CREATION OF PARTICLES IN CURVED SPACES

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Math

CERTIFICATE

I certify that the dissertation entitled **"Creation of particles in curved spaces"***submitted by* **Ms. Beena George** *in partialfulfilment of the requirements for the degree of* **Master of Philosophy** *is the outcome of a study undertaken by the candidate.*

I certify that the sources from which ideas borrowed, have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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PREFACE

In the absence of viable theory of quantum gravity, the study of influence of the gravitational field on quantum phenomena has been very much interesting, from the last decade. In the early days of quantum theory, many calculations were undertaken in which electromagnetic field was considered as a classical background field interacting with quantized matter fields. Such a semi-classical approximation yields some results that are in complete accordance with the full theory of quantum electrodynamics. A similar regime exists for quantum aspects of gravity in which the gravitational field is retained as a classical background field while the matter fields are quantized in the usual manner. Adopting Einstein's theory of general relativity, as a description of gravity, one is led to the subject of quantum field theory in a curved background of spacetime.

In Einstein's theory of gravity, curvature manifests gravitation. As many results of flat spacetime are not true in curved spacetime, vacuum state is absolute in flat spacetime, whereas it is not so in curved spacetime. As a result creation of particles is expected in curved background of the space-time.

The possibility of creation of particles, in curved background, was first discussed by S.W. Hawking, in the context of black holes. Later on, this type of approach was adopted by L. Parker, B.L. Hu, S.A. Fulling, B.K. Berger, J. Audretsch, E. Mottola, L.H. Ford and others.

This dissertation contains a systematic survey of the work done by different scientists on creation of spinless particles in different cosmological models, Chapter I is introductory and contains some basic ideas of scalar fields in flat as well as curved spacetimes, idea of creation of particles etc. In Chapter II, possibility of creation of spinless particles in different cosmological models are discussed. Chapter III contains creation of particles in anisotropically expanding universe. Chapter IV consists of idea of Quantum Equivalence Principle and discussion on possibility of creation of spinless particles, using this principle. Similar type of work in higher-dimensional Kàluza-Klein spacetimes has been included in the last Chapter.

CONTENTS

CHAPTER I

INTRODUCTION

Gravity is not an interaction which can be turned off or on at will. This property of gravity encouraged Einstein to think that it must be intrinsic to the region, where gravitating matter exists. This intrinsic property of the region was identified by its geometry. So in Einstein's theory, spacetime geometry manifests gravitation. Now if a quantum field interacts with unquantized gravitational field, it is natural to study them in curved spacetimes.

In the background of curved spacetimes it is seen that creation of particles is a spontaneous consequence of quantum field theory. It occurs in particle-antiparticle pairs in the model under consideration and does not violate local conservation laws[1]. Before going into details for the study of quantum fields in curved spacetimes, it is reasonable to know them in flat spacetime which is given as follows.

1.1 Quantized Scalar fields in Minkowski Spacetime[1,2,3]

In flat spacetime, Langrangian density for the scalar field $\phi(x)$ is given by

$$L = \frac{1}{2}\left(\eta^{ij}\,\partial_i\phi\,\partial_j\phi - m^2\phi^2\right) \tag{1.1}$$

where 'm' is the mass of the scalar field $\phi(x)$.

The action is given by

$$A = \int L(x)\, d^4x$$

varying the field $\phi(x)$ and using the action principle $\delta A = 0$, we get the field equation as

$$(\Box + m^2)\,\phi = 0 \qquad (1\cdot 2)$$

where $\Box = \eta^{ij}\partial_i \partial_j$

One set of the solution of (1.2) for the mode k is given by

$$U_k(t,x) = \left[2\omega(2\pi)^{n-1}\right]^{-\frac{1}{2}} e^{(ikx - i\omega t)} \qquad (1\cdot 3)$$

where

$$\omega = (k^2 + m^2)^{\frac{1}{2}}$$

and

$$k = |\vec{k}| = \left[\sum_{i=1}^{n} k_i^2\right]^{\frac{1}{2}} \qquad (-\infty < k_i < \infty;\ i = 1, 2, \cdot\ n-1)$$

Here $U_k(t,x)$ are positive frequency modes with respect to 't' because it is eigen-function of the operator $\frac{\partial}{\partial t}$ defined as

$$\frac{\partial}{\partial t} U_k(t,x) = -i\omega\, U_k(t,x)$$

The scalar product of two scalar fields ϕ_1 and ϕ_2 is defined as

$$(\phi_1, \phi_2) = -i \int \left\{ \phi_1 \partial_t \phi_2^* - [\partial_t \phi_1]\, \phi_2^* \right\} d^{n-1}x$$

$$= -i \int \phi_1 \overleftrightarrow{\partial_t} \phi_2^*\, d^{n-1}x\,, \qquad (1\cdot 4)$$

where t = constant defines the hypersurface enclosing n-dim. region.

Using (1.3) one gets for U_k

$$\left(U_k , U_k^{'}\right) = \delta^{n-1}\left(k-k'\right) \tag{1.5}$$

The system is quantized by treating ' ϕ ' as an operator and applying the equal-time commutation relations

$$\begin{aligned} \left[\phi(t,x), \phi(t,x')\right] &= 0 \\ \left[\pi(t,x), \pi(t,x')\right] &= 0 \\ \left[\phi(t,x), \pi(t,x')\right] &= i\,\delta^{n-1}(x-x') \end{aligned} \tag{1.6}$$

where 'Π' is the canonical conjugate momentum for ϕ defined as

$$\pi = \frac{\partial L}{\partial(\partial_t \phi)} = \partial_t \phi \tag{1.6b}$$

U_k and its complex conjugate form a complete orthonormal basis with scalar product (1.4). So $\phi(t,x)$ may be expanded in terms of U_k and U_k^* (U_k^* is the complex conjugate of U_k) as

$$\phi = \sum_k \left[a_k U_k + a_k^{\dagger} U_k^{*}\right] \tag{1.7}$$

where k's are discrete modes.

The equal-time commutation relations (1.6) imply that a_k and $a_k^{\dagger}$ in (1.7) satisfy

$$\begin{aligned} [a_k , a_{k'}] &= 0 \\ [a_k^\dagger , a_{k'}^\dagger] &= 0 \\ [a_k , a_{k'}^\dagger] &= \delta_{kk'} \end{aligned} \qquad (1 \cdot 8)$$

In the Heisenberg picture, the quantum states span a Hilbert space. A convenient basis in Hilbert space is the Fock representation. Therefore we can construct the normalized basis ket vectors $|\rangle$, from the vector $|0\rangle$ (called the vacuum or no particle state). This state $|0\rangle$ has the property that it is annihilated by all the a_k operators:

$$a_k |0\rangle = 0 \quad , \forall k \qquad (1 \cdot 9)$$

and also $|0$ has the property that it is created by all the $a_k^\dagger$ operators.

$$a_k^\dagger |0\rangle = |1_k\rangle \, , \forall k \qquad (1 \cdot 10)$$

Similarly we can also construct many particle states as

$$a_{k_1}^\dagger \; a_{k_2}^\dagger \cdots \; a_{k_j}^\dagger \; |0\rangle = | 1k_1 , 1k_2 , 1k_3 \cdots 1k_j \rangle$$

Also one gets

$$a_k^\dagger |n_k\rangle = (n+1)^{\frac{1}{2}} \; |(n+1)_k\rangle \qquad (1 \cdot 11 a)$$

$$a_k |n_k\rangle = n^{\frac{1}{2}} \; |(n-1)_k\rangle \qquad (1 \cdot 11 b)$$

To show the significance of the above stated Fock states, one can exa mine the Hamiltonian and momentum operators. These quantities are obtained from the stress-energy-momentum tensor, T_{ij} which is defined as

$$T_{ij} = \phi_{,i}\,\phi_{,j} - \frac{1}{2}\eta_{ij}\,\eta^{\lambda\delta}\,\phi_{,\lambda}\,\phi_{,\delta} + \frac{1}{2}m^2\phi^2\eta_{ij} \qquad (1.12)$$

The Hamiltonian density is given by

$$T_{tt} = \frac{1}{2}\left[(\partial_t\phi)^2 + \sum_{i=1}^{n-1}(\partial_t\phi)^2 + m^2\phi^2\right] \qquad (1.13)$$

and the momentum density

$$T_{ti} = \partial_t\phi\,\partial_i\phi\,, \qquad i = 1, \ldots, n-1 \qquad (1.14)$$

in terms of Minkowski co-ordinate (t,x). Substituting the value of ϕ from (1.7) in (1.13) one gets

$$\begin{aligned} H &= \int T_{tt}\,d^{n-1}x \\ &= \frac{1}{2}\sum_k \left(a_k^{+}a_k + a_k a_k^{+}\right)\omega \end{aligned} \qquad (1.15)$$

Similarly substituting the value of ' ϕ ' from (1.7) in (1.14) one gets the momentum as

$$p_i = \int_t T_{ti}\,d^{n-1}x = \sum_k a_k^{+}\,a_k\,k_i \qquad (1.16)$$

Again using the commutation relations (1.8) in (1.5) we can rewrite

$$H = \sum_k \left(a_k^{+}a_k + \frac{1}{2}\right)\omega \qquad (1.17)$$

Both H and P_i commute with the operator N

$$[N, H] = [N, P_i] = 0 \tag{1.18}$$

where $N = \sum_k N_k$ and $N_k = a_k^\dagger a_k$ (1.19)

From (1.9) and (1.11b) one gets

$$\langle 0 | N_k | 0 \rangle = 0 \quad \forall k \tag{1.20}$$

Similarly

$$\langle {}^1n_{k_1} {}^2n_{k_2} \cdots {}^in_{k_j} | N_{k_i} | {}^1n_{k_1} n_{k_2} \cdots {}^in_{k_j} \rangle = {}^in \tag{1.21}$$

Thus the expectation value of the operators N_k is is the integer ${}^in$. If N_k is summed up over all i

$$\langle | N | \rangle = \sum_i {}^in \tag{1.22}$$

This simple relationship between N_k & n_k suggests the name, 'number operators for the mode k', n_k and 'total number operator', N.

Also another interpretation from (1.18) is that eigenstates of N are also eigenstates of both H and P. When ${}^in$ increases by one, $\langle|H|\rangle$ and $\langle|P|\rangle$ also increase by ω_i and k_i respectively. Therefore ${}^in$ can be interpreted as the number of quanta, each of energy ω_i and momentum k_i, for a particular mode k_i. Thus the state $| {}^1n_k, {}^2m_1 \cdot {}^3m_1 \ldots \ldots {}^jn_k \rangle$ is the state containing ${}^1n$ quanta in the mode with momentum k_1, ${}^2n$ quanta in the mode with momentum k_2, etc.

From (1.11) we get a useful physical interpretation for the operators a_k and $a_k^\dagger$. In (1.11(a)) $a_k^\dagger$ increases the number of quanta by one. So the name 'creation operator'. Similarly, in (1.11b), a_k reduces the number of quanta by one. So a_k is known as the annihilation operator.

1.2 Quantized Scalar fields in Curved Space-time

Field quantization, in curved space-time, proceeds in close analogy to the Minkowski space-time. In curved space-time[2], the simplest generalization of the Klein-Gordon equation is given by replacing $\Box = \eta^{ij}\partial_i\partial_j$ (∂_i means partial derivative with respect to x^i) in Minkowski space-time by

$$\Box = \frac{1}{\sqrt{-g}}\,\partial_i\,(g^{ij}\sqrt{-g}\,\partial_j) \qquad (1.23)$$

and adding a term ξR in the mass term of the scalar field So Klein-Gordon equation for a scalar field ϕ is

$$(\Box + \xi R + m^2)\,\phi = 0 \qquad (1.24)$$

where 'ξ' is the coupling constant which couples gravity with ϕ .

If g_1 and g_2 are two solutions of (1.24), then we can define the scalar product (inner product) as

$$(g_1, g_2) = -i \int_S (g_1\,\partial^i g_2^* - g_2^*\,\partial^i g_1)\sqrt{-g}\; ds_i \;, \qquad (1.25)$$

where $ds_i = \hat{n}_i \, d\Sigma$ ($d\Sigma$ is the surface element of the three dimensional hyporsurface t = constant and $\hat{n}_i$ is a vector normal to the surface).

In the case, the Lagrangian density for the field ϕ is given by

$$L = \frac{1}{2}\sqrt{-g}\;\left[g^{ij}\,\partial_i\phi\,\partial_j\phi - (m^2 + \xi R)\phi\right] \qquad (1\cdot 26)$$

The canonical momentum, conjugate to ϕ is given by

$$\pi = \frac{\partial L}{\partial(\partial_0\phi)} = (-g)^{\frac{1}{2}}\; g^{oi}\,\partial_i\phi \qquad (1\cdot 27)$$

To quantize the field, we consider ϕ and Π as operators and impose that the equal-time canonical commutation relations

$$\begin{aligned} \left[\phi(x,t),\ \phi(x',t)\right] &= 0 \\ \left[\pi(x,t),\ \pi(x',t)\right] &= 0 \\ \left[\phi(x,t),\ \pi(x',t)\right] &= i\ \delta(x-x') \end{aligned} \qquad (1\cdot 28)$$

The Heisenbergs equation of motion are given in terms of Hamiltonian which is defined as

$$[F, H]_t + i\,\frac{\partial F}{\partial t} = i\,\frac{dF}{dt}\,, \qquad (1\cdot 29)$$

where

$$H(t) = \int d^3x\ (\pi\,\partial_0\phi - L)$$

The field operators act on state vectors $|\rangle$ which describes the possible states of system is given by the construction of a state vector space known as Fock spaces.

In the case of curved space-time,[2] it is difficult to define physically relevant positive frequency solutions and also to regularize the Hamiltonian and energy-momentum-tensor because there may exist a nonvanishing v acuum energy resulting from curvature or by the topological properties of space-time. Therefore one cannot demand that $(U_i, U_j) = \delta_{ij}$ be satisfied in curved space-time by the basis of wave functions. For example, if

$$U_i' = \alpha_{ij} U_j + \beta_{ij} U_j^* \qquad (1 \cdot 30)$$

with

$$|\alpha_{ij}|^2 - |\beta_{ij}|^2 = 1$$

Then we should have also

$(U_i', U_j') = \delta_{ij}$. Although U_i' are not purely positive frequency solution with respect to U_i. The mere fact that U_i & U_i' cannot describe the same particles can be seen as follows.

U_i' and U_i both form complete sets, so that 'ϕ' can be written as

$$\phi = \sum_i \left(a_i' u_i' + a_i'^{\dagger} u_i'^{*} \right)$$

and

$$\phi = \sum_i \left(a_i u_i + a_i^{\dagger} u_i^{*} \right) \qquad (1 \cdot 31)$$

Now,

$$a_i' = (\phi, U_i') = \sum_i \left([a_j u_j + a_j^{\dagger} u_j^*], [\alpha_{ij} u_i + \beta_{ij} u_i^*] \right)$$

$$= \alpha_j^* a_j - \beta_j^* a_j^{\dagger} \qquad (1.32)$$

Similarly

$$a_i'^{\dagger} = (\phi, U_i'^*)$$

$$= \alpha_{ij} a_j^{\dagger} - \beta_{ij} a_j \qquad (1.33)$$

The transformations of creation and annihilation operators (1.32) and (1.33) are known as Bogoliubov transformations [4,5].

By virtue of the condition $|\alpha_{ij}|^2 - |\beta_{ij}|^2 = 1$, it follows that both the primed and unprimed operators satisfy commutation relations (1.8). Hence we can construct the Fock space H' on $|0'\rangle$ where

$$a_i' |0'\rangle = 0 \quad \forall i \qquad (1.34)$$

The average number of primed particles present in the unprimed vacuum state is given by

$$N_i' = \sum_i \langle 0 | a_i'^{\dagger} a_i' | 0 \rangle = \sum_i |\beta_{ij}|^2 \qquad (1.35)$$

But the number of unprimed particles in the unprimed vacuum state is zero.

Now let us consider the Rindler coordinates[6] v,z. It is related to Minkowski coordinates (t,x) through (z 0, $-\infty < v < \infty$)

$$t+x = z e^{v} , \quad t-x = -z e^{-v} , \tag{1.36}$$

so that $$ds^2 = z^2 dv^2 - dz^2 \tag{1.37}$$

The metric coefficients are independent of the time co-ordinate 'v' and velocity is given by

$$\frac{dx}{dt} = \tanh v \tag{1.38}$$

Fulling[7] has solved the field equation (1.23) in Rindler co-ordinates (v,z) for the stationary wave function of time depending exp (jv) (j = 0,1,2,....). When m ≠ 0, in Rindler coordinates

$$\alpha_{jk} = \left[2\pi\omega_k \left(1 - e^{-2\pi j}\right)\right]^{-\frac{1}{2}} \left(\frac{\omega_k + k}{m}\right)^{ij}$$

$$\beta_{jk} = \left[2\pi\omega_k \left(e^{2\pi j} - 1\right)\right]^{-\frac{1}{2}} \left(\frac{\omega_k + k}{m}\right)^{ij}$$

with

$$\left|\alpha_{jk}\right|^2 = \left|\beta_{jk}\right|^2 e^{2\pi j} \tag{1.39}$$

From (1.32), we have

$$\begin{aligned} 1 &= \int_0^\infty dk \left(\left|\alpha_{jk}\right|^2 - \left|\beta_{jk}\right|^2\right) \\ &= \left(e^{2\pi j} - 1\right) \int_0^\infty dk \left|\beta_{jk}\right|^2 \end{aligned}$$

or

$$N_j^{\,r} = \int_0^k dk \, |\beta_{jk}|^2 = \left(e^{2\pi j} - 1\right)^{-1} \qquad (1.40)$$

which is a black body spectrum given by Unruh[9].

1.3. The Meaning of Particle Concept

It is natural to ask "What is the set of modes which furnishes the best description of a physical vacuum, ie. the state of 'no particles'?". Answer of this question depends on measurement process to detect the presence of quanta. For example, a free-falling detector will not always register the same particle density as a non-inertial, accelarating detector. In fact, this is even true in Minkowski space, where an accelarated detector will register quonta even in vacuum state defined as $a_k|0\rangle = 0 \;\forall k$.

The special feature of Minkowski space is not that there is a unique vacuum,but that the conventional vacuum state as defined in terms of mode (1.3) is the agreed vacuum for all inertial device throughout the space-time. This is because, the vacuum defined by (1.9) is invariant under Poincare' group.

