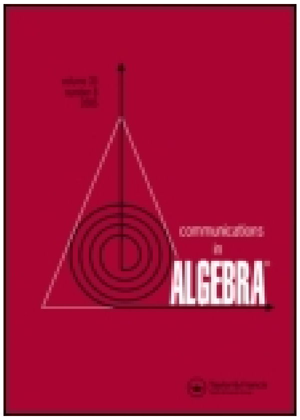

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# A Generalization of Commutativity Degree of Finite Groups

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# A GENERALIZATION OF COMMUTATIVITY DEGREE OF FINITE GROUPS

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***The commutativity degree of a finite group is the probability that two arbitrarily chosen group elements commute. This notion has been generalized in a number of ways. The object of this article is to study yet another generalization of the same notion, which further extends some of the existing generalizations.***

***Key Words:*** Commutativity degree; Finite groups; Group characters.

***2010 Mathematics Subject Classification:*** 20C15; 20D60; 20P05.

## 1. INTRODUCTION

Given a positive integer $n$, consider the free group of words (always reduced) on $n$ generators (letters) $x_1, x_2, \ldots, x_n$. A word $\omega(x_1, x_2, \ldots, x_n)$ is called *admissible* (see [1]) if it is a product of $x_{i_1}, x_{i_2}, \ldots, x_{i_m}, x_{i_1}^{-1}, x_{i_2}^{-1}, \ldots, x_{i_m}^{-1}$ in some order, where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. In other words, an admissible word need not contain all the $n$ generators, and every generator appearing in an admissible word appears exactly twice (with indices $+1$ and $-1$).

Let $G$ be a finite group and $g \in G$. Consider a nontrivial admissible word $\omega(x_1, x_2, \ldots, x_n)$, $n \geq 2$, and the corresponding word map $\alpha_\omega : G^n \to G$ induced by $\omega$. In this article, we study the ratio

$$\mathrm{Pr}_g^{\omega}(G) = \frac{\zeta_n^{\omega}(g)}{|G^n|}, \tag{1}$$

where $\zeta_n^{\omega}(g) = |\{(g_1, g_2, \ldots, g_n) \in G^n : \omega(g_1, g_2, \ldots, g_n) = g\}|$. Note that $0 \leq \mathrm{Pr}_g^{\omega}(G) \leq 1$. In fact, $\mathrm{Pr}_g^{\omega}(G)$ is the probability that an arbitrarily chosen $n$-tuple in $G^n$ is mapped to $g$ under $\alpha_\omega$. Clearly, $\mathrm{Pr}_g^{\omega}(G) = 0$ if $G$ is abelian and $g \neq 1$ (more generally, if $g \notin G'$, the commutator subgroup of $G$). Also, considering the elements of $G^n$ whose all but one coordinates are in the center $Z(G)$, we have

$$\mathrm{Pr}_1^{\omega}(G) \geq \frac{n|G : Z(G)| - n + 1}{|G : Z(G)|^n} > 0.$$

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Note that, for $\omega(x_1, x_2) = x_1x_2x_1^{-1}x_2^{-1}$, we have $\Pr_g^{\omega}(G) = \Pr_g(G)$, a notion introduced and studied extensively by Pournaki and Sobhani in [13], which coincides with the usual commutativity degree $\Pr(G)$ of $G$ if we take $g = 1$. It may be recalled (see, for example, [4]) that $\Pr(G) = \frac{k(G)}{|G|}$, where $k(G)$ denotes the number of conjugacy classes of $G$ and equals the size of $\mathrm{Irr}(G)$, the set of irreducible complex characters of $G$.

Finally, for $\omega(x_1, x_2, \ldots, x_n) = x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$, we have $\Pr_g^{\omega}(G) = \Pr_g^{n}(G)$, a detailed study of which can be found in [11].

