GENERALISATIONS OF INJECTIVITY AND PROJECTIVITY

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CERTIFICATE

I certify that the dissertation entitled " Generalisations of injectivity and projectivity " submitted by Ms. Indrakshi Choudhuri in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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INTRODUCTION

Every vector space has a basis and is thus a free module. A free module P is projective, namely if M → N is an onto homomorphism every homomorphism P → N can be 'lifted' to a homomorphism P → M. The concept of an injective module dualises that of a projective module. Baer started the study of injective modules in a classical paper [B:40][†] and proved the theorem named after him. In another important paper [ES:53] Eckmann and Schopf proved the existence of a minimal injective extension of any module. Both these concepts turned out to be of great significance in commutative and non-commutative algebra and fields like algebraic geometry where this algebra is applied. These concepts yield to natural generalisations. The concept of a quasi-injective module (a module M such that for every submodule N of M a homomorphism f:N → M can be extended to an endomorphism of M) was introduced by Johnson and Wong [JW:61] and studied by Faith, Utumi, Ann Boyle, Goodearl, Birkenmeier and others. The dual concept of a quasi-projective module was studied by Wu and Jans [WJ:67] and Rangaswamy and Vanaja [RV:72]. Azumaya [A:n.d.] defined M-injective and M-projective modules. These concepts yield quasi-injectivity and quasi-projectivity as special cases. Fuller [F:69] considered Π-quasi-injective modules (modules M for which M^I is quasi-injective for each indexing set I) and

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[†] We follow the following pattern while referring to the items in the bibliography. A book will be cited as [St] or as [XII] or as [XYZ]; here [St] or [X or [XYZ] will denote the initial letter(s) of the name(s) of the author(s). A memoir may be cited, e.g., as [XYZ:83]; here '83 gives the year in which the paper was published. This will give an approximate idea of when the research was carried out.

the dual concept of Σ-quasi-projective module.

Chapter I of this dissertation (Relative injectivity and projectivity) will be devoted to a study of these generalisations of injectivity and projectivity.

In Chapter II we study hopfian and cohopfian modules. While some results of Chapter I are indeed used there, it differs in form as well as content from Chapter I. It is a modified version of a technical report outlining some research carried out by the Ring Theory group at N.E.H.U., Shillong. Most of the results of Chapter I belong to the realm of Module Theory in the sense that they hold over arbitrary rings. Chapter II is mostly Ring Theory.
A knowledge of the basic properties of semi-simple, regular, self-injective, p-injective and strongly regular rings is required there.

For the pre-requisites on Ring Theory we refer to standard textbooks mentioned in the Bibliography. We shall usually follow the terminology of Stenström [St.].

The rest of the Introduction will be devoted to fixing some frequently used or special notation.

As usual, the symbols N, Z, Q will denote the sets of natural numbers, integers and rationals. For a module M and a natural number n, $M^{(n)}$ will denote the direct sum of n copies of M. Let M be a left R-module. For a subset S of M, by the left annihilator of S we shall mean the subset $l_R(S) = \{ r \in R \mid rx = 0 \text{ for each } x \text{ in } S\}$ of R. For a subset T of R $r_M(T)$ will denote the subset $\{ x \in M \mid tx = 0 \text{ for each } t \text{ in } T\}$ of M.

In the interest of compactness we have sometimes used some natural abbreviations and it is hoped that this will not cause any confusion. For instance, the word "map" and the symbol f: $M \longrightarrow N$ are assumed to refer to module homomorphisms unless otherwise specified. The canonical map from a module M to a quotient module M/N is given by the rule $x \longmapsto \bar{x}$ (i.e., $\bar{x}$ stands for the coset x+N). A direct sum of a family of modules $\{A_i\}_{i\in I}$ is written $\oplus A_i$ when the indexing set is clear from the context.

CONTENTS

CHAPTER - I

RELATIVE INJECTIVITY AND PROJECTIVITY

INTRODUCTION

In this chapter we shall study the notions of relative injectivity and projectivity of modules. These concepts were originally introduced by F.L. Sandomierski in his Ph. D. Thesis [S : 64] submitted to Pennsylvania State University in 1964. They were further studied by de Robert [de R : 69] and Azumaya. Two memoirs [A : n.d] and [A : 70] are generally cited while referring to Azumaya's work. Unfortunately, neither Sandomierski's Thesis nor Azumaya's memoirs are easily available. Hence a student has to study their work through other sources like [AMV : 75] [V : 76] [GV : 80] [F : 72] [FR : 86] and [FR : 87] (Apart from the treatment in [AF , pp.184-191] very little of this material has been included in text books or monographs).

As defined in §1.2 below, a module E is called M-injective if for each submodule N of M every R - homomorphism $f:N \longrightarrow E$ extends to a R-homomorphism $g:M \longrightarrow E$. M-projective modules are defined dually. A module M is called quasi-injective if it is M-injective; quasi-projective modules are defined dually. These latter concepts, which were introduced by Johnson and Wong [JW : 61] and by Wu and Jans [WJ : 67] in fact preceded the concepts of M-injective and M-projective modules. These and related concepts will be studied in other sections of this chapter. Actually, in the classical paper [B : 40] where Baer introduced the notion of an injective module, he also introduced the notion of I-completeness of a left R-module (for a left ideal I of a ring R.) This concept can be easily seen to be a particular case of the concept of (N,M)- completeness of a module which will be introduced in § 1.1 below. Although these "completeness" concepts have not been used very often this seems to be the proper setting in which a number of generalisations of injectivity can be studied. (A reference for I-completeness is [FI : p.159].) Therefore we shall begin in § 1.1 with some remarks on (N,M)-completeness and the dual concept of (N,M)-Co-completeness.

§ 1.1. Completeness and Co-completeness

1.1.1.CONVENTION. In this section the letters N,M,E,P will denote left R-modules (over a ring R)

1.1.2.DEFINITION. Let $N \leqslant M$, E a module, We say that E is

<u>(N,M) - complete</u> if for each R-homomorphism $f:N \longrightarrow E$ there exists an R - homomorphism $g:M \longrightarrow E$ extending f.

1.1.3. CONVENTION. Let $I \leq R$ (i.e. I is a left ideal of R). If E is (I,R)-complete we simply say E is <u>I-complete</u>.

1.1.4. DEFINITION. Let $\underline{S}$ be a family of submodules of M. We say E is $\underline{S}$ - complete if E is (N,M)- complete for every $N \in \underline{S}$

1.1.5. PROPOSITION. Let N be a submodule of a module M.

(I) If N is direct summand of M then every E is (N,M)-complete.

(II)Conversely, if every E is (N,M)-complete then N is a direct summand of M.

<u>Proof</u> : (I) Let $M = N \oplus N'$ where $N' \leq M$.

Let $f : N \longrightarrow E$ be a R-homomorphism. Note that each element $m \in M$ can be written uniquely in the form $m=n+n'$ where $n \in N$ and $n' \in N'$. We define $g : M \longrightarrow E$ by $g(m) = f(n)$. It is easy to see that g is R-linear and extends f.

(II) Suppose every module E is(N, M)-complete. In particular N will be (N,M)-complete. Then choosing f to be the identity map of N we see that the inclusion map $i:N \longrightarrow M$ splits, showing that N is a direct summand of M.

1.1.6. DEFINITION. We say that a is a <u>regular</u> element of R if there exists an element b belonging to R such that aba=a. We say that a ring is <u>(von Neumann) regular</u> if every element of R is regular.

1.1.7. REMARK. If a is a regular element of R then there exists an element $b \in R$ such that aba = a. Let e = ba then Ra = $Raba \leq Rba = Re \leq Ra$ showing that Re = Ra. Hence Ra is a direct summand of R, i.e. every M is Ra-complete.

1.1.8. REMARK. Let N be a submodule of M and let $h:M \longrightarrow M/N$ be the canonical quotient map. The concept of (N,M)-completeness can be dualised as follows: We shall say that P is <u>(N,M)-co-complete</u> if for each R-homomorphism $f:P \longrightarrow M/N$ there exists a "lifting" $g:P \longrightarrow M$ (i.e. an R-homomorphism g such that hog = f).

Proposition 1.1.5. has a dual:

1.1.9. PROPOSITION. (I) If N is a direct summand of M every P is (N,M)- co-complete.

(II) Conversely, if every P is (N,M)- co-complete then N is a direct summand of M.

Proof: The proof is dual to that of Proposition 1.1.5. and hence it is omitted.

§ 1.2. Basic Properties

This section is devoted to the basic properties of M-injective and M-projective modules.

1.2.1. DEFINITION. An R-modules E is called M-injective ([AMV : 75], [A, n.d]) if for every submodule N of M every R-homomorphism $f : N \longrightarrow E$ extends to a R-homomorphism from M to E.

1.2.2. DEFINITION. An R-module P is called M-projective ([AMV: 75] [A : n.d.] if for every quotient module M" of M every R-homomorphism $f : P \longrightarrow M''$ can be lifted to an R-homomorphism $g : p \longrightarrow M$.

1.2.3. REMARK. In the terminology of § 1.1 E is M-injective if and only if E is $\underline{S}(M)$ - complete, where $\underline{S}(M)$ = all submodules of M.

1.2.4. REMARK. Clearly, an R-module E is injective if and only if E is M-injective for every R-module M.

1.2.5. REMARK. An R-module E is injective if and only if E is R-injective in the sense of Azumaya. This is the well-known Baer criterion for injectivity (See [B : 40] or [L, § 4.2 Lemma 1] or [AF, 18.3] or [R, Theorem 3.19]).

1.2.6. REMARK. An R-module P is projective if and only if P is M-projective for every R-module M.

1.2.7. REMARK. P is projective over R implies that P is R-projective but the converse is not true. For example if we choose the ring to be $\mathbb{Z}$ and the module P to be any non-zero abelian group satisfying $Hom_{\mathbb{Z}}(P,\mathbb{Z}) = 0$, then P is not projective over R although P is R-projective (in the sense of Azumaya).

Let $N \leq M$ and let $j : N \longrightarrow M$ the inclusion map and $h : M \longrightarrow M/N$ the canonical quotient map. Consider for a module U, the group homomorphisms (1) $j_* : Hom_R(M,U) \longrightarrow Hom_R(N,U)$ defined by $f \longmapsto f|_N$ (restriction of f to N).

(2) $\hat{h} : Hom_R(U,M) \longrightarrow Hom_R(U, M/N)$ defined by $f \longmapsto h \circ f$

(See [AF, Proposition 16.6] [R, Theorem 2.6] for the exactness properties of the functors involved here.) We shall relate the various concepts introduced above to these mappings in the following remarks.

1.2.8. REMARK (I) E is (N,M)-complete is equivalent to saying

$$j_* : \text{Hom}_R(M,E) \longrightarrow \text{Hom}_R(N,E) \text{ is onto.}$$

(II) P is (N,M)- co-complete is equivalent to saying

$$\hat{h}: \text{Hom}_R(P,M) \longrightarrow \text{Hom}_R(P, M/N) \text{ is onto.}$$

1.2.9. REMARK. U is M-injective (M-projective) if and only if Hom (-,U) (respectively Hom (U,-)) preserves the exactness of all short exact sequences with middle term M.

1.2.10. REMARK. U is injective (projective) if and only if Hom(-,U) (respectively Hom (U,-)) preserves exactness of all short exact sequences with middle term M.

1.2.11. DEFINITION. The <u>injectivity domain</u> of a left R-module U is the collection of all modules ${}_RM$ such that U is M-injective. It is denoted by $In^{-1}(U)$ or $C^i(U)$.

1.2.12. DEFINITION. The <u>projectivity domain</u> of a left R-module U is the collection of all modules ${}_RM$ such that U is

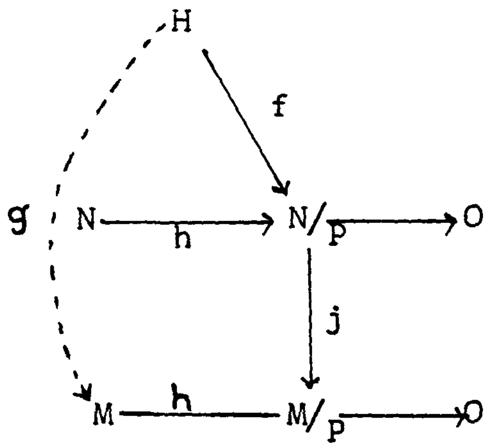
M-projective. It is denoted by P_r^{-1} (U) or C^p(U).

1.2.13. REMARK. Let V be any module and let M be a semi-simple module. Let N be a submodule of M. Then N is a direct summand of M and hence by 1.1.5.(II) U is (N,M)-complete. So by Remark 1.2.3. U is M-injective. Thus M $\in C^i(U)$ for every semi-simple module M. Similarly every semi-simple module belongs to $C^p(U)$. Also U is injective if and only if $C^i(U)$ coincides with R-mod, the class of all left R-modules and U is projective if and only if $C^p(U)$ = R-mod.

In propositions 1.2.14, 1.2.16, 1.2.19, 1.2.20 we shall show that $C^i(H)$ and $C^p(H)$ are closed under submodules and homomorphic images for each module H. Prepositions 1.2.15 and 1.2.17 follow from these preperties of $C^i(H)$ and $C^p(H)$. Two more results about $C^i(H)$ and $C^p(H)$ will be stated without proof (1.2.21 and 1.2.22).

1.2.14. PROPOSITION. <u>C^p (H) is closed under submodules.</u>

<u>Proof</u>. Let $P \leqslant N \leqslant M$ where $M \in C^p(H)$. Consider the following diagram. (We use the same letter h for the quotient maps $M \longrightarrow M/_p$ and $N \longrightarrow N/_p$).



We are given that H is M-injective for $N \leq M$

Let $f : H \longrightarrow N/_P$ be the given map. By the M-injectivity of H, there exists a R-linear map $g : H \longrightarrow M$ such that $hog = jof$, where $j : N/_P \longrightarrow M/_P$ is the inclusion map. Now for $x \in H$,

$$hog\ (x) = f(x) \in N/P$$

$\Rightarrow g(x) \in N$, so $g(x) \subset N$ and g is actually a map $H \longrightarrow N$ such that $hog = f$, hence the claim.

1.2.15. PROPOSITION. Suppose H is E-projective for every injective module E, then H is N-projective for every N i.e. H is projective.

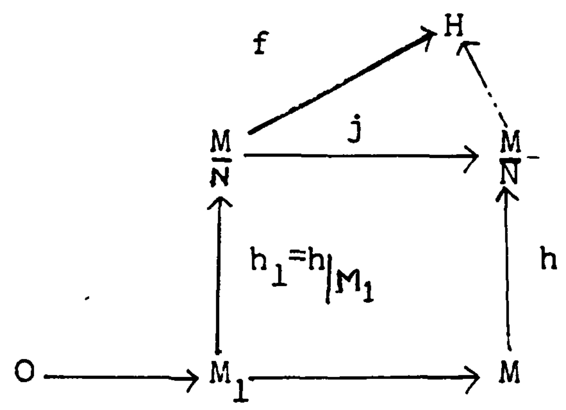
<u>Proof</u>. Let N be any module. It is well-known that there exists an injective module E_o such that $N \leq E_o$. By hypothesis, $E_o \in C^{p}(H)$. Hence Proposition 1.2.14 implies that $N \in C^{p}(H)$, i.e. H is N-projective.

1.2.16. PROPOSITION. $C^i(H)$ is closed under homomorphic images.

Proof. We are given that H is M-injective.

Claim: H is $M/_N$ injective, for every $N \leq M$

Note that any submodule of $M/_N$ can be written as M_1/N where $M_1 \leq M$. We consider the following diagram



Since H is M-injective there exists a R-linear map g : $M \longrightarrow H$ which extends $foh_1 : M_1 \longrightarrow H$.

Now for each element $n \in N$

$g(n) = f_o\ h_1(n) = f(0) = 0$

So g induces a R-homomorphism

$\bar{g} : M/_N \longrightarrow H$

Now for $\quad \bar{x} \in M_1/N$

$f(\bar{x}) = f\ (h_1(x)) = \bar{g}o\ ho\ i(x)$

$= \bar{g}\ h\ (x)$

$= \bar{g}\ (\bar{x})$

Showing that $\bar{g}$ extends 'f'. Hence H is M/N - injective.

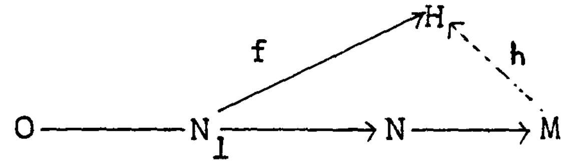
1.2.17. PROPOSITION. Suppose H is P-injective for every projective module P. Then, H is N-injective for every N i.e. H is injective.

<u>Proof</u>. Let N be any module. It is well-known that there exists a projective module P_o much that N is a homomorphic image of P_o. By hypothesis, $P_o \in C^i(H)$. Hence Proposition 1.2.16 implies that $N \in C^i(H)$, i.e. H is N-injective.

1.2.18. REMARK. Propositions 1.2.15 and 1.2.17 are reformulations of two results in Rotman's monograph ([R, Lemma 4.9] and [R, Remark after Lemma 4.9]) in the language of relative injectivity and projectivity. Rotman uses these Propositions to give a characterization of left hereditary rings, i.e. rings in which every left ideal is projective [R, Theorem 4.11].

1.2.19. PROPOSITION. <u>$C^i(H)$ is closed under submodules</u>

<u>Proof</u>. Let $N \leqslant M$ and we are given that H is M-injective. We have to prove that H is N-injective. Consider the following row exact diagram (where $N_1 \leqslant N$).



Since H is M-injective there exists a homomorphism $h : M \longrightarrow H$ which extends f. Now consider,

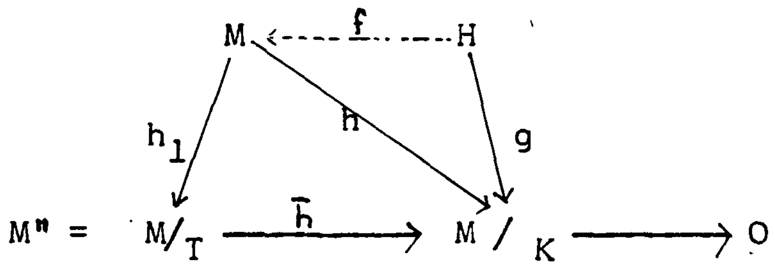
$$g = h|_N : N \longrightarrow H;$$ then this map g extends f.

Hence H is N-injective.

