A SURVEY
ON
L- FUNCTIONS OF ARITHMETIC ORDERS

Debadatta Bhattacharjee

DEPARTMENT OF MATHEMATICS

NORTH - EASTERN HILL UNIVERSITY

Submitted in partial fulfilment of the requirement

of the Degree of Master of Philosophy

TO

NORTH - EASTERN HILL UNIVERSITY

SHILLONG

APRIL 1992

CERTIFICATE

I certify that the dissertation entitled **A SURVEY ON L-FUNCTIONS OF ARITHMETIC ORDERS** submitted by Ms. Debadatta Bhattacharjee in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which the ideas have been borrowed are duly refered to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

Dr. P. K. Saikia
Supervisor
Department of Mathematics
North-Eastern Hill University
SHILLONG.

Shillong,
April 16, 1992

ACKNOWLEDGEMENTS

This work was completed under the supervision of Dr. P. K. Saikia. I am extremely grateful to him for his unfailing help during the course work and in completion of this thesis.

I am also grateful to Dr. M. B. Rege and Dr. S. S. Khare for having given us courses as per the requirement of the M. Phil. programme.

I take this oppurtunity to thank the NATIONAL BOARD OF HIGHER MATHEMATICS (NBHM), without whose financial support I probably would not have had the oppurtunity to do research and complete this work. I also express my deep gratitude to the teachers of the Mathematics Department, NEHU, Shillong, for helping me prepare to compete for the Award.

I extend my deepest thanks to the Head of the Department of Mathematics, Dr. Vijai Kumar for giving me access to the computer and using its facilities freely in typing the thesis. Thanks are due to Ms. Sima Jajodia for introducing me to the word processor. Special thanks to our research fellow Mr.Bipul Shyam Purkayastha for helping me get aquainted with the operation and various facilities of the computer which were unknown to me.

I also thank all my collegues of the Department of Mathematics, especially Ms. Amita Mitra, Ms. Sima Jajodia and Mrs. S. Dutta for always encouraging me throughout the duration of *this work.*

Mention has to be made of the non-teaching staff of the Department of Mathematics, who were very co-operative at all times.

Lastly I thank my mother for her constant inspriration, without which I would never have been where I am today.

Ms. Debadatta Bhattacharjee

PREFACE

In 1737, Euler proved that the infinitude of rational primes by using an infinite series which has come to be known as the Riemann zeta function. Riemann studied this function and showed how this function reflected the arithmetic properties of the integers. In 1837, Dirichlet, in an ingenious manner, used another infinite series, now known as the Dirichlet L-series, to prove the theorem on existence of infinitely many primes in an arithmetic progression. Dirichlet L-series was shown to have an Euler product expansion, and satisfied a functional equation. But more importantly, it was found that just as Riemann zeta function reflects the arithmetic structure of the rational field, the Dirichlet L-series does the same for the cyclotomic field.

This connection lead to further generalisations of the concepts of the L-series and the zeta function. Dedekind zeta functions for algebraic number fields and abelian L-series were introduced and their analytic continuation and functional equations were established. These series played an important role in the theory of abelian extensions of number fields.

This technique of attaching suitably defined L-series or L-functions to various mathematical objects to reflect their inherent properties continued over the years. One may mention, as examples, "non-abelian" L-functions of Artin or L-function associated by Weil to representations of "Weil groups" or more generally L-function "motives" (for a brief survey, see [JT]). In another direction, recently C. J. Bushnell and I. Reiner developed the theory of L-functions of arithmetic orders in a semisimple algebra (over $\mathbb{Q}$ or its completion $\mathbb{Q}_p$ at some prime p). They showed that these L-functions can be used to study the corresponding zeta functions introduced by L. Solomon in [LS]. As in the classical case, their L-functions satisfy functional equations.

This dissertation is an attempt to study the techniques of C. J. Bushnell and I. Reiner as developed in a series of papers [BR1], [BR2], [BR3] and [BR4]. In the following paragraphs, we outline the contents of the different chapters of this dissertation.

Chapter 2 is completely devoted to the study of the analytic treatment of the Solomon zeta function, as in [BR1]. The crux of this chapter is the developement of the Solomon zeta functions as a sum of some suitable integrals called the zeta integrals (see 2.1.2 and 2.2.6). With the help of this integral we can obtain an Euler product expansion of the Solomon zeta function (see 2.3.3).

In Chapter 3, the L-functions introduced by C. J. Bushnell and I. Reiner in [BR2] have been studied. It is shown that these L-functions are related to some standard L-function (see 3.2.17) which can be described completely in terms of classical L-functions (in the global case). This relation helps to determine the analytic continuation of the Solomon zeta functions and various partial zeta function arising out of it (see 3.3.3. and 3.3.4). Following [BR2] and [BR4], it is shown that the above mentioned function have a pole of order precisely r at s = 1, where r is the number of simple components of the semisimple algebra (see 3.3.5 and 3.3.7).

Chapter 4 has the flavour of Tate's thesis (as given in [SL]). In this chapter the functional equations of zeta integrals have been studied (see 4.1.2), which imply a functional equation for L-function (see Ch. 4, §2). The functional equation of zeta integral give rise (in the local case) to a factor which can be explained in terms of "non-abelian congruence Gauss sums" (see 4.1.10). The calculation for this relation is carried out in this chapter. This chapter follows [BR3] closely.

In the concluding chapter, some results which are closely connected with the earlier chapters have been stated without proof. We also state some problems which arise after studying the matarials contained in previous chapters.

CONTENTS

Page No.

CHAPTER 1

Introduction. We begin this chapter by briefly summarising some of the results connected with classical L- functions. This builds up the proper set-up in which the results discussed in the dissertations can be appreciated better as most of them are generalisations of classical results. Then we introduce the concepts relevant to the latter chapters and quote a few results about them.

The last section concerns itself with a brief discussion of Tate's thesis as the methods used by Bushnell and Reiner in discussing L-functions of orders follow methods introduced by Tate.

The notatins to be used are also set up in this chapter.

§.1 CLASSICAL RESULTS

We consider the Dirichlet series

1.1.1 $$\sum_{n=1}^{\infty} a_n n^{-s}$$

in which a_n are fixed complex numbers, and s is a complex variable (n^s is defined to be $e^{s\log(n)}$). Suppose $\sum_{n\leq t} a_n$ is $O(t^r)$, that is $\sum_{n\leq t} a_n t^{-r}$ is bounded as $t \longrightarrow \infty$, for some $r \geq 0$. Then the series $\sum_{n\leq t}^{\infty} a_n n^{-s}$ converges for Real $s \geq r$ and the convergence is uniform on compact subsets of the half plane Real s > r. This shows that the Riemann zeta function

1.1.2 $$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges and is analytic on the half plane Real s > 1.

The Dirichlet L-function defined via characters mod m (i.e characters of $\mathbb{Z}_m^*$) is

1.1.3 $$L(\chi, s) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}$$

where χ is a character of $\mathbb{Z}_m^*$ and is defined for all $n \in \mathbb{Z}$ as follows: If $(n, m) = 1$, $\chi(n) = \chi(n \bmod m)$ and if $(n, m) \neq 1$, set $\chi(m) = 0$. Clearly 1.1.3 converges for Real s > 1.

More generally, let R be the ring of all algebraic integers in an algebraic number field K such that (K : $\mathbb{Q}$) is finite. If X is a non-zero ideal of R, the norm NX of X is the index (R : X) of X in R. Then the Dedekind zeta function $\zeta_R(s)$ of R is defined by

1.1.4 $$\zeta_R(s) = \sum_{X \subseteq R} (NX)^{-s} = \sum_{n=1}^{\infty} j_n\, n^{-s}$$

where j_n denotes the number of ideals of R with (R:I) = n, converges to a holomorphic function of s for Real s >1. This is a straight forward generalisation of the Riemann zeta function, which occurs when R = $\mathbb{Z}$. Moreover it has analytic continuation to the whole s-plane, which is holomorphic everywhere except at s = 1, where it has a simple pole. The residue there is given by the formula

$$\lim_{s \longrightarrow 1} (s-1)\zeta_R(s) = a_k = \frac{C(k).Reg(k)}{w} d_k^{-1/2} h_R$$

where C(k) is a positive constant depending only on the behaviour of K at infinity, Reg(k) is the regulator, w is the number of roots of unity in K, d_k is the absolute discriminant and h_R is the order of the ideal class group Cl(R) of R.

We define the Hecke L-function

1.1.5 $$L(s,\psi) = \sum_{X \subseteq R} \psi(X)(NX)^{-s}$$

where $\psi : Cl(R) \longrightarrow \mathbb{C}^*$ is a linear character. We can also use the characters of ray class group for this purpose. The function $L(s,\psi)$ converges to a holomorphic function of s for $Re(s) > 1$, and admits analytic continuation to a meromorphic function on the whole s-plane. If $\psi \neq 1$, then $L(s,\psi)$ is holomorphic everywhere and $L(1,\psi) \neq 0$. In case $\psi = 1$, we have $L(s,\psi) = \zeta_R(s)$. Each L-function has an Euler product expansion

1.1.6 $$L(s,\psi) = \prod_P (1-\psi(P)(NP)^{-s})^{-1}$$

P ranging over the maximal ideals of R. This product converges absolutely and uniformly for $Re(s) > 1$.

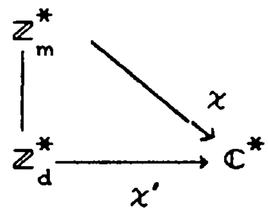
The classical Gaussian sum associated to a non-trivial character mod m (that is a character of $\mathbb{Z}_m^*$)χ, is defined as

1.1.7 $$\tau_k(\chi) = \sum_{a \in \mathbb{Z}_m^*} \chi(a)\omega^{ak} \quad \text{where } w = e^{2\pi i/m}$$

Orthogonality conditions for characters shows that

$$\tau_o(\chi) = \sum_{a \in \mathbb{Z}_m^*} \chi(a)$$

Suppose χ' is a character mod d for some $d|m$ such that the diagram



commutes. The vertical mapping is reduction mod d. Then we will say that χ' induces χ. If now χ is a character mod m which is not a character mod d, then

1.1.8 $$\tau_k(\chi) = \begin{cases} \bar{\chi}(k)\tau_1(\chi) & \text{if } (k,m) = 1 \\ 0 & \text{if } (k,m) > 1 \end{cases}$$

bar denoting complex conjugation.

§.2 SOME BASIC DEFINITIONS, NOTATIONS AND RESULTS.

We now briefly summarize some results some with proof and others without; which will be used in this dissertation. For details one can refer to [MO] for subsections (A), and (C), to [HR] for (D), to [G],[LN] and [W] for (E) and [CF] and [MO] for (F).

1.2.1 *Notation.* Throughout this dissertation

(i) $M_{(P)}$ will denote localisation of M at P and M_P the P-adic completion of M at P, where P is a maximal ideal in a Dedekind domain R.

(ii) The "*" symbol denotes the unit group of a ring.

(A) Transition from localisation to completion.

Let R be a Dedekind domain with quotient field K ≠ R and let P be a maximal ideal of R. The completion K_P of K at P is called the P-adic field and its elements the P-adic numbers. Now let M be any R-module. Then the passage from M to M_P can be accomplished in two steps: localisation (from M to M_P), and completion ($M_{(P)}$ to M_P)

Thus $M_{(P)} = R_{(P)} \otimes_R M$,

1.2.2 $$M_P = R_P \otimes_{R_{(P)}} M_{(P)}$$

If M is any R-lattice (finitely generated R-module). and V = K.M then we can recover M from M_P with P ranging over all maximal ideals of R as follows;

1.2.3 $$M = KM \cap \{ \bigcap_P M_P \},$$ P ranging over all maximal ideals of R.

Again for each P, suppose there be given a full R_P-lattice Y(P) in $K_P \otimes V = V_P$, such that $Y(P) = M_P$ almost everywhere and we define

1.2.4 $$N = V \cap \{ \bigcap_P Y(P) \},$$ P ranging over all maximal ideals R. Then N is an R-lattice in V such that KN = V. Further we have $N_P = Y(P)$ for all P.

(B) Orders.

Let R be a noetherian integral domain with field of fractions K, and let A be a finite dimensional K - algebra.

1.2.5 *Definition.* (i) For any finite dimensional K - space V, a full R - lattice in V is a finitely generated R - submodule M in V such that K.M = V

(ii) An R - order in a K -algebra A is a subring Λ of A, having the same unity element as A, and such that Λ is a full R - lattice in A. We note that Λ is both left and right noetherian, since Λ is finitely generated over a noetherian domain.

Orders can be thought of as the non-commutative generalisation of the ring of algebraic integers. Integral group rings are another example.

1.2.6 REMARKS :

(i) Let M and N be full R-lattices in A. Then $\{M:N\}_l = \left\{x \in A : Mx \subseteq N\right\}$ and $\{M:N\}_r = \left\{x \in A : xM \subseteq N\right\}$ are full R- lattices in A.

Proof: K.M =A implies that M contains a K- basis of A. So we can write $M = Re_1 \oplus \ldots\ldots\ldots \oplus Re_n$ and $N = Rf_1 \oplus \ldots\ldots\ldots \oplus Rf_n$ for some K- basis $\{e_1, \ldots\ldots, e_n\}$ and $\{f_1, \ldots\ldots, f_n\}$ of A. Again for each $y \in A$, yM is an R- lattice in A, and $ye_i = \sum \alpha_{ij} f_j$, where $\alpha_{ij} \in K$. Hence we can find an $r_i \in R$ such that $r_i(ye_i) \in N$

for $1 \leq i \leq n$. Therefore there exists $r \in R$ such that $r(yM) \subseteq N$, implying $ry \in \{ M:N \}_l$. Hence $K.\{ M:N \}_l = A$. Next as $1_r \in A$, we can find a non-zero $s \in r$ such that $s.1_r \in N$. So $\{ M:N \} \subseteq s^{-1}N$. Since R is noetherian and $s^{-1}N$ is an R- lattice, so this implies that $\{ M:N \}_l$ is also a R- lattice. Same can be done for $\{ M:N \}_r$ also. ∎

(ii) $\{ M:M \}_l = O_l(M)$ and $\{ M:M \}_r = O_r(M)$ are R-orders in A, as $1_A \in O_l(M)$ and $O_r(M)$.

(iii) For a full R- lattice M in A, and an R- order Γ in A such that $\Gamma.M \subseteq M$ we have $O_r(M) = Hom_\Gamma(M,M)$. This is because each $\phi \in Hom_\Gamma(M,M)$ extends uniquely to an element of $Hom_A(K.M \ , K.M)$ which is the same as $Hom_A(A,A)$, and hence is given by a right multiplication by an element of $O_r(M)$. We also note that this shows that $Hom_\Gamma(M,M)$ does not depend upon the choice of Γ.

(iv) If Λ is a maximal R- order in A , then for each $n \in \mathbb{N}$, $M_n(\Lambda)$ is a maximal R- order in $M_n(A)$. (Proof of this may be found in ([MO] 8.7)).

(v) Let R be a Dedekind domain, and Λ an R- order in A. Then Λ is a maximal order if and only if for each prime ideal P of R, the P-adic completion Λ_P is a maximal R_P order in A_P. This is in general true for noetherian integrally closed domains which says that Λ is a maximal R-order if and only if $\hat{\Lambda}$ is a maximal $\hat{R}$-order in $\hat{A}$ for each completion $\hat{A}$ of A. ([MO] 11.5)

(vi) Let R be a Dedekind domain and Λ an R-order in A. If Λ' is another R-order in A then there exists $\alpha, \beta \in R$ such that $\alpha\Lambda' \subseteq \Lambda \subseteq \beta\Lambda'$. Now there exists only finitely many prime ideals P of R such that either α or β or both belong to P. For all those prime ideals P such that $\alpha,\beta \notin P$ we have $(\alpha\Lambda')_{(P)} = \Lambda'$ and $(\beta\Lambda')_{(P)} = \Lambda'$ which implies that $\Lambda'_{(P)} = \Lambda_{(P)}$ for almost all P. So, $\Lambda'_P = \Lambda_P$ for almost all P because $\Lambda_P = R_P \otimes \Lambda_{(P)}$.

(vii) From (vi) we see that if Λ is any R-order in A, then Λ_P is a maximal order for almost all P, where P is a prime ideal of a Dedekind domain R.

(viii) Suppose X and Y are two $\mathbb{Z}_p$-lattices ($\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ at a prime number p) then we extend the index symbol (X:Y) so that it is defined for every pair of full $\mathbb{Z}_p$- lattices in a finite dimensional $\mathbb{Q}_p$- algebra A, by putting $(X:Y) = (X:X\cap Y)/(Y:X\cap Y)$. If $x \in A^*$ we define $\|x\|_A = (Nx:N)$. Then $\|x\|_A$ is independent of the choice of N, as $(Nx:N) = (Nx:Mx)(Mx:M)(M:N) = (Mx:M)$. Also $(Nxy:N) = (Nxy:Nx)(Nx:N)$. So $\| \ \|_A$ is a multiplicative norm on A^*. This norm map gives us a homomorphism from A^* into the cyclic subgroup of $\mathbb{Q}$ generated by p.

1.2.7 <u>LEMMA</u>. If $R = \mathbb{Z}$ or $\mathbb{Z}_p$ (where p is a prime number) and if Λ and Γ are any two R-orders in A, then $(\Lambda^*:\Gamma^*) < \infty$.

Proof: There is a positive integer n such that $n\Lambda \subseteq \Gamma$. the index $(\Lambda^*:\Gamma^*)$ is then atmost the order of the unit group $\Lambda / n\Lambda$. ■

(C) <u>Genus</u>.

Let R be a Dedekind domain with field of fractions K. Let Λ be an R-order in a K-algebra A.

1.2.8 *Definition :* Two left Λ- lattices M, N are in the same genus if for each prime ideal P of R, there is a $\Lambda_{(P)}$- isomorphism $M_{(P)} \simeq N_{(P)}$.

1.2.9 *Notation:* The Genus of $M = \{ N : N_{(P)} \simeq M_{(P)} \}$ will be denoted by $g(M)$.

1.2.10 REMARKS.

(i) Let M,N be left Λ-lattices such that $KM \cong KN$. Then M and N are in the same genus if and only if $R_P \otimes_R M \cong R_P \otimes_R N$ that is $M_P \cong N_P$. This shows that two full Λ- lattices M and N are in the same genus if $M_P \simeq N_P$.

(ii) Suppose L,M,N are left Λ-lattices in the same genus. Then there exists a left Λ-lattice X in the same genus such that $L \oplus M \cong N \oplus X$.

(iii) Let Λ be a maximal order. Then any fractional left ideal of Λ in A (always subject to the condition that $K.M = A$) is in the genus of Λ.

(D) Some basic facts about Haar measures.

Let G be a locally compact group. Then G has a non-zero measure μ such that

(a) The σ-algebra of μ measurable sets contains each open set and hence Borel sets.

(b) $\mu(A) > 0$ for every open set A.

(c) $\mu(A) < \infty$ for every compact set A.

(d) $\mu(A) < \infty$ for some open set.

(e) $\mu(A) = \mu(xA)$ for all $x \in G$ and for all μ-measurable sets. That is the measure is left invariant.

This measure is unique upto positive constant multiple. This is known as a left Haar measure on G. It arises out of a functional on the linear space

$C_c(G) \equiv \{$all continuous complex valued functions with complex support$\}$

That is if $T: C_c(G) \longrightarrow \mathbb{C}_c$ is a functional defined by $T(f) = c \in \mathbb{C}$, then we have $\int_G f d\mu = c$.

Thus all functions in $C_c(G)$ are μ-integrable. If $f \in C_c(G)$ such that $f \geq 0$ then $\int f d\mu \geq 0$. Furthermore μ-left invariant means:

$$\int_G f(g) d\mu = \int_G f(xg) d\mu \quad \text{for all } x \in G.$$

1.2.11 REMARKS.

(i) If G is compact then $\mu(G) < \infty$ and the normalised Haar measure in G is given by $\mu(G) = 1$.

(ii) One has right Haar measure too and we can convert right Haar measure to left Haar measure and vice-versa. If G is commutative they are the same.

1.2.12 Dual Measures.

Let f be a continuous function on G that is integrable with respect to the Haar measure μ. The Fourier transform $\hat{f}$ of f is a function on $\hat{G}$ defined by $\hat{f}(\hat{x}) = \int_G f(x) \langle x, \hat{x} \rangle d\mu(x)$, where $\hat{G}$ is the character group of G, and $\langle x, \hat{x} \rangle = \hat{x}(x)$ for $\hat{x} \in \hat{G}$. Now f is always continuous. If in addition $\hat{f}$ is integrable then the Fourier inversion formula.

$$f(-x) = \int_{\hat{G}} \hat{f}(\hat{x}) \langle \hat{x}, x \rangle d\hat{\mu}(\hat{x}) = \hat{\hat{f}}(x)$$

holds, for a suitable choice of Haar measure $\hat{\mu}$ on $\hat{G}$. (For this we cannonicaly identify $\hat{\hat{G}}$ with $\hat{G}$.

1.2.13 *Definition.*

(i) Such a measure $\hat{\mu}$ on $\hat{G}$ is said to be dual to μ.

(ii) Let G be a locally compact group for which $G = \hat{G}$ and μ a Haar measure on G for which $f(-x) = \hat{\hat{f}}(x)$. Then μ is called a self dual Haar measure on G. (Examples of such measures may be found in [G]).

Some facts about Haar measures on simple $\mathbb{Q}_p$ - algebras.

1.2.14 PROPOSITION Let A be a simple algebra over $\mathbb{Q}_p$ equipped with a Haar measure μ. Then for any maximal order Λ in A and $a \in A^*$, we have $\mu(a\Lambda) = \|a\|_A \, \mu(\Lambda)$, where $\| \ \|_A$ is as defined in 1.2.6 (viii).

Proof: Let $A = M_n(D)$, D a division algebra over the center of A. Then D has a unique maximal order Δ ([MO] 12). We put $\Lambda = M_n(\Delta)$. Then Λ is a maximal order in A. Since any other maximal order in A is a conjugate of Λ in A, it suffices to prove the proposition for $\Lambda = M_n(\Delta)$.

Let π be a prime element of Δ. Then for $a \in A$ we can find an integer $m \geq 0$, such that $\pi^m a \in \Lambda$. Let $b = \pi^m a$. Then $b\Lambda$ is a Λ- submodule of $a\Lambda$, and if $a \in A^*$ then $\frac{a\Lambda}{b\Lambda} \cong \frac{\Lambda}{\pi^m\Lambda}$. So $b\Lambda$ is of index $\|\pi^m\|_A$ in $a\Lambda$. Since two cosets have the same measure, so $\mu(a\Lambda) = \|\pi^m\|_A \, \mu(b\Lambda)$. Also $\frac{\mu(b\Lambda)}{\mu(\Lambda)} = (b\Lambda:\Lambda) = \|b\|_A = \|\pi\|_A^m \, \|a\|_A$. Therefore $\mu(a\Lambda) = \|\pi\|_A^{-m} \, \mu(b\Lambda) = \|a\|_A \, \mu(\Lambda)$. ∎

1.2.14. *Notation.* From now onwards, whenever we write dx, we will mean $d\mu(x)$ for the respective Haar measure μ.

1.2.15 <u>REMARK</u> We can also introduce a Haar measure on A^* by using a Haar measure on A. For this, let $f \in C_c(A^*)$ and we consider $I(f) = \int_{A^*} f(x)\,\|x\|^{-1}dx$, where $\|\ \| = \|\ \|_A$ and dx is a Haar measure on A. Now $\int_{A^*} f(xy)\|x\|^{-1}dx = \int_{A^*} f(xy)\|xy\|^{-1}\|y\|dx = \int_{A^*} f(x)\|x\|^{-1}dx$ if $\|y\|dx = d(xy)$. This is a symbolic way of writing $\mu(yE) = \|y\|\mu(E)$ for all μ-measurable sets E. To prove this we note that if we define $\mu'(E) = \mu(yE)$, y fixed, then μ' is again a Haar measure on A. Thus μ' is a constant multiple of μ and hence it suffices to prove that the constant is $\|y\|$. But we have $\mu'(\Lambda) = \mu(y\Lambda) = \|y\|\mu(\Lambda)$ from 1.2.15. Therefore $\|y\|dx = d(xy)$ is established. Hence I(f) is left invariant; also $f \geq 0$ implies that $I(f) \geq 0$. So by a well known result in measure theory I(f) can be extended to an integral on A^* which arises out of a Haar measure on A^*. This Haar measure on A^* is clearly $dx\|x\|^{-1}$.

1.2.16 <u>PROPOSOTION.</u> Let A be a finite dimensional simple $\mathbb{Q}_p$-algebra. Then the set $A \backslash A^* = \{x \in A : x \notin A^*\}$ has measure zero, with respect to any Haar measure on A.

Proof: Let $A = M_n(D)$, D a division ring over $\mathbb{Q}_p$. We assume $n \geq 2$ since, the case $n = 1$ is trivial. Let Δ be the maximal order in D, and let ξ be a prime element of Δ. Suppose $(\Delta:\Delta\xi) = q$. We fix a notation and write $\|\ \| = \|\ \|_D$. If we are given a left vector space V over D, and a D-basis $v_1, v_2, \dots, v_r$ of V, we extend $\|\ \|$ to a function on V by defining $\| \sum_{i=1}^{r} x_i v_i \| = \max_{1 \le i \le r} \|x_i\|$, $x_i \in D$.

