have already seen about $\mathbb{Z}_p$-orders in a quadratic field, we conclude that $\hat{\Lambda}_0$ is the maximal $\hat{\mathbb{Z}}$-order in $\hat{A}$ and that the chain

$$\hat{\Lambda}_0 \supset \hat{\Lambda}_1 \supset \hat{\Lambda}_2 \supset \cdots$$

gives all the $\hat{\mathbb{Z}}$-orders in $\hat{A}$. The result about isomorphism classes too carries over. So for a fixed $r \geqslant 0$, the isomorphism classes of $\hat{\Lambda}_r$-lattices in A are represented by the $\hat{\Lambda}_r$-lattices $\{\hat{\Lambda}_i : 0 \leqslant i \leqslant r\}$.

We now set $V = \hat{A}$ itself so that we can discuss $\zeta_{\hat{\Lambda}_i}(\hat{\Lambda}_r; s)$ in its proper setting. We define the partial zeta functions $\hat{Z}_{ij}$ as follows

$$\hat{Z}_{ij} = \Sigma |\hat{\Lambda}_i : M|^{-s},$$

where s is a complex variable and the sum is over all $\hat{\mathbb{Z}}$-sublattices M of $\hat{\Lambda}_i$ such that $M \cong \hat{\Lambda}_j$. Thus for fixed positive integers i and r $(0 \leqslant i \leqslant r)$, we see that the zeta function

$$\zeta_{\hat{\Lambda}_r}(\hat{\Lambda}_i, s) = \sum_{j=0}^{r} \hat{Z}_{ij}. \tag{24}$$

Thus as in § 2, we have to evaluate the functions $\hat{Z}_{ij}$. Note that $\hat{Z}_{0,0}$ is nothing but the zeta function of the maximal order $\hat{\Lambda}_0$. So we know this explicitly. If $\hat{A}$ remains a quadratic extension of $\hat{\mathbb{Q}}, \hat{Z}_{0,0}$ depends on the ramification of p in $\hat{A}$ and will be the same as in the formula found in § 2. If $\hat{A} \simeq \hat{\mathbb{Q}} \times \hat{\mathbb{Q}}$, then

$$\hat{Z}_{0,0} = (1 - p^{-s})^{-2}.$$

To evaluate $\hat{Z}_{ij}$ in general, we first express it as an integral following Bushnell and Reiner [2, § 3.4]. Our task is simplified here, for we are taking $V = \hat{A}$. Now $\hat{A}$ as a finite dimensional $\hat{\mathbb{Q}}$-algebra, inherits a topology which makes it a locally compact topological algebra; the units $\hat{A}^X$ of $\hat{A}$ form a locally compact topological group. Let a suitably normalized Haar measure $d^X x$ be chosen on $\hat{A}^X$, and let $\mu^X(S)$ denote the measure of a subset S of $\hat{A}^X$. For $x \in \hat{A}^X$, let $\|x\|$ denote its norm defined in terms of the 'generalized' index as in [2]; namely $\|x\|$ is the 'generalized' index $|Mx:M|$ for any $\hat{\mathbb{Z}}$-lattice M in $\hat{A}$ where

$$|Mx:M| = Mx:Mx \cap M|/|M:Mx \cap M|.$$

The norm $\|x\|$ independent of the lattice M, is multiplicative, and if x is a unit in some $\hat{\mathbb{Z}}$-order in $\hat{A}$, then $\|x\| = 1$. Now adopting formula (11) of [2] to our case, we obtain

$$\hat{Z}_{ij} = \mu^X(\hat{\Lambda}_j^X)^{-1} |\hat{\Lambda}_i : \hat{\Lambda}_j|^{-s} \int_{\hat{A}^X \cap \{\hat{\Lambda}_j : \hat{\Lambda}_i\}} \|x\|^s \mathrm{d}^X x, \quad 0 \leqslant j \leqslant i, \tag{25}$$

where

$$\begin{aligned}\{\hat{\Lambda}_j : \hat{\Lambda}_i\} &= \{x \in \hat{A} : \hat{\Lambda}_j x \subseteq \hat{\Lambda}_i\} \\ &= \{x \in \hat{\Lambda}_i : \hat{\Lambda}_j x \subseteq \hat{\Lambda}_i\} \\ &= p^{i-j} \hat{\Lambda}_j.\end{aligned}$$

