A STUDY OF FINITE GROUPS IN TERMS OF THEIR CENTRALIZERS

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CERTIFICATE

I certify that the dissertation entitled "A STUDY OF FINITE GROUPS IN TERMS OF THEIR CENTRALIZERS" submitted by Miss Jutirekha Dutta in partial fulfilment of the requirement of the degree of Master of Philosophy in Mathematics is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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DECLARATION

I, Jutirekha Dutta, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

This dissertation is being submitted to the North-Eastern Hill University for the degree of Master of Philosophy in Mathematics.

Signature of the Candidate

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Signature of the Head

Signature of the Supervisor

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PREFACE

Since the advent of the notion of finite groups, their classification has occupied perhaps the central position in the theory of finite groups. The finite abelian groups are the direct products of cyclic groups of prime power order, however the group structure becomes increasingly complex with decreasing abelianness. Study of finite groups in terms of their centralizers is a tool through which we can classify some of finite non-abelian groups.

Given a finite group G and $x \in G$, the set $C_G(x) = \{y \in G \mid xy = yx\}$ is called the *centralizer* of x in G. The set of all such centralizers in G is denoted by $\text{Cent}(G)$. Note that G is abelian if and only if $|\text{Cent}(G)| = 1$. At this point two questions can be posed:

(a) Given a finite group G, what can be said about $|\text{Cent}(G)|$?

(b) If the value of $|\text{Cent}(G)|$ is known then what can be said about G ?

In 1994, S. M. Belcastro and G. J. Sherman [10] proved that if a finite group G is non-abelian, then $|\text{Cent}(G)| \geq 4$. Moreover they classified all finite groups G with $|\text{Cent}(G)| = 4, 5$ and asked whether $|\text{Cent}(G)|$ could take any value different from 2 and 3. They also asked whether any value of $|\text{Cent}(G)|$ different from $1, 2, 3, 4$ and 5 characterize G. In addition they also studied the number $\text{PrCent}(G) = \dfrac{|\text{Cent}(G)|}{|G|}$.

A finite group G is called *n-centralizer* if $|\text{Cent}(G)| = n$, and *primitive n-centralizer* if $|\text{Cent}(G)| = |\text{Cent}(\frac{G}{Z(G)})| = n$. In 2000, A. R. Ashrafi [4, 5] studied n-centralizer and primitive n-centralizer groups, and answered some of the questions raised by Belcastro and Sherman. He also studied the structure of finite groups having $|\operatorname{Cent}(G)| = 6, 8$.

In 2006, A. R. Ashrafi and B. Taeri [7] studied the structure of finite groups with $|\operatorname{Cent}(G)| = 7$. They also computed $|\operatorname{Cent}(G)|$ for some finite groups using the structure of G modulo its center.

In 2007, A. Abdollahi, S. M. J. Amiri and A. M. Hassanabadi [1] gave some interesting relations between $|\operatorname{Cent}(G)|$ and the maximum number of the pairwise non-commuting elements in G. They also characterized all n-centralizer finite groups for $n = 7$ and 8, and settled a conjecture posed by A. R. Ashrafi [4] affirmatively. In 2005, A. R. Ashrafi and B. Taeri [6] obtained a characterization of A_5 in terms of the number of centralizers.

A group G is called *minimal simple* group, if it is a finite non-abelian simple group all of whose proper subgroups are solvable and G is said to be finite *semi-simple* group if it has no non-trivial normal abelian subgroup. In 2009, Mohammad Zarrin [33] calculated $|\operatorname{Cent}(G)|$ for all minimal simple groups and gave negative answer to a question raised by A. R. Ashrafi and B. Taeri [7]. They also characterized all finite semi-simple groups G with $|\operatorname{Cent}(G)| \leq 73$.

In Chapter 1, we have collected some basic definitions and results from the theory of groups which have been used in the forthcoming chapters.

For a finite group G and an element $x \in G$, the centralizer of x in G is given by $C_G(x) = \{y \in G \mid xy = yx\}$. The set of all such centralizers in G is denoted by $\operatorname{Cent}(G)$. At this point one question arises 'What can be said about $|\operatorname{Cent}(G)|$?' Clearly, G is abelian if and only if $|\operatorname{Cent}(G)| = 1$. In chapter 2, our goal is to study the possible values of $|\operatorname{Cent}(G)|$. Some of the significant results are as given below.

Proposition 2.1.2 *Let G, H be two groups, then*

$$\text{Cent}(G \times H) \;=\; \text{Cent}(G) \times \text{Cent}(H).$$

Lemma 2.1.3 *Let G be a group. Then $Z(G)$ is the intersection of all centralizers in G, i.e. $Z(G) \;=\; \underset{x\,\in\,G}{\cap} C_G(x)$.*

Lemma 2.1.4 *If G is a group, then G is the union of centralizers of all non-central elements of G, i.e. $G \;=\; \underset{x\,\in\,G-Z(G)}{\cup} C_G(x)$.*

Theorem 2.1.6 *Let G be a non-abelian group, then $|\,\text{Cent}(G)| \geq 4$.*

A group G is called *n-centralizer* if $|\,\text{Cent}(G)| \;=\; n$. We write C_n to denote a cyclic group of order n.

Proposition 2.2.1 *There exist n-centralizer groups for $n \neq 2, 3$.*

Proposition 2.2.2 *Let D_{2m} be the dihedral group of order $2m$. Then*

$$|\,\text{Cent}(D_{2m})| \;=\; \begin{cases} m+2, & \textit{if} \;\; 2 \nmid m \\ \frac{m}{2}+2, & \textit{if} \;\; 2 \mid m. \end{cases}$$

Proposition 2.2.3 *Let G be a group and p be a prime. If $\frac{G}{Z(G)} \cong C_p \times C_p$ then $|\,\text{Cent}(G)| \;=\; p+2$. If p is odd and $\frac{G}{Z(G)} \cong D_{2p}$, then $|\,\text{Cent}(G)| \;=\; p+2$.*

Proposition 2.2.9 *If G is a non-abelian p-group, then*

$$|\,\text{Cent}(G)| \geq p+2, \textit{ with equality if and only if } \frac{G}{Z(G)} \cong C_p \times C_p.$$

Theorem 2.3.1 *Let G be a group. Then $|\operatorname{Cent}(G)| = 4$ if and only if $\frac{G}{Z(G)} \cong C_2 \times C_2$; that is, G modulo its center is isomorphic to the klein four group.*

Theorem 2.4.4 *Let G be a finite group. Then $|\operatorname{Cent}(G)| = 5$ if, and only if, $\frac{G}{Z(G)} \cong C_3 \times C_3$ or $\frac{G}{Z(G)} \cong S_3$, where S_3 is the symmetric group on three symbols.*

Let G be a finite group then the ratio $\operatorname{PrCent}(G) = \frac{|\operatorname{Cent}(G)|}{|G|}$, gives an estimate for $|\operatorname{Cent}(G)|$ relative to the size of G.

Theorem 2.5.2 *Let p be the largest prime divisor of $|G|$. Then*

$$\operatorname{PrCent}(G) \leq \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ \frac{3}{4} + \frac{1}{4p}, & \text{if } p \text{ is odd.} \end{cases}$$

Given a finite group G, its commutativity degree, denoted by $\operatorname{PrCom}(G)$, is the probability that a randomly chosen pair of G commutes. Clearly, $0 \leq \operatorname{PrCom}(G) \leq 1$, and $\operatorname{PrCom}(G) = 1$ if and only if G is abelian. A non-empty subset $X = \{x_1, x_2, \ldots, x_n\}$ of a finite group G is called a set of pairwise non-commuting elements if $x_i x_j \neq x_j x_i$ for all distinct $i, j \in \{1, 2, \ldots, r\}$. A set of *pairwise non-commuting elements of G is said to have maximal size* if its cardinality is largest one among all such sets. We denote this largest cardinality by $r(G)$. In chapter 3, we study some interesting relations between $|\operatorname{Cent}(G)|$ and other invariants of G, namely the commutativity degree of G, and the maximal size of a subset of pairwise non-commuting elements of G. In this chapter, there are four sections. In the first section we discuss some results related to $\operatorname{PrCom}(G)$. Some of the results are given as follows.

Theorem 3.1.1 *Let G be a finite group. Then* $\frac{G}{Z(G)} \cong S_3$ *if and only if* $\mathrm{PrCom}(G) = \frac{1}{2}$.

Theorem 3.1.2 *If the smallest prime divisor of $|G|$ is p, then*

$$\mathrm{PrCom}(G) = \frac{p^2 + p - 1}{p^3} \textit{ if and only if } \frac{G}{Z(G)} \cong C_p \times C_p.$$

In the second section we discuss some relations between $|\operatorname{Cent}(G)|$ and $\mathrm{PrCom}(G)$. We know that for an abelian group G, $\mathrm{PrCom}(G) = |\operatorname{Cent}(G)| = 1$. Some of the important results are as mentioned below.

Theorem 3.2.1 *Let G be a finite group. Then* $\mathrm{PrCom}(G) = \frac{5}{8}$ *if and only if* $|\operatorname{Cent}(G)| = 4$.

Proposition 3.2.2 *If the smallest prime divisor of $|G|$ is p and*

$$\mathrm{PrCom}(G) = \frac{p^2 + p - 1}{p^3}, \textit{ then } |\operatorname{Cent}(G)| = p + 2.$$

In the third section we discuss some results related to $r(G)$. Some of the important results are as given below.

Proposition 3.3.1 *Let G be a finite n-centralizer group and $r(G) = n - 1$. Then every proper centralizer of G is abelian. Moreover, for every non-central elements x and y of G, $C_G(x) = C_G(y)$ or $C_G(x) \cap C_G(y) = Z(G)$.*

Proposition 3.3.2 *Let G be a finite group and $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then*

(i) $\{C_G(x_i) | i = 1, 2, \ldots, r\}$ *is an irredundant r-cover with the intersection* $Z(G) = \bigcap_{i=1}^{r} C_G(x_i)$.

(ii) $|\frac{G}{Z(G)}| \leq f(r)$.

(iii) $f(3) = 4, f(4) = 9, f(5) = 16, f(6) = 36$.

(iv) *if G is a group such that every proper centralizer is abelian, then for all $a, b \in G - Z(G)$ either $C_G(a) = C_G(b)$ or $C_G(a) \cap C_G(b) = Z(G)$.*

In the fourth section we discuss some relations between $|\operatorname{Cent}(G)|$ and $r(G)$. Some of the results are given as follows.

Proposition 3.4.1 *Let G be a finite non-abelian group, $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then,*

(i) $r \geq 3$,

(ii) $r + 1 \leq |\operatorname{Cent}(G)|$,

(iii) $r = 3$ *if and only if* $|\operatorname{Cent}(G)| = 4$,

(iv) $r = 4$ *if and only if* $|\operatorname{Cent}(G)| = 5$.

Proposition 3.4.2 *Let $X = \{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of a finite non-abelian group G having maximal size. Then,*

(a) *If $|\operatorname{Cent}(G)| < r + 4$, then*

(i) *For each element $x \in G$, $C_G(x)$ is abelian if and only if $C_G(x) = C_G(x_i)$ for some $i \in \{1, 2, \ldots, r\}$.*

(ii) *If $C_G(x_i)$ is a maximal subgroup of G for some $i \in \{1, 2, \ldots, r\}$, then $Z(G) = C_G(x_i) \cap C_G(x_j)$ for all $j \in \{1, 2, \ldots, r\} - \{i\}$. In* *particular, if* $|G|C_G(x_1)| \leq |G : C_G(x_2)| \leq 2$*, then* $|\operatorname{Cent}(G)| = 4$*, and if* $|G : C_G(x_1)| \leq |G|C_G(x_2)| = 3$*, then* $|\operatorname{Cent}(G)| = 5$*.*

(b) *If* $|\operatorname{Cent}(G)| = r + 2$*, then there exists a proper non-abelian centralizer* $C_G(x)$ *which contains* $C_G(x_{i_1}), C_G(x_{i_2})$ *and* $C_G(x_{i_3})$ *for three distinct* $i_1, i_2, i_3 \in \{1, 2, \ldots, r\}$*.*

(c) *If* $|\operatorname{Cent}(G)| = r + 3$*, then there exists a proper non-abelian centralizer* $C_G(x)$ *which contains* $C_G(x_{i_1})$ *and* $C_G(x_{i_2})$ *for two distinct* $i_1, i_2 \in \{1, 2, \ldots, r\}$*.*

Proposition 3.4.3 *Let* G *be a finite non-abelian group. Then every proper centralizer of* G *is abelian if and only if* $|\operatorname{Cent}(G)| = r + 1$*, where* r *is the maximal size of a set of pairwise non-commuting elements of* G*.*

In Chapter 4, we study some properties of $6, 7, 8$-centralizer groups. Some of the significant results are as given below.

Theorem 4.1.2 *If* G *is a* 6*-centralizer group, then*

$$\frac{G}{Z(G)} \cong D_8, A_4, C_2 \times C_2 \times C_2 \text{ or } C_2 \times C_2 \times C_2 \times C_2.$$

Theorem 4.2.3 *Let* G *be a finite group. Then* G *is* 7*-centralizer group if and only if* $\frac{G}{Z(G)} \cong C_5 \times C_5, D_{10}, \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle$*.*

Theorem 4.3.5 *Let* G *be a finite* 8*-centralizer group. Then*

$$\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2, A_4 \text{ or } D_{12}.$$

A group G is called *minimal simple* group, if it is a finite non-abelian simple group all of whose proper subgroups are solvable and G is said to be finite *semi-simple* group if it has no non-trivial normal abelian subgroup. In Chapter 5, we study some properties of what is known as a primitive n-centralizer group, i.e., a group in which the number of centralizers equals the number of centralizers of its central quotient. We also study the structure of finite groups G having large values of $|\operatorname{Cent}(G)|$. Finally, we study the values of $|\operatorname{Cent}(G)|$ for all minimal simple groups G. In this chapter, there are three sections. In the first section we study some properties of primitive n-centralizer groups. Some of the results are as follows.

Proposition 5.1.1 *There exists a primitive n-centralizer group for all odd $n \neq 3$.*

Proposition 5.1.3 *Let G be an n-centralizer group and $|G^{'} \cap Z(G)| = 1$. Then G is a primitive n-centralizer group.*

Theorem 5.1.6 *A finite group G is primitive 7-centralizer if and only if $\frac{G}{Z(G)} \cong D_{10}$ or $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle$.*

In the second section we study 22, 23-centralizer groups. Some of the important results are as mentioned below.

Theorem 5.2.3 *If G is a finite group and $\frac{G}{Z(G)} \cong A_5$, then $|\operatorname{Cent}(G)| = 22$ or 32.*

Theorem 5.2.5 *If G is a finite simple group and $|\operatorname{Cent}(G)| = 22$, then $G \cong A_5$.*

In the third section, we study the values of $|\operatorname{Cent}(G)|$ for all minimal simple groups G. Some of the significant results are as given below.

Theorem 5.3.1 *Let $G = \operatorname{PSL}(2, q)$, where q is a p-power (p-prime). Then*

(i) *If* $q \in \{2,3,5\}$ *or* $q \equiv 0$ *(mod 4). Then*

$$|\operatorname{Cent}(G)| = \begin{cases} q^2+q+2, & \textit{if} \quad q > 5, \\ 22, & \textit{if} \quad q = 4 \textit{ or } 5, \\ 6, & \textit{if} \quad q = 3, \\ 5, & \textit{if} \quad q = 2. \end{cases}$$

(ii) *If* $q > 5$ *and* $q \equiv 1$ *(mod 4). Then*

$$|\operatorname{Cent}(G)| = \frac{3q^2+3q+4}{2}.$$

(iii) *If* $q > 5$ *and* $q \equiv 3$ *(mod 4). Then*

$$|\operatorname{Cent}(G)| = \frac{3q^2+q+4}{2}.$$

Theorem 5.3.2 *Let* $G = Sz(q)(q = 2^{2m+1}, m > 0)$*. Then*

$$|\operatorname{Cent}(G)| = q^3-q^2+q+\frac{q^2(q^2+1)}{2}+\frac{q^2(q^2+1)(q-1)}{4(q+2r+1)}+\frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}$$

where $r = \sqrt{\frac{1}{2}}$.

Theorem 5.3.8 *Let* G *be a finite non-abelian simple group such that* $|\operatorname{Cent}(G)| \leq 73$*. Then* $G \cong A_5$.

Theorem 5.3.9 *If* G *is a finite semi-simple group with* $|\operatorname{Cent}(G)| \leq 73$*, then* $G \cong A_5$ *or* S_5.

We conclude the dissertation with an observation that the study of finite groups in terms of the number of their centralizers provides enough opportunity for future research even though a lot has already been done.

Contents

List of Symbols

$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of natural numbers
$H \subseteq G$	H is a subset of G
$H \subset G$	H is a proper subset of G
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$\|G\|H\|$	index of H in G
G/N	factor group
A^{-1}	$\{a^{-1} \mid a \in G\}$
HK	$\{hk \mid h \in H,\ k \in K\}$
$H \times K$	direct product of the groups H and K
$G \ltimes_\theta H$	Semidirect product of G and H with respect to θ
aH	$\{ah \mid h \in H\}$, left coset of H
Ha	$\{ha \mid h \in H\}$, right coset of H
$G \cong H$	G and H are isomorphic
$\langle a \rangle$	$\{a^n \mid n \in \mathbb{Z}\}$, the cyclic group generated by a
$\langle a_1, a_2, \cdots, a_n \rangle$	subgroup generated by $a_1, a_2, \cdots, a_n$
$o(g)$	order of g
$\|G\|$	order of the group G

$[x, y]$	$xyx^{-1}y^{-1}$, the commutator of x and y
$[H, K]$	$\langle\{[x, y] \mid x \in H, y \in K\}\rangle$
G'	$[G, G]$, the commutator subgroup of G
y^g	gyg^{-1}, conjugate of y
$\mathrm{Cl}_G(g)$	conjugacy class of g
$k(G)$	number of conjugacy classes in G
$C_G(x)$	$\{y \in G \| xy = yx\}$, centralizer of x in G
C_n	cyclic group of order n
D_{2n}	dihedral group of order $2n$
Q_{4n}	dicyclic group of order $4n$, $n \geq 2$
Q_{2^m}	generalized quaternion group of order 2^m, $m \geq 3$
S_n	symmetric group of degree n
A_n	alternating group of degree n
$Z(G)$	center of the group G
$M_n(\mathbb{F})$	set of all $n \times n$ matrices over $\mathbb{F}$
$GL(n, \mathbb{F})$	group of $n \times n$ non-singular matrices over the field $\mathbb{F}$
$SL(n, \mathbb{F})$	$\{A \in M_n(\mathbb{F}) \mid \det(A) = 1\}$
$\mathrm{PSL}(n, \mathbb{F})$	the quotient of $SL(n, \mathbb{F})$ by its center
$GF(p)$	Galois field of p elements
$\mathrm{Inv}(G)$	$\{x \in G \mid o(x) = 1 \text{ or } 2\}$
$\mathrm{PrCom}(G)$	commutativity degree of the group G
$\mathrm{Syl}_p(G)$	the set of all Sylow p-subgroups of G

Chapter 1

Preliminaries

In this chapter we recall some of the basic definitions and results from the theory of groups which have been used in the forthcoming chapters.

1.1 Group Action

Let G be a group and X be a set. Then G is said to *act* on X if there exists a map $\phi : G \times X \to X$ such that the following conditions hold:

(i) $\phi(1, x) = x \;\; \forall\, x \in X$, 1 being the identity element of G.

(ii) $\phi(gh, x) = \phi(g, \phi(h, x)) \;\; \forall\, x \in X$ and $\forall\, g, h \in G$.

If a group G acts on a set X then for each $x \in X$, the *orbit of* x denoted by $\text{orb}(x)$ is defined to be the set $\{\phi(g, x) \in X | g \in G\}$ and for each $x \in X$, the *stabilizer of* x denoted by $\text{stab}(x)$ is defined to be the subgroup $\{g \in G \mid \phi(g, x) = x\}$ of G.

Theorem 1.1.1. *Let a group G acts on a set X. Then, for each $x \in X$, the number of elements in the orbit of x equals the index of stabilizer of x, i.e.,*

$$|\operatorname{orb}(x)| \;=\; [G : \operatorname{stab}(x)].$$

1.2 Isomorphism theorems

The following theorems play crucial roles in the theory of groups.

Theorem 1.2.1. (First isomorphism theorem) ([29], page 25) *Let G and H be two groups and $\phi : G \to H$ be a homomorphism with $\operatorname{Ker}\phi = K$. Then K is a normal subgroup of G and $G/K \cong \operatorname{Im}\phi$.*

Theorem 1.2.2. (Second isomorphism theorem) ([29], page 25) *Let G be a group and let H and N be subgroups of G, and $N \trianglelefteq G$. Then*

$$\frac{H}{H \cap N} \cong \frac{HN}{N}.$$

Theorem 1.2.3. (Third isomorphism theorem) ([29], page 26) *Let G be a group and $K \subset H \subset G$, where both H and K are normal subgroups of G. Then H/K is a normal subgroup of G/K and*

$$\frac{G/K}{H/K} \cong \frac{G}{H}.$$

Theorem 1.2.4. (Correspondence Theorem) ([11], page 98)
Let $\phi : G_1 \to G_2$ be a homomorphism of a group G_1 onto a group G_2. Then the following are true:

(i) $H_1 \leq G_1 \Rightarrow \phi(H_1) \leq G_2$.

(ii) $H_2 \leq G_2 \Rightarrow \phi^{-1}(H_2) \leq G_1$.

(iii) $H_1 \trianglelefteq G_1 \Rightarrow \phi(H_1) \trianglelefteq G_2$

(iv) $H_2 \trianglelefteq G_2 \Rightarrow \phi^{-1}(H_2) \trianglelefteq G_1$

(v) *The mapping $H_1 \to \phi(H_1)$ is a 1-1 correspondace between the family of subgroups of G_1 containing* $\operatorname{Ker} \phi$ *and the family of subgroups of G_2; furthermore, normal subgroups of G_1 correspond to normal subgroups of G_2.*

Corollary 1.2.5. *Let N be a normal subgroup of G. Given any subgroup H_1 of G/N, there is a unique subgroup H of G such that $H_1 = H/N$. Further, $H \trianglelefteq G$ if and only if $H/N \trianglelefteq G/N$.*

1.3 Sylow's Theorems

Let G be a group and p be a prime such that $|G| = p^a m$ where $a, m \in \mathbb{N}$ and $p \nmid m$. Then any subgroup of G of order p^a is called a *Sylow p-subgroup* of G. The set of all Sylow p-subgroups of G is denoted by $Syl_p(G)$.

Result 1.3.1. (Sylow's first theorem) ([17], page 330) *Let G be a finite group such that $|G| = p^a m$ where $a, m \in \mathbb{N}$ and p is a prime with $p \nmid m$. Then there is a subgroup of G of order p^a.*

Result 1.3.2. (Sylow's second theorem) ([17], page 331) *Let G be a finite group and p be a prime such that p divides $|G|$. If P is a p-subgroup of G and $S \in Syl_p(G)$, then there exists $g \in G$ such that $P \subseteq S^g$.*

Result 1.3.3. (Sylow's third theorem) ([17], page 332) *Let G be a finite group and p be a prime such that p divides $|G|$. If n is a positive integer such that $|\frac{S_1}{S_1 \cap S_2}| \geq p^n$ $\forall S_1, S_2 \in Syl_p(G)$ with $S_1 \neq S_2$, then $|Syl_p(G)| \equiv 1 (\text{mod} p^n)$. In particular, we have $|Syl_p(G)| \equiv 1(\text{mod} p)$.*

1.4 Product of groups

Direct product: Let H and K be any two groups, then the *(external) direct product* of H and K, denoted by $H \times K$, is the set of all ordered pairs (h, k), where $h \in H$ and $k \in K$, with the binary operation

$$(h', k')(h'', k'') = (h'h'', k'k''), \text{ where } h', h'' \in H \text{ and } k', k'' \in K.$$

Theorem 1.4.1. ([29], page 29)
Let G be a group and H, K be two subgroups of G such that

(i) $H \cap K = \{1\}$,

(ii) $G = HK$ *and*

(iii) *Every element of H commutes with element of K, i.e.*

$$hk = kh \; \forall \; h \in H, \; \forall \; k \in K.$$

Then, $G \cong H \times K$.

Theorem 1.4.2. ([29], page 29)
Let G be a group with normal subgroups H and K. If $H \cap K = \{1\}$ and $HK = G$, then $G \cong H \times K$.

Theorem 1.4.3. ([29], page 30)
Let $G = H \times K$, *and let* $H_1 \trianglelefteq H$ *and* $K_1 \trianglelefteq K$. *Then* $H_1 \times K_1 \trianglelefteq G$ *and*

$$\frac{G}{H_1 \times K_1} \cong \frac{H}{H_1} \times \frac{K}{K_1}.$$

Semidirect product: Let X and H be any two groups and $\theta : X \to \mathrm{Aut}(H)$ be a homomorphism. Then the cartesian product $X \times H$ forms a group under the binary operation

$$(x_1,\, h_1)(x_2,\, h_2) \;=\; (x_1x_2,\, \theta(x_2)(h_1)h_2),$$

where $x_i \in X, h_i \in H, i = 1, 2$. This group is known as the *(external) semidirect product* of X with H (with respect to θ) and is denoted by $X \ltimes_\theta H$. If θ is the trivial homomorphism, then $X \ltimes_\theta H$ is the direct product of X and H.

The semidirect product of C_2 and C_3 gives S_3. The dihedral group D_{2n} is isomorphic to a semidirect product of C_2 and C_n.

1.5 Commutator Subgroup

Let $a,\ b$ be two elements of a group G, the *commutator* of a and b, denoted by $[a,\ b]$, is the element $aba^{-1}b^{-1}$. The *commutator subgroup* or *derived subgroup* of G, denoted by $[G,\ G]$ or G', is the subgroup of G generated by all the commutators in G.

Theorem 1.5.1. ([29], page 24)
The commutator subgroup is a normal subgroup, the quotient group G/G' *is*

abelian, and if H is a normal subgroup of G for which G/H is abelian then G' is contained in H.

Few Commutator Identities:

There are many commutator identities that are quite useful. Few such commutator identities are given below.

Lemma 1.5.2. ([29], page 92)
If $x,\ y,\ z \in G$ then

(i) $[x,\ y]^{-1} = [y,\ x]$

(ii) $[x,\ yz] = [x,\ y][x,\ z]^y$

(iii) $[xy,\ z] = [y,\ z]^x[x,\ z]$

1.6 Class equation

Let G be a group with *center* $Z(G) = \{x \in G|xy = yx \text{ for all } y \in G\}$. If $x \in G$, then the *centralizer* of x in G is given by $C_G(x) = \{y \in G|xy = yx\}$. Note that $Z(G) = \underset{x\in G}{\cap} C_G(x)$. Also, if G is a group and $\frac{G}{Z(G)}$ is cyclic, then G is abelian.

Let x, y be two elements of a group G. We say that x is *conjugate* to y if $x^g = gxg^{-1} = y$ for some $g \in G$. The relation '*x is conjugate to y in G*' is an equivalence relation on G. This equivalence relation yields a partition of G and each cell in the partition arising from an equivalence relation is an equivalence class. The equivalence classes are called *conjugacy classes* of G. The conjugacy class of x is denoted by $\mathrm{Cl}(x)$.

Theorem 1.6.1. *The number of conjugates of x in G is $[G : C_G(x)]$, i.e. $|\operatorname{Cl}(x)| = |G : C_G(x)|$.*

Theorem 1.6.2. (Class Equation) ([29], page 57)
Let G be a finite group, then

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

where one x_i is chosen from each conjugacy class having more than one element.

1.7 Capable groups

A group G is called *capable* if there exists a group H such that $G \cong \frac{H}{Z(H)}$.

For example All groups with trivial center, $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. All non-trivial cyclic groups are not capable groups.

Proposition 1.7.1. *[1] The groups $\langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^{-1}\rangle$, $C_3 \times D_{10}, S_3 \times C_5$ and $\langle x, y | x^6 = 1, y^2 = x^3, y^{-1}xy = x^{-1}\rangle$ are not capable.*

Proof. Suppose, for a contradiction, that there exists a group H such that

$$\frac{H}{Z(H)} \cong L = \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^{-1}\rangle.$$

So, there exist elements $h_1, h_2 \in H$ such that $|h_1 Z(H)| = 4$, $|h_2 Z(H)| = 10$ with ${h_1}^2 Z(H) = {h_2}^5 Z(H)$. Therefore $C_H(h_1)$ and $C_H(h_2)$ are subgroups of $C_H({h_1}^2)$ and so the $lcm(4, 10)$ divides $|\frac{C_H({h_1}^2)}{Z(H)}|$. This forces ${h_1}^2$ to be in $Z(H)$, which is a contradiction.

Now assume, on the contrary, that there exists a group G such that $\frac{G}{Z(G)} \cong C_3 \times D_{10}$. Then there exist two elements x and y in G such that $|xZ(G)| = 15$, $|yZ(G)| = 6$ with $x^5Z(G) = y^2Z(G)$. It follows that $C_G(x^5)$ contains both of $C_G(x)$ and $C_G(y)$. Thus $C_G(x^5) = G$, which gives $x^5 \in Z(G)$, which is not possible.

Next suppose, for a contradiction, that there exists a group K such that $\frac{K}{Z(K)} \cong S_3 \times C_5$. Thus there exist elements k_1 and k_2 in K and $|k_1Z(K)| = 15$ and $|k_2Z(K)| = 10$ with ${k_1}^3Z(K) = {k_2}^2Z(K)$. Therefore $C_K({k_1}^3)$ contains both of $C_K(k_1)$ and $C_K(k_2)$. It follows that $C_K({k_1}^3) = G$, a contradiction.

Lastly suppose, on the contrary, that there exists a group L such that

$$\frac{L}{Z(L)} \cong \langle x, y | x^6 = 1, y^2 = x^3, y^{-1}xy = x^{-1} \rangle.$$

Then there exist elements $l_1, l_2 \in L$ such that $|l_1Z(L)| = 4$, $|l_2Z(L)| = 6$ with ${l_1}^2Z(L) = {l_2}^3Z(L)$. Therefore $C_L(l_1)$ and $C_L(l_2)$ are subgroups of $C_L({l_1}^2)$ and so $lcm(4, 6)$ divides $|\frac{C_L({l_1}^2)}{Z(L)}|$. This implies that ${l_1}^2 \in Z(L)$, which cannot happen. □

Remark 1.7.2. [1] Let A be a finite abelian group. Let p be a prime number and $i > 0$ be an integer. Suppose that $r(A, p^i)$ is the number of cyclic direct summands of order p^i in the decomposition of A into cyclic groups of prime power orders. In [8], it is showed that A is capable if and only if for every prime number $p, r(A, p^i) = 1$ implies that A contains elements of order p^{i+1}.

Proposition 1.7.3. *[1]* (1) *The only capable groups of order* 12 *are* D_{12} *and* A_4.

(2) *The only capable groups of order* 20 *are*

$$D_{20} \text{ and } \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle.$$

(3) *The only capable group of order* 30 *is* D_{30}.

Proof. It is well-known that, if $n \geq 2$ is an integer then D_{2n} is a capable group. Also clearly every centerless group is capable.

(1) There are exactly four non-cyclic groups of order 12, namely

$$C_2 \times C_2 \times C_3, D_{12}, A_4 \text{ and } \langle x, y | x^6 = 1, y^2 = x^3, y^{-1}xy = x^{-1} \rangle.$$

It follows from Remark 1.7.2 and Proposition 1.7.1 that D_{12} and A_4 are the only non-cyclic capable groups of order 12.

(2) There are exactly four non-cyclic groups of order 20, namely

$C_2 \times C_2 \times C_5$, D_{20}, $T = \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle$ and $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$.

Since $Z(T) = 1$, so Remark 1.7.2 and Proposition 1.7.1 gives that D_{20} and $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle$ are the only non-cyclic capable groups of order 20.

(3) There are exactly three non-cyclic groups of order 30, namely

$$D_{30}, C_3 \times D_{10} \text{ and } S_3 \times C_5.$$

It follows from Proposition 1.7.1 that D_{30} is the only capable group of order 30. □

1.8 Solvable Groups

Normal series: A sequence $(G_0,\ G_1,\ \cdots,\ G_r)$ of subgroups of a group G is called a *normal series* of G if

$$\{1\} \ = \ G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_{r-1} \trianglelefteq G_r.$$

The factors of a normal series are the quotient groups $G_i/G_{i-1},\ 1 \le i \le r$.

Note 1.8.1. We often refer to a normal series $(G_0,\ G_1,\ \cdots,\ G_r)$ by saying that

$$\{1\} \ = \ G_0 \subset G_1 \subset \cdots \subset G_r \ = \ G$$

is a normal series of G.

As the derived subgroup G' of a group G we define the nth derived subgroup of G, written as $G^{(n)}$, as follows:

$$G^{(1)} = G', \qquad G^{(n)} = (G^{(n-1)})' \quad (n > 1).$$

A group G is said to be *solvable* if $G^{(k)} = \{1\}$ for some positive integer k.

Theorem 1.8.2. ([11], page 119)
A group G is solvable if and only if G has a normal series with abelian factors. Further, a finite group is solvable if and only if its composition factors are cyclic groups of prime orders.

Theorem 1.8.3. ([11], page 119)
Let G be a group. If G is solvable, then every subgroup of G and every homomorphic image of G are solvable.

Theorem 1.8.4. ([29], page 82)
Let N is a normal subgroup of G such that N and G/N are solvable, then G is solvable.