In many problems of interest, the spacetime can be treated as asymptotically Minkowskian in the past and future. Under these circumstances, the choice of the 'natural' Minkowskian vacuum defined by (1.9) has a well-understood physical meaning ie. the absence of particles according to all inertial observers in the asymptotic region - usually taken to be the commonly accepted idea of a vacuum. We refer to the remote past and future as the 'in' and 'out' regions respectively. This is borrowed from Minkowskian quantum field theory where it is assumed that as $t \rightarrow \pm\infty$ all the field interactions approach zero. The analogor situation here is to suppose that in the 'in' and 'out' regions spacetime admits natural particle states and a privileged quantum vacuum. This can be either Minkowski space,or some other spacetime of high symmetry such as Einstein static universe. Whether a particular spacetime constitutes a suitable 'in' or Iout' region depending on the quantum field of interest. In the case of massless, conformally coupled fields, a conformaly flat spacetime, even if not static is a good spacetime.

Since we work in the Heisenberg picture, so if we choose the state of the quantum field in the 'in' region to be the vacuum state, then it will remain in that state during its subsequent evolution. But at later times, outside the 'in' region, freely falling particle detector may still register particles in this 'vacuum' state. In particular, if there is also an 'out' region then the in-vacuum may not coincide with the out-vacuum. In that case a natural class of observers in the out-region will detect the presence of particles. We can, therefore, say that particles have been created by the time-dependant external gravitational field.

This is an especially useful description, if the 'in' and 'out' regions are Minkowskian so that all inertial observers in the out-region register the presence of quanta. Analogous processes of particle creation by external electromagnetic fields are also known[10]. The possibility of similar particle production due to spacetime curvature was discussed over forty years ago by Schrödinger and others[11-14]. The first thorough treatment of particle production by an external gravitational field was given by Parker[15] and Sexl & Urbantke[16].

1.4 Particle Creation by Black Holes

Black holes are caused by gravitational collapse of stellar objects. In 1975, S.W. Hawking[17,18] attempted to show creation of particles due to gravitational field of a black hole. In 1976, Hartle and Hawking[19] attempted the same problem through path integral approach. We shall review Hawking's original method[17,18] because of its clear and unambigious physical basis.

Consider a neutral scalar field $\phi(x)$ with mass $m \neq 0$ obeying the Klein-Gordon equation (1.24) and quantization rules (1.28). Here interest lies in observations which depend on the characteristics of the geometry outside the event horizon. The spacetime around the black hole, of mass M, is described by the Schwarzchild line element

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2 d\Omega^2 \qquad (1.41)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

One finds that the solution 'g' of (1.24) in the spacetime (1.41) is proportional to

$$g \propto r^{-1} R_{\omega l}(r)\, Y_{lm}(\Omega)\, e^{-i\omega t} \qquad (1.42)$$

where

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial R_{\omega l}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2 R_{\omega l}}{\partial\phi^2} = -l(l+1)R_{\omega l} \qquad (1.43)$$

The radial function $R_{\omega l}(r)$ satisfying

$$\frac{d^2 R_{\omega l}}{dr^{*2}} + \left(\omega^2 - \left[l(l+1)r^{-2} + 2Mr^{-3}\right]\left[1-2Mr^{-1}\right]\right)R_{\omega l} = 0 \qquad (1.44)$$

where

$$r^* = r + 2M \log\left|r(2M)^{-1} - 1\right|$$

Here $r^* \to -\infty$, when $r \to 2M$ and $r^* \to \infty$ as $r \to \infty$. Equation (1.44) is analogous to the Schrodingers equation with ω^2 playing the role of energy and $\left[l(l+1)r^{-2} + 2Mr^{-3}\right]\left[1-2Mr^{-1}\right]$ the potential. Since this potential vanishes as $r \to 2M$ and $r \to \infty$, the solutions in these regions are superpositions of $\exp(\pm i\omega r^*)$. Hence the solutions 'g' of (1.42) will be superpositions of terms proportional to

$r^{-1}e^{(-i\omega u)}Y_{lm}$ and $r^{-1}\,e^{(-i\omega v)}Y_{lm}$ in the regions $r^* \to \pm\infty$, where

$$u = t - r^* , \qquad v = t + r^* \tag{1.45}$$

are null coordinates. Here outgoing spherical null waves are characterized by u = constant and incoming waves by v=constant.

In terms of 'u' and 'v' the Schwarzschild metric becomes

$$ds^2 = (1 - 2Mr^{-1})\, du\, dv - r^2 d\Omega^2 \tag{1.46}$$

The effect of the potential in (1.44) is to scatter incoming waves or outgoing waves so that it becomes superpositions. Therefore if one sends a purely incoming wav e $r^{-1}e^{(-i\omega v)}Y_{lm}$ from infinity, a part of it will reach the horizon and be absorbed and a part will be reflected back out to infinity.

Let us consider the quantized field $\phi(x)$, in the spacetime of the black hole. To specify the initial state we are specifying the complete set of positive and negative frequency solutions of the field equation at early times. It is necessary because results at very late times depend only on the Schwarzschild metric at R = 0. Also we take

$\xi_1 = 0$ in (1.2 4). Therefore (1.24) looks as

$$\frac{1}{\sqrt{-g}}\left(\partial_i\left(g^{ij}\sqrt{-g}\,\partial_j\right)\phi\right) = 0 \qquad (1.47)$$

Denoting the complete set of positive and negative frequency by

$$\begin{array}{ll} f_{\omega' lm}(r,\theta,\phi,t) & \text{(positive frequency at early times)} \\ f^*_{\omega' lm}(r,\theta,\phi,t) & \text{(negative frequency at early times)} \end{array} \qquad (1.48)$$

Now, the field $\phi(x)$ can be written as

$$\phi = \sum_{lm}\int d\omega' \left(a_{\omega' lm} f_{\omega' lm} + a^{\dagger}_{\omega' lm} f^{*}_{\omega' lm}\right) \qquad (1.49)$$

where

$$\left(f_{\omega_1' l_1 m_1}, f_{\omega_2' l_2 m_2}\right) = \delta(\omega_1' - \omega_2')\,\delta_{l_1 l_2}\,\delta_{m_1 m_2} \qquad (1.50)$$

is the normalization of $f_{\omega' lm}$ and the conserved scalar product is defined by (1.2 5) and the commutation relations (1.2 8) becomes

$$\begin{aligned} \left[a_{\omega_1' l_1 m_1}, a^{\dagger}_{\omega_2' l_2 m_2}\right] &= \delta(\omega_1' - \omega_2')\,\delta_{l_1 l_2}\,\delta_{m_1 m_2} \\ \left[a_{\omega_1' l_1 m_1}, a_{\omega_2' l_2 m_2}\right] &= 0 \end{aligned} \qquad (1.51)$$

Here '$a_{\omega_1' l_1 m_1}$' are annihilation operators corresponding to incoming particles at early times and the specification of the state $|0\rangle$ is given by

$$a_{\omega' l m}|0\rangle = 0 \quad \text{for all } \omega' l m \tag{1.52}$$

We decompose the field into positive and negative frequency parts to find the observed facts at infinity at late times. Now let us define '$p_{\omega l m}$' such that the superposition of them over a range of 'ω' from a wave packet localized at large r* at late times, which is a superposition of outgoing positive frequency waves $r^{-1}\exp(-i\omega u)y_{lm}$. $p_{\omega lm}$ and $p^*_{\omega lm}$ f orm a complete set for expanding any solution of the field equation which is purely outgoing at late times. Since the most general solution of the wave equation has a part which is incoming at the horizon at late times. Hence we introduce a set of solutions $q_{\omega lm}$ such that a superposition of them over the range 'ω' forms a wave packet which is localized near the horizon at late times and has zero Cauchy data on future null infinity. And the normalization is also done as

$$\begin{aligned} (p_{\omega_1 l_1 m_1}, p_{\omega_2 l_2 m_2}) &= \delta(\omega_1 - \omega_2)\,\delta_{l_1 l_2}\,\delta_{m_1 m_2} \\ (q_{\omega_1 l_1 m_1}, q_{\omega_2 l_2 m_2}) &= \delta(\omega_1 - \omega_2)\,\delta_{l_1 l_2}\,\delta_{m_1 m_2} \end{aligned} \tag{1.53}$$

Since $p_{\omega lm}$ and $q_{\omega lm}$ are disjoint region at late times one gets

$$\left(q_{\omega_1 l_1 m_1}, p_{\omega_2 l_2 m_2}\right) = 0 \qquad (1.54)$$

Now the field 'ϕ' can be written as

$$\phi = \sum_{lm} \int d\omega \left\{ b_{\omega lm} p_{\omega lm} + c_{\omega lm} q_{\omega lm} + b^{+}_{\omega lm} p^{*}_{\omega lm} + c^{+}_{\omega lm} q^{*}_{\omega lm} \right\}, \qquad (1.55)$$

with

$$\begin{aligned} \left[b_{\omega_1}, b^{+}_{\omega_2}\right] &= \delta(\omega_1 - \omega_2), & \left[b_{\omega_1}, b_{\omega_2}\right] &= 0 \\ \left[b_{\omega_1}, c_{\omega_2}\right] &= 0, & \left[c_{\omega_1}, c^{+}_{\omega_2}\right] &= \delta(\omega_1 - \omega_2) \\ \left[c_{\omega_1}, c_{\omega_2}\right] &= 0, & \left[b_{\omega_1}, c^{+}_{\omega_2}\right] &= 0 \end{aligned} \qquad (1.56)$$

Here $b_{\omega lm}$ are annihilation operators for particles outgoing at late times at infinity. And the number operator is given by

$$\langle N_{\omega lm} \rangle = \langle 0 | b^{+}_{\omega lm} b_{\omega lm} | 0 \rangle \qquad (1.57)$$

Since $f_{\omega' lm}$ and $f^{*}_{\omega lm}$ are a complete set for expanding any solutions of the field equatios, one can write

$$p_{\omega lm} = \int d\omega' \left(\alpha_{\omega lm \omega' lm} f_{\omega' lm} + \beta_{\omega lm \omega' lm} f^{*}_{\omega' l -m} \right) \qquad (1.58)$$

where α and β are complex numbers, independent of the coordinates (-m on f* is necessary because we have normalized the y_{lm} such that $y^*_{lm} = y_{l,-m}$).

From (1.55) one gets.

$$b_{\omega lm} = (\phi, p_{\omega lm}) \qquad (1.59)$$

Putting the value of ϕ and $p_{\omega lm}$ in terms of 'f' and f* from (1.49) and (1.58) and also using the orthonormality relations one gets

$$b_{\omega lm} = \int d\omega' \left(\alpha^*_{\omega lm\, \omega' lm}\, a_{\omega' lm} - \beta_{\omega lm\, \omega' lm}\, a^\dagger_{\omega' l,-m} \right) \qquad (1.60)$$

From (1.52) one gets

$$\langle N_{\omega lm} \rangle = \langle 0 | b^\dagger_{\omega lm} b_{\omega lm} | 0 \rangle = \int d\omega' \left| \beta_{\omega lm\, \omega' lm} \right|^2 \qquad (1.61)$$

Similarly from (1.58) one gets

$$\begin{aligned} \beta_{\omega lm\, \omega' lm} &= (p_{\omega lm}, f^*_{\omega' l,-m}) \\ \alpha_{\omega lm\, \omega' lm} &= (p_{\omega lm}, f_{\omega' lm}) \end{aligned} \qquad (1.62)$$

Here '$p_{\omega lm}$' are positive frequency at late times at infinity, while $f_{\omega lm}$ are positive frequency at early times at infinity. Also one can notice that, if there is no mixing of positive and negative frequency solutions caused by the black-hole, the $\beta_{\omega\omega'}$ will vanish and no particles will be observed at infinity at late times.

By neglecting the scattering from the static geometry as it is not the cause of the particle creations, (1.44) can be rewritten as

$$\frac{d^2}{dr^{*2}} R_{\omega l} + \omega^2 R_{\omega l} = 0 \qquad (1.63)$$

Now it is useful to study one explicit example example which are given as under.

Consider a spherical wave starts at early times teavelling along a an incoming path v = constant, and passes through the collapsing body just before formation of the horizon. After passing r = 0 it moves along an outgoing path u = constant, which remains close to the horizon for a long time before peeling off to infinity at late times. From this idea one can conclude the following three observations.

Since the wave $r^{-1} \exp(-i\omega u)\, y_{lm}$ reaches infinity at late times with a finite frequency ω and there is a very large red shift, the incoming waves $r^{-1} \exp(\pm i\omega' v)\, y_{lm}$ at early times must have very high frequency ω'. Hence the coefficients $\beta_{\omega\omega'}$ are those with very large ω'. So one has to consider the asymptotic form of $\beta_{\omega\omega'}$ for large ω'. Since the wave has a very high frequency when it reaches the collapsing body and as it moves through the interior geometry, we can trace the propagation of the wave through the interior geometry by geometrical optics. And thirdly the probability for observing particles is built only after it enters the collapsing body after formation of the horizon and it is more when it is travelling outward near the borizon for a very long time before it moves off to infinity. Therefore firstly, one should describe the free field accurately when the wave packet reaches infinity at late times after remaining near borizon for a very long time.

Now we trace the wave packet's components $r^{-1}\exp(-i\omega u)\,y_{lm}$ at late times, backward to early times, where they can be expressed as a superposition of waves $r^{-1}\exp(\pm i\omega' v)\,y_{lm}$ with large values of ω'. Also one can observe that it's phase remains constant at the value ωu, where u is the value of u-coordinates at late times, and it has the form $r^{-1}\exp(-i\omega u(v))$ y_{lm}. Here 'u' is the constant value of u-coordinate which the wave moves along after passing r = 0 and 'v' is constant value of v coordinate which the wave moves along before reaching r = 0. Thus the problem reduces to find the value of u(v).

Let $v = v_o$ be the coordinate of the particular incoming null ray which reaches the horizon just as it is formed and let n^{μ} be the given null vector which points radially inwards at a point on the horizon outside the collapsing body. Now we choose 'ε' such that $-\varepsilon\, n^{\mu}$ connects the horizon to the ray at u and parallel transport the $-\varepsilon n^{\mu}$ along the null generator of the future horizon to the point at which the past and future horizons intersect and also choose an affine parameter λ on the past horizon such that λ vanishes at the intersection of the two horizon and increases towards the future.

Therefore $\lambda = -C e^{-\kappa u}$

with $C > 0$ and $$(1.64)$$

where $\kappa = (4M)^{-1}$

k is the surface gravity at the horizon.

Also choose c such that $n^{\mu} = dx^{\mu} | d\lambda$ at the point where the horizons intersect. When in a local inertial frame in which $d^2x^{\mu} | d\lambda^2$ vanishes because λ is affine, so that n^{μ} is constant near $\lambda = 0$, integrating $n^{\mu}\, d\lambda$ along the past horizon, from $\lambda = 0$ to $\lambda(\mu)$ one gets,

$$x^{\mu}(\lambda) - x^{\mu}(0) = -\varepsilon n^{\mu}$$

Hence

$$\varepsilon = -\lambda(\mu) = c e^{-k\mu} \qquad (1.65)$$

Now parallel transport the vector $-\varepsilon n^{\mu}$ along the null generator of the past horizon and then along the path of the radially incoming null ray at v_o out to early times and large distances from the matter. Therefore, $V - V_0 = -\varepsilon n^{V}$.

At infinity the space is nearly flat so that n^v is a constant (say D, D > 0). So,

$$V - V_0 = -\epsilon D = -CDe^{-\kappa u} \qquad (1.66)$$

$$u = -4M \log [(V_0 - V)/(CD)] \qquad (1.67)$$

The spherical wave $p_{\omega l m}$ for $r > 2M$ at late times can be written as

$$p_{\omega l m} = N\omega^{-\frac{1}{2}} r^{-1} \exp(-i\omega u) Y_{lm}, \qquad (1.68)$$

where 'N' is the real normalization constant which is given by ($N = 2^{-3/2}\pi^{-1}$). From (1.67), at early times outside the collapsing body , one gets,

$$p_{\omega l m} = \begin{cases} N\omega^{-\frac{1}{2}} r^{-1} \exp\left[i4M\omega \log\left[\frac{(V_0 - V)}{(CD)}\right]\right] Y_{lm} & \text{for } V < V_0 \\ 0 & \text{for } V > V_0 \end{cases} \qquad (1.69)$$

Here for $v > v_0$ $p_{\omega l m}$ vanishes at early times because any null ray which is incoming along $v > v_0$ will enter the horizon of the black holes and does not reach infinity at late times.

The function $f_{\omega' l m}$ at early times outside the collapsing body is given by

$$f_{\omega' l m} = N(\omega')^{-\frac{1}{2}} r^{-1} \exp(-i\omega' v) Y_{lm} \qquad (1.70)$$

By taking Fourier transforms with respect to v one gets

$$N(\omega')^{-\frac{1}{2}} \, r^{-1} Y_{\ell m} \, \alpha_{\omega\ell m\,\omega'\ell m} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dv \, e^{-i\omega' v} \, p_{\omega\ell m}$$

or

$$\alpha_{\omega\ell m\,\omega'\ell m} = \frac{1}{2\pi} \int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{+i\omega' v} \exp\left[i\,4M\omega \log\left(\frac{v_0 - v}{CD}\right)\right] \quad (1 \cdot 71)$$

and

$$\beta_{\omega\ell m\,\omega'\ell m} = \frac{1}{2\pi} \int_{-\infty}^{v_0} dv \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} e^{-i\omega' v} \exp\left[i\,4M\omega \log\left(\frac{v_0 - v}{CD}\right)\right] \quad (1 \cdot 72)$$

By evaluating these integrals one finds that,

$$\left|\alpha_{\omega\omega'}\right|^2 = \exp(8\pi M\omega) \left|\beta_{\omega\omega'}\right|^2 \quad (1 \cdot 73)$$

The average number of particles in the mode $\omega\ell m$ outgoing at infinity at late times is given by

$$\langle N_{\omega\ell m} \rangle = \int d\omega' \left|\beta_{\omega\omega'}\right|^2 = \left[\exp(8\pi M\omega) - 1\right]^{-1} \quad (1 \cdot 74)$$

Since the probability for an outgoing wave in mode $\omega\ell m$ starting from just outside the horizon to reach infinity is the same as the absorption probability $\Gamma_{\omega\ell}$ for an incoming wave starting from infinity to be absorbed by the black hole. Therefore 1.74) becomes

$$\langle N_{\omega\ell m} \rangle = \Gamma_{\omega\ell} \left[\exp(8\pi M\omega) - 1\right]^{-1} \quad (1 \cdot 75)$$

1.5 Summary of the Work included in the other Chapters and Units

Chapter II contains discussion on creation of particles in isotropically expanding models of the early universe. In isotropically expanding models, particles are created when conformal symmetry is broken. Conformal symmetry may break in different situations. Examples corresponding due to different situations are included herein. In Chapter III, idea of creation of particles, in anisotropically expanding models of the early universe, has been discussed taking massless as well as massive scalar fields. Chapter IV contains discussion of work done by Castagnino et.al. who have proposed a different method, based on Quantum Equivalence Principle, for the study of possibility of creation of particles in curved spaces. From the last decade, study of Kaluza-Klein theories have been very much interesting in the context of unification of gravity with other interactions of the Nature, for example, electromagnetic weak, strong interactions and other gauge interactions. Motivated by 5-dim. Kaluza-Klein's prescription for unifying gravity with electromagnetic interaction, higher-dim. theories have been proposed by Dewitt and others. The last Chapter contains possibility of creation of particles in the higher-dim. Kaluza-Klein spacetimes.