1.2.20. PROPOSITION. $C^p(H)$ is closed under homomorphic images.

Proof. Let H be M-projective. We shall prove that H is M" - projective where M" = M/T.

Any quotient of M" can be written as M/K where $K \geq T$. We consider the following diagram (where, as usual, the quotient maps $\bar{h}$, h_1, h satisfy the condition $h = \bar{h} \circ h_1$)



Let $g : H \longrightarrow M/K$ be given. By M-projectivity of H there exists $f : H \longrightarrow M$ such that hof = g i.e. $\bar{h}oh_1$ of = g. Thus $P = h_1of : H \longrightarrow M/T$ is a linear map satisfying

$$\bar{h}o\ p = g.$$

1.2.21. PROPOSITION. $C^i(H)$ is closed under direct sums.

Proof. See [AF, Proposition 16.13 (2)].

1.2.22. PROPOSITION. <u>$C^p(H)$ is closed under finite direct sums</u>.

<u>Proof</u>. See [AF, Proposition 16.12].

Let $\{N_i\}_{i \in I}$ be a family of modules. The following two results are "classical" :

1.2.23. PROPOSITION. The module $\prod_{i \in I} N_i$ is injective if and only if each N_i is injective.

<u>Proof</u>. See [R, Theorem 3.14] or [L, § 4.2 Proposition 2]

1.2.24. PROPOSITION. The module $\oplus N_i$ is projective if and only if each N_i is projective.

<u>Proof</u>. See [R, Theorem 3.12] or [L, § 4.1 Proposition 3]

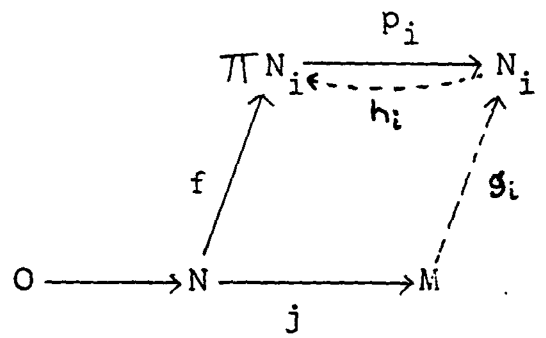
These results can be derived from analogous results for M-injective and M-projective modules. These are propositions 1.2.26, 1.2.27, 1.2.29, 1.2.30 below. Since the proofs are also similar to those of the classical results only two prooositions will be stated with proof.

1.2.25. NOTATION. We shall denote by C_i (M) the class of all M-injective modules.

1.2.26. PROPOSITION. $C_i(M)$ is closed under the formation of direct products.

<u>Proof</u>. Let $\{N_i\}_{i \in I}$ be a family of M-injective modules.

<u>Claim</u>. $\prod N_i$ is M-injective. We consider the following diagram with row exact



Let $i \in I$. Since N_i is M-injective, there exists $g_i : M \longrightarrow N_i$, which extends $p_i \circ f : N \longrightarrow N_i$. We define

$$g : M \longrightarrow \prod N_i \quad \text{by}$$
$$m \longmapsto (g_i(m))_{i \in I}$$

We claim that this map $g : M \longrightarrow \prod N_i$ extends $f : N \text{——} N_i$. (for if $n \in N$, we know that $p_i \circ f(n) = g_i \circ j(n) = p_i \circ g(n)$ for every $i \in I$; so $f(n) = g(n)$).

1.2.27. PROPOSITION. $C_i(M)$ is closed under direct factors.

1.2.28. NOTATION. We shall denote by $C_p(M)$ the class of all M-projective modules.

1.2.29 PROPOSITION. <u>$C_p(M)$ is closed under direct sums</u>.

<u>Proof</u>. Let $H = \bigoplus H_i$, where each H_i is M-projective. Let $N \leqslant M$ and let $k : M \longrightarrow M/N$ be the canonical surjection. It is sufficient to show that

$\hat{k}: Hom_R(H,M) \longrightarrow Hom_R(H,M/N)$ is onto. (See 1.2.8 (II))

So, let $f \in Hom\ (H,M/N)$ and let $f_i = f \mid H_i : H_i \longrightarrow M/N$ for each i . So $f\ (\sum hi) = \sum f(hi) = \sum f_i(hi)$, for $\sum h_i \in \oplus H_i$. Since H_i is M-projective, we get a lifting $g_i: H_i \longrightarrow M$ of the R-homomorphism $f_i : H_i \longrightarrow M/N$ i.e. an R-homomorphism $g_i : H_i \longrightarrow M$ satisfying $k_o\ g_i = f_i$. Now we consider

$$g: \oplus H_i \longrightarrow M$$
$$\sum h_i \longmapsto \sum g_i(h_i)$$

If $h_i \in H_i$, then

$$ko\ g\ (h_i) = ko\ g_i(h_i) = f_i\ (h_i)$$

$$ko\ g\ (\sum h_i) = \sum ko\ g\ (h_i) = \sum k \circ g_i\ (h_i)$$
$$= \sum f_i\ (h_i) = f\ (\sum h_i)$$
$$\therefore kog = f$$

So given $f \in Hom_R(H,\ M/N)$ there exists $g \in Hom\ (H,M)$ satisfying $kog = f$. Hence the map $\hat{k}$ is onto.

1.2.30. PROPOSITION. $C_p(M)$ is closed under direct summands.

§1.3 A Theorem on direct products

It is known that $C^p(M)$ is closed under finite direct sums (1.2.22). The following example shows that it is not closed under arbitrary direct sums, unlike $C^i(M)$ (1.2.21). (See 1.2.11 and 1.2.12 for the notations $C^i(M)$ and $C^p(M)$).

Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$, the additive group of rationals. Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0$, by Remarks 1.2.7, $\mathbb{Q}$ is $\mathbb{Z}$ - projective. Thus, $\mathbb{Z} \in C^p(\mathbb{Q})$. Now let J be an infinite set and for each $j \in J$ let $M_j = \mathbb{Z}$. Then $\mathbb{Q}$ is a quotient of $N = \bigoplus_{j \in J} M_j$. However, the identity map of $\mathbb{Q}$ cannot be lifted to a map of $\mathbb{Q}$ into N. Thus $N \notin C^p(\mathbb{Q})$.

Since $C^p(M)$ is closed under submodules (by 1.2.14) it follows from the above example that $C^p(M)$ cannot be closed under arbitrary direct products.

This section will be devoted to a theorem of Azumaya which states that if a module M has a projective cover, then $C^p(M)$ is closed under direct products. We shall follow [GV : 80] for a proof of this theorem. Since the proof is not given in any other accessible source, we shall give it in detail.

It is well-known [AF, Theorem 18:10], [L, §4.2] that every module has an injective hull (envelope). The concept of a projective cover dualises that of an injective hull.

1.3.1. DEFINITIONS. Let $K \leqslant M$. We shall say that K is <u>small</u> (or <u>superfluous</u>) in M if whenever $N \leqslant M$ and $K + N = M$, we have $N = M$. We shall write $K \ll M$ to denote K is small in M. An epimprphism $p : M \longrightarrow N$ is called <u>minimal</u> (<u>superfluous</u>) if Kerp is small in M.

1.3.2. DEFINITION. Let M be a module. By a <u>projective cover</u> [AF, p. 199] of M we mean a pair (P,p) satisfying the following conditions :

(I) P is a projective module.

(II) $P \xrightarrow{p} M$ is a minimal epimorphism.

1.3.3. REMARK (I) We also use natural variations and abbreviations of this terminology; for example, we may call P itself a projective cover of M; We may sometimes denote the projective cover of M by P(M) to emphasize its dependence on M.

(II) Projective covers need not always exist (unlike injective hulls.) Call a ring R a <u>perfect</u> if every R-module has projective cover. A ring R is called <u>semi-perfect</u> if idempotents modulo Rad R can be lifted and

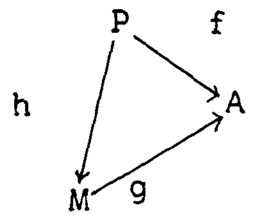
$R/_{\text{Rad } R}$ is a semi-simple ring (See [L, § 3.6] or [AF, § 27]). A ring R is semi-perfect if and only if every finitely generated left R-module has a projective cover ([L, §4.2 Exercises 13 and 15] or [AF, Theorem 27.6]). Thus if R is not semi-perfect (for example $\mathbb{Z}$) then there will exist finitely generated R-modules which do not have projective covers. This last fact can easily be seen directly (for $\mathbb{Z}$).

(III) If however a module has projective cover it is unique upto isomorphism ([L, §4.2 Exercise 10] or [AF, Lemma 17.17]).

(IV) Let $f : M \longrightarrow N$ and R - homomorphism. Let $K << M$. Then it is easy to see that $f(K) << N$. [L, §4.2 Exercise 8]

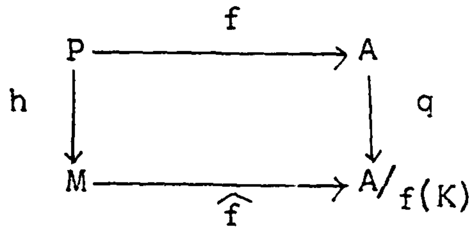
(V) Trivially, if $K' << K \leqslant M$ and K is small in M then clearly K' is small in M.

1.3.4. PROPOSITION. Let R be a ring, M,A,P left R-modules. Let M be A-projective and $h : P \longrightarrow M$ be any minimal epimorphism. Let K = Ker h. Then for any $f : P \longrightarrow A$ we have $f(K) = 0$ and hence there is a unique $g : M \longrightarrow A$ which makes the diagram

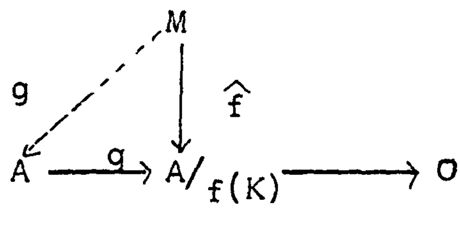


commute.

<u>Proof</u>. Let $q : A \longrightarrow A/_{f(K)}$ denote the canonical quotient map. Since $K << P$ it follows by [L, § 4.2, Exercise 8] that $f(K) << f(P)$. Also f induces a map $\hat{f}$: $M \longrightarrow A/_{f(K)}$ with



commutative. The exactness of $A \xrightarrow{q} A/_{f(K)} \longrightarrow 0$, together with the fact that M is A-projective, yields a map $g : M \longrightarrow A$ with



commutative. Let $l = f - goh : P \longrightarrow A$ we have $qol = qof - qogoh = foh - foh = 0$

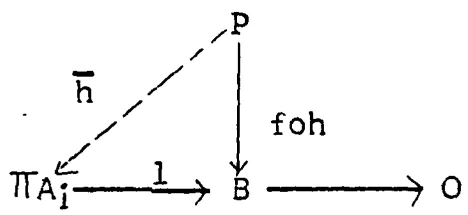
Thus $l(P) \subset f(K)$ Let $L = \ker l$. Since $l(P) \subset f(K)$, given any $u \in P$, we have $l(u) = f(k)$ for some $k \in K$. Thus $f(u)-goh(u) = f(k)$ or $f(u-k) = 0$. We get $f(u-k) = goh(u) - goh(k)$

i.e. $f(u-k) = goh(u-k)$ or $(f - goh)(u - k) = 0$ or $l(u - k) = 0$ thus $(u - k) \in Ker\ l = L$

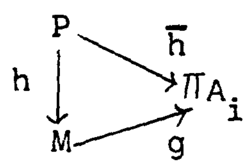
Hence $u = (u - k) + k \in L + K$ and hence $P = L + K$. From $K << P$ we get $L = P$ thus $l = 0$. This implies that $f = goh$; hence $f(K) = 0$ and $g : M \longrightarrow A$ is the map obtained from f by passing to the quotient by K.

1.3.5. PROPOSITION. Let $A_i \in C^P(M)$ for each $i \in I$. Suppose there exists a minimal epimorphism $P \xrightarrow{h} M$ with $\prod_{i \in I} A_i \in C^P(P)$. Then $\prod A_i \in C^P(M)$.

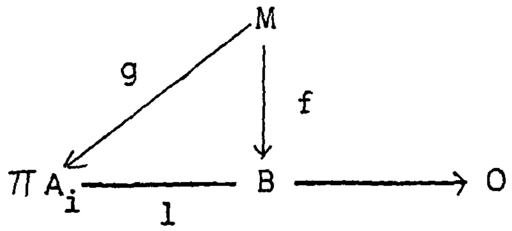
<u>Proof</u>. Let $A_i \xrightarrow{l} B \longrightarrow 0$ be exact and $f : M \longrightarrow B$ any map. Since P is $\prod A_i$ - projective, there exists a map $\bar{h} : P \longrightarrow \prod A_i$ with



commutative. For any $i_o \in I$, let $P_{i_o} : \prod A_i \longrightarrow A_{i_o}$ denote the projection. Since M is A-projective, from earlier proposition we see that $p_i o \bar{h}(K) = 0$ for all $i \in I$, where $K = Ker\ h$. Hence $\bar{h}(K) = 0$. It follows that $\bar{h}$ induces a map $g : M \longrightarrow \prod A_i$ with



commutative. Hence $f \circ h = l \circ \bar{h} = l \circ g \circ h$. Since $h : P \longrightarrow M$ is an epimorphism, we get $f = l \circ g$. Thus



is commutative. This proves that $\prod A_i \in C^P(M)$.

1.3.6. THEOREM (Azumaya) If M has a projective cover, then $C^P(M)$ is closed under direct products.

Proof. Let $P \xrightarrow{h} M$ be a minimal epimorphism with P projective. Let $A_i \in C^P(M)$ for each $i \in I$.

As P is projective, $\prod A_i \in C^P(P)$. Hence by proposition 1.3.5 $\prod A_i \in C^P(M)$.

In [A : 70] Azumaya asked whether the converse of the above Theorem 1.3.6 holds viz. whether $C^P(M)$ is closed under direct products implies that M has a projective cover. Fuller [F : 72] answered this question in the negative by giving an example (1.3.9 below).

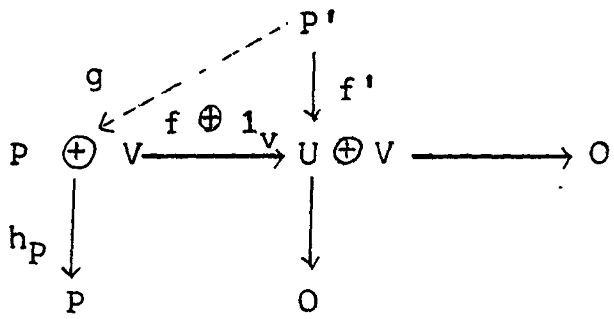
Using the theory of of additive classes due to C.L. Walker and E.A. Walker [WW : 72] Fuller proved the following result.

1.3.7. PROPOSITION. Let M be an R-module which contains direct summands isomorphic to each simple R-module then $C^P(M)$ is closed under direct products if and only if $R/_{Rad\ R}$ is a semi-simple ring.

Next he proved the following Lemma.

1.3.8. LEMMA. Let U and V be modules such that U and $U \oplus V$ have projective covers. Then V also has a projective cover.

Proof. Let $P \xrightarrow{f} V \longrightarrow 0$ and $P' \xrightarrow{f'} U \oplus V \longrightarrow 0$ be projective covers of V and $U \oplus V$ respectively. Consider the diagram



with exact row and column.

Now $Ker(f \oplus 1_V) = K \oplus 0$ where $K = Ker f$. and K small in P implies K small in $P \oplus V$. Again P' is projective hence there exists $g : P' \longrightarrow P \oplus V$ such that $(f \oplus 1_V) \circ g = f'$.

Next, we have $P \oplus V = Img + K$ [For if, $z \in P \oplus V$ then $(f \oplus 1_V)(z) = f'(p')$. But $f'(p') = (f \oplus 1_V)(g(p'))$

this implies that $z - g(p') \in K$]. So K small in $P \oplus V$ implies that $P \oplus V = \text{Im} g$. So g is onto.

Let $h_p : P \oplus V \longrightarrow P$ be the canonical projection. Then $0 \longrightarrow L \longrightarrow P' \xrightarrow{h_p o g} P \longrightarrow 0$ splits (since P is projective).

Hence $L = (h_p o g)^{-1}(0) = g^{-1}(V)$ is a direct summand of P'.

Therefore L is projective.

Notice also that $\text{Ker} g \leqslant L$.

We claim that Kerg is small in P'

(For $f'(\text{Ker} g) = (f \oplus 1_V)\, g[(\text{Ker} g)] = 0$

This implies that $\text{Ker} g \leqslant \text{Ker } f'$, and kerf' is small in P', hence Kerg is small in P').

So Ker g is small in L, a direct summand of P'.

Also

$$0 \longrightarrow \text{Ker} g \longrightarrow L = g^{-1}(V) \xrightarrow{g} V \longrightarrow 0 \text{ exact}$$

and L is projective. This shows that V has a projective cover.

1.3.9. EXAMPLE. Let K be a field, R = K [[t]], the ring of formal power series in one indeterminate t. It is known that R is a local ring with unique maximal ideal tR and thus tR is small in R. Hence the exact sequence

$$0 \longrightarrow tR \longrightarrow R \longrightarrow R/_{tR} \longrightarrow 0$$ is a projective cover of the unique simple R- module $S=R/tR$. Since R is not perfect there exists an R module Y which does not have a projective cover.

Now let $M = S \oplus Y$. Then M satisfies the hypothesis of Proposition 1.3.7 and $R/_{RadR} = R/tR$ is a field. Hence $C^P(M)$ is closed under direct products. However by Lemma 1.3.8 M cannot have a projective cover. This answer Azumaya's question in the negative.

§ 1.4 M-epimorphisms and M-monomorphisms

It is well known (and easy to see) that a module H is projective if and only if every epimorphisms $C \longrightarrow H$ splits. Dually, H is injective if and only if every monomorphism $H \longrightarrow C$ splits. In this section we shall show that these results have analogues for M-projective and M-injective modules. For this we have to replace epimorphism by "M-epimorphism" (see Definition 1.4.1 below) and monomorphism by "M-monomorphism" (see definition 1.4.2 below). These results have applications in the theory of M-projective and M-injective modules; see [FR : 86]. They are also of independent interest.

1.4.1. DEFINITION. Let $g : C \longrightarrow H$ be an epimorphism of right R-modules. The map g is called an M-epimorphism if there exists a homomorphism $f : C \longrightarrow M$ such that $\mathrm{Ker}\, g \cap \mathrm{Ker}\, f = (0)$.