Let μ_D be the (additive) Haar measure on D for which $\mu_D(\Lambda) = 1$. Let μ_V be the corresponding product measure on V because of its identification with D^r. Then for $v \in V$ and $N \ge 0$ we have $\mu_V(\{u \in V : \|u - v\| \le q^{-N}\}) = q^{-rN}$ (because $\Delta\xi^N = \{ u \in D : \|u\| \le q^{-N}\}$)

We have to show that the set $M_n(D)\backslash Gl_n(D)$ of singular matrices in A has measure zero. Now if S_i is the set of matrices with the property that the i^{th} row is a left D-linear combination of the remaining rows, then $M_n(D)\backslash Gl_n(D) = \bigcup_{i=1}^{n} S_i$. So it is enough to show that $\mu_A(S_i) = 0 \ \ \forall\, i$. To simplify notations we take $i = n$ and prove that $\mu_A(S_n) = 0$. Let V be the left D-space of vectors (x_{ij}, λ_k) for $1 \le i \le n-1$, $1 \le j \le n$, $1 \le k \le n-1$. Then $\dim_D V = n^2-1$. The map $F: V \longrightarrow S_n$ defined by

$$F((x_{ij}\ ,\ \lambda_k\ )_{i,j,k}) \ = \ \begin{bmatrix} x_{11} \cdots\cdots\cdots\cdots x_{1n} \\ \cdots\cdots\cdots\cdots \\ \sum_{i=1}^{n} \lambda_i x_{i1} \cdots\cdots \sum_{i=1}^{n} \lambda_i x_{in} \end{bmatrix}$$

is a surjection. Let L be the Δ- lattice in V consisting of the vectors with all their entries x_{ij}, λ_k lying in Δ. Since $V = \bigcup_{m \ge 0} \xi^{-m} L$, it is enough to show that $\mu_A(F(\xi^{-m}L)) = 0$ for each $m \ge 0$. Let N be a large positive integer and view M as fixed. Let $v' = (x'_{ij}, \lambda'_k) \in V$, and $v = (x_{ij}, \lambda_k) \in \xi^{-m}L$, and suppose $\| v - v' \| \le q^{-N}$.

Then, $\| F(v) - F(v') \| \le \operatorname{Max} \left\{ q^{-n}, \| \sum(\lambda_i x_{ij} - \lambda'_i\ x'_{ij}\ \ ) \| \ : \ 1 \le i \le n \right\}$

But $\| \sum (\lambda_i\ x_{ij} - \lambda'_i x'_{ij}\ \ ) \ \| \le \operatorname{Max} \left\{ \|\lambda_i - \lambda'_i\| \|x_{ij}\|,\ \|\lambda_i\| \|x_{ij} - x'_{ij}\| \right\} \le q^{m-N}$

Therefore for a fixed $v \in \xi^{-m}L$, the set $E_i = \left\{ F(v') \ : \ \|v - v'\| \le q^{-N} \right\}$

has measure $\leq q^{n^2(m-N)}$. Now let v range over a set T of coset representatives of $\xi^{-m}L$ mod $\xi^{N}L$. Then T has $(\xi^{-m}L:\xi^{N}L) = (\Delta:\xi^{m+N}\Delta)^{(n^2-1)} = q^{(m+N)(n^2-1)}$ elements. Moreover, if for $v \in V$, $X_v = \left\{v' \in V : \|v-v'\| \leq q^{-N}\right\}$ then $\xi^{-m}L = \bigcup_{v\in T} X_v$.

Hence it follows that $\mu_A(F(\xi^{-m}L)') \leq q^{(m+N)(n^2-1)}.q^{n^2(m-N)} = q^{(2n^2-1)m-N}$

Since this holds for all large positive integers N, therefore $\mu_A(F(\xi^{-m}L)) = 0$ as required. ∎

The following remark shows how Proposition 1.2.14, Remark 1.2.16 and Proposition 1.2.17 can be extended fo semi-simple algebras.

1.2.18 REMARK: If $A = \prod_{i=1}^{n} A_i$, A_i simple $\mathbb{Q}_p$- algebra, then any order Λ of A can be written as $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \ldots \oplus \Lambda_n$ (see [MO]) and hence $\|x\|_A = \prod_{i=1}^{n} \|x_i\|_A$, where $x = (x_1, \ldots\ldots, x_n)$. If μ_i is a Haar measure on A_i then μ defined by $\mu(E) = \prod_{i=1}^{n} \mu(E_i)$, for a measurable set E of A and $E_i = E \cap A_i$, is a Haar measure on A. Now since any Haar measure on A is a constant multiple of μ, therefore 1.2.14, 1.2.16 and 1.2.17 is also valid when A is a semisimple $\mathbb{Q}_p$- alebra.

(E) The space of Schwartz-Bruhat functions.

In the last section we introduced the Fourier transform $\hat{f}$ of a function f defined on a group G. Functions f for which both f and $\hat{f}$ are continous and integrable turn out to be quite important. The space of Schwartz - Bruhat functions is one such class of functions for which this is true. We shall define this class of functions when $G = \mathbb{R}^n$ and G is a vector space over

1.2.19. *Definition:*

(i) A Schwartz - Bruhat function f on $\mathbb{R}^n$ is a complex valued C^∞ function on $\mathbb{R}^n$ such that for every polynomial p on $\mathbb{R}^n$ and every differential operator D of the form $q(\partial/\partial x_1, \ldots\ldots, \partial/\partial x_n)$ where q is some polynomial in $\mathbb{C}[x_1, \ldots, x_n]$, the function Dpf is bounded.

(ii) A locally constant function on a space S is one such that every point of S has an open neighbourhood where the function is constant.

(iii) Let A be a $\mathbb{Q}_p$ - vector-space. A Schwartz - Bruhat function on A is a complex valued locally constant function with compact support.

1.2.20. <u>REMARK</u> A function f is locally constant if and only if $f^{-1}(a)$ is open for every $a \in \mathbb{C}$.

Proof: If $f^{-1}(a)$ is open for every $a \in \mathbb{C}$, then the implication is trivial. For the other way impliction, let f be locally constant, that is for $x \in A$ there exists an open neighbourhood N_x of x such that $f(N_x) = c$, c a constant. Let $f^{-1}(a) = L$ and $y \in L \subseteq A$. So there exists a neighbourhood N_y of y such that $f(N_y)$ is a constant. But $N_y \cap L$ is nonempty ,implying that $f(N_y) = a$. Therefore $N_y \subseteq L$, showing that L is open.

1.2.21. REMARKS

(i) A Schwartz - Bruhat function is continous, because if $O \subseteq \mathbb{C}$ be open, then $f^{-1}(O) = \bigcup_{x\in O} f^{-1}(x)$ is open.

(ii) A Schwartz - Bruhat function on A is a finite linear combination of characteristic function of the sets $w + X$, where $w \in A$ and X a full $\mathbb{Z}_p$- lattice in A.

Proof: The finite linear combination of characteristic function of the sets $w + X$ clearly has compact support $\bigcup(w+X)$, a finite union and so the combination is locally constant also.

Conversly, let f be a Schwartz - Bruhat function on A. Then we have $A = \bigcup_{a\in\mathbb{C}} f^{-1}(a)$, an open cover of A. Let L be the compact support of f, then since $L \subset A \subseteq \bigcup_{a\in\mathbb{C}} f^{-1}(a)$, there exists finitely many

$a_1, a_2, \ldots\ldots a_r \in \mathbb{C}$ such that $L \subseteq \bigcup_{i=1}^{r} f^{-1}(a_i)$. So Range $f \subseteq \{0, a_1, \ldots, a_r\}$ and $A_i = f^{-1}(a_i) \subset L$ if $a_i \neq 0$. Also as $f^{-1}(a_i) = A \setminus \bigcup_{k\neq 1} f^{-1}(a_k) \bigcup f^{-1}(0)$ so we see that $f^{-1}(a_i)$ is closed and hence compact (being a closed subset of a compact set L). Since $f^{-1}(a_i)$ is open and compact, there exists $x_{i1}, \ldots, x_{im}$ in $f^{-1}(a_i)$ and a full $\mathbb{Z}_p$- lattice X_i in A such that $f^{-1}(a\ ) = \bigcup_{i=1}^{m} (x_{ij} + X_i)$. The cosets are disjoint proving that $f = \sum_{i=1}^{r} \sum_{j=1}^{m} a_i$(ch.function of $(x_{ij} + X_i)$). ∎

(iii) Since $f \in S(A)$, the space of Schwartz - Bruhat function on A, f is locally constant with compact support, it follows that we can find full $\mathbb{Z}_p$- lattices X,Y in A with $X \subseteq Y$ such that f is constant on cosets mod X but is identically zero outside Y.

(iv) Given any two full $\mathbb{Z}_p$- lattice X_o, X in A, there is an integer $f \geq 0$ such that $p^f X_o \subseteq X$. So the space of Schwartz - Bruhat functions on A is spanned by characteristic function of spheres $w + p^f X_o$ for varying $w \in A$, $f \in \mathbb{Z}$, $f \geq 0$ and a fixed X_o.

1.2.22. <u>REMARK</u>. Let $A = \prod_{i=1}^{r} A_i$ is a semisimple $\mathbb{Q}_p$ - algebra. Then $S(A) = S(A_1) \otimes \ldots \otimes S(A_r)$, where $\phi_1 \otimes \ldots \otimes \phi_r \in S(A_1) \otimes \ldots \otimes S(A_r)$ is defined by $\phi_1 \otimes \ldots \otimes \phi_r(x_1, \ldots, x_r) = \prod_{i=1}^{r} \phi_i(x_i)$ for $x_i \in A_i$.

Proof: Clearly $S(A_1) \otimes \ldots \otimes S(A_r) \subseteq S(A)$. Let $\phi \in S(A)$. we choose a full $\mathbb{Z}_p$- lattice L_i in A_i for each i and put $L = \oplus_{i=1}^{r} L_i$. Then L is a finite $\mathbb{Z}_p$- lattice in A. Now ϕ is a finite $\mathbb{C}$- linear combination of characteristic functions of spheres $x + p^f L$, $f \geq 0$ and $x \in A$. This can be written as a tensor product of characteristic functions of the spheres $x_i + p^f L_i$ in A_i, $(1 \leq i \leq O)$ and where $x = (x_1, \ldots, x_r)$. Hence $S(A) = S(A_1) \otimes \ldots \otimes S(A_r)$. ∎

We now quote a theorem without proof which gives us a nice property of the space of Schwartz - Bruhat functions.

1.2.23 <u>THEOREM</u> Let G be a locally compact abelian group. If $f \in S(G)$, then f is continuous and intergable and $\hat{f} \in S(G)$. So the same is also true for $\hat{f}$.

1.2.24. *Notation:* G a locally compact group. By $S(G)$ we will mean the space of Schwartz - Bruhat function on G.

(F) <u>Reduced norms and traces.</u>

Let A be a simple algebra with centre K. Then there exists an extension E of K which splits K. This means that there is an isomorphism of E − algebra $h : E \otimes_K A \cong M_n(E)$ where $(A:K) = n^2$. Consider the characteristic polynomial of the matrix $h(1\otimes a)$.

We have the following properties:

(a) this characteristic polynomial is independent of h.

(b) this characteristic polynomial is independent of the splitting field E.

(c) characteristic. polynomial of $h(1\otimes a) \in K[x]$.

Hence for $a \in A$ we define its reduced characteristic polynomial as

1.2.25. *Definitions.*

(i) Reduced characteristic $\text{polynomial}_{A/K}(a)$

= characteristic polynomial of $h(1\otimes a)$

(ii) Reduced trace of a = $tr_{A/K}(a)$ = trace of $h(1\otimes a)$

(iii Reduced norm of a = $nr_{A/K}(a)$ = determinant of $h(1\otimes a)$

1.2.26. Relative reduced traces and norms.

Let B denote a central simple L− algebra with $(B:L) = m^2$ and let K be a subfield of L with $(L:K) = n$. For each $b \in B$, we define its reduced characteristic polynomial relative to K by

$$\text{Red. ch. poly.}_{B/K}(b) = \bar{N}_{L/K}\left(\text{red. ch. poly.}_{B/L}(b) \right)$$

$$= X^{mn} - tr_{B/K}(b)X^{mn-1} + \ldots\ldots + (-1)^{mn} nr_{B/K}(b)$$

where $\bar{N}_{L/K}f(X)$ = determinant of $\bar{f}(X)$: $\bar{f}(X)$ itself is defined as follows: If $f(X) = \sum \alpha_i x^i \in L[X]$, we define $\bar{f}(X) = \sum \bar{\alpha}_i x^i \in M_n(K[X])$ where each $\alpha \in L$ maps onto a matrix $\bar{\alpha} \in M_n(K)$ describing the action of left multiplication by α on some K- basis of L.

We call $tr_{B/K}$ the relative reduced trace and $nr_{B/K}$ the relative reduced norm, and we have

1.2.27.
$$tr_{B/K}(b) = T_{L/K}\left[tr_{B/L}(b) \right]$$
$$nr_{B/K}(b) = N_{L/K}\left[nr_{B/L}(b) \right]$$

where $T_{L/K}$ and $N_{L/K}$ are the usual trace and norm of L over K.

1.2.28. REMARKS.

(i) If $(A:K) = n^2$, then char. $poly._{A/K}(a) = \left\{ \text{red. char } poly._{A/K}(a) \right\}$ for $a \in A$. Hence $T_{A/K}(a) = n \cdot tr_{A/K}(a)$ and $N_{A/K}(a) = (nr_{A/K}(a))^n$.

(ii) If A is a simple $\mathbb{Q}$ ($\mathbb{Q}_p$) algebra with center C, then we call $tr_{A/\mathbb{Q}}$ ($tr_{A/\mathbb{Q}_p}$) and $nr_{A/\mathbb{Q}}$ ($nr_{A/\mathbb{Q}_p}$) the absolute reduced trace and norm respectively.

(iii) Let A be a semisimple algebra with center L. Then $A = \prod_{i=1}^{m} A_i$, $L = \prod_{i=1}^{m} L_i$ where A_i is a simple algebra with center L_i. The reduced norm and trace maps of A_i over L_i can be put together to give a reduced norm and trace map of A over L defined by

$$nr_{A/L}(a) = \prod_{i=1}^{m} nr_{A_i/L_i}(a_i) \quad \text{and} \quad tr_{A/L}(a) = \sum_{i=1}^{m} tr_{A_i/L_i}(a_i),$$

where $(a_i, \ldots\ldots, a_i)$. Further, if A is a semisimple $\mathbb{Q}$ - algebra with centre C, then

$$nr_{A/\mathbb{Q}}(a) = \prod_{i=1}^{m} nr_{A_i/\mathbb{Q}}(a_i) = \prod_{i=1}^{m} N_{C_i/\mathbb{Q}}\ (nr_{A_i/C_i}(a_i))$$

$$tr_{A/\mathbb{Q}}(a) = \sum_{i=1}^{m} tr_{A_i/\mathbb{Q}}(a_i) = \sum_{i=1}^{m} T_{C_i/\mathbb{Q}}\ (tr_{A_i/C_i}(a_i))$$

1.2.29. REMARKS.

(i) Let $A = M_r(D)$ where D is a skew field with centre K. Let $a \in A$ be given by $a = (a_{ij}) \in M_r(D)$. We now choose $E \supset K$ to be the splitting field for D, and let

$$\mu : D \longrightarrow E \otimes_K D \cong M_s(E) \text{ be an embedding of } A \text{ in } M_s(E).$$

Then the embedding of A in $M_{rs}(E)$ is given by

$$\mu' : A \longrightarrow E \otimes_K A \cong M_{rs}(E) \text{ defined by } \mu'(a) = (\mu(a_{ij}))_{1 \le i,j \le r}$$

Therefore $tr_{A/K}(a)$ = trace of $\mu'(a) = \sum_{i=1}^{r}$ trace of $\mu(a_{ii}) = \sum_{i=1}^{r} tr_{D/K}(a_{ii})$

(ii) Keeping the notation as in (i) if $a = (a_{ij}) \in A$ is now upper triangular then $nr_{A/K}(a) = \det.\ \mu'(a) = \prod_{i=1}^{r} \det.\ \mu(a_{ii}) = \prod_{i=1}^{r} nr_{D/K}(a_{ii})$

(iii) Notation as in (i). Let $a = (a_{ij}) \in A$. Then $nr_{A/K}(a) = \det.\ \mu'(a) = \det.\ (\mu(a_{ij}))_{i,j} = \mu(\det.\ (a_{ij})_{i,j})$. Now if we put $z = \mathrm{diag}(1,1,\ldots,\det.\ (a_{ij}))$ then we have

$$nr_{A/K}(z) = \det.\ \mu'(z) = \det.\ \mathrm{diag}(1,1,\ldots,\mu(\det.a)) = \mu(\det.a)$$
$$= nr_{A/K}(a).$$

1.2.30 PROPOSITION. Let A be a finite dimensional simple $\mathbb{Q}_p$-algebra with centre C. Let Δ be the maximal order of C, and Λ' a maximal order in B . Then we have

(i) If p is an infinite prime and $A \cong M_r(\mathbb{H})$ where $\mathbb{H}$ is the skew field of real quarternions then $nr(A) = nr(\mathbb{H}) = \mathbb{R}^+$ and $nr(A^*) = \mathbb{R}^>$.

(ii) Otherwise $nr(A) = C$ and $nr(A^*) = C^*$.

Here nr stands for $nr_{A/C}$.

(The proof of this proposition can be found in [MO], 33.3 and 33.4).

The following theorem attributed due to NAKAYAMA and MATUSHIMA tells us something about the kernel of the reduced norm map. We will quote the results without proof, but the details can be found in [NM].

1.2.31. THE NAKAYAMA MATUSHIMA THEOREM

If D is a division algebra over a p-adic number field with center C. Then every $a \in D$ with $nr_{D/C}(a) = 1$ is necessarily a product of commutators ——actually a product of atmost three commutators. Hence kernel $nr_{D/C}$ = commutator subgroup of D (say D'). Further if $A = M_r(D)$ then kernel $nr_{A/C} = A'$, the commutator subgroup of A.

§3 BRIEF DISCUSSIONS OF THE OBJECTS AND RESULTS APPEARING IN TATE'S THESIS.

Let $K = K_v$ be the completion of a number field under the absolute value v. We call K a local field. We denote by $|\ |_v$ the normalised absolute value, inducing the ordinary absolute value on the reals if v is archimedean, and the p-adic absolute value $|p|_v=1/p$ if v is $\mathfrak{p}$-adic ($\mathfrak{p}$ is the maximal ideal of the ring of integers of of the number field K and $p = \mathfrak{p} \cap \mathbb{Z}$).

If $n_v = (K_v : \mathbb{Q}_v)$ is the local degree, then we set $\|x\|_v = |x|_v^n$ If v is $\mathfrak{p}$-adic , and $N\mathfrak{p}$ denotes the number of elements in the residue class field $\mathfrak{a}/\mathfrak{p}$ of K ($\mathfrak{a}$ is the ring of integers of K), then

$$\|x\|_{\mathfrak{p}} = \|x\|_v = (N\mathfrak{p})^{-\gamma} \qquad \text{where } \gamma = \text{ord}_{\mathfrak{p}} x$$

Suppose for the moment that $K = \mathbb{Q}_v$. We define a non-trivial character on the locally compact, additive group of K as follows:

If v is archimedean, we put

$$\chi_o(x) \equiv -x \pmod{\mathbb{Z}}$$

If v is $\mathfrak{p}$-adic then there is a cannonical embedding of $\mathbb{Q}_p/\mathbb{Z}_p$ into $\mathbb{Q}/\mathbb{Z}$, namely onto that subgroup of $\mathbb{Q}/\mathbb{Z}$ having only powers of p in the denominator. Viewing $\mathbb{Q}/\mathbb{Z}$ as embedded in $\mathbb{R}/\mathbb{Z} = S^1$, we let χ_o be the compositions of these homomorphisms, sending $\mathbb{Q}_p$ into $\mathbb{R}/\mathbb{Z}$

$$\chi_o : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$

If K is a finite extension of $\mathbb{Q}_p$ and $Tr = Tr_{K/\mathbb{Q}_p}$ is the trace, then the homomorphism $\chi = \chi_o \circ Tr$ is a continuous, non-trivial homomorphism of K into the unit circle in the complex plane.

With the help of the above definitions we can now say that if K is a local field, then the bilinear map

$$(x, y) \longrightarrow e^{2\pi i \lambda(xy)}$$

induces an identification of the additive group of K with its own character group.

In choosing a Haar measure on K, we choose one which is self-dual. We shall choose

dx = ordinary Lebesgue measure on the real line if $K = \mathbb{R}$

dx = twice the ordinary Lebesgue measure if $K = \mathbb{C}$

dx = that measure for which the integers $\mathfrak{a}$ of K get measure $(ND_p)^{-1/2}$ if K is $\mathfrak{p}$-adic. Here D_p is the local different, that is the ideal of $\mathfrak{a}$ such that $D_p = \left\{ x \in K : Tr(xy) \in \mathbb{Z}_p \text{ for all } y \in \mathfrak{a} \right\}$.

If μ denotes any Haar measure on K, and if $x \in K^*$ is a non-zero element of K, then

$\mu(x\alpha) = \|x\| \mu(\alpha)$ or symbolically $d(xy) = \|x\| dy$

Notice that 1.2.14 is a clear generalisation of this result.

As in 1.2.12 if we define the Fourier transform $\hat{f}$ of a function $f \in L_1(K)$ by

$$\hat{f}(y) = \int_K f(x) \; e^{-2\pi i \lambda(xy)} \, dx$$

then with our choice of measure, the inversion formula $\hat{\hat{f}}(x) = f(-x)$ holds for f in $S(K)$.

The units $U_v = U$ of our local field K are the kernel of the homomorphism $a \longrightarrow \|a\|$, for $a \in K^*$. If v is $\mathfrak{p}$-adic, then a trivial verification shows that U is a compact open subgroup of K^*, and it is always a compact subgroup of K^*.

By a quasi-character of K^* we mean a continuous homomorphism c of K^* into the multiplicative group of complex numbers. A character is thus a quasi-character of absolute value 1. We say that c is unramified if it is trivial on U and ramified otherwise. The unramified quasi-characters of K are the maps of the form

$$c(a) = \|a\|^{s \log a}$$

where s is any complex number; s is determined by c if v is archimedean, and is determined only upto rational integral multiples of $2\pi i / \log N\mathfrak{p}$ if v is $\mathfrak{p}$-adic.

If v is $\mathfrak{p}$-adic, then the subgroups $1+\mathfrak{p}^{\gamma}$ $(\gamma \in \mathbb{Z}, \ \gamma \geq 0)$ form a fundamental system of neighbourhoods of 1 in U. Any character χ must therefore vanish on one of these subgroups; we call the ideal

$$\mathfrak{f}_{\mathfrak{p}} = \mathfrak{f}_{p,\chi} = \mathfrak{p}^m$$

the conductor of χ if m is the smallest integer for which $\chi(1+\mathfrak{p}^m) = 1$.

Now any bounded quasi-characterc is a character: for if $|c(x)| \neq 1$

for some $x \in K^*$ then x has no finite order. So without loss of generality we can assume $|c(x)| > 1$ and hence can find n large enough such that $|c(x)|^n > M$, a bound of c. From this it also follows that a quasi-character of K^* restricted to a compact subgroup of K^* is a character and hence in particular a quasi-character of K^* restricted to U is a character. Therefore if $x \in K^*$, $x = \pi^n u$ for $u \in U$ and π a local uniformizer, and c a quasi-character of K^*, then

$$c(x) = c(u)\ z^n \qquad \text{where } c(\pi) = z$$

$$= \chi(u)\ z^n \qquad \text{where } \chi \text{ is a character of } K^*.$$

Conversely given a character χ of K^* and a complex number z we get a quasi-character

$$c(x) = \chi(x)\ z^n$$

Using the above two results one can prove that the quasi-characters of K^* are the maps of the form

$$a \longmapsto c(a) = c'(a')\ \|a\|^s$$

where c' is any character of U uniquely determined by c, and a' is the U-component of $a \in K^*$. The complex number s is determined by c if v is archimedean and determine only upto rational integral multiples of $2\pi i/\log N\mathfrak{p}$ if v is $\mathfrak{p}$-adic.

Two quasi-characters c_1 and c_2 are said to be equivalent if

$$c_1(x) = \chi(x)\ \|x\|^{s_1} \qquad c_2(x) = \chi(x)\ \|x\|^{s_2}$$

for a fixed character χ of K^*. Then the equivalence class of quasi-characters determined by χ is in one-one correspondence with the "surface" obtained by identifying points in $\mathbb{C}$ which differ by rational integral multiples of $2\pi i/\log N\mathfrak{p}$. Hence we can talk about analytic properties of functions defined on that equivalence class. In other words $h(c)$ is analytic at c if it is so as a function of the complex variable s determined by $c(x) = \chi(x) \|x\|^s$.

Now let K be a local field and $f \in \mathcal{S}(K)$ and χ a character of K^*, c a quasi-character of K^* (in the equivalence class of χ). The local zeta-function of f is defined as

$$Z(f, c) = \int_{K^*} f(x)\, c(x)\, d^*x$$

(this is known as the Mellin transform of f). If $c(x) = \chi(x) \|x\|^s$ we write

$$Z(f, c) = Z(f, c, s) = \int_{K^*} f(x)\, \chi(x)\, \|x\|^s\, d^*x.$$

This local zeta-function is holomorphic for Real s > 0.

For a quasi-character c of K^*, let $\hat{c}$ be defined as

$$\hat{c}(x) = c^{-1}(x)\, \|x\|$$

If $c(x) = \chi(x) \|x\|^s$, then

$$\hat{c}(x) = \chi^{-1}(x)\, \|x\|^{1-s}$$

With the above definition of $\hat{c}$, and $f, g \in \mathcal{S}(K)$ the local zeta function satisfies a functional equation

$$Z(f, c)\, Z(\hat{g}, \hat{c}) = Z(\hat{f}, \hat{c})\, Z(g, c)$$

in the domain $0 < \text{Real } s < 1$.