Corollary 1.8.5. *If H and K are solvable, then $H \times K$ is solvable.*

Corollary 1.8.6. *Every finite p-group is solvable.*

1.9 Nilpotent Groups

We define inductively the n^{th} *center* of a group G as follows. For $n = 1$, $Z_1(G) = Z(G)$. Consider the center of the quotient group $G/Z_1(G)$. Because $Z(G/Z_1(G))$ is a normal subgroup of $G/Z_1(G)$, by Corollary 1.2.5 there is a unique normal subgroup $Z_2(G)$ of G such that

$$\frac{Z_2(G)}{Z_1(G)} = Z(G/Z_1(G)).$$

Thus, inductively we obtain a normal subgroup $Z_n(G)$ of G such that

$$\frac{Z_n(G)}{Z_{n-1}(G)} = Z(G/Z_{n-1}(G))$$

for every positive integer $n > 1$. $Z_n(G)$ is called the n^{th} center of G. Setting $Z_0(G) = \{1\}$, we have

$$\frac{Z_n(G)}{Z_{n-1}(G)} = Z(G/Z_{n-1}(G))$$

for every positive integer n. It follows immediately from the definition that

$$Z_n(G) = \{x \in G \mid xyx^{-1}y^{-1} \in Z_{n-1}(G) \ \ \forall \ \ y \in G\}.$$

Hence, $(Z_n(G))' \subset Z_{n-1}(G)$.

The ascending series

$$\{1\} = Z_0(G) \subset Z_1(G) \subset \cdots \subset Z_n(G) \subset \cdots$$

of subgroups of a group G is called the *upper central* series of G.

A group G is said to be *nilpotent* if $Z_m(G) = G$ for some m. The least such integer m is called the class of nilpotency of G.

Theorem 1.9.1. ([11], page 121) *A group G is nilpotent if and only if G has a normal series*

$$\{1\} = G_0 \subset G_1 \subset \cdots \subset G_m = G$$

such that $G_i/G_{i-1} \subset Z(G/G_{i-1})$ for all $i = 1, \cdots, m$.

1.10 Covering of a group

A group G is *covered by a family of cosets or subgroups* if G is simply the set theoretic union of the family. We say that a covering of $G = \underset{i}{\cup} X_i$ is *irredundant* if none of the subgroups X_i can be ommited; that is, $X_i \nsubseteq \underset{j \neq i}{\cup} X_j$, for each i.

The maximum value of $|G : \overset{n}{\underset{i=1}{\cap}} X_i|$ in a group G with an irredundant covering by n subgroups is denoted by $f_2(n)$. [27]

Lemma 1.10.1. *[32]* ***Tomkinson's lemma***
Let M be a proper subgroup of the (finite) group G and let $H_1, \ldots, H_k$ be subgroups with $|G : H_i| = \beta_i$ and $\beta_1 \leq \cdots \leq \beta_k$. If $G = M \cup H_1 \cup \cdots \cup H_k$, then $\beta_1 \leq k$.

Furthermore, if $\beta_1 = k$*, then* $\beta_1 = \beta_2 = \cdots = \beta_k = k$ *and* $H_i \cap H_j \leq M$*, for all* $i \neq j$*.*

Theorem 1.10.2. *[32]* ***Tomkinson's Theorem***
Let G *has an irredundant covering* $G = X_1 \cup \cdots \cup X_n$*, where* $X_1, \cdots, X_n$ *be subgroups of* G *with* $|G : X_i| = \alpha_i$ *and* $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$*; and* $D = \bigcap_{i=1}^{n} X_i$*. Then*

(i) $\alpha_2 \leq n - 1$

(ii) *If* $\alpha_2 = n - 1$ *then* $|G : D| \leq (n-1)^2(n-3)!$

(iii) *If* $\alpha_2 < n - 1$ *then* $|G : D| \leq (n-2)^3(n-3)!$.

Theorem 1.10.3. *[15]* ***Cohn's Theorem***
Let $H_1, H_2, \ldots, H_n$ *be subgroups of* G*. If* $G = \bigcup_{t=1}^{n} H_t$*, then* $|G| \leq \sum_{t=2}^{n} |H_t|$*, with equality if and only if* $H_1 H_t = G$ *for all* $t \neq 1$ *and* $H_k \cap H_l \subseteq H_1$ *for all* $k \neq l$*.*

Theorem 1.10.4. *[22]* ***Ito's Theorem***
If G *is a finite group of two class lengths, then* G *is the direct product of a* p*-group* P *with an abelian group* A*.*

1.11 Some standard groups

In this section we recall some standard examples of groups. The groups which play perhaps the most crucial role in this dissertation are the cyclic groups. We write C_n to denote a cyclic group of order n.

(A) The permutation groups

The set of all bijective maps on a set X forms a group under the composition of maps. This group is called the *symmetric group* on X and is denoted by $SymX$. In particular, if $X = \{1, 2, \dots, n\}$, then $SymX$ is also denoted by S_n and is called the symmetric group of degree n. Clearly, $|S_n| = n!$.

The group of even permutations of n symbols is called the *alternating group of degree* n and is denoted by A_n. The alternating groups of degree greater than or equal to 5 form examples of an important class of groups called the simple groups. Clearly $|A_n| = \dfrac{n!}{2}$.

(B) The dihedral groups

A *dihedral group* D_{2n} of order $2n$, $n \geq 2$, is the group of all symmetries of a regular polygon with n sides. It is presented as

$$D_{2n} = \langle x,\, y | y^n = 1,\, x^2 = 1,\, xy = y^{-1}x \rangle.$$

A *semi-dihedral group* of order $n = 2^m,\ m \geq 3$, is presented as

$$SD_n = \langle s,\, t \,| s^{2^{m-1}} = 1,\, t^2 = 1,\, ts = s^{2^{m-2}-1}t \rangle.$$

(C) The generalized quaternion groups

A *generalized quaternion group* of order $n = 2^m,\ m \geq 3$, is presented as

$$Q_n = \langle x, y \,|\, x^{2^{m-1}} = 1,\ y^2 = x^{2^{m-2}},\ yx = x^{-1}y \rangle.$$

In fact, this is a generalization of the well-known group of quaternions,

$$Q_8 = \langle i,\, j,\, k \,|\, i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

(D) The dicyclic groups

A *dicyclic group* of order $4m,\ m \geq 2$, is presented as

$$Q_{4m} = \langle a, b \mid a^{2m} = 1,\ b^2 = a^m,\ ba = a^{-1}b \rangle.$$

When $m = 2$, the dicyclic group is isomorphic to the quaternion group Q_8. More generally, when n is a power of 2, the dicyclic group is isomorphic to the generalized quaternion group.

(E) The linear groups

Let F be a field. The set of all $n \times n$ invertible matrices with entries from F forms a group under matrix multiplication. This group is known as the *general linear group* of degree n over the field F and is denoted by $GL_n(F)$ or $GL(n, F)$. For $n \geq 2$, the group $GL(n, F)$ is non-abelian.

Let F be a field. The set of all $n \times n$ matrices of determinant one with the entries from F forms a group under matrix multiplication. This group is called the *special linear group* over the field F and is denoted by $SL_n(F)$ or $SL(n, F)$. In fact, $SL(n, F)$ is a subgroup of $GL(n, F)$.

SL(2, q) is the group of all 2×2 matrices of determinant one with the entries from the finite field $\mathbb{F}_q$ of q elements, where $q = p^n$ for some prime p and for some positive integer n.

Result 1.11.1. *The cardinality of* $SL(2, q)$ *is* $q(q-1)(q+1)$.

Result 1.11.2. *The exact number of conjugacy classes in* $SL(2, q)$ *is* $q + 4$ *when* q *is odd, and* $q + 1$ *when* q *is even.*

The projective special linear group is the quotient of the group of matrices $SL(n, F)$ by its center. It is denoted by $PSL(n, F)$. The cardinality of

$PSL(2,q)$ is $(q+1)(q^2-q)$ when $q=2^n$ and $\frac{1}{2}(q+1)(q^2-q)$ when $q=p^n$, p is an odd prime and $n \geq 0$.

(F) The classification theorem for finite simple groups

Every finite simple group is one of 26 sporadic simple groups or belongs (up to isomorphism) to atleast one of the following three infinite families:

(a) A cyclic group with prime order;

(b) An alternating group of degree at least 5;

(c) A simple group of Lie type, including both

  (i) the classical Lie groups, namely the groups of projective special linear, unitary, symplectic or orthogonal transformations over a finite field;

  (ii) the exceptional and twisted groups of Lie type (including the Tits group).

Chapter 2

$|\operatorname{Cent}(G)|$, the number of centralizers

Given a finite group G and an element $x \in G$, the set $C_G(x) = \{y \in G \mid xy = yx\}$ is called the centralizer of x in G. The set of all such centralizers in G is denoted by $\operatorname{Cent}(G)$. At this point one question arises 'What can we say about $|\operatorname{Cent}(G)|$?' In this chapter, our goal is to study the possible values of $|\operatorname{Cent}(G)|$.

2.1 Definition and some properties

Let G be a finite group. Then for any element $x \in G$, the set $C_G(x) = \{y \in G \mid xy = yx\}$ is called the *centralizer* of x in G; the set of all centralizers in G is denoted by $\operatorname{Cent}(G)$ and $|\operatorname{Cent}(G)|$ denotes the *number of distinct centralizers* in G. A group G is called *n-centralizer* if

$|\operatorname{Cent}(G)| = n$. For example,

(i) $D_8 = \langle x, y \mid y^4 = x^2 = 1, xyx^{-1} = y^{-1}\rangle$ is a 4-centralizer group and the centralizers are precisely

$D_8, C_{D_8}(y) = \langle y\rangle, C_{D_8}(x) = \{1, y^2, xy^2, x\}, C_{D_8}(xy) = \{1, y^2, xy^3, xy\}$.

(ii) $S_3 = \langle a, b \mid b^3 = a^2 = 1, aba^{-1} = b^{-1}\rangle$ is a 5-centralizer group and the centralizers are precisely

$S_3, C_{S_3}(a) = \{1, a\}, C_{S_3}(b) = \langle b\rangle, C_{S_3}(ab) = \{1, ab\}, C_{S_3}(ab^2) = \{1, ab^2\}$.

Proposition 2.1.1. *[10] Let G be a group. Then G is abelian if and only if $|\operatorname{Cent}(G)| = 1$.*

Proof. We know that G is abelian if and only if $C_G(x) = G$ for each $x \in G$. This proves the proposition. □

Proposition 2.1.2. *Let G, H be two groups, then*

$$\operatorname{Cent}(G \times H) = \operatorname{Cent}(G) \times \operatorname{Cent}(H).$$

Proof. It follows easily from $C_{(G\times H)}(g, h) = C_G(g) \times C_H(h)$, for any $g \in G$ and $h \in H$. □

The following lemmas play an important role in finding lower bound for $|\operatorname{Cent}(G)|$.

Lemma 2.1.3. *Let G be a group. Then $Z(G)$ is the intersection of all centralizers in G, i.e. $Z(G) = \bigcap_{x \in G} C_G(x)$.*

Proof. Clearly, $Z(G) \subseteq \underset{x \in G}{\cap} C_G(x)$. Now, suppose $y \in \underset{x \in G}{\cap} C_G(x)$ then $xy = yx \; \forall \; x \in G$, which gives $y \in Z(G)$. Hence, the lemma follows.

□

Lemma 2.1.4. *If G is a group, then G is the union of centralizers of all non-central elements of G, i.e. $G = \underset{x \in G-Z(G)}{\cup} C_G(x)$.*

Proof. Clearly, $\underset{x \in G-Z(G)}{\cup} C_G(x) \subseteq G$. Let $g \in Z(G)$, then by using Lemma 2.1.3, $g \in C_G(x) \; \forall x \in G$. So, $g \in \underset{x \in G-Z(G)}{\cup} C_G(x)$. Let $g \in G - Z(G)$, then

$$g \in C_G(g) \Rightarrow \; g \in \underset{x \in G-Z(G)}{\cup} C_G(x).$$

Therefore $G \subseteq \underset{x \in G-Z(G)}{\cup} C_G(x)$ and the lemma follows. □

Lemma 2.1.5. *A group G cannot be written as a union of two proper subgroups.*

Proof. Suppose H, K be two proper subgroups of G such that $G = H \cup K$. Let $x \in H - K$ and $y \in K - H$. Suppose $xy \in H$, then $x^{-1}xy = y \in H$, a contradiction. Again suppose $xy \in K$, then $xyy^{-1} \in K \Rightarrow \; x \in K$, a contradiction. Therefore $xy \notin G$, a contradiction. This proves the Lemma. □

Theorem 2.1.6. *[10] Let G be a non-abelian group, then $|\operatorname{Cent}(G)| \geq 4$.*

Proof. We have that G is the union of its proper centralizers, i.e.

$$G = \underset{x \in G-Z(G)}{\cup} C_G(x) \text{ (by Lemma 2.1.4)}.$$

If $|\operatorname{Cent}(G)| = 1$ then G is abelian, a contradiction. If $|\operatorname{Cent}(G)| = 2$ then G is the proper subgroup of itself, which is not possible. Suppose

$|\operatorname{Cent}(G)| = 3$, then $\operatorname{Cent}(G) = \{G, C_G(x), C_G(y)\}$, where $C_G(x)$ and $C_G(y)$ are proper centralizers of G. Therefore $G = C_G(x) \cup C_G(y)$, which is not possible by Lemma 2.1.5.

Hence, $|\operatorname{Cent}(G)| \geq 4$. This completes the proof. □

2.2 Some computation of $|\operatorname{Cent}(G)|$

In [10] Belcastro and Sherman make two questions

(i) *Can we make $|\operatorname{Cent}(G)|$, anything we like, i.e. if n is a positive integer other than two or three, does there exists a group with $n = |\operatorname{Cent}(G)|$ centralizers?*

(ii) *Which values of $|\operatorname{Cent}(G)|$, other than 4 and 5, characterize G?*

The following proposition gives an affirmative answer to the first question. Of course the second question is still not answered completely.

Proposition 2.2.1. *[4] There exist n-centralizer groups for $n \neq 2, 3$.*

Proof. We know that any abelian group is 1-centralizer, so we can assume $n \geq 4$. Now we consider the dicyclic group $Q_{4m}, m \geq 2$. The group Q_{4m} is defined by

$$\begin{aligned} Q_{4m} &= \langle a, b | a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1} \rangle \\ &= \{1, a, a^2, \ldots, a^{2m-1}, b, ba, \ldots, ba^{2m-1}\} \end{aligned}$$

Here $C_{Q_{4m}}(1) = Q_{4m}$. Now consider $C_{Q_{4m}}(a^i), 1 \leq i \leq 2m-1$. Suppose $b \in C_{Q_{4m}}(a^i)$, then

$$ba^i = a^ib \Rightarrow a^i = ba^ib^{-1} \Rightarrow a^{2i} = 1 \Rightarrow 2m \mid 2i \Rightarrow i = mt,$$

where $t \in \mathbb{N}$.

If $t > 1$ then $i \geq 2m$, a contradiction. Therefore $t = 1 \Rightarrow i = m$. Now suppose $a^kb \in C_{Q_{4m}}(a^i)$, where $1 \leq k \leq 2n-1$. Then

$$\begin{aligned}
(a^kb)a^i &= a^i(a^kb) \\
\Rightarrow a^kba^ib^{-1} &= a^{i+k} \\
\Rightarrow a^ka^{-i} &= a^{i+k} \\
\Rightarrow a^{2i} &= 1 \\
\Rightarrow 2m &\mid 2i \\
\Rightarrow i &= ml,
\end{aligned}$$

where $l \in \mathbb{N}$. If $l > 1$ then $i \geq 2m$, a contradiction. Therefore $i = m$. Hence, $C_{Q_{4m}}(a^i) = \{1, a, \ldots, a^{2m-1}\}$, where $1 \leq i(\neq m) \leq 2m-1$.

Next consider $C_{Q_{4m}}(a^ib)$, where $1 \leq i \leq 2m-1$. Suppose $a^j \in C_{Q_{4m}}(a^ib)$, where $1 \leq j \leq 2m-1$. Then

$$a^j(a^ib) = (a^ib)a^j \Rightarrow a^{2j} = 1 \Rightarrow m \mid j \Rightarrow j = m.$$

Suppose $a^kb \in C_{Q_{4m}}(a^ib)$, where $0 \leq k \leq 2m-1$. Then

$$(a^kb)(a^ib) = (a^ib)(a^kb) \Rightarrow a^{2(k-i)} = 1 \Rightarrow 2m \mid 2(k-i) \Rightarrow k-i = mt,$$

where $t \in Z$.

If $t \geq 2$ then $k - i \geq 2m$, a contradiction. If $t \leq -2$ then $i - k \geq 2m$, a contradiction. Hence $t = 0, 1, -1$ and therefore $k = i, m+i, i-m$. Also $a^{i-m} = a^{i+m}$. Thus $C_{Q_{4m}}(a^i b) = \{1, a^m, a^i b, a^{i+m} b\}$, where $0 \leq i \leq 2m-1$. Therefore

$$\begin{aligned}
C_{Q_{4m}}(b) &= \{1, a^m, b, a^m b\} \\
C_{Q_{4m}}(ab) &= \{1, a^m, ab, a^{m+1} b\} \\
&\vdots \\
C_{Q_{4m}}(a^{m-1} b) &= \{1, a^m, a^{m-1} b, a^{2m-1} b\} \\
C_{Q_{4m}}(a^m b) &= \{1, a^m, a^m b, b\} \\
&\vdots \\
C_{Q_{4m}}(a^{2m-1} b) &= \{1, a^m, a^{2m-1} b, a^{m-1} b\}.
\end{aligned}$$

Hence $|\operatorname{Cent}(Q_{4m})| = m + 2$, where $m \geq 2$. Again since $n \geq 4$, therefore $(n-2) \geq 2$; and so $|\operatorname{Cent}(Q_{4(n-2)})| = (n-2) + 2 = n$. This proves the proposition. $\square$

Proposition 2.2.2. *[5] Let D_{2m} be the dihedral group of order $2m$. Then*

$$|\operatorname{Cent}(D_{2m})| = \begin{cases} m + 2, & \text{if } 2 \nmid m \\ \frac{m}{2} + 2, & \text{if } 2 \mid m. \end{cases}$$

Proof. The group $D_{2m}, m \geq 3$ can be presented in the form

$$\begin{aligned}
D_{2m} &= \langle x, y | x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle \\
&= \{1, x, \ldots, x^{m-1}, y, yx, \ldots, yx^{m-1}\}.
\end{aligned}$$

Now $C_{D_{2m}}(1) = D_{2m}$. Next consider $C_{D_{2m}}(x^i), 1 \leq i \leq m-1$. Suppose $yx^j \in C_{D_{2m}}(x^i)$, then

$$(yx^j)x^i = x^i(yx^j) \Rightarrow yx^i = x^iy \Rightarrow x^{2i} = 1 \Rightarrow m \mid 2i.$$

If m is odd then $m \mid i$, a contradiction. If m is even then $2i = mt, t \in \mathbb{N}$. If $t > 1$ then $2i \geq 2m \Rightarrow i \geq m$, a contradiction. So $t = 1$. Therefore $i = \frac{m}{2}$. Thus $C_{D_{2m}}(x^i) = \langle x \rangle$, if m is odd and $C_{D_{2m}}(x^k) = \langle x \rangle$, if m is even, where $1 \leq k \leq m-1$ and $k \neq \frac{m}{2}$.

Next consider $C_{D_{2m}}(yx^j), 0 \leq j \leq m-1$. Suppose $x^i \in C_{D_{2m}}(yx^j)$ then $x^{2i} = 1 \Rightarrow m \mid 2i$. If m is odd then $m \mid i$, a contradiction. If m is even then $m = 2i \Rightarrow i = \frac{m}{2}$. Therefore if m is odd then $x^i \notin C_{D_{2m}}(yx^j)$ and if m is even then $x^{\frac{m}{2}} \in C_{D_{2m}}(yx^j)$ and $C_{D_{2m}}(x^{\frac{m}{2}}) = D_{2m}$. Now suppose $yx^k \in C_{D_{2m}}(yx^j)$, where $0 \leq k(\neq j) \leq m-1$. Then

$$\begin{aligned}(yx^k)(yx^j) &= (yx^j)(yx^k)\\ \Rightarrow x^{k-j}y &= yx^{k-j}\\ \Rightarrow x^{2(k-j)} &= 1\\ \Rightarrow m &\mid 2(k-j).\end{aligned}$$

If m is odd then $m \mid k-j$, a contradiction. If m is even then $k-j = mt$, $t \in Z$. Suppose $t \geq 2$ then $k-j \geq 2m$, a contradiction. Similarly $t \leq -2$ gives $j-k \geq 2m$, a contradiction. Therefore $t = 0, 1, -1$. If $t = 0$ then $k = j$, a contradiction. So $k = j-m, j+m$. Also $yx^{j-m} = yx^{j+m}$. Hence

$$|C_{D_{2m}}(yx^j)| = \begin{cases} \{1, yx^j\}, & \text{if } m \text{ is odd} \\ \{1, x^{\frac{m}{2}}, yx^j, yx^{j+m}\}, & \text{if } m \text{ is even.} \end{cases}$$

Therefore $|\operatorname{Cent}(D_{2m})| = m+2$, if m is odd.

Again for m even,

$$\begin{aligned}
C_{D_{2m}}(y) &= \{1, x^{\frac{m}{2}}, y, yx^{m}\} \\
&\vdots \\
C_{D_{2m}}(yx^{\frac{m}{2}-1}) &= \{1, x^{\frac{m}{2}}, yx^{\frac{m}{2}-1}, yx^{m-1}\} \\
C_{D_{2m}}(yx^{\frac{m}{2}}) &= \{1, x^{\frac{m}{2}}, yx^{\frac{m}{2}}, y\} \\
&\vdots \\
C_{D_{2m}}(yx^{m-1}) &= \{1, x^{\frac{m}{2}}, yx^{m-1}, yx^{\frac{m}{2}-1}\}.
\end{aligned}$$

Therefore $|\operatorname{Cent}(D_{2m})| = \frac{m}{2}+2$, if m is even. Hence

$$|\operatorname{Cent}(D_{2m})| = \begin{cases} m+2, & \text{if } m \text{ is odd} \\ \frac{m}{2}+2, & \text{if } m \text{ is even.} \end{cases}$$

This completes the proof. □

Proposition 2.2.3. *[10] Let G be a group and p be a prime. If $\frac{G}{Z(G)} \cong C_p \times C_p$ then $|\operatorname{Cent}(G)| = p+2$. If p is odd and $\frac{G}{Z(G)} \cong D_{2p}$, then $|\operatorname{Cent}(G)| = p+2$.*

Proof. Suppose, first that $\frac{G}{Z(G)} \cong C_p \times C_p$. Then

$$\begin{aligned}
\frac{G}{Z} &= \langle Zx, Zy | (Zx)^p = (Zy)^p = Z, (Zx)(Zy) = (Zy)(Zx)\rangle, \; Z = Z(G) \\
&= \langle Zx, Zy | x^p, y^p, xyx^{-1}y^{-1} \in Z\rangle.
\end{aligned}$$

If $\frac{H}{Z} < \frac{G}{Z}$, then $\frac{|G/Z|}{|H/Z|} = p \Rightarrow |\frac{H}{Z}| = p$. Therefore $H = Z \sqcup Zt_1 \sqcup \cdots \sqcup Zt_{p-1}$, where $t_i \in H - Z$ and $i \in \{1, 2, \ldots, p-1\}$. So, the proper subgroups of G

properly containing Z are

$$\begin{aligned} H_1 &= Z \sqcup Zx \sqcup Zx^2 \sqcup \cdots \sqcup Zx^{p-1}, \\ H_2 &= Z \sqcup Zy \sqcup Zy^2 \sqcup \cdots \sqcup Zy^{p-1}, \\ H_3 &= Z \sqcup Zxy \sqcup Zx^2y^2 \sqcup \cdots \sqcup Zx^{p-1}y^{p-1}, \\ &\vdots \\ H_{p+1} &= Z \sqcup Zx^{p-1}y \sqcup Zx^{p-2}y^2 \sqcup \cdots \sqcup Zxy^{p-1}. \end{aligned}$$

Now we will show that $H_1, H_2, \ldots, H_{p+1}$ are the only proper centralizers of G. Let $a \in G - Z$ then $Za = Zk$ for some

$$k \in \{x, \ldots, x^{p-1}, y, \ldots, y^{p-1}, xy, xy^2, \ldots, xy^{p-1}, \ldots, x^{p-1}y, \ldots, x^{p-1}y^{p-1}\}.$$

Therefore $C_{\frac{G}{Z}}(Za) = C_{\frac{G}{Z}}(Zk) \Rightarrow C_G(a) = C_G(k)$ (Using Lemma 2.4.3). Again let $k \in H_i - Z$ then $C_G(k) \in \underset{\substack{j=1 \\ i \neq j}}{\overset{p+1}{\cup}} H_j$, as $H_1, H_2, \ldots, H_{p+1}$ are the only proper subgroups of G. Also $k \in C_G(k)$, therefore $C_G(k) \neq H_j, 1 \leq j \leq p+1$ and $i \neq j$. Therefore $C_G(k) = H_i$. Hence $H_1, H_2, \ldots, H_{p+1}$ are the only proper centralizers of G. Thus $|\operatorname{Cent}(G)| = p + 2$.

Next suppose that $\frac{G}{Z} \cong D_{2p}$. Then

$$\begin{aligned} \frac{G}{Z} &= \langle Zx, Zy | (Zx)^2 = (Zy)^p = Z, (Zx)(Zy)(Zx)^{-1} = (Zy)^{-1} \rangle \\ &= \{Z, Zy, \ldots, Zy^{p-1}, Zx, Zxy, \ldots, Zxy^{p-1}\}. \end{aligned}$$

If $\frac{K}{Z} < \frac{G}{Z}$ then $\frac{|\frac{G}{Z}|}{|\frac{K}{Z}|} = p$ or $2 \Rightarrow |\frac{K}{Z}| = 2$ or p. Therefore $K = Z \sqcup Zl$ or $K = Z \sqcup Zt_1 \sqcup Zt_2 \sqcup \cdots \sqcup Zt_{p-1}$, where $t_1, t_2, t_{p-1}, l \in K - Z$. So the proper subgroups of G properly containing Z are

$$K_1 = Z \sqcup Zy \sqcup \cdots \sqcup Zy^{p-1}, K_2 = Z \sqcup Zx, \ldots, K_{p+1} = Z \sqcup Zxy^{p-1}.$$

We are to show that $K_1, K_2, K_3, \ldots, K_{p+1}$ are the only proper centralizers of G. Let $b \in G - Z$ then $Zb = Zk$ for some $k \in \{y, \ldots, y^{p-1}, x, xy, \ldots, xy^{p-1}\}$. Therefore $C_{\frac{G}{Z}}(Zb) = C_{\frac{G}{Z}}(Zk) \Rightarrow C_G(b) = C_G(k)$ (Using Lemma 2.4.3). Again let $k \in K_i - Z$, then $k \notin K_j, 1 \leq j (\neq i) \leq (p+1)$. Also $k \in C_G(k)$, therefore $C_G(k) \neq K_j$. So $C_G(k) = K_i$. Hence $K_1, K_2, K_3, \ldots, K_{p+1}$ are the only proper centralizers of G. Thus $|\operatorname{Cent}(G)| = p + 2$. This completes the proof. □

Proposition 2.2.4. [5] The group U_{6n} is defined by

$$U_{6n} = \langle a, b | a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle.$$

Then for all n, $|\operatorname{Cent}(U_{6n})| = 5$.

Proof. The group U_{6n} is given by

$$U_{6n} = \langle a, b | a^{2n} = b^3 = e, a^{-1}ba = b^{-1} \rangle.$$

Now we calculate $Z(U_{6n})$. Let $a^i b^j \in Z(U_{6n}), 1 \leq i \leq 2n$ and $1 \leq j \leq 3$. Then $(a^i b^j)a = a(a^i b^j) \Rightarrow b^j a = ab^j \Rightarrow b^{2j} = 1 \Rightarrow 3 \mid 2j \Rightarrow j = 3$. Also $(a^i b^3)b = b(a^i b^3) \Rightarrow a^i b = ba^i$. Again $a^{-1}ba = b^{-1} \Rightarrow a^2 \in Z(U_{6n})$.

Case 1. i is even. In this case, $i = 2k$ for some $k \in \mathbb{N}$. So $a^{2k} \in Z(U_{6n})$. (As $a^2 \in Z(U_{6n})$)

Case 2. i is odd. In this case, $i = 2t + 1$ for some $t \in \mathbb{N}$. Suppose $a^i \in C_{U_{6n}}(b)$ then we have $a^{2t+1}b = ba^{2t+1} \Rightarrow ab = ba$, a contradiction.

Hence $\langle a^2 \rangle = Z(U_{6n})$. Therefore $|Z(U_{6n})| = |\langle a^2 \rangle| = o(a^2) = n$. So $|\frac{U_{6n}}{Z(U_{6n})}| = 6 \Rightarrow \frac{U_{6n}}{Z(U_{6n})} \cong S_3$. Hence, the proposition follows. □

Proposition 2.2.5. *[5] Let $SD_{2^n}, n > 3$ be the semi-dihedral group of order 2^n. Then $|\operatorname{Cent}(SD_{2^n})| = 2^{n-2} + 2$.*

Proof. The group SD_{2^n} is defined by

$$SD_{2^n} = \langle a, b | a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{-1+2^{n-2}} \rangle.$$

Here $ba^{2^{n-2}}b^{-1} = (bab^{-1})^{2^{n-2}} = (a^{-1+2^{n-2}})^{2^{n-2}} = a^{-2^{n-2}}(a^{2^{n-2}})^{2^{n-2}} = a^{2^{n-2}}$. Therefore $a^{2^{n-2}} \in Z(SD_{2^n})$. So $C_{SD_{2^n}}(a^{2^{n-2}}) = SD_{2^n}$.

Now consider $C_{SD_{2^n}}(a^i), 1 \leq i \leq 2^{n-1} - 1, i \neq 2^{n-2}$. Let $b \in C_{SD_{2^n}}(a^i)$ then

$$\begin{aligned} ba^i &= a^i b \\ \Rightarrow a^i &= ba^i b^{-1} \\ \Rightarrow a^i &= a^{-i}(a^{2^{n-2}i}) \\ \Rightarrow (a^{2^{n-2}})^i &= a^{2i}. \end{aligned}$$

Case 1. i is even. In this case, we get $a^{2i} = 1 \Rightarrow 2^{n-1} \mid 2i \Rightarrow 2i = 2^{n-1}k$ for some $k \in \mathbb{N}$. If $k \geq 2$ then $i \geq 2^{n-1}$, a contradiction. If $k = 1$ then $i = 2^{n-2}$, which is not the case.

Case 2. i is odd, i.e. $i = 2l + 1$ for some $l \in \mathbb{N}$. In this case, we get

$$\begin{aligned} (a^{2^{n-2}})^{2l+1} &= a^{2i} \\ \Rightarrow a^{2^{n-2}} &= a^{2i} \\ &\Rightarrow 2^{n-1} \mid 2^{n-2} - 2i \\ \Rightarrow i &= 2^{n-3} - 2^{n-2}m \\ &\Rightarrow i \text{ is even, which is not the case.} \end{aligned}$$

Hence $C_{SD_{2^n}}(a^i) = \langle a \rangle$, where $1 \leq i \leq 2^{n-1}-1, i \neq 2^{n-2}$.

Next consider $C_{SD_{2^n}}(a^i b), 1 \leq i \leq 2^{n-1}$. Let $a \in C_{SD_{2^n}}(a^i b)$ then $a(a^i b) = (a^i b)a \Rightarrow ab = ba$, a contradiction. So $\langle a \rangle \cap C_{SD_{2^n}}(a^i b) = \{1, a^{2^{n-2}}\}$. Let $a^j b \in C_{SD_{2^n}}(a^i b)$, where $0 \leq j (\neq i) \leq 2^{n-1}-1$. Then

$$\begin{aligned} (a^j b)(a^i b) &= (a^i b)(a^j b) \\ \Rightarrow b a^{i-j} b^{-1} &= a^{i-j} \\ \Rightarrow (a^{-1+2^{n-2}})^{i-j} &= a^{i-j} \\ \Rightarrow a^{-(i-j)} (a^{2^{n-2}})^{i-j} &= a^{i-j} \\ \Rightarrow a^{2(i-j)} a^{-2^{n-2}(i-j)} &= 1. \end{aligned}$$

Case i. $|i-j|$ is even. In this case, we get $a^{2(i-j)} = 1$ which gives $2^{n-1} \mid 2(i-j) \Rightarrow 2(i-j) = 2^{n-1}k$, for some $k \in \mathbb{Z}$. If $k \geq 2$ then we get $i-j \geq 2^{n-1}$, a contradiction and if $k \leq -2$ then $j-i \geq 2^{n-1}$, a contradiction. Therefore $k = 0, 1, -1$. For $k = 0$ we have $i = j$, which is not the case. For $k = 1, 2(i-j) = 2^{n-1} \Rightarrow j = i - 2^{n-2}$ and for $k = -1, j = i + 2^{n-2}$. Also $a^{i-2^{n-2}} b = a^{i+2^{n-2}} b$. Therefore $a^{i+2^{n-2}} b \in C_{SD_{2^n}}(a^i b)$, where $0 \leq i \leq 2^{n-1}-1$.