The units used, in the thesis, are natural units which are given as 8 G = c = $\hbar$ = 1, where $\hbar = (h|2\pi)$ (h is Planck's constant), c is the speed of light and G is the Newtonian gravitational constant.

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CHAPTER II

PARTICLE CREATION BY THE ISOTROPICALLY EXPANDING UNIVERSE

Here we are interested in the study of scalar field $\phi(x)$ obeying Klein-Gordon equation (1.24) in an isotropically expanding and homogeneous cosmological model. The most suited homogeneous and isotropic model is given by Robertson-Walker spacetime,

$$ds^2 = dt^2 - \frac{a^2(t)}{\left(1+\frac{kr^2}{4}\right)^2}\left[dx^2+dy^2+dz^2\right] \qquad (2.1)$$

where a(t) is the scale factor, $r^2 = x^2+y^2+z^2$ and 'k' is the spatial curvature with three possible values +1, 0, -1 for closed, flat and open models respectively. Also t is the cosmic time which corresponds to the proper time of the geodesics x^i = constant (i=1,2,3). The early universe is supposed to be spatially flat, so we shall consider the case k = 0 only.

2.1 Conformal Symmetry

Conformal symmetry is very much crucial for particle production in isotropic model. When this symmetry breaks due to some reasons one can get creation of particles.

The action is given by,

$$A = \int d^4x \; L \qquad (2\cdot2)$$

where 'L' is the Lagrangian for scalarfield $\phi(x)$ in curved spaces. The action 'A' is conformally non-invariant in three cases. (1) when $m^2 \neq 0$ and $\xi = 0$ or $\xi = \frac{1}{6}$ (ii) when $\xi \neq \frac{1}{6}$ and $m^2 = 0$ (iii) when $m^2 \neq 0$ and $\xi \neq \frac{1}{6}$

Under the conformal transformation

$$\bar{g}_{\mu\nu} = \Omega^2(x) \; g_{\mu\nu} \qquad (2\cdot3)$$

The Christoffel symbol, Ricci tensor and Ricci scalar transforms as[1]

$$\Gamma^{\rho}_{\mu\nu} \rightarrow \bar{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \Omega^{-1}\left(\delta^{\rho}_{\mu}\Omega_{;\nu} + \delta^{\rho}_{\nu}\Omega_{;\mu} - g_{\mu\nu} g^{\rho\sigma}\Omega_{;\sigma}\right) \qquad (2\cdot4)$$

$$R^{\nu}_{\mu} \rightarrow \bar{R}^{\nu}_{\mu} = \Omega^{-2} R^{\nu}_{\mu} - 2\Omega^{-1}(\Omega^{-1})_{;\mu\rho}\, g^{\rho\nu} + \tfrac{1}{2}(\Omega^{-2})(\Omega^{2})_{;\rho\sigma}\, g^{\rho\sigma}\delta^{\nu}_{\mu} \qquad (2\cdot5)$$

$$R \rightarrow \bar{R} = \Omega^{-2} R + 6\Omega^{-3}\Omega_{;\mu\nu}\, g^{\mu\nu} \qquad (2\cdot6)$$

As a result,

$$\left[\Box + \tfrac{1}{6}R\right]\phi \rightarrow \left[\bar{\Box} + \tfrac{1}{6}\bar{R}\right]\bar{\phi} = \Omega^{-3}\left[\Box + \tfrac{1}{6}R\right]\phi \qquad (2\cdot7)$$

where $\bar{\phi}(x) = \Omega^{-1}(x)\,\phi(x)$ $\qquad (2\cdot8)$

So, one can sre that the action (2.2) is conformally invariant provided that $\xi = 0$ or $\frac{1}{6}$ and $m^2 = 0$. But if $\xi \neq \frac{1}{6}$, action (2.2) is not conformally invariant even when $m^2 = 0$. Hence one can get creation of particles in the above three cases. Here examples for different cases are given for isotropically expanding cosmological models.

2.2 Case for Minimal Coupling ($\xi = 0$)

The minimal coupling[2] of the scalar field ϕ (x) with gravity is characterized by $\xi = 0$. The Klein-Gordon equation (1.2 4), in this case, is

$$a^{-3} \frac{\partial}{\partial t} \left(a^3 \frac{\partial}{\partial t} \phi \right) - \sum_{i=1,2,3} a^{-2} \frac{\partial^2 \phi}{\partial x_i^2} + m^2 \phi = 0 \qquad (2.9)$$

in the background of the Robertson-Walker line element (2.1). The Fourier expansion for ϕ (x) can be written as

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[A_k e^{ikx} \psi_k(t) + A_k^{+} e^{-ikx} \psi_k^{*}(t) \right] \qquad (2.10)$$

where k is the continuous mode ($-\infty \leq k \leq \infty$). Connecting (2 .9) and (2.10) one gets

$$\frac{d^2 \psi_k(\tau)}{d\tau^2} + a^6 \omega_k^2 \psi_k(\tau) = 0 \qquad (2.11)$$

where

$$\tau = \int^{t} \frac{dt'}{a^3(t')} \qquad (2.12)$$

and

$$\omega_k^2 = k^2 a^{-2}(\tau) + m^2$$

$A_{k's}$ are constant operators obeying commutation relations

$$\left[A_k , A_{k'}\right] = 0 \ , \quad \left[A_k , A_{k'}^{\dagger}\right] = \delta(k-k') \qquad (2.13)$$

and the Wronskian condition for consistency of the equations of (2.13) and (1.28) are so that $(f_k, f_{k'}) = \delta(k-k')$ and

where

$$f_k = (2\pi)^{-\frac{3}{2}} e^{ikx} \psi_k(\tau) \qquad (2.14)$$

is given by

$$\psi_k \frac{d\psi_k^*}{d\tau} - \psi_k^* \frac{d\psi_k}{d\tau} = i \qquad (2.15)$$

For a statically bounded expansion,

$$a \rightarrow \begin{cases} a_1 & \text{as } \tau \rightarrow -\infty \\ a_2 & \text{as } \tau \rightarrow +\infty \end{cases} \qquad (2.16)$$

When a is constant, the theory reduces to special relativity.

For large negative τ

$$(\psi_k)_{\tau\to-\infty} = (2a_1^3\omega_k)^{-\frac{1}{2}} \exp(-ia_1^3\omega_k\tau) \qquad (2.17)$$

Substituting (2.17) in (2 .10), we get

$$(\phi(x,t))_{\tau\to-\infty} = (2\pi)^{-\frac{3}{2}} \int d^3p_I \, (2\omega_k)^{-\frac{1}{2}} \Big[A_{p_I} e^{ip_I x_I} e^{-i\omega_k t} + A^{+}_{p_I} e^{-ip_I x_I} e^{i\omega_k t}\Big] \qquad (2.18)$$

where

$$p_I = \frac{k}{a_1} \quad , \quad x_I = a_1 x \qquad (2.19)$$

and

$$A_{p_I} = a_1^{\frac{3}{2}} A_k \qquad (2.20)$$

And 'x_I' is the usual Minkowski coordinate in the initial spacetime, p_I is the physical momentum, and A_{p_I} obey the boson commutation relations (2 .13) with the δ function

$$\delta(p_I - p_{I'}) = a_1^3\,\delta(k-k') \,. \qquad (2.21)$$

And for large τ, the spacetime is again flat ($a=a_2$) and one gets

$$(\psi_k)_{\tau\to\infty} = (2a_2^3\,\omega_k)^{-\frac{1}{2}} \Big[\alpha_k e^{-ia_2^2\omega_k\tau} + \beta_k e^{ia_2^3\omega_k\tau}\Big] \qquad (2.22)$$

in which ψ_k is a superposition of both positive and negative frequency components.

Putting (2.21) in (2.10) and regrouping according to positive and negative frequencies one gets a form for like (2.18) with a_1 replaced by a_2; and the annihilation operator for physical particle at late times is given by

$$a_k = \alpha_k A_k' + \beta_k A_{-k}^{\dagger} \qquad (2.23)$$

The a_k and $a_k^{\dagger}$ obeys the correct commutation relations (2.13) by virtue of the Wronskian condition (2.14) which demands

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \qquad (2.24)$$

Now suppose that no particles are present in the initial Minkowski space

$$A_k |0\rangle = 0 \quad \text{for all } 'k'. \qquad (2.25)$$

Now we will investigate whether particles are present in the final Minkowski space. For that let us impose boundary conditions on $\phi(x,t)$ in a cube of length L so that k takes on discrete values. Using (2.23) and (2.25) we get the number of particles in mode k at late times as

$$\langle N_k \rangle_{T \to \infty} = \langle 0 | a_k^{\dagger} a_k | 0 \rangle = |\beta_k|^2 \qquad (2.26)$$

Here we can observe that β_k vanishes in limit of an infinitely slow expansion. Therefore we can say that the particle number is adiabatic invariant. However, in general,

β_k does not vanish. So we can conclude that particles are present in the final Minkowski space. These particles are created during the expansion of the universe from a_1 to a_2.

Since particle number is adiabatic invariant, the initial and final periods in which $a(\tau)$ is constant can be replaced by periods in which the expansion is slow, also the WKB solutions of (2.2) are a valid approximation of the function ψ_k [3,4,5].

Now it is reasonable to understand the probability distribution and average number of created particles [2].

Let $|0\rangle$ be the vacuum in the final Minkowski space and is defined by

$$a_k|0\rangle = 0 \quad \text{for all } k \tag{2.27}$$

state $$|n,n) = (n!)^{-1} (a^{\dagger}_{-k})^n (a^{\dagger}_k)^n |0) \tag{2.28}$$

and it contains n particles in mode -k and is mode k in the final Minkowski space. Applying (2.23) and (2.25) one gets

$$a_k|0\rangle = \left(\frac{\beta_k^*}{\alpha_k^*}\right) a^{\dagger}_{-k} |0\rangle \tag{2.29}$$

Therefore

$$(n,n|0\rangle = (n!)^{-1} (\beta_k^*|\alpha_k^*)^n \langle 0|(a_{-k})^n (a^{\dagger}_{-k})^n|0\rangle$$

$$= (n!)^{-1} (\beta_k^*|\alpha_k^*)^n (0,n|(a^{\dagger}_{-k})^n|0\rangle \tag{2.30}$$

where $(0,n|$ is the state containing n particles in mode -k in the final Minkowski space. Repeating the process we get

$$(n,n|0\rangle = \left(\beta_k^*|\alpha_k^*\right)^n (0|0\rangle \qquad (2.31)$$

This equation gives the most general non-vanishing matrix element of a final state containing particles in mode -k or k with the initial vacuum because the matrix elements of the form $(m,n|0\rangle$ vanish when $m \neq n$. If $m > n$ we get $(0|a_k^{+}|0\rangle$ or $(0|a_k^{+}|0\rangle$ and if $m < n$, it vanishes in accordance with equation (2.27).

Therefore the most general non-vanishing mat ix element of a final basis with initial vacuum is given by

$$(\{n_k\}|0\rangle = \prod_k \left(\beta_k^*|\alpha_k^*\right)^{n_k} (0|0\rangle \qquad (2.32)$$

where $|\{n_k\}|$ is the final state containing n_k pairs with one particle in mode k and the other in mode -k

Using (2.31) we get,

$$1 = \langle 0|0\rangle = |(0|0\rangle|^2 \sum_{\{n_k\}} \prod_k |\beta_k|\alpha_k|^{2n_k}$$

$$= |(0|0\rangle|^2 \prod_k \sum_{n=0}^{\infty} |\beta_k|\alpha_k|^{2n}$$

$$= |(0|0\rangle|^2 \prod_k |\alpha_k|^2$$

where $|\alpha_k|^2 - |\beta_k|^2 = 1.$

Therefore,

$$|\langle 0|0\rangle|^2 = \prod_k |\alpha_k|^{-2} = \exp\left[\sum_k \log|\alpha_k|^{-2}\right] \qquad (2.33)$$

and

$$\left|\langle\{n_k\}|0\rangle\right|^2 = \prod_k \left(\left|\beta_k|\alpha_k|\right|^{2n_k}|\alpha_k|^{-2}\right) \qquad (2.34)$$

Therefore the probability , $P_n(k)$ of observing n particles in mode k at large τ is given by

$$P_n(\vec{k}) = \left|\beta_k|\alpha_k|\right|^{2n}|\alpha_k|^{-2} \qquad (2.35)$$

Using this result, the average number of particles $\langle N_k\rangle$ present in mode k in the volume $[La_2^3]$:

$$\langle N_k\rangle = \sum_{n=0}^{\infty} n\, P_n(k) = |\beta_k|^2 \qquad (2.36)$$

The average particle density,summing over all modes and taking the limit we we get,

$$\begin{aligned}|N| &= \mathrm{Lt}_{L\to\infty}[La_2]^{-3}\sum_k \langle N_k\rangle \\ &= [2\pi^2 a_2^3]^{-1}\int_0^k dk\; k^2|\beta_k|^2 \qquad (2.37)\end{aligned}$$

Now we consider a model of the very early universe in which the significant particle production occurs near the Planck time $t_p = G^{\frac{1}{2}}$ (in the unit $h = c = 1$). G is the Newtonian gravitational constant. So the average energy of the initially created particles will be of the order t_p^{-1}, which is very large with respect to the elementary pasticle rest masses. So restricting to the massless case (2.12) can be written as

$$\frac{d^2\psi_k}{d\tau^2} + k^2 a^4 \psi_k = 0 \qquad (2.38).$$

To obtain a more explicit expression for the average particle number, $\langle N_k \rangle$ and probability distribution $p_n(k)$ for a smooth expansion from a_1 to a_2, one can express $a^4(\tau)$ as a function namely,

$$a^4(\tau) = a_1^4 + e^{\xi}\left[(a_2^4 - a_1^4)\ (e^{\xi}+1) + b\right](e^{\xi}+1)^{-2} \qquad (2.39)$$

where

$$\xi = \tau s^{-1} \qquad (2.40)$$

and where b, a_1, a_2 and s are adjustable costants with $a_2 > a_1$. This function and all its derivatives are continuous, and it approaches a_1 as $\tau \to -\infty$ and a_2 as $\tau \to +\infty$.

Epstein[6] and Eckart[7] have shown that particle production can occur only when $e^{\tau s^{-1}}$ is small. So, in cosmological context, it is wise to write (2.39) effectively as

$$a^4(\tau) = a_1^4 + a_0^4 e^{\tau s^{-1}} \qquad (2.41)$$

where a_0 is an adjustable parameter, $a \rightarrow a_1$ as $\tau \rightarrow -\infty$ and $a \longrightarrow a_2$ when $\tau \rightarrow +\infty$. With this $a^4(\tau)$, the solution of (2.38) reduces to a positive frequency solution in the initial Minkowski space [8] as,

$$\psi_k = (2ka_1^2)^{-\frac{1}{2}} (k'a_0^2)^{ik'a_1^2} \, 2^{-ik'a_1^2} \, \Gamma(1-ik'a_1^2) \, J_{-ik'a_1^2}(k'a_0^2 e^{\tau'}) \quad . \tag{2.42}$$

$$\text{where } k' = 2ks \; , \quad \tau' = \tau s^{-1}/2 \tag{2.43}$$

where J is a Bessel function. For large τ, ψ_k is a super position of positive and negative frequency WKB solutions of Eq. (2.38), which is given by

$$(\psi_k)_{\tau\rightarrow\infty} = (2ka_0^2 e^{\tau'})^{-\frac{1}{2}} \left\{ \alpha_k \exp(-ik'a_0^2 e^{\tau'}) + \beta_k \exp(ik'a_0^2 e^{\tau'}) \right\} \tag{2.44}$$

where α_k and β_k satisfying (2.23). Making use of the asymptotic form of the Bessels function, one gets

$$J_\nu(z)_{z\rightarrow\infty} = \left(\frac{\pi z}{2}\right)^{-\frac{1}{2}} \cos(z - \tfrac{1}{2}\pi\nu - \tfrac{1}{4}\pi) \tag{2.45}$$

$$\alpha_k = (2k'a_1^2)^{-\frac{1}{2}} (k'a_0^2)^{ik'a_1^2} \, 2^{-ik'a_1^2} \, \Gamma(1-ik'a_1^2) \, \pi^{-\frac{1}{2}} e^{i\frac{\pi}{4}} \times e^{\frac{\pi}{2}k'a_1^2} \tag{2.46}$$

$$\beta_k = -i\alpha_k e^{-\pi k' a_1^2} \tag{2.47}$$

$$\left| \frac{\beta_k}{\alpha_k} \right|^2 = \exp(-4\pi s k a_1^2) \tag{2.48}$$

Substituting (2 .48) into the probability distribution of (2 .35) and the average number of created particles (2 .36) and making use of $|\alpha_k|^2 - |\beta_k|^2 = 1$, one finds that[9],

$$P_n(k) = \exp(-n\mu k)\left[1-\exp(-\mu k)\right] \qquad (2.49)$$

and

$$\langle N_k \rangle = \left[\exp(\mu k)-1\right]^{-1} \qquad (2.50)$$

where

$$\mu = 4\pi s a_1^2 \qquad (2.51)$$

and the average particle density of (2.37) is given by

$$\langle N \rangle = \left[2\pi^2 a_2^3\right]^{-1} \int_0^\infty dk\, k^2 \left[\exp(\mu k)-1\right]^{-1} \qquad (2.52)$$

2.3 Pertubation Calculation of the Particle Creation (Case $\xi \neq \frac{1}{6}$ and $m^2 = 0$)

Here we consider the gravitational particle creation following pertubation approach to calculate the number density of the created particles[10].

Here the line-element under consideration is a spatially flat Robertson-Walker metric,

$$ds^2 = dt^2 - a^2(t)\, dx^2 = a^2(\eta)\left(d\eta^2 - dx^2\right) \qquad (2.53)$$

where $dt = a d\eta$. The Klein-Gorden equation (1.24), with $m^2 = 0$,

$$\frac{1}{\sqrt{-g}}\, \partial_i \left(g^{ij}\sqrt{-g}\, \partial_j\right)\phi + \xi R \phi = 0 \qquad (2.54)$$

If $\xi \neq \frac{1}{6}$, this field is not conformally invariant and the particle production occurs.