In view of the functional equation, if there is a function f such that $Z(\hat{f}, \hat{c}) \neq 0$, then the quotient $Z(f, c)/Z(\hat{f}, \hat{c})$ is independent of the function f. This quotient will be denoted by $\rho(c) = \rho(\chi, s)$. Then for $f \in \mathcal{S}(K)$ and $0 < \text{Real } s < 1$ we also have the following functional equation

$$Z(f, c) = \rho(c)\, Z(\hat{f}, \hat{c})$$

The function $\rho(c)$ is defined by this equation for $0 < \text{Real } s < 1$, but can be extended to a meromorphic function to the whole s-plane by analytic continuation.

——— * ———

CHAPTER 2

Introduction: In this chapter, we discuss the analytic treatment of the Solomon's zeta function of lattices over orders in detail. The material covered here were first given in [BR1]. The treatment is in terms of idelic integrals and is in the lines of J . Tate's thesis. As a result of this discussion, not only we express Solomon's functions as zeta integrals but also establish the Euler products for such functions. All these will be basic in our discussion of L- function of orders later.

We must point out that Solomon, in his paper [LS] where he introduces his zeta functions, also establishes Euler products([LS]pg316)but then he does it in a purely algebraic manner.

§1. SOLOMON'S ZETA FUNCTIONS.

Let A be a finite dimensional semisimple $\mathbb{Q}$ or $\mathbb{Q}_p$- algebra and Λ a $\mathbb{Z}$ order if A is a $\mathbb{Q}$ -algebra or a $\mathbb{Z}_p$ order if A is a $\mathbb{Q}_p$- algebra. From now onwards we will call Λ an order in A and it will be understood that it is a $\mathbb{Z}$ or $\mathbb{Z}_p$ order according as A is a $\mathbb{Q}$ or $\mathbb{Q}_p$- algebra. Given a full Λ- lattice L in a finitely generated left A- module V, we define

2.1.1. $$\zeta_\Lambda (L ; s) = \sum_{N \subseteq L} (L:N)^{-s}$$

where the sum extends over all full Λ- lattices N in L and s is a complex variable. Thus $\zeta_\Lambda = \zeta_\Lambda(L\ ;\ s)$ counts the sublattices of L and clearly is a generalisation of the Dedekind zeta function.

We also define the partial zeta functions in case A is a $\mathbb{Q}$- algebra.

2.1.2. $$Z_\Lambda\ (L\ ,\ g(M)\ ;\ s) = \sum_{\substack{N \subseteq L \\ N \in g(M)}} (L:N)^{-s}$$

the sum extending over all full Λ- lattices in L such that N lies in the genus of M denoted by g(M).

There are only finitely many genera of full Λ- lattices in V as given in [CR]. So we can pick a finite set S of genus representatives of these genera. Then clearly

2.1.3. $$\zeta_\Lambda\ (L;\ s) = \sum_{M \in S} Z_\Lambda\ (L,\ g(M);\ s)$$

The following result establishes the region of convergence of such functions.

2.1.4. <u>PROPOSITION</u> Let Λ be any order in the $\mathbb{Q}$- algebra A, and let L, M be full Λ- lattices in a finitely generated A- module V. Then both the series $\zeta_\Lambda(L;\ s)$ and $Z_\Lambda(L,\ g(M);\ s)$ converge absolutely in the half plane Real $s > \dim_{\mathbb{Q}} V$ and define holomorphic functions there.

Proof: The proof is a modification of that of Hey ([DM] pg 130) given for zeta functions of maximal orders.

Suppose first that $A = V = \mathbb{Q}$, $\Lambda = L = \mathbb{Z}$. Then $\zeta_{\mathbb{Z}}\ (\mathbb{Z}\ ;\ s)$ is just the ordinary Riemann zeta function and here is nothing to prove. Next let $A = \mathbb{Q}$, $V = \mathbb{Q}^n$, $\Lambda = \mathbb{Z}$, $L = \mathbb{Z}^n$. If N is any full Λ- sublattice of L then N contains $\{e_1,\ldots,e_n\}$, a basis of V over $A = \mathbb{Q}$. Since L is also a full Λ- lattice in V, L contains $\{f_1,\ldots,f_n\}$ some other basis of V over $A = \mathbb{Q}$. Then for $1 \le i \le n$ we can have $e_i = \sum_{i=1}^{n} a_{ij} f_j$, with $a_{ij} \in \mathbb{Z}$ as L and M are both $\mathbb{Z}$- modules. We then have $x = (a_{ij}) \in M_n(\mathbb{Z}) \cap GL_n(\mathbb{Q})$ as $\det (a_{ij}) \neq 0$. So we can write $N = Lx$ with $x \in M_n(\mathbb{Z}) \cap GL_n(\mathbb{Q})$ for every Λ- sublattice N of N. Further, $Lx = Ly$ if and only if $xy^{-1} \in GL_n(\mathbb{Z})$. Therefore we have $\zeta_\Lambda(\mathbb{Z}^n ; s) = \sum_x (L:Lx)$, where x ranges over a full set of representatives of the right coset space $GL_n(\mathbb{Z}) \backslash \left[M_n(\mathbb{Z}) \cap GL_n(\mathbb{Q})\right]$. A set of coset representatives is given by the integral matrices in the Hermite normal form determined by the conditions (see [N] Th.II.2) $a_{ii} \ge 1$, $1 \le i \le n$, $a_{ij} = 0$ if $i > j$ and $0 \le a_{ij} \le a_{ii}$ if $i < j$. If x is in this form then $(L:Lx) = |\det x|$ and therefore $\zeta_{\mathbb{Z}}(\mathbb{Z}^n ; s) = \sum_x |\det x|$ where x ranges as above. We may rearrange $\zeta_{\mathbb{Z}}(\mathbb{Z}^n ; s)$ as a Dirichlet series $\sum^{\infty} c(m)\, m^{-s}$, where the general coefficient $c(m) = \sum_{(d_1,\ldots,d_n)} \prod_{i=1}^{n} d_i^{(i-1)}$, the sum extending over all n-tuples $(d_1,\ldots,d_n)$ of positive integers whose product is m. Now since $\sum_{d_1 \in \mathbb{N}} d_1^{-s} . \sum_{d_2 \in \mathbb{N}} d_2^{-(s-1)} \ldots \sum_{d_n \in \mathbb{N}} d^{-(s-(n-1))}$ equals $\sum_{d_1 \in \mathbb{N}} 1\backslash d_1^s . \sum_{d_2 \in \mathbb{N}} d_2 \backslash d_2^s \ldots . \sum_{d_n \in \mathbb{N}} d_n^{n-1} \backslash d_n^s$. So it follows that $\zeta_{\mathbb{Z}}(\mathbb{Z}^n; s) = \zeta_{\mathbb{Z}}(s). \zeta_{\mathbb{Z}}(s-1) \ldots \zeta_{\mathbb{Z}}(s-(n-1))$, where $\zeta_{\mathbb{Z}}(s)$ is the absolutely to a holomorphic function for Real $(s) > n$.

Returning to the general case when Λ and A are arbitrary the comparision test yields the proposition because all Λ- lattices will be $\mathbb{Z}$ lattices and hence the number of Λ- lattices will be less than that of $\mathbb{Z}$- lattices. ∎

In the local case if A is a $\mathbb{Q}_p$ - algebra, we have already defined $\zeta_\Lambda(L; s)$ where Λ is a $\mathbb{Z}_p$ order in A and L is a full Λ- lattice in a finitely generated left A- module V. Now we define a local analogue of the partial zeta function. Given a full Λ- lattice M in V, we modify the definition 2.1.2 as follows :

2.1.5. $$Z_\Lambda(L, M; s) = \sum_{\substack{N \subseteq L \\ N \cong M}} (L:N)^{-s}$$

the sum extending over all full Λ- sublattices N of L such that $N \cong M$ (this is a correct analogue of 2.1.2 , since in the local case only one prime is involved). As in 2.1.3 we can again express $\zeta_\Lambda(L;s)$ as a finite sum $\sum_M Z_\Lambda(L, M; s)$, where now M ranges over a full set of representatives of isomorphism classes of full Λ- lattices in V. Also Proposition 2.1.4 is valid in this case with $\mathbb{Q}$ replaced by $\mathbb{Q}_p$ and $Z_\Lambda(L, g(M); s)$ replaced by $Z_\Lambda(L, M; s)$.

§2 THE TRANSITION TO IDELES

Our aim in this section is to express the zeta functions defined in the preceeding section in terms of suitable integrals.

(A) Local case

Suppose that A is a $\mathbb{Q}_p$- algebra and Λ a $\mathbb{Z}_p$-order in A. Let V be a finitely generated left A- module and let L, M, N denote full Λ- lattices in V.

Let $B = \mathrm{End}_A V$ and $B^* = \mathrm{Aut}_A V$ = Units of B

We can view B as an (A,B)- bimodule. We now define

$$\{M : N\}_r = \{M : N\} = \left\{ x \in B : Mx \subseteq N \right\}$$

This is a full $\mathbb{Z}_p$- lattice in B, and furthermore, $\{M : M\} = O_r(M)$ is a $\mathbb{Z}_p$ order in B and its group of units is precisely $Aut_\Lambda M$ (see Remark 1.2.6 (i) and (ii)).

Now note the following remark:

2.2.1. <u>REMARK</u> L, M, N as above. $N \subseteq L$ and $N \cong M$, then $N = Mx$ for some $x \in \{M : L\} \cap B^*$ with x unique mod $Aut_\Lambda M$.

Proof: For the proof of this remark, note that as $A = \mathbb{Q}_p\Lambda$ and $V = \mathbb{Q}_pN = \mathbb{Q}_pM$, the isomorphism between N and M can be extended to an A- isomorphism of V, say x. Thus $N = Mx$ with $x \in Aut_\Lambda V = B^*$. Since $N \subseteq L$ it follows that $x \in \{M : L\}$. Uniqueness of x is clear. ∎

The above remark at once gives us

2.2.2. $$Z_\Lambda (L, M; s) = \sum_x (L:Mx)^{-s}$$

the sum extending over all $x \in Aut_\Lambda M \setminus \left[\{M : L\} \cap B^* \right]$ and (L:Mx) is defined as in Remark 1.2.6 (viii). We now put $\|x\|_V = (Nx:N)$ for $x \in B^*$, where n is any full $\mathbb{Z}_p$- lattice in V. As shown in 1.2.6 (viii), this norm $\| \|_V$ is independent of the chioce of N and multiplicative. The norm map therefore gives a multiplicative homomorphism from B^* into the cyclic subgroup generated by p. Further, if x is unit in some $\mathbb{Z}_p$ order N in B, then $Nx = N$ showing that $\|x\|_v = 1$.

We now have $(L:Mx) = (L:M)\ (M:Mx) = (L:M)\ \|x\|_v$ and therefore

2.2.3 $$Z_\Lambda(L,\ M\ ;\ s) = (L:M)^{-s} \sum_x \|x\|_v^s$$

where x ranges over the orbit space $\mathrm{Aut}_\Lambda M \setminus \left(\{M : L\} \cap B^* \right)$ as before.

We now discuss the topology on B and B^* and show how to introduce a suitable Haar measure on B^* so as to write $Z_\Lambda(L, M; s)$ as an integral.

A basic theorem given in [W], which says that if V is a topological finite dimensional vector space over a local field K, then the mapping $(x_1,\ldots,x_n) \longmapsto \sum_{i=1}^{n} x_i v_i$ of K^n onto V is a topological isomorphism for the structures of K^n and V, for any basis $(x_1,\ldots,x_n)$ of V over K, and further V is locally compact.

This shows that the finite dimensional $\mathbb{Q}_p$- algebra B is a locally compact algebra.

Now let M be any full $\mathbb{Z}_p$- lattice in B . Then $M = b_1\mathbb{Z}_p \oplus \ldots \oplus b_n\mathbb{Z}_p$ for some $\mathbb{Q}_p$- basis $(b_1,\ldots,b_n)$ of B. Since $\mathbb{Z}_p$ is a compact open neighbourhood of zero in $\mathbb{Q}_p$, it follows that $b_i\mathbb{Z}_p$ is a compact open neighbourhood of zero in $\mathbb{Q}_p$ for $1\le i\le n$ whence $\sum_{i=1}^{n} b_i\mathbb{Z}_p$ is a compact open neighbourhood of $(0,\ldots,0)$ in B. So we conclude that any full $\mathbb{Z}_p$- lattice in B is a compact neighbourhood of zero in B. Also the units B^* of B form an open subset of B as $B^* = \mathbb{Q}_p^n \setminus \left\{ \bigcup_{i=1}^{n} (\mathbb{Q}_p \times \ldots \times \underset{(i\text{TH place})}{0} \times \ldots\ldots \times \mathbb{Q}_p^n) \right\}$ as vector spaces and $\mathbb{Q}_p \times \ldots \times \underset{(i\text{TH place})}{0} \times \ldots \times \mathbb{Q}_p$ is closed in $\mathbb{Q}_p^n$ for $1\le i\le n$. With the subspace topology, B^* forms a locally compact topological group. Similarly note that if Γ is any $\mathbb{Z}_p$ order in B, its unit group is a compact open subgroup of B^* in the subspace topology as Γ is open compact subgroup of B.

We therefore can choose a Haar measure $d\mu^*(x) = d^*x$ on B^*. This measure can be chosen of the form $dx/\ \|x\|_B$ for a Haar measure $d\mu(x) = dx$ on B. (It is possible to prove that these measures are both left and right invariant). Now for each full $\mathbb{Z}_p$- lattice M in V, we have $End_\Lambda M \cong O_r(M)$ is a $\mathbb{Z}_p$ order in B and its unit group $Aut_\Lambda M$ is a compact open subgroup of B^*. Hence by elementary results in measure theory quoted in Chapter 1, $Aut_\Lambda M$ has finite nonzero measure $\mu^*(\ Aut_\Lambda M\ )$. Now we consider the following integral

$$\int_{B^*} \phi(x)\ \|x\|_V^s\ d^*x$$

, where $\phi = \phi_{\langle M:L\rangle}$ is the characteristic function in B of the lattice $\{M:L\}$. Since $B^* = \bigcup_y (Aut_\Lambda M)y$, where y ranges over $Aut_\Lambda M \setminus B^*$ and so this is a finite union as $(\ B^*\ :\ Aut_\Lambda M\ )$ is finite by 1.2.8. Hence it follows by Fubini's theorem ([RH] pg 269) that in the domain of absolute convergence of the integral,

$$\int_{B^*} \phi(x)\ \|x\|_V^s\ d^*x \ = \ \sum_y \int_{(Aut_\Lambda M)y} \phi(x)\ \|x\|_V^s\ d^*x$$

the sum extending over all $y \in Aut_\Lambda M \setminus B^*$. Note that if $x \in (Aut_\Lambda M)y$ for $y \in \{M:L\}$, then $May \subseteq My \subseteq L$, so that $x \in \{M:L\}$. Thus ϕ being the characteristic function of $\{M:L\}$ for $y \in \{M:L\}$ we have

$$\int_{(Aut_\Lambda M)y} \phi(x)\ \|x\|_V^s\ d^*x \ = \ \int_{(Aut_\Lambda M)y} \|x\|_V^s\ d^*x$$

$$= \ \|y\|_V^s\ \mu^*(Aut_\Lambda M)$$

The last equality follows as $\|x\|_v = 1$ if $x \in \mathrm{Aut}_\Lambda M$, $\mathrm{Aut}_\Lambda M$ being a unit group for the $\mathbb{Z}_p$- order $\mathrm{End}_\Lambda M$. Also, if $y \notin \{M:L\}$ then

$$\int_{(\mathrm{Aut}_\Lambda M)y} \phi(x) \ \|x\|_v^s \ d^*x = 0.$$

Thus $$\int_{B^*} \phi(x) \ \|x\|_v^s \ d^*x = \mu^*(\mathrm{Aut}_\Lambda M) \ \Sigma \ \|y\|_v^s$$

where y ranges over $\mathrm{Aut}_\Lambda M \setminus \left[B^* \cap \{M:L\} \right]$. Comparing with 2.2.3 this yields

2.2.4 $$Z_\Lambda(L, M; s) = \mu^*(\mathrm{Aut}_\Lambda M)^{-1} \ (L : M)^{-s} \int_{B^*} \phi(x) \|x\|_v^s \ d^*x$$

where ϕ is the characteristic function of $\{M:L\}$ in B.

In the special case of $L = \Lambda$, M a full left ideal of Λ, this can be used to prove that $Z_\Lambda(\Lambda, M; s)$ converges at $s = 1$. See ([O] pg 62).

(B) Global case

We now extend the treatment of the local case to the global case. For this it will be necessary to introduce ideles first.

To start with, let B be any finite dimensional semisimple $\mathbb{Q}$-algebra, and let Γ be a $\mathbb{Z}$-order in B. The ring $\mathrm{Ad}(B)$ of finite adeles of B is the topological restricted direct product of the algebras B_p with respect to Γ_p , p ranging over all rational primes. This definition is independent of the choice of the order Γ. To see this, let Γ and Γ' are two $\mathbb{Z}$-orders in B. Then there exists $r, s \in \mathbb{Z}$ such that $r\Gamma \subseteq \Gamma' \subseteq s\Gamma$. Now for all primes which do not divide both r and s we have $(r\Gamma)_p = \Gamma_p$ and $(s\Gamma)_p = \Gamma_p$, so that $\Gamma_p = \Gamma'_p$. Since there are only finitely many primes which divide either r or s, we have $\Gamma'_p = \Gamma_p$ for almost all primes p. By generalities of topological restricted direct product, Ad(B) is a locally compact topological ring. We set $Ad(\Gamma) = \prod_p \Gamma_p$, p ranging over all rational primes. Then it is clear that $Ad(\Gamma)$ is a compact open subring of Ad(B). Similarly for any $\mathbb{Z}$- lattice X in B, we define $Ad(X) = \prod_p X_p$, p ranging over all prime numbers. So Ad(X) is a compact open additive subgroup of Ad(B).

Likewise we can form the group J(B) of finite ideles of B as the topological restricted direct product of the group B_p^* with respect to the subgroups Γ_p^*. Again J(B) is independent of the choice of Γ, and is a locally compact abelian group. We write $U(\Gamma) = \prod_p \Gamma_p^*$, p ranging over all prime numbers. Then $U(\Gamma)$ is a open compact subgroup of J(B).

We now suppose that A is a semisimple $\mathbb{Q}$- algebra as before, and V a finitely generated A- module. Set $B = End_A V$. Then the group J(B) acts on the set of full Λ- lattices in V. We describe this action explicitly now. Define the lattice Mx, for M a full Λ- lattice in V and $x = (x_p) \in J(B)$, by the requirement that $(Mx)_p = M_p x_p$ for all rational primes p. To prove that such a lattice exists, we start with the identification of B_p with $End_{A_p} V_p$. If $\Gamma = End_\Lambda M$ then Γ is an order in B and $\Gamma_p = End_{\Lambda_p} M_p$ is an order in B_p. We have $x_p \in \Gamma_p^*$ for almost all p and then $M_p x_p = M_p$ for each such p. Then $Mx = V \cap \left\{ \bigcap_p M_p x_p \right\}$ is a full Λ- lattice in V. (We have used the results 1.2.4 quoted in Chapter I freely here).

This action of the idele group $J(B)$ on the set of full Λ-lattices in V is transitive on the lattices of each genus. The lattices in the genus of M are precisely the lattices Mx with $x \in J(B)$. This is because if $N \in g(M)$, then $M_p \cong N_p$ as Λ_p-modules, which shows that there is a Λ_p- isomorphism x_p such that $M_p x_p = N_p$. So $x_p \in B_p^*$. Now as in 1.2.6 (vi) for any two full $\mathbb{Z}$- lattices M and N in V $M_p = N_p$ for almost all p, say for $p \notin S$ a finite set. Therefore for all $p \notin S$ $M_p x_p = M_p$, implying $x_p \in \Gamma_p^*$. Hence taking $x = (x_p) \in J(B)$ we have $Mx = N$. Further $Mx = My$ if and only if $M_p x_p = M_p y_p$ for all p, if and only if $x_p y_p^{-1} \in \Gamma_p^*$ for all p, if and only if $xy^{-1} \in U(\Gamma)$. We now define $\|x\|_V = (Mx:M)$, for $x \in J(B)$, where M is any full $\mathbb{Z}$- lattice in V. Then $\|x\|_V$ is independent of the choice of M. We now consider

$\int_{J(B)} \phi(x)\ \|x\|_V^s\ d^*x$, where ϕ is the characteristic function of $Ad(\{M:L\})$ in $Ad(B)$ and d^*x is a multiplicative Haar measure on $J(B)$. Since $J(B) = \bigcup_y U(\Gamma)y$, where y ranges over the coset representatives of $U(\Gamma) \setminus J(B)$. Therefore

$\int_{J(B)} \phi(x)\ \|x\|_V^s\ d^*x = \sum_y \int_{U(\Gamma)y} \phi(x)\ \|x\|_V^s\ d^*x$, where y ranges over the orbit space $U(\Gamma) \setminus J(B)$.

Now if $y = (y_p) \in Ad(\{M:L\})$, then $M_p y_p \subseteq L_p$ as $Ad(\{M:L\}) = \prod_p \{M_p:L_p\}$ where p ranges over all prime numbers. Also $a = (a_p) \in U(\Gamma)$ implies that $M_p a_p y_p = M_p y_p \subseteq L_p$, so that $ay \in Ad(\{M:L\})$. Thus for $y \in Ad(\{M:L\})$ we have

$$\int_{U(\Gamma)y} \phi(x)\ \|x\|_V^s\ d^*x = \int_{U(\Gamma)} \|xy\|_V^s\ d^*x = \|y\|_V^s \int_{U(\Gamma)} \|x\|_V^s\ d^*x$$

$$= \|y\|_V^s\ \mu^*(U(\Gamma))$$

where μ^* denotes the measure of a set with respect to the Haar measure d^*x on $J(B)$. The last equality follows as $x \in U(\Gamma)$ implies $(Mx)_p = M_p x_p = M_p$ for all p, implying that $Mx = M$ and hence $\|x\|_v = 1$. On the other hand for $y \notin Ad(\{M:L\})$ we have

$$\int_{U(\Gamma)y} \phi(x) \ \|x\|_v^s \ d^*x = 0$$

Putting both these together, we obtain that

$$\int_{J(B)} \phi(x) \ \|x\|_v^s \ d^*x = \mu^* \ (U(\Gamma)) \sum_y \|y\|_v^s$$, where y ranges over the orbit space $U(\Gamma) \setminus \left[J(B) \cap Ad(\{M:L\}) \right]$.

To connect this integral with Z_Λ (L, g(M); s), we start with $Z_\Lambda(L, g(M); s) = \sum_{\substack{N \subseteq L \\ N \in g(M)}} (L:N)^{-s}$. But $N \subseteq L$ and $N_p \cong M_p$ for all p implies that $N = Mx$ for $x \in U(\Gamma) \setminus \left[J(B) \cap Ad(\{M:L\}) \right]$. Hence

2.2.5 $$Z_\Lambda(L, g(M); s) = \sum_x (L:Mx)^{-s} = (L:M)^{-s} \sum_x \|x\|_v^s$$

where x ranges over the orbit space $U(\Gamma) \setminus \left[J(B) \cap Ad(\{M:L\}) \right]$.

Therefore we have

2.2.6 $$Z_\Lambda(L, g(M); s) = \mu^*(U(\Gamma))^{-1} \ (L:M)^{-s} \int_{J(B)} \phi(x) \ \|x\|_v^s \ d^*x$$

Here, $\Gamma = End_\Lambda M$, ϕ is the characteristic function of $Ad(\{M:L\})$ in $Ad(B)$, and μ^* denotes the measure of a set with respect to the Haar measure d^*x on $J(B)$. 2.2.6 is the global analogue of 2.2.4 .

§3 THE EULER PRODUCT

We may relate formulas 2.2.4 and 2.2.6 with an Euler product.

For this we have to first construct a Haar measure on $J(B)$. Returning for the time being to the general notation, let B be a $\mathbb{Q}$- algebra and Γ an order in B. A basis for open sets in $J(B)$ is provided by sets of the form $E = \prod_p E_p$, where E_p is open in B_p^* for all p, and $E_p = \Gamma_p$ for almost all p. We can construct a Haar measure d^*x on $J(B)$ by choosing a Haar measure d^*x_p on B_p^* for each p in such a manner that $\int_{\Gamma_p^*} d^*x_p = 1$ for almost all p. We then put $\int_E d^*x = \prod_p \int_{E_p} d^*x_p$. This product has only finitely many factors different from 1. The measure d^*x is thereby uniquely determined, and any Haar measure on $J(B)$ can be put in this form.

We write $d^*x = \prod_p d^*x_p$, p ranging over all prime numbers.

To apply this in §2, let $B = \mathrm{End}_A V$ and $\Gamma = \mathrm{End}_\Lambda V$, where A, V, Λ and M are as before. We choose d^*x_p on B_p^* as above. Now we consider a finite set S of primes which include all those p for which either $\{M_p:L_p\} \neq \Gamma_p$ or $\int_{\Gamma_p^*} d^*x_p \neq 1$.