Case ii. $|i-j|$ is odd. In this case, we get

$$\begin{aligned} a^{2(i-j)} a^{-2^{n-2}} &= 1 \\ \Rightarrow 2^{n-1} &\mid 2(i-j) - 2^{n-2} \\ \Rightarrow 2(i-j) - 2^{n-2} &= 2^{n-1}k, \text{ for some } k \in \mathbb{N} \\ \Rightarrow i - j &= 2^{n-3}l, \text{ where } 1 + 2k = l, n > 3 \\ \Rightarrow |i-j| &\text{ is even, which is not the case.} \end{aligned}$$

Thus $C_{SD_{2^n}}(a^ib) = \{1, a^{2^{n-2}}, a^ib, a^{i+2^{n-2}}b\}$, where $1 \leq i \leq 2^{n-1}$. Therefore

$$\begin{aligned}
C_{SD_{2^n}}(ab) &= \{1, a^{2^{n-2}}, ab, a^{1+2^{n-2}}b\} \\
C_{SD_{2^n}}(a^2b) &= \{1, a^{2^{n-2}}, a^2b, a^{2+2^{n-2}}b\} \\
&\vdots \\
C_{SD_{2^n}}(a^{2^{n-2}}b) &= \{1, a^{2^{n-2}}, a^{2^{n-2}}b, b\} \\
C_{SD_{2^n}}(a^{2^{n-2}+1}b) &= \{1, a^{2^{n-2}}, a^{1+2^{n-2}}b, ab\} \\
&\vdots \\
C_{SD_{2^n}}(a^{2^{n-1}}b) &= \{1, a^{2^{n-2}}, b, a^{2^{n-2}}b\}.
\end{aligned}$$

Hence $|\operatorname{Cent}(SD_{2^n})| = 1 + 1 + 2^{n-2} = 2 + 2^{n-2}$. This completes the proof.

□

Proposition 2.2.6. *[4] Let $n \geq 2$ be an integer and let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2n}$. Then $|\operatorname{Cent}(G)| = n + 2$.*

Proof. Given $\frac{G}{Z} \cong D_{2n}$, where $Z = Z(G)$. Then

$$\frac{G}{Z} = \langle xZ, yZ | y^2Z = x^nZ = Z, (yZ)(xZ)(yZ)^{-1} = (xZ)^{-1}\rangle$$

Therefore $G = Z \sqcup xZ \sqcup \ldots \sqcup x^{n-1}Z \sqcup yZ \sqcup yxZ \sqcup \ldots \sqcup yx^{n-1}Z$. Now let us we consider $C_{\frac{G}{Z}}(x^iZ)$, $0 \leq i \leq n-1$. Let $y^kx^lZ \in C_{\frac{G}{Z}}(x^iZ)$ where $0 \leq l \leq n-1, k = 0, 1$; then $(y^kZ)(x^iZ) = (x^iZ)(y^kZ)$. If $k = 0$ then it is obvious. If $k = 1$ then

$$\begin{aligned}
&(yZ)(x^iZ) = (x^iZ)(yZ) \\
&\Rightarrow x^{2i}Z = Z \\
&\Rightarrow n \mid 2i.
\end{aligned}$$

For n odd, $n \mid i$, a contradiction. For n even, $2i = nt$ where $t \in \mathbb{N}$. If $t \geq 2$ then $n \leq i$, a contradiction. So $t = 1$ that is $i = \frac{n}{2}$. Thus $C_{\frac{G}{Z}}(x^iZ) = \{Z, xZ, \ldots, x^{n-1}Z\}$, if n is odd; and for $i \neq \frac{n}{2}$, we have $C_{\frac{G}{Z}}(x^iZ) = \{Z, xZ, \ldots, x^{n-1}Z\}$, if n is even.

Now consider $C_{\frac{G}{Z}}(yx^jZ), 0 \leq j \leq n-1$. Let $y^kx^lZ \in C_{\frac{G}{Z}}(yx^jZ)$ where $0 \leq l \leq n-1; k = 0, 1$. Then $(y^kx^lZ)(yx^jZ) = (yx^jZ)(y^kx^lZ)$.

Case 1. $k = 0$. In this case,

$$(x^lZ)(yx^jZ) = (yx^jZ)(x^lZ) \Rightarrow x^{2l}Z = Z \Rightarrow n \mid 2l.$$

For n odd, $n \mid l$, a contradiction. For n even, $2l = np$ where $p \in \mathbb{N}$. If $p \geq 2$ then $n \leq l$, a contradiction. So $p = 1$ and hence $l = \frac{n}{2}$.

Case 2. $k = 1$. In this case,

$$(yx^lZ)(yx^jZ) = (yx^jZ)(yx^lZ) \Rightarrow x^{2(l-j)}Z = Z \Rightarrow n \mid 2(l-j).$$

For n odd, $n \mid l-j$, a contradiction. For n even, $2(l-j) = nt$, for some $t \in \mathbb{Z}$. If $t \geq 2$ then $l - j \geq n$, a contradiction. If $t \leq -2$ then $j - l \geq n$, a contradiction. Therefore, $t = 0, 1, -1$. If $t = 0$ then $l = j$. If $t = 1$ then $l = \frac{n}{2} + j$. If $t = -1$ then $l = j - \frac{n}{2}$. Again $yx^{j-\frac{n}{2}} = yx^{j+\frac{n}{2}}$.

Thus $C_{\frac{G}{Z}}(yx^jZ) = \{Z, yx^jZ\}$, if n is odd. Also $\frac{C_G(yx^j)}{Z} \subseteq C_{\frac{G}{Z}}(yx^jZ)$, so $\frac{C_G(yx^j)}{Z} = C_{\frac{G}{Z}}(yx^jZ) = \{Z, yx^jZ\}$. Therefore $C_G(yx^j) = Z \sqcup yx^jZ$, where $0 \leq j \leq n-1$.

Again, from above we have $C_{\frac{G}{Z}}(yx^jZ) = \{Z, yx^jZ, yx^{j+\frac{n}{2}}Z\}$, if n is even. Here $C_{\frac{G}{Z}}(yx^{j+\frac{n}{2}}) = C_{\frac{G}{Z}}(yx^jZ)$ (Since $x^{\frac{n}{2}}Z \in Z(\frac{G}{Z})$) and using a simple calculation we can see that $\frac{C_G(yx^j)}{Z} \subseteq C_{\frac{G}{Z}}(yx^jZ)$. Suppose

$\frac{C_G(yx^j)}{Z} = C_{\frac{G}{Z}}(yx^jZ)$, for some j, $0 \leq j \leq n-1$. Then

$$\frac{C_G(yx^{j+\frac{n}{2}})}{Z} \subseteq C_{\frac{G}{Z}}(yx^{j+\frac{n}{2}}Z) = C_{\frac{G}{Z}}(yx^j) = \frac{C_G(yx^j)}{Z}.$$

Therfore $C_G(yx^{j+\frac{n}{2}}) \subseteq C_G(yx^j)$. So $yx^{j+\frac{n}{2}} \in C_G(yx^j) \Rightarrow x^{\frac{n}{2}} \in Z$, which is a contradiction. Therefore $\frac{C_G(yx^j)}{Z} < C_{\frac{G}{Z}}(yx^jZ)$, where $0 \leq j \leq n-1$. Hence $|\frac{C_G(yx^j)}{Z}| = 2$ and so $C_G(yx^j) = \{Z, yx^jZ\}, 0 \leq j \leq n-1$.

Again $\frac{C_G(x^i)}{Z} \subseteq C_{\frac{G}{Z}}(x^iZ)$, where $1 \leq i \leq n-1$. If $\frac{C_G(x^i)}{Z} < C_{\frac{G}{Z}}(x^iZ)$ for some $i \in \{1, 2, \ldots, n-1\}$, then $x^kZ \notin \frac{C_G(x^i)}{Z}$ for some $k \in \{1, 2, \ldots, n-1\}$, $k \neq i \Rightarrow x^k \notin C_G(x^i)$, which is not true. Hence $\frac{C_G(x^i)}{Z} = C_{\frac{G}{Z}}(x^iZ)$. Therefore, $C_G(x^i) = \{Z, xZ, \ldots, x^{n-1}Z\}, 1 \leq i \leq n-1$.

Thus, $\text{Cent}(G) = \{G, C_G(x), C_G(y), C_G(yx), \ldots, C_G(yx^{n-1})\}$. Hence $|\text{Cent}(G)| = n+2$. This completes the proof.

□

Proposition 2.2.7. *[4] Let $F = GF(2^n)$, a finite field of order 2^n and θ be an automorphism of F. Consider the group $A(n, \theta)$ consisting of all matrices of the form*

$$U(a, b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \theta(a) & 1 \end{bmatrix}$$

with $a, b \in F$. Then $|\text{Cent}(A(n, \theta))| = 2^n$.

Proof. We know that $A(n, \theta)$ is a group under the matrix multipication, i.e., $U(a, b)U(a', b') = U(a + a', b + b' + a'\theta(a))$ where $a, a', b, b' \in F$. We will show that if θ is non-trivial then $Z(A(n, \theta)) = \{U(0, b)|b \in F\}$. Let

$U(x,y) \in Z(A(n,\theta))$ then for all $U(a,b) \in A(n,\theta)$, we have

$$\begin{aligned} U(x,y)U(a,b) &= U(a,b)U(x,y) \\ \Rightarrow U(x+a, y+b+a\theta(x)) &= U(a+x, b+y+x\theta(a)) \\ \Rightarrow a\theta(x) &= x\theta(a) \\ \Rightarrow \theta(xa^{-1}) &= xa^{-1} \\ \Rightarrow xa^{-1} &= 0 \text{ (Since } \theta \text{ is non-trivial)} \\ \Rightarrow x &= 0. \end{aligned}$$

Therefore, $Z(A(n,\theta)) = \{U(0,b) | b \in F\}$. Now let $\theta : F \longrightarrow F$ be an automorphism such that $\theta(x) = x^2 \ \forall\ x \in F$. Suppose $a \neq 0$ and therefore $U(a,b) \notin Z(A(n,\theta))$.

Now consider $C(U(a,b))$. Let $U(r,s) \in C(U(a,b))$ where $r,s \in F$. Then

$$\begin{aligned} U(r,s)U(a,b) &= U(a,b)U(r,s) \\ \Rightarrow U(r+a, s+b+a\theta(r)) &= U(a+r, b+s+r\theta(a)) \\ \Rightarrow s+b+a\theta(r) &= b+s+r\theta(a) \\ \Rightarrow \theta(ra^{-1}) &= ra^{-1} \\ \Rightarrow (ra^{-1})^2 &= ra^{-1} \\ \Rightarrow ra^{-1} &= 1 \\ \Rightarrow r &= a. \end{aligned}$$

Therefore, $C(U(a,b)) = \{U(0,s) | s \in F\} \cup \{U(a,s) | a,s \in F\}$. Similarly if $a' \neq 0$ then $C(U(a',b')) = \{U(0,s') | s' \in F\} \cup \{U(a',s') | a',s' \in F\}$. Therefore

$$C(U(a,b)) = C(U(a',b')) \text{ if and only if } a = a'. \tag{2.2.a}$$

Now

$$\begin{aligned}\text{Cent}(A(n,\theta)) &= \{C(U(a,b))|U(a,b) \in A(n,\theta)\} \\ &= \{C(U(a,b))|a \in F\}. \text{ (Using (2.2.a))}\end{aligned}$$

Therefore, $|\text{Cent}(A(n,\theta))| = |\{C(U(a,b))|a \in F\}| = |F| = 2^n$, which completes the proof. □

Proposition 2.2.8. *[4] Let* $F = GF(q), q = p^n$ *is a prime power, a finite field of order* $q = p^n$*. Consider the group* $A(n,p)$ *consisting of all matrices of the form*

$$V(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

with $a,b,c \in F$*. Then* $|\text{Cent}(A(n,p))| = p^n + 2$*.*

Proof. We know that $A(n,p)$ is a group under the matrix multipication. i.e. $V(a,b,c)V(a',b',c') = V(a+a',b+b'+ca',c+c')$ where $a,b,c,a',b',c' \in F$. We will show that $Z(A(n,p)) = \{V(0,r,0)|r \in F\}$. Let $V(x,y,z)$ be an arbitrary element of $Z(A(n,p))$ then for all $V(a,b,c) \in A(n,p)$, we have

$$\begin{aligned} V(x,y,z)V(a,b,c) &= V(a,b,c)V(x,y,z) \\ \Rightarrow V(x+a,y+b+za,z+c) &= V(a+x,b+y+cx,c+z) \\ \Rightarrow y+b+za &= b+y+cx \\ \Rightarrow za &= cx. \end{aligned}$$

In particular for $a = 1, c = 0$ we get $z.1 = 0.x \Rightarrow z = 0$. Also for $a = 0, c = 1$ we get $z.0 = 1.x \Rightarrow x = 0$ and for $a = 1, c = 1$ we get

$z = x$. Hence $z = x = 0$. Therefore

$$Z(A(n,p)) = \{V(0,r,0) | r \in F\}.$$

Using similar arguments to those in Proposition 2.2.7, we can see that $|\operatorname{Cent}(A(n,p))| = p^n + 2$. This completes the proof. $\square$

Proposition 2.2.9. *[4] If G is a non-abelian p-group, then*

$$|\operatorname{Cent}(G)| \geq p+2, \textit{ with equality if and only if } \frac{G}{Z(G)} \cong C_p \times C_p.$$

Proof. We will prove this using induction on p.

We know that, if G is a non-abelian group then by Proposition 2.1.6, $|\operatorname{Cent}(G)| \geq 4 = 2+2$. So for $p = 2$, the result is true. Assume that p is an odd prime. Suppose $p = 3$ and $|\operatorname{Cent}(G)| = 4$. Then by Theorem 2.3.1, we have

$$\begin{aligned} \frac{G}{Z(G)} &\cong C_p \times C_p \\ \Rightarrow \left|\frac{G}{Z(G)}\right| &= 4 \\ \Rightarrow 4|Z(G)| &= |G| \\ \Rightarrow 4 &\mid |G|, \end{aligned}$$

which is a contradiction, as G is a 3-group. Therefore, $|\operatorname{Cent}(G)| \geq 5 = 3+2$.

Now let $p \geq 5$ and G be a non-abelian group with $r = |\operatorname{Cent}(G)| \leq p+1$. Suppose $X_1, \ldots, X_r$ are distinct centralizers of G such that $|X_1| \geq \cdots \geq |X_r|$ and $X_1 = G$. So $G = \bigcup_{i=2}^{r} X_i$ and by Cohn's Theorem 1.10.3 we have

$$|G| \leq \sum_{i=3}^{r} |X_i|.$$

Also, $|X_i| \leq \frac{|G|}{p}$, since G is a p-group and so $|X_i|$ is at most $\frac{|G|}{p}$, where $i \neq 1$. Hence

$$|G| \leq \underbrace{\frac{|G|}{p} + \cdots + \frac{|G|}{p}}_{(r-2)-\text{times}}$$

$$\Rightarrow \ |G| \leq (r-2)\frac{|G|}{p}.$$

Therefore, $p \leq r-2 \Rightarrow \ r \geq p+2$, a contradiction. Therefore

$$|\operatorname{Cent}(G)| \ = \ r > p+1$$

$$\Rightarrow |\operatorname{Cent}(G)| \geq p+2.$$

Second part: Suppose that $\frac{G}{Z(G)} \cong C_p \times C_p$ then by Proposition 2.2.3 we have $|\operatorname{Cent}(G)| \ = \ p+2$.

Conversely, assume that $r = |\operatorname{Cent}(G)| = p+2$ so that $\frac{r-2}{p} = 1$. Suppose $X_1, X_2, \ldots, X_r$ are distinct centralizers of G such that $|X_1| \geq \cdots \geq |X_r|$ and $X_1 \ = \ G$. So $G \ = \ \bigcup_{i\,=\,2}^{r} X_i$ and by Cohn's Theorem 1.10.3 we have

$$|G| \leq \sum_{i\,=\,3}^{r} |X_i|.$$

Also, $|X_i| \leq \frac{|G|}{p}$, since G is a p-group and so $|X_i|$ is at most $\frac{|G|}{p}$, where $i \neq 1$.

Suppose, there exists an X_i such that $|X_i| < \frac{|G|}{p}$ for $3 \leq i \leq r$ then

$$|G| < \underbrace{\frac{|G|}{p} + \cdots + \frac{|G|}{p}}_{(r-2)-\text{times}}$$

$$\Rightarrow \ |G| < (r-2)\frac{|G|}{p} \ = \ |G|,$$

a contradiction. Hence $|X_3| = \frac{|G|}{p}, \ldots, |X_r| = \frac{|G|}{p}$. Also $|X_2| \geq \cdots \geq |X_r|$ therefore $|X_2| \geq \frac{|G|}{p}$. But $|X_i| \leq \frac{|G|}{p}, 2 \leq i \leq r$ so $|X_i| = \frac{|G|}{p}$ where $2 \leq i \leq r$. Hence $\sum_{i=3}^{r} |X_i| = (r-2)\frac{|G|}{p} = |G|$. Therefore $\sum_{i=3}^{r} |X_i| = |G|$ if and only if $X_2X_t = G \ \forall\, t \neq 2$ and $X_k \cap X_l \subseteq X_2$ for all $k \neq l$ (By Cohn's Theorem 1.10.3). Therefore, by interchanging X_i's, we have $X_2 \cap X_3 = Z(G)$. Thus

$$\begin{aligned} |X_2X_3| &= \frac{|X_2||X_3|}{|X_2 \cap X_3|} \\ \Rightarrow |G| &= \frac{\frac{|G|}{p} \times \frac{|G|}{p}}{|Z(G)|} \\ \Rightarrow \left|\frac{G}{Z(G)}\right| &= p^2 \\ \Rightarrow \frac{G}{Z(G)} &\cong C_p \times C_p, \end{aligned}$$

since G is non-abelian. This completes the proof. □

PROBLEM 2.2.10. If p is the smallest prime dividing $|G|$, then is it true that $|\operatorname{Cent}(G)| \geq p$?

Proposition 2.2.11. *[5] Let G be an n-centralizer group, $n - 2 = p$, p a prime and $X_1, X_2, \ldots, X_n$ be distinct centralizers of elements of G such that $X_1 = G$. If $|G : X_2| = |G : X_3| = \cdots = |G : X_n| = n - 2$, then $\frac{G}{Z(G)} \cong C_p \times C_p$.*

Proof. Given $X_1, X_2, \ldots, X_n$ are distinct centralizers of G such that $X_1 = G$ and $|G : X_2| = |G : X_3| = \cdots = |G : X_n| = p = n - 2$, so G has only two class lengths 1 and p, as index of the centralizer of an element is order of the conjugacy class of that element. Also by Ito's theorem 1.10.4, G is the direct product of a p-group P with an abelian group A. Therefore

$$|\operatorname{Cent}(G)| = |\operatorname{Cent}(P \times A)| = |\operatorname{Cent}(P)| \times |\operatorname{Cent}(A)| = |\operatorname{Cent}(P)|.$$

Without loss of generality we can assume that G is a p-group. Furthermore it is clear that $G = \bigcup_{i=2}^{n} X_i$, $\bigcap_{i=2}^{n} X_i = Z(G)$ and $|G| = \sum_{i=3}^{n} |X_i|$. Suppose one of X_i's is non-normal subgroup of G, say X_2. Then by [7, Theorem 1], $Z(G) = \mathrm{Core}_G X_2$ and $|\frac{G}{Z(G)}| = p^2$. Since $|\frac{G}{X_2}| = p$, hence $\frac{G}{Z(G)} = \frac{G}{\mathrm{Core}_G X_2}$ is isomorphic to a subgroup of S_p and so we must have $p^2 \mid p!$, which is a contradiction. Therefore all of X_i's are normal in G and so $\frac{G}{Z(G)} \cong C_p \times C_p$. This completes the proof. □

Proposition 2.2.12. *[4] Let G be an n-centralizer group, $n - 2 = p^2$, p a prime, and $X_1, X_2, \ldots, X_n$ be distinct centralizers of elements of G such that $X_1 = G$. If $|G : X_2| = |G : X_3| = \cdots = |G : X_n| = n - 2$, and two of X_i's are normal in G, then $\frac{G}{Z(G)} \cong C_p \times C_p \times C_p \times C_p$.*

Proof. Given $X_1, X_2, \ldots, X_n$ are distinct centralizers of G such that $X_1 = G$ and $|G : X_2| = |G : X_3| = \cdots = |G : X_n| = p^2 = n - 2$. So, G has only two class lengths 1 and p^2, as index of the centralizer of an element is order of the conjugacy class of that element. Also by Ito's theorem 1.10.4, G is the direct product of a p-group P with an abelian group A. Therefore

$$|\mathrm{Cent}(G)| = |\mathrm{Cent}(P \times A)| = |\mathrm{Cent}(P)| \times |\mathrm{Cent}(A)| = |\mathrm{Cent}(P)|.$$

Without loss of generality we can assume that G is a p-group. Again $G = \bigcup_{i=2}^{n} X_i$ and $\bigcap_{i=2}^{n} X_i = Z(G)$. Also $\sum_{i=3}^{n} |X_i| = |G|$. Suppose X_2 and X_3 are normal in G. Now by Cohn's Theorem 1.10.3, $X_2 X_t = G$, for all $t \neq 2$ and since $|G : X_i| = n - 2, 2 \leq i \leq n$, therefore $X_2 \cap X_3 = Z(G)$.

Therefore

$$\begin{aligned} |X_2X_3| &= \frac{|X_2||X_3|}{|X_2 \cap X_3|} \\ \Rightarrow |G| &= \frac{\frac{|G|}{p^2} \times \frac{|G|}{p^2}}{Z(G)} \\ \Rightarrow |G : Z(G)| &= p^4. \end{aligned}$$

Now it is clear that $\frac{G}{Z(G)} = \frac{X_2X_3}{Z(G)} = \frac{X_2}{Z(G)}\frac{X_3}{Z(G)}$ and $\frac{X_2}{Z(G)} \cap \frac{X_3}{Z(G)} = \{Z(G)\}$. So by Theorem 1.4.2, $\frac{G}{Z(G)} \cong (\frac{X_2}{Z(G)}) \times (\frac{X_3}{Z(G)})$. Since $|\frac{X_2}{Z(G)}| = |\frac{X_3}{Z(G)}| = p^2$, therefore

$$\frac{G}{Z(G)} \cong C_{p^2} \times C_{p^2} \text{ or } C_{p^2} \times C_p \times C_p \text{ or } C_p \times C_p \times C_p \times C_p.$$

If $\frac{G}{Z(G)} \cong C_{p^2} \times C_p \times C_p$ then $C_{p^2} \times C_p \times C_p$ is capable. But by Corollary 8.20 of [8], $C_{p^2} \times C_p \times C_p$ is not a capable group. Therefore $\frac{G}{Z(G)} \ncong C_{p^2} \times C_p \times C_p$.

Next suppose that $\frac{G}{Z(G)} \cong C_{p^2} \times C_{p^2}$. Here we will show that G has exactly $p(p+1)$ cyclic subgroups of order p^2. First our aim is to calculate the number of elements of $C_{p^2} \times C_{p^2}$ of order p^2.

Let $(a, b) \in C_{p^2} \times C_{p^2}$ then $o((a, b)) = p^2$, if $o(a) = p^2$ or $o(b) = p^2$. Now number of elements of order p^2 in C_{p^2} is $\phi(p^2) = p^2 - p = p(p-1)$. So number of elements (a, b) in $C_{p^2} \times C_{p^2}$ with $o(a) = p^2$ is $p(p-1)p^2 = p^3(p-1)$ (Since $o(b)$ may be equal to $1, p$ or p^2, so b can be chosen p^2-times). Also number of elements (a, b) in $C_{p^2} \times C_{p^2}$ with $o(a) \neq p^2$ and $o(b) = p^2$ is $(p^2 - p(p-1))\phi(p^2) = p^2(p-1)$. Hence total number of elements in $C_{p^2} \times C_{p^2}$ of order p^2 is $p^3(p-1) + p^2(p-1)$, that is we get $p^3(p-1) + p^2(p-1)$ cyclic subgroups (not distinct) of order p^2. But if $o((a, b)) = p^2$ then number of generators of $\langle (a, b) \rangle$ will be $\phi(p^2) = p(p-1)$.

Therefore, total number of distinct cyclic subgroups in $C_{p^2} \times C_{p^2}$ is

$$\frac{p^3(p-1)+p^2(p-1)}{p(p-1)} \;=\; p(p+1).$$

Hence, there are exactly $p(p+1)$ cyclic subgroups of order p^2 in $C_{p^2} \times C_{p^2}$, say $\frac{A_1}{Z(G)}, \frac{A_2}{Z(G)}, \ldots, \frac{A_{p(p+1)}}{Z(G)}$. Now, we choose $a_i \in A_i - Z(G)$ for $1 \leq i \leq p(p+1)$. Then by assumption, $C(a_i) \;=\; A_i, 1 \leq i \leq p(p+1)$ and A_i's are distinct centralizers of G, which is impossible. Therefore, $\frac{G}{Z(G)} \ncong C_{p^2} \times C_{p^2}$. Hence $\frac{G}{Z(G)} \cong C_p \times C_p \times C_p \times C_p$. This completes the proposition. $\square$

In this direction, we pose the following problem

PROBLEM 2.2.13. Let G be an n-centralizer group, $n-2 \;=\; p^k$, where $2 < k < n, p$ a prime, and $X_1, X_2, \ldots, X_n$ be distinct centralizers of G such that $X_1 \;=\; G$. If $|G : X_2| \;=\; |G : X_3| \;=\; \cdots \;=\; |G : X_n| \;=\; n-2$, and k number of X_i's are normal in G, then is it true that $\frac{G}{Z(G)} \cong \underbrace{C_p \times \cdots \times C_p}_{2^k-\text{times}}$?

Suppose p is a prime number. Then $N(p)$ denotes the set of all integers n such that $n \;=\; |\operatorname{Cent}(G)|$ for some p-group G.

Proposition 2.2.7 shows that $2^n \in N(2)$,

Proposition 2.2.8 shows that $p^n + 2 \in N(p)$ and

Lemma 2.2.9 shows that $N(p) \neq N - \{2, 3\}$

Define $N(\pi) \;=\; \underset{p \textbf{ is a prime}}{\cup} N(p)$ (See [4])

PROBLEM 2.2.14. [4] Is it true that $N(\pi) \;=\; N - \{2, 3\}$?

2.3 4-centralizer groups

In this section, we study the structure of 4-centralizer groups.

Theorem 2.3.1. *Let G be a group. Then $|\operatorname{Cent}(G)| = 4$ if and only if $\frac{G}{Z(G)} \cong C_2 \times C_2$; that is, G modulo its center is isomorphic to the klein four group.*

Proof. Suppose $|\operatorname{Cent}(G)| = 4$, then $\operatorname{Cent}(G) = \{G, C(p), C(q), C(r)\}$, where p, q and r are non-central elements of G. Therefore $G = C(p) \cup C(q) \cup C(r)$, since G is the union of its proper centralizers.

Now consider $C(pq)$. Then $C(pq)$ will be one of $G, C(p), C(q)$ or $C(r)$.

Case 1. $C(pq) = G$. In this case, $pq \in Z(G) \Rightarrow p = zq^{-1}$ for some $z \in Z(G)$. Let $t \in C(p)$ then

$$tp = pt \Rightarrow tzq^{-1} = zq^{-1}t \Rightarrow t \in C(q).$$

Therefore $C(p) \subseteq C(q)$. Thus $G = C(p) \cup C(q)$, a contradiction.

Case 2. $C(pq) = C(p) = p^{-1})$. In this case, $pq \in C(p^{-1}) \Rightarrow pq = qp$. Let $g \in C(p) = C(pq)$, then

$$gpq = pqg \Rightarrow pgq = pqg \Rightarrow g \in C(q).$$

Therefore $C(p) \subseteq C(q)$. Hence $G = C(q) \cup C(r)$, a contradiction.

Case 3. $C(pq) = C(q)$. Let $t \in C(q) = C(pq)$, then

$$tpq = pqt \Rightarrow tpq = ptq \Rightarrow t \in C(q).$$

Therefore $C(q) \subseteq C(p)$. Hence $G = C(p) \cup C(r)$, a contradiction.

Thus $C(pq) = C(r)$ and so

$$G = C(p) \cup C(q) \cup C(pq).$$

In the similar way we can show that $C(pq) = C(qp)$.

Now we will show that, $C(p) \cap C(q) = Z(G)$. Clearly, $Z(G) \subseteq C(p) \cap C(q)$. Let $g \in C(p) \cap C(q)$,then

$$gp = pg \Rightarrow (gp)q = (pg)q \Rightarrow gpq = pqg \Rightarrow g \in C(pq).$$

Therefore $g \in C(p) \cap C(q) \cap C(pq) = Z(G)$ and so $C(p) \cap C(q) \subseteq Z(G)$. Thus

$$C(p) \cap C(q) = Z(G).$$

Similarly we can show that $C(q) \cap C(pq) = Z(G)$ and $C(p) \cap C(pq) = Z(G)$.

Next we will show $G = Z \sqcup Zp \sqcup Zq \sqcup Zpq$, where $Z = Z(G)$. First we will show that Z, Zp, Zq, Zpq all are disjoint.

Supppose $Z = Zp \Rightarrow p \in Z$, a contradiction. Therefore $Z \neq Zp$. Similarly $Z \neq Zq, Zpq$.

Suppose $Zp = Zq \Rightarrow p = zq$ for some $z \in Z$. Let $g \in C(p)$ then

$$gzq = zqg \Rightarrow g \in C(q).$$

Therefore $C(p) \subseteq C(q)$ and so $G = C(q) \cup C(pq)$, a contradiction. Again suppose $Zp = Zpq$ then

$$(pq)^{-1}p \in Z \Rightarrow q \in Z, \text{ a contradiction.}$$

Similarly $Zq \neq Zpq$. Thus all Z, Zp, Zq, Zpq are disjoint.

Now clearly, $Z \sqcup Zp \sqcup Zq \sqcup Zpq \subseteq G$.

Let $g \in Z$, then $g \in Z \sqcup Zp \sqcup Zq \sqcup Zpq$. Suppose $g \in G - Z$ and without any loss we can assume that $g \in C(p)$.

We consider $C(gq)$, then $C(gq)$ is one of $G, C(p), C(q)$ or $C(pq)$.

Case 1. $C(gq) = G$. In this case $gq \in C(x)\ \forall\ x \in G$. In particular $gqq = qgq \Rightarrow g \in C(q)$. Therefore $g \in C(p) \cap C(q) = Z$, a contradiction.

Case 2. $C(gq) = C(p)$.

In this case $gqp = pgq \Rightarrow p \in C(q)$. Therefore $p \in C(p) \cap C(q) = Z$, a contradiction.

Case 3. $C(gq) = C(q)$.

In this case $C(gq) = C(q) = C(q^{-1}) \Rightarrow gqq^{-1} = q^{-1}gq \Rightarrow g \in C(q)$. Therefore $g \in C(p) \cap C(q) = Z$, a contradiction. Hence

$$C(gq) = C(pq).$$

Thus we have

$$\begin{aligned} & C(gq) = C(pq) = C((pq)^{-1}) \\ \Rightarrow\ & (gq)(pq)^{-1} = (pq)^{-1}(gq) \\ \Rightarrow\ & gp^{-1} = q^{-1}p^{-1}gq \\ \Rightarrow\ & gp^{-1} \in C(q). \end{aligned}$$

Now our aim is to show that,

$$g = zp \text{ for some } z \in Z$$

i.e. to show that $gp^{-1} \in Z$

i.e. to show that $gp^{-1} \in C(p) \cap C(q)$

i.e. to show that $g \in C(p)$ and $gp^{-1} \in C(q)$,

which are true.

Hence, $g = zp$ for some $z \in Z$. Therefore $g \in Z \sqcup Zp \sqcup Zq \sqcup Zpq$. Hence $G = Z \sqcup Zp \sqcup Zq \sqcup Zpq$, i. e.

$$\left|\frac{G}{Z(G)}\right| = 4$$
$$\Rightarrow \frac{G}{Z(G)} \cong C_2 \times C_2.$$

Conversely let $\frac{G}{Z(G)} \cong C_2 \times C_2$, then we are to show that $|\operatorname{Cent}(G)| = 4$.

Given $\frac{G}{Z(G)} \cong C_2 \times C_2 \Rightarrow \left|\frac{G}{Z(G)}\right| = 4$. Let Z, Zp, Zq, Zr be the elements of $\frac{G}{Z(G)}$, where p, q, r are distinct non-central elements of G and $Z = Z(G)$. Then

$$G = Z \sqcup Zp \sqcup Zq \sqcup Zr.$$

So, $g \in G$ which gives $g \in Z$ or $g \in Zp$ or $g \in Zq$ or $g \in Zr$. If $g \in Z$ then $C_G(g) = G$. If $g \in Zp$ then $g = zp$ for some $z \in Z$, therefore $C_G(g) = C_G(zp) = C_G(p)$. If $g \in Zq$ then $g = z_1q$ for some $z_1 \in Z$, therefore $C_G(g) = C_G(z_1q) = C_G(q)$. If $G \in rZ$ then $g = z_2r$ for some $z_2 \in Z$ and so $C_G(g) = C_G(z_2r) = C_G(r)$. Hence G has atmost four centralizers namely $G, C_G(p), C_G(q)$ and $C_G(r)$. Again by Theorem 2.1.6, $|\operatorname{Cent}(G)| \geq 4$.

Hence $|\operatorname{Cent}(G)| = 4$. This completes the theorem. □

2.4 5-centralizer groups

In this section, we study the structure of 5-centralizer groups. Given any two non-empty subsets A and B of a group G, we write $AB = \{ab \mid a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} \mid a \in G\}$.

Lemma 2.4.1. *[13] Let* $G = A \cup B \cup C$, *where* A, B, C *are the proper distinct subgroups of* G. *Put* $K = A \cap B \cap C, L = A \cap B - K, M = A \cap C - K$, $N = B \cap C - K$ *and* $\tilde{A} = A - (B \cup C)$, $\tilde{B} = B - (A \cup C)$, $\tilde{C} = C - (A \cup B)$. *Then*

(i) $L = M = N = \phi$,

(ii) $\tilde{A}^{-1} \subseteq \tilde{A}, \tilde{B}^{-1} \subseteq \tilde{B}$ *and* $\tilde{C}^{-1} \subseteq \tilde{C}$,

(iii) $\tilde{A}\tilde{B} \subseteq \tilde{C}, \tilde{B}\tilde{C} \subseteq \tilde{A}$ *and* $\tilde{C}\tilde{A} \subseteq \tilde{B}$,

(iv) $\tilde{A}\tilde{A} \subseteq K, \tilde{B}\tilde{B} \subseteq K$ *and* $\tilde{C}\tilde{C} \subseteq K$,

(v) K *is a normal subgroup of* G.