This definition can be dualised as follows:

1.4.2. DEFINITION. Let $g : B \longrightarrow A$ be a monomorphism of right R-modules. The map g is called an M-monomorphism if there exists a homomorphism $f : M \longrightarrow A$ such that $\mathrm{Im}\, g + \mathrm{Im}\, f = A$.

First, we shall define push-outs and prove a proposition about them. This will be used to relate M-injectivity to M-monomorphisms in Theorem 1.4.6.

1.4.3. DEFINITION. Let $f : B \longrightarrow H$ and $g : B \longrightarrow A$ be maps. Consider the submodule $K = \{(-g(b),\ f(b)) \,/\, b \in B\}$ of $A \times H$. Let $Q(A,H) = (A \times H)/K$.

Let $v : H \longrightarrow Q(A,H)$ be defined by $v(h) = \overline{(o,h)}$

And let $u : A \longrightarrow Q(A,H)$ be defined by $u(a) = \overline{(a,o)}$

Then $(v \circ f)(b) = v(f(b)) = \overline{(o,f(b))} = \overline{(g(b),o)} = u \circ g(a)$

(Since $\overline{(o,f(b)) - (g(b),\ o)} = 0$).

Thus the diagram

$$\begin{array}{ccc} B & \xrightarrow{\ g\ } & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ H & \xrightarrow[\ v\ ]{} & Q\ (A,H) \end{array}$$

commutes. This diagram is called the push-out of $f : B \longrightarrow H$ and $g : B \longrightarrow A$.

In 1.4.4 and 1.4.5 below we follow the notation of 1.4.3

1.4.4. REMARK. Suppose g is a monomorphism then $0 = v(h) = \overline{(o,h)}$. This implies $(0,h) \in K$; so, $(0,h) = (-g(b),\ f(b))$ for some $b \in B$.

Hence $g(b) = 0$ yielding $b = 0$. Hence $h = f(b) = 0$

Thus v is also a monomorphism.

1.4.5. PROPOSITION. (Mac Lane) Assume g is a monomorphism. Then v splits if and only if there exists $l : A \longrightarrow H$ satisfying $log = f$.

Proof. ($\Leftarrow$) Suppose $l : A \longrightarrow H$ satisfies $log = f$. We define $r : A \times H \longrightarrow H$ by $r(a,h) = l(a) + h$. Let $k \in K$

then $k = (-g(b), f(b))$ for some $b \in B$.

Now, $r(k) = f(b) - log(b)) = f(b)-f(b) = 0$

Hence r induces a map $\bar{r} : Q(A,H) \longrightarrow H$ defined by

$\bar{r}(\overline{a,h}) = l(a) + h$.

Now $\bar{r}_O v(h) = \overline{r(o,h)} = l(0) + h = h$,

This shows that v is split by $\bar{r}$.

($\Rightarrow$) Let v be split by s. We define $r = sou : A \longrightarrow H$.

So that $rog = souog = sovof = 1_H of = f$. Hence if v splits then there exists $r: A \longrightarrow H$ satisfying $rog = f$.

1.4.6. THEOREM. Let H and M be R-modules.

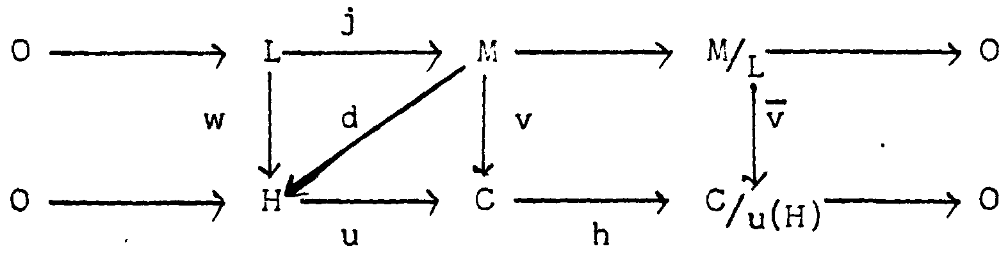
Then the following conditions are equivalent:

(i) H is M-injective.

(ii) Every M-monomorphism $u : H \longrightarrow C$ splits.

(iii) Given an M monomorphism $g: B \longrightarrow A$ and any homomorphism $f: B \longrightarrow H$ There exists $l: A \longrightarrow H$. Such that $log = f$.

<u>Proof</u>. (i) $\Rightarrow$(ii) Let $v : M \longrightarrow C$ be such that $\text{Im}u + \text{Im}v = C$. Consider the row exact diagram.



-(1)

Here $L = v^{-1}(\text{Im}(u))$, by definition.

So that we have $v(L) \leq \text{Im}(u)$.

So, there exists $w : L \longrightarrow H$ such that $u_o w = v|_L$ and $M \longrightarrow M/_L$ is the natural map. The map $v : M \longrightarrow C$ has the property that $hov(L) = h_o u_o w(L) = h(u(H)) = 0$.

Hence $\bar{v} : M/_L \longrightarrow C/_{u(H)}$ given by $\bar{m} \longmapsto h(v(m))$ is well-defined.

Clearly, both squares of figure (I) commute. Since H is M-injective there exists $d : M \longrightarrow H$ satisfying $doj = w$.

Let $\bar{c} \in C/_{u(H)}$

write $c = u(h) + v(m)$ for some $h \in H$, $m \in M$.

Hence $\bar{c} = \overline{v(m)} = \bar{v}(\bar{m})$ in $C/_{u(H)}$.

Thus $\bar{v}$ is an epimorphism.

Next $\bar{v}(\bar{m}) = 0 = v(m) \in u(H) = \text{Im } u$.

So $m \in v^{-1}(\text{Im } u) = L$ i.e. $\bar{m} = 0$ in M/L.

Thus $\bar{v}$ is a monomorphism. So $\bar{v}$ is an isomorphism.

Next, we define a map $M/_L \longrightarrow C$ as follows : Note that $(v-u_od)\ (l) = v(l)-u_ow(l) = 0$ by hypothesis on w. So, $(v-u_od)\ (L) = 0$ and induces a map $q : M/_L \longrightarrow C$ defined by $q(\bar{m}) = (v - uod)\ (m)$.

Notice that, $hoq(\bar{m}) = h\ (v - uod)\ (m)$

$$= h(v(m)) = \bar{v}(\bar{m}).$$

Hence, $h : C \longrightarrow C/_{u(H)}$ splits. This implies that $u : H \longrightarrow C$ also splits.

(ii)$\Rightarrow$(iii) Consider the push-out $Q(A, H)$ of $f : B \longrightarrow H$ and $g : B \longrightarrow A$ and the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \xrightarrow{\ u\ } & Q(A,H) & \longrightarrow & Q(A,H)/_{v(H)} & \longrightarrow & 0 \\
 & & f\ \uparrow & & u\ \uparrow & & \bar{u}\ \uparrow & & \\
0 & \longrightarrow & B & \xrightarrow[g]{} & A & \longrightarrow & A/_{g(B)} & \longrightarrow & 0
\end{array}$$

Hence, now $uog(B) = v(f(B)) = v(H)$. Hence u induces $\bar{u}$: $A/_{g(B)} \longrightarrow Q(A,H)/v(H)$ making the diagram commute.

The map $g : B \longrightarrow A$ is a M-monomorphism. So, there exists $w : M \longrightarrow B$ such that $\text{Im } g + \text{Im } w = B$.

Hence $(\overline{a,h}) = (\overline{a,0}) + (\overline{0,h}) = u(a) + v(h)$; further, letting $a = w(m) + g(b)$ we get $(\overline{a,h}) = u(w(m)) + u(g(b)) + v(h)$

$$= u_ow\ (m) + v(f(a)) + v(h)$$

$$= u_ow(m) + v[f(a)+h] \in \text{Im}(u_ow) + \text{Im} v$$

Hence, $Q(A,H) = \text{Im}v + \text{Im}(u_o w)$ showing v is a Monomorphism. Now, by hypothesis v splits. Hence by Maclane's Proposition (1.4.5) there exists $l : A \longrightarrow H$ such that $log = f$. Hence the result O

(iii)$\Rightarrow$(i). Let $N \leq M$ and $r : N \longrightarrow H$ any map. Now the inclusion $j_N : N \longrightarrow M$ is a M-monomorphism (for $1_M : M \longrightarrow M$ is such that $\text{Im } j_N + \text{Im } 1_M = M$). So by hypothesis ((iii)) there exists $l : M \longrightarrow H$ such that $loj_N = r$. Hence H is M-injective.

Now we shall define pull-backs as duals of push-outs and prove a proposition about them. This will be used to relate M-epimorphisms to M-projectivity in Theorem 1.4.10.

1.4.7. DEFINITION. Let $f : H \longrightarrow B$ and $g : A \longrightarrow B$ be maps. Consider the submodule $P(A,H) = \{(a,h) \in A \times H / g(a) = f(h)\}$ of $A \times H$. Let $p_H : P(A,H) \longrightarrow H$ be defined by $p_H(a,h) = h$ and $p_A : P(A, H) \longrightarrow A$ be defined by $p_A(a,h) = a$. Then for $(a,h) \in P(A,H)$, we have,

$fop_H(a,h) = f(h) = g(a) = g_o p_A(a,h)$

Thus the diagram

$$\begin{array}{ccc} P(A,H) & \xrightarrow{p_H} & H \\ \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle r} \\ A & \xrightarrow[g]{} & B \end{array}$$

commutes. This diagram is called the <u>pull-back</u> of $f : H \longrightarrow B$ and $g : A \longrightarrow B$.

We follow the notation of 1.4.7 in 1.4.8 and 1.4.9 below.

1.4.8. REMARK. Suppose g is an epimorphism. Let $h \in H$; then $f(h) \in B$. Since g is onto there exists an element $a \in A$ such that $g(a) = f(h)$. This implies that $(a,h) \in P(A,H)$.

Now, $p_H(a,h) = h$, shows that p_H is an epimorphism.

1.4.9. PROPOSITION. (MacLane). Assume g is an apimorphism. Then p_H splits if and only if there exists a homomorphism $u : H \longrightarrow A$ satisfying $gou = f$.

Proof ($\Leftarrow$) Suppose $u : H \longrightarrow A$ satisfies $gou = f$. We define $r : H \longrightarrow P(A,H)$ by

$$h \longmapsto (u(h),h)$$

Since $gou = f$, it follows that $(u(h),h) \in P(A,H)$

Then it is easy to see that r is R-linear and $p_H or (h) = p_H(u(h), h) = h$

Hence p_H is split by r.

Conversely, suppose that p_H has a splitting, say r, We will show that there exists a homomorphism $u : H \longrightarrow A$ satisfying $gou = f$.

Let $r : H \longrightarrow P(A,H)$ be such that $p_H or = 1_H$

Let $u : H \longrightarrow A$ be defined by $u = p_A or$ where $p_A : P(A,H) \longrightarrow A$ is the first projection. Then, $r(h) = (u(h),h)$ and $g(u(h)) = f(h)\ \forall h \in H$ [since $(u(h),h) \in P(A,H)$]

Hence, $gou = f$.

1.4.10. THEOREM. Let H and M be right R-modules. Then the following are equivalent (i). H is M-projective

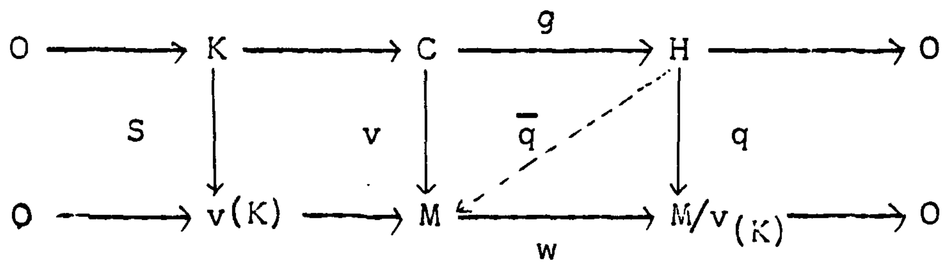
(ii) Every M-epimorphism $g : C \longrightarrow H$ splits.

(iii) Given an M-epimorphism $g : A \longrightarrow B$ and any homomorphism $f : H \longrightarrow B$, there exists $u : H \longrightarrow A$ such that $gou = f$.

<u>Proof</u>. (i) $\Rightarrow$ (ii) Suppose that H is M-projective and let $g : C \longrightarrow H$ bean M-epimorphism. Then $\operatorname{Ker} g \cap \operatorname{Ker} v = 0$ for some $v : C \longrightarrow M$.

Let $K = \operatorname{Ker} g$ and $v_o = v|_K$, the restriction of v to K. Then for $k \in K$, $v_o(k) = 0$ implies that $k \in \operatorname{Ker} v \cap \operatorname{Ker} g = 0$, by assumption. Thus v_o is a monomorphism. It follows that $K \cong v(K) \leq M$.

Now consider the following commutative diagram with exact rows. (The map w is the canonical epimorphism and q is defined by $q(h) = w_o v\ (c)$ for an element $c \in C$ satisfying $g(c) = h$).



The map 'q' is well defined, for if

$$g(c_1) = g(c_2) = h$$

$$\Rightarrow (c_1 - c_2) \in \text{Ker } g = K$$

$$\Rightarrow v(c_1 - c_2) \in v(K)$$

$$\Rightarrow wv\ (c_1) = wv\ (c_2).$$

By M-projectivity of H there exists $\bar{q} : H \longrightarrow M$ satisfying $wo\bar{q} = q$.

Now if $c \in C$, $wo\,[\,v - \bar{q}og\,](c)\,]$

$$= w\,[v(c) - \bar{q}og\ (c)]$$

$$= wov(c) - qog(c)$$

$$= q(h) - q(h) = 0.$$

Therefore, $\text{Im}\,[v - \bar{q}og] \subseteq \text{Ker } w = v(K)$

also $v(K) = [v - \bar{q}og]\ (K) \subseteq \text{Im}\ (v - \bar{q}og)$.

hence, $\text{Im}\,[v - \bar{q}og] = v(K)$.

Now, consider the composite

$$u: C \xrightarrow{v-\bar{q}og} V(K) \xrightarrow{v_o^{-1}} K$$

for $k \in K$

$v_o^{-1} [(v - qog)\,(k)] = v_o^{-1} v(k) = k.$

So, the inclusion

$K \longrightarrow C$ is split by $u : C \longrightarrow K$

hence by a standard result $g : C \longrightarrow H$ also splits.

(ii) $\Rightarrow$ (iii) Let $g : A \longrightarrow B$ be an M-epimorphism and let $f : H \longrightarrow B$ be a homomorphism. Then we have the row exact diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & P(A,H) & \xrightarrow{p_H} & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow p_A & & \downarrow f & & \\
0 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{g} & B & \longrightarrow & 0
\end{array}$$

Let $v : A \longrightarrow M$ be a homomorphism satisfying $\text{Ker } g \cap \text{Ker } v = (0)$.

Let $\bar{v} = vop_A : P(A,H) \longrightarrow M$. Clearly then, $\text{Ker } p_H \cap \text{Ker } \bar{v} = (0)$; for if $(a,h) \in P(A,H)$ such that $p_H\,(a,h) = 0$ and $\bar{v}(a,h) = vp_A(a,h) = 0$ then $h = 0$ and $v(a) = 0$.

So $a \in \text{Ker} v$.

Also $g(a) = f(h) = 0$ implies $a \in \mathrm{Ker}\, g$

So $a = 0$. Thus $(a,h) = 0$

Hence p_H is an M-epimorphism and therefore it splits.

Hence we are in the situation of Proposition 1.4.9.

So there exists a homomorphism $u : H \longrightarrow A$ satisfying

$f = gou$

i.e. (iii) holds.

(iii) $\Rightarrow$ (i) Note that any epimorphism of the form $g : M \longrightarrow M''$ is automatically an M-epimorphism. (Let $f = 1_M : M \longrightarrow M$, then $\mathrm{Ker}\, f \cap \mathrm{Ker}\, g = 0$). Therefore, given an epimorphism $g : M \longrightarrow M''$ and a homomorphism $f : H \longrightarrow M''$ there exists $u : H \longrightarrow M$ satisfying $gou = f$. Hence H is M-projective.

§ 1.5 V-modules and p-V-modules

Recall that a ring is called a left V-ring (after Villamayor) if every simple left R-module is injective. It was shown by Kaplansky [RZ : 59] that for a commutative ring R this condition is equivalent to von Neumann regularity. V-rings were studied in depth by Michler and Villamayor [MV : 73] and many other authors. There is an important class of rings, the class of left p-V-rings which contains all left V-rings and all regular rings. To define this class we first introduce the concept of p-M-injectivity for a module.

1.5.1. DEFINITION. Let M and N be left R-modules. We say N is <u>p-M-injective</u> if N is $(R_m M)$- complete for each element m of M.

A special case of the above concept is the following :

1.5.2. DEFINITION. We say a module N is <u>p-injective</u> if it is p-R-injective in the sense of 1.5.1. (Thus N is called p-injective if N is Ra-complete for each element a of R in the sense of 1.1.3.)

1.5.3. DEFINITION. A ring R is called a left <u>p-V-ring</u> if every simple left R-module is p-injective.

1.5.4. REMARKS. (I) Trivially M-injective implies p-M-injective and injective implies p-injective.

(II) As seen in 1.1.7. if R is a regular ring then every left R-module is Ra-complete for each a in R. So every left R-module is p-injective. In particular, R is a left (and right) p-V-ring.

(III) Trivially, left V-rings are left p-V-rings.

Regular rings, V-rings and p-V-rings have been extensively studied. (See [G], [St, Chapter I, §12], [R :86], [M:74], [MV:73]). The concepts of a V-ring and a p-V-ring have the following natural extensions to modules.

1.5.5. DEFINITION. Let ${}_RM$ be a module. We say M is a <u>V-module</u> if every simple left R-module is M-injective. We say M is a <u>p-V-module</u> if every simple left R-module is p-M-injective.

1.5.6. REMARKS. (I) In [H : 81] Y.Hirano has credited Tominaga [T : 76] with the definition of a V-module. These modules have also been called co-semisimple modules in [F : 72, § 3] and [AF, Exercises 9.14 and 18.23]. In line with this terminology Anderson and Fuller have called left V-rings as left co-semisimple rings [AF, Exercises 13.10 and 18.23]. However, the term "V-ring" has been used by most other authors and has become standard.

(II) In the notation of § 1.2 M is a V-module if and only if $M \in C^i(\underline{S})$ where $\underline{S}$ is the family of all simple left R-modules.