Now for $x = (x_p) \in J(B)$ and ϕ the characteristic function of $\{M:L\}$ we have $\phi(x) = \prod_p \phi_p(x_p)$, where ϕ_p is the characteristic function of $\{M_p:L_p\}$, $(L:M) = \prod_p (L_p:M_p)$ and so $\|x\|_V = \prod_p \|x_p\|_{V_p}$, and $\mu^*(U(\Gamma)) = \prod_p \mu_p^*(\Gamma_p)$ where μ_p^* is the measure of a set with respect to d^*x_p on B_p^*. Let $J_S(B) = \prod_{p\in S} B_p^* \prod_{p\notin S} \Gamma_p^*$ an open subgroup of $J(B)$. Then we have

$$\int_{J_S(B)} \phi(x)\ \|x\|_V^s\ d^*x$$

$$= \prod_{p\in S} \int_{B_p^*} \phi(x)\ \|x_p\|_{V_p}^s\ d^*x_p \prod_{p\notin S} \int_{\Gamma_p^*} \|x_p\|_{V_p}^s\ d^*x_p$$

$$= \prod_{p \in S} \mu^*(\Gamma_p^*)\ (L_p : M_p)^s\ Z_{\Lambda_p}(L_p, M_p; s).$$

Taking the limit over increasing sequences of such sets S, we obtain (in any domain of absolute convergence)

$$\int_{J(B)} \phi(x)\ \|x\|_V^s\ d^*x = \mu^*(U(\Gamma))\ (L:M)^s \prod_p Z_{\Lambda_p}(L_p, M_p; s),$$ that is

2.3.1 $$Z_\Lambda(L, g(M); s) = \prod_p Z_{\Lambda_p}(L_p, M_p; s)$$

One can now assemble these product formulas 2.3.1 for various M's, to obtain an Euler product for $\zeta_\Lambda(L;s)$. First of all, Λ_p is a maximal order in A_p for almost all p. For each such p, any two Λ_p- lattices in V_p are isomorphic ([MO] 18.10) and hence

$$Z_{\Lambda_p}(L_p, M_p; s) = \zeta_{\Lambda_p}(L_p; s) \quad \text{for any } L \text{ and } M.$$

2.3.2 <u>REMARK</u> Let $\left\{ M_1, \ldots, M_h \right\}$ be a complete set of genus representatives of all full Λ- lattices in V. Then there exists r_i and s_i in $\mathbb{Z}$ for $i = 1,\ldots,h$ such that $r_i M_i \subseteq L \subseteq s_i M_i$ for a full Λ- lattice L in V. So we can choose a finite set S' such that for $s \notin S'$ $L_p = (M_i)_p$ for $i = 1,\ldots,h$. Now let S'' be the finite set such that for all $p \notin S''$, we have Λ_p is a maximal order. Then if $S = S' \cup S''$, we take $x = (x_p) \in J(B)$ such that for all $p \notin S$, $x_p \in$ where we now take $\Gamma_p = \mathrm{End}_{\Lambda_p} L_p$ an order in B_p^*.

So $(M_i x)_p = (M_i)_p x_p = L_p x_p = L_p$ for all $p \notin S$.

Also since $(M_i)_p x_p \cong (M_i)_p$ Hence we can choose the complete set of genus representatives of all full Λ- lattices in V to be $\left\{ M_1 x, \ldots, M_h x \right\}$. ∎

Let R be the complete set of genus representatives as in the remark above, that is for $M \in R$, $M_p = L_p$ for all $p \notin S$ as above. Let R_p be a subset of $\left\{ M_p : M \in R \right\}$ for each p, such that R_p is a complete set of representatives of isomorphism classes of full Λ_p-lattices in V_p. Then there is a natural bijection

$$\theta : R \longrightarrow \prod_{p \in S} R_p$$

$$M \longmapsto \{ Y_p \} \quad \text{where } M_p \cong Y_p \quad \text{for}$$

all $p \in S$.

θ is one-one as $\theta(M) = \theta(N)$ implies that $M_p \cong N_p$ for all $p \in S$. For all $p \notin S$ we already have $M_p \cong N_p$ as Λ_p is a maximal order then. Therefore N and M are in the same genus. Since $M, N \in R$ so $M = N$. θ is onto for if $\{ Y_p \} \in \prod_{p \in S} R_p$, then by the theorem ([MO] 5.3) quoted in the first chapter (1.2.2) we can have a full Λ- lattice N in V such that $N = V \cap \left\{ \bigcap_p Y_p \right\}$ where for $p \notin S$ $Y_p = L_p$. Take the genus representative of N say $\tilde{N} \in R$, then $\theta(\tilde{N}) = \{ Y_p \}$.

Therefore we obtain

$$\prod_p \zeta(L_p; s) = \prod_p \sum_{M_p \in R_p} Z(L_p, M_p; s)$$

$$= \prod_{p \notin S} \zeta(L_p; s) \quad \prod_{p \in S} \sum_{M_p \in R_p} Z(L_p, M_p; s)$$

$$= \prod_p \zeta(L_p; s) \quad \sum_{M \in R} \prod_{p \in S} Z(L_p, M_p; s)$$

$$= \sum_{M \in R} \prod_p Z(L_p, M_p; s) = \sum_{M \in R} Z(L, g(M); s). \text{ So}$$

2.3.3 $$\prod_p \zeta(L_p; s) = \zeta(L; s)$$

Thus in analogy to Dedekind zeta functions it is shown that the Solomon zeta functions also satisfies an Euler product identiy. It should be noted that formula 2.3.3 can be obtained without using adeles, see [LS] Pg316 Lemma 6.

——— * ———

CHAPTER 3

Introduction : We shall deal with the analytic theory of L-functions of orders in a finite dimensional semisimple $\mathbb{Q}$- algebras in this chapter. Such L- functions will be defined in terms of Solomon's zeta functions that were studied in the last chapter. Standard L- functions (they appear in [GJ]) which can be defined in terms of classical L- functions will be introduced and it will be shown that these two types of L- functions differ by some elementary functions which can be described in purely local term. The main aim of this chapter is to establish the analytic continuation of the L- functions of orders and to discuss their behaviour around s = 1. However, we also study partial zeta functions defined in the last chapter —— they are best studied via L- functions.

We have the following convention : we consider only those (left) full ideals of an order whose indices are finite

§1 GLOBAL AND LOCAL L- FUNCTIONS AND EULER PRODUCTS

(A) Global L- functions

Let Λ be a $\mathbb{Z}$ order in a finite dimensional semisimple $\mathbb{Q}$- algebra A. Consider the Solomon zeta function

3.1.1 $$\zeta_\Lambda(s) = \sum_X (\Lambda:X)^{-s}$$

where X ranges over all left ideals X of Λ, and s a complex variable. This converges at least for Real s > 1 (see [LS])). To study the analytic properties of ζ_Λ we introduce partial zeta functions as explained below.

Let M be a (full) left ideal of Λ.

3.1.2 *Definition:* Two lattices X, Y in the genus g(M) of M is said to be stably isomorphic if $X \oplus M \cong Y \oplus M$.

In most cases, stable isomorphism implies ordinary isomorphism. This is true if the algebra A satisfies "Eichler condition" ([MO]38.1). It was shown by Jacobinski [JH] and Frohlich [FA], that the genus g(M) can be partitioned into a finite number of stable isomorphism classes.

We now define the partial zeta function $Z_\Lambda(g(M); s)$, $Z_\Lambda([M]; s)$, $Z_\Lambda(M ; s)$ by a series such as 3.1.1 but where X is restricted to the genus of M, the stable isomorphism classes [M] in g(M) and the isomorphism classes of M respectively. All these series converge well for Real s > 1 by comparison with $\zeta_\Lambda(s)$. It is clear that

3.1.3 $$\begin{cases} \zeta_\Lambda(s) = \sum_M Z_\Lambda(g(M); s) \\ Z_\Lambda(g(M); s) = \sum_{i=1}^{h} Z_\Lambda([M_i]; s) \end{cases}$$

where M ranges over a (finite) set of representatives of the genera of left ideals of Λ, and where for a fixed M, the stable isomorphism classes in $g(M)$ are $[M_1], \ldots, [M_h]$.

In the special case of $M = \Lambda$, the stable isomorphism classes in $g(\Lambda)$ form a finite abelian group $Cl(\Lambda)$, called the locally free class group of Λ. Addition in $Cl(\Lambda)$ is given by $[X] + [X'] = [X'']$ for $X, X' \in g(\Lambda)$, where $[X'']$ is defined by the condition $X \oplus X' \cong X'' \oplus \Lambda$ (such an X'' will exist by the result in [MO] 27.3). That $Cl(\Lambda)$ is actually an abelian group is verified in [MO] 35.5 and [MO] 38 P343.

In the general case where M is an arbitrary left ideal of Λ, we set $\Gamma = End_{\Lambda}(M)$. Then Γ is a $\mathbb{Z}$-order in A, and we may view M as a right Γ-module. As shown by Jacobinski [JH], there is a bijection between $Cl(\Gamma)$ and the set of stable isomorphism classes in $g(M)$ given by $[X] \longmapsto [MX]$ for each $X \in Cl(\Gamma)$. Now via this bijection we can impose a structure of finite abelian group on the set of stable isomorphism classes in $g(M)$. The addition is therefore given by the formula $[M_1] + [M_2] = [M_3]$ where $M_1 \oplus M_2 \cong M_3 \oplus M$. This group is denoted by $Cl(M)$ and will be called the genus class group of M.

We note that this group structure on $Cl(M)$ depends on the choice of M in its genus because it is given via the isomorphism $Cl(\Gamma) \cong Cl(M)$. But as we shall see this will not be an obstruction to our calculations ahead.

Let ψ be a character of $Cl(M)$, that is, a homorphism from $Cl(M)$ to S', the unit circle in $\mathbb{C}$. We now define the global L-function by the formula

3.1.4 $$L_\Lambda(M, s, \psi) = \sum_{\substack{X \subseteq \Lambda \\ X \in g(M)}} \psi[x](\Lambda:X)^{-s}$$

where s is a complex variable. This series converges absolutely for Real s > 1, uniformly on compact subsets, by comparison with the series 3.1.1. We show how to express the zeta functions in 3.1.3 in terms of this L-function. It is clear that

$$L_\Lambda(M, s, \psi) = \sum_{[N] \in Cl(M)} \psi[N]\ Z_\Lambda([N]; s)$$

Therefore $$\sum \psi^{-1}[P]\ L_\Lambda(M, s, \psi) = \sum_{\psi} \sum_{[N]} \psi^{-1}[P]\psi[N]\ Z_\Lambda([N];s)$$

$$= \sum_{\psi} \sum_{[N]} \psi([N]-[P])\ Z_\Lambda([N]; s)$$

$$= \sum_{[N]} \left\{ \sum_{\psi} \psi([N]-[P]) \right\} Z_\Lambda([N]; s)$$

Now orthogonality conditions for characters of finite groups([G] say that $\sum \psi([N]-[P]) = \begin{cases} 0 \text{ if } [N] \neq [P] \\ h = |Cl(M)| \text{ if } [N] = [P] \end{cases}$

and hence $\sum \psi^{-1}[P]\ L_\Lambda(m, s, \psi) = h\ Z_\Lambda([P]; s)$. This implies that

3.1.5 $$Z_\Lambda([P]; s) = h^{-1} \sum_{\psi} \psi^{-1}[P]\ L_\Lambda(M, s, \psi)$$

for each stable isomorphism class [P] in g(M). This suggests that we can use information about L-functions to study the partial zeta function $Z_\Lambda([N]; s)$.

Now we will show that a character ψ of Cl(M) can be viewed as an idele class character of A. This is done as follows.

Let ψ be a character of Cl(M). Then ψ may also be viewed as a character of $Cl(\Gamma)$ via the Jacobinski isomorphism $Cl(\Gamma) \cong Cl(M)$, where $\Gamma = End_{\Lambda}(M)$. Let $J_f(A)$ denote the full idele group of A, formed by taking both finite and infinite primes of $\mathbb{Q}$. (Note: this is different from the finite idele group of $\mathbb{Q}$-algebras introduced in Chapter 2 for the finite idèle group was formed by taking only the finite primes of $\mathbb{Q}$.). Let $J_f'(A)$ be the closure of the commutator subgroup of $J_f(A)$. The group A^* of invertible elements of A sits inside $J_f(A)$ because for $x \in A^*$, $x\Gamma$ is a full lattice in A and hence for almost all p, $x_p\Gamma_p = \Gamma_p$. The image of A^* inside $J_f(A)$ is called the group of principal ideles in $J_f(A)$.

Frolich in [FA] showed that

3.1.6 $$Cl(\Gamma) \cong J_f(A)/J_f'(A).A^*.U_f(\Gamma)$$

where $U_f(\Gamma)$ is the group of unit ideles of the order Γ, defined by

3.1.7 $$U_f(\Gamma) = (\mathbb{R} \otimes_{\mathbb{Q}} A) \times \prod_p \Gamma_p^*$$

with p ranging over all prime numbers.

Now following Frolich [FA] we can describe $Cl(\Gamma)$ in an alternative way as follows.

Since A is semisimple $\mathbb{Q}$-algebra with center C we can write $A = \prod_{i=1}^{n} A_i$ and $C = \prod_{i=1}^{n} C_i$, where each A_i is a semisimple algebra with center C_i. It is routine verification that

$$J_f(A) = \prod_{i=1}^{n} J_f(A_i)$$

$$J_f(C) = \prod_{i=1}^{n} J_f(C_i)$$

Now the reduced norm $A_i^* \longrightarrow C_i^*$ can be put together as shown in chapter 1 to give a reduced norm map $nr_{A/C} : A^* \longrightarrow C^*$. On completion we get the reduced norm $nr_{A_p/C_p} : A_p^* \longrightarrow C_p^*$ and hence a continuous homomorphism

$$nr : J_f(A) \longrightarrow J_f(C)$$
$$(x_p) \longmapsto (nr_{A_p/C_p}(x_p))$$

which we again call the reduced norm map. Now Ker(nr) is a closed subgroup of $J_f(A)$ and $J_f'(A) \subseteq \mathrm{Ker}(nr)$. By Nakayama-Matushima theorem which was quoted in Chapter 1 we have $\mathrm{Ker}\left[nr_{A/C}\right]$ and $\mathrm{Ker}\left[nr_{A_p/C_p}\right]$ are $A^{*\prime}$ and $A_p^{*\prime}$ respectively where $A^{*\prime}$, $A_p^{*\prime}$ denotes the respective commutator subgroups.

The commutator subgroup of $J_f(A)$ is $J_f(A)' = J_f(A) \cap \prod_p A_p^{*\prime}$. Since now for $x = (x_p) \in \mathrm{Ker}(nr)$ $nr_{A_p/C_p}(x_p) = 1$ which implies that $x_p \in A_p^{*\prime}$ which further implies that $(x_p) \in \prod_p A_p^{*\prime}$. Therefore $\mathrm{Ker}(nr) \subset J_f(A)'$ and hence $J_f(A)' = \mathrm{Ker}(nr)$.

By HASSE-SCHILLING-MAASS norm theorem which is quoted below we have $nr(A^*) = C^* \cap nr(J_f(A))$ and $C^*.nr(J_f(A)) = J_f(C)$ and so we have $J_f(C) \setminus C^* = C^*.nr(J_f(A)) \setminus C^* \cong nr(J_f(A)) \setminus nr(A^*)$. Therefore nr induces isomorphisms

3.1.8
$$\begin{cases} J_f(A) \setminus J_f(A)'.A^* \cong J_f(C) \setminus C^* \\ J_f(A) \setminus J_f(A)'.A^*.U_f(\Gamma) \cong J_f(C) \setminus C^*.nr(U_f(\Gamma)) \end{cases}$$

Now we suppose that M is a left ideal of an order Λ in a $\mathbb{Q}$-algebra A, and let $\Gamma = \mathrm{End}_\Lambda(M)$. Because of the Jacobinski isomorphism $Cl(M) \cong C$ and the Frolich isomorphism of 3.1.6, each character ψ of $Cl(M)$ can be viewed as a continuous character (also denoted by ψ) of the idele group $J_f(A)$, which is trivial on $J_f(A)'.A^*.U_f(\Gamma)$. Furthermore by 3.1.8 we may write

3.1.9 $\quad \psi(x) = \tilde{\psi}\,(nr(x)),\ x \in J_f(A)$

for some uniquely determined continuous character $\tilde{\psi}$ of $J_f(C)$ such that ψ is trivial on $C^*.nr(U_f(\Gamma))$.

We conclude this subsection by a result which describes the characters of $Cl(M)$.

3.1.10 THEOREM. (HASSE-SCHILLING-MAASS [MO],pg 289)

Let A be a central simple K-algebra, where K is a global field. Let $\alpha \in K^*$. Then α is the reduced norm of an element of A if and only if $\alpha_P > 0$ at every real prime P of K ramified in A. (a prime P of K is said to be ramified in A if $A_P = M_{k_P}(S)$, S a skew field over K_P and $(S : K_P) = m_P^2 > 1$)

(B) Local L-functions.

Let A be a semisimple $\mathbb{Q}_p$- algebra and Λ is a $\mathbb{Z}_p$ order in A. Let M be a left ideal of Λ, of finite index in Λ and we set $\Gamma = End_\Lambda(M)$. Suppose that ψ is a continuous character of A^* which is trivial on Γ^*. We then define the local L- function

3.1.11 $\quad L_\Lambda(M, s, \psi) = \sum_X \psi(X)\ (\Lambda : X)^{-s}$

where the sum extends over all left ideals X of Λ such that $X \cong M$. This is the correct local analogue for 3.1.4 as here only one prime p is involved. This sum converges for Real s > 1.

If we now identify A with $End_{A}A$, then as in Chapter 2 we can find $x \in A^*$ such that $X = Mx$ and this x is unique modΓ^*. We now define $\psi(X) = \psi(x)$. ψ is independent of the choice of x in the above formula as ψ is trivial on Γ^*. $L_\Lambda(M, s, \psi)$ is then a power series in p^{-s}, with coefficients in the ring $\mathbb{Z}[\psi]$ generated by adjoining to $\mathbb{Z}$ the values of ψ (which are of course roots of unity).

(C) Euler product formula.

In this subsection let A be a semisimple $\mathbb{Q}$-algebra again. Let J(A) be the finite idele group of A. Then we may view J(A) as a subgroup of $J_f(A)$. Similiarly the finite adele ring Ad(A) is a subring of $Ad_f(A)$. As in Chapter 2, if X is a full $\mathbb{Z}$-lattice in A, we put $Ad(X) = \prod_p X_p$ where p ranges over rational primes. Then Ad(X) can be viewed as an additive subgroup of $Ad_f(A)$. If $U(\Gamma) = \prod_p \Gamma_p^*$ p ranging over rational primes, then $U(\Gamma) = J(A) \cap U_f(\Gamma)$. Let ϕ be the characteristic function of $Ad(\{M:\Lambda\})$ in Ad(A) and ϕ_p the characteristic function of $\{M_p:\Lambda_p\}$ in A_p. Let ψ be a character of the genus class group Cl(M). Then as shown in Chapter 2 §1 (A), ψ also defines a continuous character of $J_f(A)$, also written as ψ. We now put $\psi_f = \psi|\ J(A)$, and $\psi_p = \psi|\ A_p^*$ for each prime number p. We then obtain $\psi_f = \prod_p \psi_p$. The character ψ of $J_f(A)$ is trivial on the subgroup $\left(\mathbb{R} \otimes_{\mathbb{Q}} A\right)^*$ of $J_f(A)$ as it is trivial $U_f(\Gamma)$. Therefore just as in the proof 2.3.14, we obtain

3.1.11 $$L_\Lambda(M, s, \psi) = \mu^*(U(\Gamma))^{-1}. (\Lambda:M)^{-s} \int_{J(A)} \phi(x)\ \psi_f(x)\ \|x\|^s\ d^*x$$

where d^*x is a Haar measure on $J(A)$, μ^* denotes the measure of a set with respect to d^*x, and $\|x\| = (N:Nx)^{-1}$ for any full $\mathbb{Z}$- lattice N in A.

The character ψ is trivial on $U(\Gamma)$ and so ψ_p is trivial on Γ_p^* for all rational primes p. So we can form $L_{\Lambda_p}(M_p, s, \psi_p)$ for each rational prime p, and just as in Chapter 2 we obtain

$$3.1.12 \quad L_{\Lambda_p}(M_p, s, \psi_p) = \mu_p^*(\Gamma_p^*)^{-1}. (\Lambda_p:M_p)^{-s} \int_{A_p^*} \phi_p(x)\psi_p(x)\|x\|_p^s \, d^*x$$

where d^*x is a measure A_p^* as defined in Chapter 2 and $\|x\| = (N:Nx)^{-1}$ for any full $\mathbb{Z}_p$- lattice in A_p.

Proceeding now in the same way as in Chapter 2 §3, the Euler product formula for $L_\Lambda(M, s, \psi)$

$$3.1.13 \qquad L_\Lambda(M, s, \psi) = \prod_p L_{\Lambda_p}(M_p, s, \psi_p), \quad \text{Real } s > 1$$

results.

3.1.14 <u>REMARK</u> All these above calculations can be carried out even when A is an F- algebra, F an algebraic number field with ring of integers R, and Λ an R-order in A, and p replaced by the nonzero prime ideals of R.

§2 STANDARD L- FUNCTIONS

In this section we will construct the local standard L-functions on a semisimple $\mathbb{Q}_p$- algebra A, and develop some of their interesting properties. The global counterpart will then be defined via the Euler product (which we will show converges). We will also show the connection between L- functions defined in 3.1.4 and 3.1.10, and the standard L- functions.

We begin with the construction of the local standard L-functions.

(A) Local Standard L- functions.

Let A be a $\mathbb{Q}_p$- algebra and ψ a continuous character of finite order of the group A^*. Let $S(A)$ be the space of Schwartz-Bruhat functions on A (see Chapter 1 for definition). If $\phi \in S(A)$, then ϕ is a $\mathbb{C}$- linear combination of a finite number of characteristic functions of spheres ($w + p^f\Gamma$) for an order Γ in A , f is a fixed positive integer and w ranges over a finite set of elements of A (see 1.2.21 ii and iv). For $\phi \in S(A)$ we define

3.2.1 $$Z(\phi, s, \psi) = \int_{A^*} \phi(x)\, \psi(x)\, \|x\|^s$$

where d^*x is a multiplicative Haar measure on A^*.

Now if ϕ is the characteristic function of $p^f\Gamma$, for an order Γ in A then $p^f\Gamma\setminus\{0\} = \bigcup_{n=f}^{\infty} A_n$, where $A_n = p^n\Gamma \setminus p^{n+1}\Gamma$. Therefore if we define $\|x\| = 0$ for $x \in A\setminus A^*$ then we have

$$| Z(\phi, s, \psi) | \le \int_{(p^f\Gamma \cap A^*)} \|x\|^\sigma d^*x \qquad \text{where } \sigma = \text{Real } s$$

$$= \int_{p^f\Gamma} \|x\|^{\sigma-1} dx \qquad \text{(see 1.2.16)}$$

$$= \sum_{n=f}^{\infty} \int_{A_n} \|x\|^{\sigma-1} dx$$

$$= \sum_{n=f}^{\infty} \int_{p^n\Gamma \setminus p^{n+1}\Gamma} \|x\|^{\sigma-1}\, dx$$

$$= \sum_{n=f}^{\infty} \int_{\Gamma \setminus p\Gamma} \|p^n x\|^{\sigma-1}\, dx$$

$$= \left(\int_{\Gamma \setminus p\Gamma} \|x\|^{\sigma-1} dx \right) \sum_{n=f}^{\infty} \|p\|^{n(\sigma-1)}$$

$$\leq \left(\int_{\Gamma \setminus p\Gamma} dx \right) \sum_{n=f}^{\infty} \|p\|^{n(\sigma-1)} \text{for if } x \in \Gamma \text{ then } \|x\| \leq 1$$

$$\leq \mu\ (\Gamma\ )\ .\ \sum_{n=f}^{\infty} p^{-n(\sigma-1)l},\ \text{for } l > 0.$$

Then $Z(\phi, s, \psi)$ conveges absolutely to a holomorphic function of s, for Real $s > 1$. Hence for all $\phi \in S(A)$, $Z(\phi, s, \psi)$ converges to a holomorphic function of s for Real $s > 1$.

We now introduce the standard L- functions $L_A(s, \psi)$. They are in some sense the "least common denominator" of the zeta integrals $Z(\phi, s, \psi)$ and are characterised by the following axioms:

(C1) There is a polynomial $f(x) \in \mathbb{C}[x]$ with $f(0) = 1$, such that $L_A(s, \psi) = f(p^{-s})^{-1}$.

(C2) There exists $\phi \in S(A)$ such that $L_A(s, \psi) = Z(\phi, s, \psi)$.

(C3) For any $\phi \in S(A)$, the function $L_A(s, \psi)^{-1}.\ Z(\phi, s, \psi)$ is a polynomial in $\mathbb{C}[p^s, p^{-s}]$.

The idea of such a "standard L- function" occurs in [GJ].

The existence of such a function satisfying (C1), (C2) and (C3) will be shown by producing an explicit one. But once such a function exists, its uniqueness can be shown as follows:

Suppose there is another $L'_A(s, \psi)$ satisfying the axioms. That is

(i) for some $h(x) \in \mathbb{C}[x]$, $L'_A(s, \psi) = h(p^{-s})^{-1}$ and $h(0) = 1$.

(ii) $L'_A(s, \psi) = Z(\phi', s, \psi)$ for some $\phi' \in S(A)$.

(iii) For any $\phi \in S(A)$ is such that $L'_A(s, \psi)^{-1}.\ Z(\phi, s, \psi) \in \mathbb{C}[p^s, p^{-s}]$.

If $\phi \in S(A)$ is such $L_A(s, \psi) = Z(\phi, s, \psi)$, then by (iii) $L'_A(s, \psi)^{-1}.\ L_A(s, \psi) = g(p^s, p^{-s}) \in \mathbb{C}[p^s, p^{-s}]$. Also from (C1) we have $L'_A(s, \psi)^{-1}.\ L_A(s, \psi) \in \mathbb{C}(p^{-s})$, the space of complex rational functions in p^{-s}. Therefore $g(p^s, p^{-s}) \in \mathbb{C}(p^{-s}) \cap \mathbb{C}[p^s, p^{-s}] = \mathbb{C}[p^{-s}]$. Also note that $g(0) = 1$ as $f(0) = 1 = h(0)$. Repeating the arguement for $L_A(s, \psi)^{-1}.L'_A(s, \psi) = g^{-1}(p^s, p^{-s})$, we see that $g^{-1}(p^s, p^{-s}) \in \mathbb{C}[p^{-s}]$ and $g^{-1}(0) = 1$. This, along with the conclusion about g yeilds that $g = 1$ identically. Hence the uniqueness.