Now to calculate the number density and energy density of the created particles, we take a pertubation calculation treating $|\xi - \frac{1}{6}|$ as a small parameter. This was dòne by Zel'dovich and Starobinsky[11] and by Birrel and Davies[12].

In the spacetime (2.53), the scalar curvature is given by

$$R = 3\, C^{-1}\left(\dot{D} + \tfrac{1}{2} D^2\right) \qquad (2.55)$$

where $c = a^2$ and $D = \dot{c}\,|\,c$. And the field 'ϕ' possess mode solutions of the form

$$\phi_k(x) = (2\omega)^{-\frac{1}{2}}\, e^{ikx}\, \psi_k(\eta) \qquad (2.56)$$

where

$$\psi_k(\eta) = e^{-i\omega\eta} + \frac{1}{\omega}\int_{-\infty}^{\eta} V(\eta_1)\, \sin\omega(\eta-\eta_1)\, \psi_k(\eta_1)\, d\eta_1 \qquad (2.57)$$

with

$$V(\eta) = \left(\tfrac{1}{6} - \xi\right) R(\eta)\, c(\eta)$$

when $\eta \to \infty$, (2.57) can be written as

$$\psi_k = \alpha_\omega e^{-i\omega\eta} + \beta_\omega e^{i\omega\eta} \qquad (2.58)$$

where

$$\alpha_\omega = 1 + \frac{i}{2\omega}\int_{-\infty}^{\infty} V(\eta)\, d\eta$$

and

$$\beta_\omega = \frac{-i}{2\omega} \int_{-\infty}^{\infty} e^{-2i\omega\eta} \, V(\eta) \, d\eta \qquad (2.59)$$

The number density of the created particles is given by

$$n = \frac{1}{2\pi^2 a^3} \int_0^{\infty} |\beta_\omega|^2 \omega^2 \, d\omega \qquad (2.60)$$

which can be written in terms of co-ordinate space integrals as

$$n = \frac{1}{16\pi a^3} \int_{-\infty}^{\infty} V^2(\eta) \, d\eta \qquad (2.61)$$

Also, the energy density of the created particles is given by

$$\rho = \frac{1}{2\pi^2 a^4} \int_0^{\infty} \omega^3 |\beta_\omega|^2 d\omega \qquad (2.62)$$

or

$$\rho = \frac{-1}{32\pi^2 a^4} \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \, \log(|\eta_1 - \eta_2|\mu) \, \dot{V}(\eta_1) \, \dot{V}(\eta_2) \qquad (2.63)$$

2.4. Particle Creation in de Sitter space (case $m^2 \neq 0$ for all ξ)

Here we are interested in the study of scalar field $\phi(x)$ obeying the Klein-Gordon equation (1.24) in a spatially closed de Sitter space which is given by [13]

$$ds^2 = -dt^2 + \frac{1}{H^2} \cosh^2 Ht \left(d\lambda^2 + \sin^2\lambda \, d\Omega^2\right) \qquad (2.64)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$.

The Klein-Gordon equation (1.24), in this case, is

$$\left\{\frac{1}{\cosh^3 Ht}\frac{\partial}{\partial t}\left[\cosh^3 Ht\frac{\partial}{\partial t}\right]-\frac{H^2\Delta_3}{\cosh^2 Ht}+M^2\right\}\phi=0 \qquad (2.65)$$

where

$$M^2=m^2+\xi_\varphi R \qquad (2.66)$$

and

$$\Delta_3=\frac{1}{\sin^2\lambda}\left\{\frac{\partial}{\partial\lambda}\left[\sin^2\lambda\frac{\partial}{\partial\lambda}\right]+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right]+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}$$

$R = 12H^2$. (2.67)

The solutions to (2.65) can be written in terms of standard functions analysis

$$\Phi(t,\lambda,\theta,\phi) = y_k(t)\,Y_{k\ell m}(\lambda,\theta,\phi) \qquad (2.68)$$

where Y_{klm} are S_3 spherical harmonics obeying

$$-\Delta_3 Y_{k\ell m} = k(k+2)Y_{k\ell m} \qquad (2.69)$$

which are $(K+1)^2$ fold degenerate (m= - l,..., +l; l=0,1,...k).

So, (2.65) becomes

$$\frac{1}{\cosh^3 Ht}\frac{d}{dt}\left[\cosh^3 Ht\frac{dy_k}{dt}\right]+\frac{H^2k(k+2)}{\cosh^2 Ht}y_k+M^2y_k=0 \qquad (2.70)$$

which is a form of Legendre's equation.

The solutions are given by[14]

$$y_k^{\pm}(t)\sim\cosh^k(Ht)\exp\left[\left(-k-\tfrac{3}{2}\pm i\gamma\right)Ht\right]{}_2F_1\left(k+\tfrac{3}{2};\,k+\tfrac{3}{2}\pm i\gamma;\,1\pm i\gamma;\,-e^{-2Ht}\right)$$

(2.71)

and

$$y_{k(\pm)}(t) \sim \cosh^{3}(Ht) \exp\left\{\left[k+\frac{3}{2} \pm i\gamma\right] Ht\right\} {}_2F_1\left(k+\frac{3}{2}; k+\frac{3}{2} \pm i\gamma; 1+i\gamma; -e^{2Ht}\right) \tag{2.72}$$

where ${}_2F_1$ is the hyper geometric Function[15]. As $t \to +\infty$ the first set of solutions are

$$y_k^{(\pm)}(t) \to \exp\left[-\frac{3}{2} Ht \pm i\gamma Ht\right], \quad t \to +\infty \tag{2.73}$$

As $t \to -\infty$,

$$y_{k(\pm)}(t) \to \exp\left[\frac{3}{2} Ht \pm i\gamma Ht\right] \quad t \to -\infty \tag{2.74}$$

And the four solutions are given by,

$$\left.\begin{aligned} y_k^{(-)}(t) &= \left[y_k^{+}(t)\right]^{*} \\ y_{k(-)}(t) &= \left[y_{k(+)}(t)\right]^{*} \end{aligned}\right\} \tag{2.75}$$

$$\& \quad y_{k(\pm)}(t) = y_k^{(\pm)}(-t) \tag{2.76}$$

From (2.73) and (2.74) we can observe that, $y_k^{(\pm)}$ and $y_{k(\pm)}$ can be used as basis function for the field $\phi(x)$. Using $y_{k(\pm)}$, when $t \to -\infty$, we can write

$$\phi = \sum_{k\ell m} b_{k\ell m}\, \phi_{(+)k\ell m} + b^{+}_{k\ell m}\, \phi_{(-)k\ell m} \tag{2.77}$$

Similarly, $y_k^{(\pm)}$ functions define a similar decomposition at $t \to \infty$

$$\phi = \sum_{k\ell m} b_{k\ell m}\, \phi^{(+)}_{k\ell m} + b^{(-)}_{k\ell m}\, \phi^{(-)}_{k\ell m} \tag{2.78}$$

The Bogoliubov coefficients, which are given by

$$\begin{pmatrix} \phi_{\lambda(+)} \\ \phi_{\lambda(-)} \end{pmatrix} = \begin{pmatrix} \alpha_\lambda & \beta_\lambda \\ \beta_\lambda^* & \alpha_\lambda^* \end{pmatrix} \begin{pmatrix} \phi_\lambda^{(+)} \\ \phi_\lambda^{(-)} \end{pmatrix} . \qquad (2.79)$$

are found from (2.71) and (2.72) and using the inversiton transformation for the hyper geometric function.

$$\alpha_k = \frac{\Gamma(1-i\gamma)\ \Gamma(-i\gamma)}{\Gamma(k+\frac{3}{2}-i\gamma)\ \Gamma(k-\frac{1}{2}-i\gamma)}$$

$$(2.80)$$

$$\beta_k = \frac{\Gamma(1-i\gamma)\ \Gamma(i\gamma)}{\Gamma(k+\frac{3}{2})\ \Gamma(-\frac{1}{2}-k)} = \frac{i(-)^k}{\sinh \pi\gamma}$$

These coefficients α_k & β_k satisfy $|\alpha_k|^2 - |\beta_k|^2$ for all k;
The magnitudes $|\alpha_k|$ and $|\beta_k|$ are independent of k.
Therefore one can write

$$\alpha_k = e^{-2i\delta_k} \cosh 2\theta$$

$$\beta_k = i(-)^k \sinh 2\theta$$

where θ is a fixed constant given by

$$\sinh 2\theta = \operatorname{cosec} h\pi\gamma \qquad (2.81)$$

The relative amplitude for producing a pair of particles in the final states (klm) and (kl,-m). If none were present in the initial state is

$$\frac{\langle out | b_{k\ell m} \, b_{k\ell -m} | in \rangle}{\langle out | in \rangle}$$

Using (2.81) one gets the relative probability of creating a pair of particles as

$$\omega_{k\ell m} = \left| \frac{\beta_k}{\alpha_k} \right|^2 = \tanh^2 2\theta = \operatorname{sech}^2 \pi r \qquad (2.82)$$

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CHAPTER III

PARTICLE CREATION BY THE ANISOTROPICALLY EXPANDING UNIVERSE

So far, only isotropic models have been discussed. In 1970, Zel'dovich [1] speculated that the universe might have begun in an anisotropic state, but rapidly isotropized as a result of quantum effects. Robertson-Walker spaces (which are homogeneous and isotropic) are conformally flat, but anisotropic homogeneous models are not conformally flat. So anisotropy is a very significant factor in a conformally invariant theory. In the massless limit of the scalar field theory with conformal coupling $\xi_\varphi = \frac{1}{6}$, there will be no particle creation in an isotropic space-time because breaking of conformal symmetry causes creation of particles and presence of massive term caused breaking of conformal symmetry. On the otherhand, one finds that in anisotropic model, conformal symmetry is broken even in massless limit. So in the absence of mass also, in anisotropic model, particles are created. Some examples are given below.

3.1 Massless Scalar fields in Bianchi type I model.

Here a spatially flat, homogeneous and anisotropic Bianchi type I model is considered [2], which is a generalised Kasner model. This model is given by

$$ds^2 = dt^2 - \sum a_j^2(t)\, dx^{j\,2} \qquad (3.1)$$

The Klein-Gordon equation (1.24) with $\xi = 0$, $m = 0$ in this model is[3]

$$(a_1 a_2 a_3)^{-1} \frac{\partial}{\partial t}\left(a_1 a_2 a_3 \frac{\partial}{\partial t}\phi\right) - a_1^{-2}\frac{\partial^2\phi}{\partial x_1^2} - a_2^{-2}\frac{\partial^2\phi}{\partial x_2^2} - a_3^{-2}\frac{\partial^2\phi}{\partial x_3^2} = 0 \tag{3.2}$$

The Fourier expansion for ϕ in the metric (3.1) can be written as (2.10) with A_k and $A_k^{\dagger}$ obeying the commutation relations (2.13). Here ψ_k which appears in (2.10) satisfies

$$\frac{d^2\psi_k}{d\tau^2} + v^2\omega^2\psi_k = 0 \tag{3.3}$$

where

$$\left.\begin{aligned} v &= \sqrt{-g} = a_1 a_2 a_3 \\ \tau &= \int^t v^{-1}(t')\,dt' \end{aligned}\right\} \tag{3.4}$$

and

$$\omega^2 = \sum k_j^2 \,|\, a_j^2 \tag{3.5}$$

The Wronskian condition is the same as (2.14)

As in the isotropic case, for large negative τ,

$$\psi_k(\tau)_{\tau\to-\infty} = (2\omega_I v_I)^{-\frac{1}{2}} \exp(-i\omega_I v_I \tau) \tag{3.6}$$

where ω_I and v_I are the initial value of ω and v. Also one can take the state vector to be $|0\rangle$ characterized by the absence of particles in the initial space-time is

$$A_K |0\rangle = 0 \quad \forall K \tag{3.7}$$

At late times, one gets again

$$\Psi_K(\tau) \underset{\tau \to \infty}{=} (2\omega_F v_F)^{-\frac{1}{2}} \left\{ \alpha_K e^{-i\omega_F v_F \tau} + \beta_K e^{-i\omega_F v_F \tau} \right\} \tag{3.8}$$

with the same condition as (2.2 3) and the annihilation operators at late times are the same as

$$a_K = \alpha_K A_K + \beta_K^{*} A_{-K}^{+} \tag{3.9}$$

For large negative values of τ , the quantity $v^2\omega^2$ appearing in (3.3) smoothly approaches $v_I\omega_I^2$, and if one supposes that its behaviour sufficiently smooth, then as in the isotropic case, one gets

$$\left| \frac{\beta_K}{\alpha_K} \right|^2 \simeq \exp(-4\pi s v_I \omega_I) \tag{3.10}$$

where s is the characteristic τ -time interval during which significant particle production occurs. If most of the particle creation occurs in a t-time interval of order t_p, during which time $v = v_I$, then (3.5) gives

$$S = v_I^{-1} t_p \qquad (3\cdot 11)$$

The probability of observing 'n' particles in mode 'k' at late times is

$$P_n(k) = \left|\frac{\beta_k}{\alpha_k}\right|^{2n} |\alpha_k|^{-2}$$

$$= \exp(-n 4\pi t_p \omega_I)\left[1 - \exp(-4\pi t_p \omega_I)\right] \qquad (3\cdot 12)$$

And the average number of particles per unit volume in the state $|0\rangle$ at late times, when $v = v_F$, is

$$\langle N \rangle = (8\pi^3 v_F)^{-1} \int d^3k\, |\beta_k|^2$$

$$= (8\pi^3 v_F)^{-1} \int d^3k \left[\exp(4\pi t_p \omega_I) - 1\right]^{-1} \qquad (3\cdot 13)$$

3.2 Approximation Method in Anisotropic models

It is very difficult to solve Klein-Gordon equation in anisotropic models which is needed for proper understanding of the number density and energy density of created particles during the course of expansion of the model with increasing time. So, some simple cases have been attempted by Fulling,[4] Parker and Hu,[5] and Nariari.[6] In view of this difficulty, It is natural to investigate approximate methods for solving it.

One such method was proposed by Zel'dovich and Starobinski.[7] In this method, anisotropy is assumed as small pertubation in Minkowski metric. So, the line element looks like

$$ds^2 = dt^2 - a_1(t)\,dx^2 - a_2(t)\,dy^2 - a_3(t)\,dz^2$$

where

$$a_1(t) = 1 + h_1(t) \quad , \quad a_2(t) = 1 + h_2(t)$$

$$a_3(t) = 1 + h_3(t)$$

such that

$$\max |h_i(t)| << 1 \quad (i = 1, 2, 3)$$

Later on this method was developed by Birrel and Davies[8] taking the metric,

$$ds^2 = a^2(\eta)\left(d\eta^2 - \sum_{i=1}^{3} (1 + h_i(\eta))\,dx_i^2\right) \tag{3.14}$$

where

$$\max |h_i(\eta)| << 1, \quad (i = 1, 2, 3) \tag{3.15}$$

3.3. <u>Birell & Davies approach for Anisotropic model</u>

The Klein-Gordon equation (1.24) has mode solutions, $\phi(x)$ in the spacetime (3.14) of the form

$$\phi_k(x) = (2\omega)^{-\frac{1}{2}}\, a^{-1}\, e^{+ikx}\, \psi_k(\eta) \tag{3.16}$$

where

$$\psi_k(\eta) = e^{-i\omega\eta} + \frac{1}{\omega}\int_{-\infty}^{\eta} V_k(\eta_1)\, \sin\omega(\eta - \eta_1)\, \psi_k(\eta_1)\, d\eta_1 \tag{3.17}$$

with

$$V_k(\eta) = \sum_i h_i(\eta)\, k_i^2 + m^2\left[a^2(\infty) - a^2(\eta)\right] - \left(\xi - \tfrac{1}{6}\right) R(\eta)\, a^2(\eta) \tag{3.18}$$

and $\omega^2 = k^2 + m^2 a^2(\infty)$ (3.19)

provided by the restriction on h_i as

$$\sum_i h_i(\eta) = 0 \tag{3.20}$$

In the late times, when $\eta \to \infty$, (3.17) becomes,

$$\psi_k(\eta) = \alpha_\omega e^{-i\omega\eta} + \beta_\omega e^{i\omega\eta} \tag{3.21}$$

where the Bogoliubov coefficients are given by

$$\alpha_\omega = 1 + \frac{i}{2\omega}\int_{-\infty}^{\infty} e^{i\omega\eta}\, V_k(\eta)\, \psi_k(\eta)\, d\eta \tag{3.22}$$

$$\beta_\omega = -\frac{i}{2\omega}\int_{-\infty}^{\infty} e^{-i\omega\eta}\, V_k(\eta)\, \psi_k(\eta)\, d\eta \tag{3.23}$$

One can see that, when $\eta \to -\infty$, $\alpha_\omega = 1$ and $\beta_\omega = 0$. Substituting in (3.21) we get $\psi_k(\eta) = e^{-i\omega\eta}$ and putting this value of $\psi_k(\eta)$ in (3.2 2) and (3.2 3) one gets,

$$\alpha_\omega = 1 + \frac{i}{2\omega}\int_{-\infty}^{\infty} V_k(\eta)\, d\eta \tag{3.24}$$

$$\beta_{\omega} = \frac{-i}{2\omega}\int_{-\infty}^{\infty} e^{-2i\omega\eta}\, V_K(\eta)\, d\eta \qquad (3.25)$$

The number density (per unit proper volume) in out region $(\eta \to +\infty)$ is

$$n = (2\pi a)^{-3} \int |\beta_{\omega}|^2 d^3k \qquad (3.26)$$

for the quantum state chosen corresponding to the 'in' vacuum.

The energy density is given by

$$\rho = \frac{1}{(2\pi)^3 a^4} \int |\beta_{\omega}|^2 \omega \, d^3k \qquad (3.27)$$

where $h_i(\eta)$, $a^2(\eta)R(\eta) \to 0$ as $\eta \to \pm\infty$.