(vi) $K\tilde{a} = \tilde{A}, K\tilde{b} = \tilde{B}, K\tilde{c} = \tilde{C}$.

Proof. (i) Consider $l \in L$ and $\tilde{c} \in \tilde{C}$. Then $\tilde{c}l \in A$ or B or C. If $\tilde{c}l \in A$ then $\tilde{c}ll^{-1} = \tilde{c} \in A$, a contradiction. If $\tilde{c}l \in B$ then $\tilde{c}ll^{-1} = \tilde{c} \in B$, a contradiction. If $\tilde{c}l \in C$ then $\tilde{c}^{-1}\tilde{c}l = l \in C$, a contradiction. Since $\tilde{C} \neq \phi$, we must have $L = \phi$. Similarly $M = N = \phi$.

(ii) Let $\tilde{a} \in \tilde{A}$, then $\tilde{a} \in A \Rightarrow \tilde{a}^{-1} \in A \Rightarrow \tilde{a}^{-1} \in K$ or $\tilde{A}$. If $\tilde{a}^{-1} \in K$ then $\tilde{a} \in K$, a contradiction. Hence $\tilde{a}^{-1} \in \tilde{A}$.

Similarly, if $\tilde{b} \in \tilde{B}$ then $\tilde{b}^{-1} \in \tilde{B}$ and if $\tilde{c} \in \tilde{C}$ then $\tilde{c}^{-1} \in \tilde{C}$.

(iii) Suppose $\tilde{a} \in \tilde{A}, \tilde{b} \in \tilde{B}$, then $\tilde{a}\tilde{b} \in K$ or $\tilde{A}$ or $\tilde{B}$ or $\tilde{C}$. If $\tilde{a}\tilde{b} \in \tilde{A} \subseteq A$, then $\tilde{a}^{-1}\tilde{a}\tilde{b} \in A$, a contradiction. If $\tilde{a}\tilde{b} \in \tilde{B} \subseteq B$, then $\tilde{a}\tilde{b}\tilde{b}^{-1} \in B \Rightarrow \tilde{a} \in B$, a contradiction. If $\tilde{a}\tilde{b} \in K$, then $\tilde{a}\tilde{b} \in A$. So $\tilde{a}^{-1}\tilde{a}\tilde{b} \in A \Rightarrow \tilde{b} \in A$, a contradiction. Hence $\tilde{a}\tilde{b} \in \tilde{C}$.

(iv) Given $\tilde{a}, \tilde{a_1} \in \tilde{A} \subseteq A$. So $\tilde{a}\tilde{a_1} \in A \Rightarrow \tilde{a}\tilde{a_1} \in \tilde{A}$ or K.

Suppose $\tilde{a}\tilde{a_1} \in \tilde{A}$. Consider $\tilde{b}\tilde{a}\tilde{a_1}$ for some $\tilde{b} \in \tilde{B}$. Then, by third part we have $\tilde{b}(\tilde{a}\tilde{a_1}) \in \tilde{C}$ and $(\tilde{b}\tilde{a})\tilde{a_1} \in \tilde{B}$. Since $\tilde{a}, \tilde{b}, \tilde{a_1} \in G$ so by associativity in G, $\tilde{b}(\tilde{a}\tilde{a_1}) = (\tilde{b}\tilde{a})\tilde{a_1} = \tilde{b}\tilde{a}\tilde{a_1} \in \tilde{B} \cap \tilde{C}$, a contradiction. Similarly, we can show the other two.

(v) Let $g \in G$ then $g \in K$ or $g \in \tilde{A}$ or $g \in \tilde{B}$ or $g \in \tilde{C}$. If $g \in K$ then $g^{-1}kg \in K \ \forall \ k \in K$.

If $g \in \tilde{A}$ then $g^{-1} \in \tilde{A}$. Also $g \in \tilde{A} \subseteq A, k \in K \subseteq A$, so

$$kg \in A = \tilde{A} \sqcup K \Rightarrow kg \in \tilde{A} \,\text{or}\, K.$$

Suppose $kg \in K$ then $k^{-1}kg = g \in K$, a contradiction. Therefore $kg \in \tilde{A}$. So $g^{-1}kg \in K \ \forall \ k \in K$. If $g \in \tilde{B}$ then $g^{-1} \in \tilde{B}$. Also $g \in \tilde{B} \subseteq B, k \in K \subseteq B$, so

$$kg \in B = \tilde{B} \sqcup K \Rightarrow kg \in \tilde{B} \,\text{or}\, K.$$

Suppose $kg \in K$ then $k^{-1}kg = g \in K$, a contradiction. Therefore $kg \in \tilde{B}$. So $g^{-1}kg \in K \ \forall \ k \in K$. If $g \in \tilde{C}$, then $g^{-1} \in \tilde{C}$. Also $g \in \tilde{C} \subseteq C, k \in K \subseteq C$, so

$$kg \in C = \tilde{C} \sqcup K \Rightarrow kg \in \tilde{C} \,\text{or}\, K.$$

Suppose $kg \in K$ then $k^{-1}kg = g \in K$, a contradiction. Therefore $kg \in \tilde{C}$. So $g^{-1}kg \in K \ \forall \ k \in K$.

Thus $g^{-1}kg \in K \ \forall \ g \in G, \ \forall \ k \in K$ and hence $K \trianglelefteq G$.

(vi) Let $k\tilde{a} \in K\tilde{a}$, where $\tilde{a} \in \tilde{A} \subseteq A, k \in K \subseteq A$. So $k\tilde{a} \in A = \tilde{A} \sqcup K$. If $k\tilde{a} \in K$ then $k^{-1}k\tilde{a} \in K \Rightarrow \tilde{a} \in K$, a contradiction. Therefore $k\tilde{a} \in \tilde{A}$ and so $K\tilde{a} \subseteq \tilde{A}$. Conversely let $\tilde{a_1} \in \tilde{A}$, then $\tilde{a_1}\tilde{a}^{-1} \in K$. So $\tilde{a_1}\tilde{a}^{-1} = k$ for some $k \in K \Rightarrow \tilde{a_1} = k\tilde{a} \in K\tilde{a}$. Therefore $\tilde{A} \subseteq K\tilde{a}$. Hence, $K\tilde{a} = \tilde{A}$. Similarly, $K\tilde{b} = \tilde{B}$ and $K\tilde{c} = \tilde{C}$.

□

Lemma 2.4.2. *[13] A group G can be written as the union of three proper subgroups A, B, C if and only if $K = A \cap B \cap C$ is a normal subgroup of G and $\frac{G}{K} \cong C_2 \times C_2$.*

Proof. Suppose $G = A \cup B \cup C$ where A, B, C are three proper subgroups of G. We are to show that $K = A \cap B \cap C$ is a normal subgroup of G and $\frac{G}{K} \cong C_2 \times C_2$.

If $A \subseteq B$ then $G = B \cup C$, a contradiction. If $B \subseteq C$ then $G = A \cup C$, a contradiction. If $A \subseteq C$ then $G = B \cup C$, a contradiction. Hence all A, B, C are distinct. suppose $\tilde{A} = A - (B \cup C)$, $\tilde{B} = B - (A \cup C)$, $\tilde{C} = C - (A \cup B)$ and $K = A \cap B \cap C$.

Then by using Lemma 2.4.1, we have that the quotient group

$$\frac{G}{K} = \{Kx | x \in G\} = \{K, K\tilde{a}, K\tilde{b}, K\tilde{c}\}.$$

Now, $o(K\tilde{a}) = o(K\tilde{b}) = o(K\tilde{c}) = 2$ (since $(K\tilde{a})(K\tilde{a}) = K(\tilde{a}\tilde{a}) = K$).

Thus $|\frac{G}{K}| = 4$ and order of each non-central element is 2. Therefore $\frac{G}{K} \cong C_2 \times C_2$.

Conversely let $\frac{G}{K} \cong C_2 \times C_2$, where $K \trianglelefteq G$ and K is the intersection of three proper subgroups of G. Given $\frac{G}{K} \cong C_2 \times C_2 \Rightarrow |\frac{G}{K}| = 4$. Let $\frac{G}{K} = \{K, X, Y, Z\}$, where $o(X) = o(Y) = o(Z) = 2$.

Consider $P = X \sqcup K, Q = Y \sqcup K, R = Z \sqcup K$. We are to show that P is a subgroup of G. As $o(X) = 2$, so $XX = K \Rightarrow x_1x_2 \in K,\ \forall\, x_1, x_2 \in X$ and K is the identity element of $\frac{G}{K}$, so $XK = X \Rightarrow xk \in X$, where $x \in X, k \in K$. And since $K \leq G$, so for $k_1, k_2 \in K$ we have $k_1k_2 \in K$. Therefore P is closed. Let $p \in P = X \cup K \Rightarrow p \in X$ or $p \in K$. If $p \in X$ then $p = x$ for some $x \in X \Rightarrow p^{-1} = x^{-1} \in X^{-1} = X$. Therefore $p^{-1} \in X \subseteq P$. If $p \in K$ then $p^{-1} \in K \subseteq P$. Hence P is a subgroup of G. Similarly, $Q, R \leq G$. Also $P \cap Q \cap R = K$, as $X \cap Y \cap Z = \phi$; therefore

$$P \cup Q \cup R = (X \cup K) \cup (Y \cup K) \cup (Z \cup K) = K \cup X \cup Y \cup Z = G,$$

which completes the proof. □

Lemma 2.4.3. *[6] Let $x \in G$ such that $|C_{\frac{G}{Z}}(xZ)| = p, p$ a prime, where $Z = Z(G)$. Then for all $y \in G$ with $C_{\frac{G}{Z}}(xZ) = C_{\frac{G}{Z}}(yZ)$, we have $C_G(x) = C_G(y)$.*

Proof. We have that

$$\frac{C_G(x)}{Z} \leqslant C_{\frac{G}{Z}}(xZ).$$

Suppose that $\frac{C_G(x)}{Z} < C_{\frac{G}{Z}}(xZ)$. Since $|C_{\frac{G}{Z}}(xZ)| = p$ and $|\frac{C_G(x)}{Z}|$ divides $|C_{\frac{G}{Z}}(xZ)|$ so $|\frac{C_G(x)}{Z}| = 1 \Rightarrow C_G(x) = Z \Rightarrow x \in Z$, a contradiction. Therefore $\frac{C_G(x)}{Z} = C_{\frac{G}{Z}}(xZ)$.

Clearly, $\frac{C_G(y)}{Z} \leqslant C_{\frac{G}{Z}}(yZ) = C_{\frac{G}{Z}}(xZ)$. Therefore $|C_{\frac{G}{Z}}(xZ)| = |\frac{C_G(y)}{Z}|$ and so $\frac{C_G(y)}{Z} = \frac{C_G(x)}{Z}$. Thus

$$\frac{C_G(x)}{Z} = \frac{C_G(y)}{Z} = \{Z, t_1Z, t_2Z, \ldots, t_{p-1}Z\},$$

where $\{t_1, \ldots, t_{p-1}\} \in C_G(x) \cap C_G(y) - Z$. So $C_G(x) = C_G(y)$. Hence, the lemma follows. □

Now, we will prove the main theorem of this section which characterizes 5-centralizer groups.

Theorem 2.4.4. *[10] Let G be a finite group. Then $|\operatorname{Cent}(G)| = 5$ if, and only if, $\frac{G}{Z(G)} \cong C_3 \times C_3$ or $\frac{G}{Z(G)} \cong S_3$, where S_3 is the symmetric group on three symbols.*

Proof. Suppose first that $\frac{G}{Z(G)} \cong C_3 \times C_3$. Then

$$\begin{aligned}\frac{G}{Z} &= \langle Zx, Zy | (Zx)^3 = (Zy)^3 = Z, (Zx)(Zy) = (Zy)(Zx)\rangle \\ &= \{Zx, Zy | x^3, y^3, xyx^{-1}y^{-1} \in Z\}.\end{aligned}$$

If $\frac{H}{Z} < \frac{G}{Z}$, then $|\frac{\frac{G}{Z}}{\frac{H}{Z}}| = 3 \Rightarrow |\frac{H}{Z}| = \frac{|\frac{G}{Z}|}{3} = 3$. Therefore $H = Z \sqcup Zk \sqcup Zl$, where $k, l \in H - Z$. So the proper subgroups of G properly containing Z are $H_1 = Z \sqcup Zx \sqcup Zx^2, H_2 = Z \sqcup Zy \sqcup Zy^2$, $H_3 = Z \sqcup Zxy \sqcup Zx^2y^2$, $H_4 = Z \sqcup Zxy^2 \sqcup Zx^2y$.

Now we are to show that H_1, H_2, H_3, H_4 are the only proper centralizers of G. Let $a \in G - Z$ then $Za = Zk$ for some $k \in \{x, x^2, y, y^2, xy, x^2y^2, xy^2, x^2y\}$. Therefore $C_{\frac{G}{Z}}(Za) = C_{\frac{G}{Z}}(Zk) \Rightarrow C_G(a) = C_G(k)$ (Using Lemma 2.4.3). Again let $k \in H_i - Z$, then $C_G(k) \in \overset{4}{\underset{i=1, j\neq i}{\cup}} H_j$ as H_1, H_2, H_3, H_4 are the only proper subgroups of G. Also $k \in C_G(k)$ gives $C_G(k) \neq H_j, 1 \leqslant j (\neq i) \leqslant 4$. Therefore $C_G(k) = H_i$. Hence H_1, H_2, H_3, H_4 are the only proper centralizers of G. Thus $|\operatorname{Cent}(G)| = 5$.

Next assume that $\frac{G}{Z} \cong S_3$. Therefore

$$\begin{aligned}\frac{G}{Z} &= \langle Zx, Zy | (Zx)^2 = (Zy)^3 = Z, (Zx)(Zy)(ZX)^{-1} = (Zy)^{-1}\rangle \\ &= \{Z, Zy, Zy^2, Zx, Zxy, Zxy^2\}.\end{aligned}$$

If $\frac{K}{Z} < \frac{G}{Z}$ then $|\frac{\frac{G}{Z}}{\frac{K}{Z}}| = 3$ or $2 \Rightarrow |\frac{K}{Z}| = 2$ or 3. Therefore $K = Z \sqcup Zt_1 \sqcup Zt_2$ or $K = Z \sqcup Zl$, where $t_1, t_2, l \in K - Z$. So the proper subgroups of G properly containing Z are

$$K_1 = Z \sqcup Zy \sqcup Zy^2, K_2 = Z \sqcup Zx, K_3 = Z \sqcup Zxy, H_4 = Z \sqcup Zxy^2.$$

Now we are to show that K_1, K_2, K_3, K_4 are the only proper centralizers of G. Let $b \in G - Z$, then $Zb = Zk$ for some $k \in \{y, y^2, x, xy, xy^2\}$. Therefore $C_{\frac{G}{Z}}(Zb) = C_{\frac{G}{Z}}(Zk) \Rightarrow C_G(b) = C_G(k)$ (Using Lemma 2.4.3). Again let $k \in K_i - Z$ then $C_G(k) \in \bigcup_{\substack{i=1,\\ i \neq j}}^{4} K_j$, as K_1, K_2, K_3, K_4 are the only proper subgroups of G. Also $k \in C_G(k)$ gives $C_G(k) \neq K_j, 1 \leqslant j \leqslant 4, 1 \leqslant i \leqslant 4$ and $i \neq j$. Therefore $C_G(k) = K_i$. Hence K_1, K_2, K_3, K_4 are the only proper centralizers of G. Thus $|\operatorname{Cent}(G)| = 5$.

Conversely suppose $|\operatorname{Cent}(G)| = 5$. Let P, R, S, T be the four proper centralizers of G and we choose $p \in P - (R \cup S), r \in R - (P \cup S)$ and $s \in S - (P \cup R)$.

Now we will show that $C(p) = P, C(r) = R$ and $C(s) = S$. If $C(p) \neq P$ then $C(p) = R, S$ or T. But $p \notin R, S$; so $C(p) \neq R, S$. So $p \in T - (R \cup S)$ and this gives $P - (R \cup S) \subseteq T - (R \cup S) \Rightarrow P \subseteq T$. Next if we interchange the roles of P and T, then we have $T \subseteq P$. Hence $P = T$, a contradiction. Therefore $C(p) = P$. Similarly $C(r) = R, C(s) = S$.

Now we will use the following lemmas to prove this part.

Lemma 1. No one of P, R, S or T is contained in the union of the other three.

proof: Suppose to the contrary and without loss of generality that T is a subset of $P \cup R \cup S$. Then $G = P \cup R \cup S \cup T = P \cup R \cup S$. Now by Proposition 2.4.2, we have $\frac{G}{P \cap R \cap S} \cong C_2 \times C_2$. Now we will show that $P \cap R \cap S = Z$. Let $\exists x \in (P \cap R \cap S) - Z$ and consider $C(x)$:

(i) $C(x) \neq G$, since $x \notin Z$.

(ii) $C(x) \neq T$, since if $C(x) = T$ then $x \in T$. Also $x \in P \cap R \cap S$, so $x \in P \cap R \cap S \cap T = Z \Rightarrow x \in Z$, contradiction.

(iii) $C(x) \neq P, R, S$. As $x \in P \cap R \cap S$, therefore $p, r, s \in C(x)$. But $r \notin C(p), p \notin C(r)$ and $r \notin C(s)$.

This means if $\exists\, x \in P \cap R \cap S - Z$ then $C(x) \neq G, P, R, S, T$. That is $|\operatorname{Cent}(G)|$ must be at least 6, a contradiction. Hence $P \cap R \cap S - Z = \phi$, therefore $P \cap R \cap S = Z$. Thus by Theorem 2.3.1,

$$\frac{G}{Z} \cong C_2 \times C_2 \Rightarrow |\operatorname{Cent}(G)| = 4,$$

which is a contradiction.

Lemma 2. No element of G is in exactly two proper centralizers.

proof: Let $g \in (P \cap R) - (S \cup T)$ then $g \in C(p) \cap C(r) \Rightarrow p, r \in C(g)$. But $p \notin C(r)$, so $C(g) \neq P, R$. Again $g \notin S, T$; so $C(g) \neq S, T$. Also $C(g) \neq G$, since $g \in G - Z$. Therefore $C(g) \neq G, P, R, S, T$. That is $|\operatorname{Cent}(G)|$ must be at least 6, a contradiction. Hence $(P \cap R) - (S \cup T) = \phi$.

Lemma 3. No element of G is in exactly three proper centralizers.

proof: Suppose $g \in (P \cap R \cap S) - T$ then $g \in C(p) \cap C(s) \cap C(s)$. Therefore $p, r, s \in C(g)$. But $r \notin C(p), s \notin C(r)$. So $C(g) \neq P, R, S$. Also $C(g) \neq T, G$; as $g \notin T$ and $g \notin Z$. Therefore $C(g) \neq G, P, R, S, T$. This means $|\operatorname{Cent}(G)|$ must be at least 6, a contradiction. Hence $P \cap R \cap S - T = \phi$.

Therefore the above lemmas show that G is atmost a disjoint union of its four proper centralizers. The arithmetic summary is

Lemma 4. $|G| \; = \; |P \cup R \cup S \cup T| \; = \; |P| + |R| + |S| + |T| - 3|Z|$.

Next we compute the value of $|Z|$.

Lemma 5. If X and Y are distinct proper centralizers of G, then

$$\frac{|X||Y|}{|G|} \leq |Z| \leq \frac{|G|}{6}.$$

proof: Let X and Y be proper subgroups of G. Then $\frac{|X||Y|}{|XY|} \; = \; |X \cap Y|$. Also $XY \subseteq G$, so $\frac{1}{|XY|} \geq \frac{1}{|G|}$. Therefore $|X \cap Y| \geq \frac{|X||Y|}{|G|} \Rightarrow \; |Z| \geq \frac{|X||Y|}{|G|}$.

Also by Lemma 4,

$$\begin{aligned} |G| \; &= \; |P| + |R| + |S| + |T| - 3|Z| \\ &\geq 2|Z| + 2|Z| + 2|Z| + 2|Z| - 3|Z| \\ &= \; 5|Z| \\ \Rightarrow \; \left|\frac{G}{Z}\right| &\geq 5. \end{aligned}$$

If $\left|\frac{G}{Z}\right| \; = \; 5$ then $\frac{G}{Z}$ is cyclic $\Rightarrow$ G is abelian, a contradiction. Therefore $|\frac{G}{Z}| \geq 6 \Rightarrow \; |G| \geq 6|Z| \Rightarrow \; |Z| \leq \frac{|G|}{6}$. So $\frac{|X||Y|}{|G|} \leq |Z| \leq \frac{|G|}{6}$.

Now our aim is to get more near lower bound for $|Z|$. We may assume without loss of generality that $|P| \geq |R| \geq |S| \geq |T|$. Suppose $|P| < \frac{|G|}{3}$, as $1 < |P| \leq \frac{|G|}{2}$. That is $|P| \leq \frac{|G|}{4}$. Now by Lemma 4,

$$\begin{aligned} |G| \; &= \; |P| + |R| + |S| + |T| - 3|Z| \\ &\leq \frac{|G|}{4} + \frac{|G|}{4} + \frac{|G|}{4} + \frac{|G|}{4} - 3|Z| \\ &= \; |G| - 3|Z| \\ \Rightarrow \; |G| &< |G|, \end{aligned}$$

a contradiction. Hence $|P| = \frac{|G|}{2}$ or $|P| = \frac{|G|}{3}$.

Case 1. $|P| = \frac{|G|}{2}$. In this case Lemma 4 gives,

$$\begin{aligned} |G| &= |P| + |R| + |S| + |T| - 3|Z| \\ &= \frac{|G|}{2} + |R| + |S| + |T| - 3|Z| \\ \Rightarrow \frac{|G|}{2} &< |R| + |S| + |T| \\ \Rightarrow \frac{|G|}{2} &< 3|R| \text{ (since } |R| \geq |S| \geq |T| \text{)} \\ \Rightarrow \frac{|G|}{6} &< |R|. \end{aligned}$$

Also applying Lemma 5 on P and R we have $\frac{|P||R|}{|G|} \leq \frac{|G|}{6} \Rightarrow |R| \leq \frac{|G|}{3}$. That is $\frac{|G|}{6} < |R| \leq \frac{|G|}{3}$. So $|R|$ is one of $\frac{|G|}{3}$, $\frac{|G|}{4}$ or $\frac{|G|}{5}$. Again reapplying Lemma 5 on P and R we have,

$$\begin{aligned} &\frac{|P||R|}{|G|} \leq |Z| \leq \frac{|G|}{6} \\ \Rightarrow &\frac{|R|}{2} \leq |Z| \leq \frac{|G|}{6} \\ \Rightarrow &\frac{|G|}{10} \leq |Z| \leq \frac{|G|}{6} \text{ (} as \ \frac{|G|}{5} < \frac{|G|}{4} < \frac{|G|}{5}). \end{aligned}$$

Thus $|Z|$ is one of $\frac{|G|}{6}, \frac{|G|}{7}, \frac{|G|}{8}, \frac{|G|}{9}$ or $\frac{|G|}{10}$. Now suppose $|Z| = \frac{|G|}{7}$. Here $|Z|$ divides $|P| \Rightarrow \frac{|G|}{7} \mid \frac{|G|}{2} \Rightarrow 2 \mid 7$, which is not possible. Again suppose $|Z| = \frac{|G|}{9}$, then $\frac{|G|}{9}$ divides $\frac{|G|}{2} \Rightarrow 2 \mid 9$, which is not possible. Now let $|Z| = \frac{|G|}{6} \Rightarrow |\frac{G}{Z}| = 6 \Rightarrow \frac{G}{Z} \cong S_3$. Next let $|Z| = \frac{|G|}{8}$ then $|Z|$ divides $|R| \Rightarrow \frac{|G|}{8} \mid |R|$. If $|R| = \frac{|G|}{3}$ then $3 \mid 8$, a contradiction. If $|R| = \frac{|G|}{5}$ then $5 \mid 8$, a contradiction. Therefore $|R| = \frac{|G|}{4}$.

Also by Lemma 4, we have $|G| = |P| + |R| + |S| + |T| - 3|Z|$ which gives $\frac{5|G|}{8} = |S| + |T|$. Also $|R| \geq |S| \geq |T|$. So $|S|, |T| \leq \frac{|G|}{4}$. Therefore

$|S|+|T| \leq \frac{|G|}{2} < \frac{5|G|}{8} = |S|+|T|$, a contradiction. So $|Z|$ cannot be equal to $\frac{|G|}{8}$.

Next suppose $|Z| = \frac{|G|}{10}$, so $|Z|$ divides $|R| \Rightarrow \frac{|G|}{10} \mid |R|$. If $|R| = \frac{|G|}{3}$ then $3 \mid 10$, a contradiction. If $|R| = \frac{|G|}{4}$ then $4 \mid 10$, a contradiction. Therefore $|R| = \frac{|G|}{5}$. Now Lemma 4 gives, $|G| = |P|+|R|+|S|+|T|-3|Z|$ which gives $|S|+|T| = \frac{6|G|}{10}$. Also $|R| \geq |S| \geq |T|$, therefore

$$|S|+|T| \leq \frac{2|G|}{5} < \frac{6|G|}{10} = |S|+|T|,$$

a contradiction. Hence $|Z| \neq \frac{|G|}{10}$.

Case 2. $|P| = \frac{|G|}{3}$. In this case Lemma 4 gives,

$$\begin{aligned} |G| &= |P|+|R|+|S|+|T|-3|Z| \\ \Rightarrow \frac{2|G|}{3} &< |R|+|S|+|T| \\ \Rightarrow \frac{2|G|}{3} &< 3|R| \\ \Rightarrow \frac{2|G|}{8} &\leq |R| \\ \Rightarrow |R| &\geq \frac{|G|}{4}. \end{aligned}$$

Also $|P| \geq |R|$, so $\frac{|G|}{3} \geq |R| \geq \frac{|G|}{4}$. Therefore $|R| = \frac{|G|}{3}$ or $\frac{|G|}{4}$. Again applying Lemma 5 on P and R we get,

$$\begin{aligned} \frac{|P||R|}{|G|} &\leq |Z| \leq \frac{|G|}{6} \\ \Rightarrow \frac{|R|}{3} &\leq |Z| \leq \frac{|G|}{6} \\ \Rightarrow \frac{|G|}{12} &\leq |Z| \leq \frac{|G|}{6}, \text{ (since } \frac{|G|}{4} < \frac{|G|}{3}). \end{aligned}$$

Therefore $|Z|$ is one of $\frac{|G|}{6}, \frac{|G|}{7}, \frac{|G|}{8}, \frac{|G|}{9}, \frac{|G|}{10}, \frac{|G|}{11}$ or $\frac{|G|}{12}$. Now if $|Z| = \frac{|G|}{7}$ then $|Z|$ divides $|P| = \frac{|G|}{3} \Rightarrow 3 \mid 7$, a contradiction. Similarly $|Z| \neq \frac{|G|}{8}, \frac{|G|}{10}, \frac{|G|}{11}$.

Now let $|Z| = \frac{|G|}{6}$ then $|\frac{G}{Z}| = 6 \Rightarrow \frac{G}{Z} \cong S_3$. Let $|Z| = \frac{|G|}{9}$ then $\frac{G}{Z} \cong C_3 \times C_3$. Let $|Z| = \frac{|G|}{12}$ and $|R| = \frac{|G|}{3}$ then applying Lemma 5 on P and R we have,

$$\begin{aligned} \frac{|P||R|}{|G|} &\leq |Z| \\ \Rightarrow \frac{|R|}{3} &\leq |Z| \\ \Rightarrow \frac{|G|}{9} &\leq \frac{|G|}{12}. \end{aligned}$$

Which gives a contradiction. If $|R| = \frac{|G|}{4}$ then Lemma 4 gives,

$$\begin{aligned} |G| &= |P| + |R| + |S| + |T| - 3|Z| \\ \Rightarrow |S| + |T| &= \frac{4|G|}{6}. \end{aligned}$$

Also $|S|, |T| \leq \frac{|G|}{4}$, so $|S| + |T| \leq \frac{3|G|}{6} < \frac{4|G|}{6} = |S| + |T|$, which is not possible. Hence $|Z| \neq \frac{|G|}{12}$.

Thus $\frac{G}{Z} \cong C_3 \times C_3$ or S_3, where S_3 is the symmetric group on three symbols. □

In this direction we can pose the following problems:

PROBLEM 2.4.5. If G is a finite group such that $\frac{G}{Z} \cong S_n$, then what can we say about $|\operatorname{Cent}(G)|$?

PROBLEM 2.4.6. If every non-trivial subgroup of G is a centralizer then is $G \cong D_{2p}$?

2.5 An estimate for $|\operatorname{Cent}(G)|$ relative to $|G|$

Let G be a finite group. Consider the ratio

$$\operatorname{PrCent}(G) = \frac{|\operatorname{Cent}(G)|}{|G|}.$$

Then $\operatorname{PrCent}(G)$ gives an estimate for $|\operatorname{Cent}(G)|$ relative to the size of G. We now discuss about bounds of $\operatorname{PrCent}(G)$ for a finite group G. The following lemma is useful in Theorem 2.5.2.

Lemma 2.5.1. *[26] If G is a finite group and not all elements of G are involutions, then*

$$\operatorname{PrInv}(G) = \frac{|\operatorname{Inv}(G)|}{|G|} \leq \frac{p+1}{2p},$$

where p is the largest prime divisor of $|G|$ and $|\operatorname{Inv}(G)|$ denotes the number of elements x in G such that $x = e$ or x is an involution.

Theorem 2.5.2. *[10] Let p be the largest prime divisor of $|G|$. Then*

$$\operatorname{PrCent}(G) \leq \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ \frac{3}{4} + \frac{1}{4p}, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. First our aim is to find a partition of G that is created by assigning elements in their centralizers.

If $p = 2$ then G is a 2-group and since it is a p-group so it has non-trivial center. Again elements in the same coset of the center have the same centralizer. Therefore $|\operatorname{Cent}(G)|$ will be atmost $|G:Z|$ where $Z = Z(G)$, as G can be written as the union of right (left) cosets on the center. Hence

$$\operatorname{PrCent}(G) = \frac{|\operatorname{Cent}(G)|}{|G|} \leq \frac{|G:Z|}{|G|} = \frac{1}{|Z|}.$$

As G is a 2-group, so

$$|Z| \geq 2$$
$$\Rightarrow \frac{1}{|Z|} \leq \frac{1}{2}.$$

Therefore $\text{PrCent}(G) \leq \frac{1}{2}$.

If p is odd, then we cannot say about the existence of non-trivial center. By Cauchy's theorem, there exists an element $x \in G$ such that $o(x) = p$. So such an element is distinct from its inverse. Also $C_G(x) = C_G(x^{-1})$. So we partition G by taking sets $\{x, x^{-1}\}$ as its components. Again some of these components may consist of only one element if $x^2 = e$ that is $x = e$ or x is an involution. We denote the number of such elements in G by $|\operatorname{Inv}(G)|$, so $|\operatorname{Inv}(G)| < |G|$. Again since for $x \in \operatorname{Inv}(G)$, either $x = e$ or $x = x^{-1}$, so number of centralizers of those elements (whose order is not equal to one or two) will be atmost

$$\frac{|G - \operatorname{Inv}(G)|}{2} = \frac{|G| - |\operatorname{Inv}(G)|}{2} > 0.$$

Therefore

$$|\operatorname{Cent}(G)| \leq \frac{|G| + |\operatorname{Inv}(G)|}{2}. \tag{2.5.a}$$

Hence

$$\begin{aligned}
&\operatorname{PrCent}(G) \leq \frac{|G| + |\operatorname{Inv}(G)|}{2|G|}\\
\Rightarrow\ &\operatorname{PrCent}(G) \leq \frac{1}{2} + \frac{|\operatorname{Inv}(G)|}{2|G|}\\
\Rightarrow\ &\operatorname{PrCent}(G) \leq \frac{1}{2} + \frac{|\operatorname{Inv}(G)|}{2|G|}\\
\Rightarrow\ &\operatorname{PrCent}(G) \leq \frac{1}{2} + \frac{1}{2} \times \frac{p+1}{2p} \text{ (Using Lemma 2.5.1)}\\
\Rightarrow\ &\operatorname{PrCent}(G) \leq \frac{3}{4} + \frac{1}{4p}.
\end{aligned}$$

This completes the proof. □

We use the following lemma in proving Theorem 2.5.4.

Lemma 2.5.3. *[26] Let p be the largest prime divisor of $|G|$ and let* $\operatorname{PrInv}(G) = \frac{p+1}{2p}$*, then* $G \cong D_p \times \underbrace{C_2 \times \cdots \times C_2}_{j-times}$ *where j is the number of times C_2 occures as a direct factor in G.*

Theorem 2.5.4. *[10] Let G be a finite group. Then* $\operatorname{PrCent}(G) \leq \frac{5}{6}$ *and* $\operatorname{PrCent}(G) = \frac{5}{6}$ *if and only if $G \cong S_3$.*

Proof. First part: Theorem 2.5.2 gives, if p is the largest prime divisor of $|G|$ then

$$\operatorname{PrCent}(G) \leq \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ \frac{3}{4} + \frac{1}{4p}, & \text{if } p \text{ is odd.} \end{cases}$$

For $p = 3, \frac{3}{4} + \frac{1}{4.3} = \frac{5}{6}$. For $p > 3, \frac{3}{4} + \frac{1}{4p} < \frac{5}{6}$. For $p = 2, \operatorname{PrCent}(G) \leq \frac{1}{2} < \frac{5}{6}$. Hence $\operatorname{PrCent}(G) \leq \frac{5}{6}$.