We begin the construction of $L_A(s, \psi)$. The general case may be reduced to that of simple algebras by using the following proposition:

3.2.2 PROPOSITION Let $A = \prod_{i=1}^{r} A_i$ be a finite dimensional semisimple $\mathbb{Q}_p$- algebra with simple components A_i, and let $\psi_i = \psi|A_i^*$, $1 \le i \le r$. Suppose that, for each i, there exists a standard L- function $L_{A_i}(s, \psi_i)$ satisfying (C1), (C2) and (C3). Then the standard L- function $L_A(s, \psi)$ also exists and is given by $L_A(s, \psi) = \prod_{i=1}^{r} L_{A_i}(s, \psi_i)$.

Proof: We only need to check that the above product satisfies (C1), (C2) and (C3). Now (C1) and (C2) are obviously satisfied. To check (C3), we note that $S(A) = S(A_1) \otimes_C \ldots\ldots \otimes_C S(A_r)$ where C is the center of A (see 1.2.22). Then any $\phi \in S(A)$ has the form $\phi = \phi_1 \otimes \ldots \otimes \phi_r$ and now (C3) is obvious. ∎

It therefore suffices to define $L_{\Lambda}(s, \psi)$ when A is a simple $\mathbb{Q}_p$-algebra. The first step is to describe the characters of A^*: this is done in the following local version of 3.1.9.

3.2.3 LEMMA Let A be a finite dimensional simple $\mathbb{Q}_p$- algebra with center with center C. Let R be the valuation ring of the field C, and let Λ'be a maximal order in A. Then for each continuous character ψ of A^* of finite order, there is a unique character $\tilde{\psi}$ of C^* such that $\psi = \tilde{\psi} \circ nr$, where nr is the reduced norm map from A^* to C^*. Further,

3.2.4
$$\begin{cases} nr(A^*) = C^* & \text{.......(i)} \\ nr(\Lambda'^*) = R^* & \text{.......(ii)} \end{cases}$$

Thus ψ is trivial on (Λ'^*) if and only if $\tilde{\psi}$ is trivial on R^*.

Proof: Statement 3.2.4 (i) is just the local norm theorem ([MO] 33.4) quoted in 1.2.30. To see 3.2.4 (ii) we let $A = M_k(D)$, D a division algebra over $\mathbb{Q}_p$, with center C. Then D has a unique maximal order Δ, which is a discrete valuation ring. Since $nr(\Delta) \subseteq R$ so if ξ is a prime element of Δ, then $nr(\xi) = \pi$, where π is a prime element of R. So $nr(\Delta) = R$. But $nr(\Delta) = nr(\Lambda)$ by 1.2.29 (iii). Since nr is multiplicative and all maximal orders Λ' in A are conjugates of Λ, we have $nr(\Lambda') = R$ implying that $nr(\Lambda'^*) = R^*$.

The NAKAYAMA-MATSUSHIMA theorem stated in 1.2.31 says that the kernel of the reduced norm map coincides with the commutator subgroup of A^*. Thus there is an isomorphism $A^*/A^{*\prime} \cong C^*$, $A^{*\prime}$ is the commutator subgroup of A^*. This is induced by the reduced norm map. For any continuous character ψ of A^*, we thus have that

ψ is trivial on $A^{*\prime}$. Now we define a character $\tilde{\psi}$ of C^* as follows: for $c \in C^*$, let $\tilde{\psi}(c) = \psi(a)$ where $nr(a) = c$ and $a \in A^*$. Then this is well defined as ψ is trivial on $A^{*\prime}$. Therefore $\psi = \tilde{\psi} \circ nr$ for a uniquely determined character $\tilde{\psi}$ of C^*. The remaining statement is now clear. ∎

3.2.5 *Definition*: Notations as in Lemma. A character $\tilde{\psi}$ of C^* is called unramified if it is trivial on R^* and ramified otherwise. We can extend this terminology to ψ, a character of A^* by saying that ψ is unramified if and only if $\tilde{\psi}$ is unramified, that is if and only if ψ is trivial on Λ^*, for some (or equivalently, any) maximal order Λ in A.

We are now ready to to give a formula for $L_A(s,\psi)$ when A is a simple $\mathbb{Q}_p$- algebra.

3.2.6 <u>THEOREM</u> Let a be a simple $\mathbb{Q}_p$- algebra with center C. Let R be the valuation ring of C and $\rho = \pi R$ the maximal ideal of R. We put $N\rho = (R:\rho)$, so that N is the usual counting norm. We write $A = M_k(D)$, where D is a division algebra wth center C, and let

$$\dim_C D = e^2 \ , \ \dim_C A = n^2 \ , \ n = ke.$$

Given a continuous character ψ of A^* of finite order, let $\tilde{\psi}$ be the character of C^* such that $\psi = \tilde{\psi} \circ nr$ as in 3.2.3. Define

$$3.2.7 \qquad L_C(s,\tilde{\psi}) = \begin{cases} (1 - \tilde{\psi}(\pi)\, N\rho^{-s})^{-1} & \text{if } \tilde{\psi} \text{ is unramified} \\ 1 & \text{otherwise.} \end{cases}$$

and

3.2.8 $$L_A(s,\psi) = \prod_{\substack{j=0 \\ j\equiv 0 (\mathrm{mod}\ e)}}^{n-1} L_C(ns-j, \tilde{\psi})$$

Then $L_A(s,\psi)$ is the standard L- function of A satisfying the conditions (C1), (C2) and (C3).

Proof: The proof can be divided into two distinct parts. Case I, when ψ is unramified and Case II, when ψ is ramified. For the Case I we require the following Lemma.

3.2.9 <u>LEMMA</u> Let Λ' be a maximal order in A, and let ψ be any unramified character of A^*. Then

$$L_{\Lambda'}(\Lambda', s, \psi) = L_A(s,\psi)$$

where $L_{\Lambda'}(\Lambda', s, \psi)$ is the L- function of Λ' defined in 3.1.10 and $L_A(s,\psi)$ as in 3.2.8.

Proof: Before going into the proof we first recall some facts required in the proof.

(a) If D is a division ring whose center contains a local field, then there exists a unique maximal R- order Δ in D ([MO] 12.8).

(b) Δ is a non commutative discrete valuation ring and is actually the integral closure of R in D ([MO] 12.6).

(c) Let ξ denote a prime element of Δ and hence powers of $\Delta\xi$ will give all non zero one sided ideals of Δ. These ideals are necessarily two sided ideals ([MO] 17.3).

(d) If $A = M_k(D)$ be a central simple C- algebra (C contains a local field) then $M_k(\Delta)$ and its conjugates give all possible maximal R- orders in A ([MO] 17.3).

(e) If $(D:C) = e^2$ then the index of ramification $e(D/C) = e$

whence if $\rho = \pi R$ is the maximal ideal of R then $\rho\Delta = (\Delta\xi)^e$ ([MO] 14.3)

(f) Let $= M_k(D)$. Then $\Lambda = M_k(\Delta)$ is a maximal order and every other maximal order Λ' in A is a conjugate of Λ in A.

Coming back to the proof of the Lemma (notations used will be the same as in 3.2.6), let Δ denote the unique maximal R- order in D. Then Δ is a non commutative discrete valuation ring and we may choose its prime element ξ so that $nr_{D/C}(\xi) = \pi$, where π is a prime element of R. We then have $\Delta\xi = \xi\Delta$ and $(\Delta:\Delta\xi) = N\rho^e$, where $\rho = \pi R$. Any two maximal orders are conjugates in A and hence it follows that $L_{\Lambda'}(\Lambda', s, \psi)$ is independent of the choice of Λ'. Thus we may without of loss of generality assume that $\Lambda' = M_k(\Delta)$. Since every left ideal of Λ' is principal, and ψ is trivial on Λ'^* the L- function $L_{\Lambda'}(\Lambda', s, \psi)$ can be written as

$$L_{\Lambda'}(\Lambda', s, \psi) = \sum_x \psi(x) \ (\Lambda':\Lambda'x)^{-s} \qquad \text{for Real } s > 1$$

where x ranges over a full set of representatives of right coset space $\Lambda'^* \setminus \Lambda' \cap A^*$.

Since $\Lambda'^* = Gl_k(\Delta)$, we may choose these orbit representatives in the "Hermite normal form" ([N] pg 15) $x = a_{ij} \in M_k(\Delta)$, where a_{ij} satisfy

$$\begin{cases} a_{ii} = \xi^r, \ r_i \geq 0, \ 1 \leq i \leq k. \\ a_{ij} = 0 \quad \text{if } j < i \\ a_{ij} \text{ ranges over a full set of coset representatives of } \Delta \bmod \xi^r \Delta \text{ if } j>i \end{cases}$$

We note that for $i < j$, there are $(\Delta:\xi^r \Delta)$ choices for a_{ij}, that is $N\rho^{er}$ choices. Now we have by 1.2.29 (ii)

$$nr_{A/C}(x) = \left[nr_{D/C}(\xi)\right]^{r_1+\dots+r_k} = \pi^{r_1+\dots r_k}$$

We further have

$$(\Lambda':\Lambda' x) = (\Delta:\Delta\xi)^{k(r_1+...+r_k)} = Np^{n\Sigma r_i}$$

by considering Bx ($B \in \Lambda'$) to be the sum of the column vectors of the product of B and the upper triangular matrix x.

Also having chosen $r_1, r_2, ..., r_k$, there are $Np^{e(r_2+2r_3+...+(k-1)r_k)}$ choices for the coset representatives x, with each of them in distinct cosets $\mathrm{mod}(\Lambda' \cap A^*)$. Then $L_{\Lambda'}(\Lambda', s, \psi)$ is equal to

$$\sum_{r_1,...,r_k>0} \tilde{\psi}(\pi)^{r_1+...+r_k} Np^{e(r_2+2r_3+...+(k-1)r_k)} Np^{-ne(r_1+...+r_k)}$$

This multiple series when summed gives us

3.2.10 $$L_{\Lambda'}(\Lambda', s, \psi) = \prod_{\substack{j=0 \\ j\equiv 0 \ \mathrm{mod} e}}^{n-1} \left[1-\tilde{\psi}(\pi) \quad Np^{-(ne-j)}\right]^{-1} \quad \text{for Real } s > 1$$

Hence the Lemma. ■

3.2.11 <u>COROLLARY</u> Keeping the above notation, for some $\varepsilon > 0$ the function $L_A(s,\psi)$ is holomorohic and non zero for Real $s > 1-\varepsilon$.

Proof: Immediate from the Lemma and the fact that $L_{\Lambda'}(\Lambda', s, \psi)$ is holomorphic and non zero for Real $s > 1$. ■

Now coming back to the proof of the Theorem 3.2.6 for the case when ψ is unramified, we must show that the function $L_A(s,\psi)$ in 3.2.7 satisfies (C1), (C2) and (C3). Of these (C1) is obviously satisfied. Condition (C2) holds because by Lemma 3.2.9 we have $L_A(s,\psi) = L_{\Lambda'}(\Lambda', s, \psi) = \mu^*(\Lambda'^*)^{-1} \; Z(\phi, s, \psi)$, where ϕ is the characteristic function of $\{\Lambda':\Lambda'\}$ in A. Since a constant multiple of a Schwartz- Bruhat function is again a Schwartz- Bruhat function, so (C2) is also satisfied.

We now proceed to verify (C3) for the case when ψ is unramified. Let $\phi \in S(A)$. We may express ϕ as a finite $\mathbb{C}$-linear combination of characteristic function of spheres $w + p^t\Lambda'$, t is some fixed positive integer depending on ϕ and w ranges over some finite set of elements of A. We can choose $\Lambda' = M_k(\Delta)$ as in (f) of 3.2.9 without loss of generality. Therefore it suffices to verify (C3) when ϕ is the characteristic function of a typical sphere $w + p^t\Lambda'$. Next we choose $d \in \mathbb{Z}$, $d \geq 0$ such that $p^d w \in \Lambda'$.

$$\text{Then } \int_{A^*} \phi(x)\ \psi(x)\ \|x\|^s\ d^*x \;=\; \int_{A^*} \phi(p^d x)\ \psi(p^d x)\ \|x\|^s\ d^*x$$

$$= \|p\|^{ds}\psi(p^d) \int_{A^*} \phi(p^d x)\ \|x\|^s\ \psi(x)\ d^*x$$

$$= p^{ls}\ \psi(p^d) \int_{A^*} \phi(p^d x)\ \|x\|^s\ \psi(x)\ d^*x$$

The support of the function $x \longrightarrow \phi(p^d x)$ is p^{-d} times the support of ϕ. Hence at the cost of introducing a factor cp^{ls} for some $l \in \mathbb{Z}$ and $c \in \mathbb{C}$, we can assume that the support of ϕ is contained in the lattice Λ'. It follows that we can assume that ϕ is the characteristic function of the sphere $\alpha + p^f\Lambda'$ with $\alpha \in \Lambda'$ and $f \geq 0$. Hence it suffices to show that

$$3.2.12 \qquad L_A(s,\psi)^{-1} \int_{(\alpha+p^f\Lambda')\,\cap\, A^*} \psi(x)\ \|x\|^s\ d^*x$$

lies in $\mathbb{C}[p^s,p^{-s}]$. We now break up the remaining part of the proof into the following steps.

Step I Since ψ is unramified, we can change the range of integration on either sides by a factor from Λ'^{*} without changing the value of the above integral. This allows us to reduce α to the "Smith normal form" ([N])

$\alpha = \text{diag}\ (\xi^{j}, \xi^{j}, \ldots, \xi^{j}, 0, \ldots, 0)$

with $j_1 \le j_2 \le \ldots \le j_r$ and $\xi^{j}\Delta$ properly contains $p^f\Delta$ for $1 \le i \le r$

Step II Now let $\beta = \text{diag}\ (\beta_1, \beta_2, \ldots, \beta_k)$ where $\beta_i = \xi^{-j}$ for $1 \le i \le r$, and $\beta_j = 1$ for $j > r$. Then we have

$$\int_{(\beta\alpha+\beta p^f\Lambda')\cap A^*} \psi(x)\ \|x\|^s\ d^*x = \int_{(\alpha+p^f\Lambda')\cap A^*} \psi(\beta x)\ \|\beta x\|^s\ d^*x$$

$$= \|\beta\|^s\ \psi(\beta)\int_{(\alpha+p^f\Lambda')\cap A^*} \psi(x)\,\|x\|^s\ d^*x$$

So our integral is a power of p^{-s} multiplied by a constant complex multiple of the same integral taken over the set $(\beta\alpha + \beta p^f\Lambda') \cap A^*$. Clearly $\beta p^f\Lambda' \subseteq \Lambda'$. We decompose $(\beta\alpha + \beta p^f\Lambda')$ into a disjoint union of spheres $\gamma + p^f\Lambda'$. Since γ looks like $\text{diag}\ (\gamma_1, \ldots, \gamma_r, 0, \ldots, 0)$ where $\gamma_i = 1 + p^f\lambda_i$ for $\lambda_i \in \Lambda'$, so reducing γ in the same way as was done for α , we can make γ look like

$\gamma = \text{diag}\ (1, 1, \ldots, 1, 0, \ldots, 0)$ with r ones on the diagonal.

It is now clear that we only need to show that

$$L_A(s,\psi)^{-1} = \int_{(\gamma+p^f\Lambda')\cap A^*} \psi(x)\ \|x\|^s\ d^*x$$

lies in $\mathbb{C}[p^s, p^{-s}]$ (where γ is as above).

Step III Let U be the set of all $x \in \Lambda'^{*}$ such that $x(\gamma+p^f\Lambda') = \gamma+p^f\Lambda'$ This is an open subgroup of λ' of finite index as U is precisely $1 + p^f\Lambda'$. We then have

$$\int_{(\gamma+p^f\Lambda')\cap A^*} \|x\|^s \, d^*x = (\Lambda'^*:U)^{-1} \int_{\Lambda'^*.(\gamma+p^f\Lambda')\cap A^*} \|x\|^s \, d^*x$$

We now decompose $\Lambda'^*.(\gamma + p^f\Lambda'^*) \cap A^*$ as a disjoint union of Λ'^* orbits ,say Λ'^*y. Each orbit contains a unique element y in the "Hermite normal form" looking like

$$y = \left[\begin{array}{ccc|ccc} 1 & \cdots & & & * & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & \xi^j & & * \\ & 0 & & & \ddots & \\ & & & 0 & & \xi^j \end{array}\right]$$

It follows that the integral of the R.H.S of the last equality can be written as $\mu^*(\Lambda'^*) \sum_y \psi(y) \|y\|^s$, where the sum is over the representatives of distinct orbits Λ'^*y and each y is in the "Hermite normal form". Therefore it can be summed as in the proof of the lemma 3.2.9 to give a series in $\mathbb{C}[p^s,p^{-s}]$. This completes the unramified case.

Case II Let ψ be ramified, that is ψ is not trivial on Λ'^* for a maximal order in A. In this case the assertion of 3.2.7 is that $L_A(s,\psi)$ is identically 1. So (C1) surely holds. Now we verify (C2). First note that any small neighbourhood N of $1 \in S^1$ contains no subgroup except {1}. Since ψ is continuous on A^* so $\psi|\Lambda'^*$ is continuous on Λ'^*. Hence there exists a neighbourhood N' of $1 \in \Lambda'^*$ such that $\psi(N') \subseteq N$. We can choose f sufficiently large such that $(1 + \rho^f\Lambda') \subseteq N'$. This is possible as $1 + \rho^f\Lambda'$ is an open neighbourhood of $1 \in A$ for all f. Now $1 + \rho^f\Lambda'$ is a subgroup of Λ'^*, so $\psi(1 + \rho^f\Lambda')$ is a subgroup of N. Therefore we must have $\psi(1 + \rho^f\Lambda') = \{1\}$ and hence $1 + \rho^f\Lambda' \subseteq \ker \psi$.

Let ϕ be the characteristic function of the set $1 + \rho^f\Lambda'$ in A. Then $\phi \in S(A)$ and $Z(\phi, s, \psi) = \mu^*(1 + \rho^f\Lambda')$. Since $1 + \rho^f\Lambda'$ is open and compact in A so its measure is non zero and finite. This shows that the constant function 1 is of the form $Z(\phi', s, \psi)$ for some $\phi' \in S(A)$ as desired. Now we establish that $Z(\phi, s, \psi) \in \mathbb{C}[p^s, p^{-s}]$ for every $\phi \in S(A)$. As in Case I it suffices to do this when ϕ is the characteristic function of the set $\gamma + p^l\Lambda'^*$ $(\rho^f = p^l R)$, where $\gamma = \text{diag}\ (1, 1, \ldots, 1, 0, 0, \ldots, 0) \in \Lambda' = M_k(\Delta)$. Suppose that the first r diagonal entries are 1 and the rest 0. If $r = k$ then $\gamma = 1_A$ and we have just now seen that then $Z(\phi, s, \psi)$ is a constant. On the other hand if $r < k$, then we choose $y \in \Lambda'^*$ of the form $y = \text{diag}\ (1, 1, \ldots, 1, u)$ with $u \in \Delta^*$ chosen such that

$$\psi(y) = \tilde{\psi}(\, nr_{A/C}y) = \tilde{\psi}(\, nr_{D/C}u) \neq 1.$$

We can find such a "u" as $\tilde{\psi}$ is ramified and $nr(\Delta^*) = R^*$ by 3.2.4 . Then since $\phi(xy) = \phi(x)$ for $x \in A$ and $\|y\| = 1$ so we have

$$Z(\phi, s, \psi) = \int_{A^*} \phi(x)\ \psi(x)\ \|x\|^s\ d^*x$$

$$= \int_{A^*} \phi(xy)\ \psi(xy)\ \|xy\|^s\ d^*x$$

$$= \psi(y)\ Z(\phi, s, \psi)$$

But $\psi(y) \neq 1$, which forces $Z(\phi, s, \psi)$ to be zero. Hence (C3) is satisfied and the proof of 3.2.6 is complete. ∎

Having given an explicit formula for $L_A(s,\psi)$ in the local case, we are now in a position to compare $L_A(s,\psi)$ with $L_\Lambda(M,s,\psi)$ as defined in 3.1.10. The following theorem gives us the connection between the two.

3.2.13 <u>THEOREM</u>. Let Λ be an order in a finite dimensional semisimple $\mathbb{Q}_p$-algebra, M a left ideal of Λ, and ψ a character of A^* which is trivial on $\mathrm{Aut}_\Lambda M$. Then there exists a polynomial $f(s) \in \mathbb{Z}[\psi][p^{-s}]$, ($\mathbb{Z}[\psi]$ is the ring obtained by adjoining to $\mathbb{Z}$ all the values of ψ), such that

$$L_\Lambda(M,s,\psi) = f(s)\ L_A(s,\psi)$$

Further, f(s) is identically 1 if Λ is a maximal order.

Proof: Note that the values of ψ are roots of unity. For Real $s > 1$, we set $f(s) = L_\Lambda(M,s,\psi)/L_A(s,\psi)$. Both these functions lie in the ring $\mathbb{Z}[\psi][[p^{-s}]]$ of formal power series in p^{-s} with coefficients in the ring $\mathbb{Z}[\psi]$. Further,

$$L_A(s,\psi)^{-1} \in \mathbb{Z}[\psi][p^{-s}] \quad \text{.......... (i)}$$

On the other hand we have by 3.1.11

$$L_\Lambda(M,s,\psi) = \mu^*(\Gamma^*).(\Lambda:M) \int_{A^*} \phi(x)\ \psi(x)\ \|x\|^s\ d^*x$$

where ϕ is the characteristic function of $\{M:\Lambda\}$ in A, $\Gamma = \mathrm{End}_\Lambda M$. Thus we can find a $\phi' \in S(A)$ such that $L_\Lambda(M,s,\psi) = Z(\phi',s,\psi)$ and therefore by (C3) we have

$$f(s) \in \mathbb{C}[p^s,p^{-s}] \quad \text{.......... (ii)}$$

From (i) and (ii) we get

$$f(s) \in \mathbb{C}[p^s,p^{-s}] \cap \mathbb{Z}[\psi][[p^{-s}]] = \mathbb{Z}[\psi][p^{-s}]$$

For the second assertion, if Λ is a maximal order then $M \cong \Lambda$ as Λ-modules ([MO] 18.10) and so $L_\Lambda(M,s,\psi) = L_\Lambda(\Lambda,s,\psi)$. Also when Λ is a maximal order, $\mathrm{End}_\Lambda M$ is also a maximal order and so ψ is trivial on $\mathrm{Aut}_\Lambda M$ implies that $\psi|A_i^*$ is also unramified ($A = \prod_{i=1}^{r} A_i$, A_i are the simple components of A). Therefore by 3.2.3 and 3.2.10 we get $L_A(s,\psi) = L_\Lambda(\Lambda,s,\psi) = L_\Lambda(M,s,\psi)$, and so $f(s) = 1$. ■

3.2.13 <u>COROLLARY</u>. Each local L-function admits analytic continuation to a meromorphic function on the whole s-plane. Further, there exists $\varepsilon > 0$ such that $L_\Lambda(M,s,\psi)$ is holomorphic for Real $s \geq 1-\varepsilon$ for all $\psi \in S(A)$.

Proof: For Real $s > 1$ we have

$$L_\Lambda(M,s,\psi) = f(s)\, L_A(s,\psi) \quad \text{and} \quad L_A(s,\psi) = g(p^{-s})^{-1}$$

where $f(s) \in \mathbb{Z}[\psi][p^{-s}]$ and $g(p^{-s}) \in \mathbb{C}[p^{-s}]$ with $g(0) = 1$.

Since $f(s)/g(p^{-s})$ is meromorphic in the whole s-plane, it is possible to define $L_\Lambda(M,s,\psi)$ for Real $s \leq 1$ as $f(s)/g(p^{-s})$. This is the desired analytic continuation. Also since there exists $\varepsilon > 0$ such that $L_A(s,\psi)$ is holomorphic for Real $s \geq 1-\varepsilon$ the last statement follows. ■

Similiarly one can prove

3.2.14 <u>COROLLARY</u>. Let A be a finite dimensional semi-simple $\mathbb{Q}_p$-algebra and ψ a continuous character of A^* of finite order. Then the function $Z(\phi,s,\psi)$ defined as in 3.2.1 admits analytic continuation to a meromorphic function of s. Moreover, there exists $\varepsilon > 0$ such that for all $\phi \in S(A)$ the integral 3.2.1 converges to a holomorphic function of s in the region Real $s \geq 1-\varepsilon$. ■

(B) <u>Global standard L- function</u>.

In this subsection we define a global standard L- function. Thus assume for this that A is a finite dimensional semisimple $\mathbb{Q}$- algebra and ψ a continuous character of $J_f(A)$ of finite order which is trivial on A^*. The idea is to form Euler product of the various local standard L- functions $L_{A_p}(s,\psi_p)$ and show that the product converges in a certain region. Then the product can be defined to be the global standard L- function $L_{\mathbb{A}}(s,\psi)$ as usual. For the convergence one has to link up our L- function with the classical Hecke L- function. The precise formulation of the result is the following.