Substituting (3.23) in (3.26) we get,

$$n = \frac{1}{32\pi^3 a^3}\int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \int \frac{d^3k}{\omega^2}\Big[\sum_i \sum_j \left(k_i^2 k_j^2 h_i(\eta_1)\, h_i(\eta_2)\right)$$

$$+ V(\eta_1)\, V(\eta_2)\Big] \exp\left[2i\omega(\eta_1 - \eta_2)\right] \qquad (3.28)$$

where $V(\eta) = m^2\left[a^2(\infty) - a^2(\eta)\right] - \left(\xi - \frac{1}{6}\right) R(\eta)\, a^2(\eta)$ (3.29)

And also,under the angular integration in k space, yields,

$$\sum_i h_i \int k_i^2 dk = \frac{4\pi}{3} k^2 \sum_i h_i = 0 \qquad (3\ 30)$$

$$\int k_i^2 k_j^2 d\hat{k} = \frac{4\pi}{5} k^4 \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \qquad (3\ 31)$$

Using (3.20) we have,

$$\sum_i \sum_j \int k_i^2 k_j^2 h_i(\eta_1) h_j(\eta_2) d\hat{k} = \frac{8\pi}{15} k^4 \sum_i h_i(\eta_1) h_i(\eta_2) \qquad (3\ 32)$$

Assuming that h_i', $h_i'' \rightarrow 0$ as $\eta \rightarrow \pm\infty$, Integrating the Fourier transform of h_i twice by parts one can replace $h_i(\eta)$ by $(- h_i''(\eta) / 4\omega^2)$. Also replacing the exponential in (3.2 8) by its real part (3.28) becomes

$$\eta = \frac{1}{960\pi^2} \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \left[\sum_i h_i''(\eta_1)\, h_i''(\eta_2) \int_0^{\infty} \left(\frac{k}{\omega}\right)^6 \cos 2\omega(\eta_1-\eta_2)\, dk \right.$$

$$\left. + 120\, V(\eta_1)\, V(\eta_2) \int_0^{\infty} \left(\frac{k}{\omega}\right)^2 \cos 2\omega(\eta_1-\eta_2)\, dk \right] \qquad (3\ 33)$$

using the following, integral,

$$\left\{ I = \int_0^{\infty} \cos \sigma \left(k^2+c^2\right)^{\frac{1}{2}} dk \right.$$

$$= \frac{\partial}{\partial \alpha} \int_0^\infty \frac{\sin \alpha (k^2+c^2)^{\frac{1}{2}}}{\sqrt{k^2+c^2}} dk$$

$$= \int_0^\infty dk \frac{\partial}{\partial \alpha} \int_0^\infty dx \; J_0 \left[c(\alpha^2 - x^2)^{\frac{1}{2}} \right] \cos kx$$

$$= \int_0^\infty dk \left(\cos k\alpha + \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\partial}{\partial \alpha} J_0 [c(\alpha^2 - x^2)^{\frac{1}{2}}] \cos kx \right)$$

$$= \pi \, \delta(\alpha) + \frac{\pi}{2} \frac{\partial}{\partial \alpha} J_0(c|\alpha|)$$

$$\left. = \pi \, \delta(\alpha) + \frac{\pi}{2} \frac{\partial}{\partial \alpha} J_0(c\alpha) = \pi \, \delta(\alpha) - \frac{\pi}{2} c \, J_1(c\alpha) \right\}$$

one gets,

$$\begin{aligned} n &= \frac{1}{960 \pi a^3} \int_{-\infty}^{\infty} a^4 \left(c^2(\eta) + 60 \, V^2(\eta) \right) d\eta \\ &+ \frac{\bar{m}}{960 a^3} \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \Big\{ F[\bar{m}(\eta_1 - \eta_2)] \Big[-60 \, V(\eta_1) V(\eta_2) \\ &+ (8\bar{m}^4 - 6\bar{m}^2 a_1 a_2 + \tfrac{3}{2} a_1^2 a_2^2) \sum_i h_i(\eta_1) h_i(\eta_2) \Big] \qquad (3\cdot 34) \\ &- J_1[2\bar{m}(\eta_1 - \eta_2)] \left(60 \, V(\eta_1) V(\eta_2) + \tfrac{1}{2} \sum_i h_i''(\eta_1) h_i''(\eta_2) \right) \Big\} \end{aligned}$$

where

$$\partial_1 = \frac{\partial}{\partial \eta_1}, \quad c^2 = c^{\alpha\beta\gamma\delta} c_{\alpha\beta\gamma\delta} = \frac{1}{2} \sum_i (h_i'')^2,$$

$$\bar{m} = m \, a(\infty)$$

and

$$F(x) = \frac{-1}{\pi} + x\left(J_0(2x) H_{-1}(2x) + H_0(2x) J_1(2x)\right)$$

H_v are Sturve functions, J_v are Bessel functions.

Assuming h_i''' vanish as $\eta \longrightarrow \pm\infty$, and performing k-integrations one obtains

$$\rho = \frac{1}{38400\pi^2 a^4} \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \; \mathrm{Re}\, k_0\left[2i\bar{m}(\eta_1 - \eta_2)\right]\left(\partial_1\partial_2 - 4\bar{m}^2\right)$$

$$\left[120\, v(\eta_1)\, v(\eta_2) + \left(\partial_1\partial_2 - 4\bar{m}^2\right)^2 \sum_i h_i(\eta_1)\, h_i(\eta_2)\right] \quad (3.35)$$

where k_0 is the modified Bessel function of second kind. In the massless limit, (3.35) reduces to

$$\rho = \frac{1}{38400\pi^2 a^4} \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \log\left[2i\bar{m}(\eta_1 - \eta_2)\right] \times$$

$$\left(120\, v'(\eta_1)\, v'(\eta_2) + \sum_i h_i'''(\eta_1)\, h_i'''(\eta_2)\right) \quad (3.36)$$

3.4 Explicit Examples

Following are some examples of few choices of $h_i(\eta)$ or $V(\eta)$ for which the number density and energy density can be calculated by pertubation technique.

(a) Isotropic Spacetime, Conformally coupled massive field

In this example we are considering a conformally coupled ($\xi = \frac{1}{6}$) massive field ($m \neq 0$). In this case conformal symmetry breaks due to the presence of mass and in turn causes creation of particles.

$$a^2(\eta) = 1 - A \exp(-\alpha^2\eta^2), \tag{3.37}$$

where α and A are constants corresponding to a universe which contracts rapidly bounces at $\eta = 0$ and expands out again symmetrically. From (3.29) one gets

$$V(\eta) = m^2 A \exp(-\alpha^2\eta^2) \tag{3.38}$$

Substituting in (3.25) one gets,

$$\beta_\omega = \frac{-i m^2 A\sqrt{\pi}}{2\omega\alpha} \exp(-\omega^2/\alpha^2) \tag{3.39}$$

where $\omega^2 = k^2 + m^2$ from (3.19) (3.40)

Substituting in (3.26) one gets,

$$n = \frac{m^4 A^2}{8\pi\alpha^2} \exp\left(-\frac{2m^2}{\alpha^2}\right) \left[\frac{\alpha\sqrt{\pi}}{2\sqrt{2}} - m^2 \int_0^\infty \frac{e^{-\frac{2k^2}{\alpha^2}}}{k^2+m^2} dk\right] \tag{3.41}$$

Similarly substituting in (3.27) one gets

$$\rho = \frac{m^4 A^2}{8\pi\alpha^2} \exp\left(-\frac{2m^2}{\alpha^2}\right) \int_0^\infty \frac{k^2 \exp(-2k^2/\alpha^2)}{\sqrt{k^2+m^2}} dk \tag{3.42}$$

Again using,

$$\frac{1}{(k^2+m^2)^{\frac{1}{2}}} = \frac{2}{\pi} \int_0^\infty K_0(mx) \cos kx \, dk \tag{3.43}$$

and by interchanging the order of integration, and using Wittaker function one obtains,

$$\rho = \frac{m^4 A^2}{128\pi} \exp\left(-\frac{m^2}{\alpha^2}\right)\left[2 K_0\left(\frac{m^2}{\alpha^2}\right) - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\alpha}{m} W_{-1,0}\left(\frac{2m^2}{\alpha^2}\right)\right] \tag{3.44}$$

Here we can observe that n and ρ becomes '0' as m $\longrightarrow$ 0. Also n & $\rho \longrightarrow 0$ as $m \longrightarrow \infty$. This case happens when m/α satisfies (m/α) $\ll 1$ and (m/α) $\gg 1$ respectively. We can use this special case to conclude that a Freidmann universe which expands from the Planck era with a decelarating rate ($a(t) \propto t^{\frac{1}{2}} \propto \eta$) will not produce conformally coupled particles appreciably until $\dot{a}/a \sim m$ ie. $t = m^{-1}$.

(b) Isotropic spacetime, Non-conformally coupled massive field

In this case, we take a field which is both massive and non-conformally coupled , and the influence of the mass is neglected. ($\xi \neq \frac{1}{6}$) in a realistic cosmological model. Consider the model,

$$a(\eta) = 1 - \frac{\alpha^2}{2(\alpha^2+\eta^2)} \tag{3.45}$$

where

$$R = 6\,(3\alpha^2\eta^2 - \alpha^4) \Big/ (\eta^2+\alpha^2/2)^3 \tag{3.46}$$

From (3.26)

$$\beta_\omega = \frac{i\pi}{16\omega\alpha}\left[(A\omega+B)\,e^{-2\alpha\omega} - C\,e^{-\sqrt{2}\,\alpha\omega}\right] \tag{3.47}$$

where

$$A = 2\alpha^2 m^2 + 384\alpha\Lambda, \qquad B = 672\Lambda - 7\alpha^2 m^2,$$

$$C = 480\sqrt{2}\,\Lambda, \qquad \Lambda = (\xi - \tfrac{1}{6}) \tag{3.48}$$

From (3.27) and (3.46) one gets,

$$\rho = \frac{m}{512\alpha^2}\left[\frac{A^2}{64}\frac{\partial^2}{\partial\alpha^2}\left(\frac{K_1(4m\alpha)}{\alpha}\right) + A\frac{\partial}{\partial\alpha}\left(\frac{C}{(3+2\sqrt{2})\alpha}\right.\right.$$

$$\left. K_1[(2+\sqrt{2})m\alpha] - \frac{BK_1(4m\alpha)}{8\alpha}\right) + \frac{B^2}{4\alpha}K_1(4m\alpha) \tag{3 49}$$

$$\left. + \frac{C^2}{2\sqrt{2}\alpha}K_1(2\sqrt{2}m\alpha) - \frac{2BC}{(2+\sqrt{2})\alpha}K_1[(2+\sqrt{2})m\alpha]\right]$$

when $n \longrightarrow \infty$, $\rho \longrightarrow 0$. But in the massless limit (3.49) does not vanish for nonzero Λ , and it is given by

$$\rho = 1 \cdot 843\,\Lambda^2 / \alpha^4 = \Lambda^2 R^2_{max}, \tag{3 50}$$

where R_{max} is the maximum value of the curvature.

(c) Anisotropic spacetime, Massive field

Here the conformal symmetry is broken by the departure of the background spacetime from conformal flatness and causes particle production.

Considering the metric (3.14) with

$$h_i(\eta) = \exp(-\alpha\eta^2)\ \cos(\beta\eta^2 + \delta_i) \tag{3 51}$$

where δ_i differ by '$\frac{2\pi}{3}$' so that, $\sum_i h_i = 0$

From (3.26) one gets,

$$\beta_\omega = \frac{i\sqrt{\pi}}{2\omega} \sum_i k_i^2 \; \mathrm{Re}\left(\frac{\exp\left[-\omega^2/(\alpha+i\beta)\right]}{(\alpha+i\beta)^{\frac{1}{2}}} \exp(-i\delta_i) \right) \qquad (3.52)$$

From (3.27) one gets,

$$\rho = \frac{m^2 a^2(\omega)}{1536\sqrt{\pi}\, a^4} \; \frac{(\alpha^2+\beta^2)^{\frac{3}{2}}}{\alpha^2} \exp\left[-3\alpha m^2 a^2(\omega) / (\alpha^2+\beta^2)\right]$$

$$W_{-\frac{3}{2},\frac{3}{2}} \left(\frac{2\alpha m^2 a^2(\omega)}{\alpha^2+\beta^2} \right) \qquad (3.53)$$

for the contribution of the anisotropic perturbations to the energy density. In the massless limit this reduces to

$$\rho = \frac{1}{2880\pi} \; \frac{(\alpha^2+\beta^2)^{\frac{5}{2}}}{\alpha^3 a^4} \qquad (3.54)$$

As a second similar example consider

$$h_i(\eta) = \frac{\alpha^2}{\alpha^2+\eta^2} \cos(\beta\eta+\delta_i) \qquad (3.55)$$

Then

$$\beta_\omega = \frac{i\alpha\pi}{4\omega} \sum_i k_i^2 \Big(e^{i\delta_i} e^{-|\alpha(2\omega-\beta)|} + e^{-i\delta_i} e^{-\alpha(2\omega+\beta)} \Big) \qquad (3.56)$$

and

$$\rho = \frac{\alpha^2 \cosh 2\alpha\beta}{120 a^4} \int_0^\infty \theta(\omega - \beta/2) \frac{k^6}{\omega} e^{-4\alpha\omega} dk$$

$$+ \frac{\alpha^2 e^{-2\alpha\beta}}{120 a^4} \int_0^\infty \theta(\beta/2 - \omega) \frac{k^6}{\omega} \cosh 4\alpha\omega \, dk \qquad (3.57)$$

when $\beta < 2m\, a(\infty)$, one gets

$$\rho = \frac{m\, a^3(\infty) \cosh 2\alpha\beta}{512 \alpha a^4} K_3 [4 m\, a(\infty) \alpha] \qquad (3.58)$$

and for $\beta > 2m$ the integrals cannot be performed in terms ofi known functions. However, in the massless limit,

$$\rho = \left[4(\alpha\beta)^5 + 20(\alpha\beta)^3 + 3\alpha\beta + 15 e^{-2\alpha\beta}\right] \Big/ \left(61440 \alpha^4 a^4\right) \qquad (3.59)$$

Now a natural question is that whether anisotropic oscillations of the form (3.51) and (3.55) is possible in realistic cosmological models without producing so many particles.as to be in conflict with observation.This is particularly relevant since it seems inevitable that the universe will emerge from the quantum gravity era, prior to Planck time ($t_p = 10^{-43}$s), with 'random' oscillations of this form. These oscillations will persumably be damped by a variety of mechanism [8] including back reaction, by the created particles. But we are not in a position to study this back reaction, especially since vacuum polarisation effects are of first order in the anisotropic pertubation [9,10] and thus are of more importance to back reaction than the second-order particle production calculated as follows (we shall take $\rho_0 = 10^{-30}$).

If we consider the oscillations about a radiation dominated Friedmann model with

$$a(t) = r t^{\frac{1}{2}} = \frac{1}{2} r^2 \eta \qquad (3.60)$$

substituting in (3.51) one gets,

$$h_i(t) = exp\left(\frac{-4\alpha t}{r^2}\right) \cos\left(4\beta t / r^2 + \delta_i\right) \qquad (3.61)$$

and the frequency 'ν' of oscillation is given by

$$\nu = 4\beta / r^2 \qquad (3.62)$$

with a measure of the isotropisation time t_I is given by

$$t_I = r^2 / (4\alpha) \qquad (3.63)$$

and the energy density (3.54) at the present time $t_0 = 10^{17}$ s and in terms of ν and t_I is given by

$$P(t_0) = \frac{\hbar}{46080\pi c^5} \frac{t_1^3}{t_0^2} \left(\nu^2 + t_1^{-2}\right)^{5/2}, \qquad (3.64)$$

Putting $y = \nu t_1$ the values of known constants and rearranging one gets (3.64) as

$$y^{10} + 5y^8 + 10y^6 + 10y^4 + 5y^2 + 1 \simeq 10^{176} (y/\nu)^4 \qquad (3.65)$$

when $y = 1$,

$\nu^{-4} \simeq 10^{-175}$, so that $\nu \simeq 10^{44}\ s^{-1}$

and $t_I = 10^{-44} s \simeq t_p$.

If $1 \ll y \ll 10^{44}\ \nu^{-1}$, (3.65) gives

$y^{10} = 10^{176} (y/\nu)^4$, implies that

$$t_I = 10^{29}\ \nu^{-5/3} \qquad (3.66)$$

Now we consider a model with oscillations (3.55), which is given by

$$h_i(t) = \frac{\alpha^2}{\alpha^2 + \frac{4t}{\nu^2}} \cos\left(2\beta t^{\frac{1}{2}}/\nu + \delta_i\right) \qquad (3.67)$$

where isotropization time is given by

$$t_I = \alpha^2 r^2 / 4 \tag{3.68}$$

while the frequency of oscillations are given by

$$\nu(t) = \nu_I (t_I / t)^{-\frac{1}{2}} \tag{3.69}$$

where ν_I is the frequency at the isotropisation time,

$$\nu_I = \frac{2\beta}{\gamma t_I^{1/2}} \tag{3.70}$$

From (3.59) one gets

$$\rho(t_0) = \frac{\hbar}{983040c^5} \left(\frac{\nu_I}{t_0}\right)^2 \frac{1}{y^2} \left(4y^5 + 20y^3 + 3y + 15e^{-2y}\right), \tag{3.71}$$

where $y = \nu_I t_I$

Inserting the known constants, the condition $\rho (t_0) = \rho_0$ becomes

$$4y^5 + 20y^3 + 3y + 15e^{-2y} \simeq 10^{88} (y/\nu_I)^2 . \tag{3.72}$$

when $y = 1$,

$v_I^{-2} = 10^{-87}$ so that $v_I = 10^{-43}$, $s = t_p$ & $t_I = t_p$.

Thus, as in the previous example, Planck frequency oscillations damp within few Planck times.

If $\ll \; y \ll 10^{88} v_I^{-2}$ (3.73)

From (3.72) one gets

$$y^3 = 10^{88} \; v_I^{-2} \; ,$$

hence $y \simeq 10^{29} v_I^{-2/3}$, and thus, for $v_I \ll t_p^{-1}$,

$$t_I \simeq 10^{29} \; v_I^{-5/3}, \qquad (3.74)$$

which is the same as (3.66).

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CHAPTER IV

CREATION OF PARTICLES IN AN EXPANDING UNIVERSE USING QUANTUM EQUIVALENCE PRINCIPLE

Most of the workers, studying the possibility of creation of particles in curved spaces, do not prefer to decompose the fields into positive and negative frequency parts during the expansion of the universe for defining the particle concept. They prefer to confine themselves to the determination of particle number only in asymptotically static situations ie. in - and out - states which has been discussed at length in previous chapters in different models. In this chapter, on the contrary, the approach of M. Castagnino, D. Harari and L. Chimento [1] for evaluating the creation of particles in an expanding universe as a function of time, has been discussed. This approach is based on the particle model proposed by M. Castagnino and R. Weder [2] called Quantum Equivalence Principle (Q.E.P.). Later on, Q.E.P. was reformulated by M. Castagnino, A.Foussats, R.Laura and O. Zandron [3] in a more precise and consistent manner and so they obtain an exact particle model which coincides with the mode of reference [2] .