Second part: Suppose that $G \cong S_3$. Then

$$\begin{aligned}\mathrm{PrCent}(G) &= \mathrm{PrCent}(S_3)\\ &= \frac{|\,\mathrm{Cent}(S_3)|}{|S_3|}\\ &= \frac{3+2}{6}\\ &= \frac{5}{6}.\end{aligned}$$

Conversely, suppose $\mathrm{PrCent}(G) = \frac{5}{6} < \frac{1}{2}$. Therefore, by theorem 2.5.2 if p is the largest prime divisor of $|G|$ then p is odd. Therefore

$$\begin{aligned}&\mathrm{PrCent}(G) \leq \frac{3}{4} + \frac{1}{4p}\\ &\Rightarrow \frac{5}{6} \leq \frac{3}{4} + \frac{1}{4p}\\ &\Rightarrow p \leq 3.\end{aligned}$$

Hence $p = 3$. So for $p = 3, \mathrm{PrCent}(G) = \frac{5}{6} = \frac{1}{2} + \frac{1+p}{2.2p}$. Therefore using Equation (2.5.a) from Theorem 2.5.2, we have that for $p = 3, \mathrm{PrInv}(G)$ is maximum i. e.

$$\mathrm{PrInv}(G) = \frac{3+1}{2.3}.$$

Hence by Lemma 2.5.3, $G \cong D_3 \times \underbrace{C_2 \times \cdots \times C_2}_{j-\text{times}}$. Therefore by Proposition 2.1.2, $|\,\mathrm{Cent}(G)| = |\,\mathrm{Cent}(D_3)| = 5$. So

$$\begin{aligned}&\mathrm{PrCent}(G) = \frac{5}{2.3.2^j}\\ &\Rightarrow \frac{5}{6} = \frac{5}{6.2^j}\\ &\Rightarrow j = 0.\end{aligned}$$

Thus $G \cong D_3 \cong S_3$. This completes the proof. □

Using the GAP system, we can see all the groups G of order $n \leq 100$ such that $\text{PrCent}(G) \geq \frac{1}{2}$. In the following table we list all the groups of order $n \leq 100$ and $\text{PrCent}(G) \geq \frac{1}{2}$ except dihedral groups; as we have computed the number of distinct centralizers of dihedral groups.

Let $S(m,n)$ denote the *solvable group* (m,n) in the library of GAP [34].

Table 2.1: [5] Non-dihedral groups G of order $n \leq 100$ and $\text{PrCent}(G) \geq \frac{1}{2}$

$\lvert\text{Cent}(S(12,5))\rvert = 6$	$\lvert\text{Cent}(S(18,3))\rvert = 11$	$\lvert\text{Cent}(S(18,5))\rvert = 11$
$\lvert\text{Cent}(S(24,15))\rvert = 14$	$\lvert\text{Cent}(S(30,4))\rvert = 14$	$\lvert\text{Cent}(S(32,42))\rvert = 16$
$\lvert\text{Cent}(S(32,43))\rvert = 16$	$\lvert\text{Cent}(S(36,13))\rvert = 25$	$\lvert\text{Cent}(S(42,6))\rvert = 23$
$\lvert\text{Cent}(S(50,3))\rvert = 27$	$\lvert\text{Cent}(S(50,5))\rvert = 27$	$\lvert\text{Cent}(S(54,13))\rvert = 29$
$\lvert\text{Cent}(S(54,14))\rvert = 29$	$\lvert\text{Cent}(S(54,15))\rvert = 29$	$\lvert\text{Cent}(S(60,11))\rvert = 35$
$\lvert\text{Cent}(S(66,4))\rvert = 35$	$\lvert\text{Cent}(S(70,4))\rvert = 37$	$\lvert\text{Cent}(S(72,47))\rvert = 47$
$\lvert\text{Cent}(S(72,48))\rvert = 38$	$\lvert\text{Cent}(S(78,6))\rvert = 41$	$\lvert\text{Cent}(S(82,2))\rvert = 43$
$\lvert\text{Cent}(S(84,14))\rvert = 45$	$\lvert\text{Cent}(S(90,9))\rvert = 47$	$\lvert\text{Cent}(S(90,10))\rvert = 47$

Conjecture 2.5.5. *[4] If* $\mathrm{PrCent}(G) \geq \frac{2}{3}$*, then* G *is isomorphic to* $S_3, S_3 \times S_3$ *or a dihedral group of order* 10.

PROBLEM 2.5.6. To study the ratio,

$$CPR(G) = \frac{\text{Number of centralizers in } G}{\text{Total number of subgroups in } G}$$

i.e., the probability of randomly chosen subgroup of G to be a centralizer in G.

Chapter 3

Connecting $|\operatorname{Cent}(G)|$ with other invariants

Recall that, given a finite group G, the number of centralizers in G is denoted by $|\operatorname{Cent}(G)|$. In this chapter, we study some interesting relations between $|\operatorname{Cent}(G)|$ and other invariants of G, namely the commutativity degree of G, and the size of a maximal subset of pairwise non-commuting elements of G.

3.1 Commutativity degree of finite groups

Given a finite group G, its commutativity degree, denoted by $\operatorname{PrCom}(G)$, is the probability that a randomly chosen pair of G commutes. That is

$$\operatorname{PrCom}(G) = \frac{\text{Number of ordered pairs } (x, y) \in G \times G \text{ such that } xy = yx}{\text{Total number of ordered pairs } (x, y) \in G \times G}.$$

Clearly, $0 \leq \operatorname{PrCom}(G) \leq 1$, and $\operatorname{PrCom}(G) = 1$ if and only if G is abelian. It may be recall that (see, for example, [16]) that

$\text{PrCom}(G) = k(G)/|G|$ where $k(G)$ denotes the number of conjugacy classes in G and by using pages 329 and 330 of [17] we have that for a non-abelian group G, $\text{PrCom}(G) \leq \frac{5}{8}$.

Theorem 3.1.1. *[10] Let G be a finite group. Then $\frac{G}{Z(G)} \cong S_3$ if and only if* $\text{PrCom}(G) = \frac{1}{2}$.

Proof. Suppose that $\frac{G}{Z(G)} \cong S_3$, then

$$\frac{G}{Z(G)} = \langle Zy, Zx | Zx^3 = Zy^2 = Z, Zyxy^{-1} = Zx^{-1} \rangle,$$

where $Z = Z(G)$. Therefore, $G = Z \sqcup Zx \sqcup Zx^2 \sqcup Zy \sqcup Zyx \sqcup Zyx^2$. Hence the only four centralizers of G are

$$C_G(zx) = Z \sqcup Zx \sqcup Zx^2,\ C_G(zy) = Z \sqcup Zy,$$
$$C_G(zyx) = Z \sqcup Zyx,\ C_G(zyx^2) = Z \sqcup Zyx^2,$$

where $z \in Z$. Let $a \in Zx \sqcup Zx^2$ then $a = z_1 x$ or $z_2 x^2$, for some $z_1, z_2 \in Z$. Now we consider $\text{Cl}_G(z_1 x)$. Then for any $zy \in Zy$,

$$(zy)(z_1 x)(zy)^{-1} = z_1(yxy^{-1}) = z_3 x^2,$$

for some $z_3 \in Z$ (Since $Zyxy^{-1} = Zx^2$); for any $zyx \in Zyx$,

$$(zyx)(z_1 x)(zyx)^{-1} = z_1(yxy^{-1}) = z_3 x^2,$$

for some $z_3 \in Z$; for any $zyx^2 \in Zyx^2$,

$$(zyx^2)(z_1 x)(zyx^2)^{-1} = z_1(yxy^{-1}) = z_3 x^2,$$

for some $z_3 \in Z$ and for any $zx^2 \in Zx^2$,

$$(zx^2)(z_1x)(zx^2)^{-1} \;=\; z_1x.$$

Therefore $\mathrm{Cl}_G(z_1x) \;=\; \{z_1x, z_3x^2\}$. Next consider $\mathrm{Cl}_G(z_2x^2)$. Then for any $zy \in Zy$,

$$(zy)(z_2x^2)(zy)^{-1} \;=\; (z_2yxy^{-1})(yxy^{-1}) \;=\; z'x,$$

for some $z' \in Z$; for any $zyx \in Zyx$,

$$(zyx)(z_2x^2)(zyx)^{-1} \;=\; z_2yx^2y^{-1} \;=\; z'x,$$

for some $z' \in Z$; for any $zyx^2 \in Zyx^2$,

$$(zyx^2)(z_2x^2)(zyx^2)^{-1} \;=\; z_2yx^2y^{-1} \;=\; z'x,$$

for some $z' \in Z$ and for any $zx \in Zx$,

$$(zx)(z_2x^2)(zx)^{-1} \;=\; z_2x^2.$$

Therefore, $\mathrm{Cl}_G(z_2x^2) \;=\; \{z'x, z_2x^2\}$. Hence $\mathrm{Cl}_G(a) \subseteq Zx \sqcup Zx^2$ and each non-central element of $Zx \sqcup Zx^2$ forms a conjugacy class of order 2, so number of conjugacy classes of $Zx \sqcup Zx^2$ is $\frac{|Zx \sqcup Zx^2|}{2} \;=\; |Z|$.

Again for any $zy \in Zy$,

$$|\mathrm{Cl}_G(zy)| \;=\; \frac{|G|}{|C_G(zy)|} \;=\; \frac{6|Z|}{2|Z|} \;=\; 3,$$

Similarly, for any $zyx \in Zyx$, $|\mathrm{Cl}_G(zyx)| \;=\; 3$; and for any $zyx^2 \in Zyx^2$, $|\mathrm{Cl}_G(z_2yx^2)| \;=\; 3$. Now each conjugacy classes of $Zy \sqcup Zyx \sqcup Zyx^2$ is

$$\frac{|Zy \sqcup Zyx \sqcup Zyx^2|}{3} \;=\; \frac{3|Z|}{3} \;=\; |Z|.$$

Hence

$$\begin{aligned} \mathrm{PrCom}(G) &= \frac{k(G)}{|G|}, \text{where } k(G) \text{ is the number of conjugacy classes in } G. \\ &= \frac{|Z| + |Z| + |Z|}{|G|} \\ &= \frac{3|Z|}{6|Z|} \\ &= \frac{1}{2} \end{aligned}$$

Converse part is proved by using Lemma 3 and Corollary of [25]. □

Theorem 3.1.2. *[10] If the smallest prime divisor of $|G|$ is p, then*

$$\mathrm{PrCom}(G) = \frac{p^2 + p - 1}{p^3} \textit{ if and only if } \frac{G}{Z(G)} \cong C_p \times C_p.$$

Proof. Suppose first that $\frac{G}{Z(G)} \cong C_p \times C_p$. Then

$$\begin{aligned} \frac{G}{Z} &= \langle Zx, Zy | (Zx)^p = (Zy)^p = Z, (Zx)(Zy) = (Zy)(Zx) \rangle, \ Z = Z(G) \\ &= \langle Zx, Zy | x^p, y^p, xyx^{-1}y^{-1} \in Z \rangle. \end{aligned}$$

If $\frac{H}{Z} < \frac{G}{Z}$ then $|\frac{\frac{G}{Z}}{\frac{H}{Z}}| = p \Rightarrow |\frac{H}{Z}| = p$. Therefore $H = Z \sqcup Zt_1 \sqcup \cdots \sqcup Zt_{p-1}$, where $t_i \in H - Z$ and $i \in \{1, 2, \ldots, p-1\}$. So, the proper subgroups of G

properly containing Z are

$$\begin{aligned}
H_1 &= Z \sqcup Zx \sqcup Zx^2 \sqcup \cdots \sqcup Zx^{p-1},\\
H_2 &= Z \sqcup Zy \sqcup Zy^2 \sqcup \cdots \sqcup Zy^{p-1},\\
H_3 &= Z \sqcup Zxy \sqcup Zx^2y^2 \sqcup \cdots \sqcup Zx^{p-1}y^{p-1},\\
&\vdots\\
H_{p+1} &= Z \sqcup Zx^{p-1}y \sqcup Zx^{p-2}y^2 \sqcup \cdots \sqcup Zxy^{p-1},
\end{aligned}$$

and each of which is the centralizer of non-central elements of G.

Let $a \in G - Z$ then

$$|\operatorname{Cl}_G(a)| = \frac{|G|}{|C_G(a)|} = \frac{p^2|Z|}{p|Z|} = p.$$

Therefore each conjugacy class of $G - Z$ has p elements. So number of conjugacy classes of $G - Z$ is $\frac{(p^2-1)|Z|}{p}$. Therefore

$$\begin{aligned}
\operatorname{PrCom}(G) &= \frac{k(G)}{|G|}, \text{where } k(G) \text{ is the number of conjugacy classes in } G\\
&= \frac{|Z| + \frac{(p^2-1)|Z|}{p}}{p^2|Z|}\\
&= \frac{p^2+p-1}{p^3}.
\end{aligned}$$

Conversely, suppose $\operatorname{PrCom}(G) = \frac{p^2+p-1}{p^3}$, where p is the smallest prime divisor of $|G|$. Then

$$\frac{p^2+p-1}{p^3} = \frac{|Z|}{|G|} + \frac{k(G)-|Z|}{|G|}, \tag{3.1.a}$$

noting that $\mathrm{PrCom}(G) = \frac{k(G)}{|G|}$. Since p is the smallest prime divisor of $|G|$, so $|\frac{G}{Z}| > p \Rightarrow |\frac{G}{Z}| \geq p^2$. Suppose $|\frac{G}{Z}| > p^2 \Rightarrow \frac{|Z|}{|G|} < \frac{1}{p^2}$. Therefore Equation (3.1.a) gives,

$$\frac{p^2+p-1}{p^3} < \frac{1}{p^2} + \frac{k(G)-|Z|}{|G|}$$
$$\Rightarrow \frac{p^2-1}{p^3} < \frac{k(G)-|Z|}{|G|}.$$

Now using class equation, $|G| = |Z| + \sum_x |G|C_G(x)|$, where x runs through a complete set of representatives of all the distinct conjugacy classes having more than one element, we have

$$|G| \geq |Z| + p\sum_x 1$$
$$\Rightarrow |G| \geq |Z| + p(k(G) - |Z|)$$
$$\Rightarrow |G| \geq k(G) + (p-1)k(G) + (1-p)|Z|$$
$$\Rightarrow 1 \geq \frac{k(G)}{|G|} + \frac{(p-1)(k(G)-|Z|)}{|G|}$$
$$\Rightarrow 1 \geq \frac{p^2+p-1}{p^3} + (p-1)\frac{k(G)-|Z|}{|G|}$$
$$\Rightarrow \frac{p^2-1}{p^3} \geq \frac{k(G)-|Z|}{|G|},$$

which is a contradiction. Hence $|\frac{G}{Z}| = p^2 \Rightarrow \frac{G}{Z(G)} \cong C_p \times C_p$. This completes the proof. □

Putting $p = 3$ in Theorem 3.1.2 we get the following:

Corollary 3.1.3. *[10] If* $\frac{G}{Z(G)} \cong C_3 \times C_3$ *then* $\mathrm{PrCom}(G) = \frac{11}{27}$ *and if* $\mathrm{PrCom}(G) = \frac{11}{27}$ *and the smallest prime divisor of* $|G|$ *is three, then*

$$\frac{G}{Z(G)} \cong C_3 \times C_3.$$

3.2 Relation between $|\operatorname{Cent}(G)|$ and $\operatorname{PrCom}(G)$

Recall that for any finite group G, $|\operatorname{Cent}(G)|$ denotes the number of distinct centralizers in G and $\operatorname{PrCom}(G)$ denotes the commutativity degree of G. We know that for an abelian group G, $\operatorname{PrCom}(G) = |\operatorname{Cent}(G)| = 1$.

Theorem 3.2.1. *[10] Let G be a finite group. Then* $\operatorname{PrCom}(G) = \frac{5}{8}$ *if and only if* $|\operatorname{Cent}(G)| = 4$.

Proof. Suppose that $|\operatorname{Cent}(G)| = 4$. Then by Theorem 2.3.1, we have

$$\frac{G}{Z(G)} \cong C_2 \times C_2 \Rightarrow \left|\frac{G}{Z(G)}\right| = 4.$$

Now for $a_1 \in G - Z(G)$, there exists $a_2 \in G - Z(G)$ such that $a_1a_2 \neq a_2a_1$; otherwise $a_1 \in Z(G)$, a contradiction. Next consider $a_3 = a_1a_2$. Then $a_3 \in G - Z(G)$, because for $a_3 \in Z(G)$ we have $a_1a_3 = a_3a_1 \Rightarrow a_1a_2 = a_2a_1$, a contradiction.

Now we will show that $Z(G), a_1Z(G), a_2Z(G), a_3Z(G)$ all are disjoint. If $Z(G) = a_1Z(G)$ then $a_1 \in Z(G)$, a contradiction. So $a_1Z(G) \neq Z(G)$. Similarly, $a_2Z(G) \neq Z(G), a_3Z(G) \neq Z(G)$. Suppose $a_1Z = a_2Z$ where $Z = Z(G)$, then

$$\begin{aligned} &{a_2}^{-1}a_1 \in Z \\ &\Rightarrow {a_2}^{-1}a_1 = z_1 \text{ for some } z_1 \in Z \\ &\Rightarrow a_1 = a_2z_1 \text{ for some } z_1 \in Z. \end{aligned}$$

Therefore $a_1a_2 = a_2z_1a_2 = a_2a_2z_1 = a_2a_1$, a contradiction. Now, suppose

$a_1 Z = a_3 Z$ then

$$\begin{aligned} & a_3{}^{-1} a_1 \in Z \\ \Rightarrow & (a_1 a_2)^{-1} a_1 \in Z \\ \Rightarrow & a_2 \in Z, \end{aligned}$$

a contradiction. Suppose $a_2 Z = a_3 Z$ then

$$\begin{aligned} & a_3 a_2{}^{-1} \in Z \\ \Rightarrow & a_1 a_2 a_2{}^{-1} \in Z \\ \Rightarrow & a_1 \in Z, \end{aligned}$$

a contradiction. Therefore $\frac{G}{Z(G)} = \{Z(G), a_1Z(G), a_2Z(G), a_3Z(G)\}$, where $a_3 = a_1 a_2$ and $a_1{}^2, a_2{}^2, a_3{}^2 \in Z(G)$. Again, if $x \in a_i Z(G)$ and $y \in a_j Z(G)$ then $xy \neq yx$ where $1 \leq i(\neq j) \leq 3$, because for $xy = yx$ we have

$$\begin{aligned} (a_i z_1)(a_j z_2) &= (a_j z_2)(a_i z_1), \text{ where } z_1, z_2 \in Z(G) \\ \Rightarrow a_i a_j &= a_j a_i, \end{aligned}$$

a contradiction. Thus $x, y \in G$ will commute if

(i) $x \in Z(G), y \in G$.

(ii) $x \in a_1 Z(G), y \in Z(G)$ or $y \in a_1 Z(G)$.

(iii) $x \in a_2 Z(G), y \in Z(G)$ or $y \in a_2 Z(G)$.

(iv) $x \in a_3 Z(G), y \in Z(G)$ or $y \in a_3 Z(G)$.

Now

$$
\begin{aligned}
|\operatorname{Com}(G)| &= |\{(x,y) \in G \times G | xy = yx\}| \\
&= |\{(x,y) \in G \times G | x \in Z(G), y \in G\}| \\
&\quad + 3|\{(x,y) \in G \times G | x \in a_1 Z(G), y \in Z(G)\}| \\
&\quad + 3|\{(x,y) \in G \times G | x, y \in a_1 Z(G)\}| \\
&= |Z(G)||G| + 3|a_1 Z(G)||Z(G)| + 3|a_1 Z(G)||a_1 Z(G)| \\
&= |Z(G)||G| + 6|Z(G)|^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\operatorname{PrCom}(G) &= \frac{|\operatorname{Com}(G)|}{|G|^2} \\
&= \frac{|Z(G)||G| + 6|Z(G)|^2}{|G|^2} \\
&= \frac{|Z(G)|}{|G|} + 6\frac{|Z(G)|^2}{|G|^2} \\
&= \frac{1}{4} + 6\frac{1}{4^2} \\
&= \frac{5}{8}.
\end{aligned}
$$

Conversely, suppose that $\operatorname{PrCom}(G) = \frac{5}{8}$. Then G is non-abelian and so by Theorem 2.1.6, $|\operatorname{Cent}(G)| \geq 4$. Also it is clear that $|\operatorname{Cent}(G)| \leq |\frac{G}{Z(G)}|$.

Therefore $|\frac{G}{Z(G)}| \geq 4$. Assume $|\frac{G}{Z(G)}| > 4 \Rightarrow \frac{|Z(G)|}{|G|} < \frac{1}{4}$. Now

$$\begin{aligned}
\text{PrCom}(G) &= \frac{5}{8} \\
\Rightarrow \frac{k(G)}{|G|} &= \frac{5}{8}, \text{ where } k(G) \text{ is the number of conjugacy classes in } G \\
\Rightarrow \frac{|Z(G)| + k(G) - |Z(G)|}{|G|} &= \frac{5}{8} \\
\Rightarrow \frac{|Z(G)|}{|G|} + \frac{k(G) - |Z(G)|}{|G|} &= \frac{5}{8} \\
\Rightarrow \frac{5}{8} < \frac{1}{4} + \frac{k(G) - |Z(G)|}{|G|} \\
\Rightarrow \frac{3}{8} < \frac{k(G) - |Z(G)|}{|G|}.
\end{aligned}$$

Using class equation, $|G| = |Z(G)| + \sum_{x} |G : C_G(x)|$, where x runs through a complete set of representatives of all the distinct conjugacy classes having more than one element, we have

$$\begin{aligned}
&|G| \geq |Z(G)| + 2\sum_{x} 1 \\
&\Rightarrow |G| \geq |Z(G)| + 2(k(G) - |Z(G)|) \\
&\Rightarrow |G| \geq 2k(G) - |Z(G)| \\
&\Rightarrow 1 \geq \frac{k(G)}{|G|} + \frac{k(G) - |Z(G)|}{|G|} \\
&\Rightarrow 1 \geq \frac{5}{8} + \frac{k(G) - |Z(G)|}{|G|} \\
&\Rightarrow \frac{k(G) - |Z(G)|}{|G|} \leq \frac{3}{8},
\end{aligned}$$

which is a contradiction. Hence $|\frac{G}{Z(G)}| = 4 \Rightarrow |\text{Cent}(G)| = 4$. This completes the proof. □

Thus, combining Theorem 3.1.2 and Proposition 2.2.3, we get the following proposition.

Proposition 3.2.2. *If the smallest prime divisor of* $|G|$ *is* p *and* $\text{PrCom}(G) = \frac{p^2+p-1}{p^3}$ *, then* $|\text{Cent}(G)| = p+2$.

The following question arises trivially.

PROBLEM 3.2.3. Suppose p is the smallest prime divisor of $|G|$ and $|\text{Cent}(G)| = p+2$. Then is it true that $\text{PrCom}(G) = \frac{p^2+p-1}{p^3}$?

Note that, for p-group the answer is true by Proposition 2.2.9 and Theorem 3.1.2, for $p = 2$ the answer is true by Theorem 3.2.1; and for $p = 3$ the answer is true by Theorem 2.4.4 and Theorem 3.1.2.

3.3 Maximal subset of non-commuting elements

A non-empty subset X of a finite group G is called *a set of pairwise non-commuting elements* if $xy \neq yx$ for all $x, y \in X$ with $x \neq y$. A set of *pairwise non-commuting elements of* G *is said to have maximal size* if its cardinality is largest one among all such sets. We denote this largest cardinality by $r(G)$.

Proposition 3.3.1. *[7] Let* G *be a finite* n*-centralizer group and* $r(G) = n-1$*. Then every proper centralizer of* G *is abelian. Moreover, for every non-central elements* x *and* y *of* G, $C_G(x) = C_G(y)$ *or* $C_G(x) \cap C_G(y) = Z(G)$.

Proof. Let $\{x_1, x_2, \ldots, x_{n-1}\}$ be a set of pairwise non-commuting elements of G. Then we have

$$\text{Cent}(G) = \{G, C_G(x_1), \ldots, C_G(x_{n-1})\}.$$

Now suppose for some $i, 1 \leq i \leq n-1$, $X = C_G(x_i)$ is non-abelian. We choose elements $a, b \in X$ such that $ab \neq ba$ (since, X is non-abelian such elements exists). So $a, b \notin Z(G)$. Therefore $C_G(a), C_G(b) \neq G$. Also $C_G(a) \neq C_G(b)$, as $b \notin C_G(a)$. Without any loss, we can assume that $C_G(a) = C_G(x_j)$ for some $j \neq i, 1 \leq j \leq n-1$.

Also $x_i \in C_G(a)$, since $a \in X = C_G(x_i)$. Which implies $x_i \in C_G(x_j)$, a contradiction. Hence every proper element centralizer of G is abelian.

For the second part, we assume that x and y are non-central elements of G and $C_G(x) \neq C_G(y)$. Let $T = C_G(x) \cap C_G(y)$. Choose an element $a \in T - Z(G)$, i.e. $C_G(a) \neq G$.

Let $b \in C_G(x) \cup C_G(y)$ be an arbitrary element. If $b \in C_G(x)$ then $ab = ba$, since $a \in C_G(x)$ and $C_G(x)$ is abelian. Therefore $b \in C_G(a)$. Similarly, if $b \in C_G(y)$ then $b \in C_G(a)$. Hence

$$C_G(x) \cup C_G(y) \subseteq C_G(a)$$
$$\Rightarrow C_G(x) \subsetneq C_G(a) \text{ and } C_G(y) \subsetneq C_G(a).$$

But since $C_G(x) \neq C_G(y)$ and x and y are non-central elements of G, so for some $i \neq j$, $1 \leq i, j \leq n-1$, we have $C_G(x) = C_G(x_i)$ and $C_G(y) = C_G(x_j)$

and $C_G(a) = C_G(x_k)$ for some $k \neq i, j; 1 \leq k \leq n-1$. So

$$\begin{aligned} C_G(x) &= C_G(x_i) \subset C_G(a) \\ \Rightarrow C_G(x_i) &\subset C_G(x_k) \\ \Rightarrow x_i x_k &= x_k x_i, \text{a contradiction.} \end{aligned}$$

Hence

$$\begin{aligned} T - Z(G) &= \phi \\ \Rightarrow C_G(x) \cap C_G(y) &= Z(G). \end{aligned}$$

This completes the proof. □

Proposition 3.3.2. *[1] Let G be a finite group and $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then*

(i) *$\{C_G(x_i) | i = 1, 2, \ldots, r\}$ is an irredundant r-cover with the intersection $Z(G) = \bigcap_{i=1}^{r} C_G(x_i)$.*

(ii) *$|\frac{G}{Z(G)}| \leq f(r)$.*

(iii) *$f(3) = 4, f(4) = 9, f(5) = 16, f(6) = 36$.*

(iv) *if G is a group such that every proper centralizer is abelian, then for all $a, b \in G - Z(G)$ either $C_G(a) = C_G(b)$ or $C_G(a) \cap C_G(b) = Z(G)$.*

Proof. (i) If there were an element $x \in G - (\bigcup_{i=1}^{r} C_G(x_i))$, then $\{x_1, x_2, \ldots, x_r, x\}$ would be a set of $(r+1)$ pairwise non-commuting elements. This contradicts the maximality of the set $\{x_1, x_2, \ldots, x_r\}$ of pairwise non-commuting elements of G. Hence

$$G - (\overset{r}{\underset{i=1}{\cup}} C_G(x_i)) = \phi$$
$$\Rightarrow G = \overset{r}{\underset{i=1}{\cup}} C_G(x_i).$$

Now, we suppose that for some $i, 1 \leq i \leq r$, $C_G(x_i) \subset \underset{j \neq i}{\cup} C_G(x_j)$ where $1 \leq j \leq r$, then $x_i \in C_G(x_j)$, a contradiction. Therefore

$$C_G(x_i) \nsubseteq \underset{j \neq i}{\cup} C_G(x_j), 1 \leq j \leq r.$$

Next suppose that there is an element $a \in (\overset{r}{\underset{i=1}{\cap}} C_G(x_i)) - Z(G)$. Then there is an element b such that $[a, b] \neq 1$. Now for each $i = 1, 2, \ldots, r$, define

$$y_i = \begin{cases} x_i, & \text{if } [x_i, b] \neq 1 \\ ax_i, & \text{if } [x_i, b] = 1. \end{cases}$$

Then $\{y_1, y_2, \ldots, y_r, b\}$ is a set of $(r+1)$ pairwise non-commuting elements contradicting the maximality of r. Therefore

$$(\overset{r}{\underset{i=1}{\cap}} C_G(x_i)) - Z(G) = \phi$$
$$\Rightarrow \overset{r}{\underset{i=1}{\cap}} C_G(x_i) = Z(G).$$

Hence $\{C_G(x) | i = 1, 2, \ldots, r\}$ is an irredundant r-cover with the intersection $Z(G) = \overset{r}{\underset{i=1}{\cap}} C_G(x_i)$.

(ii) See Corollary 5.2 of [32]

(iii) See [30], [18], [14] and [2] respectively

(iv) Suppose $C_G(a) \cap C_G(b) \neq Z(G)$, which gives there exists an element $z \in (C_G(a) \cap C_G(b)) - Z(G)$. Then $C_G(z)$ contains both $C_G(a)$ and $C_G(b)$, since $C_G(a)$ and $C_G(b)$ are abelian. Since z is not in $Z(G)$, $C_G(z) \leq C_G(a)$ and $C_G(z) \leq C_G(b)$. Thus $C_G(z) = C_G(a) = C_G(b)$. This completes the proof.

□

3.4 Relation between $|\,\mathrm{Cent}(G)|$ and $r(G)$

For any finite group G, $|\,\mathrm{Cent}(G)|$ denotes the number of centralizers in G and $r(G)$ denotes the maximum possible size of a set of pairwise non-commuting elements of G. In this section we discuss about some relations between $|\,\mathrm{Cent}(G)|$ and $r(G)$.

Proposition 3.4.1. *[1] Let G be a finite non-abelian group, $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then,*

(i) $r \geq 3$

(ii) $r + 1 \leq |\,\mathrm{Cent}(G)|$

(iii) $r = 3$ *if and only if* $|\,\mathrm{Cent}(G)| = 4$

(iv) $r = 4$ *if and only if* $|\,\mathrm{Cent}(G)| = 5$.

Proof. (i) Since G is not abelian, there exists elements $x, y \in G$ such that $xy \neq yx$. Thus $\{x, y, xy\}$ is a set of pairwise non-commuting elements of G, and so r is atleast 3, as required.

(ii) Given $\{x_1, x_2, \ldots, x_r\}$ is a set of pairwise non-commuting elements of G, so $C_G(x_1), C_G(x_2), \ldots, C_G(x_r)$ all are distinct. Therefore

$$r + 1 \leq |\operatorname{Cent}(G)|.$$

(iii) Suppose that $r = 3$. Then

$$\begin{aligned}
&\left|\frac{G}{Z(G)}\right| \leq f(3) \ \text{(Using Proposition 3.3.2)}\\
\Rightarrow &\left|\frac{G}{Z(G)}\right| \leq 4 \ \text{(Using Proposition 3.3.2)}\\
\Rightarrow &\left|\frac{G}{Z(G)}\right| = 4\\
\Rightarrow &\frac{G}{Z(G)} \cong C_2 \times C_2\\
\Rightarrow &|\operatorname{Cent}(G)| = 4.
\end{aligned}$$

Conversely let $|\operatorname{Cent}(G)| = 4$. Then by first part $r \geq 3$ and second part forces to be $r = 3$.

(iv) Suppose first that $r = 4$ then $|\frac{G}{Z(G)}| \leq f(4) = 9$. Since $r = 4$, so $\{x_1, x_2, x_3, x_4\}$ is a set of pairwise non-commuting elements of G having maximal size. Then $C_G(x_1), C_G(x_2), C_G(x_3), C_G(x_4)$ all are distinct centralizers in G and so $G = C_G(x_1) \cup C_G(x_2) \cup C_G(x_3) \cup C_G(x_4)$. Suppose $|G : C_G(x_i)| = \alpha_i$ for $1 \leq i \leq 4$ and $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$.

Now we will show that $\frac{G}{Z(G)}$ is not a 2-group. Assume on the contrary that, $\frac{G}{Z(G)}$ is a 2-group. Then $\left|\frac{G}{Z(G)}\right| = 4$ or 8 and so $C_G(x)$ is abelian for each $x \in G - Z(G)$, this is because $\left|\frac{C_G(x)}{Z(C_G(x))}\right|$ is cyclic. Hence by using Proposition 3.3.2(iv), we have $Z(G) = C_G(x) \cap C_G(y)$ for distinct proper centralizers $C_G(x)$ and $C_G(y)$.

Again by Theorem 1.10.2,

$$\alpha_2 \leq 4 - 1$$
$$\Rightarrow |G : C_G(x_2)| \leq 3.$$

Also $|G : C_G(x_2)| = \frac{|\frac{G}{Z(G)}|}{|\frac{C_G(x_2)}{Z(G)}|}$ therefore $|G : C_G(x_2)|$ divides $\left|\frac{G}{Z(G)}\right|$. But we assumed that $\frac{G}{Z(G)}$ is a 2-group, so $|\frac{G}{C_G(x_2)}| = 2$ that is $\alpha_2 = 2$. Therefore $\alpha_1 = 2$.