The motivation for the present work comes from the works of Garion and Shalev in [3, 14] on the equidistribution and measure preservation of certain word maps, especially, the commutator map. They made extensive use of a probability distribution associated to these maps and settled some open problems regarding $T$-systems and the *PRA* graph. For nontrivial admissible words, which are obvious generalizations of commutator words, this probability distribution coincides with $\Pr_g^{\omega}(G)$ given by (1). In [14, Section 2], Shalev also talks about the interest in the study of the kernel of word maps and the probability that an $n$-tuple of group elements is mapped to the identity element under a word map.

## 2. BASIC PROPERTIES

In this section, we study some basic properties of $\Pr_g^{\omega}(G)$. We begin with the following lemma.

**Lemma 2.1.** *Let $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Then*

$$\omega(x_1, x_2, \ldots, x_n) = \omega_1\, u\, \omega_2\, v\, \omega_3\, u^{-1}\, \omega_4\, v^{-1}\, \omega_5,$$

*where $u = x_i$ or $x_i^{-1}$ and $v = x_j$ or $x_j^{-1}$ for some $i$, $j$ with $1 \leq i \neq j \leq n$, and each $\omega_k$, $1 \leq k \leq 5$, is a word in the remaining $n - 2$ letters.*

*Proof.* Clearly, the lemma holds for $n = 2$. Assume that $n > 2$. Then

$$\omega(x_1, x_2, \ldots, x_n) = y\, \omega_6\, y^{-1}\, \omega_7, \tag{2}$$

where $y = x_r$ or $x_r^{-1}$ for some $r$ with $1 \leq r \leq n$, and $\omega_6, \omega_7$ are words in the remaining $n - 1$ letters. Note that we may have $\omega_7 = 1$, but we always have $\omega_6 \neq 1$ as $\omega(x_1, x_2, \ldots, x_n)$ is a reduced word. If $\omega_6$ is an admissible word, then, by induction hypothesis, the lemma holds for $\omega_6$, and hence for $\omega(x_1, x_2, \ldots, x_n)$. On the other hand, if $\omega_6$ is not an admissible word, then

$$\omega_6 = \omega_8\, z\, \omega_9 \qquad \text{and} \qquad \omega_7 = \omega_{10}\, z^{-1}\, \omega_{11}, \tag{3}$$

where $z = x_s$ or $x_s^{-1}$ for some $s$ with $1 \leq s \neq r \leq n$, and $\omega_8, \omega_9, \omega_{10}, \omega_{11}$ are words in the $n - 2$ letters other than $x_r$ and $x_s$. Hence, it follows, from (2) and (3), that the lemma holds for $\omega(x_1, x_2, \ldots, x_n)$. This completes the proof. □

**Proposition 2.2.** *Let $G$ be a finite group and $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Then*:

(a) $\Pr_1^{\omega}(G) = 1$ *if and only if $G$ is abelian*;

(b) $\Pr_g^{\omega}(G) = \Pr_h^{\omega}(G)$ *if* $g, h \in G'$ *are conjugate in* $G$;

(c) $\Pr_g^{\omega}(G) = \Pr_h^{\omega}(G)$ *if* $g, h \in G'$ *generate the same cyclic subgroup of* $G$. *Consequently,*

$$\Pr_g^{\omega}(G) = \frac{1 - \Pr_1^{\omega}(G)}{p-1} \tag{4}$$

*if* $|G'| = p$, *a prime, and* $1 \neq g \in G'$.

***Proof.*** (a) Let $\Pr_1^{\omega}(G) = 1$ and $a, b \in G$. Then, by Lemma 2.1 (with $x_i = a$, $x_j = b$ and $x_t = 1$ for other values of $t$), we have $ab = ba$. Converse is obvious.

(b) By [1, Theorem 2.4], $\zeta_n^{\omega}$ is a character (in particular, a class function) of $G$. So, the result follows from (1).