(III) Generalising the element-wise definition of a regular ring Zelmanowitz [Z : 72] defines a right R-module M to be regular if for every element m of M, there exists an R-homomorphism $f : M \longrightarrow R$ such that $mf(m) = m$. Suppose M_R is a regular module. Then it is easy to see that if m is an element of M then mR is a direct summand of M. Hence by 1.1.5. every right R-module N will be (mR, M)-complete. So every N_R will be p-M-injective. This shows that if M is a regular module, then M is a p-V-module. These remarks extend Remark 1.5.4 (II).

(IV) Clearly, we can use the concept of co-completeness (1.1.8) for dualising some concepts introduced above. For example, we can dualise the condition in Definition 1.5.1 by considering the condition "N is (Rm, M)-co-complete for each $m \in M$".

(V) It is well-known that all simple left R-modules are projective if and only if R is semi-simple (See [MV:77, p.566] for a proof.) This extends to the result : simple modules are M-projective if and only if M is a semi-simple module (See [AF, Exercise 16.8]). Thus replacing injective by projective in the definition of V-module does not yield a new class of modules.

Next we prove a theorem giving a necessary and sufficient condition for a module to be a V-module. Putting M = R in Theorem 1.5.8, we obtain the result that R is a left V-ring if and only if every left ideal I of R is the intersection of maximal left ideals containing it, a well-known characterization of left V-rings due to Michler and Villamayor [MV:73 Theorem 2.1].

1.5.7. NOTATION. For a sub-module N of M, N* will denote the intersection of all members of the family $\mathfrak{F}_N$ of maximal sub-modules of M containing N. If N = M, then $M_M = \emptyset$ and M* = M. Thus O* is the intersection of all maximal sub-modules of M and thus equals J(M), the Jacobson radical of M. Clearly N = N* if and only if O* = O in $M/_N$ if and only if $J(M/_N) = O$.

1.5.8. THEOREM. The following conditions are equivalent for a module ${}_RM$.

(i) M is a V-module.

(ii) M/N has zero Jacobson radical for each sub-module N of M.

<u>Proof</u>. In the proof $\bar{x}$ will denote the coset x + Q for different Q'S. The meaning will be clear from the context.

(i) $\Rightarrow$ (ii) Let N be a sub-module of M. It is required to prove that for $\bar{x} \in M/N$, $\bar{x} \neq 0$ there exists a maximal sub-module Zo/N (say) of M/N such that $\bar{x} \notin Zo/N$.

Consider $\mathcal{S} = \{ N_o \leq M_o = M/N$ such that $\bar{x} \notin N_o\}$. Since $0 \in \mathcal{S}$, $\mathcal{S}$ is nonempty. It is easy to see that $\mathcal{S}$ is an inductive poset.

Let $Y_o = Zo/N$ be a maximal element of $\mathcal{S}$. This exists by Zorn's lemma. Since $x \notin Y_o$. we have $x \notin Z_o$. Let $\mathcal{F} = \{Y / Y_o \lneq Y \leq M_o\}$; note that $M_o \in \mathcal{F}$. Then $\bar{x} \in Y$ for every Y in this family $\mathcal{F}$ (by the maximality of Y_o).

Let $D = \bigcap_{\mathcal{F}} Y$; then $\bar{x} \in D$ and $D \geqslant Y_o$

As $x \notin Y_o$ so $D \ngtr Y_o$ and $D/Y_o \neq 0$.

Claim: D/Y_o is a simple R-module. For, if $D_1/Y_o \leq D/Y_o$ with $D_1/Y_o \neq 0$ then $Y_o \lneq D$, and $D_1 \in \mathcal{F}$.

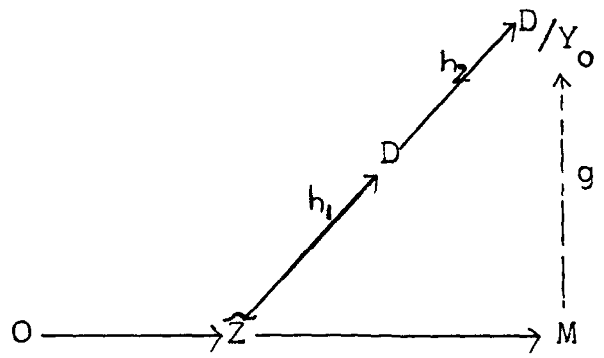
So, $D \leq D_1$. This implies $D = D_1$. Thus D/Y_o is a simple R-module.

(Hence the claim)

Hence D/Y_o is M-injective, by hypothesis.

As $D \leq M_o = M/_N$ we can write $D = \widetilde{Z}/N$ for some $\widetilde{Z} \leq M$.

Consider the following diagram (where h_1, h_2 are canonical quotient maps)



By the M-injectivity of $D/_{Y_o}$, there exists a map $g : M \longrightarrow D/Y_o$ extending $h = h_2 o\ h_1 : \widetilde{Z} \longrightarrow D/Y_o$.

Now $Y_o \leq D$ i.e. $Z_o/N \leq \widetilde{Z}/_N$ and

$D/_{Y_o}$ is simple implies that $\widetilde{Z}/_{Z_o}$ is simple.

Now for, $z_o \in Z_o$

$g(z_o) = h_2 o h_1(z_o) = h_2(\overline{z_o}) = 0$

Hence g induces a map $\overline{g} : M/Z_o \longrightarrow D/_{Y_o} \xrightarrow{\sim} \widetilde{Z}/_{Z_o}$.

Also for $p \in \widetilde{Z}$, $\overline{g}(\overline{p}) = g(p) = h_2 o h_1(p) = h_2(\overline{p}) = \overline{p}$,

(Where $\overline{p}$ denotes the coset of p in appropriate quotients).

So g splits the inclusion $\widetilde{Z}/_{Z_o} \hookrightarrow M/_{Z_o}$, and $M/Z_o = Z/Z_o \oplus T/Z_o$ for some submodule T/Z_o of M/Z_o

We have, $Y_o = Z_o/N \leqslant T/N$.

Now $x \notin T/Z_o$ since $x \in \widetilde{Z}/Z_o$ and $\bar{x}$ is nonzero as $x \notin Z_o$

So $x \notin T$, this implies that $\bar{x} \notin T/N$

Now $Z_o/N = Y_o \lneqq T/N$ is impossible by the maximality of Y_o. Hence $Z_o = T$.

So $M/Z_o = Z/Z_o$ is simple, which shows that Z_o is a maximal submodule of M.

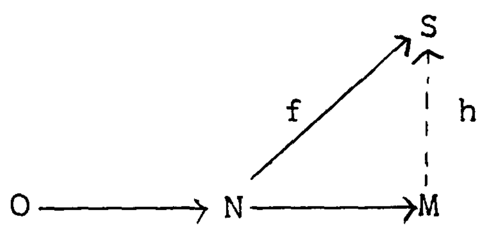
Now $Y_o = Z_o/N \leqslant M/N = M_o$

So $M_o/Y_o \xrightarrow{\sim} M/Z_o$ is simple.

i.e. Y_o is a maximal submodule of M/N. Thus there exists a maximal submodule Z_o/N of M/N such that $\bar{x} \notin Z_o/N$.

(ii) $\Rightarrow$ (i) Let ${}_RS$ be a simple R-module.

Let us consider the following diagram



Case (1) : $f=0$; then $h = 0$ extends f.

Case (2) : $f \neq 0$; then $f(N) \neq 0$ so $f(N) = S$ (since S is simple).

and $\bar{f} : N/Kerf \xrightarrow{\sim} S$ defined by $x+K \longmapsto f(x)$ (where $K = Kerf$) is an isomorphism.

Now since f is nonzero, there exists $n \in N$ such that

$n \notin K$ i.e. $\bar{n} \neq 0$ in $N/_K \leqslant M/_K$

By hypothesis, M/K has zero radical; so there exists a

maximal submodule M_1/K of M/K such that $\bar{n} \notin M_1/K$ i.e. $n \notin M_1$.

We consider

$g : N \hookrightarrow M \longrightarrow M/_{M_1}$

Note that $n \longmapsto n \longrightarrow n + M_1 = \bar{n} \neq 0$ under g and $g(K)=0$.

The map g induces a R-linear isomorphism

$\bar{g} : N/_K \longrightarrow M/_{M_1}$

Also $n + K \longmapsto n + M_1 \neq 0$ under $\bar{g}$.

So this map has an inverse

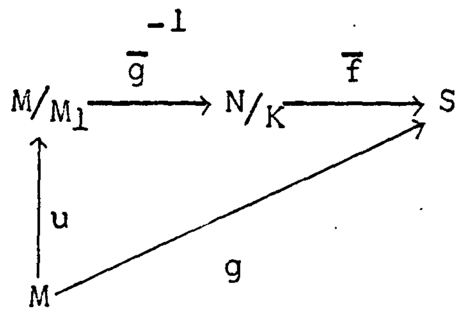
$\bar{g}^{-1} : M/_{M_1} \longrightarrow N/_K$

For $z \in N$, we have,

$g(x) = \bar{f} o \bar{g}^{-1} o u\ (x)$

$= \bar{f} o \bar{g}^{-1}\ (x + M_1)$

$= \bar{f}\ (x + K) = f(x)$



So g extends f, showing that every simple R-module is M-injective.

As in Theorem 1.5.8 a number of properties of V-rings extend to V-modules. Similarly many properties of p-V-rings extend to p-V-modules. In Theorem 1.5.9 we shall prove an analogue for p-V-modules of a characterisation of p-V-rings due to Ming [M:80, Theorem 1]. Since the result for p-V-modules has not appeared in the literature we shall give a detailed proof.

1.5.9. THEOREM. For a R-module M, the following conditions are equivalent.

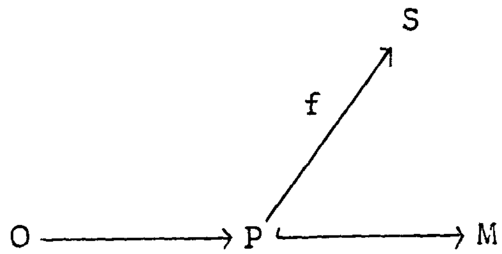
(1) M is a p-V-module.

(2) If K is a maximal submodule of a cyclic submodule P of M then $K^* \neq P^*$

Proof. (2) $\Rightarrow$ (1) Let S be a simple left R-module.

It is required to show that S is p-M-injective.

Let P be a cyclic submodule of M. We consider the following diagram



Case (1) f = O. Then $g = 0 : M \longrightarrow S$ extends f.

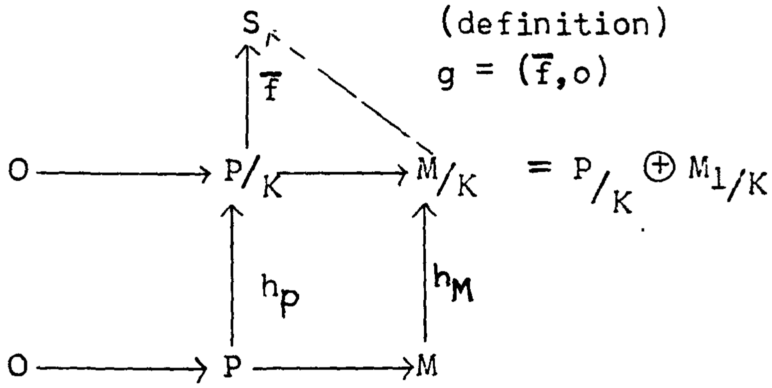
Case (2) : $f \neq 0$. Then $P/_{Kerf} \xrightarrow{\sim} S$ is simple. So $K = Kerf$ is a maximal submodule of P, hence $K^* \neq P^*$ (by hypothesis). i.e. there exists a maximal submodule M_1 of M such that $M_1 \geqslant K$ but $M_1 \not\geqslant P$; therefor $M_1 + P = M$. Notice that $K \leqslant M_1 \cap P \lneqq P$ (for otherwise, $P = M_1 \cap P \leqslant M_1$ which is is not the case).

Hence $K = M_1 \cap P$ (as P/K is simple)

and $M_{1/K} \cap P/K = (M_1 \cap P)/K$

This implies that $M/K = M_{1/K} \oplus P/K$.

Consider the following diagram (with usual notation)



For $p \in P$ we have $g_o h_M(p) = g_o h_P(p) = g(\bar{p})$

$= (\bar{f}, o)(\bar{p})$

$= \bar{f}(\bar{p})$

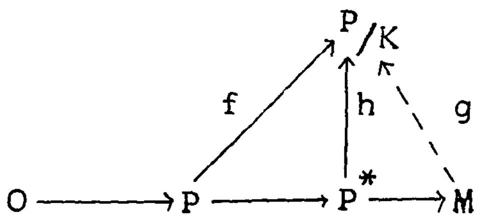
$= f(p)$

This shows that S is p-M-injective.

(1) $\Rightarrow$ (2)

Suppose, if possible P is a cyclic submodule of M and K is a maximal submodule of P such that $K^* = P^*$

We consider the following diagram (where f is the canonical quotient map)



Since P/K is simple and hence M-injective, there exists a map $g : M \longrightarrow P/K$ which extends $f : P \longrightarrow P/K$

Let $h = g|_{P^*} : P^* \longrightarrow P/K$, then h extends f and $h(K) = f(K) = 0$.

This implies that $K \subset H = \mathrm{Ker}h$ —(A)

Notice that,

$$H \leqslant P^* = K^* \leqslant M \text{ and } K \leqslant P \leqslant P^* = K^* \leqslant M \quad \text{- (B)}$$

We shall use the following two easily verifiable remarks:

Remark (1) $H_1 \leqslant H_2$ implies that $H_1^* \leqslant H_2^*$

Remark (2) $H^* = H^{**}$

Claim (1) $H^* = K^*$

$H^* \leqslant (K^*)^* = K^*$ by (B) and Remarks (1) and (2).

Also $K^* \leqslant H^*$ by (A) and Remark (1), Hence claim(1).

Let $G = \mathrm{Ker}g$.

Then $M/G \cong P/K$ shows that G is a maximal submodule of M.

Claim (2) $G \cap P^* = H$.

We have, $H = \{z \in P^* / h(z) = 0\} = \{z \in P^* / g(z) = 0\}$

$= P^* \cap \mathrm{Ker}g = P^* \cap G \leqslant G$ and so $H^* \leqslant G$

Now $H^* = K^*$ (by claim (1))

$= P^*$ (Assumption)

So, $P^* \leqslant G$. But $H = P^* \cap G$, this implies that $P^* = H$.

So, $h = 0$ i.e. $f = 0$ and $P = K$, absurd.

Hence $K^* \neq P^*$.

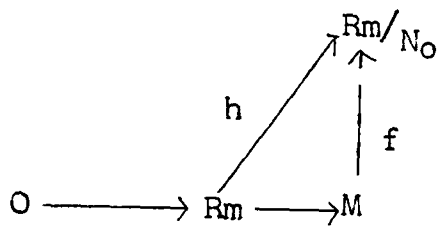
As mentioned by Zelmanowitz [Z :72,(2.5)] if M is a regular module, then the Jacobson radical J(M) of M equals zero. A similar result holds for V-modules by Theorem 1.5.8.

The following extension of these facts does not seem to have appeared in the literature.

1.5.10. PROPOSITION. If $_RM$ is a p-V-module then J(M) = 0.

<u>Proof</u>. If possible, let $0 \neq m \in J(M)$.

Since $0 \neq Rm \leqslant M$, there exists a maximal submodule N_o of Rm so that Rm/N_o is simple (and, in particular, non-zero). We consider the following row exact diagram (where h is the canonical quotient map)



Since Rm/N_o is simple and hence p-M-injective there exists a map $f : M \longrightarrow Rm/N_o$ extending $h : Rm \longrightarrow Rm/N_o$. Then f is an epimorphism and so $M/\ker f \xrightarrow{\sim} Rm/N_o$. So Kerf is a maximal submodule of M. Hence $m \in J(M) \leqslant$ Kerf $\Rightarrow f(m) = 0$

Now $\bar{m} = h(m) = f(m) = 0$ in Rm/N_o

So $Rm/N_o = 0$ which is impossible. This contradiction shows that J(M) = 0.

Many results about V-modules and p-V-modules have been proved by Hirano [H: 81]. These classes of modules have also been studied by Yousif. We shall conclude this section by stating without proof a result for projective modules over commutative rings (See [Y: 86, Proposition 1.1] where reference is made to [H: 81] and [Y: n.d].)

1.5.11 PROPOSITION. If R is a commutative ring and M is a projective module then the following are equivalent

(i) M is a regular module

(ii) M is a V-module.

(iii) M is a V'-module (i.e. every simple singular R-module is M-injective)

(iv) M is a p-V-module

(v) M is a p-V'-module (i.e. every simple singular module is p-M-injective).

§1.6. A theorem on direct sums

We know that a direct product of M-injective modules is always M-injective (1.2.26) and that of injective modules injective ([AF. 16.11]) Direct sums of injective modules need not, in general, be injective (Let $\{K_i\}_{i \in IN}$ be an infinite sequence of fields and $R = \prod K_i$. Then R is a self injective and von Neumaun regular ring, each K_i is an injective R-module but the ideal $A = \oplus K_i$ of R is not R-injective). It was shown by Bass ([B : 62]) that R is left noetherian if and only if the direct sum of a family of injective left R-modules is always injective. (See [FI, Exercise 3.48.3]) or ([AF, Proposition 18.13]) or ([R, Theorem 4.27].) In this section we follow the argument of Azumaya, Mbuntum and Varadarajan to prove a result giving a necessary and sufficient condition for a direct sum of M-injective modules to be an M-injective module. Bass' result follows as a special case of this theorem [AMV:75, Theorem 2.5].

1.6.1. NOTATION. For any module ${}_RA$ and any $x \in A$, we denote the left annihilator $\{\lambda \in R \mid \lambda x = 0\}$ of x by L_x; L_x is a left ideal of R.

1.6.2. DEFINITION. An element $x \in A$ is said to be dominated by M if $L_x \supset L_m$ for some $m \in M$.

Given a family $\{A_i\}_{i \in J}$ of modules let $\underline{x}$ be the element of $\prod_{i \in J} A_i$ whose i-component is x_i.

Let $I_{\underline{x}} = \{\lambda \in R \,/\, \lambda \underline{x} \in \oplus_{i \in J} A_i\}$

1.6.3. DEFINITION. We call $\underline{x} \in \prod_{i \in J} A_i$ a special element if $I_{\underline{x}} x_i = 0$ for almost all i. In other words there exists a finite subset F of J such that $\lambda x_i = 0$ for all $\lambda \in I_{\underline{x}}$ and for all $i \notin F$.