Also

$$\begin{aligned}
|C_G(x_1)C_G(x_2)| &= \frac{|C_G(x_1)||C_G(x_2)|}{|C_G(x_1) \cap C_G(x_2)|} \\
\Rightarrow |G| &\geq \frac{|C_G(x_1)||C_G(x_2)|}{|Z(G)|} \\
\Rightarrow \left|\frac{G}{Z(G)}\right| &\leq 4 \\
\Rightarrow \left|\frac{G}{Z(G)}\right| &= 4 \\
\Rightarrow \frac{G}{Z(G)} &\cong C_2 \times C_2 \\
\Rightarrow |\operatorname{Cent}(G)| &= 4.
\end{aligned}$$

So, by third part $r = 3$, a contradiction. Hence $\frac{G}{Z(G)}$ is not a 2-group. Thus

$$\left|\frac{G}{Z(G)}\right| = 6, 9.$$

If $\left|\frac{G}{Z(G)}\right| = 6$ then $\frac{G}{Z(G)} \cong S_3 \Rightarrow |\operatorname{Cent}(G)| = 5$. If $\left|\frac{G}{Z(G)}\right| = 9$ then $\frac{G}{Z(G)} \cong C_3 \times C_3 \Rightarrow |\operatorname{Cent}(G)| = 5$.

Conversely, suppose $|\operatorname{Cent}(G)| = 5$. Then by second part, $r \leq 4$ and by first part, $r \geq 3$. On the other hand $r \neq 3$ by third part. Thus $r = 4$. This completes the proof.

□

Proposition 3.4.2. *[1] Let $X = \{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of a finite non-abelian group G having maximal size. Then,*

(a) *If $|\operatorname{Cent}(G)| < r + 4$, then*

(i) *For each element $x \in G$, $C_G(x)$ is abelian if and only if $C_G(x) = C_G(x_i)$ for some $i \in \{1, 2, \ldots, r\}$.*

(ii) *If $C_G(x_i)$ is a maximal subgroup of G for some $i \in \{1, 2, \ldots, r\}$, then $Z(G) = C_G(x_i) \cap C_G(x_j)$ for all $j \in \{1, 2, \ldots, r\} - \{i\}$. In particular, if $|G : C_G(x_1)| \leq |G : C_G(x_2)| \leq 2$, then $|\operatorname{Cent}(G)| = 4$, and if $|G : C_G(x_1)| \leq |G : C_G(x_2)| = 3$, then $|\operatorname{Cent}(G)| = 5$.*

(b) *If $|\operatorname{Cent}(G)| = r + 2$, then there exists a proper non-abelian centralizer $C_G(x)$ which contains $C_G(x_{i_1}), C_G(x_{i_2})$ and $C_G(x_{i_3})$ for three distinct $i_1, i_2, i_3 \in \{1, 2, \ldots, r\}$.*

(c) *If $|\operatorname{Cent}(G)| = r + 3$, then there exists a proper non-abelian centralizer $C_G(x)$ which contains $C_G(x_{i_1})$ and $C_G(x_{i_2})$ for two distinct $i_1, i_2 \in \{1, 2, \ldots, r\}$.*

Proof. **(a)** (i) Suppose, for a contradiction that there exists an index i such that $K = C_G(x_i)$ is non-abelian. Then

$$|\operatorname{Cent}(K)| = |\operatorname{Cent}(C_G(x_i))| \geq 4.$$

So $C_G(x_i)$ contains at least three proper centralizers, say $C_G(y_j)$, $j = 1, 2, 3$. By hypothesis, $C_G(x_t) \neq C_G(y_j)$ for every $1 \leq t \leq r$ and $1 \leq j \leq 3$. Therefore

$$(r+1) + 3 \leq |\operatorname{Cent}(G)|$$
$$\Rightarrow r + 4 \leq |\operatorname{Cent}(G)| < r + 4,$$

which is not possible. Hence, each $C_G(x_i), 1 \leq i \leq r$ is abelian.

Conversely suppose that for each element $x \in G, C_G(x)$ is abelian. Since $\{x_1, x_2, \ldots, x_r\}$ is a set of pairwise non-commuting elements of G having maximal size, so there exists an index j such that $x \in C_G(x_j)$ and by assumption, $C_G(x_j)$ is abelian.

Now, we will show that $C_G(x_j) = C_G(x)$. Let $t \in C_G(x_j)$ then

$$tx = xt, \text{ since } x \in C_G(x_j) \text{ and } C_G(x_j) \text{ is abelian}$$
$$\Rightarrow t \in C_G(x).$$

Therefore $C_G(x_j) \subseteq C_G(x)$. Similarly $C_G(x) \subseteq C_G(x_j)$. Thus $C_G(x_j) = C_G(x)$.

(ii) First part: Let $x \in C_G(x_i) \cap C_G(x_j)\ \forall\ j \in \{1, 2, \ldots, r\} - \{i\}$. By previous part, $C_G(x_i)$ is abelian. Now we will show that $C_G(x_i) \leq C_G(x)$. Let $l \in C_G(x_i)$. Then

$$lx = xl, \text{ since } x \in C_G(x_i) \text{ and } C_G(x_i) \text{ is abelian}$$
$$\Rightarrow l \in C_G(x).$$

Suppose, $C_G(x_i) = C_G(x)$. Then $x_j \in C_G(x) = C_G(x_i)$, a contradiction. Therefore $C_G(x_i) < C_G(x)$. Since $C_G(x_i)$ is a maximal subgroup of G, so $C_G(x) = G \Rightarrow x \in Z(G)$. Hence

$$Z(G) = C_G(x_i) \cap C_G(x_j) \text{ for all } j \in \{1, 2, \ldots, r\} - \{i\}.$$

Second part: Given $|G : C_G(x_1)| \leq |G : C_G(x_2)| \leq 2$, therefore $|G : C_G(x_1)| = |G : C_G(x_2)| = 2$. So, $C_G(x_1)$ is a maximal subgroup of G and by first part we have, $C_G(x_1) \cap C_G(x_2) = Z(G)$. Therefore

$$\begin{aligned}
|C_G(x_1)C_G(x_2)| &= \frac{|C_G(x_1)||C_G(x_2)|}{|C_G(x_1) \cap C_G(x_2)|} \\
\Rightarrow |G| \geq \frac{|C_G(x_1)||C_G(x_2)|}{|Z(G)|} \\
\Rightarrow \left|\frac{G}{Z(G)}\right| \leq 4 \\
\Rightarrow \left|\frac{G}{Z(G)}\right| &= 4 \\
\Rightarrow \frac{G}{Z(G)} \cong C_2 \times C_2 \\
\Rightarrow |\operatorname{Cent}(G)| &= 4.
\end{aligned}$$

Third part: If $|G : C_G(x_1)| \leq |G : C_G(x_2)| = 3$, then $C_G(x_2)$ is a maximal subgroup of G and $Z(G) = C_G(x_1) \cap C_G(x_2)$, by the first part. So

$$\begin{aligned}
|C_G(x_1)C_G(x_2)| &= \frac{|C_G(x_1)||C_G(x_2)|}{|C_G(x_1)\cap C_G(x_2)|}\\
\Rightarrow |G| \geq \frac{|C_G(x_1)||C_G(x_2)|}{|Z(G)|}\\
\Rightarrow \left|\frac{G}{Z(G)}\right| \leq 9.
\end{aligned}$$

Also $|G : C_G(x_2)| = 3$, therefore

$$\begin{aligned}
3 \mid \left|\frac{G}{Z(G)}\right|\\
\Rightarrow \left|\frac{G}{Z(G)}\right| &= 3, 6, 9\\
\Rightarrow \frac{G}{Z(G)} \cong S_3 \text{ or } \frac{G}{Z(G)} \cong C_3 \times C_3\\
\Rightarrow |\operatorname{Cent}(G)| &= 5.
\end{aligned}$$

(b) Given $|\operatorname{Cent}(G)| = r + 2$. So by part (a)(i), there exists $x \in G$ such that $C_G(x) \neq C_G(x_i), 1 \leq i \leq r$ and $K = C_G(x)$ is non-abelian. Therefore,

$$|\operatorname{Cent}(K)| \geq 4.$$

So $K = C_G(x)$ contains at least three proper centralizers, say $C_K(y_i)$, $i = 1, 2, 3$. Therefore $\{C_K(y_i) | i = 1, 2, 3\}$ is a set of three proper centralizers of K, then $\{C_G(y_i) | i = 1, 2, 3\}$ is a set of three proper centralizers of G.

Since $|\operatorname{Cent}(G)| = r + 2$, we may assume that $C_G(y_i) = C_G(x_{j_i})$ for three distinct $j_1, j_2, j_3 \in \{1, 2, \ldots, r\}$. Then $C_G(y_i)$ is abelian for $i = 1, 2, 3$, by part (a)(i).

Next we will show that $C_G(y_i) \subseteq C_G(x)$ for $i = 1, 2, 3$. Let $t \in C_G(y_i)$. Therefore

$$\begin{aligned} tx &= xt, \text{ since } x \in C_K(y_i) \subseteq C_G(y_i) \text{ and } C_G(y_i) \text{ is abelian} \\ &\Rightarrow t \in C_G(x). \end{aligned}$$

So, $C_G(y_i) \subseteq C_G(x)$ for $i = 1, 2, 3$.

(c) Given $|\operatorname{Cent}(G)| = r + 3$. So by part (a)(i), there exists $x \in G$ such that $H = C_G(x)$ is non-abelian. Therefore

$$|\operatorname{Cent}(H)| \geq 4.$$

So H contains at least three proper centralizers, say $C_H(w_i), i = 1, 2, 3$. Therefore $\{C_H(w_i)|i = 1, 2, 3\}$ is a set of three proper centralizers of H, then $\{C_G(w_i)|i = 1, 2, 3\}$ is a set of three proper centralizers of G.

Since $|\operatorname{Cent}(G)| = r + 3$, we may assume that $C_G(w_i) = C_G(x_{j_i})$ for two distinct $j_1, j_2 \in \{1, 2, \ldots, r\}$. Then $C_G(w_i)$ is abelian for $i = 1, 2$, by part (a)(i).

Now we will show that $C_G(w_i) \subseteq C_G(x)$ for $i = 1, 2$. Let $n \in C_G(w_i)$ then

$$\begin{aligned} nx &= xn, \text{ since } x \in C_H(w_i) \subseteq C_G(w_i) \text{ and } C_G(w_i) \text{ is abelian} \\ &\Rightarrow n \in C_G(x). \end{aligned}$$

So, $C_G(w_i) \subseteq C_G(x)$ for $i = 1, 2$. This completes the proof.

□

Proposition 3.4.3. *[1] Let G be a finite non-abelian group. Then every proper centralizer of G is abelian if and only if $|\operatorname{Cent}(G)| = r+1$, where r is the maximal size of a set of pairwise non-commuting elements of G.*

Proof. Suppose that every proper centralizer of G is abelian and let $X = \{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements having maximal size. Consider $C_G(x)$ where $x \in G - Z(G)$. Then there exists an $i \in \{1, 2, \ldots, r\}$ such that $x \in C_G(x_i)$.

We will show that $C_G(x_i) = C_G(x)$. Let $t \in C_G(x_i)$ then

$$\begin{aligned} tx &= xt, \text{ since } x \in C_G(x_i) \text{ and } C_G(x_i) \text{ is abelian} \\ \Rightarrow & t \in C_G(x). \end{aligned}$$

So, $C_G(x_i) \subseteq C_G(x)$. Similarly $C_G(x) \subseteq C_G(x_i)$. Thus $C_G(x_i) = C_G(x)$. Hence

$$\begin{aligned} \operatorname{Cent}(G) &= \{G, C_G(x_1), C_G(x_2), \ldots, C_G(x_r)\} \\ \Rightarrow |\operatorname{Cent}(G)| &= r+1. \end{aligned}$$

By using Lemma 3.3.1, we can prove the converse part of this Lemma. □

Chapter 4

Groups with $|\mathrm{Cent}(G)| = 6,\ 7,\ 8$

In this chapter we study some properties of $6, 7, 8$-centralizer groups.

4.1 6-centralizer groups

In this section we study the structure of 6-centralizer groups. The following lemmas are useful in proving the main theorem in this section.

Lemma 4.1.1. *[4] Let G be a 6-centralizer group and $X_1, X_2, \ldots, X_6$ be distinct centralizers of G such that $X_1 = G$. If $|G : X_2| = \cdots = |G : X_6| = 4$, then all of X_i's are normal in G and $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \times C_2$.*

Proof. Given $X_1, X_2, \ldots, X_6$ be distinct centralizers of G such that $X_1 = G$.

So $G = \overset{6}{\underset{i=2}{\cup}} X_i$ and $\overset{6}{\underset{i=2}{\cap}} X_i = Z(G)$. Also

$$\begin{aligned}\sum_{i=3}^{6}|X_i| &= |X_3|+|X_4|+|X_5|+|X_6| \\ &= \frac{|G|}{4}+\frac{|G|}{4}+\frac{|G|}{4}+\frac{|G|}{4} \\ &= |G|.\end{aligned}$$

Therefore by Cohn's Theorem 1.10.3, $X_2.X_t = G$ for all $t \neq 2$ and by interchanging X_i's we have, $X_2 \cap X_t = Z(G)$. So

$$\begin{aligned}|X_2.X_t| &= \frac{|X_2||X_t|}{|X_2 \cap X_t|} \\ \Rightarrow\ 4^2|G| &= \frac{|G|^2}{|Z(G)|} \\ \Rightarrow |G|Z(G)| &= 16.\end{aligned}$$

Suppose X_i is a non-normal subgroup of G for some $i, 2 \leq i \leq 6$, then $\mathrm{Core}_G(X_i) = Z(G)$. Since $|G : X_i| = 4$, therefore $\frac{G}{Z(G)} = \frac{G}{\mathrm{Core}_G(X_i)}$ is isomorphic to a subgroup of S_4. Therefore $16 \mid 4!$, which is impossible. So, all of X_i's are normal in G. Hence by Lemma 2.2.12, $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \times C_2$. This completes the proof. □

Now we will prove the main result of this section.

Theorem 4.1.2. *[5] If G is a 6-centralizer group, then*

$$\frac{G}{Z(G)} \cong D_8, A_4, C_2 \times C_2 \times C_2 \ \text{or}\ C_2 \times C_2 \times C_2 \times C_2.$$

Proof. Let G be a 6-centralizer group and $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then $C_G(x_1), \ldots, C_G(x_r)$

are distinct proper centralizers of G. Suppose $X_i = C_G(x_i), 1 \leq i \leq r$, and $|G : X_1| \leq |G : X_2| \leq \cdots \leq |G : X_r|$.

Also by using Proposition 3.4.1, we have $r \geq 3$ and $r + 1 \leq |\operatorname{Cent}(G)|$, so $3 \leq r \leq 5$. Suppose $r = 3$ then by Proposition 3.4.1, $|\operatorname{Cent}(G)| = 4$, a contradiction. Suppose $r = 4$ then by Proposition 3.4.1, $|\operatorname{Cent}(G)| = 5$, a contradiction. Therefore $r = 5$ and so by Proposition 3.3.2, G has an irredundant covering by proper centralizers and by Lemma 3.3.1, $X_i \cap X_j = Z(G)$ where $1 \leq i,\ j \leq 5; i \neq j$. Also by Proposition 3.3.2,

$$\left|\frac{G}{Z(G)}\right| \leq f(5) = 16.$$

Again by Theorem 1.10.2,

$$|G : X_2| \leq 5 - 1 = 4.$$

So our main proof will consider a number of cases.

Case i. $|G : X_2| = 2$. It is easy to see that $C_G(x_1) \cap C_G(x_2) \subseteq C_G(x_1x_2)$, hence $|G : C_G(x_1x_2)| = 1, 2, 4$.

Suppose that $C_G(x_1x_2) = G$, then $x_1x_2 \in Z(G)$. So in particular

$$\begin{aligned} x_1x_1x_2 &= x_1x_2x_1 \\ \Rightarrow \quad x_1x_2 &= x_2x_1, \end{aligned}$$

which gives contradiction to our assumption. Also if $|G : C_G(x_1x_2)| = 4$ then $C_G(x_1x_2) = C_G(x_1) \cap C_G(x_2)$, which is also a contradiction. Lastly, if $|G : C_G(x_1x_2)| = 2$ then we can see that two of x_i's commute, which is a contradiction.

Case ii. $|G : X_2| = 3$. Since $3 \mid |\frac{G}{Z(G)}|$ and $|\frac{G}{Z(G)}| \leq 16$, therefore $|\frac{G}{Z(G)}| = 12$. But A_4 is the only 6-centralizer group of order 12, so $\frac{G}{Z(G)} \cong A_4$.

Case iii. $|G : X_2| = 4$. In this case by Theorem 1 in [15], we have $|G : X_2| = |G : X_3| = |G : X_4| = |G : X_5| = 4$ and hence $|G : X_1| = 2, 3$ or 4.

If $|G : X_1| = 4$ then by Lemma 4.1.1, $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \times C_2$. If $|G : X_1| = 3$ then $12 \mid |\frac{G}{Z(G)}|$ and so $\frac{G}{Z(G)} \cong A_4$. If $|G : X_1| = 2$ then $8 \mid |\frac{G}{Z(G)}|$ and so $|\frac{G}{Z(G)}| = 8, 16$. Suppose $|G : Z(G)| = 8$ then $\frac{G}{Z(G)} \cong D_8, C_2 \times C_4$ or $C_2 \times C_2 \times C_2$. But by Remark 1.7.2, $C_2 \times C_4$ is not capable, so $\frac{G}{Z(G)} \ncong C_2 \times C_4$. Next suppose $|G : Z(G)| = 16$. Here let X_2 be non-normal, then $\frac{G}{Z(G)}$ is isomorphic to a subgroup of S_4, therefore $16 \mid 24$, a contradiction. Thus $\frac{G}{Z(G)}$ is abelian. So, $\frac{G}{Z(G)} \cong C_2 \times C_8, C_2 \times C_2 \times C_4, C_2 \times C_2 \times C_2 \times C_2$ or $C_4 \times C_4$. But using Corollary 8.20 in [24], we have $C_2 \times C_8$ and $C_2 \times C_2 \times C_4$ are not capable.

Suppose that $\frac{G}{Z(G)} \cong C_4 \times C_4$, then $\frac{G}{Z(G)}$ has exactly six cyclic subgroups of order 4, say $\frac{A_1}{Z(G)}, \frac{A_2}{Z(G)}, \ldots, \frac{A_6}{Z(G)}$. Choose $a_i \in A_i - Z(G), 1 \leq i \leq 6$. We can assume that $A_i = C(a_i), 1 \leq i \leq 6$ and A_i's are distinct centralizers of the group G. Therefore $|\operatorname{Cent}(G)| = 7$, which is impossible. This completes the proof. □

Corollary 4.1.3. *[4, 5] If $\frac{G}{Z(G)} \cong A_4$, then $|\operatorname{Cent}(G)| = 6$ or 8.*

Proof. We first assume that there exists an element $x \in G$ such that $|\frac{C(x)}{Z(G)}| = 4$. We will now show that there is no $y \in G$ such that $|\frac{C(y)}{Z(G)}| = 2$, that is $C(y) = Z \sqcup yZ$ where $o(yZ) = 2$ and $Z = Z(G)$. To see this assume that $|\frac{C(y)}{Z(G)}| = 2$, then since $\frac{G}{Z(G)} \cong A_4$ we have $\frac{C(y)}{Z(G)} \subseteq \frac{C(x)}{Z(G)}$ and so $C(y) \subseteq C(x)$. Now since $C(y)$ is an abelian subgroup of G, therefore $Z(C(x)) \subseteq Z(C(y)) = C(y)$. Hence $Z(C(x)) = Z(G)$, that is, $x \in Z(G)$, which is a contradiction. Therefore, if there exists an element $x \in G$ such that $|\frac{C(x)}{Z(G)}| = 4$, then for all $y \in G$, we have $|\frac{C(y)}{Z(G)}| = 3, 4, 12$. But, A_4 has exactly six subgroups with orders $3, 4, 12$. Thus $|\operatorname{Cent}(G)| = 6$.

Now we assume that there is no such element x, that is, for any x with the conditon $|C(xZ(G))| = 4$, then we have $|\frac{C(x)}{Z(G)}| = 2$. If $xZ(G), yZ(G)$ and $tZ(G)$ are three involutions in $\frac{G}{Z(G)}$, then $\frac{C(x)}{Z(G)} = \{Z(G), xZ(G)\}$, $\frac{C(y)}{Z(G)} = \{Z(G), yZ(G)\}$ and $\frac{C(t)}{Z(G)} = \{Z(G), tZ(G)\}$. Therefore, the all of $C(x), C(y)$ and $C(t)$ are distinct. Now if we consider the sylow 3-subgroups of A_4 and itself A_4, then by using Sylow's second and third theorem we have $|\operatorname{Cent}(G)| = 8$. This completes the proof. □

Corollary 4.1.4. *[7] If* $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$*, then* $|\operatorname{Cent}(G)| = 6$ *or* 8.

Proof. Clearly, $|\operatorname{Cent}(G)| \leq |G|Z(G)| = 8$. Suppose

$$\frac{G}{Z} = \{Z, x_1Z, x_2Z, x_3Z, x_4Z, x_5Z, x_6Z, x_7Z\}$$

where $Z = Z(G)$ and $|\operatorname{Cent}(G)| < 8$. Suppose $C(x_i) \neq C(x_j)$ for all i, j. Then $|\operatorname{Cent}(G)| = 8$, which is a contradiction to our assumption. Therefore, there are i and j such that $1 \leq i, j \leq 7, i \neq j$ and $C(x_i) = C(x_j) \neq G$.

Now we consider, $\frac{C(x_i)}{Z} = \{yZ | y \in C(x_i)\}$. Then possible orders of $\frac{C(x_i)}{Z}$ are $1, 2, 4, 8$. If $|\frac{C(x_i)}{Z}| = 1, 8$ then $x_i \in Z$, a contradiction. If $|\frac{C(x_i)}{Z}| = 2$ then $\frac{C(x_i)}{Z} = \{Z, x_iZ\}$. But $x_j, x_ix_j \in C(x_i)$, therefore $|\frac{C(x_i)}{Z}| \neq 2$ and so, $|\frac{C(x_i)}{Z}| = 4$ and $\frac{C(x_i)}{Z} = \{Z, x_iZ, x_jZ, x_ix_jZ\} = \frac{C(x_j)}{Z}$.

Next consider $\frac{C(x_ix_j)}{Z}$. Then possible orders of $\frac{C(x_ix_j)}{Z}$ are $1, 2, 4, 8$. If $|\frac{C(x_ix_j)}{Z}| = 1, 8$ then $x_ix_j \in Z$, which is not possible. If $|\frac{C(x_ix_j)}{Z}| = 2$ then

$x_i = x_j = x_i x_j$ as $x_i, x_j \in C(x_i x_j)$, a contradiction. Hence $|\frac{C(x_i x_j)}{Z}| = 4$ and $\frac{C(x_i x_j)}{Z} = \{Z, x_i Z, x_j Z, x_i x_j Z\}$. So

$$C(x_i) = C(x_j) = C(x_i x_j) = Z \sqcup x_i Z \sqcup x_j Z \sqcup x_i x_j Z.$$

This shows that $|\operatorname{Cent}(G)| \leq 6$. Again, since G is non-abelian, therefore $|\operatorname{Cent}(G)| \geq 4$. If $|\operatorname{Cent}(G)| = 4$ then $\frac{G}{Z(G)} \cong C_2 \times C_2$, a contradiction. If $|\operatorname{Cent}(G)| = 5$ then $\frac{G}{Z(G)} \cong C_3 \times C_3$ or S_3, which is a contradiction. Thus $|\operatorname{Cent}(G)| = 6$. □

Remark 4.1.5. [4, 5] If G is a 6-centralizer group such that $X_1 = G$ and $|G : X_2| = |G : X_3| = \ldots = |G : X_6| = 4$, then $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \times C_2$. The converse is not true.

Assume that G is a direct product of two copies of D_8 that is $G = D_8 \times D_8$. Then

$$\frac{G}{Z(G)} \cong \frac{D_8}{Z(D_8)} \times \frac{D_8}{Z(D_8)}$$
$$\Rightarrow \frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2 \times C_2.$$

and $|\operatorname{Cent}(G)| = |\operatorname{Cent}(D_8) \times \operatorname{Cent}(D_8)| = |\operatorname{Cent}(D_8)| \times |\operatorname{Cent}(D_8)| = 16$. Therefore the converse is not true.

4.2 7-centralizer groups

In this section we study the structure of 7-centralizer groups.

Lemma 4.2.1. *[1] Let G be a finite 7-centralizer group. Then $\frac{G}{Z(G)}$ is not a 2-group.*

Proof. Suppose on the contrary that $\frac{G}{Z(G)}$ is a 2-group.

Let $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size such that $|G : C_G(x_i)| = \alpha_i$ with $\alpha_1 \leq \alpha_2 \cdots \leq \alpha_r$. Then $\{C_G(x_i) | i = 1, 2, \ldots, r\}$ is an irredundant cover for G with intersection $Z(G)$. Also $r + 1 \leq |\operatorname{Cent}(G)|$ and $r \geq 3$. Then $3 \leq r \leq 6$. Again since for $r = 3, |\operatorname{Cent}(G)| = 4$ and for $r = 4, |\operatorname{Cent}(G)| = 5$, therefore $r \neq 3, 4$. Thus $r = 5$ or 6.

Suppose that $r = 5$. Then by Theorem 1.10.2, $\alpha_2 \leq 4$. If $\alpha_2 \leq 2$ then by Proposition 3.4.2 (a)(ii), $|\operatorname{Cent}(G)| = 4$, a contradiction. If $\alpha_2 = 3$ then by Proposition 3.4.2 (a)(ii), $|\operatorname{Cent}(G)| = 5$, a contradiction. Therefore $\alpha_2 = 4$. Now Proposition 3.4.2 (b) shows that there exists a proper centralizer $C_G(x)$ which is not abelian and contains at least three $C_G(x_i)$, say $i = 2, 3, 4$. Therefore

$$G = C_G(x_1) \cup C_G(x_5) \cup C_G(x).$$

So, by Lemma 1.10.1, $\alpha_5 \leq 2 \Rightarrow \alpha_5 = 2$, which is not possible.

Next suppose that $r = 6$. Then by Lemma 3.3.1, for every non-central elements x and y of $G, C_G(x) = C_G(y)$ or $C_G(x) \cap C_G(y) = Z(G)$. Hence, in such a group $G, \{C_G(x) | x \in G - Z(G)\}$ forms a partition with kernal $Z(G)$. It follows that $\{\frac{C_G(x)}{Z(G)} | x \in G - Z(G)\}$ forms a partition whose kernal is the trivial subgroup. We assume that $|\frac{C_G(x_i)}{Z(G)}| = n_i$ for $1 \leq i \leq 6$. Since

$\frac{G}{Z(G)} = \bigcup_{i=1}^{6} \frac{C_G(x_i)}{Z(G)}$, therefore

$$\begin{aligned} \left|\frac{G}{Z(G)}\right| &= \left|\frac{C(x_1)}{Z(G)}\right| + \cdots + \left|\frac{C(x_6)}{Z(G)}\right| - 5 \\ \Rightarrow \left|\frac{G}{Z(G)}\right| &= \sum_{i=1}^{6} n_i - 5 \\ \Rightarrow \sum_{i=1}^{6} n_i &= \left|\frac{G}{Z(G)}\right| + 5. \end{aligned}$$

As $\frac{G}{Z(G)}$ is a 2-group, so $\sum_{i=1}^{6} n_i$ is an odd integer, which is a contradiction. Hence $\frac{G}{Z(G)}$ is not a 2-group. □

PROBLEM 4.2.2. If $|\operatorname{Cent}(G)| = p$, a prime, then is it true that $\frac{G}{Z(G)}$ is not a 2-group ?

Now we will prove the main theorem of this section which characterizes 7-centralizer group.

Theorem 4.2.3. *[1] Let G be a finite group. Then G is 7-centralizer group if and only if $\frac{G}{Z(G)} \cong C_5 \times C_5, D_{10}, \langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle$.*

Proof. Suppose that G is a 7-centralizer group and $\{x_1, x_2, \ldots, x_r\}$ is a set of pairwise non-commuting elements of G having maximal size. Let $|G : C_G(x_i)| = \alpha_i$ such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$. Then the set $\{C_G(x_i) | i = 1, 2, \ldots, r\}$ is an irredundant r-cover for G with intersection $Z(G)$. So it follows from Proposition 3.4.1 that $r = 5$ or 6.

If $r = 5$ then by Theorem 1.10.2, $\alpha_2 \leq 4$. If $\alpha_2 \leq 3$ then by Proposition 3.4.2, $|\operatorname{Cent}(G)| = 4$ or 5, a contradiction. Therefore $\alpha_2 = 4$ and so by Lemma 1.10.1, $\alpha_i = 4$ for $i \geq 2$.

Now Proposition 3.4.2 gives that there exists a proper centralizer $C_G(x)$ which is not abelian and contains $\underset{i \in S}{\cup} C_G(x_i)$ for some $S \subset \{1, 2, \ldots, 5\}$ with $|S| = 3$. Thus $G = (\underset{i \in T}{\cup} C_G(x_i)) \cup C_G(x)$, where $T = \{1, \ldots, 5\} - S$. Therefore Lemma 1.10.1 gives that $\alpha_i = 2$ for some $2 \leq i \leq 5$, which is a contradiction. Therefore $r = 6$. Then by Lemma 3.3.1, all proper centralizers of G are abelian. Therefore $\{C_G(x_i) | i = 1, 2, \ldots, 6\}$ is a partition with kernal $Z(G)$ for G. So it follows from Theorem 1.10.2 that $\alpha_2 \leq 5$. Also Proposition 3.4.2 implies that $\alpha_2 \neq 2, 3$. If $\alpha_2 = 4$ then $4 \mid |\frac{G}{Z(G)}|$. Again

$$\begin{aligned} &|G| \geq |C_G(x_1).C_G(x_2)| \\ \Rightarrow &|G| \geq \frac{|C_G(x_1)||C_G(x_2)|}{|C_G(x_1) \cap C_G(x_2)|} \\ \Rightarrow &|G| \geq \frac{|G||G|}{16|Z(G)|} \\ \Rightarrow &|G : Z(G)| \leq 16. \end{aligned}$$

Now Lemma 4.2.1 implies that $|\frac{G}{Z(G)}| = 12$ and so $\frac{G}{Z(G)} \cong D_{12}$ or A_4, which is impossible. Therefore $\alpha_2 = 5$ and so by Lemma 1.10.1, we have $\alpha_i = 5$ for $i \geq 2$. Thus $|G| = \sum_{i=2}^{6} |C_G(x_i)|$ and so by Cohn's Theorem 1.10.3, $G = C_G(x_1).C_G(x_i)$ for all $i \neq 1$. If $\alpha_1 = 2$ then

$$\begin{aligned} &|G| = |C_G(x_1).C_G(x_2)| \\ \Rightarrow &|G| = \frac{|C_G(x_1)||C_G(x_2)|}{|C_G(x_1) \cap C_G(x_2)|} \\ \Rightarrow &|G| = \frac{|G||G|}{5 \times 2|Z(G)|} \\ \Rightarrow &|G : Z(G)| = 10, \end{aligned}$$

so $\frac{G}{Z(G)} \cong D_{10}$. If $\alpha_1 = 3$ then as above $|\frac{G}{Z(G)}| = 15$ and so it is cyclic.

Thus in this case $|\operatorname{Cent}(G)| = 1$, which is not possible. If $\alpha_1 = 4$ then $|\frac{G}{Z(G)}| = 20$. Therefore by Proposition 1.7.3, there are only two non-cyclic groups of order 20, namely D_{20} and $\langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$. Therefore $\frac{G}{Z(G)} \cong D_{20}$ or $\langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$. If $\frac{G}{Z(G)} \cong D_{20}$ then $|\operatorname{Cent}(G)| = 12$, a contradiction. Hence

$$\frac{G}{Z(G)} \cong \langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^2\rangle.$$

Next assume that $\alpha_1 = 5$ then $|\frac{G}{Z(G)}| = 25$ and so $\frac{G}{Z(G)} \cong C_5 \times C_5$.

Conversely, suppose $\frac{G}{Z(G)} \cong C_5 \times C_5, D_{10}, \langle x, y|x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$. If $\frac{G}{Z(G)} \cong C_5 \times C_5$ or D_{10} then by Proposition 2.2.3, $|\operatorname{Cent}(G)| = 7$. Next suppose that $\frac{G}{Z(G)} \cong \langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$, then $|gZ(G)| \in \{1, 2, 4, 5\}$ for each $g \in G$.

If $|gZ(G)| = 4$ or 5 then $C_G(g) = \langle g\rangle Z(G)$ and if $|gZ(G)| = 2$ then there exists $g_1 \in G$ such that $|g_1 Z(G)| = 4$ and $g = {g_1}^2 z$ for some $z \in Z(G)$. So $C_G(g) = C_G({g_1}^2) = C_G(g_1)$. On the other hand, G has five sylow 2-subgroups and one sylow 5-subgroup which are all cyclic. Thus $|\operatorname{Cent}(G)| = 7$. This completes the proof of the theorem. □

4.3 8-centralizer groups

In this section we study the structure of groups having 8 centralizers.