(c) Since $\zeta_n^{\omega}$ is a rational valued character of $G$, we have $\zeta_n^{\omega}(g) = \zeta_n^{\omega}(h)$, using [6, Lemma 5.22]. So, the first part follows from (1). Now, using the definition of probability, we have

$$1 = \sum_{h \in G'} \Pr_h^{\omega}(G) = \Pr_1^{\omega}(G) + (p-1)\Pr_g^{\omega}(G).$$

This proves the second part. □

$\Pr_g^{\omega}(G)$ respects the Cartesian product in the following sense.

**Proposition 2.3.** *Let* $H$ *and* $K$ *be two finite groups,* $(h, k) \in H' \times K'$ *and* $\omega(x_1, x_2, \ldots, x_n)$ *be a nontrivial admissible word. Then*

$$\Pr_{(h,k)}^{\omega}(H \times K) = \Pr_h^{\omega}(H)\Pr_k^{\omega}(K).$$

***Proof.*** Since the elements $((h_1, k_1), \ldots, (h_n, k_n)) \in (H \times K)^n$ satisfying the equation $\omega((h_1, k_1), \ldots, (h_n, k_n)) = (h, k)$ are in one-to-one correspondence with the elements $((h_1, \ldots, h_n), (k_1, \ldots, k_n)) \in H^n \times K^n$ satisfying the equation $(\omega(h_1, \ldots, h_n), \omega(k_1, \ldots, k_n)) = (h, k)$, the result follows from (1). □

A group $G$ is said to be *isoclinic* (see [5, p. 133]) to a group $H$ if there exist two isomorphisms $\phi : G/Z(G) \longrightarrow H/Z(H)$ and $\psi : G' \longrightarrow H'$ such that $a_H \circ (\phi \times \phi) = \psi \circ a_G$, where the maps $a_G : G/Z(G) \times G/Z(G) \longrightarrow G'$ and $a_H : H/Z(H) \times H/Z(H) \longrightarrow H'$ are induced by the commutator maps of $G$ and $H$, respectively. The following result shows that $\Pr_g^{\omega}(G)$ is an invariant under isoclinism of finite groups.

**Proposition 2.4.** *Let* $G$ *and* $H$ *be two finite groups and* $(\phi, \psi)$ *be an isoclinism from* $G$ *to* $H$. *If* $g \in G'$ *and* $\omega(x_1, x_2, \ldots, x_n)$ *is a nontrivial admissible word, then*

$$\Pr_g^{\omega}(G) = \Pr_{\psi(g)}^{\omega}(H).$$

***Proof.*** Given $g_1, g_2, \ldots, g_n \in G$, it is easy to see from the definition of an admissible word that $\omega(g_1, g_2, \ldots, g_n)$ lies in the commutator subgroup of

$\langle g_1, g_2, \ldots, g_n \rangle$, and hence, it is a product of commutators in the products of $g_i$. Consequently, it follows from the definition of isoclinism of groups that

$$\psi(\omega(g_1, g_2, \ldots, g_n)) = \omega(h_1, h_2, \ldots, h_n),$$

where $h_i \in H$ such that $\phi(g_iZ(G)) = h_iZ(H)$, $1 \leq i \leq n$. Thus, one can see that there is an one to one correspondence between the $n$-tuples $(g_1Z(G), \ldots, g_nZ(G)) \in (G/Z(G))^n$ satisfying the equation $\omega(g_1, \ldots, g_n) = g$ and the $n$-tuples $(h_1Z(H), \ldots, h_nZ(H)) \in (H/Z(H))^n$ satisfying the equation $\omega(h_1, \ldots, h_n) = \psi(g)$.