1.6.4. PROPOSITION. A is M-injective $\Leftrightarrow$ A is Rm-injective for all $m \in M$.

<u>Proof</u>. ($\Rightarrow$) Follows from the closedness of $C^i(A)$ under submodules (see Proposition 1.2.19)

($\Leftarrow$) we know that $C^i(A)$ is closed under direct sums (Proposition 1.2.21). Hence it follows that A is $\oplus_{m \in M}$ Rm-injective. M is a homomorphic image of $\oplus_{m \in M}$ Rm and since $C^i(A)$ is closed under homomorphic images (Proposition 1.2.16) it follows that A is M-injective.

1.6.5. THEOREM. The module $\oplus_{i \in J} A_i$ is M-injective $\Leftrightarrow$ each

A_i is M-injective and every element of $\prod_{i \in J} A_i$ is dominated by M is special.

Proof. ($\Rightarrow$) Let $\underline{x} = (x_i)_{i \in J} A_i$ be dominated by M, that is, there is an $m \in M$ such that $L_m \subset L_{\underline{x}}$ this implies that the mapping $\lambda m \longmapsto \lambda \underline{x}$ $(\lambda \in R)$ is well-defined and gives a homomorphism $f : Rm \longrightarrow \prod A_i$. The image of the submodule $I_{\underline{x}}m$ by f is clearly $I_{\underline{x}}\underline{x}$ $(\subset \oplus A_i)$. Thus the restriction of f to $I_{\underline{x}}m$ by f is regarded as a homomorphism $I_{\underline{x}}m \longrightarrow \oplus A_i$. Since $\oplus A_i$ is Rm-injective (by 1.6.4) this homomorphism can be extended to a homomorphism $Rm \longrightarrow \oplus A_i$ which means that there exists an $\underline{u} \in \oplus A_i$ such that $\lambda \underline{x} = \lambda \underline{u}$ for all $\lambda \in I_{\underline{x}}$, Let $\underline{u} = (u_i)_{i \in J}$ It follows then that $I_{\underline{x}} x_i = I_{\underline{x}} u_i$ for all $i \in J$. But since $u_i = 0$ for almost all i, it follows that $I_{\underline{x}}(\underline{x})_i = 0$ for almost all i too, i.e., $\underline{x}$ is special.

($\Leftarrow$) Let $m \in M$ and consider the cyclic submodule Rm of M. Let I be a left ideal of R. Then Im is a submodule of Rm. (Conversely, every submodule of Rm is of the form of Im with a suitable left ideal I). Let there be given a homomorphism $h : Im \longrightarrow \oplus A_i$ Then since $\oplus A_i \subset \prod A_i$ and $\prod A_i$ is M-injective and hence Rm-injective, h can be extended to a homomorphism $k : Rm \longrightarrow \prod A_i$.

Let $\underline{x} \in \prod A_i$ be the image of m. Then the homomorphism k is given by $\lambda m \longmapsto \lambda \underline{x}$ ($\lambda \in R$). Therefore, it follows that $I\underline{x} = h(Im) \subset \oplus A_i$ and hence $I \subset I_{\underline{x}}$. On the other hand, since clearly $L_m \subset L_{\underline{x}}$, $\underline{x}$ is dominated by M and thus $\underline{x}$ is special by assumption, i.e. $I_{\underline{x}} x_i = 0$ whence $I_{\underline{x}} x_i = \theta$ for almost all i. Let $\underline{u}$ be the element of $\oplus A_i$ whose i-component is x_i or 0 according as $I_{\underline{x}} x_i \neq 0$ or $I_{\underline{x}} x_i = 0$.

Then it is clear that $\lambda \underline{u} = \lambda \underline{x}$ for all $\lambda \in I$. Further, it is also clear that $L_m \subset L_{\underline{x}} \subset L_{\underline{u}}$ and therefore the mapping $\lambda m \longmapsto \lambda \underline{u}$ ($\lambda \in R$) is well-defined. This mapping gives a homomorphism $f : Rm \longrightarrow \oplus A_i$ which is an extension of h, because $f(\lambda m) = \lambda \underline{u} = \lambda \underline{x}$ for all $\lambda \in I$. This implies that $\oplus A_i$ is Rm-injective for each $m \in M$ and so is M-injective (by Proposition 1.6.4).

1.6.6. THEOREM. The direct sum of any family of M-injective modules is M-injective $\Leftrightarrow$ every cyclic submodule of M is noetherian.

Proof. ($\Leftarrow$) Let $\{A_i\}$ be a family of M-injective modules. Let $\underline{x}$ be an element of $\prod A_i$ dominated by M; thus there is an $m \in M$ such that $L_m \subset L_{\underline{x}}$. Consider $I_{\underline{x}} m$. Since

$L_{\underline{x}} \subset I_{\underline{x}}$ whence $L_m \subset I_{\underline{x}}$. It follows that $I_{\underline{x}}/L_m \cong I_{\underline{x}}m$. On the other hand, $I_{\underline{x}}m$ is a submodule of the Noetherian module Rm. Hence $I_{\underline{x}}/L_m$ is finitely generated, i.e. there exists a finite number of elements $\lambda_1, \lambda_2, \ldots \lambda_n$ of $I_{\underline{x}}$ such that

$I_{\underline{x}} = R\lambda_1 + R\lambda_2 + \ldots\ldots + R\lambda_n + L_m$

It follows therefore $I_{\underline{x}}\, x_i = R\lambda_1 x_i + R\lambda_2 x_i + R\lambda_3 x_i + \ldots + R\lambda_n x_i$ for all components x_i. Since, however for each j, $\lambda_j x_i = 0$ for almost all i, it follows that $I_{\underline{x}} x_i = 0$ for almost all i, that is $\underline{x}$ is special. Thus $\oplus A_i$ is M-injective (by Theorem 1.6.5).

($\Rightarrow$) Let Rm, $m \in M$, be any cyclic submodule of M. Then $R/L_m \cong Rm$, and there is a one to one correspondence between the left ideals of R containing L_m and submodules of Rm. Thus in order to show that Rm is Noetherian it is sufficient to prove that there is no properly ascending infinite sequence of left ideals of $\underline{R}$ containing L_m. Suppose there exists an infinite sequence $L_m \leqslant I_1 \leqslant I_2 \leqslant I_3 \ldots$ of left ideals with $I_k \neq I_{k+1}$ for every $k \geqslant 1$. Let $B_k = R/I_k$, $h_k : R \longrightarrow B_k$ the cononical quotient map. Let E_k be the injective hull of B_k. Then each E_k is M-injective also. By assumption, $I_1 \supset L_m$. The element

$x = (x_k)_{k \geqslant 1}$ of $\prod_{k \geqslant 1} E_k$ where $x_k = h_k(1)$ is clearly dominated by M. For any $\lambda \in I_k$ we have $\lambda x_p = 0$ for $p \geqslant k$. Hence $I_k \leqslant I_{\underline{x}}$ for all $k \geqslant 1$. Let λ_k be any element of I_{k+1} which is not in I_k. Then $\lambda_k x_k \neq 0$ and $\lambda_k \in I_{\underline{x}}$. This proves that $I_{\underline{x}} x_k \neq 0$ for every $k \geqslant 1$.

This means $\underline{x}$ is not a special element and hence by Theorem 1.6.5, $\bigoplus_{k \geqslant 1} E_k$ is not M-injective.

1.6.7. REMARK. Letting M = R we get Bass' theorem mentioned in the introduction to this section.

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§ 1.7 Quasi-injectivity and quasi-projectivity

1.7.1. DEFINITION. An R-module M is called a <u>quasi-injective</u> module if for every submodule N of M every R-homomorphism $f : N \longrightarrow M$ can be extended to a R-homomorphism from M to M.

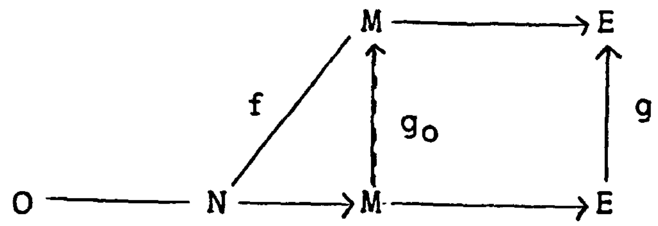
1.7.2. DEFINITION. An R-module M is called a <u>quasi-projective</u> module if for every quotient module M" of M every R-homomorphism $f : M \longrightarrow M"$ can be lifted to an R-homomorphism $g : M \longrightarrow M$.

1.7.3. REMARK. Clearly, M is quasi-injective if and only if M is $\underline{S}(M)$-complete (where $\underline{S}(M)$ = all submodules of M) in the terminology of § 1.2. Dually, quasi-projectivity can also be described in terms of co-completeness and M-projectivity.

1.7.4. DEFINITION. Let $M \leqslant N_R$. We shall say M is <u>invariant under all endomorphisms</u> of N if for every R-endomorphism $f:N \longrightarrow N$, we have $f(M) \leqslant M$. We shall also say that M is a <u>fully invariant submodule</u> of N in this case.

1.7.5. PROPOSITION. Let E be an injective module and $M \leqslant E$. If M is invariant under all endomorphisms of E then M is quasi-injective.

<u>Proof</u> : Let us consider the following diagram



Since E is injective there exists a map $g:E \longrightarrow E$ extending $f:N \longrightarrow M$. Now $g \in End_R(E)$ implies $g(M) \leq M$ (by hypothesis) So $g_o = g|_M$ is an endomorphism of M. For $n \in N$, $g_o(n) = g(n) = f(n)$ (since g extends f). So g_o extends f. Hence M is quasi-injective.

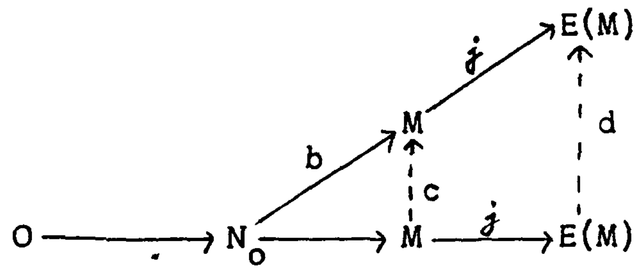
1.7.6. PROPOSITION. Let E(M) be the injective hull of M. If M is quasi-injective then M is invariant under all endomorphisms of E(M).

<u>Proof</u>. We shall denote the canonical inclusion $M \longrightarrow E(M)$ by j.

Let $a:E(M) \longrightarrow E(M)$ be an endomorphism of E(M).

Let $N = a^{-1}(M) \leq E(M)$ and $N_o = N \cap M$. Hence we have a map $b = a|N_o : N_o \longrightarrow M$.

Now consider the diagram



As M is quasi-injective there exists an extension $c: M \longrightarrow M$ of b. Since E(M) is injective there exists a map $d: E(M) \longrightarrow E(M)$ extending c.

Consider $a-d : E(M) \longrightarrow E(M)$.

Let if possible $(a-d)(M) \neq 0$, then there exists $m \in M$ such that $0 \neq (a-d)(m) \in E(M)$.

As $M \triangleleft E(M)$, so there exists $r \in R$ such that

$0 \neq r(a-d)(m) \in M \Rightarrow 0 \neq (a-d)(rm) \in M$ ———— (I)

Now $d(rm) = c(rm) \in M$

$\Rightarrow$ $a(rm) \in M$ hence, $rm \in N$ (since $a^{-1}(M) = N$)

So $rm \in N \cap M = N_o$

Therefore $a(rm) = b(rm) = d(rm)$ contradicting (I)

(since c extends b and d extends c)

So $(a-d)(M) = 0$

i.e. $a(m) = d(m)\ \forall\ m \in M$; $a(m) = c(m) \in M\ \ \forall\ m \in M$

i.e. $a(M) \leqslant M$.

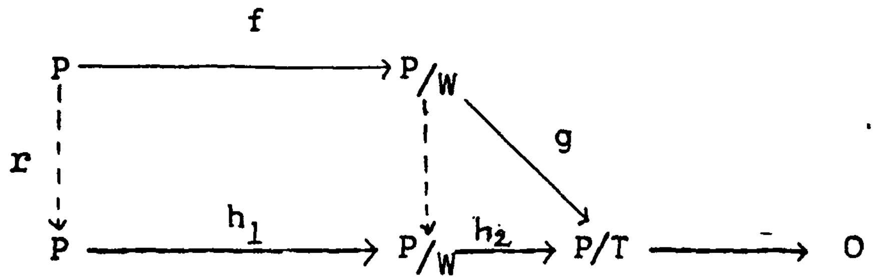
Showing that M is invariant under all endomorphisms of E(M).

1.7.7. THEOREM. An R-module M is quasi-injective if and only if M is invariant under all endomorphisms of E(M).

Proof. This follows from Propositions 1.7.5 and 1.7.6.

1.7.8. PROPOSITION. If P is projective, $W \leqslant P$ and W is invariant under all endomorphisms of P, then $P_{/W}$ is quasi-projective.

<u>Proof</u>. Let us consider the following diagram (where h_1 and h_2 are the natural quotient maps)



Since P is projective there exists a map $r : P \longrightarrow P$ satisfying $gof = h_2 o\, h_1 or$.

Also W is invariant under all endomorphisms of P, hence $r(W) \leqslant W$. So r induces a map $\bar{r} : P_{/W} \longrightarrow P_{/W}$ defined by $\bar{r}\,(\bar{p}) = \overline{r(p)}$ for each $\bar{p} \in P_{/W}$ (By $\bar{p}$ we mean the canonical image of p in $P_{/W}$)

$$\begin{aligned} \text{Now } h_2 o\bar{r}\,(\bar{p}) &= h_2(\overline{r(p)}) \\ &= h_2 h_1(r(p)) \\ &= h_2 o h_1 o r(p) \\ &= gof(p) \\ &= g(\bar{p}), \text{ for each } \bar{p} \in P_{/W} \end{aligned}$$

Hence $h_2 o\bar{r} = g$

i.e. $P_{/W}$ is quasi-projective.

1.7.9. PROPOSITION. If M is quasi-projective and has a projective cover $0 \longrightarrow \text{Kerp} \longrightarrow P(M) \xrightarrow{p} M \longrightarrow 0$ then Kerp is invariant under all endomorphisms of P(M).

<u>Proof</u>. Let $0 \longrightarrow \text{Kerp} \longrightarrow P(M) \xrightarrow{p} M \longrightarrow 0$ -(I)

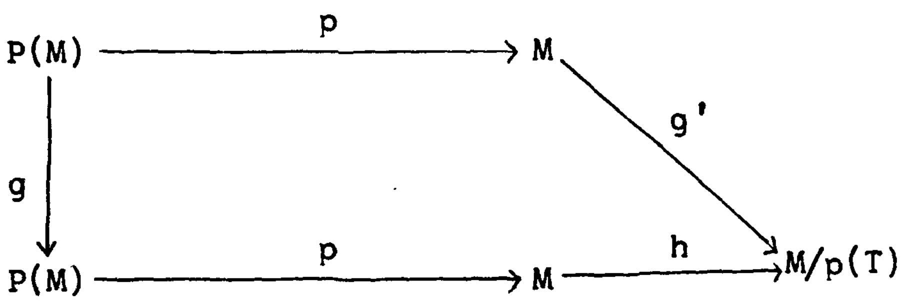
be the projective cover of M with M quasi-projective and let $g \in \text{End}(P(M))$. We shall show that $g(\text{Kerp}) \leqslant \text{Kerp}$.

Let $T = (\text{Kerp}) + g(\text{Kerp})$

Now, consider the map $g' : M \longrightarrow M/p(T)$ defined as follows :

Let $m \in M$, then there exists, $x \in P(M)$ such that $m = p(x)$.

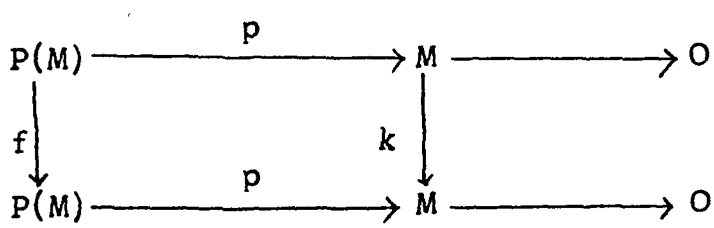
Define $g'(m) = hopog(x)$. Since $g(\text{Kerp}) \leqslant T$. The map g' is well-defined. Moreover, the following diagram commutes



-(II)

Now, since M is quasi-projective there exists $k : M \longrightarrow M$ satisfying $hok = g'$. Again, using the projectivitiy of P(M), we get the following commutative diagram

(with $f \in \text{End}(P(M))$) :



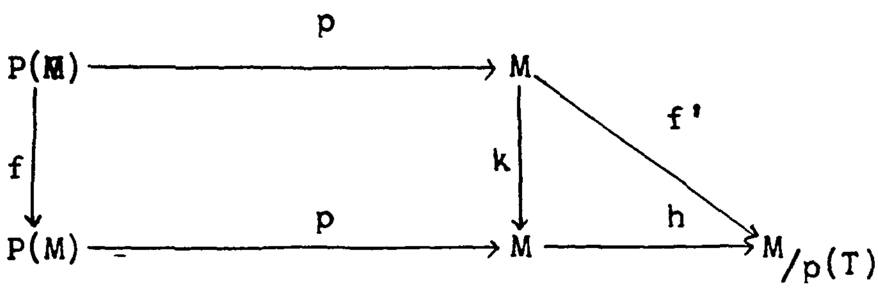
(III)

Notice, that the commutativity of (III) implies that

f(Kerp) $\leqslant$ Kerp. -(IV)

Next, consider the map $f':M \longrightarrow M/p(T)$ defined by

$f' = hok$. Then the following diagram



-(V)

commutes. Since g' = f' using diagrams (II) and (IV) we

get hop $[(g-f)(x)]$ = 0 for every $x \in P(M)$. Hence

$(g-f)(P(M)) \subset Ker(hop) \leqslant T$, as shown below (If $z \in Ker(hop)$

then 0 = hop(z) ; this implies $p(z) \in p(T)$.