We affect a change of variables in the integral of (25) by letting $x = p^{i-j} y, y \in \hat{\Lambda}_j$. Then

$$\mathrm{d}^X x = \mathrm{d}^X y, \quad \|x\| = (p^{-2})^{i-j} \|y\|$$

so

$$\int_{\hat{A}^X\cap\{\hat{\Lambda}_j:\hat{\Lambda}_i\}} \|x\|^s \mathrm{d}^X x = p^{-2s(i-j)} \int_{\hat{A}^X\cap\hat{\Lambda}_j} \|y\|^s \mathrm{d}^X y.$$

Also note that $|\hat{\Lambda}_i:\hat{\Lambda}_j|^{-1} = p^{i-j}$. Thus we have the following form of (25)

$$\hat{Z}_{ij} = \mu^X(\hat{\Lambda}_j^X)^{-1} p^{-s(i-j)} \int_{\hat{A}^X\cap\hat{\Lambda}_j} \|x\|^s \mathrm{d}^X x. \tag{26}$$

One easily concludes that

$$\hat{Z}_{ij} = p^{-s(i-j)}\hat{Z}_{jj}, \quad 0 \leqslant j \leqslant i. \tag{27}$$

This is precisely our formula (ii) of (16). It looks better as there is no extraneous matrix to keep track of.

We now look for a recursive formula for $\hat{Z}_{jj}$, which according to (26) has the form

$$\hat{Z}_{jj} = \mu^X(\hat{\Lambda}_j^X)^{-1} \int_{\hat{A}^X\cap\hat{\Lambda}_j} \|x\|^s \mathrm{d}^X x, \quad j \geqslant 0. \tag{28}$$

For $j=0$, we already know $\hat{Z}_{00}$, for it is the zeta function of the maximal order $\hat{\Lambda}_0$. So we assume $j \geqslant 1$. Now to handle the integral in (28), we break up the range of integration $\hat{A}^X \cap \hat{\Lambda}_j$ into two non-overlapping subsets, $\hat{A}^X \cap \operatorname{Rad}\hat{\Lambda}_j$ and $\hat{A}^X \cap (\hat{\Lambda}_j - \operatorname{Rad}\hat{\Lambda}_j)$. Note that for $j \geqslant 1$, $\hat{\Lambda}_j$ is a local ring; in fact

$$\hat{\Lambda}_j = \hat{\mathbb{Z}} \oplus p^j\mathbb{Z}\theta, \quad \operatorname{Rad}\hat{\Lambda}_j = p\hat{\mathbb{Z}} \oplus p^j\hat{\mathbb{Z}}\theta. \tag{29}$$

Since $\operatorname{Rad}\hat{\Lambda}_j = p\hat{\Lambda}_{j-1} (j \geqslant 1)$, it follows that

$$\begin{aligned}\int_{\hat{A}^X\cap \operatorname{Rad}\hat{\Lambda}_j} \|x\|^s \mathrm{d}^X x &= p^{-2s} \int_{\hat{A}^X\cap\hat{\Lambda}_{j-1}} \|y\|^s \mathrm{d}^X y \\ &= p^{-2s}\mu^X(\hat{\Lambda}_{j-1}^X)\hat{Z}_{j-1,j-1}.\end{aligned} \tag{30}$$

To evaluate the integral over the other set, we note from (29) $\hat{\Lambda}_j/\operatorname{Rad}\hat{\Lambda}_j \cong \mathbb{Z}/p\mathbb{Z}$, whence comparing the unit groups of these rings (and using the fact that $\hat{\Lambda}_j$ is local) we obtain

$$\hat{\Lambda}_j^X/(1+\operatorname{Rad}\hat{\Lambda}_j) \cong (\mathbb{Z}/p\mathbb{Z})^X$$

so that

$$\mu^X(\hat{\Lambda}_j^X) = (p-1)\mu^X(1+\operatorname{Rad}\hat{\Lambda}_j).$$

Also for any $x \in \Lambda_j$ such that $x \notin \operatorname{Rad}\hat{\Lambda}_j$, x is a unit in $\hat{\Lambda}_j$ as well as in $\hat{A}$. Thus

$$\begin{aligned}\int_{\hat{A}^X\cap(\hat{\Lambda}_j - \operatorname{Rad}\hat{\Lambda}_j)} \|x\|^s \mathrm{d}^X x &= (p-1)\mu^X(1+\operatorname{Rad}\hat{\Lambda}_j) \\ &= \mu^X(\hat{\Lambda}_j^X).\end{aligned} \tag{31}$$