Again, since $\omega(x_1, x_2, \ldots, x_n)$ is admissible, the map $\alpha_\omega : G^n \longrightarrow G$ factors through the quotient group $G^n/Z(G^n) = (G/Z(G))^n$. So, we have

$$\begin{aligned} &|\{(g_1, g_2, \ldots, g_n) \in G^n : \omega(g_1, g_2, \ldots, g_n) = g\}| \\ = \ &|\{(g_1Z(G), \ldots, g_nZ(G)) \in (G/Z(G))^n : \omega(g_1, \ldots, g_n) = g\}| \times |Z(G)|^n. \end{aligned}$$

Similarly, we have the corresponding equation for $H$. Hence, noting that $|G : Z(G)| = |H : Z(H)|$, we have

$$\begin{aligned} \mathrm{Pr}_g^\omega(G) &= \frac{\zeta_n^\omega(g)}{|G^n|} \\ &= \frac{|\{(g_1Z(G), \ldots, g_nZ(G)) \in (G/Z(G))^n : \omega(g_1, \ldots, g_n) = g\}|}{|G : Z(G)|^n} \\ &= \frac{|\{(h_1Z(H), \ldots, h_nZ(H)) \in (H/Z(H))^n : \omega(h_1, \ldots, h_n) = \psi(g)\}|}{|H : Z(H)|^n} \\ &= \frac{\zeta_n^\omega(\psi(g))}{|H^n|} = \mathrm{Pr}_{\psi(g)}^\omega(H). \end{aligned}$$

This completes the proof. □

## 3. SOME BOUNDS

In this section, we obtain some bounds for $\mathrm{Pr}_g^\omega(G)$. We begin with the following result.

**Proposition 3.1.** *Let $G$ be a finite group, $g \in G'$, and $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Then*

$$\frac{\mathrm{Pr}_g(G)}{|G : Z(G)|^{n-2}} \leq \mathrm{Pr}_g^\omega(G)$$

*with equality if $n = 2$.*

*Proof.* If $g$ is not a commutator, then $\mathrm{Pr}_g(G) = 0$, and so there is nothing to prove. Let $a$, $b \in G$ be such that $aba^{-1}b^{-1} = g$. Then, using Lemma 2.1, with $x_i = a$ or $a^{-1}$, $x_j = b$ or $b^{-1}$ (so that $u = a$ and $v = b$) and choosing $x_t \in Z(G)$ for all $t$ other than $i$ and $j$, we see that $g$ lies in the image of the word map $\alpha_\omega$. Hence, the result follows from (1). □

**Remark 3.2.** Let $G$ be a finite group and $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Then some of the consequences of Proposition 3.1 may be listed as follows:

(a) If $G$ is a finite non-abelian simple group, then, using Ore conjecture (see [12]), which has been established recently in [9, Theorem 1], we have $\mathrm{im}(\alpha_\omega) = G$, that is, every element of $G$ is of the form $\omega(g_1, g_2, \ldots, g_n)$ for some $g_1, g_2, \ldots, g_n \in G$.
(b) If $|\mathrm{cd}(G)| = 2$, then, using [13, Theorem 2.2], we have $G' \subseteq \mathrm{im}(\alpha_\omega)$.
(c) In [8], a number of conditions have been listed under each of which every element of $G'$ is a commutator. It follows that under these conditions $G' \subseteq \mathrm{im}(\alpha_\omega)$.

Let $G$ be a finite group and $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Since $\zeta_n^\omega$ is a character of $G$, we have

$$\zeta_n^\omega = \sum_{\chi \in \mathrm{Irr}(G)} [\zeta_n^\omega, \chi]\chi,$$

where $[\,,\,]$ denotes the inner product of class functions on $G$. So, it follows from (1) that

$$\Pr_g^\omega(G) = \sum_{\chi \in \mathrm{Irr}(G)} \frac{[\zeta_n^\omega, \chi]}{|G|^n}\chi(g). \tag{5}$$

This formula, in light of the following lemmas, plays a crucial role in the study of $\Pr_g^\omega(G)$.