Lemma 4.3.1. *[7] Let G be a finite group and $\frac{G}{Z(G)} \cong R$, where $R = \langle a, b|a^6 = 1, a^3 = b^2, bab^{-1} = a^{-1}\rangle$. Then $|\operatorname{Cent}(G)| = 8$.*

Proof. Given $\frac{G}{Z(G)} \cong R = \langle a, b | a^6 = 1, a^3 = b^2, bab^{-1} = a^{-1}\rangle$. Therefore

$$\begin{aligned}\frac{G}{Z(G)} &= \langle aZ, bZ | a^6 Z = Z, a^3 Z = b^2 Z, (bZ)(aZ)(bZ)^{-1} = (aZ)^{-1}\rangle \\ &= \langle aZ, bZ | a^6 Z = Z, a^3 Z = b^2 Z, bab^{-1}Z = a^{-1}Z\rangle.\end{aligned}$$

Now we will show that $\langle aZ\rangle \subseteq \frac{C_G(a^i)}{Z}$ where $0 \le i \le 5$. It is clear that $\langle aZ\rangle \subseteq \frac{C_G(a^i)}{Z}$ for all i. Suppose for some $i, \langle aZ\rangle \subset \frac{C_G(a^i)}{Z}$, then

$$\begin{aligned}& a^j bZ \subseteq \frac{C_G(a^i)}{Z} \quad \text{for} \quad \text{some } j,\ j \in \{1, 2, \ldots, 5\} \\ \Rightarrow\ & a^j b \in C_G(a^i) \\ \Rightarrow\ & b \in C_G(a^i) \\ \Rightarrow\ & a^i \in Z,\end{aligned}$$

a contradiction. Thus for every $i, 0 \le i \le 5$, we have

$$\begin{aligned}\langle aZ\rangle &= \frac{C_G(a^i)}{Z} \\ \Rightarrow \frac{C_G(a^i)}{Z} &= \{Z, aZ, \ldots, a^5 Z\} \\ \Rightarrow C_G(a^i) &= Z \sqcup aZ \sqcup \cdots \sqcup a^5 Z.\end{aligned}$$

Next consider $C_{\frac{G}{Z(G)}}(a^i bZ), 0 \le i \le 5$. Let $a^k b^l Z \in C_{\frac{G}{Z(G)}}(a^i bZ)$, where $0 \le k \le 5, l = 1, 2$.

Case 1. $l = 1$. Then

$$\begin{aligned}& (a^k bZ)(a^i bZ) = (a^i bZ)(a^k bZ) \\ \Rightarrow\ & a^{2(k-i)} Z = Z \\ \Rightarrow\ & 6 \mid 2(k-i) \\ \Rightarrow\ & k - i = 3t, t \in \mathbb{Z}.\end{aligned}$$

If $t > 1$ then $k - i \geq 6$, a contradiction. If $t < -1$ then $i - k \geq 6$, a contradiction. Therefore $t = 0, 1, -1$. If $t = 0$ then $k = i$. If $t = 1$ then $k = i + 3$ and if $t = -1$ then $k = i - 3$. Again $a^{i+3}bZ = a^{i-3}bZ$. Therefore $a^{i+3}bZ \in C_{\frac{G}{Z(G)}}(a^i bZ)$.

Case 2. $l = 2$.

In this case,

$$\begin{aligned} & a^k Z \in C_{\frac{G}{Z(G)}}(a^i bZ) \\ \Rightarrow\ & (a^k Z)(a^i bZ) = (a^i bZ)(a^k Z) \\ \Rightarrow\ & a^{2k} Z = Z \\ \Rightarrow\ & 6 \mid 2k \\ \Rightarrow\ & k = 0, 3. \end{aligned}$$

Therefore $C_{\frac{G}{Z(G)}}(a^i bZ) = \{Z, a^3 Z, a^i bZ, a^{i+3} bZ\}$ where $0 \leq i \leq 5$. Also $\frac{C_G(a^i b)}{Z} \subseteq C_{\frac{G}{Z(G)}}(a^i bZ)$. Suppose $\frac{C_G(a^i b)}{Z} = C_{\frac{G}{Z(G)}}(a^i bZ)$ for some i, $0 \leq i \leq 5$. Then

$$\frac{C_G(a^{i+3}b)}{Z} \subseteq C_{\frac{G}{Z(G)}}(a^{i+3}bZ) = C_{\frac{G}{Z(G)}}(a^i bZ) = \frac{C_G(a^i b)}{Z}.$$

Therefore

$$\begin{aligned} & \frac{C_G(a^{i+3}b)}{Z} \subseteq \frac{C_G(a^i b)}{Z} \\ \Rightarrow\ & C_G(a^{i+3}b) \subseteq C_G(a^i b) \\ \Rightarrow\ & (a^{i+3}b)(a^i b) = (a^i b)(a^{i+3}b) \\ \Rightarrow\ & a^3 b = b a^3 \\ \Rightarrow\ & a^3 \in Z, \end{aligned}$$

a contradiction. Hence $\frac{C_G(a^ib)}{Z} \subsetneq C_{\frac{G}{Z(G)}}(a^ibZ)$. Therefore $|\frac{C_G(a^ib)}{Z}| = 1$ or 2. But, if $|\frac{C_G(a^ib)}{Z}| = 1$ then $a^ib \in Z$, a contradiction. Thus $|\frac{C_G(a^ib)}{Z}| = 2$ and so

$$\frac{C_G(a^ib)}{Z} = \{Z, a^ibZ\}$$
$$\Rightarrow C_G(a^ib) = Z \sqcup a^ibZ,$$

for every i, $0 \leq i \leq 5$. Hence

$$|\operatorname{Cent}(G)| = 8.$$

This completes the proof of the lemma. □

Lemma 4.3.2. *[1] Let G be a finite 8-centralizer group. Then $\frac{G}{Z(G)}$ is a $\{2,3\}$-group.*

Proof. Let $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then $\{C_G(x_i) | i = 1, 2, \ldots, r\}$ is an irredundant r-cover with intersection $Z(G)$. Therefore it follows from Proposition 3.4.1 that $r = 5, 6$ or 7.

Assume that $|G : C_G(x_i)| = \alpha_i$ where $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$ and consider a p-element x such that p is a prime number greater than 5. Then by Lemma 2.1 of [14], $x \in Z(G)$, a contradiction and so $\frac{G}{Z(G)}$ is $\{2,3,5\}$-group.

Now by Lemma 1.10.1, $\alpha_2 \leq 6$. If $r \leq 6$ then by Proposition 3.3.2,

$|\frac{G}{Z(G)}| \leq 36$. If $r = 7$ then by Lemma 3.3.1, $C_G(x_1) \cap C_G(x_2) = Z(G)$. So

$$\begin{aligned} |G| &\geq |C_G(x_1)C_G(x_2)| \\ \Rightarrow |G| &\geq \frac{|C_G(x_1)||C_G(x_2)|}{|C_G(x_1) \cap C_G(x_2)|} \\ \Rightarrow |G| &\geq \frac{|G|^2}{36|Z(G)|} \\ \Rightarrow \left|\frac{G}{Z(G)}\right| &\leq 36. \end{aligned}$$

If 5 is a divisor of $|\frac{G}{Z(G)}|$, then $|\frac{G}{Z(G)}| \in \{10, 15, 20, 25, 30, 35\}$. Since every group of order 15 or 35 is cyclic and $\frac{G}{Z(G)}$ is not cyclic, therefore

$$\left|\frac{G}{Z(G)}\right| \in \{10, 20, 25, 30\}.$$

If $|\frac{G}{Z(G)}| = 10$ or 25, then $\frac{G}{Z(G)} \cong D_{10}$ or $C_5 \times C_5$ and so by Theorem 4.2.3 $|\operatorname{Cent}(G)| = 7$, a contradiction.

If $|\frac{G}{Z(G)}| = 20$ or 30, then $\frac{G}{Z(G)} \cong \langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle$, D_{20} or D_{30}. So by Theorem 4.2.3 and Proposition 2.2.6, $|\operatorname{Cent}(G)| = 7, 12, 17$, a contradiction. Hence $\frac{G}{Z(G)}$ is not a 5-group. This completes the proof. □

Lemma 4.3.3. *[1] Let G be a finite 8-centralizer group. If $\frac{G}{Z(G)}$ is a 2-group, then $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$.*

Proof. Suppose that $X = \{x_1, x_2, \ldots, x_r\}$ is a set of pairwise non-commuting elements of G having maximal size. Assume that $|G : C_G(x_i)| = \alpha_i$ such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$. Then $\{C_G(x_i) | i = 1, 2, \ldots, r\}$ is an irredundant cover with intersection $Z(G)$. Therefore it follows from Proposition 3.4.1 that $r \neq 3$ or 4.

If $r = 5$ then $\alpha_2 \leq 4$. But by Proposition 3.4.2 and Theorem 1.10.2, $\alpha_i = 4$ for $i \geq 2$. Therefore by Proposition 3.4.2(c), there exists a proper centralizer $K = C_G(x)$ which is not abelian and contains at least two $C_G(x_i)$s, say for $i = 1, 2$. Thus $G = C_G(x_3) \cup C_G(x_4) \cup C_G(x_5) \cup C_G(x)$ which implies from Lemma 1.10.1 that $\alpha_i \leq 3$ for some $i \in \{3, 4, 5\}$, a contradiction.

Now suppose that $r = 6$ then by Proposition 3.4.2 and Lemma 4.3.2, $\alpha_2 = 4$. Also by Proposition 3.4.2(b), there exists a proper centralizer $C_G(x)$ which is not abelian and contains at least three $C_G(x_i)$s, say for $i = 1, 2, 3$. Thus $G = C_G(x_4) \cup C_G(x_5) \cup C_G(x_6) \cup C_G(x)$ which implies from Lemma 1.10.1 that $\alpha_i \leq 3$ for at least one $i \in \{4, 5, 6\}$. If $\alpha_1 = 2$ then $|\frac{G}{Z(G)}| \leq 8$. This gives $|\frac{C_G(x)}{Z(G)}| \leq 4$ and so $C_G(x)$ is abelian, a contradiction. Thus $\alpha_i \geq 4$ for every $i \in \{1, 2, \ldots, 6\}$, which is not possible. Therefore $r \neq 6$.

Finally suppose that $r = 7$. Then by Proposition 3.4.2, $C_G(x_i)$ is abelian for each $i \in \{1, 2, \ldots, 7\}$ and $\{\frac{C_G(x_i)}{Z(G)} | 1 \leq i \leq 7\}$ forms a partition whose kernel is the trivial subgroup. Also by Theorem 1.10.2 and Lemma 4.3.2, $\alpha_2 = 4$ and so $|\frac{G}{Z(G)}| \leq \alpha_1\alpha_2 \leq 16$, then $|\frac{G}{Z(G)}| = 8$ or 16.

If $|\frac{G}{Z(G)}| = 8$ then $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$ or D_8. But $\frac{G}{Z(G)} \ncong D_8$, so $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$.

If $|\frac{G}{Z(G)}| = 16$ then $\alpha_1 = 4$, otherwise $|\frac{G}{Z(G)}| \leq 8$ or 12, a contradiction. If $C_G(x_i)$ is not a normal subgroup of G for $i = 1$ or 2, then $Z(G) = \mathrm{Core}_G(C_G(x_i))$ and so, $|\frac{G}{Z(G)}|$ divides $4!$, which is not possible. Hence $C_G(x_i)$ is a normal subgroup of G for $i = 1, 2$. It follows that $\frac{G}{Z(G)}$ is abelian and so $\frac{G}{Z(G)} \cong C_4 \times C_4, C_2 \times C_2 \times C_2 \times C_2, C_2 \times C_8$ or $C_2 \times C_2 \times C_4$. But $C_2 \times C_2 \times C_4$ and $C_2 \times C_8$ are not capable. Again $C_4 \times C_4$ cannot have such a partition. For, if $|\frac{C_G(x_i)}{Z(G)}| = n_i$, then

$$\begin{aligned}\sum_{i=1}^{7}(n_i - 1) &= \left|\frac{G}{Z(G)}\right| - 1 \\ \Rightarrow \sum_{i=1}^{7} n_i &= 22 \\ \Rightarrow \sum_{i=3}^{7} n_i &= 14.\end{aligned}$$

It yields that $n_3 = n_4 = 4$ and $n_5 = n_6 = n_7 = 2$. This is impossible, since $C_4 \times C_4$ has only three elements of order 2. Finally, we will show that $\frac{G}{Z(G)} \not\cong C_2 \times C_2 \times C_2 \times C_2$.

Consider $\mathbf{P} = \{\frac{C_G(x_i)}{Z(G)} | i = 1, 2, \ldots, 7\}$ and $A_i = \frac{C_G(x_i)}{Z(G)}$ for $1 \leq i \leq 7$. Then $\mathbf{P}$ is a partition of $\frac{G}{Z(G)}$ with trivial kernal having exactly three members of size 2 and four members of size 4. So for each $x \in G$, let $\overline{x} = xZ(G)$, and let $A_1 = \{\overline{1}, \overline{a}, \overline{b}, \overline{ab}\}$ and $A_2 = \{\overline{1}, \overline{c}, \overline{d}, \overline{cd}\}$ for some $a, b, c, d \in G - Z(G)$.

Now using the fact that $\mathbf{P}$ is a partition of $\frac{G}{Z(G)}$ with trivial kernal, we can show that $\mathbf{P}$ is equal to one of the following:

(i) $\{A_1, A_2, \{\overline{1}, \overline{ac}\}, \{\overline{1}, \overline{bc}, \overline{ad}, \overline{abcd}\}, \{\overline{1}, \overline{bd}, \overline{abc}, \overline{acd}\}, \{\overline{1}, \overline{abd}\}, \{\overline{1}, \overline{bcd}\}\}$.

(ii) $\{A_1, A_2, \{\overline{1}, \overline{ac}\}, \{\overline{1}, \overline{bc}, \overline{abd}, \overline{acd}\}, \{\overline{1}, \overline{ad}, \overline{abc}, \overline{bcd}\}, \{\overline{1}, \overline{abcd}\}, \{\overline{1}, \overline{bc}\}\}$.

(iii) $\{A_1, A_2, \{\overline{1}, \overline{ac}, \overline{bd}, \overline{abcd}\}, \{\overline{1}, \overline{bc}\}, \{\overline{1}, \overline{ad}, \overline{abc}, \overline{bcd}\}, \{\overline{1}, \overline{abd}\}, \{\overline{1}, \overline{acd}\}\}$.

(iv) $\{A_1, A_2, \{\overline{1}, \overline{ac}, \overline{bd}, \overline{abcd}\}, \{\overline{1}, \overline{bc}, \overline{abd}, \overline{acd}\}, \{\overline{1}, \overline{ad}\}, \{\overline{1}, \overline{bcd}\}, \{\overline{1}, \overline{abc}\}\}$.

(v) $\{A_1, A_2, \{\overline{1}, \overline{ac}, \overline{abd}, \overline{bcd}\}, \{\overline{1}, \overline{bd}, \overline{abc}, \overline{acd}\}, \{\overline{1}, \overline{bc}\}, \{\overline{1}, \overline{ad}\}, \{\overline{1}, \overline{abcd}\}\}$.

(vi) $\{A_1, A_2, \{\overline{1}, \overline{ac}, \overline{abd}, \overline{bcd}\}, \{\overline{1}, \overline{bc}, \overline{ad}, \overline{abcd}\}, \{\overline{1}, \overline{bd}\}, \{\overline{1}, \overline{acd}\}, \{\overline{1}, \overline{abc}\}\}$.

Again, if $\overline{x}, \overline{y} \in A_i$ then $[x, y] = 1$, and if $\overline{1} \neq \overline{x} \in A_i, \overline{1} \neq \overline{y} \in A_j$ and $i \neq j$, then $[x, y] \neq 1$. Now we reach to a contradiction in each of the above cases. For example in the case (i), we have $1 = [bc, ad] = [b, d][c, a]$ and $1 = [bd, abc] = [b, c][d, a][d, b]$ which yield that $[abd, bcd] = 1$, a contradiction. Hence $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$. □

Lemma 4.3.4. *[1] Let G be a finite 8-centralizer group. Then $|\frac{G}{Z(G)}| \neq 24, 36$.*

Proof. Let $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then $\{C_G(x_i)|i = 1, 2, \ldots, r\}$ is an irredundant r-cover with intersection $Z(G)$. Assume that $|G : C_G(x_i)| = \alpha_i$ where $\alpha_1 \leq \cdots \leq \alpha_r$.

Suppose, for a contradiction, that $|\frac{G}{Z(G)}| = 36$. Then by Proposition 3.3.2 and Proposition 3.4.1, $r = 6$ or 7. If $r = 6$ then by Lemma 4.3.2 and Lemma 1.10.1, $\alpha_2 \leq 4$. Now Proposition 3.4.2 implies that $\alpha_2 \neq 2$ or 3 and so $\alpha_2 = 4$. If $\alpha_1 \leq 3$ then $|\frac{G}{Z(G)}| \leq 12$, which is not so. Thus $\alpha_1 = 4$ and so $\alpha_i \geq 4$ for each $i \geq 1$.

On the other hand by Proposition 3.4.2, there exists a proper centralizer $C_G(x)$ which is not abelian and contains atleast three subgroups $C_G(x_i)$ for $i \in \{1, 2, \ldots, 6\}$. Thus $G = \underset{i \in T}{\cup} C_G(x_i) \cup C_G(x)$ for some $T \subset \{1, 2, \ldots, 6\}$ such that $|T| = 3$. Therefore by Lemma 1.10.1, there exists at least one of $C_G(x_i)$s whose index is smaller than 4, which is a contradiction.

Therefore $r = 7$ and $\{C_G(x_i)|i = 1, 2, \ldots, 7\}$ forms a partition with kernal $Z(G)$. Since $|\frac{G}{Z(G)}| = 36$, therefore by Lemma 1.10.1 and Proposition

3.4.2, we have $\alpha_1 = \cdots = \alpha_7 = 6$. This gives that

$$\left|\frac{C_G(x_i)}{Z(G)}\right| = 6$$
$$\Rightarrow \frac{C_G(x_i)}{Z(G)} \cong C_6 \text{ or } S_3.$$

But $\frac{C_G(x_i)}{Z(G)}$ is abelian as $C_G(x_i)$ is abelian, so $\frac{C_G(x_i)}{Z(G)} \ncong S_3$. Thus $\frac{C_G(x_i)}{Z(G)}$ is a cyclic subgroup of order 6 for every $i \in \{1, 2, \ldots, 7\}$ and so $\frac{G}{Z(G)}$ has 14 elements of order 3 (as, there are exactly two elements of order 3 in each $\frac{C_G(x_i)}{Z(G)}, i \in \{1, 2, \ldots, 7\}$), this contradicts the fact that every group of order 36 has a unique sylow 3-subgroup.

Now suppose that $|\frac{G}{Z(G)}| = 24$. If $r = 6$ then by Proposition 3.4.2$(a)(ii)$ and Lemma 1.10.1 we have, $\alpha_2 = 4$. If $\alpha_1 \leq 3$ then by Proposition 3.4.2$(a)(ii)$, $|\frac{G}{Z(G)}| \neq 24$, which is not so. Thus $\alpha_i \geq 4$ for each $i \in \{1, 2, \ldots, 6\}$. On the other hand by Proposition 3.4.2(b), there exists a proper centralizer $C_G(x)$ which is not abelian and contains atleast three proper centralizers $C_G(x_i)$ for $i \in \{1, 2, \ldots, 6\}$. Therefore by Lemma 1.10.1, there exists at least one of $C_G(x_i)$s whose index is smaller than 4, which is a contradiction.

If $r = 7$ then by Theorem 1.10.2, $\alpha_2 \leq 6$. Now Proposition 3.4.2(a)(ii) implies that $\alpha_2 \neq 2$ or 3. If $\alpha_2 = 4$ then $|\frac{G}{Z(G)}| \leq \alpha_1\alpha_2 \leq 16$, which is not possible. Therefore by Lemma 4.3.2, $\alpha_i = 6$ for $2 \leq i \leq 7$. If $\alpha_1 = 6$ then by inclusion-exclusion principal, $|\frac{G}{Z(G)}| = 36$, a contradiction. Thus $\alpha_1 = 4$ and so $\frac{C_G(x_1)}{Z(G)}$ is the unique cyclic subgroup of order 6. Therefore $\overline{G} = \frac{G}{Z(G)}$ has the unique sylow 3-subgroup $\overline{U} = \langle \overline{w} \rangle$, where $\overline{w} = wZ(G)$.

Since $G = \bigcup_{i=1}^{7} \frac{C_G(x_1)}{Z(G)}, |\frac{C_G(x_i)}{Z(G)} \cap \frac{C_G(x_j)}{Z(G)}| = 1$ for distinct i and $j; \alpha_1 = 4$

and $\alpha_i = 6$ for $2 \le i \le 7, \overline{G}$ has 20 elements of order 2 or 4. It follows that the number of sylow 2-subgroups of $\overline{G}$ is three, namely $\overline{P}, \overline{P}^{\overline{w}}$ and $\overline{P}^{\overline{w}^2}$. Also $|\overline{P} \cup \overline{P}^{\overline{w}} \cup \overline{P}^{\overline{w}^2}| = 20$ so that $|\overline{P} \cap \overline{P}^{\overline{w}} \cap \overline{P}^{\overline{w}^2}| = 2$. Thus $|\mathrm{Core}_{\overline{G}}(\overline{P})| = 2$.

Next since $\overline{G} = \overline{U}\,\overline{P}$, we have $C_{\overline{G}}(\overline{U}) = \overline{U}\mathrm{Core}_{\overline{G}}(\overline{P})$, so that $|C_{\overline{G}}(\overline{U})| = 6$. But $\frac{\overline{G}}{C_{\overline{G}}(\overline{U})}$ embeds into $\mathrm{Aut}(\overline{U}) \cong C_2$ and so $|G| \le 12$, which is a contradiction. Hence $|\frac{G}{Z(G)}| \neq 24, 36$. □

Now using above lemmas we will prove the main theorem of this section.

Theorem 4.3.5. *[1] Let G be a finite 8-centralizer group. Then*

$$\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2, A_4 \text{ or } D_{12}.$$

Proof. Let $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then $\{C_G(x_i) | i = 1, 2, \ldots, r\}$ is an irredundant r-cover with intersection $Z(G)$. Assume that $|G : C_G(x_i)| = \alpha_i$ where $\alpha_1 \le \cdots \le \alpha_r$. Then by Lemma 3.4.1, $r = 5, 6$ or 7.

Now, suppose that $r = 5$. Then by Proposition 3.3.2, $|\frac{G}{Z(G)}| \le f(5) = 16$. If $\alpha_2 \le 3$ then $|\mathrm{Cent}(G)| = 4$ or 5, a contradiction. Therefore by Lemma 1.10.1, $\alpha_2 = \cdots = \alpha_5 = 4$. If $\alpha_1 = 2$ then by Proposition 3.4.2$(a)(ii)$, $|\frac{G}{Z(G)}| \le 8$. Now using Theorem 2.3.1 and Theorem 2.4.4, we have $|\frac{G}{Z(G)}| \neq 4$ or 6. Again Lemma 4.3.3 gives that $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$. If $\alpha_1 = 3$ then $|\frac{G}{Z(G)}| = 12$ and so $\frac{G}{Z(G)} \cong A_4$ or D_{12}. Also since $\alpha_i = 4$ for $i = 2, 3, 4, 5$, we have $\frac{G}{Z(G)} \cong A_4$.

Assume that $\alpha_1 = 4$, then Proposition 3.4.2 implies that there exists a proper centralizer $C_G(x)$ which is not abelian and contains atleast two of $C_G(x_i)$s for $1 \le i \le 5$. Thus $G = \underset{i \in T}{\cup} C_G(x_i) \cup C_G(x)$ for some

$T \subset \{1, 2, \ldots, 6\}$ such that $|T| \leq 3$. Therefore by Lemma 1.10.1, there exists at least one of $C_G(x_i)$s whose index is smaller than 3, which is not possible.

Suppose that $r = 6$. Then $\alpha_2 \leq 5$ and so by Lemma 4.3.2 and Proposition $3.4.2(a)(ii), \alpha_2 = 4$. If $\alpha_1 = 2$ then $|\frac{G}{Z(G)}| = 8$ from which it follows that $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$, a contradiction. If $\alpha_1 = 3$ then $|\frac{G}{Z(G)}| = 12$. Therefore $\frac{G}{Z(G)} \cong A_4$ or D_{12}, which is not possible. Therefore $\alpha_1 = 4$ and $\alpha_i \geq 4$ for $i \in \{1, 2, \ldots, 6\}$. On the other hand there exists a proper centralizer $C_G(x)$ which is not abelian and contains at least three subgroups $C_G(x_i)$ for $i \in \{1, 2, \ldots, 6\}$. Thus $G = \underset{i \in T}{\cup} C_G(x_i) \cup C_G(x)$ for some $T \subset \{1, 2, \ldots, 6\}$ such that $|T| \leq 3$. Therefore by Lemma 1.10.1, there exists at least one of $C_G(x_i)$s whose index is smaller than 3, which is a contradiction.

Finally suppose that $r = 7$. Then $\{C_G(x_i) | 1 \leq i \leq 7\}$ is a partition with kernel $Z(G)$ for G by Lemma 3.3.1. Also by Theorem 1.10.2 $\alpha_2 \leq 6$. Now by Proposition 3.4.2 (a)(ii), we have that $\alpha_2 \geq 4$ and so by Lemma 4.3.2, $\alpha_2 = 4$ or 6. Now assume that $\alpha_2 = 4$.

If $\alpha_1 = 2$, then $|\frac{G}{Z(G)}| \leq 8$ and so $|\frac{G}{Z(G)}| = 8$. Thus by Lemma 4.3.3, $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$ which is not possible, since $C_2 \times C_2 \times C_2$ does not have an irredundant 7-cover with $\alpha_1 = 2$.

If $\alpha_1 = 3$ then by Proposition 3.4.2(a)(ii), $|\frac{G}{Z(G)}| = 12$. So $\frac{G}{Z(G)} \cong A_4$ or D_{12}. This is impossible since every irredundant cover of A_4 with $\alpha_1 = 3$ has five members and D_{12} has an element of order 6 which implies that $\alpha_1 = 2$.

Thus $\alpha_1 = 4$ and so $|\frac{G}{Z(G)}| \leq \alpha_1\alpha_2 \leq 16$. It follows that $|\frac{G}{Z(G)}| = 8, 12$ or 16. Since $\alpha_1 = \alpha_2 = 4$, $\frac{G}{Z(G)} \ncong D_{12}$. Hence, by Lemma 4.3.3 and properties of capable groups, $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$ or A_4.

Now suppose that $\alpha_2 = 6$. Then Lemma 1.10.1 implies that $\alpha_i = 6$

for $2 \leq i \leq 7$. Since $\frac{G}{Z(G)} = \bigcup_{i=1}^{7} \frac{C_G(x_i)}{Z(G)}$ and $|\frac{C_G(x_i)}{Z(G)} \cap \frac{C_G(x_j)}{Z(G)}| = 1$ for $1 \leq i \neq j \leq 7$, we have $|\frac{G}{Z(G)}| = 6\alpha_1$.

If $\alpha_1 = 2$, then $|\frac{G}{Z(G)}| = 12$ and so $\frac{G}{Z(G)} \cong D_{12}$ or A_4. Now since A_4 does not contain a subgroup of index 2, we have $\frac{G}{Z(G)} \cong D_{12}$.

If $\alpha_1 = 3$, then $|\frac{G}{Z(G)}| = 18$ and $\overline{K} = \frac{C_G(x_1)}{Z(G)}$ is a unique subgroup of $\frac{G}{Z(G)}$ of order 6. Therefore there exists an element $\overline{y} = yZ(G) \in \overline{K}$ such that $o(\overline{y}) = 2$ and so $\langle \overline{y} \rangle$ is a subgroup of $Z(\frac{G}{Z(G)})$. Thus $Z(\frac{G}{Z(G)}) = \langle \overline{y} \rangle \overline{P}$ where $\overline{P}$ is a normal Sylow 3-subgroup of $Z(\frac{G}{Z(G)})$. Hence $\frac{G}{Z(G)}$ is abelian and we have $\frac{G}{Z(G)} \cong C_2 \times C_3 \times C_3$, which is not possible.

Finally if $\alpha_1 = 4$ or 6 then $|\frac{G}{Z(G)}| = 24$ or 36, which is not possible by Lemma 4.3.4. This completes the proof. □

Remark 4.3.6. [4] There exists a group G with exactly eight centralizer subgroups such that $\frac{G}{Z(G)} \cong A_4$.

Assume $H = \langle x \rangle$ is the cyclic group of order 3 and $N = Q_8$ is the quarternion group of order 8. Then $\mathrm{Aut}(N)$ has a unique conjugacy type of automorphism of order 3.

Consider the semidirect product $G = H \ltimes_\theta N$, where $\theta(x)$ is an automorphism of order 3 of the group Q_8. Then G is a group of order 24, $\frac{G}{Z(G)} \cong A_4$, and $|\,\mathrm{Cent}(G)| = 8$.

PROBLEM 4.3.7. To study the characterization of finite groups having 9 through 21 centralizers.

Chapter 5

Groups with $|\operatorname{Cent}(G)| = |\operatorname{Cent}(\frac{G}{Z(G)})|$, and with $22 \leq |\operatorname{Cent}(G)| \leq 73$

In this chapter, we study some properties of what is known as a primitive n-centralizer group, i.e., a group in which the number of centralizers equals the number of centralizers of its central quotient. We also study the structure of finite groups G having large values of $|\operatorname{Cent}(G)|$. Finally, we study the values of $|\operatorname{Cent}(G)|$ for all minimal simple groups G.

5.1 Primitive n-centralizer group

Recall that for a finite group G, $|\operatorname{Cent}(G)|$ denotes the number of centralizers in G. A group G is called *primitive n-centralizer* if

$$|\operatorname{Cent}(G)| = \left|\operatorname{Cent}\left(\frac{G}{Z(G)}\right)\right| = n.$$

Proposition 5.1.1. *[4] There exists a primitive n-centralizer group for all odd $n \neq 3$.*

Proof. We first assume that $n \geq 5$ is an odd integer.

Now consider the dihedral group D_{2m} of order $2m$. Then by Proposition 2.2.2,

$$|\operatorname{Cent}(D_{2m})| = \begin{cases} m+2, & \text{if } m \text{ is odd} \\ \frac{m}{2}+2, & \text{if } m \text{ is even.} \end{cases}$$

Since $n \geq 5$ is an odd integer, so $|\operatorname{Cent}(D_{2(n-2)})| = (n-2)+2 = n$. Again $Z(D_{2(n-2)}) = \{1\}$. Therefore

$$|\operatorname{Cent}(D_{2(n-2)})| = \left|\operatorname{Cent}\left(\frac{D_{2(n-2)}}{Z(D_{2(n-2)})}\right)\right| = n.$$

This proves the proposition. □

Corollary 5.1.2. *[1] Let G be a finite group and $\frac{G}{Z(G)} \cong D_{2n}$. Then G is primitive n-centralizer if and only if $n > 1$ is an odd integer.*

Proof. Given, $n > 1$ is an odd integer. Since there is no 3-centralizer group, so $n \geq 5$ is an odd integer that is, $(n-2) \geq 3$ is an odd integer. Therefore by Proposition 2.2.2,

$$\left|\operatorname{Cent}\left(\frac{G}{Z(G)}\right)\right| = |\operatorname{Cent}(D_{2(n-2)})| = n-2+2 = n.$$

Again by Proposition 2.2.6, we have $|\operatorname{Cent}(G)| = n-2+2 = n$. Hence $|\operatorname{Cent}(G)| = |\operatorname{Cent}(\frac{G}{Z(G)})| = n$. Therefore G is primitive n-centralizer group.

Conversely, suppose that G is primitive n-centralizer group. We know that $|\operatorname{Cent}(D_{2(n-2)})| = (n-2)+2 = n$, if $n-2$ is odd. Therefore $\frac{G}{Z(G)} \cong D_{2(n-2)}$, if n is odd. This completes the proof. □

In the following simple proposition, we obtain a class of primitive n-centralizer groups.

Proposition 5.1.3. *[4] Let G be an n-centralizer group and $|G' \cap Z(G)| = 1$. Then G is a primitive n-centralizer group.*

Proof. Suppose $|\operatorname{Cent}(G)| = n$ and $\operatorname{Cent}(G) = \{C_G(x_1), C_G(x_2), \ldots, C_G(x_n)\}$. Consider an element $xZ(G) \in \frac{G}{Z(G)}$.

Now, we will show that $C_{\frac{G}{Z(G)}}(xZ(G)) = \frac{C_G(x)}{Z(G)}$. Let $yZ(G)$ be an element of $C_{\frac{G}{Z(G)}}(xZ(G))$. Then

$$\begin{aligned}(yZ(G))(xZ(G)) &= (xZ(G))(yZ(G))\\ \Rightarrow y^{-1}x^{-1}yx &\in Z(G).\end{aligned}$$

Also $y^{-1}x^{-1}yx \in G'$. Therefore

$$\begin{aligned}y^{-1}x^{-1}yx \in G' \cap Z(G) &= \{1\}\\ \Rightarrow xy &= yx\\ \Rightarrow y &\in C_G(x).\end{aligned}$$

Therefore $yZ(G) \in \frac{C_G(x)}{Z(G)}$ and so $C_{\frac{G}{Z(G)}}(xZ(G)) \leq \frac{C_G(x)}{Z(G)}$. Again, clearly $\frac{C_G(x)}{Z(G)} \leq C_{\frac{G}{Z(G)}}(xZ(G))$. Hence $C_{\frac{G}{Z(G)}}(xZ(G)) = \frac{C_G(x)}{Z(G)}$.

Next, it is enough to show that, for any $1 \leq i,j \leq n, i \neq j$; $C_{\frac{G}{Z(G)}}(x_iZ(G)) \neq C_{\frac{G}{Z(G)}}(x_jZ(G))$. We assume that there exists some $1 \leq i,j \leq n,\ i \neq j$ such that

$$\begin{aligned} C_{\frac{G}{Z(G)}}(x_iZ(G)) &= C_{\frac{G}{Z(G)}}(x_jZ(G)) \\ \Rightarrow \frac{C_G(x_i)}{Z(G)} &= \frac{C_G(x_j)}{Z(G)}. \end{aligned}$$

Suppose $y \in C_G(x_i)$. Then

$$yZ(G) \in C_{\frac{G}{Z(G)}}(x_iZ(G)) = C_{\frac{G}{Z(G)}}(x_jZ(G)) = \frac{C_G(x_j)}{Z(G)},$$

Therefore $y \in C_G(x_j)$. Thus $C_G(x_i) \leq C_G(x_j)$. Similarly, we can show that $C_G(x_j) \leq C_G(x_i)$. Hence $C_G(x_i) = C_G(x_j)$, a contradiction. Hence $|\operatorname{Cent}(G)| = |\operatorname{Cent}(\frac{G}{Z(G)})| = n$. This completes the proof. □

Result 5.1.4. [28] Let G be a finite group and any sylow subgroup of G is abelian, then $|G' \cap Z(G)| = 1$.