The new formulation is based on the fact that in a local inertial frame (where gravitational force vanish), the particle model must be similar to the model of flat spacetime. Usually general relativity translates flat spacetime theories to curved space ones by replacing ordinary partial derivative by covariant derivatives. In Quantum field theory, particles are supposed to be linear combinations of plane wave function, $\exp[i(\omega t - kx)]$ (with $\omega > 0$). So it does not contain any partial derivative to be replaced by covariant ones. This is the main difficulty. In ref. [2] as a panacea to this problem, a particle model has been prescribed which behaves as the flat spacetime particle model in normal co-ordinates at each spacetime point. To make such a definition possible, a new global property of spacetime has been imposed which is not only globally hyperbolic but also has normal Cauchy surface (each point on the Cauchy surface can be linked with a unique space - like geodesic). In this kind of manifold, Q.E.P. can be formulated yielding satisfactory consequences.

So, one can try set of solutions $\{\psi_k(x)\}$ with the generalization of properties of $(2\pi)^{-\frac{3}{2}}(2\omega_k)^{-\frac{1}{2}}\exp[i(\omega t - kx)]$ where t,x are normal cartisian coordinates in flat spacetime, but in curved spacetime such coordinates do not exist. Moreover, we do not know which of the linear combinations must be generalized to obtain $\psi_k(x)$ in curved space. Both ambiguities will be eliminated if the particle model were described by two kernels G(x,y) and $G_1(x,y)$ first introduced by Lichnerowicz [4].

Details of normal coordinates and Q.E.P. are given below.

4.1 Normal Co-ordinates

We use normal Riemannian co-ordinates for representing the points of curved spacetime[5,6]. Let x = $(\tau, \bar{x})$ and y = $(t, \bar{y})$ be two points of the curved spacetime, which is endowed with normal Cauchy hyper surface and let 's' be the length of the geodesic between x and y and v_i is the unitary vector tangent to the geodesic at x. Then the normal *co-ordinates of y with respect to the origin 'x' is given by*

$$y^{i'} = \delta^{i'}_{i} s v^{i}, \qquad (4.1)$$

The transformation of ordinary coordinates to normal coordinates is given by

$$y^{i'} = \delta^{i'}_{i} \left\{ \Delta x^{i} + \frac{1}{2!} \Lambda^{i}_{jk}(x) \Delta x^{j} \Delta x^{k} + \frac{1}{3!} \Lambda^{i}_{jk\ell}(x) \Delta x^{j} \Delta x^{k} \Delta x^{\ell} + \cdots \right\} \qquad (4.2)$$

$$\Lambda^{i}_{jk} = \Gamma^{i}_{jk}, \quad \Lambda^{i}_{jk\ell} = \Gamma^{i}_{jk\ell} + 3\Gamma^{\gamma}_{(jk} \Gamma^{i}_{\ell)\gamma} \qquad (4.3)$$

and

$$\begin{cases} \Gamma^{i}_{jk\ell} = \partial_{j} \Gamma^{i}_{k\ell} - 2\Gamma^{m}_{(jk} \Gamma^{i}_{\ell)m} \\ \Gamma^{i}_{j_p \cdots j_1 k} = \partial_{j_p} \Gamma^{i}_{j_{p-1} \cdots j_1 k} - p\,\Gamma^{m}_{(j_p j_{p-1}} \Gamma^{i}_{j_{p-2} \cdots j_1 k)m} \end{cases} \qquad (4.4)$$

When we consider the metric

$$ds^2 = dt^2 - a^2(t)\ (dx^2 + dy^2 + dz^2) \qquad (4.5)$$

In this case, the transformation from ordinary co-ordinates to normal coordinates is given by using (4.2) - (4.4).

$$\bar{y}^1 = \Delta\bar{x} + H\Delta\bar{x}\Delta t + \frac{1}{6}a^2H^2(\Delta\bar{x})^2 + \frac{1}{3}\left(\frac{R}{6} - H^2\right)\Delta t^2\Delta\bar{x} + \bar{O}(H^3, \Delta x^4), \qquad (4.6)$$

$$\bar{t} = \Delta t + \frac{1}{2}a^2 H(\Delta\bar{x})^2 + \frac{1}{3}a^2\left(\frac{R}{12} + H^2\right)\Delta\bar{x}^2\Delta t + O(H^3, \Delta x^4)$$

where $R = -(6H^2 + \dot{a}^2/a)$

and $O(H^3, \Delta x^4)$ and $O(H^3, \Delta x^4)$ are negligible when $H \to 0$ and $\Delta x \to 0$

In normal co-ordinates

$$\Delta_1(x^1 = 0, y^1) = \frac{1}{(2\pi)^3}\int \frac{\cos\omega_k^c t^1}{\omega_k^c}\exp\left[-iky^1a\right]d^3k \qquad (4.7)$$

where

$$\omega_k^c = (k^2 + m_c^2)^{\frac{1}{2}}, \qquad m_c^2 = m^2 + \xi R$$

4.2 Quantum Equivalence Principle (QEP)

The QEP completes a study begun by Lichnerowicz[4] devoted to the generalization to curved spacetime of the propagators $\Delta(x-x')$ and $\Delta_1(x-x')$ used in flat spacetime to make the decomposition of free fields into positive and negative frequency parts. It has been proved[7] that Lichnerowicz conditions allow the unique definition of the function $G(x,x')$, the generalization of $\Delta(x-x')$, but not of a function $G_1(x,x')$, except in a static case. In ref. [3] the possibility is then considered of the existence of a different G_1 at each Cauchy surface in every globally hyperbolic universe and a method is proposed to find such a function in a unique manner. From Ref. [2], one can get the boundary conditions in the form of an expansion in powers of Hubble coefficients. QEP states that "The kernel $G_1(x_0,x)$ has the same Cauchy data on the normal surface Σ_τ as $\Delta_1(0,x')$ written in normal co-ordinates with origin at x_0".

From ref. [2] applying QEP to the case of a spatially flat universe up to the second order in the Hubble coefficients to the following Cauchy data for negative frequency solutions are given by[3]

$$p^{T}_{k}\frac{e^{-ikx}}{(2\pi a)^{3/2}} = \left(\frac{1}{2\omega_k} - \frac{5}{16}\frac{H^2 m^4}{\omega_k^7} - \frac{\xi_\phi R}{2\omega_k^3}\right)^{\frac{1}{2}} \times \frac{e^{-ikx}}{(2\pi a)^{3/2}} \tag{4.8}$$

$$\dot{p}^{T}_{k}\frac{e^{-ikx}}{(2\pi a)^{3/2}} = \frac{-1}{2}\left(\frac{1}{2\omega_k} - \frac{5}{16}\frac{H^2 m^4}{\omega_k^7} - \frac{\xi_\phi R}{2\omega_k^3}\right)^{-1/2} \times \left(i + \frac{H(2\omega_k^2 + m^2)}{2\omega_k^3}\right)\frac{e^{-ikx}}{(2\pi a)^{3/2}}, \tag{4.9}$$

where

$$\omega_k = \left(\frac{k^2}{a^2} + m^2\right)^{\frac{1}{2}}$$

4.3 General Formalism

Here we consider the particle model obtained from the QEP and develop the general formalism to find the Bogoliubov transformation between the appropriate basis at two different Cauchy surfaces, and the number of created particles in the time elapsed between them, which is evaluated up to second order in an expansion in powers of the Hubble coefficients.

Consider a scalar field $\phi(x)$ satisfying the Klein-Gordon Equation (1.2 4) in the background of spatially flat Robertson-Walker universe, for which the matric is given by (2.1) with $k = 0$.

We use separation of variables in (1.24). Letting,

$$\psi_k(x,t) = f_k(t) \frac{e^{ikx}}{(2\pi a)^{3/2}} \tag{4.10}$$

as the solution of (1.24) one gets $f_k(t)$, satisfying the equation,

$$\frac{1}{a^3}\frac{d}{dt}\left[a^3\dot{f}_k(t)\right] + \left(\frac{k^2}{a^2} + m^2 + \xi R\right) f_k(t) = 0, \tag{4.11}$$

where $k = |\kappa|$

Now let us propose

$$f_k(t) = \frac{\exp\left[-i\int^t \Omega(k,t')\,dt'\right]}{\left[2\Lambda(k,t)\right]^{1/2}} \tag{4.12}$$

where Ω and Λ are real functions which can be calculated using (4.11). Here this procedure has no loss of generality as we have expressed $f_k(t)$ in terms of its argument and absolute value. One can construct the functions $\psi_k(x)$ and its complex conjugates, a basis for the space Klein-Gordon equation solutions by imposing the orthonormality condition on ψ_k of (4.10), which is given by

$$\langle \psi_k, \psi_h \rangle = -\delta_{kh} \tag{4.13}$$

and the inner product is the space of solutions of the Klein-Gordon equation, which is defined as

$$\langle \psi_k, \psi_h \rangle = i \int_{\Sigma} [(\partial_j \psi_k^*) \psi_h - \psi_k^* (\partial_j \psi_h)] d\sigma_j \qquad (4.14)$$

which is independent of the Cauchy surface on which it is evaluated. Using (4.13) we get $\Omega = \Lambda$ and

$$\langle \psi_k^*, \psi_{-h}^* \rangle = \delta_{kh} \qquad (4.15)$$

Therefore one can take $\{\psi_k(x), \psi_h^*(x)\}$ as an orthonormal basis for the Klein-Gordon equation, since ψ_k and its complex congugate form a complete orthonormal basis with the inner product (4.14), so the Klein-Gordon equation solution can be ex essed as

$$\phi(x) = \int d^3k \left[a_k \psi_k(x,t) + b_{-k} \psi_{-k}^*(x,t) \right] \qquad (4.16)$$

where $\psi_k(x,t)$, is given by

$$\psi_k(x,t) = \frac{\exp\left[-i \int^t \Omega(k,t')\, dt'\right]}{[2\Omega(k,t)]^{1/2}} \frac{e^{ikx}}{(2\pi a)^{3/2}} \qquad (4.17)$$

Substituting (4.12) into (4.11), we have Ω satisfies the equation,

$$\Omega^2 - \frac{1}{4}\left(\frac{\dot{\Omega}}{\Omega}\right)^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\dot{\Omega}}{\Omega}\right) = \frac{k^2}{a^2} + m^2 + \xi R - \frac{3}{2}\left(\dot{H} + \frac{3}{2}H^2\right) \tag{4.18}$$

where $H = \dot{a}/a$, the Hubble constant and $R = 6\ (\dot{H} + 2H^2)$.

We have already seen in section [2.2] of Chapter II that if coefficients of the dxpansion of the field in a particular basis are interpreted as creation and annihilation operators, the vacuum state associated with a particular basis *would be a many-particle state respective to other basis.*

Here using QEP, the appropriate linear combination of $\psi_k(x)$ and $\psi_k^*(x)$ over each Cauchy surface can be found out. Here we equate the normal coordinates over the hyper-surface of the Green functions with their analog in flat space. From this one gets the boundary conditions at that particular Cauchy surface such that the solutions of the type (4.17) are considered as negative frequency solutions and positive for its complex conjugates. And we take the boundary conditions over the surface Σ_r by

$$-\frac{p_k^\tau e^{-ikx}}{(2\pi a)^{3/2}} \qquad \text{and} \qquad \frac{\dot{p}_k^\tau e^{-ikx}}{(2\pi a)^{3/2}}$$

Now we will find the Bogoliubov transformation between the basis at two different Cauchy surfaces $\Sigma\, r$ and $\Sigma\, r_0$ in terms of P_K^{τ} and $\dot{P}_K^{\tau}$ Now we write,

$$P_K^{\tau}\,\frac{e^{-ikx}}{(2\pi a)^{3/2}} = A_K^{\tau}\,\psi_K(x,t) + B_{-K}\,\psi_{-K}(x,t) \qquad (4\cdot19)$$

$$\dot{P}_K^{\tau}\,\frac{e^{-ikx}}{(2\pi a)^{3/2}} = A_K^{\tau}\,\dot{\psi}_K(x,t) + B_{-K}^{\tau}\,\psi_{-K}^{*}(x,t) \qquad (4\cdot20)$$

where $\left|A_K^{\tau}\right|^2 - \left|B_{-K}^{\tau}\right|^2 = 1$

solving (4.19)) and (4.20) one gets

$$A_k^{\tau} = \frac{1}{[2\Omega(k,\tau)]^{1/2}}\left[-P_k^{\tau}\,F^{*}(k,\tau) + \dot{P}_K^{\tau}\right] \times \exp\left[i\int^{\tau}\Omega(k,t)\,dt\right] \qquad (4\cdot21)$$

$$B_k^{\tau} = \frac{i}{[2\Omega(k,\tau)]^{1/2}}\left[P_k^{\tau}\,F(k,\tau) - \dot{P}_K^{\tau}\right] \times \exp\left[-i\int^{\tau}\Omega(k,t)\,dt\right] \qquad (4\cdot22)$$

where $F(k,t) = -i\Omega - \frac{1}{2}(3H + \dot{\Omega}/\Omega)$.

and the negative frequency function over $\Sigma\, r$ is given by

$$\phi_k^{\tau}(x,t) = A_k^{\tau}\,\psi_k(x,t) + B_k^{\tau}\,\psi_{-k}(x,t) \qquad (4\cdot23)$$

Therefore there exist a Bogoliubov transformation related to the surface $\Sigma\tau_0$

$$\phi^{\tau}_{k}(x,t) = \alpha_k(\tau_0,\tau)\,\phi^{\tau_0}_{k}(x,t) + \beta_k(\tau_0,\tau)\,\phi^{\tau_0 *}_{-k}(x,t) \tag{4.24}$$

Then one gets

$$A^{\tau}_{k} = \alpha_k(\tau_0,\tau)\,A^{\tau_0}_{k} + \beta_k(\tau_0,\tau)\,B^{\tau_0 *}_{k}$$

$$B^{\tau}_{k} = \alpha_k(\tau_0,\tau)\,B^{\tau_0}_{k} + \beta_k(\tau_0,\tau)\,A^{\tau_0 *}_{k}$$

Solving the above two equations again one gets

$$\beta_k(\tau_0,\tau) = \frac{1}{[2\Omega(\tau_0)]^{1/2}[2\Omega(\tau)]^{1/2}} \times \left\{ \left[\dot{P}^{\tau_0}_{k} - P^{\tau_0}_{k} F^{*}(\tau_0)\right]\left[P^{\tau}_{k} F(\tau) - \dot{P}^{\tau}_{k}\right] \exp\left(-i\int_{\tau_0}^{\tau}\Omega\, dt\right) - \left[\dot{P}^{\tau}_{k} - P^{\tau}_{k} F^{*}(\tau)\right]\left[P^{\tau_0}_{k} F(\tau_0) - \dot{P}^{\tau_0}_{k}\right] \exp\left(i\int_{\tau_0}^{\tau}\Omega\, dt\right) \right\} \tag{4.25}$$

Decomposing the field $\phi(x,t)$ in terms of the basis $\{\phi^{\tau}_{k}, \phi^{\tau *}_{k}\}$ one gets,

$$\phi(x,t) = \int d^3k \left[a^{\tau}_{k}\,\phi^{\tau}_{k}(x,t) + a^{\tau\dagger}_{-k}\,\phi^{\tau\dagger}_{-k}(x,t)\right], \tag{4.26}$$

where a_k^{τ} , $a_{-k}^{\tau\dagger}$ are the annihilation and creation operators at the surface Σ_τ . Similarly one can write,

$$\phi(x,t) = \int d^3k \left[a_k^{\tau_0} \phi_k^{\tau_0}(x,t) + a_{-k}^{\tau_0\dagger} \phi_{-k}^{\tau_0\dagger}(x,t) \right] \tag{4.27}$$

and the particle number operator, with eigenvalue $N_k(\tau_0)$ is given by

$$a_k^{\tau_0\dagger} a_k^{\tau_0} |\tau_0\rangle = N_k(\tau_0) \tag{4.28}$$

From (4.26) and (4.27),one gets

$$a_k^{\tau} = \alpha_k^*(\tau_0,\tau)\, a_k^{\tau_0} - \beta_k^*(\tau_0,\tau)\, a_{-k}^{\tau_0\dagger} \tag{4.29}$$

and the particle number operator at Σ_τ is given by

$$\begin{aligned} N_k(\tau) &= \langle\tau_0| a_k^{\tau\dagger} a_k^{\tau} |\tau_0\rangle \\ &= N_k(\tau_0) + |\beta_k|^2 \left[1 + 2N_k(\tau_0)\right] \end{aligned} \tag{4.30}$$

Now to evaluate $\beta_k(\tau_0,\tau)$ and particle production number by the expanding universe to second power in H, one can propose for Ω(k,t) as,

$$\Omega(k,t) = \omega_k + \epsilon(t) H + \gamma(t) H^2 \tag{4.31}$$

for which H is propotional to H^2, such as $a(t) = t^{\epsilon}$ with 'ϵ' real. Substituting (4.31) in (4.18) and equating the coefficients in the expansion and taking $\dot{\epsilon}$ and $\dot{\gamma}$ proportional to H, we obtain

$$\pm\Omega(k,t) = \omega_k + \frac{5}{8}\frac{H^2 m^4}{\omega_k^5} - \frac{H^2 m^2}{4\omega_k^3} - \frac{R}{12\omega_k^3}\left(\frac{m^2}{2} + \omega_k^2\right) + \frac{\epsilon_f R}{2\omega_k} \tag{4.32}$$

Substituting in (4.25) for β_k, and taking only up to second powers of H, one gets

$$\beta_k(\tau_0,\tau) = L(k,\tau)\exp\left(-i\int_{\tau_0}^{\tau}\omega_k dt\right) - L(k,\tau_0)\exp\left(i\int_{\tau_0}^{\tau}\omega_k dt\right), \tag{4.33}$$

with

$$L(k,t) = \frac{1}{4W_k^4}\left[\frac{m^2}{2}H^2 + \frac{R}{6}\left(\frac{m^2}{2} + \omega_k^2\right)\right]$$

Then

$$\left|\beta_k(\tau_0,\tau)\right|^2 = L^2(\tau) + L^2(\tau_0) - 2L(\tau)L(\tau_0)\cos\left(2\int_{\tau_0}^{\tau}\omega_k dt\right) \tag{4.34}$$

Since in (4·34), there is no term in ξ_ρ , we can say that, there is no difference in minimal coupling and conformal coupling. Also it depends only in k = $|k|$, one can conclude that particles are created in pair having opposite momentum.