So let p(z) = p(t) for some $t \in T$; then p(z-t) = 0

showing that $z - t \in ker\ p \leqslant T$. Thus $z \in T$)

Let $X = \{x \in P(M) / g(x)-f(x) \in Kerp\}$ ——— (VI)

We shall show that X = P(M)

Now let $x \in P(M)$ then,

$g(x) - f(x) = k_1 + g(k_2)$ with $k_i \in$ Kerp.

or $g(x-k_2) = k_1 + f(x)$ ——— (VII)

consider,

$g(x-k_2) - f(x-k_2) = k_1 + f(x) - f(x) + f(k_2)$ by (VII)

$= k_1 + f(k_2) \in$ Kerp by (IV)

This implies $(x - k_2) \in X$, so $X + \text{Kerp} = P(M)$.

Now p is a minimal epimorphism, hence $X = P(M)$.

Let $x \in \text{Kerp} \leqslant P(M) = X$. Then by the definition (VI) of X;

$g(x) - f(x) \in \text{Kerp}$ ——— (VIII).

By (IV) since $x \in \text{Kerp}$, $f(x) \in \text{Kerp}$ So (VIII) gives

$g(x) \in \text{Kerp}$. Thus we get $g(\text{Kerp}) \leqslant \text{Kerp}$, the desired conclusion.

1.7.10. THEOREM. Let M be any module. Suppose

$$0 \longrightarrow \text{Kerp} \longrightarrow P(M) \xrightarrow{p} M \longrightarrow 0$$

be a projective cover of M. Then M is quasi-projective if and only if Kerp is invariant under all endomorphisms of P(M).

Proof. Follows from Propositions 1.7.8 and 1.7.9.

1.7.11. EXAMPLES (1) Clearly injective (projective) modules are quasi-injective (quasi-projective).

(11) Let M be a semi-simple module and $N \leqslant M$. Then N is a direct summand of M and hence by Proposition 1.1.5 every module E is (N,M)-complete. So every E is M-injective. In particular, M is M-injective i.e. M is quasi-injective. A dual argument shows that semi-simple modules are quasi-projective. Examples of semi-simple modules are

simple modules and vector spaces over division rings.

(III) Let R be a commutative self-injective ring and A any ideal of R. Then each R-linear map from R to R is a homothecy $f_r: R \longrightarrow R$, which takes x to rx. So, $f_r(A) = rA \subseteq A$ for every $r \in R$. Hence A is quasi-injective over R (Proposition 1.7.5).

More generally, let R be a right self-injective rings, A any ideal of R. Then A is a quasi-injective right R-module. (For, $A \leq R_R$, since A is a right ideal. Also A is a left ideal, So $f_r(A) = rA \leq A$ for every $r \in R$. Hence by Proposition 1.7.5 A_R is quasi-injective).

(IV) It follows from Proposition 1.7.8 that if R is a ring and A an ideal of R, then $R/_A$ is quasi-projective as a left R-module as well as right R-module.

Let M be a left S-module and $f: R \longrightarrow S$ a ring homomorphism. Then M can be always regarded as a left R-module "via f" : $r.m = f(r).m$ for all $r \in R$, $m \in M$. In particular, if A is an ideal of R, taking the canonical quotient map $f: R \longrightarrow R/_A$, any left $R/_A$ module M becomes a left R-module "via f". It is easy to see that M being injective over $R/_A$ does not imply that M is injective over R. (Take $R = \mathbb{Z}$, $A = p\mathbb{Z}$, p a prime , $M = \mathbb{Z}/p\mathbb{Z}$.)

However, for quasi-injectivity we have the following result.

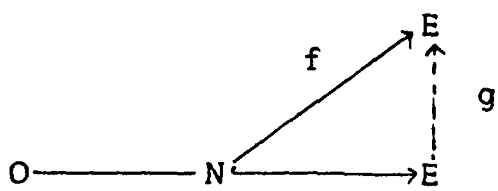
1.7.12. PROPOSITION. Let $\bar{R} = R/_A$ and M a left $\bar{R}$-module. Then

(I) E is quasi-injective over $\bar{R}$ if and only if E is quasi-injective over R.

(II) M is quasi-projective over $\bar{R}$ if and only if M is quasi-projective over R.

<u>Proof</u>. (I)($\Rightarrow$) Trivial.

($\Leftarrow$) Consider the following diagram (where N is any $\bar{R}$ submodule of E and f: N $\longrightarrow$ E is $\bar{R}$-linear).



Notice that for every $n \in N$, $r.n = \bar{r}.n \in N$. So, N is a R-submodule of E and f is R-linear.

Since E is quasi-injective over R there exists a R-linear map g : E $\longrightarrow$ E extending f: N $\longrightarrow$ E.

Now $g(\bar{r}.e) = g(r.e) = r.g(e) = \bar{r}.g(e)$

i.e. g is $\bar{R}$-linear extension of f. Hence E is quasi-injective over $\bar{R}$.

(II) Proof of this is dual to that of part (I).

1.7.13. REMARK. In the notation of 1.7.12 clearly $\overline{R}$ is projective as a left (as well as right) $\overline{R}$-module and hence quasi-projective as a left (as well as right) R-module. In particular, $\mathbb{Z}/_{n\mathbb{Z}}$ is always a quasi-projective $\mathbb{Z}$-module (≡ abelian group).

1.7.14. EXAMPLE. It is well-known that if R is a P.I.D and A a nonzero ideal of R, then $S = R/_A$ is a self-injective ring [R, Theorem 4.28]

In particular, for a prime p, $\mathbb{Z}/(p^n)$ is always a self injective ring. Letting $R = \mathbb{Z}$, $\overline{R} = \mathbb{Z}/_{(p^n)}$, $n \geqslant 2$ we get from Proposition 1.7.12 that since $\overline{R}$ is a self-injective and so quasi-injective over $\overline{R}$, it is also quasi-injective as a $\mathbb{Z}$-module. However, it is neither injective, nor semi-simple as a $\mathbb{Z}$-module.

1.7.15. REMARK. $\mathbb{Z}/_{(p^2)}$ is quasi-projective over $\mathbb{Z}$ (by 1.7.13) but is neither semi-simple nor projective over $\mathbb{Z}$.

Let N be a sub-module of a module M. In general, N can be isomorphic to a direct summand of M without being itself a direct summand of M (e.g. $R = \mathbb{Z}$, $M = \mathbb{Z}$, $N = 2\mathbb{Z}$; Clearly $N \cong M$.) However, this cannot happen in a quasi-injective module M. We shall prove this in Proposition 1.7.16. A dual result (Proposition 1.7.17) will be stated without proof.

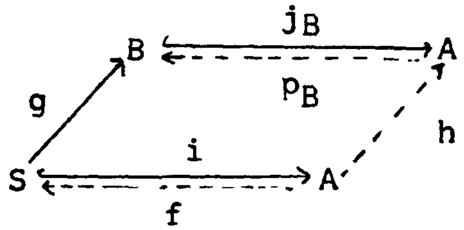
1.7.16. LEMMA. [RV:72, Lemma 4.3']

Let S be a submodule of a quasi-injective module A. Then S will be a summand of A if and only if S is isomorphic to a summand of A.

Proof. ($\Longrightarrow$) trivial.

($\Longleftarrow$) Let $S \xrightarrow[\sim]{g} B$ where $B \oplus C = A$; $B \hookrightarrow A$ is the natural inclusion and $A \xrightarrow{p_B} B$ is the natural projection.

We consider the following diagram



As A is quasi-injective there exists a map $h : A \longrightarrow A$ extending $j_B o g : S \longrightarrow A$ i.e. $h o i = j_B o g$

or $p_B o\, h o i = p_B o j_B o g$

or $g^{-1} o\, p_B o\, h o i = g^{-1} o g$ (since 'g' is an isomorphism)

or $(g^{-1} o\, p_B o\, h o i) o i = 1$

Hence there exists a map $f : A \longrightarrow S$ satisfying $f \circ i = 1_S$ i.e. the inclusion splits and hence S is a direct summand of A.

1.7.17. LEMMA. Let S be a submodule of a quasi-projective module A. Then S is a summand of A if and only if A/S is isomorphic to a summand of A.

Proof. See [RV:72; Lemma 4.3]

The two results below appear in [RV:72]. However, our proofs use the concepts of relative injectivity and projectivity and as a result, are somewhat simpler than those in [RV:72].

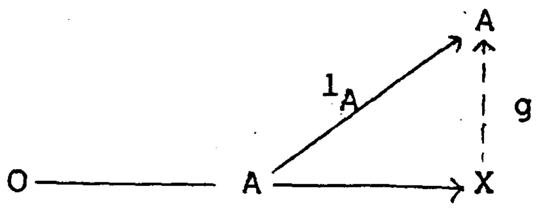
1.7.18. LEMMA. [RV:72; Lemma 4.4']

If A is quasi-injective, then the exact sequence

$$0 \longrightarrow A \overset{i}{\hookrightarrow} X \longrightarrow Y \longrightarrow 0$$ splits

whenever X is a quotient of A.

Proof. It is given that A is A-injective hence A is X-injective (for a quotient X of A) by 1.2.16. Hence given a row exact diagram



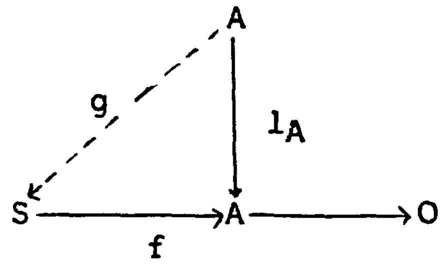
there exists a map $g:X \longrightarrow A$ which extends $1_A:A \longrightarrow A$ i.e. $g \circ i = 1_A$ i.e. the monomorphism $i:A \longrightarrow X$ splits.

1.7.19. LEMMA. [RV:72; Lemma 54.4]

Let A be a quasi-projective module. Then the exact sequence $0 \longrightarrow T \longrightarrow S \longrightarrow A \longrightarrow 0$ splits, whenever S is a submodule of A.

Proof. Given A is A-projective. So A is S-projective (for a submodule S of A) by 1.2.14.

We now consider the following row exact diagram

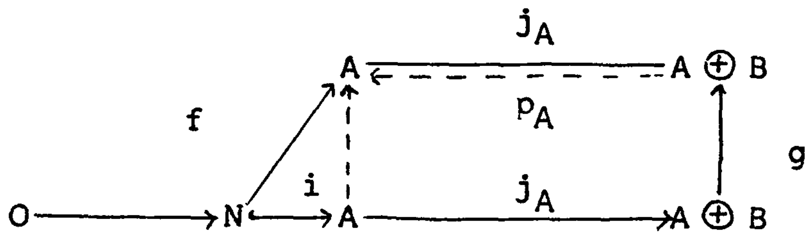


then there exists a map $g: A \longrightarrow S$ such that $f \circ g = 1_A$.

So f splits.

1.7.20. PROPOSITION. If $A \oplus B$ is quasi-injective then A and B are also quasi-injective.

Proof. We consider the following diagram (with usual notations)



Since $A \oplus B$ is quasi-injective there exists a map $g : A \oplus B \longrightarrow A \oplus B$ satisfying $g_o j_A o\ i = j_A o f$.

Now consider $h = p_A o g o j_A : A \longrightarrow A$ then

$hoi = p_A o\ g\ o\ j_A oi = p_A o j_A of = 1_A of = f$.

Hence the above map $h: A \longrightarrow A$ extends $f: N \longrightarrow A$. So A is quasi-injective. Similarly we can also show that B is quasi-injective.

Dually we have :

1.7.21. PROPOSITION. If $A \oplus B$ is quasi-Projective then A and B are also quasi-projective.

1.7.22. REMARK (I) A,B quasi-injective does not imply that $A \oplus B$ is quasi-injective. For example, we know that ${}_{\mathbb{Z}}Q$ is injective and hence quasi-injective; and for a prime p , $\mathbb{Z}/p\mathbb{Z}$ is simple/$_{\mathbb{Z}}$ and hence quasi-injective. We shall show that $M = Q \oplus (\mathbb{Z}/p\mathbb{Z})$ is not quasi-injective/$\mathbb{Z}$.

If possible let $M = Q \oplus (\mathbb{Z}/p\mathbb{Z})$ be quasi-injective. Let us consider the submodule $N = \mathbb{Z} \oplus 0$ of $Q \oplus (\mathbb{Z}/p\mathbb{Z})$ and define a map $f : \mathbb{Z} \oplus 0 \longrightarrow Q \oplus (\mathbb{Z}/p\mathbb{Z})$ by $f(a,o) = (o,\bar{a})$.
Since ${}_{\mathbb{Z}}M$ is quasi-injective there exists a map $g: M \longrightarrow M$ extending $f : N \longrightarrow M$.

$$\begin{aligned} \text{Now } f(1,0) = g(1,0) = g(p\tfrac{1}{p}, 0)) &= p\, g(\tfrac{1}{p}, 0) \\ &= p(x, \bar{y}) \text{ for } x \in Q \text{ and } \bar{y} \in \mathbb{Z}/p\mathbb{Z} \\ &= (px, p\bar{y}) \\ &= (px, \bar{0}) \end{aligned}$$

But $f(1,0) = (0,\bar{1})$ and $(0,\bar{1}) = (px,\bar{0})$

This implies $\bar{1} = \bar{0}$, which is not possible. So, M is not quasi-injective.

(II) In contrast to the situation in Remark (I) if A and B are isomorphic quasi-injective modules, then $A \oplus B$ is also quasi-injective. More generally, we have the following Theorem by Harada.

1.7.23. THEOREM. (Harada) Let n be a natural number. If M is quasi-injective then $M^{(n)} = M \oplus \ldots \oplus M$ (n copies) is also quasi-injective.

<u>Proof</u>. Let E(M) be the injective hull of M then $E(M)^{(n)} = E(M) \oplus \ldots \oplus E(M)$ (n copies) is the injective hull of $M^{(n)}$.

Now, $M^{(n)} \leqslant E(M^{(n)}) = E(M)^{(n)} = E_o$ (Say).

Since n is finite, E_o is injective. Then End $(E(M)^{(n)})$ can be viewed as the total n X n matrix ring over E(M).

And any $f \in$ End $(E(M)^{(n)})$ is of the form

$f = (f_{ij})\ {}^{i = 1 \ldots n}_{j = 1 \ldots n}$ where $f_{ij} : E \longrightarrow E$ and E_i denotes the i^{th} copy of E(M).

Now $f_{ij}(M) \leqslant M \quad \forall \quad i, j$

$$f(M^{(n)}) \leqslant M^{(n)}.$$

Thus M is invariant under all endomorphisms of $E(M^{(n)})$.

Hence, by Theorem 1.7.5 $M^{(n)}$ is also quasi-injective.

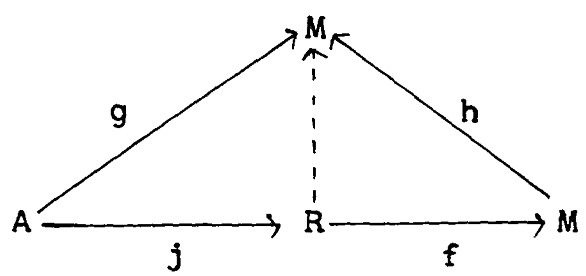
1.7.24. PROPOSITION. If M has a projective cover and if M is quasi-projective then $M^{(n)}$ is also quasi-projective.

<u>Proof</u>. Let $0 \longrightarrow \text{Kerp} \longrightarrow P(M) \xrightarrow{p} M \longrightarrow 0$

be the projective cover of M. The projective cover of $M^{(n)}$ is $P(M)^{(n)}$, the direct sum of n copies of P(M) with appropriate projection. By Proposition 1.7.8 it is sufficient to show the kernel of $P(M)^{(n)} \longrightarrow M^{(n)} \longrightarrow 0$ is an End $(P(M)^{(n)})$ submodule of $P(M)^{(n)}$. The endomorphism ring of $P(M)^{(n)}$ can be viewed as the total n X n matrix ring over End(P(M)). Also the kernel of the map is $\mathrm{Kerp}^{(n)}$. Since it is clear that any map f_{ij}: $P_i(M) \longrightarrow P_j(M)$ (from the i^{th} copy of P(M) to the j^{th}) must carry Kerp_i into Kerp_j (because Ker p is an End (P(M))- sub-module), it follows that Kerp is an End $(P(M)^{(n)})$ submodule of $P(M)^{(n)}$. Hence the proof.

1.7.25. PROPOSITION. Let M_R be such that M contains a copy of R (i.e. there exists a R-monomorphism f from R to M). Then M is quasi-injective over R implies that M is injective over R.

<u>Proof</u>. Let A be a right ideal of R. Consider the diagram



As M is quasi-injective there exists h:M $\longrightarrow$ M such that hofoj = g or (hof) oj = g. Hence hof : $R \longrightarrow M$ extends $g : A \longrightarrow M$. So, by Baer's criterion (1.2.5) M is injective over R.

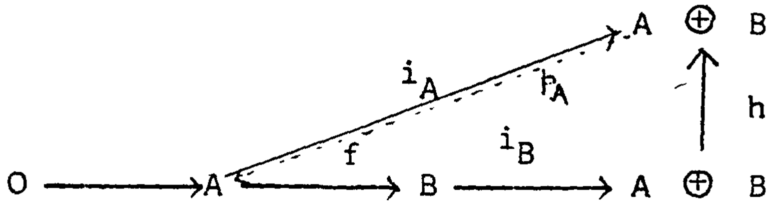
1.7.26.REMARKS(I) Let R be a ring, then R_R quasi-injective implies that R is a *right self-injective ring*.

(II) The fact that $M = \mathbb{Q} \oplus (\mathbb{Z}/p\mathbb{Z})$ is not quasi-injective can be deduced from 1.7.25. If M is quasi-injective over $\mathbb{Z}$ then M is injective over $\mathbb{Z}$ and so $\mathbb{Z}/p\mathbb{Z}$ is injective over $\mathbb{Z}$, which is impossible.

We know that R is semi-simple if and only if every left R-module is injective (projective). In the following Propositions we shall generalise this to quasi-injective (quasi-projective) modules.

1.7.27.PROPOSITION. If $A \oplus B$ is quasi-injective and $f : A \longrightarrow B$ is a monomorphism then f splits.

<u>Proof</u>. Consider the diagram (with usual notation)



Since $A \oplus B$ is quasi-injective there exists a map $h : A \oplus B \longrightarrow A \oplus B$ extending f.

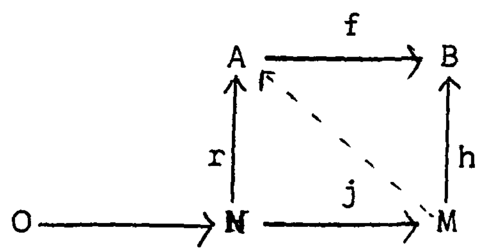
i.e. $hoi_B of = i_A$.