3.2.16 <u>THEOREM</u> Let $A = \prod_{i=1}^{r} A_i$, where A_i are the simple components of the algebra A. Let $\psi_i = \psi|J_f(A_i)$, $1 \le i \le r$. Then

(i) The Euler product $\prod_p L_{A_p}(s,\psi_p)$ converges absolutely and uniformly on compact sets in the region Real $s > 1$, to a holomorphic function $L_{\mathbb{A}}(s,\psi)$. Moreover,

3.2.17 $$L_{\mathbb{A}}(s,\psi) = \prod_{i=1}^{r} L_{A_i}(s,\psi_i)$$

(ii) The function $L_{\mathbb{A}}(s,\psi)$ admits analytic continuation to a meromorphic function of s on the whole s- plane.

(iii) Let $t = t(\psi)$ be the number of indices i for which ψ_i is the trivial character. Then $L_{\mathbb{A}}(s,\psi)$ has a pole of order t at $s = 1$. If $t(\psi) = 0$ then $L_{\mathbb{A}}(s,\psi)$ is an entire function of s, and $L_{\mathbb{A}}(1,\psi) \neq 0$.

Proof: We claim that once the simple case is proved, the semisimple case is easy (except possibly for iii), for then by 3.2.2

$$L_{A_p}(s,\psi_p) = \prod_{i=1}^{r} L_{A_{i,p}}(s,\psi_{i,p})$$

By the simple case we know that the Euler products for the $L_{A_i}(s,\psi_i)$ converge absolutely for Real $s > 1$ and so we can rearrange them to give

$$\prod_{i=1}^{r} L_{A_p}(s,\psi_i) = \prod_{i=1}^{r} \prod_{p} L_{A_{i,p}}(s,\psi_{i,p}) = \prod_{p} \prod_{i=1}^{r} L_{A_{i,p}}(s,\psi_{i,p}) = \prod_{p} L_{A_p}(s,\psi_p)$$

and so $\prod_p L_{A_p}(s,\psi_p)$ is also convergent absolutely and uniformly on compact sets for Real $s > 1$. Since

3.2.18 $$\prod_p L_{A_p}(s,\psi_p) = L_A(s,\psi) \qquad \text{Real } s > 1$$

by definition, the theorem follows in the general case by its validity in the case of simple algerba.

Therefore, it is enough to prove the theorem for simple $\mathbb{Q}$-algebra A. As usual let C denote the centre of A, and R the valuation ring of C. Suppose $\dim_C A = n^2$ and $\tilde{\psi}$ the idele class character of C such that $\psi = \tilde{\psi} \circ nr_{A/C}$ as in 3.1.9. For each prime number p, we have $A_p = A \otimes \mathbb{Q}_p = \prod_{\rho | p} A_\rho$, where the product is taken over all maximal ideals ρ of R lying over p. As A is simple each A_ρ is therefore a central simple C_ρ- algebra of dimension n^2 and is isomorphic to a full ring of matrices over some central C_ρ- division algebra of dimension e_ρ^2 over C_ρ. Then $e_\rho > 1$ only for finitely many ρ ([MO] 32.1). We also have by applying 3.2.3 to the C_p- algebra A_p

3.2.19 $$L_{A_p}(s,\psi_p) = \prod_{\rho | p} L_{A_\rho}(s,\psi_\rho)$$

where as before $\psi_\rho = \psi_p \mid A_\rho^*$.

Then we have the Euler product

3.2.20 $$\prod_{p} L_{A_p}(s,\psi_p) = \prod_{\rho} L_{A_\rho}(s,\psi_\rho)$$

where ρ ranges over all the maximal ideals of R and p over all rational primes. We have seen in the local case 3.2.8 that

3.2.21 $$L_{A_\rho}(s,\psi_\rho) = \prod_{\substack{j=0 \\ j\equiv 0 \bmod e}}^{n-1} L_{C_\rho}(ns-j,\tilde{\psi}_\rho)$$

with $L_{C_\rho}(s,\tilde{\psi}_\rho)$ defined as 3.2.7. It is natural to consider the product

3.2.22 $$L_C(s,\tilde{\psi}) = \prod_{\rho} L_{C_\rho}(s,\tilde{\psi}_\rho)$$

where ρ ranges over all maximal ideals of R. Then $L_C(s,\psi)$ is the classical Hecke L- function attached to the idele class character $\tilde{\psi}$ of C ([SL] Ch. XII §1).

We note the following properties of $L_C(s,\tilde{\psi})$ and $\prod_{\rho} L_{C_\rho}(s,\psi_\rho)$.

(Li) The product $\prod_{\rho} L_{C_\rho}(s,\tilde{\psi}_\rho)$ converges absolutely and uniformly on compact sets, in the region Real s > 1 ([SL] Ch. XIII §1, Ch.VIII §3 Th.7)

(Lii) The function $L_C(s,\psi)$ defined by $\prod_{\rho} L_{C_\rho}(s,\psi_\rho)$ admits analytic continuation to a meromorphic function of s, with no zeros in the region Real s > 1 ([SL] Ch. XIII §3 Th.1, Ch. VIII §3 Th.7).

(Liii) If $\tilde{\psi}$ is the trivial character, then $L_C(s,\tilde{\psi})$ has a pole at s = 1, while otherwise $L_C(s,\tilde{\psi})$ is entire and $L_C(1,\tilde{\psi}) \neq 0$ ([SL]

Ch. VIII §3 Th.8, §2 Th.5c).

From (Li) we see that $\prod_{\rho} L_{A_\rho}(s,\psi_\rho)$ converges uniformly and absolutely on compact sets for Real $s > 1$ and thus $L_{\mathbb{A}}(s,\psi) = \prod_{p} L_{A_p}(s,\psi_p)$ where p ranges over all rational primes.

To complete the proof of the theorem we now go from the other direction to link up $L_{\mathbb{A}}(s,\psi)$ with the various L_C we have introduced. To facilitate matters we introduce the following "auxiliary functions".

3.2.23 $$K_A(s,\psi) = \prod_{j=0}^{n-1} L_C(ns-j, \tilde{\psi})$$

3.2.24 $$K_{A_\rho}(s,\psi) = \prod_{j=0}^{n-1} L_{C_\rho}(ns-j, \tilde{\psi}_\rho)$$

The properties (Li), (Lii) and (Liii) imply that

(Ki) The Euler product $\prod_{\rho} K_{A_\rho}(s,\psi_\rho)$ converges absolutely, uniformly on compact sets to $K_A(s,\psi)$ in the region Real $s > 1$.

(Kii) $K_A(s,\psi)$ admits analytic continuation to a meromorphic function of s on the whole s- plane.

(Kiii) If ψ is the trivial character (equivalently $\tilde{\psi}$ is trivial), then $K_A s,\psi)$ has a simple pole at $s = 1$. Otherwise $K_A(s,\psi)$ is an entire function of s and $K_A(1,\psi) \neq 0$.

On the other hand, comparing 3.2.23 and 3.2.24 with 3.2.21 and 3.2.22 we see that

$$L_{\mathbb{A}}(s,\psi) = \prod_{\rho} L_{A_\rho}(s,\psi_\rho) = K_A(s,\psi) \prod_{\rho} f_\rho(s)$$

where for each ρ, $f_\rho(s)$ is defined as follows:

$$f_{\wp}(s) = \prod_{\wp} \left(1 - \tilde{\psi}_{\wp}(\pi_{\wp})\ N\wp^{-(ns-j)}\right) \quad \text{if } \tilde{\psi}_{\wp} \text{ is unramified}$$

$$f_{\wp}(s) = 1 \quad \text{if } \tilde{\psi}_{\wp} \text{ is ramified}$$

(Here $\pi_{\wp}$ denotes some prime element of the discrete valuation ring $R_{\wp}$) Thus $f_{\wp}(s)$ is a polynomial in p^{-s}, where p is a prime number lying below $\wp$. For each maximal ideal $\wp$ of R, therefore $f_{\wp}(1) \neq 0$. Further, $f_{\wp}(s)$ is identically 1 unless both conditions $e_{\wp} > 1$ and $\tilde{\psi}_{\wp}$ unramified are satisfied. But there are only finitely many $\wp$ for which $e_{\wp} > 1$ ([MO] 32.1). And so for almost all $\wp$, $f_{\wp}(s) = 1$ identically. Hence it follows that $L_{A}(s,\psi)/K_{A}(s,\psi)$ is a finite product of polynomials $f_{\wp}(s)$ and therefore can be continued to an entire function of s which does not vanish at s = 1. Consequently the relation $L_{A}(s,\psi) = K_{A}(s,\psi) \prod_{\wp} f_{\wp}(s)$ allows us to analytically continue $L_{A}(s,\psi)$ to a meromorphic function on the whole of s- plane with a simple zero at s = 1 if ψ is trivial, and otherwise to an entire function of s with $L_{A}(1,\psi) \neq 0$. This completes the proof for the simple case.

Property (iii) for semisimple case now follows. For, note that $L_{A_i}(s,\psi_i)$ has a pole of order 1 at s = 1 whenever $\psi_i = \psi|A_i^*$ is trivial; otherwise it is entire with $L_{A_i}(1,\psi_i) \neq 0$. So if $t = t(\psi)$ be the number of indices for which $\psi_i = \psi|A_i^*$ is trivial, then $L_{A}(s,\psi) = \prod_{i=1}^{r} L_{A_i}(s,\psi_i)$ implies property (iii). This ends the proof of Theorem 3.2.16. ■

§3 BEHAVIOUR OF GLOBAL L- FUNCTIONS, $\zeta_{\Lambda}(s)$, $Z_{\Lambda}(g(M), s)$ AND $Z_{\Lambda}([M], s)$ AT s = 1.

By now we have accumulated enough knowledge about the behaviour of the global standard L- functions to undertake the study of the behaviour mentioned at the title of the section, because all such funtions are related to the global L- function in a very simple manner.

Therefore assume that Λ is a $\mathbb{Z}$-order in a finite dimensional semisimple $\mathbb{Q}$- algebra A, M a left ideal of Λ, and ψ is a continuous character of $J_f(A)$ of finite order, which is trivial on $J_f'(A).A^*.U_f(\Gamma)$. In this situation we have

3.3.1 THEOREM

(i) The global L- function $L_\Lambda(M, s, \psi)$ admits analytic continuation to a meromorphic function on the whole s- plane.

(ii) Let $A = \prod_{i=1}^{r} A_i$, where A_i are simple components of the algebra A and let ψ_i be the restrictions of ψ to $J_f(A)$ for $1 \le i \le r$. If exactly $t = t(\psi)$ of the characters ψ_i are trivial, then $L_\Lambda(M, s, \psi)$ has a pole of order atmost t at $s = 1$.

(iii) If the character ψ is non-trivial, then $L_\Lambda(M, s, \psi)$ has a pole of order atmost $r-1$ at $s = 1$. If $t(\psi) = 0$, then $L_\Lambda(M, s, \psi)$ is an entire function.

(iv) If Λ is a maximal order then $\psi_p = \psi|A_p^*$ is unramified for all p and $L_\Lambda(M, s, \psi) = L_A(s,\psi)$.

Proof: It is clear that the proof depends on the relation between $L_A(s,\psi)$ and $L_\Lambda(M, s, \psi)$. From the Euler products 3.1.13 and 3.2.18 we have for Real $s > 1$

$$L_\Lambda(M, s, \psi)/ L_A(s,\psi) = \prod_p f_p(s)$$

where for each prime number p,

$$f_p(s) = L_{\Lambda_p}(M_p, s, \psi_p) / L_{A_p}(s, \psi_p)$$

By 3.2.13 each $f_p(s)$ is a polynomial in p^{-s} with co-efficients in $\mathbb{Z}[\psi]$. Further $f_p(s) = 1$ whenever Λ_p is a maximal order. Therefore $f_p(s) = 1$ for almost all p. Thus we obtain

3.3.2 $$L_\Lambda(M, s, \psi) = L_A(s,\psi) \prod_p f_p(s)$$

where the product $\prod f_p(s)$ is a finite product. So in view of the properties of $L_A(s,\psi)$ in 3.2.16, the identity 3.3.2 immediately implies the assertions (i), (ii) and (iii) of the theorem. For assertion (iv) we know that if Λ is a maximal order then $End_\Lambda M$ is also a maximal order and hence ψ_p is trivial on $Aut_{\Lambda_p} M_p$ the unit group of the maximal order $End_{\Lambda_p} M_p$. Therefore each ψ_p is unramified. Now (iv) follows from the Euler products of $L_\Lambda(M, s, \psi)$ and $L_A(s,\psi)$ and 3.2.13. ∎

3.3.3 <u>COROLLARY</u> Notations as in the preceeding theorem. The partial zeta function $Z_\Lambda([M], s)$ defined in §1 admits analytic continuation to a meromorphic function of s, with a pole of order atmost r at s = 1.

Proof: By 3.1.5 we have $Z_\Lambda([N], s) = h^{-1} \sum_\psi \psi^{-1}([N]) L_\Lambda(M, s, \psi)$ for each stable isomorphism class [N] in g(M). Here ψ ranges over all characters of Cl(M) and $h = |Cl(M)|$. So by Theorem 3.3.1 the corollary follows. ∎

Note that this treatment does not give any information about the continuation of $Z_\Lambda(M, s)$. But in 1984 BUSHNELL and REINER in [BR4] showed in a completely different way that

3.3.4 <u>THEOREM</u> The partial zeta function $Z_{\Lambda}(M, s)$ defined in §1 admits analytic continuation to a meromorphic function of s to the whole s-plane.

For the proof of this theorem refer to [BR4]. The proofs of two special cases, namely when the algebra satisfies Eichler condition and when A is a totally definite quaternion algebra has been done in [BR2]. ■

Next we come to the order of the poles of the various zeta functions considered till now at s = 1. It turns out that the order of the pole at s = 1 of each $\zeta_{\Lambda}(s)$, $Z_{\Lambda}(g(M), s)$ and $Z_{\Lambda}([M], s)$ is exactly r.

Let Λ′ denote a maximal order in A. Since every left ideal of Λ′ lies in the genus g(Λ′), we have

$$\zeta_{\Lambda'}(s) = Z_{\Lambda'}(g(\Lambda'), s) = L_{\Lambda'}(\Lambda', s, \psi_o)$$

where ψ_o is the trivial character of Cl(Λ′) (or of $J_f(A)$). But by 3.3.1 (iv) we have $L_{\Lambda'}(\Lambda', s, \psi_o) = L_{A}(s,\psi_o)$ and so from 3.2.15 it follows that $\zeta_{\Lambda'}(s)$ has a pole of order r at s = 1. Also for any order Λ in A we have

$$Z_{\Lambda}(g(M), s) = \sum_{i=1}^{h} Z_{\Lambda}([M_i], s)$$

where $[M_1], \ldots, [M_h]$ are the stable isomorphism classes in g(M) and so by 3.3.3. we get that $Z_{\Lambda}(g(M), s)$ has a pole of order atmost r at s = 1. Here r denotes the number of simple components of A. Therefore

$$d_M = \lim_{s \to 1} Z_{\Lambda}(g(M), s)/ \zeta_{\Lambda'}(s)$$

is a non-negative constant and independent of the choice of Λ′.

The following theorem gives an explicit and nice formula for d_M, which shows that $d_M > 0$. The fact that $d_M > 0$ will then at once give us that the order of the pole of $Z_\Lambda(g(M), s)$ at $s = 1$ is exactly r.

3.3.5 <u>THEOREM</u> Let Λ be an order in the finite dimensional semisimple $\mathbb{Q}$- algebra A, Λ' a maximal order in A, and M a left ideal of Λ. Then

3.3.6 $$d_M = (\Lambda:M)^{-1} (\{M:\Lambda\}:\Lambda') \prod_p (\Lambda'^{*}:\mathrm{Aut}_{\Lambda_p} M_p)$$

so that $d_M > 0$.

Proof: Since $\mathrm{End}_\Lambda M$ is an order in A, so for almost all p, $\mathrm{End}_{\Lambda_p} M_p = \Lambda'_p$. Consequently $(\Lambda'^{*}:\mathrm{Aut}_{\Lambda_p} M_p) = 1$ for almost all p showing that the product $\prod_p (\Lambda'^{*}:\mathrm{Aut}_{\Lambda_p} M_p)$ is well defined. On the other hand for each rational prime p we have $Z_{\Lambda_p}(M_p, s) = \zeta_{\Lambda'_p}(s)\, f_p(s)$ for some polynomial $f_p(s) \in \mathbb{Z}[p^{-s}]$ (by 3.2.13) which is identically 1 if Λ_p is a maximal order. Taking the Euler products of $Z_\Lambda(g(M), s)$ and $\zeta_{\Lambda'}(s)$ for Real $s > 1$, we therefore get that

$$Z_\Lambda(g(M), s)/\zeta_{\Lambda'}(s) = \prod_p f_p(s)$$

and the product is a finite one . By analytic continuation, the same formula holds for all s, and we have $d_M = \prod_p f_p(1)$. Therefore it is sufficient to prove that for each p,

$$f_p(1) = (\Lambda_p:M_p)^{-1} (\{M_p:\Lambda_p\}:\Lambda'_p)\ (\Lambda'^{*}: \mathrm{Aut}_{\Lambda_p} M_p)$$

For this we need the following formula of 3.1.12

$$Z_{\Lambda_p}(M_p, s) = \mu^{*}(\mathrm{Aut}_{\Lambda_p} M_p)^{-1} (\Lambda_p:M_p)^{-s} \int_{A_p^{*}} \phi_p(x)\ \|x\|^{s}\ d^{*}x$$

where ϕ_p is the characteristic function of $\{M_p:\Lambda_p\}$. This is valid in the region Real $s > 1-\varepsilon$ for some $\varepsilon > 0$ (see 3.2.14)

Our aim is to evaluate this integral at $s = 1$. We begin by changing the measure d^*x into the additive Haar measure dx of A_p (by the formula $d^*x = \|x\|^{-1}dx$) and then using the Proposition 1.2.16 of Chapter 1 which says that for any finite dimensional semisimple $\mathbb{Q}_p$- algebra A the measure of $A\backslash A^*$ is zero with respect to any measure. Thus we have (ϕ_p same as before)

$$\int_{A_p^*} \phi_p(x) \; \|x\|^s \; d^*x = \int_{\{M_p:\Lambda_p\}\cap A^*} \|x\|^{s-1} \, dx = \int_{\{M_p:\Lambda_p\}} \|x\|^{s-1}dx$$

so that the value of the integral is $\mu(\{M_p:\Lambda_p\})$ at $s = 1$ (μ denotes the measure of a set with respect to dx). Therefore

$$Z_{\Lambda_p}(M_p, 1) = \mu^*(\mathrm{Aut}_{\Lambda_p} M_p)^{-1} \; (\Lambda_p:M_p)^{-1} \; \mu(\{M_p:\Lambda_p\})$$

$$\zeta_{\Lambda'_p}(s) = \mu^*(\Lambda'_p)^{-1} \int_{A_p^*} \phi(x) \; \|x\|^s \; d^*x$$

where ϕ is the charateristic function of Λ'. We similarly have

$$\zeta_{\Lambda'_p}(1) = \mu^*(\Lambda'_p)^{-1} \; \mu(\Lambda'_p)$$

These two formulas allow us to compute $f_p(1)$ as shown below:

$$\begin{aligned} f_p(1) &= Z_{\Lambda_p}(M_p, 1) \;/\; \zeta_{\Lambda'_p}(1) \\ &= \mu^*(\Lambda'^*_p) \; \mu^*(\mathrm{Aut}_{\Lambda_p} M_p)^{-1} \; \mu(\{M_p:\Lambda_p\}) \; \mu(\Lambda'_p)^{-1} \; (\Lambda_p:M_p)^{-1} \end{aligned}$$

and by a basic fact in elenentary measure theory we get

$$\begin{cases} \mu^*(\Lambda'^*_p)/\;\mu^*(\mathrm{Aut}_{\Lambda_p} M_p) = (\; \Lambda'^*_p : \mathrm{Aut}_{\Lambda_p} M_p \;) \\ \mu(\Lambda'_p)/\;\mu(\{M_p:\Lambda_p\}) \quad = (\Lambda'_p \; : \; \{M_p:\Lambda_p\}) \end{cases}$$

Hence the Theorem ∎

The following proposition tells us about the order of the pole of $Z_\Lambda(M, s)$ at $s=1$.

3.3.7 PROPOSITION The function $Z_\Lambda(M, s)$ has a pole of order precisely r at $s=1$, where r is the number of simple components of the algebra A.

Proof: Let Λ and Δ be orders in A, and let M be a left ideal of Λ and N a left ideal of Δ. Then for Real $s > 1$

$$Z_\Lambda(M, s) = \sum_{\substack{X \subseteq \Lambda \\ X \cong M}} (\Lambda:X)^{-s} = (\Lambda:M)^{-s} \sum_x \|x\|^s$$

where x ranges over the orbits of $\{M:\Lambda\} \cap A^*$ modulo $\mathrm{Aut}_\Lambda M$ (the action considered here is the left one). Similarly

$$Z_\Delta(N, s) = (\Delta:N)^{-s} \sum_y \|y\|^s$$

the sum extending over orbit representatives y of $\{N:\Delta\} \cap A^*$ modulo $\mathrm{Aut}_\Delta N$. Now we consider the group

$$W = \mathrm{Aut}_\Lambda M \cap \mathrm{Aut}_\Delta N$$

Since W is the group of units of the order $\mathrm{End}_\Lambda M \cap \mathrm{End}_\Delta N$, it follows from 1.2.7 that W is of finite index in both $\mathrm{Aut}_\Lambda M$ and $\mathrm{Aut}_\Delta N$. So we get

$$Z_\Lambda(M, s) = c_1 (\Lambda:M)^{-s} \sum_{x \in W \backslash (\{M:\Lambda\} \cap A^*)} \|x\|^s_*$$

for some positive constant c_1. Similarly there is a $c_2 > 0$ such that

$$Z_\Delta(N, s) = c_2 (\Delta:N)^{-s} \sum_{y \in W \backslash (\{N:\Delta\} \cap A^*)} \|y\|^s_*$$

Since $\{N:\Delta\}$ and $\{M:\Lambda\}$ are full $\mathbb{Z}$-lattices we can find a positive integer q such that $q\{N:\Delta\} \subseteq \{M:\Lambda\}$. Then if s is real and $s > 1$ and $\dim_{\mathbb{Q}} A = n$ then

$$Z_\Delta(N, s) = c_2 q^{ns} (\Delta:N)^{-s} \sum_{y \in W \setminus (\{N:\Delta\} \cap A^*)} \|qy\|^s$$

$$= c_2 q^{ns} (\Delta:N)^{-s} \sum_{y \in W \setminus (q\{N:\Delta\} \cap A^*)} \|y\|^s$$

$$\leq c_2 q^{ns} (\Delta:N)^{-s} c_1^{-1} (\Lambda:M)^s \; Z_\Lambda(M, s)$$

Therefore given $\varepsilon > 0$ we can find $c > 0$ such that

3.3.8 $$Z_\Delta(N, s) \leq c \, Z_\Lambda(M, s) \quad \text{for } 1 < s < 1+\varepsilon$$

Now let $\{M_i\}$ be any finite collection of left ideals of Λ and $\{N_j\}$ a finite collection of left ideals of Δ. Summing the inequalities 3.3.8 for M_i and N_j we get $c' > 0$ such that

$$\sum Z_\Delta(N_j, s) \leq c' \sum Z_\Lambda(M_i, s)$$

The reverse inequality, with a different constant factor holds by symmetry and hence any two such sums $\sum Z_\Delta(N_j, s)$ and $\sum Z_\Lambda(M, s)$ which are meromorphic at s=1 must have the same number of poles. But we already know that if Λ any order then there are only a finite number of isomorphism classes of left Λ-ideals (by Jordan Zassenhaus theorem). So in particular if Λ' is a maximal order then

$$\zeta_{\Lambda'}(s) = \sum_{i=1}^{k} Z_{\Lambda'}(M_i, s)$$

and $\zeta_{\Lambda'}(s)$ has a pole of order exactly r at s=1. The proposition now follows at once. ■

We complete this chapter by investigating the behaviour of $\zeta_{\Lambda'}(s)$ and $Z_\Lambda(g(M), s)$ near $s = 1$.

3.3.9 <u>THEOREM</u> Let Λ be an order in a finite dimensional semisimple $\mathbb{Q}$- algebra A, and suppose A has r simple components. Let Λ' be a maximal order in A, and let

$$\zeta_{\Lambda'}(s) = c_A (s-1)^{-r} + c_A^1 (s-1)^{1-r} + \dots\dots\dots$$

be the Laurent expansion of $\zeta_{\Lambda'}(s)$ about $s = 1$. That is

$$c_A = \lim_{s \to 1} (s-1)^r \zeta_{\Lambda'}(s)$$

Let M be a left ideal of Λ, and h_M be the number of stable isomorphism in the genus $g(M)$. Then

(i) $Z_\Lambda(g(M), s)$ and $\zeta_\Lambda(s)$ have a pole of order exactly r at $s = 1$ and the Laurent expansion of $Z_\Lambda(g(M), s)$ about $s = 1$ has leading term $d_M\, c_A\, (s-1)^{-r}$.

(ii) For each stable isomorphism class $[N]$ in $g(M)$ the partial zeta function $Z_\Lambda([N], s)$ has a pole of order r at $s = 1$, and its Laurent expansion about $s = 1$ has leading term $c_A\, d_M\, h_M^{-1}\, (s-1)^{-r}$.