**Lemma 3.3.** *Let $G$ be a finite group and $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. If $\chi \in \mathrm{Irr}(G)$, then*

$$\left(\frac{|G|}{\chi(1)}\right)^{n-1} \le [\zeta_n^\omega, \chi] \le \frac{|G|^{n-1}}{\chi(1)}.$$

*Proof.* The proof is essentially same as that of [1, Theorem 2.4], except that the induction here starts at $n = 2$, when, by the classical formula of Frobenius (see [2]), we have

$$\zeta_2^\omega = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|}{\chi(1)}\chi.$$

So, for all $\chi \in \mathrm{Irr}(G)$, we have

$$[\zeta_2^\omega, \chi] = \frac{|G|}{\chi(1)}.$$

Assume that the result is true for all $n < k$, where $k \ge 3$. Then, using the induction hypothesis, it is a routine matter to see that, for each of the expressions of $\zeta_k^\omega$ derived in the proof of [1, Theorem 2.4] (see Subcase 1.1, Subcase 1.2, and Case 2), we have

$$\left(\frac{|G|}{\chi(1)}\right)^{k-1} < [\zeta_k^\omega, \chi] \leq \frac{|G|^{k-1}}{\chi(1)} \qquad \text{for all } \chi \in \mathrm{Irr}(G).$$

This completes the proof. ☐

**Lemma 3.4.** *Let* $\sum_{i=1}^n r_i(a_i - 1) = 0$, *where* $r_i$*'s are positive rational numbers, and* $a_i \in \mathbb{C}$, $|a_i| \leq 1$ *for all* $i = 1, 2, \ldots, n$. *Then* $a_i = 1$ *for* $1 \leq i \leq n$.

*Proof.* Note that

$$0 = \sum_{i=1}^n \mathrm{Re}(r_i(a_i - 1)) = \sum_{i=1}^n r_i(\mathrm{Re}(a_i) - 1) \leq \sum_{i=1}^n r_i(|a_i| - 1) \leq 0.$$

So, for $1 \leq i \leq n$, we have $\mathrm{Re}(a_i) = |a_i| = 1$, whence $a_i = 1$. ☐

**Lemma 3.5** ([7, Proposition 2.6(iv)]). *Let G be a finite non-abelian group and p be the smallest prime divisor of* $|G|$. *Then*

$$\mathrm{Pr}(G) \leq \frac{1}{p^2}\left(1 + \frac{p^2 - 1}{|G'|}\right).$$

*Proof.* Since $|G : G'| = |\{\chi \in \mathrm{Irr}(G) : \chi(1) = 1\}|$ and $\chi(1)$ divides $|G|$ for all $\chi \in \mathrm{Irr}(G)$, we have, using the degree equation,

$$|G| = |G : G'| + \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) \neq 1}} \chi(1)^2 \geq |G : G'| + p^2(|\mathrm{Irr}(G)| - |G : G'|)$$

Hence, the lemma follows. ☐

**Proposition 3.6.** *Let G be a finite group,* $g \in G'$, *and* $\omega(x_1, x_2, \ldots, x_n)$ *be a nontrivial admissible word. Then*:

(a) $\mathrm{Pr}_g^\omega(G) \leq \mathrm{Pr}_1^\omega(G) \leq \mathrm{Pr}(G)$;
(b) $\mathrm{Pr}_g^\omega(G) = \mathrm{Pr}_1^\omega(G)$ *if and only if* $g = 1$.

*Consequently,* $\mathrm{Pr}_g^\omega(G) = 1$ *if and only if* $g = 1$ *and G is abelian.*

*Proof.* (a) Using (5), Lemma 3.3, and the fact that $|\chi(g)| \leq \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, we have

$$\begin{aligned}\mathrm{Pr}_g^\omega(G) &\leq \sum_{\chi \in \mathrm{Irr}(G)} \frac{[\zeta_n^\omega, \chi]}{|G|^n}|\chi(g)| \\ &\leq \sum_{\chi \in \mathrm{Irr}(G)} \frac{[\zeta_n^\omega, \chi]}{|G|^n}\chi(1) = \mathrm{Pr}_1^\omega(G) \\ &\leq \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|^{n-1}}{\chi(1)}\frac{\chi(1)}{|G|^n} = \frac{|\mathrm{Irr}(G)|}{|G|} = \mathrm{Pr}(G).\end{aligned}$$