Hence $p_A ohoi_B o\ f = p_A o\ i_A$

or $(p_A ohoi_B)of = 1_A$. Hence f splits.

1.7.28. COROLLARY. [RV:72, Lemma 3.2']. If $A \oplus B$ is quasi-injective, B is injective and $f:A \longrightarrow B$ is a monomorphism then A is injective.

Proof. Since $A \oplus B$ is quasi-injective and $f:A \longrightarrow B$ is a monomorphism the map f splits (by 1.7.27), i.e. There exists a map $g:B \longrightarrow A$ Such that $gof = 1_A$. Now, let us consider the following diagram with row exact



Since B is injective there exists $h : M \longrightarrow B$ extending $for:N \longrightarrow B$.

Thus, we have $for = hoj$.

Let $q = goh:M \longrightarrow A$

Then $qoj = gohoj = gofor = 1_A or = r$, which shows that A is injective.

Dually we have the following results:

1.7.29. PROPOSITION. If $A \oplus B$ is quasi-projective and $f: B \longrightarrow A$ is an epimorphism then f splits.

1.7.30. COROLLARY [RV:72, Lemma 3.2]. If $A \oplus B$ is quasi-projective, B is projective and $f: B \longrightarrow A$ is an epimorphism then A is projective.

1.7.31. REMARK. A, B quasi-projective does not imply that $A \oplus B$ is quasi-projective.

For example, Let p be a prime number. If $\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})$ is quasi-projective/$\mathbb{Z}$, it will follow from 1.7.30 (on considering the canonical quotient map $\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$) that $\mathbb{Z}/p\mathbb{Z}$ is projective/$\mathbb{Z}$, which is absurd. This shows that direct sum of quasi-projective modules need not be a quasi-projective module.

1.7.32. PROPOSITION. R is semi simple if and only if every left R-module is quasi-injective.

Proof. ($\Rightarrow$) Trivial

($\Leftarrow$) Let us consider any R-module M. Then M can be embedded in an injective R-module E. $M \oplus E$ is quasi-injective (by hypothesis). Hence by 1.7.28 M is an injective R-module. So, R is semi-simple.

Dually we have:

1.7.33. PROPOSITION. R is semi-simple if and only if every left R-module is quasi-projective.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Let us consider an R-module M. Then there exists a free R-module F with an R-epimorphism $p : F \longrightarrow M$. Now $F \oplus M$ is quasi-projective (by hypothesis). Hence by 1.7.30 ${}_RM$ is projective. So, R is semi-simple.

1.7.34. PROPOSITION. Submodules of quasi-injective are quasi-injective if and only if every module is quasi-injective.

Proof. ($\Leftarrow$) Trivial.

($\Rightarrow$) Let ${}_RA$ be any module. Then ${}_RA$ can be embedded in an injective module ${}_RE$. Now E is injective, hence quasi-injective. Hence A is quasi-injective (by hypothesis).

1.7.35. PROPOSITION. R is semi-simple if and only if every quasi-projective left R-module is projective.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) We know that all simple modules are quasi-projective So, by hypothesis, all simple modules are projective. Hence R is semi-simple.

In the remainder of this section we shall state two theorems without proof.

Using the standard techniques of localisation of rings and modules Faticoni has proved the Theorem 1.7.36 stated below :

1.7.36. THEOREM [F:84, Theorem 2] Let C be a commutative ring and let R be a C-algebra. Let M be a finitely presented R-module. Then M is quasi-projective R-module if and only if for each prime (maximal) ideal I of C, M_I is a quasi-projective R_I- module.

In [FR:70] Fuchs and Rangaswamy explicitly described the quasi-projective abelian groups in the following theorem :

1.7.37. THEOREM [FR:70, Theorem] An abelian group A is quasi-projective if, and only if, it is (i) free or (ii) a torsion group such that every p-component A_p is a direct sum of cyclic groups of the same order p^n.

....

§ 1.8. The Π- and Σ- properties

Let M be a right R-module. We shall say M is Π-x (Σ-x) if every direct product (sum) of copies of M has property x. Injective modules are Π-injective by 1.2.23. Similarly, the terms Σ-projective and Σ-flat are clearly redundant. This section will be devoted to a survey of the research on Π- and Σ-properties in the context of quasi-injectivity and quasi-projectivity that has been carried out in the last three decades.

In [CR:71] Colby and Rutter gave a characterization of Π-flat modules which follows the characterization of rings for which direct products of flat modules are flat given by Chase in [C:60] They also considered Π-projective modules. However, it will take us too far a field to consider Π-flatness and Π-projectivity here.

Faith characterised Σ-injective modules in [F:66] Fuller ([F:69]) and [F:70]) studied Π- Σ-quási-projectives and quasi-injectives. These concepts were also considered by Hill [H:74] and Rangaswamy [R:79].

We begin with Faith's theorem on Σ-injective modules.

For a module $_RM$ left ideals of the form $l_R(X)$ (X a subset of M) will be called M-annihilator left ideals.

1.8.1. THEOREM (Faith) The following conditions on an injective R-module M are equivalent.

(i) M is Σ-injective.

(ii) The M-annihilator left ideals satisfy the ascending chain condition.

(iii) $M^{(\mathbb{N})}$ is injective.

Proof. See [AF, Theorem 25.1] or [F:66].

Next we shall use the above Theorem to derive a result of Faith and Walker.

1.8.2. THEOREM (Faith and Walker). If every injective module is Σ-injective then R is left noetherian. (Note that the converse follows by a theorem of Bass (§1.6 Introduction).

Proof. Let $\mathcal{S}$ be the family of all left ideals of R. Consider $M = \bigoplus_{A \in \mathcal{S}} R/_A$. Then if $E = E(M)$ the set of E-annihilator left ideals is precisely $\mathcal{S}$. As E is Σ-injective, it follows by 1.8.1 that $\mathcal{S}$ satisfies the ascending chain condition. So R is left noetherian.

We shall now consider Π-quasi-injective modules. Clearly

M-injective implies M^J injective for each indexing set J (by 1.2.23) and hence M must be π-quasi-injective. In the Theorem that follows we shall prove that ${}_RM$ is π-quasi-injective if and only if M is injective over $R/l_R(M)$. In the following Remarks we shall record some results we need.

1.8.3. REMARKS (I) Let us recall the following notation. For a module ${}_RM$ and an ideal I of R, by $r_M(I)$ we mean the subset $\{x \in M / Ix = 0\}$ of M. Notice the following :

for $x \in r_M(I)$, $r \in R$, $Irx \subset Ix = 0$.

This implies that $rx \in r_M(I)$. Thus $r_M(I) \leqslant {}_RM$.

(II) We shall show that $r_M(I)$ is invariant under all endomorphisms of M. For, let $f:M \longrightarrow M$ be any R-linear map. Let $x \in r_M(I)$. Then $Ix = 0 \Rightarrow f(Ix) = 0$ i.e. $If(x) = 0$ i.e. $f(x) \in r_M(I)$. So $f(r_M(I)) \leqslant r_M(I)$.

(III) If E is injective R-module then $X = r_E(I)$ is quasi-injective. For, $f(X) \leqslant X$, whenever $f \in \mathcal{E}nd_R(E)$ (using Remark II). This shows that X is invariant under all endomorphisms of E. Hence by 1.7.7 X is quasi-injective over R.

1.8.4. THEOREM. The following conditions are equivalent for a left R-module M.

(i) M is Π-quasi-injective

(ii) M is injective over $\bar{R} = R/l_R(M)$

(iii) $M = r_{E(M)}(I)$ for some ideal I in R.

Proof. (i) $\Rightarrow$ (ii) Consider the map $f : \bar{R} \longrightarrow M^M$ defined by $f(s) = (sm)_{m \in M}$. The map f is a monomorphism as M is $\bar{R}$-faithful. Thus there exists an indexing set J such that $\bar{R} \longrightarrow M^J$ is a monomorphism. So, M^J is quasi-injective over R i.e. M^J is injective over $\bar{R}$ (by 1.7.12). Hence M^J is injective over $\bar{R}$ (by 1.7.30). This implies that M is injective over $\bar{R}$.

(ii) $\Rightarrow$ (i) Let $M_o = M^J$, J any indexing set. Now M is injective over $\bar{R}$ implies that M_o is injective over $\bar{R}$ which in turn implies that M_o is quasi-injective over $\bar{R}$. Hence, M_o is quasi-injective over R (by 1.7.12). This shows that M is Π-quasi-injective over R.

(iii) $\Rightarrow$ (i) $M = r_{E(M)}(I)$. So $M^J = r_{E(M)^J}(I)$
Since $E(M)^J$ is injective over R, $r_{E(M)^J}(I)$ is quasi-injective over R (by 1.8.3). So M^J is quasi-injective over R. Thus M is Π-quasi-injective.

(ii) $\Rightarrow$ (iii) Let $Q = r_{E(M)}(l_R(M)) \subset E(M)$.

Clearly if $m \in M$, $x \in l_R(M)$ Then $xm = 0$. So $m \in Q$ which implies $M \leqslant Q$. Also Q is an essential extension of M with same annihilator (For $xM = 0 \Rightarrow x \in l_R(M) \Rightarrow xQ = 0$). Thus Q is an $\overline{R}$-module, M is essential in Q and M is $\overline{R}$-injective. Hence $M = Q$.

So $M = r_{E(M)}(I)$ for some ideal I of R.

1.8.5. REMARK. Take $R = \mathbb{Z}$, $M = \mathbb{Z}/p\mathbb{Z}$, where p is a prime. Then $l_R(M) = p\mathbb{Z}$. As M is injective over $\mathbb{Z}/p\mathbb{Z}$, M is π-quasi-injective over $\mathbb{Z}$ by Theorem 1.8.4. However, M is not injective over $\mathbb{Z}$.

1.8.6. REMARK. Tisseron [T:70] has proved that if every quasi-injective left R-module is π-quasi-injective then $R/RadR$ is a finite-dimensional left V-ring. Fuller [F:72, Theorem 3.3] has proved that if every semi-simple left R-module is π-quasi-injective then R must be a left noetherian left V-ring (Cf. [H:74, Theorem 1.5]).

Now we shall record some results on Σ-quasi-injective modules.

1.8.7. REMARK (I) Clearly if M is semi-simple then $M^{(J)}$ is semi-simple and so quasi-injective for every indexing set J. Thus M is Σ-quasi-injective.

(II) Suppose R is a left noetherian ring and M an injective left R-module Then M is Σ-injective (see introduction to 1.6) and a fortiori, Σ-quasi-injective Thus over left noetherian rings injective modules are necessarily Σ- quasi-injective.

(iii) Conversely, if every injective module is Σ-quasi-injective then R is left noetherian: For let E be any injective left R-module and E(R) an injective module containing R.Let $E_o = E(R) \oplus E$. Then E_o (being injective) is Σ-quasi-injective, by hypothesis. Let A be any indexing set. Then $E_o^{(A)}$ is quasi-injective and so injective (for $A = \emptyset$. This is trivial; for $A \neq \emptyset$ This follows by 1.7.30 since E_o and so $E_o^{(A)}$ contains a copy of E(R) and hence of R.) Since $E_o^{(A)} = E(R)^{(A)} \oplus E^{(A)}$ it follows that $E^{(A)}$ is injective. Hence by the theorem of Faith and Walker (1.8.2) R must be left noetherian.

(iv) A ring R is left noetherian if and only if every quasi-injective module is Σ-quasi-injective;"If" follows from (III). "Only if": assume that R is left noetherian and M is quasi-injective, then M is M-injective and the condition in Theorem 1.6.6. is satisfied. Hence $M^{(A)}$ is M-injective for each indexing set A. So $M \in C^{i}(M^{(A)})$ implying, by 1.2.21 that $M^{(A)} \in C^{i}(M^{(A)})$ i.e. $M^{(A)}$ is quasi-injective.

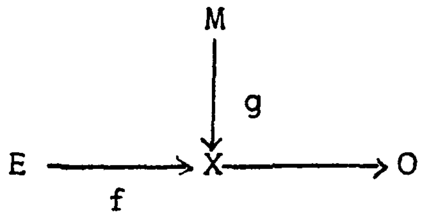
(v) The following theorem [H:74, Theorem 1.4] improves (IV): A ring R is left noetherian if and only if for every pair M, N of left R-modules and every set A, M is N-injective implies $M^{(A)}$ is $N^{(A)}$-injective. This can be proved using Theorem 1.6.6.

If P is a projective module, Then by 1.2.24 $P^{(J)}$ is also projective for each indexing set J. Hence projective modules are clearly Σ-quasi-projective. If M is semi-simple then for each indexing set J, $M^{(J)}$ is semi-simple and so quasi-projective. Thus M is Σ-quasi-projective. Thus Σ-quasi-projective modules need not be projective, as shown by examples of simple non-projective modules.

The following result gives a sufficient condition for Σ-quasi-projective modules to be projective.

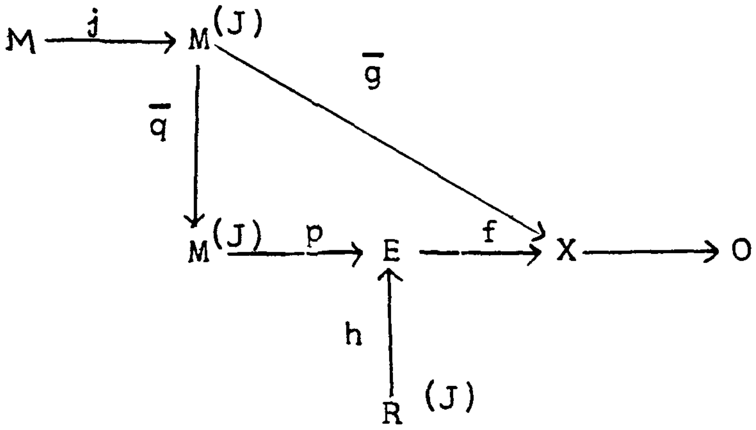
1.8.8. PROPOSITION. If M is a Σ-quasi-projective module and M contains a sub-module isomorphic to ${}_RR$ Then M is projective.

<u>Proof</u>. Consider the row-exact diagram with E injective:



(I)

Now there exists an indexing set J such that $R^{(J)} \xrightarrow{h} E$ is a surjection. Since, ${}_RE$ is injective this gives a map $M^{(J)} \xrightarrow{p} E$ which is also a surjection. Now consider the diagram (where j and $\bar{q}$ are defined below)



(II)

As M is Σ-quasi-projective $M^{(J)}$ is quasi-projective. So there exists a map $\bar{q} : M^{(J)} \longrightarrow M^{(J)}$ satisfying $fop o\bar{q} = \bar{g}$. As $\bar{g}oj = g$ where $j : M \longrightarrow M^{(J)}$ identifies M with a copy of M in $M^{(J)}$. Now we have $fopo\bar{q}oj = \bar{g}oj = g$.

Thus $po\bar{q}oj : M \longrightarrow E$ is the required map which makes the diagram (I) commute. Hence M is projective, by 1.2.15.

We conclude this section by stating without proof the following result of Fuller and Hill concerning π-quasi-projectives.

1.8.9. THEOREM ([FH:70, Corollary 2.2]) If the left R-module $_RR$ is π-quasi-projective, then every direct product of projective left R-modules is projective.

CHAPTER – II

HOPFIAN AND COHOPFIAN MODULES

§ 2.1. Introduction. Let V be a finite-dimensional vector space. It is well-known that a linear map $f:V \longrightarrow V$ is one-to-one if and only if it is onto. This fact has led to the introduction of the following concepts: a module M is cohopfian [B:76] if every one-to-one endomorphism of M is an automorphism; a module is hopfian ([B:76], [HM:86] [H:86]) if every onto endomorphism of M is an automorphism. Noetherian modules are hopfian and artinian modules are cohopfian. Orzech [O:71] considered the following property stronger than hopfianness: for each submodule N of M every epimorphism $N \longrightarrow M$ is an isomorphism. He showed that noetherian modules satisfy this stronger condition.

In § 2 of this chapter we fix conventions and state a number of results, most of them essentially known. In § 3 we prove some results involving character modules, change of rings, localisation and direct sums. In § 4 we consider Orzech's property and its dual; we also introduce a number of related properties. Of these "epimorphism properties" are studied in § 5 and "monomorphism properties" in § 6. In the final section we give examples of commutative rings with non-hopfian ideals, and thereby answer a query in [H: 86]. This leads us to a study of rings all of whose left ideals are hopfian (cohopfian). We show, inter alia, that strongly regular rings have both these properties (on left as well as on right).

§2.2. Known Results. For ease of reference we collect a number of definitions and results. A module is called directly finite if it has no proper direct summand isomorphic to it ([G, p.165], [P: 75])

2.2.1. REMARK. *Hopfian (cohopfian) modules are directly finite.*

2.2.2. PROPOSITION. ([G , Lemma 5.1])

A module M is directly finite if and only if $fg = 1_M$ implies $gf = 1_M$ in $\mathrm{End}_R(M)$.

2.2.3. COROLLARY. R_R is directly finite if and only if $xy = 1$ implies $yx = 1$ in R.

If R satisfies the left-right symmetric condition of Corollary 2.2.3 R is called a directly finite ring (or "von Neumann finite ring" [P: 75] or "Dedekind finite ring" [F II, p.85]).

2.2.4. PROPOSITION. (Utumi [U: 65]) If R is left and right self-injective then R is directly finite.

Recall that for a module M, and natural number n, $M^{(n)}$ denotes the direct sum of n copies of M. Proposition 2.2.2 yields:

2.2.5. PROPOSITION. Let n be a natural number. The right R-module $R^{(n)}$ is directly finite if and only if $M_n(R)$ is a directly finite ring.

2.2.6. PROPOSITION. Let M be a quasi-injective, directly finite module. Then M is cohopfian.

Proof. Let f : M ⟶ M be a monomorphism. As M is quasi-injective, there exists g : M ⟶ M such that gof = 1_M. Proposition 2.2. implies that fog = 1_M and so f is an epimorphism.

2.2.7. COROLLARY. [B:76, Corollary 2] If M is quasi-injective and hopfian then it is cohopfian.

The following are the duals of 2.2.6 and 2.2.7.

2.2.8. PROPOSITION. Let M be a quasi-projective, directly finite module. Then M is hopfian.

2.2.9. COROLLARY. [B:76, Remark on p.102] If M is quasi-projective and cohopfian then it is hopfian.

A module $_RM$ is called p-injective if for each principal left ideal I of R every R-homomorphism f: I ⟶ M can be extended to a R-homomorphism

$g: R \longrightarrow M$. Left self-injective rings and (von Neumann) regular rings are left p-injective, i.e., the module ${}_RR$ is p-injective.