Proof: (i) Since $\zeta_\Lambda(s) = \sum_M Z_\Lambda(g(M), s)$ where M ranges over a finite set of representatives of the genera of left ideals of Λ, and each $Z_\Lambda(g(M), s)$ has a pole of order exactly r at $s = 1$, so the same is true for $\zeta_\Lambda(s)$. The second assertion of (i) follows as

$$d_M = \lim_{s \to 1} Z_\Lambda(g(M), s) / \zeta_{\Lambda'}(s)$$

(ii) We have by 3.1.5

$$Z_\Lambda([N], s) = h_M^{-1} \sum_\psi \psi^{-1}([N])\, L_\Lambda(M, s, \psi)$$

where ψ ranges over all characters of $Cl(M)$. On the other hand $L_\Lambda(M, s, \psi_0) = Z_\Lambda(g(M), s)$ where ψ_0 is the trivial character of $Cl(M)$. Therefore $Z_\Lambda([N], s)$ has a pole of order exactly r at $s = 1$ by (i) above and 3.3.1. The last assertion of (ii) is now clear. ∎

3.3.10 <u>COROLLARY</u> Notations as in the theorem. Then we have

$$\lim_{s \to 1} (s - 1)^r \zeta_\Lambda(s) = c_A \sum_M d_M$$

where M ranges over a finite set of representatives of the genera of left ideals of Λ. ∎

— * —

CHAPTER 4

Introduction: In this chapter, we continue working with the zeta integrals introduced in the last chapter in order to establish their functional equation. The functional equations of L-functions of orders will be a consequence. The functional equation for a zeta integral gives rise to a "local constant" or symmetry factor. This factor will be evaluated in terms some "non-abelian congruence Gauss sum". As in earlier chapters, the subject matter is a generalisation of the results related to zeta integrals in the local field case (as in Tate's thesis).

§1 LOCAL FUNCTIONAL EQUATIONS AND GAUSS SUMS

Let A be a finite dimensional semisimple $\mathbb{Q}_p$- algebra and ψ a continuous character of A^*. Our main aim here is to show that $Z(\phi, s, \psi)$ satisfies a functional equation.

We begin with the special case of simple algebras. Let A be a finite dimensional simple $\mathbb{Q}_p$- algebra, with centre C. Let θ_A denote the cannonical continuous character of the additive group of A defined by

4.1.1 $$\theta_A(x) = \exp(2\pi i \, tr_{A/\mathbb{Q}_p}(x)) \quad ; x \in A$$

When there is no chance of confusion we will denote θ_A by just θ.

In 4.1.1 $tr_{A/\mathbb{Q}_p}$ is the absolute reduced trace as defined in 1.2.26. For $\alpha \in \mathbb{Q}_p$, we interpret exp ($2\pi i\alpha$) as exp ($2\pi i\alpha_o$), where $\alpha_o \in \mathbb{Q}$ is the principal part of α. The pairing $A \times A \longrightarrow \mathbb{C}^*$ give by $(x, y) \longrightarrow \theta_A(xy)$, $x, y \in A$ is nondegenerate and symmetric as $tr_{A/\mathbb{Q}_p}$ is so. So as in [G] 7.1.10 it allows us to identify the locally compact abelian group A with its Pontrjagin dual $\hat{A}$ via $\theta_A(x)$ for a fixed $x \in A$. Let dx be the self dual Haar measure on A for this identification. As already pointed out in Chapter I, a self dual measure is one for which Fourier inversion formula

$$\hat{\hat{\phi}}(x) = \phi(-x) \ , \ \phi \in S(A), \ x \in A$$

holds.

We can now prove the desired functional equation.

4.1.2 <u>THEOREM</u> Let A be a finite dimensional simple $\mathbb{Q}_p$- algebra, and let ψ be a continuous character of A^* of finite order. For $f \in S(A)$ consider the zeta integral

$$Z(f, s, \psi) = \int_{A^*} f(x)\, \psi(x)\, \|x\|^{s} d^* x$$

Then for $g \in S(A)$ and for all s

$$Z(f, s, \psi)\, Z(\hat{g}, 1-s, \bar{\psi}) = Z(\hat{f}, 1-s, \bar{\psi})\, Z(g, s, \psi)$$

where $\bar{\psi}$ is the complex conjugate of ψ, and $\hat{f}$ (similarly $\hat{g}$) is the Fourier transform of f given by

$$\hat{f}(y) = \int_A f(x)\, \theta(xy)\, dy \quad , \ y \in A$$

Proof: The proof is broken up into two cases.

<u>CASE I ψ is unramified</u> This case is a consequence of the proof in([GJ] 9.6 pg 125) by taking in the notation of [GJ], π to be the unramified character ψ of A^* and replacing s by ns, where $n^2 = (A:C)$

C being the centre of A.

<u>CASE II ψ is ramified</u> The above proof fails in the ramified case because then some of the functions $Z(\phi, s, \psi)$ arising in the proof becomes identically zero. But if ψ is ramified then as $L_A(s,\psi) = 1$, the condition (C3) of the standard L-function shows that $Z(\phi, s, \psi)$ converge uniformly for all s. Hence both sides of the above functional equation converge for all s and can be compared directly following the idea of Tate's thesis as shown below.

Let dx be the self dual Haar measure on A. We choose a multiplicative measure on A^* as $d^*x = \|x\|^{-1}dx$.

Then $Z(f, s, \psi)\, Z(\hat{g}, 1-s, \bar{\psi})$

$$= \int_{A^*} f(x)\, \psi(x)\, \|x\|^s\, d^*x \int_{A^*} \hat{g}(y)\, \bar{\psi}(y)\, \|y\|^{1-s}\, d^*y$$

$$= \int_{A^*} \left\{ f(xy)\, \psi(xy)\, \|xy\|^s \int_A \hat{g}(y)\, \bar{\psi}(y)\, \|y\|^{1-s}\, d^*y \right\} d^*x$$

$$= \int_{A^*} \int_A f(xy)\, \hat{g}(y)\, \psi(x)\, \|x\|^s\, d^*x\, dy$$

as by 1.2.16 the measure of $A \setminus A^*$ is zero with respect to any Haar measure. Thus

$$Z(f, s, \psi)\, Z(\hat{g}, 1-s, \bar{\psi}) = \int_{A^*} \left\{ \psi(x)\, \|x\|^s \int_A f(xy)\, \hat{g}(y)\, dy \right\} d^*x$$

Similarly the right hand side of the functional equation becomes

$$Z(\hat{f}, 1-s, \bar{\psi})\, Z(g, s, \psi) = \int_{A^*} \left\{ \psi(x)\, \|x\|^{s-1} \int_A \hat{f}(yx^{-1})\, g(y)\, dy \right\} d^*x$$

where we replace x by x^{-1} after the first step. Now we apply the definition of Fourier transform to obtain

$$\int_A f(xy)\, \hat{g}(y)\, dy = \int_A f(xy) \int_A g(z)\, \theta(yz)\, dz\, dy$$

$$= \int_A g(z) \int_A f(xy)\, \theta(yz)\, dy\, dz$$

But $$\hat{f}(zx^{-1}) = \int_A f(xy)\, \theta(zx^{-1}xy)\, dxy$$

$$= \|x\| \int_A f(xy)\, \theta(yz)\, dy$$

Therefore for all s we have

$$Z(f,\ s,\ \psi)\ Z(\hat{g},\ 1-s,\ \bar{\psi}) = \int_{A^*} \left\{ \psi(x)\ \|x\|^{s-1} \int_A f(yx^{-1})\ g(y)\ dy \right\} d^*x$$

$$= Z(\hat{f},\ 1-s,\ \bar{\psi})\ Z(g,\ s,\ \psi)$$

By our remark about the convergence of zeta integrals, now the functional equation holds for all s. ∎

Now let us define for $\phi \in S(A)$

$$Y(\phi,\ s,\ \psi) = L_A(s,\psi)^{-1}\ Z(\phi,\ s,\ \psi)$$

Then from the defining property (C3) of $L_A(s,\psi)$ (see 3.2.) we see that $Y(\phi,\ s,\ \psi) \in \mathbb{C}[p^s, p^{-s}]$. The functional equation 4.1.2 and shows that there is a meromorphic function $\varepsilon_A(s,\psi)$. independent of ϕ such that for all $\phi \in S(A)$ we have

4.1.3 $$Y(\hat{\hat{\phi}},\ 1-s,\ \bar{\psi}) = \varepsilon_A(s,\psi)\ Y(\phi,\ s,\ \psi)$$

We also observe that $\varepsilon_A(s,\psi)$ is independent of the choice of the multiplicative Haar neasure d^*x.

Our aim, as explained at the outset, is to examine the nature of $\varepsilon_A(s,\psi)$. To that end we need more concepts and notations which we introduce now.

Let $A = M_k(D)$, D a skew field with centre C and $\dim_C D = e^2$. Let Λ be a maximal order in A. We will show that all our calculations are independent of the chioce of Λ, so we take $\Lambda = M_k(\Delta)$, where Δ is the unique maximal order in D. Let β be the unique maximal ideal of Δ. Then the two sided fractional ideals of Λ in A form a free abelian group generated by $\beta\Lambda$ (see([MO] 17.1)). The absolute inverse different D_A^{-1} of A is defined below generalising the same concept for the ring of algebraic integers in an algebraic number field.

4.1.4 *Definition.* $D_A^{-1} = \left\{ x \in A : tr_{A/\mathbb{Q}_p}(xy) \in \mathbb{Z}_p, \text{ for all } y \in \Lambda \right\}$

Note that since Λ is finitely generated over $\mathbb{Z}_p$ by say a basis $\{e_i\}$ of A over $\mathbb{Q}_p$, so D_A^{-1} is also finitely generated over $\mathbb{Z}_p$ by a basis dual to $\{e_i\}$ with respect to the reduced trace form. By symmetry of the reduced trace map, D_A^{-1} is also a two sided Λ-submodule of A containing Λ. So D_A^{-1} is actually a fractional Λ ideal in A. Therefore $D_A^{-1} = (\beta\Lambda)^{-m}$ for some $m \geq 0$. The ideal $D_A = (\beta\Lambda)^m$ of Λ is called the absolute different of A. We also note that with θ_A defined in 4.1.1 we have

$$D_A^{-1} = \left\{ x \in A : \theta_A(xy) = 1, \text{ for all } y \in \Lambda \right\}$$

Now we choose a Haar measure on A such that $\int_\Lambda dx = N\mathrm{D}^{-1/2}$, where if δ is a fractional Λ ideal in A we put $N_A\delta = N\delta = (\Lambda : \delta)$. This Haar measure is independent of the choice of the maximal orders Λ

in A because all maximal orders in A are conjugates of $M_k(\Delta)$ and hence have the same measure.

In the next theorem we show that this measure is self-dual.

4.1.5 <u>THEOREM</u> The Haar measure on A for which $\int_\Lambda dx = ND_A^{-1/2}$ is self dual.

Proof: We are to prove that if $f \in S(A)$ then $f(x) = \hat{\hat{f}}(-x)$ with respect to the given haar measure. now by a well known result in Fourier analysis we have $f(x) = c\,\hat{\hat{f}}(-x)$, where c is a constant depending on f and the chosen Haar measure. The theorem will be proved if we show that c = 1 for the chosen Haar measure on A. Hence it suffices to show that c = 1 for a particular $f \in S(A)$. So we take f = ch. function of Λ. Then

$$\hat{f}(y) = \int_\Lambda f(x)\,\theta_A(xy)\,dx = \int_\Lambda \theta_A(xy)\,dx = \begin{cases} 0 \text{ if } y \notin D_A^{-1} \\ \mu(\Lambda) = ND_A^{-1/2} \text{ if } y \in D_A^{-1} \end{cases}$$

Therefore

4.1.6 $$\hat{f} = ND_A^{-1/2}\ .\ \text{ch. function of } D_A^{-1}$$

Again

$$\hat{\hat{f}}(-x) = \int_A \hat{f}(y)\,\theta_A(yx)\,dy = \int_{D_A^{-1}} ND_A^{-1/2}\,\theta_A(yx)\,dy$$

$$= ND_A^{-1/2}\int_{D_A^{-1}} \theta_A(yx)\,dy$$

$$= \begin{cases} ND_A^{-1/2}\mu(D_A^{-1}) \text{ if } x \in \Lambda \\ 0 \text{ if } x \notin \Lambda \end{cases}$$

But by a 1.2.14, since D_A^{-1} is a fractional Λ ideal in A we have

$$\mu(D_A^{-1}) = ND_A\ \mu(\Lambda) = ND_A^{-1}\ ND_A^{-1/2} = ND_A^{1/2}$$

Hence

$$\hat{\hat{f}}(-x) = \begin{cases} 1 \text{ if } x \in \Lambda \\ 0 \text{ if } x \notin \Lambda \end{cases}$$

So $\hat{\hat{f}}(-x) = f(x)$ holds. Hence the theorem.

We next generalise the concept of the conductor

4.1.7 *Definition:* Let ψ be a character of A^*. Then we define the conductor $f(\psi)$ of ψ by setting $f(\psi) = \Lambda$ if ψ is unramified. Otherwise we define $f(\psi)$ to be the largest two sided ideal $\mathcal{U}$ of Λ such that ψ is trivial on the subgroup $1 + \mathcal{U}$ of Λ^*. In other words, in the case of ψ ramified, if for an ideal $\mathcal{B}$ of Λ such that $\psi(1+\mathcal{B}) = 1$ then $\mathcal{B} \subseteq f(\psi)$.

Now we come to the main object in our list of definition. The generalised Gauss sum $\tau(\psi)$ (for classical definition see 1.1.6) associated with a character ψ of A is defined by

4.1.8 $$\tau(\psi) = \sum_x \psi(c^{-1}x)\ \theta_A(c^{-1}x)$$

where x ranges over a full set of coset representatives of Λ^* modulo $(1+f(\psi)) \cap \Lambda^*$, and c is any element of Λ such that $c\Lambda = D_A f(\psi)$. Such a c always exists as $D_A f(\psi)$ is a two sided ideal in Λ. Further c can be chosen to be in A^*. Again we note that if ψ is unramified, then $\Lambda^* \subseteq 1+f(\psi)$ and hence in this case we obtain $\tau(\psi) = \psi(c^{-1})$ since θ_A is trivial on D_A^{-1}.

4.1.9 REMARKS

(i) $\tau(\psi)$ is independent of the choice of the coset representatives x for if $x_1(1+f(\psi)) = x_2(1+f(\psi)$ then $x_2^{-1}x_1 \in 1+f(\psi)$, whence $\psi(x_2^{-1}x_1) = 1$. Therefore $\psi(x_2^{-1}c\ c^{-1}x_1) = 1$ implies that $\psi(c^{-1}x_2) = \psi(c^{-1}x_1)$. Also $c\Lambda = D_A f(\psi)$ implies that $f(\psi)\ c^{-1} = D_A^{-1}$. Therefore we conclude that if $x_1-x_2 \in f(\psi)$ then $c^{-1}(x_1-x_2) \in D_A^{-1}$ and hence $\theta_A(c^{-1}x_1) = \theta_A(c^{-1}x_2)$.

(ii) $\tau(\psi)$ is independent of the choice of c. To see this we notice that if $c_1\Lambda = D_A f(\psi) = c_2\Lambda$ then $c_2^{-1}c_1 \in \Lambda^*$. Now for $x \in \Lambda^*$, $c_1^{-1}x = c_2^{-1}c_2\ c_1^{-1}x = c_2^{-1}y$ with $y \in \Lambda^*$. But $x_1^{-1}x_2 \in 1+f(\psi)$ if and only if $(c_2c_1^{-1}x_1)^{-1}\ (c_2c_1^{-1}x_2) = x_1^{-1}x_2 \in 1+f(\psi)$. Since $\tau(\psi)$ is independent of the choices of the coset representatives x by (i) it follows that $\tau(\psi)$ is also independent of the choice of c.

(iii) This is to justify our choice of the maximal order as $\Lambda = M_k(\Delta)$ in the definition of the Gauss sum $\tau(\psi)$. If Λ is relaced by another maximal order Λ', then Λ and Λ' are conjugates in A. Consequently the ideals $f(\psi)$ and D_A are replaced by their corresponding conjugates. Since both ψ and θ_A are invariant under cunjugation by elements of A^*, it follows at once that $\tau(\psi)$ is independent of the choice of the maximal order.

Now we are ready to relate the local constant factor $\varepsilon_A(s,\psi)$ of 4.1.3 to the sum $\tau(\psi)$

4.1.10 THEOREM The function $\varepsilon_A(s,\psi)$ is given by

$$\varepsilon_A(s,\psi) = N(D_A f(\psi))^{1/2-s}\ \tau(\bar{\psi}) / \ Nf(\psi)^{1/2}$$

where notation is as above.

Proof: Through out this proof whenever we say a Haar measure dx on A we will mean a Haar measure $dx = d\mu(x)$, that is the measure of a set E with respect to dx is $\mu(E)$. Similarly for $d^*x = d^*\mu(x)$.

CASE I- ψ unramified, that is $f(\psi) \neq \Lambda$

We have seen in 3.2. that $L_A s,\psi) = L_\Lambda(\Lambda, s, \psi)$ and so by 3.1. $L_A(s,\psi) = \mu^*(\Lambda^*)^{-1} Z(\phi, s, \psi)$ where ϕ is the characteristic function of Λ in A. For this ϕ we therefore have $Y(\phi, s, \psi) = \mu^*(\Lambda^*)$. Also by 4.1.6 we have $\hat{\phi} = \mu(\Lambda)$. ch, function of D_A^{-1}. We choose $c \in \Lambda \cap A^*$ such that $c\Lambda = D_A$, and then we have

$$\hat{\phi}(x) = ND_A^{-1/2} \phi(x) \qquad \text{for } x \in A$$

Therefore
$$Z(\hat{\phi}, s, \bar{\psi}) = ND_A^{-1/2} \int_{A^*} \phi(cx) \bar{\psi}(x) \|x\|^s d^*x$$
$$= ND_A^{-1/2} \psi(c) \|c\|^{-s} Z(\phi, s, \bar{\psi})$$
$$= ND_A^{-1/2} \psi(c) \|c\|^{-s} \mu^*(\Lambda^*) L_A(s,\psi)$$

But D_A is a two sided ideal of Λ and hence $\|c\| = (\Lambda:\Lambda c)^{-1} = ND_A^{-1}$.

So
$$L_A(s,\bar{\psi})^{-1} Z(\hat{\phi}, s, \bar{\psi}) = ND_A^{s-1/2} \psi(c) \mu^*(\Lambda^*).$$

Relpacing s by 1-s in the above expression we have

$$Y(\hat{\phi}, 1-s, \bar{\psi}) = ND_A^{1/2-s} \psi(c) \mu^*(\Lambda^*)$$

Therefore

$$\varepsilon_A(s,\psi) = ND_A^{1/2-s} \bar{\psi}(c^{-1}) = ND_A^{1/2-s} \tau(\psi)$$

as required. Now we turn our attention to the case when ψ is ramified.

CASE II- ψ is ramified, that is $f(\psi) \neq \Lambda$.

From now onwards we will write $f = f(\psi)$ whenever there is no chance of confusion. If ϕ_1 is the characteristic function of $1+f$ then we have $Z(\phi_1, s, \psi) = \mu^*(1+f)$ and so

4.1.11 $$Y(\phi_1, s, \psi) = \mu^*(1+f)$$

We will now calculate $Y(\hat{\phi}_1, s, \bar{\psi})$. For this we notice that $\mathfrak{f}(\bar{\psi}) = \mathfrak{f}$. Let ϕ be the characteristic function Λ and ϕ_0 the characteristic function of $\mathfrak{f}$. Then

$$\hat{\phi}_1(y) = \int_A \phi_1(x)\, \theta_A(xy)\, dx = \int_A \phi_0(x-1)\, \theta_A(xy)\, dx$$

$$= \int_A \phi_0(x)\, \theta_A((x+1)y)\, dx$$

$$= \theta_A(y)\, \hat{\phi}_0(y)$$

Now imitating the proof of 4.1.6 we have

$$\hat{\phi}_0 = \mu(\mathfrak{f}).\text{ch.function of } D_A^{-1}\mathfrak{f}^{-1}$$

and so

$$\hat{\phi}_0(y) = ND_A^{-1/2}.N\mathfrak{f}^{-1}\phi(cy), \qquad \text{for } y \in A$$

where $c \in \Lambda \cap A^*$ is such that $c\Lambda = D_A\mathfrak{f}$. Therefore

$$Z(\hat{\phi}_0, s, \bar{\psi}) = ND_A^{-1/2}.N\mathfrak{f}^{-1} \int_{A^*} \phi(cy)\theta_A(y)\, \|y\|^s\, d^*y$$

$$= N(D_A\mathfrak{f})^{s-1/2}N\mathfrak{f}^{-1/2} \int_{A^*} \phi(y)\, \theta_A(c^{-1}y)\, \bar{\psi}(c^{-1}y)\, \|y\|^s\, d^*y$$

since $\|c\| = N(D_A\mathfrak{f})^{-1}$.

Our aim will now be to evaluate this integral. Denote it by I for brevity and write $I = I_1 + I_2$ where

$$I_1 = \int_{\Lambda^*} \bar{\psi}(c^{-1}y)\, \theta_A(c^{-1}y)\, d^*y$$

$$I_2 = \int_{(\Lambda\setminus\Lambda^*)\cap A} \bar{\psi}(c^{-1}y)\, \theta_A(c^{-1}y)\, \|y\|^s\, d^*y$$

We will prove that $I_2 = 0$ and $I_1 = \mu^*(1+\mathfrak{f}).\tau(\bar{\psi})$. The ranges of integration in both I_1 and I_2 are invariant under translation by elements of Λ^*, and it follows as in 4.1.9(ii) that both the integrals are independent of the choice of c where c is such that $c\Lambda = D_A\mathfrak{f}$.

Consider I_2. It is possible to write $(\Lambda\backslash\Lambda^*) \cap A^*$ as a countable disjoint union of sets $x\Lambda^*$, for $x \in (\Lambda\backslash\Lambda^*) \cap A^*$. It therefore suffices to show that

$$I(x) = \int_{x\Lambda^*} \bar{\psi}(c^{-1}y)\ \theta_A(c^{-1}y)\ \|y\|^s\ d^*y = 0$$

whenever $x \in \Lambda \cap A^*$ but $x \notin \Lambda^*$. By 4.1.3, the integral is unchanged when the range of integration is replaced by $\Lambda^* x$. Thus it is possible to replace x by $u_1 x u_2$ for any $u_1,\ u_2 \in \Lambda^*$. This means that we can assume x to be of the Smith-Normal form, that is

$$x = \text{diag}\left(\xi^{r_1}, \xi^{r_2}, \ldots, \xi^{r_k}\right)$$

where ξ is a prime element of Δ and r_i's are non-negative integers with $r_k > 0$ ([N],15II9). Now suppose $\mathfrak{f} = \xi^l\Lambda$ with $l > 1$. Then there exists $w = (a_{ij}) \in \xi^{l-1}\Lambda\backslash\xi^l\Lambda$ such that $\psi(1 + w) \neq 1$. Let $\tilde{\psi}$ be the continuous character of C^* such that $\psi = \tilde{\psi} \circ nr_{A/C}$. Then

$$\psi(1 + w) = \tilde{\psi}(nr_{A/C}(1 + w)) \neq 1.$$

Since $nr_{A/C}(1 + w) = nr_{D/C}(\det(1 + w))$ and $\tilde{\psi}(nr_{D/C}(\det(1 + w))) \neq 1$ by definition of $\mathfrak{f}$, we have $1-\det(1 + w) \in \xi^{l-1}\Delta\backslash\xi^l\Delta$. Hence we can find $z = \text{diag}(0,0,\ldots,\lambda\xi^{l-1})$ with $\lambda \in \Delta^*$ such that $\psi(1 + z) \neq 1$. Clearly $1 + z \in \Lambda^*$. Then because $\|1 + z\| = 1$ and the measure μ^* is invariant under translation, we have

$$\begin{aligned} I(x) &= \int_{x\Lambda^*} \bar{\psi}(c^{-1}y(1+z))\ \theta_A(c^{-1}y(1+z))\ \|y\|^s\ \|1+z\|^s\ d^*y \\ &= \bar{\psi}(1 + z) \int_{x\Lambda^*} \bar{\psi}(c^{-1}y)\ \theta_A(c^{-1}y)\ \theta_A(c^{-1}yz)\ \|y\|^s\ d^*y. \end{aligned}$$

Now if we write $y = xu$ with $u \in \Lambda^*$, then

$\theta_A(c^{-1}yz) = \theta_A(c^{-1}xuz) = \theta_A(c^{-1}xu)$ (by symmetry of reduced trace). Since $zxu \in \xi^l\Lambda = \mathfrak{f}$, so $c^{-1}zxu \in D_A^{-1}$. Hence $\theta_A(c^{-1}yz) = 1$ and $I(x) = \bar{\psi}(1 + z)\ I(x)$ implying $I(x) = 0$.

If $l = 1$ then in the same way as in the case $l > 1$ we can choose $\lambda \in \Delta^*$ and set $z = \mathrm{diag}(0,0,\ldots,\lambda) \in \Lambda$ such that $\psi(1 + z) \neq 1$. But notice that such a choice is possible unless $(\Delta : \xi\Delta) = 2$. For then $1 + \lambda \in \xi\Delta$ and so $\psi(1 + z) = 1$, a contradiction to our choice of λ. But when this case arises $\mathfrak{f} = \xi\Lambda$ is not possible because of the following reason: As $\tilde{\psi}$ is ramified we can choose $u \in \Delta^*$ such that $\tilde{\psi}(nr_{D/C}(u)) \neq 1$ (by 3.2.4). But $(\Delta : \xi\Delta) = 2$ implies that $u - 1 \in \xi\Delta$. Hence $\mathrm{diag}(0,0,\ldots,u-1) \in \xi\Lambda$ which implies that $y = \mathrm{diag}(1,1,\ldots,u) \in 1 + \xi\Lambda$ and since $\xi\Lambda = \mathfrak{f}$ so $\tilde{\psi}(nr_{D/C}(u)) \neq 1$, a contradiction to our choice. Hence the case $(\Delta : \xi\Delta) = 2$ and $\mathfrak{f} = \xi\Lambda$ cannot arise together.