4.4. Alternative Method

Here we use the basis for the decomposition of the field into its positive and negative frequency parts, which are functions, that are not solutions of the Klein-Gordon equation but which satisfy at every Cauchy surface the boundary conditions imposed by the QEP. Parker[8] was the first who noticed that, the particle-production process involves a mixing of positive and negative frequencies using another criterion to define the particle model : the so called minimization principle.

Now when we propose the following decomposition of the scalar field in a spatially flat expanding universe:

$$\phi(x,t) = \int d^3k \left(a_k(t) \frac{\exp\left[-i\int^t W(k,t')\,dt'\right]}{[2\,W(k,t)]^{\frac{1}{2}}} \frac{e^{ikx}}{(2\pi a)^{3/2}} + H.C \right) \qquad (4.35)$$

where W(k,t) is a real function, to be determined in a physical way interpreting the coefficient $a_{k(t)}$ and $a^{\dagger}_{k(t)}$ as annihilation and creation operators of particles at every instant. Here we use QEP and take as functions of negative frequency the following:

$$\phi_k(x,t) = \frac{\exp\left[-i\int^t W(k,t')\,dt'\right]}{\left[2\,W(k,t)\right]^{1/2}} \; \frac{e^{ikx}}{(2\pi a)^{3/2}} , \tag{4.36 a}$$

and

$$\dot{\phi}_k(x,t) = -i\left(i\,W + \frac{3}{2}H + \frac{\dot{W}}{W}\right) \frac{\exp\left(-i\int^t W\,dt'\right)}{(2W)^{1/2}} \; \frac{e^{ikx}}{(2\pi a)^{3/2}} \tag{4.36 b}$$

and imposing (4.36a) and (4.36b) to satisfy the boundary conditions (4.8) and (4.9) at every spatial hypersurface and neglecting higher powers, one gets

$$W(k,t) = \omega_k \left(1 + \frac{5}{8}\,\frac{H^2 m^4}{\omega_k^6} + \frac{\xi R}{2\omega_k^2}\right) \tag{4.37}$$

Comparing this expression with (4.32), one can conclude that (4.36a) is not a Klein-Gordon solutions. If $W = \Omega$ the Klein-Gordon solutions are the same as (4.17) satisfying at every time the boundary conditions of the QEP. Therefore there is no mixing of positive and negative frequency while passing from one Cauchy surface to another, and there is no particle creation.

Now, if one propose the following relation,

$$a_k(\tau) = \alpha_k^*(\tau_0,\tau)\, a_k(\tau_0) - \beta_k^*(\tau_0,\tau)\, a_k^\dagger(\tau_0) \qquad (4.38)$$

with the condition $\alpha_k^*(\tau_0,\tau) = 1$ and $\beta_k(\tau_0\ \tau_0) = 0$. Keeping (4.35) as the Klein-Gordon solution, one obtains

$$\alpha_k = 1 - i\int_{\tau_0}^{\tau} dt\ \ S(t)\left[\alpha_k + \beta_k \exp\left(-2i\int_{\tau_0}^{\tau} W dt'\right)\right] \qquad (4.39a)$$

$$\beta_k = i\int_{\tau_0}^{\tau} dt\ S(t)\left[\beta_k + \alpha_k \exp\left(2i\int_{\tau_0}^{\tau} W dt'\right)\right] \qquad (4.39b)$$

where the function $S(k,t)$ is defined from W as

$$2WS = W^2 - W^{1/2}\frac{d^2}{dt^2}\left(W^{-1/2}\right) - \omega_k^2 - \xi R + \frac{3}{2}\left(\dot{H} + \frac{3}{2}H^2\right) \qquad (4.40)$$

Putting (4.40) in (4.37) one gets

$$S = \frac{1}{4}\frac{m^2}{\omega_k^3}\left(\dot{H} + 3H^2\right) + \frac{R}{6}\frac{1}{2\omega_k} \qquad (4.41)$$

Taking in the integrand a zero-order approximation: $\alpha_k^{(0)} = 1$, $\beta_k^{(0)} = 0$, (4.39b) becomes

$$\beta_k^{(1)}(\tau_0, \tau) = i \int_{\tau_0}^{\tau} dt \;\; S(t) \exp\left(2i \int_{\tau_0}^{\tau} W \, dt'\right) \qquad (4.42)$$

Here $S(t)$ is itself of second order, we can stop the iteration method,and taking $W = \omega_k$ in the exponential one gets,

$$\beta_k(\tau_0, \tau) = \frac{S(t)}{2\omega_k} \exp\left(2i \int_{\tau_0}^{\tau} \omega_k dt'\right) \Bigg|_{\tau_0}^{\tau} \qquad (4.43)$$

$$= L(\tau) \exp\left(2i \int_{\tau_0}^{\tau} \omega_k \, dt'\right) - L(\tau_0),$$

From this one can observe that when $S(t)$ vanishes, there is no particle creation from (4.39b) which gives $\beta_k = 0$ for every t. Also when $S_t = 0$ (4.40) and (4.18) coincides and $\Omega = W$, resulting no particle creation.

4.5 Future developments

Now a natural interests is in facing the problem of comparing particle model with experimental evidence. Which kind of astronomical observations can one try to explain with this particle-creation mechanism? Of course, the ultimate aim of these types of theories is to formulate a "quantum cosmology" explaining the existing data of the universe evolution. But one can also search for observable effects of this particle-creation process without formulating a self-consistent cosmological model, for example, assuming the standard hot big-bang model of evolution of the universe, and evaluating the production by this background. If the creation is not so great such as to have a considerable "back-reaction" effect in the universe evolution, but if it is not completely negligible, one can search for observable consequences. Of course, the kind of observations which this model would eventually explain must be related with background, isotropic distributions of particles, with no local origin (i.e., that are not created in our solar system, by the explosion of a s , etc.). Diffuse cosmic x and γ rays might be an ... ple,[9] or the high-energy electron component of cosmic rays.[10,11]

Thus one can evaluate the present flux of the created particles by the expansion of the universe per unit energy interval, per unit solid angle, in order to compare it with cosmicray experimental data. The particle model considered will be valid only if it is verified that the condition $H/\omega_k < 1$ is satisfied. Otherwise, the expansions made to obtain the boundary conditions from the QEP and to solve Eq. (4.18) would be meaningless. For most common cosmological models $H \sim 1/t$, thus one is restricted $\tau_0 > 1/m$ for massive particles. For massless particles the minimum possible value for τ_0 will depend upon the energy E wished to be considered at the time τ : one needs $\tau_0 > [a(\tau_0)/a(\tau)](1/E)$. One can observe that the particle model is perfectly valid in the present stage of the universe evolution, leading to an absolutely negligible production compared to the number of already existing particles.

One can also note from expression (4.34) that for every evolution of the type $a(t) = At^{\epsilon}$ with $\epsilon < 1$ it can be verified that $L(\tau_0) \gg L(\tau)$ if $\tau \gg \tau_0$. Then the number of created particles between two very widely separated instants τ_0 and τ does not depend upon the particular evolution considered throughout the time range, but only on the evolution in the neighbourhood of the initial instant (ie., on R_0 and a_0, the scalar curvature and the radius of the universe at the initial instant). The evolution at the times near the initial instant τ_0 will determine the rate at which the process takes place.

Imposing the condition that $H/\omega_k < 1$, and considering all the parameters R_0 and a_0/a, the ratio between the radius of the universe at the initial instant and the present one, instead of assuming a particular evolution for a (t) between

τ_0 and τ . For the type of evolutions when $\tau >> \tau_0$ and $L(\tau_0) >> L(\tau)$, one can approximate,

$$\left|\beta_k(\tau_0,\tau)\right|^2 = \frac{1}{4\omega_k^8(\tau_0)}\left\{\frac{m^2}{2}H_0^2 + \frac{R_0}{6}\left[\frac{m^2}{2} + \omega_k^2(\tau_0)\right]\right\}^2 \qquad (4.44)$$

Another approximation can also be made for the ratio of the present kinetic energy at time τ_0 is the same as the ratio, a_0/a which in turn less than unity. Neglecting 'm' with respect to k/a_0 one gets,

$$\left|\beta_k(\tau_0,\tau)\right|^2 = \frac{1}{16}\left(\frac{R_0}{6}\right)^2 \frac{a_0^4}{k^4} \qquad (4.45)$$

and the density of the created particles is given by

$$\int d^3k\, \rho(k) = \int dk\; 4\pi k^2 \frac{|\beta_k|^2}{(2\pi c a)^3} \qquad (4.46)$$

Flux Φ is given by

$$\int \Phi(k)\, dk = \int \rho(k)\, v_k\, dk \qquad (4.47)$$

where v_k is the velocity of the particle with linear momentum k/a. The energy of the created particles is

$$E = \frac{m\hbar}{\left(1 - V_k^2/c^2\right)^{\frac{1}{2}}} = \hbar\,\omega_k = \hbar\left(\frac{k^2}{a^2} + m^2\right)^{\frac{1}{2}} \qquad (4\ 48)$$

with m expressed in frequency dimensions [m(1/s) = m(gr) (c^2/h)] . Then

$$V_k = \frac{c}{ma}\,k\left(1 + \frac{k^2}{m^2a^2}\right)^{-1/2}$$

Denoting by T the kinetic energy of the particle, T = E-mh, one gets,

$$k = a\left(\frac{T^2}{\hbar^2} + \frac{2mT}{\hbar}\right)^{\frac{1}{2}}$$

connecting (4.47) and (4.45) one can write

$$\Phi(T)\,dT = \frac{\hbar}{16(2\pi)^3c^2}\left(\frac{R_o}{6}\right)^2\frac{a_o^4}{a^4}\,\frac{dT}{T^2 + 2m\hbar T} \qquad (4.49)$$

giving the present flux per kinetic energy interval per unit solid angle.

Experimental evidence for the diffuse cosmic 'x' and 'γ' ray background reveals a spectrum given by $\Phi = 25\ E^{-2}$ keV/cm^2 s, which is exactly the same as (4.49) in the massless limit. If one assume the standard hot big-band model for the evolution of the universe, with a radiation-dominated era continued by a matter-dominated one, and take the age of the universe as the Hubble time $t_H = H^{-1}\ 10^{18}$sec, then one can put $a_0 / a = (T_0^{1/2} / t_H^{2/3})\ (T_d^{2/3} / T_d^{1/2})$, where T_d denotes the time when radiation and matter became decoupled, approximately 10^{13}sec. For dimensional reasons one can take $R_0 \sim 1/T_0^2$, without specifying strictly the time dependence of the radius of the universe in the neighbourhood of the initial instant of the creation process. Then equating (4.49) with the experimental data for cosmic x and y ray one obtains $T_0 \sim 3\times10^{-43}$sec, which is of the order of magnitude of Planck time $t_p = (Gh/c^5)^{1/2} \sim 10^{-43}$ sec.

Heuristic arguments [12] show that this semiclassical theory can be applied just until Planck time, but before this time it would be necessary to quantify gravitation because the energy density of the created particles would be of the order of that producing the spacetime curvature. For the present considered energy of the photons (E > 1 keV) it is verified $R_0 (k/a_0)^{-2} \sim 10^{-4}$, so the particle model in an expansion in powers of H/ω_k is valid. Thus this theory can be applied to such an extremely little time.

Nevertheless, if the particles would have been created before the decoupling of radiation with matter, the created photons would become thermalized, and would be present today at the 3 K background radiation. Although this produces a negative result,the coincidence in the shape ofthe spectrum encourages us to further improve the model, and to continue this study with higher spins, in order to eventually explain in a more realistic way the extagalactic isotropic component of cosmic rays as created by the expansion of the universe.

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CHAPTER V

QUANTUM PARTICLE PRODUCTION DUE TO A CONFORMALLY COUPLED SCALAR FIELD PLUS GRAVITY IN N + 4 DIMENSIONS USING KALUZA-KLEIN THEORIES

In the previous four chapters we had seen the possibility of particle creation under the influence of gravity due to a scalar field in 4 dimensions. But Nature has some other symmetries also. Under the unification scheme of gravity with other interactions of nature (electromagnetism, weak interaction and strong interaction), Kaluza-Klein type[1] theories are very important. In these theories, observed gauge symmetries of nature are a result of the proper description of nature being general relativity in more than 4-dimensions. The extra dimensions are assumed to be compact with the distances around these dimensions being so small[2] that they cannot be directly perceived in the history of universe. Using the Einstein-Hilbert action in (n+4) dimensions as a starting point, one may obtain an effective four dimensional action by integrating over the n - compact dimensions. In this chapter the influence of gauge symmeteries of Nature can be seen in the four dimensional theory thus obtained. The metric for the gravitational field[3] is taken as a generalization of Robertson-Walker metric which is given by

$$ds^2 = -dt^2 + R^2(t)\,\delta_{ij}\,dx^i dx^j + A^2(t)\,\delta_{AB}\,dy^A dy^B \qquad (5\cdot1)$$

where x^i are the coordinates of the usual three space while y^A are the coordinates of the compact space and this compact space is taken as n-torus.

5.1 General Formalism

In (n+4) dimensions the Lagrangian density is given by [4]

$$L = -\frac{1}{2}\sqrt{-g}\left(g^{ij}\,\partial_i\phi\,\partial_j\phi + (m^2+\xi R)\phi^2\right) \qquad (5\cdot2)$$

where i,j = 1,2... n+4 with x^4 being the time variable, R is scalar curvature [5]. Variation of L with respect to ϕ gives

$$\left[\frac{1}{\sqrt{g}}\,\partial_i\sqrt{-g}\,g^{ij}\,\partial_j - (m^2-\xi R)\right]\phi = 0 \qquad (5\cdot3)$$

as the equation of motion for ϕ . To simplify the equations, one can put

$$\frac{d\eta}{dt} = a^{-1} = (R^3A^n)^{-1/n+3} \qquad (5\cdot4)$$

Using (5.4) and substituting (5.1) in the Klein-Gordon equation (5.3), one gets

$$\ddot{\phi} + (n+2)\left[\frac{\dot{a}}{a}\right]\dot{\phi} + a^2(m^2 + \xi R - R^{-2}\partial_i^{\,2} - A^{-2}\partial_B^{\,2})\phi = 0 \qquad (5.5)$$

where i = 1,2,3, B = 5, ..., n+4 and the differentiation is with respect to η .

Since ϕ is periodic in the y^A, we can impose the periodic boundary condition

$$\phi(x,y,\eta) = \phi(x + mL_3, y + nL_{n+4}, \eta); \quad (L_3 \to \infty)$$

where m and n are vectors with three and n integer components, respectively and 'r' represents the pair (x,y). Now,

$$\phi(r,\eta) = \sum_k \left[a_k \phi_k(r,\eta) + a_k^{\dagger}\phi_k^{*}(r,\eta)\right] \qquad (5.6)$$

where

$$\phi_k(r,\eta) = \left(L_3^3 L_{n+4}^n\right)^{-1/2} a(\eta)^{-(n+2)/2} e^{-ik_x x}$$

$$\chi_{k_y}(y)\,\psi_k(\eta)$$

and $\chi_{ky}(y)$ are the eigen-functions of the Laplacian on the compact space. a_k and $a_k^\dagger$ obey the usual commutation relations,

$$\left[a_k, a_{k'}\right] = 0, \quad \left[a_k, a_{k'}\right] = \delta_{k,k'}$$

From (5.4) and the above commutation relations we get the differential equation for $\psi_k(\eta)$ as

$$\ddot{\psi}_k(\eta) + \Big[a^2(\eta)\Big\{m^2 + k_x^2 / R^2(\eta) + k_y^2 / A^2(\eta) + [\xi - \xi(n)]R(\eta)\Big\}$$
$$+ \frac{3n}{n+3}\,\xi(n)\left[\dot{R} / R(\eta) - \dot{A} / A(\eta)\right]^2\Big]\psi_k(\eta) = 0, \qquad (5.7)$$

where $\xi(n) = \frac{1}{4}(n+2) / (n+3)$, and $-k_y^2$ are the eigenvalues of the Laplacian on the compact space. Defining $\omega_k^2(\eta)$

$$\omega_k^2(\eta) = a^2(\eta)\Big\{m^2 + k_x^2 / R^2(\eta) + k_y^2 / A^2(\eta) + [\xi - \xi(n)]R(\eta)\Big\}$$
$$+ (3n / n+3)\,\xi(n)\left[\dot{R} / R(\eta) - \dot{A} / A(\eta)\right]^2$$

So (5.7) becomes

$$\ddot{\psi}_k(\eta) + \omega_k^2(\eta_0)\,\psi_k(\eta) = [\omega_k^2(\eta_0) - \omega_k^2(\eta)]\,\psi_k(\eta)$$
$$= V_k(\eta)\,\psi_k(\eta) \qquad (5.8)$$

where $\eta = \eta_0$ *is the time when* $a_k^\dagger$ *and* a_k *corresponds to* physical creation and annihilation operators.Converting(5.8) into integral form one gets,

$$\psi_k(\eta) = \psi_k^{in}(\eta) + \frac{1}{\omega_k(\eta_0)} \int_{\eta_0}^{\eta} V_k(\eta') \sin[\omega_k(\eta_0)(\eta-\eta')]\,\psi_k(\eta')\,d\eta' \qquad (5.9)$$

where $\psi_k^{in}(\eta)$ is the positive frequency solution to (5.8),and it is related to $\psi_k(\eta)$ by a Bogoliubov transformation,

$$\psi_k(\eta) = \alpha_k(\eta,\eta_0)\,\psi_k^{in}(\eta) + \beta_k(\eta,\eta_0)\,\psi_{-k}^{in}(\eta)$$

when we substitute in (5.5) one gets,

$$a_k(\eta) = \alpha_k^*(\eta,\eta_0)\,a_k + \beta_k(\eta,\eta_0)\,a_{-k}^\dagger$$

where

$$\alpha_k(\eta,\eta_0) = 1 + i\int_{\eta_0}^{\eta} V_k(\eta')\,\psi_k^{in*}(\eta')\,\psi_k(\eta')\,d\eta' \quad . \qquad (5.10)$$

$$\beta_k(\eta,\eta_0) = -i\int_{\eta_0}^{\eta} V_k(\eta')\, \psi_k^{in}(\eta')\, \psi_k(\eta')\, d\eta' \qquad (5\cdot 11)$$

The vacuum is defined by

$$a_k\,|0\rangle = 0 \quad \text{for all } k \qquad (5\cdot 12)$$

The particle number per mode at time η is given by

$$\begin{aligned} \langle N_k(\eta,\eta_0)\rangle &= \langle 0|\, a_k^+(\eta)\, a_k(\eta)\,|0\rangle \\ &= |\beta_k(\eta,\eta_0)|^2 \end{aligned} \qquad (5\cdot 13)$$

The total particle number $\langle N(\eta,\eta_0)\rangle$ is given by

$$\begin{aligned} \langle N(\eta,\eta_0)\rangle &= \sum_k \langle N_k(\eta,\eta_0)\rangle \\ &= \sum_k |\beta_k(\eta,\eta_0)|^2 \end{aligned} \qquad (5\cdot 14)$$

and the number density $n(\eta,\eta_0)$ is given by

$$n(\eta,\eta_0) = \left[L_3 R(\eta)\right]^{-3} \langle N(\eta,\eta_0)\rangle \qquad (5.15)$$

When $L_3 \to \infty$, $n(\eta,\eta_0)$ goes to

$$\lim_{L_3\to\infty} n(\eta,\eta_0) = \left[2\pi^2 R^3(\eta)\right]^{-1} \sum_{k_y} \int_0^\infty dk_x\, k_x^2 \left|\beta_k(\eta,\eta_0)\right|^2 \qquad (5.16)$$

Here one can observe that "topological inflation" viewpoint espoused by Kaku and Lykken[6] are the same as above equations.