Corollary 5.1.5. *[4] If any sylow subgroup of G is abelian and $n = |\operatorname{Cent}(G)|$, then G is a primitive n-centralizer group.*

Proof. Given any sylow subgroup of G is abelian, then by Result 5.1.5, $|G' \cap Z(G)| = 1$. So by Proposition 5.1.3, G is a primitive n-centralizer group. □

Theorem 5.1.6. *[7] A finite group G is primitive 7-centralizer if and only if $\frac{G}{Z(G)} \cong D_{10}$ or $\langle x,y | x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$.*

Proof. Using Theorem 4.2.3, we have that G is 7-centralizer group if and only if $\frac{G}{Z(G)} \cong C_5 \times C_5, D_{10}, \langle x,y | x^5 = y^4 = 1, y^{-1}xy = x^2\rangle$.

If $\frac{G}{Z(G)} \cong C_5 \times C_5$ then $|\operatorname{Cent}(\frac{G}{Z(G)})| = |\operatorname{Cent}(C_5 \times C_5)| = 1 \neq |\operatorname{Cent}(G)|$, which yields that G is primitive 7-centralizer if and only if $\frac{G}{Z(G)} \cong D_{10}$, $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle$. □

Remark 5.1.7. [10, 1]

(a) Suppose G is a 4-centralizer group. Then by Theorem 2.3.1, $\frac{G}{Z(G)} \cong C_2 \times C_2$ and so $|\operatorname{Cent}(\frac{G}{Z(G)})| = 1$. Hence, there is no primitive 4-centralizer group.

(b) Suppose that G is a finite 8-centralizer group. Then by Theorem 4.3.5, $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2, A_4$ or D_{12}. If $\frac{G}{Z(G)} \cong C_2 \times C_2 \times C_2$ then $|\operatorname{Cent}(\frac{G}{Z(G)})| = 1$. If $\frac{G}{Z(G)} \cong A_4$ then $|\operatorname{Cent}(\frac{G}{Z(G)})| = 6$ and if $\frac{G}{Z(G)} \cong D_{12}$ then by Proposition 2.2.2, $|\operatorname{Cent}(\frac{G}{Z(G)})| = 5$. Hence, there is no primitive 8-centralizer group.

The above discussion gives the following conjecture.

Conjecture 5.1.8. *[5] For all positive integer n, there is no primitive 2^n-centralizer group.*

PROBLEM 5.1.9. To determine the positive even integers for which there is a primitive n-centralizer group, equivalently, for which positive even integers there is no primitive n-centralizer group.

5.2 $22, 23$-centralizer groups

In this section we prove that, if G is a finite group such that $\frac{G}{Z(G)} \cong A_5$ then $|\operatorname{Cent}(G)| = 22$ or 32. Moreover, we prove that A_5 is the only finite simple group with 22-centralizers.

First we prove some lemmas which will be used in proving, if G is a finite group and $\frac{G}{Z(G)} \cong A_5$, then $|\operatorname{Cent}(G)| = 22$ or 32.

Lemma 5.2.1. *[6] Let x be an element of a finite group G such that $|C_{\frac{G}{Z(G)}}(xZ)| = pq$, where $Z = Z(G)$; p, q are primes not necessarily distinct and $\frac{C_G(x)}{Z} = C_{\frac{G}{Z(G)}}(xZ)$. If $\frac{C_G(g)}{Z} \leq \frac{C_G(x)}{Z}$ for some $g \in G$, then $C_G(g) = C_G(x)$.*

Proof. Suppose on the contrary that $C_G(g) \lneq C_G(x)$. Then $\frac{C_G(g)}{Z} < \frac{C_G(x)}{Z}$, otherwise $C_G(g) = C_G(x)$. Therefore $|\frac{C_G(g)}{Z}| = p$ or q, as $|\frac{C_G(x)}{Z}| = |C_{\frac{G}{Z(G)}}(xZ)|$.

Suppose $|\frac{C_G(g)}{Z}| = p$ then $\frac{C_G(g)}{Z}$ is cyclic. Therefore

$$\begin{aligned} \frac{C_G(g)}{Z} &= \langle tZ \rangle, \text{ where } t \in C_G(g) - Z \\ \Rightarrow \frac{C_G(g)}{Z} &= \{Z, tZ, \ldots, t^{p-1}Z\} \\ \Rightarrow \frac{C_G(g)}{Z} &= Z \sqcup tZ \sqcup \cdots \sqcup t^{p-1}Z \\ &\Rightarrow C_G(g) \text{ is abelian and } Z(C_G(g)) = C_G(g). \end{aligned}$$

Again,

$$\begin{aligned} & Z(C_G(x)) \subseteq Z(C_G(g)) \\ \Rightarrow \; & Z(C_G(x)) \subseteq C_G(g). \end{aligned}$$

Now for if $u \in Z(C_G(x))$, then $[u, v] = 1$ for all $v \in C_G(x)$. In particular when $v = g$, we have $u \in C_G(g)$. Also, for all $v \in C_G(g), [u, v] = 1$, so $u \in Z(C_G(g))$. Therefore $Z \subseteq Z(C_G(x)) \subseteq C_G(g)$ and $|C_G(g)|Z| = p$. Hence either $Z(C_G(x)) = C_G(g)$ or $Z(C_G(x)) = Z$. If $Z(C_G(x)) = C_G(g)$ then for all $u \in C_G(x)$ we have $[u, g] = 1$, and so $u \in C_G(g)$. Therefore

$C_G(x) = C_G(g)$, a contradiction. If $Z = Z(C_G(x))$ then $x \in Z$, which is a contradiction. This completes the proof. $\square$

Lemma 5.2.2. *[6] Let G be a finite group, $\frac{G}{Z(G)} \cong A_5$; and let $x \in G$ such that $|C_{\frac{G}{Z(G)}}(xZ)| = 4$, where $Z = Z(G)$. If $\frac{C_G(x)}{Z} = C_{\frac{G}{Z(G)}}(xZ)$, then for all $y \in G$ with $|C_{\frac{G}{Z(G)}}(yZ)| = 4$, we have $\frac{C_G(y)}{Z} = C_{\frac{G}{Z(G)}}(yZ)$.*

Proof. Given $\frac{G}{Z(G)} \cong A_5$ and $|C_{\frac{G}{Z(G)}}(xZ)| = 4$. Therefore, $C_{\frac{G}{Z(G)}}(xZ)$ is a sylow 2-subgroup of $\frac{G}{Z(G)}$, and for every $yZ \in C_{\frac{G}{Z(G)}}(xZ)$, we have

$$C_{\frac{G}{Z(G)}}(xZ) = C_{\frac{G}{Z(G)}}(yZ).$$

Therefore

$$\frac{C_G(y)}{Z} \leq C_{\frac{G}{Z(G)}}(yZ) = C_{\frac{G}{Z(G)}}(xZ)$$
$$\Rightarrow \frac{C_G(y)}{Z} \leq \frac{C_G(x)}{Z}.$$

So by Lemma 5.2.1, $C_G(x) = C_G(y)$. Hence

$$\frac{C_G(y)}{Z} = \frac{C_G(x)}{Z} = C_{\frac{G}{Z(G)}}(xZ) = C_{\frac{G}{Z(G)}}(yZ).$$

Now suppose $yZ \notin C_{\frac{G}{Z(G)}}(xZ)$. Then $C_{\frac{G}{Z(G)}}(yZ)$ is a sylow 2-subgroup of $\frac{G}{Z(G)}$ different from $C_{\frac{G}{Z(G)}}(xZ)$. Since A_5 acts transitively, by conjugation, on the set of its sylow 2-subgroups, there exists $u \in G$ such that $u^{-1}xuZ \in C_{\frac{G}{Z(G)}}(yZ)$. Therefore $C_{\frac{G}{Z(G)}}(yZ) = C_{\frac{G}{Z(G)}}(u^{-1}xuZ)$ and

$$\frac{C_G(y)}{Z} \leq C_{\frac{G}{Z(G)}}(yZ) = C_{\frac{G}{Z(G)}}(xZ)^u = \frac{C_G(x)^u}{Z} = \frac{C_G(x^u)}{Z}.$$

Now by Lemma 5.2.1, $C_G(y) = C_G(x^u)$, which gives

$$\frac{C_G(y)}{Z} = \frac{C_G(x^u)}{Z} = C_{\frac{G}{Z(G)}}(yZ).$$

This completes the proof. $\square$

Theorem 5.2.3. *[6] If G is a finite group and $\frac{G}{Z(G)} \cong A_5$, then $|\operatorname{Cent}(G)| = 22$ or 32.*

Proof. Let n_p denote the number of sylow p-subgroups of A_5. Then $n_2 = 5, n_3 = 10, n_5 = 6$. Let $C_{\frac{G}{Z(G)}}(x_iZ), i = 1, 2, \dots, 21$ be proper and distinct centralizer of $\frac{G}{Z(G)}$, where

$$|C_{\frac{G}{Z(G)}}(x_iZ)| = 4, i = 1, 2, \dots, 5$$

$$|C_{\frac{G}{Z(G)}}(x_iZ)| = 3, i = 6, 7, \dots, 15$$

$$|C_{\frac{G}{Z(G)}}(x_iZ)| = 5, i = 16, 17, \dots, 21.$$

Clearly, $\frac{C_G(x_i)}{Z} = C_{\frac{G}{Z(G)}}(x_iZ), i = 6, 7, \dots, 21$. Thus $\frac{C_G(x_i)}{Z}, i = 6, 7, \dots, 21$ are all of the sylow subgroups of $\frac{G}{Z(G)}$ of order $3, 5$. Now if $y \in C_G(x_i) - Z$ for some $6 \leq i \leq 21$, so by Lemma 2.4.3, $C_G(x_i) = C_G(y)$.

Again if $\frac{C_G(x_i)}{Z} = C_{\frac{G}{Z(G)}}(x_iZ)$ for some $1 \leq i \leq 5$, then by Lemma 5.2.2, $\frac{C_G(x_i)}{Z} = C_{\frac{G}{Z(G)}}(x_iZ)$ for all $1 \leq i \leq 5$. Therefore in this case, we have 5 distinct centralizers $C_G(x_i), i = 1, 2, \dots, 5$. Hence $|\operatorname{Cent}(G)| = 22$.

Now suppose $\frac{C_G(x_i)}{Z} < C_{\frac{G}{Z(G)}}(x_iZ)$ for all i for $1 \leq i \leq 5$. Let

$$C_{\frac{G}{Z(G)}}(x_iZ) = \{Z, x_iZ, y_iZ, t_iZ\}, i = 1, 2, \dots, 5.$$

Then $C_{\frac{G}{Z(G)}}(x_iZ) = C_{\frac{G}{Z(G)}}(y_iZ) = C_{\frac{G}{Z(G)}}(t_iZ), i = 1, 2, \dots, 5$. Therefore $\frac{C_G(x_i)}{Z}, \frac{C_G(y_i)}{Z}, \frac{C_G(t_i)}{Z} < C_{\frac{G}{Z(G)}}(x_iZ)$ where $i = 1, 2, \dots 5$ and so

$$\left|\frac{C_G(x_i)}{Z}\right| = \left|\frac{C_G(y_i)}{Z}\right| = \left|\frac{C_G(t_i)}{Z}\right| = 2.$$

That is, $C_G(x_i) = Z \sqcup x_iZ$, $C_G(y_i) = Z \sqcup y_iZ$, $C_G(t_i) = Z \sqcup t_iZ$ where $i = 1, 2, \dots 5$. Therefore $C_G(x_i), C_G(y_i), C_G(t_i), i = 1, 2, \dots, 5$ are all distinct and we have 15 distinct centralizers of G. Hence $|\operatorname{Cent}(G)| = 32$. □

Now we give an example which verifies the above theorem:

Example 5.2.4. [6] We assume that G is a group generated by two permutations

$(1, 20, 17, 5, 12)(2, 3, 9, 19, 10)(4, 14, 22, 11, 6)(7, 8, 15, 13, 16)$ and

$(2, 18)(5, 1)(6, 21)(7, 24)(9, 17)(10, 16)(12, 23)(13, 20)(14, 19)(15, 22)$.

Then $|G| = 240$, $\frac{G}{Z(G} \cong A_5$ and $|\operatorname{Cent}(G)| = 32$.

The following theorem can be proved by using classification of finite simple groups which proves that A_5 is the only finite simple group with 22-centralizers.

Theorem 5.2.5. [6] If G is a finite simple group and $|\operatorname{Cent}(G)| = 22$,then $G \cong A_5$.

Combining Theorem 5.2.3 and Theorem 5.2.5, it can be proved that A_5 is the only finite simple group with 22 centralizers. We can also prove that if G is a finite group and $|\operatorname{Cent}(G)| \leq 21$, then G is not simple. Moreover all proper subgroups of G are not simple.

Though Ashrafi and Taeri [6] claimed that if G is a finite group and $|\operatorname{Cent}(G)| \leq 21$, then G is solvable, but it does not come from the above two theorems. As such we leave it as an open problem.

PROBLEM 5.2.6. If G is a finite group and $|\operatorname{Cent}(G)| \leq 21$, then is it true that G is solvable?

Remark 5.2.7. [6] It may seem that if G and H are finite simple groups and $|G| \leq |H|$, then $|\operatorname{Cent}(G)| \leq |\operatorname{Cent}(H)|$. But this is not true. For example, $|\operatorname{Cent}(PSL(2, 7))| = 79 > 74 = |\operatorname{Cent}(PSL(2, 8))|$.

In this direction Ashrafi and Taeri [6] posed the following question:

Let G and H be finite simple groups. Is it true that if $|\operatorname{Cent}(G)| = |\operatorname{Cent}(H)|$, *then G is isomorphic to H ?*

However, the above question has a negative answer as we shall see in the next chapter.

5.3 Distinct centralizers in minimal simple group

Let G be a minimal simple group, i.e. finite non-abelian group all of whose proper subgroups are solvable. In this section we compute the value of $|\operatorname{Cent}(G)|$.

Theorem 5.3.1. [33] Let $G = \operatorname{PSL}(2, q)$, where q is a p-power (p-prime). Then

(i) If $q \in \{2, 3, 5\}$ or $q \equiv 0$ $(mod\ 4)$. Then

$$|\operatorname{Cent}(G)| = \begin{cases} q^2 + q + 2, & \text{if } q > 5, \\ 22, & \text{if } q = 4 \text{ or } 5, \\ 6, & \text{if } q = 3, \\ 5, & \text{if } q = 2. \end{cases}$$

(ii) If $q > 5$ and $q \equiv 1$ $(mod\ 4)$. Then

$$|\operatorname{Cent}(G)| = \frac{3q^2 + 3q + 4}{2}.$$

(iii) If $q > 5$ and $q \equiv 3$ $(mod\ 4)$. Then

$$|\operatorname{Cent}(G)| = \frac{3q^2+q+4}{2}.$$

Proof. (i) We have that

$$\mathrm{PSL}(2,2) \cong S_3, \mathrm{PSL}(2,3) \cong A_4,\ \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong A_5$$

and $|\operatorname{Cent}(S_3)| = 5, |\operatorname{Cent}(A_4)| = 6, |\operatorname{Cent}(A_5)| = 22$, therefore

$$|\operatorname{Cent}(G)| = \begin{cases} 22, & \text{if } \ q = 4 \text{ or } 5, \\ 6, & \text{if } \ q = 3, \\ 5, & \text{if } \ q = 2. \end{cases}$$

If $q > 5$ then G is an AC-group (i.e. the centralizer of every non-central element is abelian). Now this part follows from Proposition 3.4.3 and Lemma 4.4 of [3].

(ii) From Proposition 3.21 of [3], the group G contains subgroups A, B and P such that the set

$$\mathcal{P} = \{A^x - \{1\}, B^x - \{1\}, P^x - \{1\} | x \in G\}$$

is a partition for the set $G - \{1\}$ and the number of conjugates of A, B and P in G are respectively $\alpha = \frac{(q+1)(q-1)q}{2(q-1)}, \beta = \frac{(q+1)(q-1)q}{2(q+1)}$ and $\gamma = q+1$ and also for any $a \in G - \{1\}$ we have

$$C_G(a) = \begin{cases} N_G(\langle a \rangle), & \text{if } \ a^2 = 1 \text{ and } a \in A^x \text{ for some } x \in G, \\ A^x, & \text{if } \ a^2 \neq 1 \text{ and } a \in A^x \text{ for some } x \in G, \\ B^x, & \text{if } \ a \in B^x \text{ for some } x \in G, \\ P^x, & \text{if } \ a \in P^x \text{ for some } x \in G. \end{cases}$$

As $\mathcal{P}$ is a partition, the element a lies in the conjugate of one the subgroups P^x, A^x or B^x. Now if $a \in P^x$ or B^x, for some $x \in G$. Then $C_G(a) = P^x$ or B^x (indeed $|C_G(a)| = q, \frac{q+1}{2}$ respectively). That is, corresponding to each element in $\mathcal{P} - \{A^x - \{1\} | x \in G\}$ we have exactly one element centralizer of G.

Assume that a lies in the subgroup A^x for some $x \in G$. Again since $q > 5$ and A is cyclic, there exists an element $a \in A^x$ with $a^2 \neq 1$ and also since $q \equiv 1$ $(mod\ 4)$ and A is cyclic, there exists exactly one element of order 2. If $a^2 \neq 1$, then $C_G(a) = A^x$ so $|C_G(a)| = \frac{q-1}{2}$. If $a^2 = 1$ then $C_G(a) = N_G(\langle a \rangle)$, so by part (2) of Proposition 3.21 of [3], $|C_G(a)| = q - 1$. It follows that corresponding to each $A^x - \{1\}$ of $\mathcal{P}$ we have exactly two distinct centralizers. Therefore,

$$\begin{aligned} |\operatorname{Cent}(G)| &= 2\alpha + \beta + \gamma + 1 \\ &= \frac{3q^2 + 3q + 4}{2}. \end{aligned}$$

(iii) Using part (6) of Proposition 3.21 of [3], and a similar argument as part (ii) mentioned for A^x, we can see that corresponding to each $B^X - \{1\}$ of $\mathcal{P}$ we have exactly one centralizer. It follows that

$$\begin{aligned} |\operatorname{Cent}(G)| &= (q+1) + \frac{(q+1)(q-1)\frac{q}{2}}{\frac{2(q-1)}{2}} + 2 \times \frac{(q+1)(q-1)\frac{q}{2}}{\frac{2(q+1)}{2}} + 1 \\ &= \frac{3q^2 + q + 4}{2}. \end{aligned}$$

This completes the proof. □

Theorem 5.3.2. [33] Let $G = Sz(q)(q = 2^{2m+1}, m > 0)$. Then

$$|\operatorname{Cent}(G)| = q^3 - q^2 + q + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}$$

where $r = \sqrt{\frac{1}{2}}$.

Proof. Given, G be a Suzuki group. Then by [[21], pp. $192-193$, Theorems 3.10 and 3.11], G contains subgroups F, A, B and C such that $|F| = q^2$, $|A| = q-1, |B| = q-2r+1$ and $|C| = q+2r+1$ and the set

$$\mathcal{P} = \{A^x, B^x, C^x, F^x | x \in G\}$$

is a partition for G and A, B, C are cyclic and F is a sylow 2-subgroup of G and also $C_G(b) \leq M$ for all $b \in M - \{1\}$ for every $M \in \mathcal{P}$ (**).

Now [[21], Chapter XI, Theorems 3.10 and 3.11], implies that the number of conjugates of C, B, A and F in G are respectively,

$$k = \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)}, l = \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}, m = \frac{q^2(q^2+1)}{2}, n = q^2+1.$$

From [[21], Chapter XI, proof of Lemma 5.9], we have $|C_F(g)|Z(F)| = 2$, for all $g \in F - Z(F)$. If $C_F(g) = H$, then $|\frac{F}{Z(F)}| = 2$ which implies that F is abelian, namely F is an AC-group. Assume that

$$\operatorname{Cent}(F) = \{C_F(x_1), C_F(x_2), \ldots, C_F(x_n)\} \cup \{F\}.$$

Since F is an AC-group, we have that for every $1 \leq i < j \leq n, x_i x_j \neq x_j x_i$. It follows that $F = C_F(x_1) \cup \cdots \cup C_F(x_n)$ and the set $\{\frac{C_F(x_i)}{Z(F)} | i = 1, 2, \ldots, n\}$ forms a partition for the group $\frac{F}{Z(F)}$. Therefore $|\operatorname{Cent}(G)| = q$, which implies that corresponding to every element of G in $\{F^x - \{1\} | x \in G\}$ we have exactly $q-1$ element centralizer of G.

On the other hand as A, B and C are cyclic groups and also by $(**)$, we have that for every non-trivial $a \in A, B$ and C, $C_G(a) = A, B$ and C, respectively. It follows that corresponding to each element of $\mathcal{P} - \{F^x | x \in G\}$ we have just one element centralizer of G. Thus

$$\begin{aligned} |\operatorname{Cent}(G)| &= n(q-1)+k+l+m+1 \\ &= q^3 - q^2 + q + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}. \end{aligned}$$

This completes the proof. □

Note that by using GAP, we get $|\operatorname{Cent}(\mathrm{PSL}(3,3))| = 1237$. Therefore in view of the above theorems and also by [31], we have obtained $|\operatorname{Cent}(G)|$ for all minimal simple groups.

Finally, we give the negative answer to the question raised by Ashrafi and Taeri with the help of following proposition:

Proposition 5.3.3. [33] $|\operatorname{Cent}(\mathrm{PSL}(2,23))| = |\operatorname{Cent}(A_7)| = 807$, but they are not isomorphic.

Proof. It is easy to see by GAP system, that $|\operatorname{Cent}(A_7)| = 807$ and also by Theorem 5.3.1(*iii*), $|\operatorname{Cent}(\mathrm{PSL}(2,23))| = 807$. Clearly, $\mathrm{PSL}(2,23) \not\cong A_7$ □

Remark 5.3.4. [33] By an easy computation we can see that the answer to the question, which is raised by Ashrafi and Taeri, is affirmative for every two projective special linear groups of degree 2, namely,

if $|\operatorname{Cent}(\mathrm{PSL}(2,q_1))| = |\operatorname{Cent}(\mathrm{PSL}(2,q_2))|$, then $q_1 = q_2$.

Now we prove the following lemma which is useful in Theorem 5.3.9.

Lemma 5.3.5. [33] Let G_i be a finite group with $|\operatorname{Cent}(G_i)| = n_i$, $i \in \{1, 2, \ldots, m\}$. Then $|\operatorname{Cent}(G_1 \times G_2 \times \cdots \times G_m)| = \prod_{i=1}^{m} n_i$.

Proof. We put $K = G_1 \times G_2 \times \cdots \times G_m$. It is clear that

$$C_K(x_1, x_2, \ldots, x_m) = C_{G_1}(x_1) \times C_{G_2}(x_2) \times \cdots \times C_{G_m}(x_m)$$

for all $(x_1, x_2, \ldots, x_m) \in K$. It follows that for every $1 \leq i \leq m$ we have $C_K(x_1, x_2, \ldots, x_m) = C_K(y_1, y_2, \ldots, y_m)$ if and only if $C_{G_i}(x_i) = C_{G_i}(y_i)$. This implies that

$$|\operatorname{Cent}(G_1 \times G_2 \times \cdots \times G_m)| = \prod_{i=1}^{m} n_i.$$

Hence the lemma follows. □

We also need the following results in the proof of Theorem 5.3.9.

Theorem 5.3.6. [[20], Theorem 1] If G is a finite non-abelian simple group then G contains a subgroup which is a minimal simple group.

Lemma 5.3.7. [33] Let G be a finite group and $H \leq G$. Then

$$|\operatorname{Cent}(H)| \leq |\operatorname{Cent}(G)|.$$

Proof. Let $X_i = C_H(h_i), i = 1, 2, \ldots, m$ be distinct centralizers in H. Then $X_i = H \cap C_G(h_i)$, and so $C_G(h_i) \neq C_G(h_j)$, for all $i \neq j$. Hence the lemma follows. □

Let G be finite group, p a prime divisor of the order of G. We denote by $v_p(G)$, the *number of Sylow p-subgroups of G which pairwise intersect trivially.*

Theorem 5.3.8. [33] Let G be a finite non-abelian simple group such that $|\operatorname{Cent}(G)| \leq 73$. Then $G \cong A_5$.

Proof. Suppose on the contrary that there exists a non-abelian finite simple group G, not isomorphic to A_5 and of the least possible order such that $|\operatorname{Cent}(G)| \leq 73$. Now by Proposition 3 of [28], Theorem 5.3.6 and Lemma 5.3.7, it is enough to consider the following groups:

$\mathrm{PSL}(2, 2^p), p = 4$ or a prime;

$\mathrm{PSL}(2, 3^p), \mathrm{PSL}(2, 5^p), p$ a prime;

$\mathrm{PSL}(2, p), p$ a prime and $p \geq 7$;

$\mathrm{PSL}(3, 3)$;

$\mathrm{PSL}(3, 5)$;

$\mathrm{PSU}(3, 4)$(the projective special unitary group of degree 3 over the finite field of order 4^2) or

$Sz(2^p), p$ an odd prime.

If G is isomorphic to either $\mathrm{PSL}(2, 2^p), p = 4$ or a prime, or isomorphic to $\mathrm{PSL}(2, q)$, where $q = 3^p, 5^p$ or $p(p$ is a prime), then by Theorem 5.3.1, we can see that $|\operatorname{Cent}(G)| \geq 74$, a contradiction. Therefore we must consider the groups $\mathrm{PSL}(3, 3), \mathrm{PSL}(3, 5), \mathrm{PSU}(3, 4)$ and $Sz(2^p), p$ an odd prime.

If $G \cong \mathrm{PSL}(3, 3)$. then $|\operatorname{Cent}(\mathrm{PSL}(3, 3))| = 1237$, a contradiction.

If $G \cong \mathrm{PSL}(3, 5)$, then $|G| = 5^3 \times 2^5 \times \times 3 \times 31$ and so by GAP system, we can see $v_{31}(G) = 4000$ which is not possible by Lemma 4 of [9].

If $G \cong \mathrm{PSU}(3, 4)$ then by Theorem 10.12(d) in [20], it has order $2^6 \times 5^2 \times 13$. Therefore $v_{13}(L) = 1 + 13k > 21$ for some $k > 0$. Since $v_{13}(G)$

divides $|G|$, we have $v_{13}(G) > 79$, a contradiction.

If $G \cong Sz(2^p)$ then $|G| = 2^{2p} \times (2^p - 1) \times (2^{2p} + 1)$. Now by Theorem 5.3.2, we have $73 < |\operatorname{Cent}(G)|$. Thus in each case we obtain a contradiction. This completes the proof. □

If G is a finite group then we call the product of all minimal normal non-abelian subgroups of G, the centerless *CR-radical* of G. It is a direct product of non-abelian simple groups. Let G be a finite group, then $\operatorname{Sol}(G)$ denote the *solvable radical* (that is, the largest normal solvable subgroup of G).

Theorem 5.3.9. [33] If G is a finite semi-simple group and $|\operatorname{Cent}(G)| \leq 73$, then $G \cong A_5$ or S_5.

Proof. Assume that G is a semi-simple group and R is its centerless CR-radical. Then R is a direct product of a finite number, t, of finite non-abelian simple groups, say $R = S_1 \times S_2 \times \cdots \times S_t$.

By Lemma 6 in [9], we have $|\operatorname{Cent}(S_i)| \leq 73$. Also $|\operatorname{Cent}(R)| \leq 73$. Therefore by Theorem 5.3.8, $S_i \cong A_5$ for $i \in \{1, 2, \ldots, t\}$.

On the other hand from Lemma 5.3.5 and since $|\operatorname{Cent}(R)| \leq 73$, we have $t = 1$. Therefore $R \cong A_5$ and so $\operatorname{Aut}(R) \cong S_5$. But we know that $C_G(R) = 1$ and so G is embedded into $\operatorname{Aut}(R)$. Hence $G \cong A_5$ or S_5. □

Corollary 5.3.10. [33] Let G be a finite non-solvable group such that $|\operatorname{Cent}(\frac{G}{\operatorname{Sol}(G)})| \leq 73$. Then $\frac{G}{\operatorname{Sol}(G)} \cong A_5$ or S_5.

Proof. Since for any finite group H, $\frac{H}{\operatorname{Sol}(H)}$ has no non-trivial and proper normal abelian subgroup, the proof follows from Theorem 5.3.9. □

We conclude the chapter and also the dissertation with an observation that even though lots of work have been done as the number of centralizers of finite groups, lots of gaps still remain to be filled up. This provides enough opportunity for future research. For example, what can be said about the solvability and the nilpotency of a finite group G if the value of $|\operatorname{Cent}(G)|$ is known.

Bibliography

[1] A. Abdollahi, S. M. J. Amiri and A. M. Hassanabadi, *Groups with specific number of centralizers* , Houston Journal of Mathematics, **33** (1) (2007), 43–57.

[2] A. Abdollahi, M. J. Ataei, S. M. J. Amiri and A. M. Hassanabadi, *On groups with a maximal irredundant 6-cover* , Comm. Algebra, **33** (2005), 3225–3238.

[3] A. Abdollahi, S. Akbari, and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra, **298** (2006), 468–492.

[4] A. R. Ashrafi, *On finite groups with a given number of centralizers*, Algebra Colloquium, **7** (2) (2000), 139–146.

[5] A. R. Ashrafi, *Counting the Centralizers of some Finite Groups*, Korean J. Comput. & Appl. Math, **7** (1) (2000), 115–124.

[6] A. R. Ashrafi and B. Taeri, *On finite groups with a certain number of Centralizers* , J. Appl. Math. & Computing, **17** (1–2) (2005), 217–227.

[7] A. R. Ashrafi and B. Taeri, *On finite groups with exactly seven element centralizers* , J. Appl. Math. & Computing, **22** (1–2) (2006), 403–410.

[8] R. Baer, *Groups with preassigned central and central quotient group*, Trans. Amer. Math. Soc. , **44** (1938), 387–412.

[9] M. J. J. Barry and M. B. Ward, *Simple groups contain minimal simple groups*, Publ. Mat. , **41** (1997), 411–415.

[10] S. M. Belcastro and G. J. Sherman, *Counting Centralizers in Finite Groups* , Math. Magazine, **67** (5) (1994), 366–374.

[11] P. B. Bhattacharya, S. K. Jain, S. R. Nagpal *"Basic abstract algebra"*, second edition, Cambridge University Press, Cambridge, 1997.

[12] R. D. Blyth and D. J. S. Robinson, *Insoluble groups with the rewriting property* P_8, J. Pure Appl. Algebra, **72** (1991), 251–263.

[13] M. Bruckheimer, A. C. Bryan, and A. Muir, *Groups which are the union of three subgroups* , Amer. Math. Monthly, **77** (1970), 52–57.

[14] R. A. Bryce, V. Fedri and L. Serena, *Covering groups with subgroups* , Bull. Austral. Math. Soc. , **55** (1997), 469–476.

[15] J. H. E. Cohn , *On n-sum groups* , Math. Scand, **75** (1994), 44–58.

[16] P. Erdös and P. Turan, *On some problems of a statistical group-theory, IV* , Acta Math. Acad, Sci. Hung. **19** (1968), 413–415.

[17] J. A. Gallian, *Contemporary Abstract Algebra* , 2nd edition, D. C. Health and Company, Laxington, MA, 1990.

[18] D. Greco, *Sui gruppi che sono somma di quattro o cinque sottogruppi* , Rend. Accad. delle Scienze di Napoli (4) , **23** (1956), 49–56.

[19] I. N. Herstein, *Topics in Algebra*, Second Edition, John Wiley & Sons, 1999.

[20] B. Huppert, *Endliche Gruppen*, I, Springer - Verlag, Berlin, 1967.

[21] B. Huppert and N. Blackburn, *Finite groups*, III, Springer - Verlag, Berlin, 1982.

[22] N. Ito, *On the degrees of irreducible representations of a finite group* , Nagoya Math. J. , **3** (1951), 5–6.

[23] G. D. James and M. W. Liebeck, *Representations and characters of groups* , Cambridge University Press, 1993.

[24] G. Karpilovsky, *Group Representations*, Volume 2, North-Holland Mathematical Studies, Vol. 177, Amesterdam-New York-Oxford-Tokyo.

[25] J. Leavitt, G. J. Sherman, and M. J. Walker, *Rewriteability in finite groups* , Amer. Math. Monthly , **99** (1992), 446–452.

[26] G. A. Miller, *Groups containing the largest possible number of operators of order two* , Amer. Math. Monthly , **12** (1905), 149–151.

[27] B. H. Neumann, *Groups covered by permutable subsets* , J. London Math. Soc. , **29** (1954), 236–248.

[28] D. J. S. Robinson, *A Course in the theory of groups*, Springer-Verlag New York, 1980.

[29] J. J. Rotman, *An Introduction to the Theory of Groups*, 3rd edition, Allyn and Bacon, Inc, 1984.

[30] G. Scorza, *Gruppi che possono pensarsi come somma di tre sottogruppi*, Boll. Un. Mat. Ital , **5** (1926), 216–218.

[31] J. G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable (Part I)*, Bull. Amer. Math. Soc. (NS), **74** (1968), 383–437.

[32] M. J. Tomkinson, *Groups covered by finitely many cosets or subgroups* , Comm. Algebra, **15** (1987), 845–859.

[33] Mohammad Zarrin, *On element-centralizers in finite groups* , Arch. Math. , **93** (2009), 497–503.

[34] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.4.12; 2008, (`http://www.gap-system.org`).

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