Now the proof for the case $l = 1$ carries out word for word as in the previous case and we get $I(x) = 0$.

We can now write $I = I_1$, that is

$$Z(\hat{\phi}_1, s, \bar{\psi}) = N(D_A\mathfrak{f})^{s-1/2}\, N\mathfrak{f}^{-1/2} \int_{\Lambda^*} \bar{\psi}(c^{-1}y)\, \theta_A(c^{-1}y)\, d^*y$$

We may write $\Lambda^* = \bigsqcup_x x(1 + \mathfrak{f})$, where x ranges over a set of coset representatives of $\Lambda^* \bmod (1+\mathfrak{f})$. Then, since $\bar{\psi}$ is trivial on $1+\mathfrak{f}$ and θ_A is trivial on $c^{-1}x\mathfrak{f}$, for $x \in \Lambda$ (as $c^{-1}\Lambda = \mathfrak{f}^{-1}D_A^{-1}$ and θ_A is symmetric), we obtain

$$Z(\hat{\phi}_1, s, \bar{\psi}) = N(D_A\mathfrak{f})^{s-1/2}\, N\mathfrak{f}^{-1/2}\, \mu^*(1+\mathfrak{f}) \sum_x \bar{\psi}(c^{-1}x)\, \theta_A(c^{-1}x)$$

x ranging over a full set of coset representatives of $\Lambda^* \bmod (1+\mathfrak{f})$. Therefore

$$Y(\hat{\phi}_1, s, \bar{\psi}) = N(D_A\mathfrak{f})^{s-1/2}\, N\mathfrak{f}^{-1/2}\, \mu^*(1+\mathfrak{f})\, \tau(\bar{\psi})$$

as ψ is ramified. Therefore by 4.1.11 we have

$$\frac{Y(\hat{\phi}_1,\ 1-s,\ \bar{\psi})}{Y(\phi_1,\ s,\ \psi)} = N(D_A f)^{1/2-s}\ Nf^{-1/2}\ \ \tau(\bar{\psi})$$

and hence $\varepsilon_A(s,\psi) = N(D_A f)^{1/2-s}\ \ Nf^{-1/2}\ \tau(\bar{\psi})$ as required. ■

4.1.12 <u>REMARK</u> Notations as in theorem. Let ψ be any character of A^*. We now choose $\phi \in S(A)$ such that $Y(\phi,\ s,\ \psi)$ is not identically zero. In fact we can take $Y(\phi,\ s,\ \psi) = 1$ for a suitable choice of $\phi \in S(A)$. From 4.1.3 we have

$$Y(\hat{\phi},\ 1-s,\ \bar{\psi}) = \varepsilon_A(s,\psi)\ Y(\phi,\ s,\ \psi)$$

and

$$Y(\hat{\hat{\phi}},\ s,\ \psi) = \varepsilon_A(1-s,\bar{\psi})\ Y(\hat{\phi},\ 1-s,\ \bar{\psi})$$

Now we have $\hat{\hat{\phi}}(-x) = \phi(x)$ as dx is self dual, and we obtain

$$Y(\hat{\hat{\phi}},\ s,\ \psi)\ =\ \psi(-1)\ Y(\phi,\ s,\ \psi)$$

Therefore theorem 4.1.10 yields the identity

4.1.13 $$\tau(\bar{\psi})\ \tau(\psi) = \psi(-1)\ Nf(\psi)$$

Also from 4.1.8 we have

4.1.14 $$\overline{\tau(\psi)} = \sum_x \overline{\psi(c^{-1}x)}\ \overline{\theta_A(c^{-1}x)} = \sum_x \bar{\psi}(c^{-1}x)\ \theta_A(-c^{-1}x)$$
$$= \bar{\psi}(-1)\ \sum_x \bar{\psi}(c^{-1}x)\ \theta_A(c^{-1}x) = \psi(-1)\ \tau(\bar{\psi})$$

everywhere x ranges over a full set of coset representatives of Λ^* mod $\Lambda^* \cap\ (1+f)$. Therefore 4.1.13 and 4.1.14 imply the following corollary to theorem 4.1.10.

4.1.15 <u>COROLLARY</u> The Gauss sum $\tau(\psi)$ is a complex number of absolute value $N\mathfrak{f}(\psi)^{1/2}$. In particular $\tau(\psi) \neq 0$. ∎

We will conclude this section by briefly outlining how we can go over from the simple case to he semisimple case. Suppose A is semisimple $\mathbb{Q}_p$– algebra, and $A = \prod_{i=1}^{r} A_i$, where each A_i is a simple algebra. Since

$$tr_{A/\mathbb{Q}_p}(a) = \sum_{i=1}^{r} tr_{A_i/\mathbb{Q}_p}(a_i) \text{ , where } a = (a_1,\ldots a_r)$$

so we can define a cannonical character θ_A of A by $\theta_A = \prod_{i=1}^{r} \theta_{A_i}$, for the canonical character θ_{A_i} of A_i defined in 4.1.1.

The self dual Haar measure $dx = d\mu(x)$ on A is the product of the self dual measures on A_i. Defining the Fourier transform in the usual manner, we at once see that Theorem 4.1.2 remains valid using the fact that $S(A) = S(A_1) \otimes \ldots \otimes S(A_r)$.

Defining $Y(\phi, s, \psi)$ in the same way as in the simple case (for $\phi \in S(A)$ and a character ψ of A^*) we see that 4.1.3 remains valid. Further choosing $\phi = \prod_{i=1}^{r} \phi_i$, $\phi_i \in S(A_i)$, we obtain

4.1.16 $$\varepsilon_A(s,\psi) = \prod_{i=1}^{r} \varepsilon_{A_i}(s,\psi_i)$$

where ψ_i is the restriction of ψ to A_i^* . One can define differents and conductors as before, relative to the choice of the maximal order Λ in A. These can now be expressed as corresponding objects for the A_i. To be presise if $\psi_i = \psi|\, A_i^*$ we write

$$\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \ldots \oplus \Lambda_r \quad \text{where } \Lambda_i \text{ is a maximal order of } A_i$$

$D_A = D_{A_1} \oplus D_{A_2} \oplus \ldots \oplus D_{A_r}$ where D_{A_i} is the different of A_i

$f(\psi) = f(\psi_1) \oplus f(\psi_2) \oplus \ldots \oplus f(\psi_r)$ where $f(\psi_i)$ is the conductor of ψ_i

The formula 4.1.8 will now define the Gauss sum (when A is a semisimple $\mathbb{Q}_p$- algebra) and it is clear that

4.1.17 $$\tau(\psi) = \prod_{i=1}^{r} \tau(\psi_i)$$

because if $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots y_r)$ with $x, y \in \Lambda^*$ are such that $xy^{-1} \notin (1+f(\psi)) \cap \Lambda^*$ then there exists i such that $x_i y_i^{-1} \notin (1+f(\psi_i)) \cap \Lambda_i^*$ and vice versa.

Theorem 4.1.10, Remark 4.1.12 and Corollary 4.1.15 now hold for A semisimple $\mathbb{Q}_p$- algebra. The semisimple version of Theorem 4.1.10 can now be stated as follows:

4.1.18 <u>THEOREM</u> Let A be a finite dimensional semisimple $\mathbb{Q}_p$- algebra, and ψ a continuous character of A^*. Let Λ be a maximal order in A, and let D_A be the absolute different of A, $f(\psi)$ the conductor of ψ defined relative to Λ. Then for each $\phi \in S(A)$ we have

4.1.19 $$L_A(1-s,\bar{\psi})^{-1} Z(\hat{\phi}, 1-s, \bar{\psi})$$

$$= N_A(D_A f(\psi))^{1/2-s} \tau(\bar{\psi}) N_A f(\psi)^{-1/2} L_A(s,\psi)^{-1} Z(\phi, s, \psi)$$

where for a two sided fractional ideal δ of A we put $N_A \delta = (\Lambda:\delta)$. ■

This theorem can be used to work out the functional equations for L-functions as shown in the next section.

§2 FUNCTIONAL EQUATIONS FOR L-FUNCTIONS (LOCAL CASE)

In this section we will briefly show how 4.1.18 can be used to work out the functional equations for L-functions. To start with let A be a finite dimensional semi-simple $\mathbb{Q}_p$- algebra and let Λ be a $\mathbb{Z}_p$ order in A. Let M be a left ideal of Λ of finite index $(\Lambda:M)$ in Λ. Suppose θ_A is as defined earlier (4.1.1). Let

4.2.1 $$M^{\perp} = \left\{ x \in A : \theta_A(yx) = 1 \text{ for all } y \in M \right\}$$

By symmetry of θ_A, it follows that $M^{\perp}$ is a right Λ-module. Also $M^{\perp}$ is a full Λ-lattice as M is so (proof goes in the same way as for the proof that D_A is a full Λ-lattice).

Now letting $\Gamma = \mathrm{End}_{\Lambda}(M)$ can view M as a (Λ, Γ)-bimodule. Then $\Gamma \subseteq \mathrm{End}_{\Lambda}(M^{\perp})$ and so $M^{\perp}$ is (Γ, Λ)-bimodule. Repeating the process we get $\mathrm{End}_{\Lambda}(M^{\perp}) \subseteq \mathrm{End}_{\Lambda}(M^{\perp\perp})$ and since $M^{\perp\perp} = M$, we obtain $\Gamma = \mathrm{End}_{\Lambda}(M^{\perp})$. Now suppose ψ is a character of A^* which is trivial on Γ^*. Then we may define another L-function based on the right ideals of Λ (note that we have been till now defining L-functions based on the left ideals of Λ). We define this as follows:

4.2.2 $$L_{\Lambda}^{(r)}(M^{\perp}, s, \psi) = \sum_{Y} \psi(Y)\ (\Lambda:Y)^{-s}$$

where the sum is taken over all right ideals Y of Λ which are isomorphic to $M^{\perp}$. ($\psi(Y)$ is defined as $\psi(Y) = \psi(y)$ where $y \in A^*$ is such that $Y = yM^{\perp}$. Since ψ is trivial on $\mathrm{Aut}_{\Lambda}M^{\perp}$ this is well defined). This L-function converges for Real $s > 1$. We are interested in establishing a relation between L-functions via left ideals of Λ and L-functions via right ideals of Λ. For this we proceed in the following way by inverting $L_{\Lambda}^{(r)}(M^{\perp}, s, \psi)$ into an integral.

Let ϕ be the characteristic function of $\{M:\Lambda\}$. Then

$$\hat{\phi}(y) = \int_{A} \phi(x)\, \theta_{A}(xy)\, dx = \int_{\{M:\Lambda\}} \theta_{A}(xy)\, dx$$

$$= \begin{cases} 0 \text{ if } y \notin \{M:\Lambda\}^{\perp} \\ \mu(\{M:\Lambda\}) \text{ if } y \in \{M:\Lambda\}^{\perp} \end{cases}$$

This implies that $\hat{\phi} = \mu(\{M:\Lambda\})$ times the ch.function of $\{M:\Lambda\}^{\perp}$ Now as since $M \subseteq M^{\perp}$ it follows that

$$\{M^{\perp}:\Lambda\} \subseteq \{M:\Lambda\} \subseteq \{M:\Lambda\}^{\perp} \qquad \text{(i)}$$

Also if $z \in \{M:\Lambda\}$ then $zM \subseteq \Lambda$ implying that $\theta(zm) = 1$ for all $m \in M$. This shows that $\{M:\Lambda\} \subseteq M^{\perp}$. $xM^{\perp} \subseteq \Lambda$ shows that $\theta(xy) = 1$ for all $y \in M^{\perp}$ and so in particular for all $y \in \{M:\Lambda\}$. Hence $x \in \{M:\Lambda\}^{\perp}$. Therefore $\{M^{\perp}:\Lambda\} \subseteq \{M:\Lambda\}^{\perp}$ and this together with (i) yields $\{M:\Lambda\}^{\perp} = \{M^{\perp}:\Lambda\}$.

If now $\| \;\|_{r}$ denotes the norm defined in 1.2.26 (viii) with right Λ-lattices replacing left Λ-lattices, then we can write

$$L_{\Lambda}^{(r)}(M^{\perp}, s, \psi) = \sum_{y} \|y\|_{r}^{s}\, \psi(y)$$

where y ranges over a full set of coset representatives of $\{M^{\perp}:\Lambda\} \cap A^{*}$ mod $\mathrm{Aut}_{\Lambda}M^{\perp}$. As in 2.2.4 we connect this with the integral $\int_{A^{*}} \hat{\phi}(x)\, \psi(x)\, d^{*}x$, where ϕ is the characteristic function of $\{M:\Lambda\}$. Proceeding in the in the same way as in Chapter 2 §2 and using the preceeding facts, one obtains

$$L_\Lambda^{(r)}(M^\perp,\ s,\ \psi) = \mu^*(\Gamma)^{-1}\ (\Lambda:M)^{-1}\ \mu^*(\{M:\Lambda\})^{-1} \int_{A^*} \hat{\phi}(x)\ \psi(x)\,\|x\|_r^s\ d^*x$$

This together with the functional equation 4.1.18 gives us a relation between

$$L_\Lambda(M,\ s,\ \psi)\ /\ L_A(s,\psi) \quad \text{and} \quad L_\Lambda^{(r)}(M^\perp,\ 1-s,\ \bar{\psi})\ /\ L_A(1-s,\bar{\psi}).$$

In general this relation will be too complicated, however in case A is a group ring it takes a neater form. For then we can identify $L_\Lambda^{(r)}(M^\perp,\ s,\ \psi)$ with an L- function defined via the left ideals of Λ with the help of the standard anti- automorphism of Λ. Let F be a finite extension of $\mathbb{Q}_p$ with with valuation ring R. Let G be a finite group of order n and put $A = FG$, $\Lambda = RG$. Let the suffix α denote the standard anti automorphism of A defined by

$$\Big(\sum_{g\in G} \alpha_g\ g \Big)_\alpha = \sum_{g\in G} \alpha_g\ g^{-1}\ , \quad \alpha_g \in F$$

For a function f on A we denote by f_α defined by $f_\alpha(x) = f(x_\alpha)$ for $x \in A$. Let M be a left ideal of Λ. Then we define the contragradient of M as

$$\check{M} = \Big(\{M:\Lambda\} \Big)_\alpha$$

With all these notations above now the functional equation of $L_\Lambda(M,\ s,\ \psi)$, when A is a group ring looks like

4.2.3 $\quad L_A(1-s,\bar{\psi}_\alpha)^{-1}\ L_\Lambda(\check{M},\ 1-s,\ \bar{\psi}_\alpha)$

$$= \Big\{(\Lambda':\Lambda)\ N_A f(\psi)^{-1/2}\Big\}^{2s-1}\ \bar{\psi}(nc_F\ b^{-1})\ \tau(\bar{\psi})\ Nf(\psi)^{-1/2}\ L_A(s,\psi)^{-1} L_\Lambda(M,s,\psi)$$

where $c_F \in R$ such that $c_F R = D_F$, the absolute different of F and Λ' is some maximal order in A containing Λ.

4.2.4 <u>REMARK</u> The global counterpart of 4.2.3 or the relation between $L_\Lambda(M, s, \psi)/ L_A(s,\psi)$ and $L_\Lambda^{(r)}(M^\perp, 1-s, \bar{\psi})/ L_A(1-s,\bar{\psi})$ can be obtained by taking the product of the functional equation 4.2.3 (with A replaced by A_p and the other objects appearing accordingly) over all rational primes p. It is a trivial verification that the products are well defined.

——————— * ———————

CHAPTER 5

<u>Introduction</u>: In this concluding chapter, we will briefly state some results (without proofs) which are closely connected with the material covered in the earlier chapters. We will also state some problems which arise in connection with the results we have discussed so far.

§1 (A) <u>Some results connected with the earlier chapters.</u>

In the classical case, we know that the Dirichlet L- series was used to prove the distribution of primes in arithmetic progression of rational integers. We now discuss a similar result in terms of the L- functions of orders in semisimple algebras (for number field case refer to [SL] Ch.XV). To see this let Λ be an order in a finite dimensional semisimple $\mathbb{Q}$- algebra A with r simple components. Let Λ' be a maximal order in A. Let

$$c_A = \lim_{s \to 1} (s-1)\, \zeta_{\Lambda'}(s)$$

Let M be a left ideal of Λ, and let h_M be the number of isomorphism classes in the genus $g(M)$. Then, for a positive T tending to ∞ and each stable isomorphism class $[N]$ in $g(M)$ (see 3.1.2) , the number of left ideals X of Λ such that

$$(\Lambda:X) \le T, \quad X \in g(M), \quad [X] = [N]$$

is asymptotically equal to

5.1.1 $$\frac{c_A d_M}{h_M} \frac{1}{(r-1)!} T(\log T)$$

In particular, the left ideals of Λ in a given genus $g(M)$ are asymptotically, uniformly distributed among the stable isomorphism classes in that genus. For the proof of this result refer to ([BR2] 6.10).

In Chapter 4 we have discussed the functional equations for zeta integrals and L- functions of orders in the local case extensively. For the global case, let A be a finite dimensional semisimple $\mathbb{Q}$- algebra and Λ a maximal order in A, ψ a character of $J_f(A)$ which is trivial on A^*. Then we can write $\psi = \prod_p \psi_p$, where $\psi_p = \psi|A_p^*$. The space $S(A)$ of functions on $Ad_f(A)$ is defined as all those functions which are spanned by functions of the form

$$\phi = \prod_p \phi_p : x \longrightarrow \prod_p \phi_p(x_p) \ , \ x = (x_p) \in Ad_f(A)$$

where $\phi_p \in S(A_p)$ for all p and ϕ_p is the characteristic function of Λ_p for almost all p. It is a trivial verification to see that the definition is independent of the choice of Λ. We now form

5.1.2 $$Z(\phi, s, \psi) = \int_{J(A)} \phi(x) \psi(x) \|x\|^s d^*x$$

where as usual $\|x\| = \prod_p \|x_p\|$.

If now $\phi \in S(A)$ is of the form $\phi = \prod_p \phi_p$, $\phi_p \in S(A_p)$, then

5.1.3 $$Z(\phi, s, \psi) = \prod_p Z(\phi_p, s, \psi_p), \quad \text{Real } s > 1$$

Bushnell and Reiner in [BR3] showed that in the global case

5.1.4 $\quad Z(\hat{\phi}, 1-s, \bar{\psi}) = Z(\phi, s, \psi), \quad$ for all $\phi \in S(A)$.

We now change the notation and let A be a finite dimensional semisimple $\mathbb{Q}_p$- algebra, Λ a $\mathbb{Z}_p$- order in A, Λ' a maximal order in A and M a left ideal of Λ. We have already seen in 3.2.13 that

$$f(s) = Z_\Lambda(M,s) / \zeta_{\Lambda'}(s) \quad \in \mathbb{Z}[p^{-s}].$$

The functional equation 4.1.2 yields some information about $f(s)$ and it has been shown in([BR3] 12.8) that

5.1.5 $$f(s) = c_M \|\alpha\|^{s-1} (\Lambda:M)^{-s} \, p^{-\alpha(N)(1-s)}$$
$$+ \quad \text{(terms contaning lower powers of } p^{-s})$$

where

(i) c_M is the positive constant

$$c_M = \mu^*(\mathrm{Aut}_\Lambda M)^{-1} \, \chi(N) \, (\{M:\Lambda\}:\Lambda')$$

(ii) $\alpha(L) = \mu^*(\, S_{-\alpha(L)}(L)\,)$ where for an integer k we set

$$S_k(L) = \left\{ x \in L \cap A^* : \|x\| = p^k \right\}$$

(iii) the integer $\alpha(L)$ is defined by

$$p^{-\alpha(L)} = \mathrm{Max} \left\{ \|x\| : x \in L \cap A^* \right\}$$

for any arbitrary full $\mathbb{Z}_p$- lattice L in A.

(iv) $N := \left\{ y \in A : \theta(xy) = 1, \text{ for all } x \in \{M:\Lambda\} \right\}$, that is $N = \{M:\Lambda\}^\perp$ (using the notation introduced in Chapter 4).

(v) $\alpha \in A^*$ is such that $\Lambda'^\perp = \Lambda'\alpha$ (such a choice is possible as $\Lambda'^\perp$ is a left Λ'- lattice).

(B) <u>Some open problems.</u>

(I) As usual, the calculation of L- functions of orders in concrete cases present a lot difficulty. Of immediate concern is the calculation of orders in number fields. In this connection, let us point out that even the simplest case of quadratic fields have not been tackled so far. (L-functions for group rings have been calculated in [BR2].).

(II) Another problem worth venturing into is to see whether $L_{\Lambda}^{(r)}(M^{\perp}, s, \psi)$ can be expressed as an L- series via left of Λ in general (it has been shown in Chapter 4 that this is possible for integral group rings). It is too much expect a neat relation between $L_{\Lambda}(M, s, \psi) \,/\, L_{A}(s,\psi)$ and $L_{\Lambda}^{(r)}(M^{\perp}, 1-s, \bar{\psi}) \,/\, L_{A}(1-s, \bar{\psi})$ in general. However, it may not be impossible to get such a relation in special case.

(III) If A is a finite dimensional semisimple $\mathbb{Q}_p$- algebra then with usual notations we know that

$$L_{\Lambda}(M, s, \psi) = g(s)\, L_{A}(s,\psi)$$

where $g(s) \in \mathbb{Z}[\psi][p^{-s}]$. If $\psi = \psi_o$ is the trivial character, then

$$L_{\Lambda}(M, s, \psi_o) = Z_{\Lambda}(M,s)$$

and in 5.1.5 we have given a somewhat explicit expansion of $f(s) = Z_{\Lambda}(M,s) \,/\, \zeta_{\Lambda'}(s)$, Λ' a maximal order. It would be nice to see if such a formula could be given for

$$g(s) \quad L_{\Lambda}(M, s, \psi) \,/\, L_{A}(s,\psi) .$$

when ψ is any character of A^{*}.

———— * ————

BIBLIOGRAPHY

[AM] M. F. Atiyah and I. G. Macdonald, Introduction to commutative algebra, Addison-Wesley Publishing company, Inc.(1969).

[BR1] C. J. Bushnell and I. Reiner, Zeta functions of arithmetic orders and Solomon's Conjectures, Math. Zeit 173 (1980), 135-161.

[BR2] C. J. Bushnell and I. Reiner, L-functions of arithmetic orders and asymptotic distribution of ideals, J. reine angew. Math 327 (1981) 156-183.

[BR3] C. J. Bushnell and I. Reiner, Functional equations for L-functions of arithmetic orders, J. reine angew. Math 329 (1981), 88-124.

[BR4] C. J. Bushnell and I. Reiner, Analytic continuation of partial zeta functions of arithmetic orders,

[CF] J. W. S. Cassels and A. Frohlich (ed.), Algebraic number theory, London (1967).

[CR] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras, Interscience, New York (1962).

[DM] M. Deuring, " Algebren ", Springer-Verlag, Berlin (1935).

[FA] A. Frohlich, Locally free modules over arithmetic orders, J. reine agnew., Math 274/275 (1975) 112-138.

[G] L. J. Goldstein, Analytic number theory, Prentice-Hall Inc., Englewood cliffs, New Jersy (1971).

[GJ] R. Godement and H. Jacquet, Zeta functions of simple algebras, Lecture Notes in Mathematics No. 26, Springer-Verlag, Berlin - Heidelberg - New York (1972).

[HR] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis Vol I, Springer-Verlag, Berlin - Heidelberg - New York (1964).

[J1] N. Jacobson, Basic algebra Vol. I, Hindustan Publishing Corporation (1991).

[J2] N. Jacobson, Basic algebra Vol. II, Hindustan Publishing Corporation (1991).

[JH] H. Jocobinski, Genera and decomposition lattices, Acta Mathematica, 121 (1968), 1-29.

[JT] J. Tate Proceedings of Symposia in Pure Mathematics, Vol.33 (1979), Part 2, 3-26.

[K] Neal Koblitz, p-adic numbers, p-adic analysis and zeta functions, Springer-Verlag, New York Inc. (1977).

[LN] L. Nachbin, The Haar integral, D. Van Nostrand Company, Inc. (1965).

[LS] L. Solomon, Zeta functions and integral representation theory, Advances in Mathematics Vol. 29 (1977) 306-326.

[M] Daniel A. Marcus, Number Fields, Springer-Verlag, New York - Heidelberg - Berlin (1977).

[MO] I. Reiner, Maximal Orders, Academic Press Inc. London (1975).

[N] M. Newmann, Integral matrices, Academic Press, New York and London (1972).

[NM] Nakayama and Matushima, Uber die multiplicative gruppe einer p-adishen division algebra, Proceedings Imp. Acad. Tokyo Vol. 19 (1943) 622-628.

[O] Orders and their Application, Proceedings Oberwolfach 1984, Lecture otes in Mathematics, Springer-Verlag, erlin - Heidelberg - New York - Tokyo (1985).

[RR] I. Reiner and Klaus W. Roggenkamp, Integral representations, Lecture Notes in Mathematics, Springer-Verlag, Berlin - Heidelberg - New York (1979).

[RH] H. L. Royden, Real Analysis, second edition, Macmillan Publishing Co. Inc., New York, Collier Macmillan Publishers, London (1968).

[SJP] Jean Piere Serre, A course in Arithmetic, Springer-Verlag, New York Inc., (1973).

[S] R. G. Swan, K-theory of finite groups ang orders, Lecture Notes in Mathematics No. 149, Springer-Verlag, Berlin - Heidelberg - New York (1970).

[SL] S. Lang, Algebraic Number Theory, Addison-Wesley Publishing Co. (1970).

[W] Andre' Weil, Basic Number Theory, Springer-Verlag, Berlin - Heidelberg - New York (1973).