5. Calculation of Number density

Here, the particle number density is calculated to lowest order about conformal invariance in a similar manner to that used by Birrel and Davies[7] for Bianchi type-I spacetimes. From (5.9) $\psi_k(\eta)$ in lowest order is given by $\psi_k^{in}(\eta)$ .And $\psi_k^{in}(\eta)$ is given by

$$\psi_k^{in}(\eta) = \left[2\omega_k(\eta_0)\right]^{-1/2} e^{-i\omega_k(\eta_0)\eta} .$$

and substituting this in (5.11) one gets,

$$\beta_k(\eta,\eta_0) = \frac{-i}{2\omega_k(\eta_0)} \int_{\eta_0}^{\eta} V_k(\eta')\, e^{-2i\omega_k(\eta_0)\eta'}\, d\eta' \qquad (5.17)$$

Now let us write,

$$\begin{aligned} V_k(\eta) &= h(\eta)\, k_x^2 + \tilde{h}(\eta)\, k_y^2 + g(\eta) \qquad (5.18) \\ &= h(\eta)\, k_x^2 + g(\eta) \end{aligned}$$

where

$$h(\eta) = \left[\frac{a}{R}\right]^2(\eta_0) - \left[\frac{a}{R}\right]^2(\eta) = 1 - \left[\frac{a}{R}\right]^2(\eta), \qquad (5.19a)$$

$$\tilde{h}(\eta) = \left[\frac{a}{A}\right]^2(\eta_0) - \left[\frac{a}{A}\right]^2(\eta) = 1 - \left[\frac{a}{A}\right]^2(\eta), \qquad (5.19b)$$

$$f(\eta) = a^2(\eta')\left[\xi - \xi(n)\right] R(\eta') + \frac{3n}{n+3}\,\xi(n)\left[\frac{\dot{R}}{R} - \frac{\dot{A}}{A}\right]^2(\eta')\Bigg|_{\eta_0}^{\eta}, \qquad (5.19c)$$

$$\bar{m}^2(\eta) = a^2(\eta)\left[m^2 + k_y^2 / A^2(\eta)\right], \qquad (5.19d)$$

$$g(\eta) \doteq \bar{m}^2(\eta_0) - \bar{m}^2(\eta) - f(\eta), \qquad (5.19e)$$

$$\bar{g}(\eta) = \left[a^2(\eta_0) - a^2(\eta)\right] m^2 - f(\eta). \qquad (5.19f)$$

and we have assumed that $R(\eta_0) = A(\eta_0)$ and that $a^2(\eta_0) < \infty$ for $m \neq 0$.

Putting (5.18) in (5.11) one gets, (5.17) as,

$$n(\eta,\eta_0) = \frac{1}{2\pi^2 R^3(\eta)} \sum_{k_y} \int_0^\infty dk_x \left[\frac{k_x}{2\omega_0}\right]^2 \int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta} d\eta_2\, e^{2i\omega_0(\eta_1-\eta_2)}$$

$$\left[h(\eta_1)k_x^2 + g(\eta_1)\right]\left[h(\eta_2)k_x^2 + g(\eta_2)\right] \tag{5.20}$$

where ω_0 denotes $\omega_k(\eta_0)$. Integrating by η_1 and η_2 by parts, one obtains

$$n(\eta,\eta_0) = \left[2\pi^2 R^3(\eta)\right]^{-1} \sum_{k_y} \int_0^\infty dk_x \,(k_x/2\omega_0)^2 \;\times$$

$$\int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta} d\eta_2 \cos 2\omega_0(\eta_1-\eta_2) \Big\{ (k_x/2\omega_0)^4 \ddot{h}(\eta_1)\ddot{h}(\eta_2) \tag{5.21}$$

$$+ (k_x/2\omega_0)^2 \left[\dot{h}(\eta_1)\dot{g}(\eta_2) + \dot{h}(\eta_2)\dot{g}(\eta_1)\right] + g(\eta_1)g(\eta_2) \Big\}$$

Using the help of cosine transform in Oberhettinger[8] (5.21), becomes,

$$
\begin{aligned}
\eta(\eta, \eta_0) = {} & \frac{\pi}{64} \left[2\pi^2 R^3(\eta) \right]^{-1} \times \\
& \sum_{kq} \Bigg[\int_{\eta_0}^{\eta} d\eta' \left[\ddot{h}^2(\eta') + 8 \dot{h}(\eta') \dot{g}(\eta') + 16 g^2(\eta') \right] \\
& + \bar{m}(\eta_0) \int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta} d\eta_2 \, F(2\bar{m}(\eta_0) |\eta_1 - \eta_2|) \\
& \times \Big\{ 3 \left[\partial_1^2 \partial_2^2 - 12 \bar{m}^2(\eta_0) \partial_1 \partial_2 + 16 \bar{m}^4(\eta_0) \right] h(\eta_1) h(\eta_2) \\
& + 8 \left[\partial_1 \partial_2 - 2\bar{m}^2(\eta_0) \right] \left[h(\eta_1) g(\eta_2) + h(\eta_2) g(\eta_1) \right] + 16 g(\eta_1) g(\eta_2) \Big\} \\
& - \bar{m}(\eta_0) \int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta} J_1(2\bar{m}(\eta_0) |\eta_1 - \eta_2|) \\
& \times \Big\{ \ddot{h}(\eta_1) \ddot{h}(\eta_2) + 4 \left[\dot{h}(\eta_1) \dot{g}(\eta_2) + \dot{h}(\eta_2) \dot{g}(\eta_1) \right] + 16 g(\eta_1) g(\eta_2) \Big\} \\
& - 16 \bar{m}(\eta_0) \int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta} d\eta_2 \Big\{ \bar{m}^4(\eta_0) h(\eta_1) h(\eta_2) \\
& - \bar{m}^2(\eta_0) \left[h(\eta_1) g(\eta_2) + h(\eta_2) g(\eta_1) \right] + 16 g(\eta_1) g(\eta_2) \Big\} \Bigg]
\end{aligned}
$$

(5-22)

where $F(x) = (\pi/2)\, x\, [J_0(x)\, H_{-1}(x) + J_1(x)\, H_0(x)]$. $J_\mu(x)$ and $H_\mu(x)$ are Bessel and Sturve functions respectively.

To perform k_y sums, we choose an 'n' torus so that, k_y are given by $k_y = (2\pi/L_{n+4})\,1$, where the components of 1 are integers. By the repeated application of the Abel-Plana summation formula and for m = 0, the number density can be written as[9],

$$n(\eta,\eta_0) = \frac{-\pi}{8}\left[2\pi^2 R^3(\eta)\right]^{-1}\left[\frac{2\pi}{L_{n+4}}\right]\Bigg[\zeta_E\left[\frac{-1}{2},n\right]\left[\int_{\eta_0}^{\eta} d\eta' f(\eta')\right]^2$$

$$-2\left[\frac{2\pi}{L_{n+4}}\right]^2 \zeta_E\left[\frac{-3}{2},n\right]\int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta_1} d\eta_2\, f(\eta_1)\left[\bar{h}(\eta_2) - h(\eta_2)\right]$$

$$+\left[\frac{2\pi}{L_{n+4}}\right]^4 \zeta_E\left[\frac{-5}{2},n\right]\left[\int_{\eta_0}^{\eta} d\eta' \left[h(\eta') - \bar{h}(\eta')\right]\right]^2\Bigg] \qquad (5.23)$$

where $\zeta_E(-ns/2,n)$ is the analytic continuation of the Epstein ζ function given by[10,11]

$$\zeta_E(ns/2,n) = \sum_{l_1\to-\infty}^{\infty} \cdots \sum_{l_n\to-\infty}^{\infty} (l_1^2 + \cdots + l_n^2)^{-ns/2}.$$

As a particular case, when $n = 1$ and with $\zeta_E(s/2,1) = 2\zeta(s)$, the number density can be written as

$$n(\eta,\eta_0)\Big|_{n=1} = \frac{\pi}{48}\left[2\pi^2 R^3(\eta)\right]^{-1}\left[\frac{2\pi}{L_5}\right]$$
$$\times\left\{\left[\frac{1}{10}\left[\frac{79}{21}\right]^{1/2}\left[\frac{2\pi}{L_5}\right]^2\int_{\eta_0}^{\eta} d\eta'\left[\bar{h}(\eta')-h(\eta')\right]\right]^2\right.$$
$$\left.+\left[\int_{\eta_0}^{\eta} d\eta'\left[\frac{1}{10}\left[\frac{2\pi}{L_5}\right]^2\left[\bar{h}(\eta')-h(\eta')\right]+f(\eta')\right]\right]^2\right\} \quad (5\text{-}24)$$

Since the values of ns/2 are fixed and the zeros of the Epstein ζ functions are $ns/2 = -\ell$, ℓ is a positive integer, the functions never change its sign as n varied. The ζ functions do not cross as functions of n. Therefore, the number density can be written as,

$$n(\eta,\eta_0) = \frac{\pi}{8}\left[2\pi^2 R^3(\eta)\right]^{-1}\left[\frac{2\pi}{L_{n+4}}\right]$$
$$\times\left\{\left[\left[-\zeta_E(-\tfrac{5}{2},n)+\zeta_E^2(-\tfrac{3}{2},n)\Big/\zeta_E(-\tfrac{1}{2},n)\right]^{1/2}\right.\right.$$

$$\times\left[\frac{2\pi}{L_{n+4}}\right]^2 \int_{\eta_0}^{\eta} d\eta' \left[\tilde{h}(\eta') - h(\eta')\right]\Bigg]^2$$

$$+\left[\left[\int_{\eta_0}^{\eta} d\eta' \left\{\left[-\zeta_E\left(-\tfrac{1}{2},n\right)\right]^{1/2} f(\eta')\right.\right.\right.$$

$$\left.\left.\left.+\frac{\zeta_E\left(\tfrac{3}{2},n\right)}{\left[-\zeta_E\left(\tfrac{-1}{2},n\right)\right]^{1/2}}\left[\frac{2\pi}{L_{n+4}}\right]^2\left[\tilde{h}(\eta')-h(\eta')\right]\right\}\right]^2\right\} \qquad (5\cdot25)$$

when $m \neq 0$, number density can be expressed in terms of the analytic continuation of a ζ function like quantity denoted by $Z(ns/2,nt/2,n)$, and it is given by

$$Z\left[\frac{ns}{2},\frac{nt}{2},n\right] = \sum_{\ell_1=-\infty}^{\infty}\cdots\sum_{\ell_n=-\infty}^{\infty}\left(\ell_1^2+\cdots+\ell_n^2+\mu^2\right)^{-ns/2}\left(\ell_1^2+\cdots+\ell_n^2\right)^{-nt/2} \qquad (5\cdot26)$$

For $m \neq 0$, the number density is given by

$$n(\eta_\iota, \eta_0) = -\frac{\pi}{8}\left[2\pi^2 R^3(\eta_\iota)\right]^{-1}\left[\frac{2\pi}{L_{n+4}}\right]$$

$$\times \int_{\eta_0}^{\eta_\iota} d\eta_{\iota_1} \int_{\eta_0}^{\eta_\iota} d\eta_{\iota_2} \left\{ \left[\frac{2\pi}{L_{n+4}}\right]^4 \left\{ Z(-\tfrac{5}{2},0,n)\, h(\eta_{\iota_1}) h(\eta_{\iota_2}) \right.\right.$$

$$- Z(-\tfrac{3}{2},-1,n)\left[h(\eta_{\iota_1})\bar{h}(\eta_{\iota_2}) + h(\eta_{\iota_2})\bar{h}(\eta_{\iota_1})\right]$$

$$\left. + Z(-\tfrac{1}{2},-2,n)\,\bar{h}(\eta_{\iota_1})\,\bar{h}(\eta_{\iota_2}) \right\}$$

$$+ \left[\frac{2\pi}{L_{n+4}}\right]^2 \left\{ Z(-\tfrac{1}{2},-1,n)\left[\bar{h}(\eta_{\iota_1})\,\tilde{g}(\eta_{\iota_2}) + \bar{h}(\eta_{\iota_2})\tilde{g}(\eta_{\iota_1})\right]\right.$$

$$\left. - Z(-\tfrac{3}{2},0,n)\left[h(\eta_{\iota_1})\tilde{g}(\eta_{\iota_2}) + h(\eta_{\iota_2})\,\tilde{g}(\eta_{\iota_1})\right] \right\}$$

$$\left. + Z(-\tfrac{1}{2},0,n)\,\tilde{g}(\eta_{\iota_1})\;\tilde{g}(\eta_{\iota_2}) \right\} \qquad (5.27)$$

5.3 Explicit Examples

Here we take the five-dimensional massless (n=1, m=0) case. This is the simplest case.[12] For $R(\eta_\iota) = 1$ it is also the same case as the one considered in Appelquist and Chodos.

For $n = 1$ and $m = 0$, the number density is given by,

$$n(\eta, \eta_0) = \frac{\pi}{48}\left[2\pi^2 R^3(\eta)\right]^{-1}\left[\frac{2\pi}{L_5}\right]\left\{\frac{1}{21}\left[\frac{2\pi}{L_5}\right]^4\left[\int_{\eta_0}^{\eta} d\eta'\left[\bar{h}(\eta') - h(\eta')\right]\right]^2\right.$$

$$\left. + \frac{1}{5}\left[\frac{2\pi}{L_5}\right]^2\int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta} d\eta_2\, f(\eta_1)\left[\bar{h}(\eta_2) - h(\eta_2)\right] + \left[\int_{\eta_0}^{\eta} d\eta' f(\eta')\right]^2\right\} \tag{5.28}$$

Now suppose that $R(\eta)$ and $A(\eta)$ are given by

$$R(\eta) = 1, \quad A(\eta) = 1 + \varepsilon\, \theta(\eta)\, e^{-\alpha\eta} \sin\omega\eta \tag{5.29}$$

with $|\varepsilon| << 1$. Evaluating (5.28) for this choice of $R(\eta)$ and $A(\eta)$ to $O(\varepsilon^2)$ over the interval (η_0, ∞), with $\eta_0 < 0$ gives

$$n\,(\infty, \eta_0 < 0) = (504\pi)^{-1}\left[\frac{\varepsilon\omega}{\alpha^2+\omega^2}\right]^2\left[\frac{2\pi}{L_5}\right]^5 \tag{5.30}$$

Since we could imagine the oscillations to be initially driven by some mechanism one might be interested in $R(\eta)$ and $A(\eta)$ of the form

$$R(\eta) = 1 \quad , \qquad A(\eta) = 1 + \varepsilon\, e^{-\alpha|\eta|} \cos\omega\eta \tag{5.31}$$

To lowest order in ε the number density produced over $(-\infty, \infty)$ is given by,

$$n(\infty, -\infty) = (126\pi)^{-1} \left[\frac{\epsilon\alpha}{\alpha^2+\omega^2} \right]^2 \left[\frac{2\pi}{L_5} \right]^5 \tag{5.32}$$

If one had instead chosen $A(\eta) = 1+\epsilon\, e^{-\alpha|\eta|} \sin\omega\eta$ one would have found that no particles were produced to $O(\epsilon^2)$. This is because the perturbation in $A(\eta)$ is an odd function of η. Here one can observe that the number density does not depend on ξ_φ. To see this consider $A(\eta)$ given by

$$A(\eta) = R(\eta)\left[1+\epsilon(\eta)\right] \tag{5.33}$$

where sup $\epsilon(\eta) << 1$. To $O(\epsilon^2)$ the resulting number density is given by

$$\begin{aligned} n(\eta,\eta_0) = {} & \frac{\pi}{48}\left[2\pi^2 R^3(\eta)\right]^{-1} \left[\frac{2\pi}{L_5}\right] \left\{ \frac{1}{21}\left[\frac{2\pi}{L_5}\right]^4 \left[\int_{\eta_0}^{\eta} d\eta'\, 2\,\epsilon(\eta')\right]^2 \right. \\ & + \frac{1}{5}\left[\frac{2\pi}{L_5}\right]^2 \left[\xi_\varphi - \frac{3}{16}\right] \int_{\eta_0}^{\eta} d\eta_1 \left[8\frac{\ddot{R}}{R} + 4\left[\frac{\dot{R}}{R}\right]^2 \right. \\ & \left. + 6\left[\frac{\dot{R}}{R}\right]\dot{\epsilon} + 2\ddot{\epsilon}\right](\eta_1) \left[\int_{\eta_0}^{\eta} d\eta_2\, 2\,\epsilon(\eta_2)\right] \\ & \left. + \left[\xi_\varphi - \frac{3}{16}\right]^2 \left\{ \int_{\eta_0}^{\eta} d\eta' \left[8\frac{\ddot{R}}{R} + 4\left[\frac{\dot{R}}{R}\right]^2 + 6\left[\frac{\dot{R}}{R}\right]\dot{\epsilon} + 2\ddot{\epsilon}\right](\eta') \right\} \right\} \end{aligned} \tag{5.34}$$

where $a^2 R = 8\ddot{R}/R + 4(\dot{R}/R)^2 + 6(\dot{R}/R)\dot{\varepsilon} + 2\ddot{\varepsilon}$ to $O(\varepsilon)$. This

shows that to $O(\varepsilon^2)$ the number density is independent to whenever $R(\eta)$ is a constant and $\dot{\varepsilon}(\eta)$ vanishes at the limits of integration, which was the situation in the above examples. Equation (5·34) also shows that the choice $R = 1$, $\varepsilon(\eta) = \varepsilon\, e^{-\alpha|\eta|}$ gives no ε^2 contribution to the number density because $\int_{-\infty}^{\infty} d\eta\, \varepsilon(\eta) = 0$.

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