Therefore, combining (30) and (31), and using (28), we obtain

$$\begin{aligned}\hat{Z}_{jj} &= 1 + p^{-2s}\{\mu^X(\hat{\Lambda}_{j-1}^X)/\mu^X(\hat{\Lambda}_j^X)\}\hat{Z}_{j-1,j-1} \\ &= 1 + p^{-2s}\{|\hat{\Lambda}_{j-1}^X:\hat{\Lambda}_j^X|\}\hat{Z}_{j-1,j-1}.\end{aligned}$$

Finally, we show how to compute $|\hat{\Lambda}_{j-1}^X:\hat{\Lambda}_j^X|$. Note that $\hat{\Lambda}_{j-1}^X$ acts transitively on the set of all $\hat{\mathbb{Z}}$-sublattices between $\hat{\Lambda}_{j-1}$ and $p\hat{\Lambda}_{j-1}$ and having their order (or coefficient ring) as $\hat{\Lambda}_j$. Since the stabilizer of any such sublattice is $\hat{\Lambda}_j^X$, we see that the index $|\hat{\Lambda}_{j-1}^X:\hat{\Lambda}_j^X|$ is precisely the number of sublattices between $\hat{\Lambda}_{j-1}$ and $p\hat{\Lambda}_{j-1}$ having $\hat{\Lambda}_j$ as their order. From Theorem 2, we conclude that

$$\begin{aligned} |\hat{\Lambda}_{j-1}^X:\hat{\Lambda}_j^X| &= p, \qquad j>1 \\ &= p-\varepsilon(p), \quad j=1. \end{aligned}$$

Thus the final form of the recursive relations for $\hat{Z}_{jj}$ is as follows:

$$\begin{aligned} \hat{Z}_{jj} &= 1+p^{-2s+1}\hat{Z}_{j-1,j-1}, \quad j>1 \\ \hat{Z}_{1,1} &= 1+p^{-2s}(p-\varepsilon(p))\hat{Z}_{00}. \end{aligned} \tag{32}$$

Note that in §2, in relations (iii) of (16) and (15) we had found these relations by Solomon's determinant method. It is now a routine work to compute various $\hat{Z}_{ij}$ (by using (27)) and then Solomon's zeta functions (using (25) and (23)).

Now that the two methods of calculating zeta functions of quadratic orders (one by evaluating determinants and the other by integration) may be compared, it may appear that the method of integration is superior due to its simplicity. But here it should be pointed out that Solomon's papers raise some other questions in addition to the straightforward calculation of zeta functions. See for example his conjecture about the determinant $\mathscr{A}(t)$ [9]. It seems certain that Solomon's combinatorial method will continue to be useful in questions relating to the determinant $\mathscr{A}(t)$.

Acknowledgements

The material presented here, except that in §3, formed part of the thesis submitted by the author to the Graduate School of the University of Wisconsin, Madison, in August 1980. The author expresses his gratitude to Prof. Louis Solomon under whose supervision this research was carried out.

References

[1] Borevich Z I and Shafarevich I R, *Number theory* (New York: Academic Press) (1986)
[2] Bushnell C J and Reiner I, Zeta functions of arithmetic orders, *Math. Z.* **173** (1980) 135–161
[3] Bushnell C J and Reiner I, *L*-functions of arithmetic orders, *J. Reine Angew. Math.* **327** (1981) 156–183
[4] Curtis C W and Reiner I, *Methods of representation theory* (New York: John Wiley) Vol. 1 (1981)
[5] Galkin V M, ζ-functions of some one-dimensional rings, *Izv. Akad. Nauk SSR* **37** (1973) 9–19
[6] Lang S, *Elliptic functions* (Reading: Addison Wesley) (1964) p. 98
[7] Reiner I, *Maximal orders* (New York: Academic Press) (1975)
[8] Rota G C, On the foundation of combinatorial theory, *Z. Wahrscheinlichkeits theorie and Verw. Gabiete* **2** (1964) 340–368
[9] Solomon L, 1979 Partially ordered sets with colors, *Proc. Symp. Pure Math. XXXIV* (Providence, RI: Am. Math. Soc.) (ed.) Roy-Choudhury 309–329
[10] Solomon L, Zeta functions and integral representation theory, *Adv. Math.* **26** (1977) 306–326