(b) Using (5), Lemma 3.3, Lemma 3.4, and the fact that $|\chi(g)| \leq \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, we have

$$\begin{aligned}
\mathrm{Pr}_g^{\omega}(G) &= \mathrm{Pr}_1^{\omega}(G) \\
&\iff \sum_{\chi\in\mathrm{Irr}(G)} \frac{[\zeta_n^{\omega},\chi]\chi(1)}{|G|^n}\left(\frac{\chi(g)}{\chi(1)}-1\right)=0 \\
&\iff \chi(g)=\chi(1) \qquad \forall\chi\in\mathrm{Irr}(G) \\
&\iff g=1.
\end{aligned}$$

This completes the proof. □

The following result gives a universal lower bound for $\mathrm{Pr}_1^{\omega}(G)$.

**Proposition 3.7.** *Let G be a finite group and* $\omega(x_1, x_2, \ldots, x_n)$ *be a nontrivial admissible word. Then*

$$\mathrm{Pr}_1^{\omega}(G) \geq \frac{1}{|G'|}\left(1+\frac{|G'|-1}{|G:Z(G)|^{n/2}}\right). \tag{6}$$

*In particular,* $\mathrm{Pr}_1^{\omega}(G) > \frac{1}{|G'|}$ *if G is non-abelian.*

*Proof.* Using (5), Lemma 3.3 and [10, Theorem 1] together with the facts that $|G : G'| = |\{\chi \in \mathrm{Irr}(G) : \chi(1) = 1\}|$ and $\chi(1) \leq |G : Z(G)|^{\frac{1}{2}}$ for all $\chi \in \mathrm{Irr}(G)$, we have

$$\begin{aligned}
\mathrm{Pr}_1^{\omega}(G) &\geq \sum_{\chi\in\mathrm{Irr}(G)}\left(\frac{|G|}{\chi(1)}\right)^{n-1}\frac{\chi(1)}{|G|^n} \\
&= \frac{1}{|G'|}+\frac{1}{|G|}\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi(1)\neq 1}}\frac{1}{\chi(1)^{n-2}} \\
&\geq \frac{1}{|G'|}+\frac{1}{|G:Z(G)|^{\frac{n-2}{2}}}\left(\mathrm{Pr}(G)-\frac{1}{|G'|}\right) \\
&\geq \frac{1}{|G'|}\left(1+\frac{|G'|-1}{|G:Z(G)|^{n/2}}\right)
\end{aligned}$$

This completes the proof. □

Note that, for $n = 2$, the equality holds in (6) under a number of conditions listed in [10, Theorem 2]. As noted in [11, Proposition 6.1], the equality also holds in (6) if $G$ is a finite group of central type with $|\mathrm{cd}(G)| = 2$ and $\omega(x_1, x_2, \ldots, x_n) = x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$.

We conclude our discussion with the following result, which generalizes [13, Proposition 5.2].

**Proposition 3.8.** *Let $G$ be a finite non-abelian group, $g \in G'$, and $p$ be the smallest prime divisor of $|G|$. If $\omega(x_1, x_2, \ldots, x_n)$ is a nontrivial admissible word and $g \neq 1$, then $\Pr_g^{\omega}(G) < \frac{1}{p}$. In particular, we have $\Pr_g^{\omega}(G) < \frac{1}{2}$.*

*Proof.* Let $\Pr_g^{\omega}(G) \geq \frac{1}{p}$. Then, by Proposition 3.6, $\Pr(G) > \frac{1}{p}$ . Therefore, by Lemma 3.5, we have $|G'| < p + 1$. This means that $|G'| = p$. Hence, by (4), $\Pr_1^{\omega}(G) \leq \frac{1}{p}$. This contradicts Proposition 3.7. □

In [11, Proposition 6.7], it has been proved that for each $\varepsilon > 0$ and for each prime $p$, there exists a finite group $G$ such that $\left|\Pr_g^{n}(G) - \frac{1}{p}\right| < \varepsilon$ for each $g \in G'$. In this sense, the bound mentioned in Proposition 3.8 is the best possible.

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