2.2.10. PROPOSITION. ([cf. B:76, Remark on p.103]) Let R be a directly finite, left p-injective ring. Then ${}_RR$ is cohopfian.

Proof. Analogous to that of Proposition 2.2.6.

2.2.11. REMARK. It follows from Propositions 2.2.4 and 2.2.10 that if R is left and right self-injective then ${}_RR$ and R_R are cohopfian.

2.2.12. REMARK. It follows from Remark 2.2.1 and Proposition 2.2.8 that the conditions (a) R is a directly finite ring (b) ${}_RR$ is hopfian and (c) R_R is hopfian are equivalent.

A special case of Corollary 2.2.9. is

2.2.13. REMARK. If ${}_RR$ is cohopfian then it is hopfian.

More examples are given in

2.2.14. PROPOSITION. (I) Noetherian modules are hopfian ([L , p.23]; [AF, p.138]). (II) Artinian modules are cohopfian (Loc.cit. in Remark (I)). (III) If M is a finitely generated module over a commutative ring then M is hopfian (a theorem of Vasconcelos [V: 69]).

2.2.15. EXAMPLE. The 2-Prufer group $\mathbb{Z}(2^{\infty})$ is an example of an artinian, and therefore cohopfian, $\mathbb{Z}$-module which is not hopfian, as multiplication by 2 is an onto endomorphism, but not an isomorphism.

Let us recall some (more or less) standard terminology: an ideal means a two-sided ideal; a ring is called left (right) duo if every left (right) ideal is an ideal; it is called left(right) quasi-duo if every maximal left (right) ideal is an ideal; a duo (quasi-duo) ring means a left or right duo (quasi-duo) ring. By an abelian ring we mean a ring such that every idempotent is central, by a local ring a ring such that R/Rad R is a division ring. A local ring is left and right quasi-duo.

2.2.16. REMARKS. (I) Suppose xy = 1 in a left duo ring R. Then 1 belongs to the ideal Rx and so 1 = zx for some z. Then y = z and so yx = 1. So R is directly finite. This result will be extended in section 2.5.

(II)R is directly finite if and only if R/RadR is so. [F.II, p.85].

(III)It is a consequence of (II) that local rings are directly finite.

The next result extends remarks 2.2.16(I) and (III). It may be known but we could not find a reference in the literature.

2.2.17. PROPOSITION. A quasi-duo ring R is directly finite.

<u>Proof</u>. By Proposition 4.4 in [R:86] R/Rad R is reduced and, in fact, a subring of a product of division rings. So R/Rad R is directly finite. Hence by Remark 2.2.16(II) R is directly finite.

A ring R is called left π-regular if for each element a of R there exists an integer $n \geq 1$ and an element b of R such that $b.a^{n+1} = a^n$. Right π-regular rings are defined similarly. In [D:76] Dischinger attributes this definition to Azumaya . Dischinger defines a ring to be strongly π-regular if it is left as well as right π-regular.

2.2.18. REMARK. Left π-regularity is equivalent to the d.c.c. on principal left ideals Ra^n, $n \geq 1$, for each a in the ring.

2.2.19. REMARK. Right perfect rings satisfy d.c.c. on principal left ideals [AF, § 28] and so are left π-regular.

2.2.20. REMARK. Recall that a ring R is called strongly regular if for each element a of R there exists an element b such that $a = a^2b$. Strong regularity is a left-right symmetric condition [St.,p.40]. Clearly, strongly regular rings are strongly π-regular (with n = 1).

2.2.21. PROPOSITION. Let R be commutative. Then R is strongly π-regular if and only if R is zero-dimensional.

The above proposition was proved by Storrer [S:68, Lemma 5.6]. The result below was stated without proof in [D:76]. However, a modification of the argument in [S:68, Lemma 5.6, (e) $\Rightarrow$ (a)] yields a proof.

2.2.22. PROPOSITION. A ring R is right π-regular if and only if every cyclic right R-module is cohopfian.

2.2.23. THEOREM. [D:76, Theorem 1]. A ring R is left π-regular if and only if it is right π-regular.

2.2.24. COROLLARY. Every cyclic left R-module is cohopfian if and only if every cyclic right R-module is cohopfian.

§ 2.3 Hopfian and Cohopfian modules

In this section we record some results on hopfian and co-hopfian modules and answer one of Hiremath's queries.

Recall that if M is a right R-module then its character module is defined to be the left R-module $M^{*} = \mathrm{Hom}_{Z}(M, Q/Z)$. It is well known that the sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

of right R-modules is exact if and only if the sequence

$$0 \longrightarrow M_3^{*} \longrightarrow M_2^{*} \longrightarrow M_1^{*} \longrightarrow 0$$

of left R-modules is exact. ([L, p.127], [R, p.57])

An application of this result will yield

2.3.1. PROPOSITION. Suppose ${}_R M^{*}$ is hopfian, then M_R is co-hopfian.

<u>Proof</u>. Let $f: M \longrightarrow M$ be a monomorphism. Consider the exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \xrightarrow{g} M/f(M) \longrightarrow 0$$

where g is the natural quotient map.

Then

$$0 \longrightarrow (M/f(M))^{*} \xrightarrow{g^{*}} M^{*} \xrightarrow{f^{*}} M^{*} \longrightarrow 0$$

is also exact.

As M^{*} is hopfian f^{*} is an isomorphism implying that $(M/f(M))^{*} = 0$. Therefore, $M/f(M) = 0$ and so f is an epimorphism.

Dually, we have

2.3.2. PROPOSITION. Suppose ${}_R M^*$ is co-hopfian. Then M_R is hopfian.

Proof. Dual to that of proposition. 2.3.1.

Next we study the situation under a change of rings. Let $\theta : R \longrightarrow T$ be a unitary ring homomorphism. Let M be a left T-module. Regard M as a left R-module via θ.

2.3.3. PROPOSITION. (I) ${}_R M$ cohopfian implies that ${}_T M$ is cohopfian.

(II) ${}_R M$ hopfian implies that ${}_T M$ is hopfian.

(III) If θ is onto then the converses of (I) and (II) hold.

2.3.4. EXAMPLE. The converses of (I) and (II) in 2.3.3. do not hold generally. Let R be a field and $S = \prod_{\mathbb{N}} R_i$ where each $R_i = R$. Regard R as a subring of S via the map $a \longmapsto (a, a, a, \ldots)$. As S is a regular ring, by 2.2.10 and 2.2.12, S is cohopfian and hopfian over itself. As S is an infinite-dimensional R-vector space it is neither hopfian nor cohopfian over R.

In the next Proposition we use the terminology of [St., Chapter 2].

2.3.5. PROPOSITION. Let S be a right denominator set in a ring R. Let M be a S-torsion-free right R-module. If $M[S^{-1}]$ is hopfian as a right $R[S^{-1}]$-module then M is hopfian as an R-module.

Proof. Let $f: M \longrightarrow M$ be an R-epimorphism with kernel K. As $R[S^{-1}]$ is flat as a left R-module, the induced map $M[S^{-1}] \longrightarrow M[S^{-1}]$ is an epimorphism. The hypothesis then implies that this map is an isomorphism. So its kernel $K[S^{-1}] = 0$. As K is S-torsion-free it follows that K = 0.

2.3.6. REMARK. The first example of a domain T such that $M_2(T)$ is not a directly finite ring is generally attributed to Shepherdson [S:51] (See also [H, pp.33-34]). It follows that $T \oplus T$ is not a d.f. (left) T-module. and hence not a hopfian T-module. However, since T is a domain $_TT$ is a hopfian module. This answers a query in [H:86, observation after Proposition 6]: whether M,N hopfian implies that $M \oplus N$ hopfian. The infinite analogue of this question [H:86, same observation] has trivially a negative answer: an infinite dimensional vector space, a non-hopfian module, is a direct sum of 1-dimensional spaces.

§ 2.4. Orzech's condition and related properties

Orzech's results (Propositions 2.5.1 and 2.5.2 below) suggest the introduction of a property (called as E_1 by us) stronger than hopfianness. This property has a natural dual, M_1. In the definitions below we shall introduce a number of related properties. For a fixed module U, $\underline{C}$ will denote a class of submodules of U.

2.4.1. DEFINITIONS. A module U has property $E_1\underline{C}$ (Respectively, $E_2\underline{C}$, $E_3\underline{C}$) if, whenever $f: W \longrightarrow U$ is an epimorphism with W belonging to $\underline{C}$ then f is an isomorphism (Resp. $W=U$, $W \xrightarrow{\sim} U$).

2.4.2. DEFINITIONS. A module U has property $M_1\underline{C}$ (Respectively, $M_2\underline{C}$, $M_3\underline{C}$) if, whenever $f: U \longrightarrow U/W$ is a monomorphism with W belongig to $\underline{C}$ then f is an isomorphism (Resp. $W = 0$, $U/W \xrightarrow{\sim} U$).

2.4.3. NOTATION. When $\underline{C}$ denotes the class of all submodules of a module we shall drop the suffix $\underline{C}$ and talk simply of properties E_1 , M_1 , E_2 etc.

In the rest of this section we shall record some immediate consequences of these definitions. No conditions on the rings are assumed in any of these results.

2.4.4. REMARK. Each of the properties $E_1\underline{C}$ and $E_2\underline{C}$ implies $E_3\underline{C}$. Similarly, $M_1\underline{C}$ and $M_2\underline{C}$ each implies $M_3\underline{C}$.

2.4.5. REMARK. If a hopfian module has $E_2\underline{C}$ then it has $E_1\underline{C}$. If a cohopfian module has $M_2\underline{C}$ then it has $M_1\underline{C}$.

2.4.6. REMARK. Let $\underline{D}$ denote the direct summands of U. Then each of the conditions $E_2\underline{D}$ and $M_2\underline{D}$ is equivalent to the direct finiteness of the module U.

2.4.7. REMARK. Let X = E_1, E_2, M_1 or M_2. If a direct sum $U \oplus V$ of modules has property X then each of them has property X.

2.4.8. REMARK. Every module of finite length has each of the properties E_1, E_2, M_1, M_2.

Next we extend Corrollary 2.2.9.

2.4.9. THEOREM. Let $\underline{C}$ be a class of submodules of a cohopfian module U. Suppose that U is W-projective for each W in $\underline{C}$. Then U has properties $E_1\underline{C}$ and $E_2\underline{C}$.

Proof. Let $f: W \longrightarrow U$ epic, where $W \in \underline{C}$. Then by the W-projectivity of U we get a map $g: U \longrightarrow W$ such that $fog = 1_U$. Now $g(U) \leqslant W \leqslant U$ and g is monic. As U is cohopfian, we get $g(U) = W = U$. Thus g is an isomorphism and hence f also is an isomorphism. Thus U has the asserted properties.

2.4.10. COROLLARY. A quasi-projective, cohopfian module has properties E_1 and E_2.

Proof. Note that since U is U-projective U is W-projective for each submodule W of U (See Proposition 1.2.14).

This theorem has a dual :

2.4.11.THEOREM. Let $\underline{C}$ be a class of submodules of a hopfian module U. Suppose that U is U/W-injective for each $W \in \underline{C}$. Then U has properties $M_1\underline{C}$ and $M_2\underline{C}$.

Proof. Let $W \in \underline{C}$ and $f:U \longrightarrow U/W$ monic. Then we get a map $g:U/W \longrightarrow U$ such that $gof = 1_U$. Let $h:U \longrightarrow U/W$ be the natural quotient map. Then $goh:U \longrightarrow U$ is epic and so, as U is hopfian, monic. So h is monic, W = 0 and $h = 1_U$. Hence g and f are both isomorphisms.

2.4.12.COROLLARY. A quasi-injective, hopfian module has properties M_1 and M_2.

Proof. Note that since U is U-injective, U is U/W-injective for each submodule W of U (See Proposition 1.2.16).

2.4.13.NOTATION. In the next two propositions $\underline{F} = \underline{F}(U)$ denotes the class of fully invariant submodules of a module U. (See Definition 1.7.4).

2.4.14.PROPOSITION. A quasi-injective module has property $E_2\underline{F}$.

Proof. Let $W \in \underline{F}(U)$ and $f:W \longrightarrow U$ an epimorphism. As

U is quasi-injective there exists an endomorphism g of U extending f. As W is fully invariant in U, it follows that $U = f(W) = g(W) \leqslant W$. Hence $W = U$.

We have the following dual result.

2.4.15. PROPOSITION. A quasi-projective module has property $M_2\underline{F}$.

Proof. Let $W \in \underline{F}(U)$ and $f: U \longrightarrow U/W$ a monomorphism. Let $h: U \longrightarrow U/W$ be the natural quotient map. As U is quasi-projective there exists an endomorphism g of U satisfying $h \circ g = f$. As W is a fully invariant submodule of U we have $g(W) \leqslant W$. Hence $f(W) = h(g(W)) = 0$. Since f is a monomorphism we get $W = 0$.

§ 2.5. The Epimorphism properties

In this section we shall consider the "epimorphism properties" E_1, E_2, E_3, usually in the context of particular rings.

First, we recall Orzech's results. He extended 2.2.19(I) as follows:

2.5.1. PROPOSITION. [O:71, Lemma 1] Every noetherian module has property E_1.

He deduced, using the technique of noetherian reduction, the following extension of Vasconcelos' theorem.

2.5.2. PROPOSITION. [O:71, Theorem 1] Every finitely generated (f.g.) module over a commutative ring has E_1.

Two results obtained by Flanders in [F:67] follow as corollaries of 2.5.2.

2.5.3. COROLLARY. Let R be a commutative ring, M a free R-module of finite rank and N a free R-submodule of M. Then $\text{rank}_R N \leqslant \text{rank}_R M$.

2.5.4. COROLLARY. Let R be a commutative ring and M an R-module generated by n elements. Suppose N is a free submodule of M with $\text{rank}_R N = n$. Then M is a free R-module and $\text{rank}_R M = n$.

(There is an obvious misprint in the statement of this result in Orzech's paper which we have corrected.)

Next, we look at a few examples.

2.5.5. EXAMPLE. Let V be a vector space. Let W be a subspace of V and $f:W \longrightarrow V$ be an epimorphism. An application of Schröder-Bernstein theorem to the bases of V and W shows that $V \xrightarrow{\simeq} W$. So V has property E_3. (A similar argument shows that V has M_3.)

In the next two examples $\underline{P}$ denotes the set of principal left ideals of a ring R.

2.5.6. EXAMPLE. Let R be a non-zero domain which is not a division ring. Then there exists $a \in R$ such that $Ra \neq R$ and $Ra \xrightarrow{\simeq} R$. Thus ${}_RR$ does not have $E_2\underline{P}$.

2.5.7. EXAMPLE. Let R be a directly finite ring and $f:Ra \longrightarrow R$ an epimorphism. Then there exists b in R such that $1 = f(ba) = bf(a)$. As R is directly finite we have $f(a)b = 1$. Therefore, for each y in Ker(f) we have: $y = za$, $zf(a) = f(za) = 0$, and so $z = zf(a)b = 0$. Thus, f is an isomorphism and ${}_RR$ has $E_2\underline{P}$.

2.5.8. EXAMPLE. Let R be a left artinian ring. Then every f.g. left R-module M is artinian and noetherian and hence of finite length. So M has properties E_1 and E_2 (also M_1 and M_2).

In the rest of this section we shall be mainly interested in conditions under which ${}_RR$ has properties E_1 and E_2.

2.5.9. PROPOSITION. If ${}_RR$ is a cohopfian module, then ${}_RR$ has properties E_1 and E_2.

Proof. This is a special case of 2.4.10.

2.5.10. COROLLARY. Let R be a directly finite, left p-injective ring. Then ${}_RR$ has properties E_1 and E_2.

Proof. This follows from 2.2.10.

2.5.11. COROLLARY. Let R be a directly finite, regular ring. Then for each element a of R the left ideal Ra has properties E_1 and E_2.

Proof. In a regular ring every principal left ideal is a direct summand of ${}_RR$. So this follows from 2.5.10. and 2.4.7.

2.5.13. COROLLARY. Let R be right and left self-injective. Then ${}_RR$ has properties E_1 and E_2.

Proof. This follows from 2.5.10 and 2.2.4.

2.5.14 COROLLARY. Let R be strongly π-regular. Then ${}_RR$ has properties E_1 and E_2.

Proof. This follows from 2.2.22 and 2.2.23.

We shall denote the set of ideals of a ring R by $\underline{I}$; in notation 2.4.13 $\underline{I} = \underline{F}({}_R R) = \underline{F}(R_R)$. There is an analogy between quasi-injective modules and p-injective rings. This was noticeable in propositions 2.2.6 and 2.2.10; it is again evident on comparing 2.5.15(II) with 2.4.14. Part (I) of 2.5.15 is actually Corollary 2.5.10. However, we shall give a self-contained proof.

2.5.15. PROPOSITION. Let R be a left p-injective ring, A a left ideal of R and $f: A \longrightarrow R$ an epimorphism.

(I). If R is directly finite, then A = R (and f is an isomorphism).

(II). If $A \in \underline{I}$, then A = R. (Thus the module ${}_R R$ (for R a left p-injective ring) has $E_2\underline{I}$ property.)

<u>Proof</u>. Let g be a splitting of f and let a = g(1). Then the restriction h of f to Ra is an isomorphism from Ra to R. As R is a left p-injective ring the map h is given by the right multiplication by an element z of R. Hence we have 1 = h(a) = az.

Case (I). If R is directly finite, then $1 = za \in Ra$. So A = R. This proves part (I).

Case (II). If A is an ideal of R, then $1 = az \in A$. So A = R. This proves part (II).

Next we consider duo rings in the context of the property E_1; the E_2 property for ${}_RR$ does not hold for any commutative domain which is not a field. (2.5.6). We prove that if R is a left duo ring then every cyclic left R-module has property E_1. This will be deduced from a proposition giving sufficient conditions for a "duo" left ideal to have property E_1. The left annihilator of a subset S of R will be denoted by $l(S)$.

2.5.16. PROPOSITION. Let A be a left ideal satisfying $A \cap l(A) = 0$. Suppose that, considered as a ring (possibly) without identity, A is left duo. Then ${}_RA$ has property E_1.

<u>Proof</u>. Let $B \leqslant A$ and $f: B \longrightarrow A$ an epimorphism with kernel K. As A is left duo, $KB \leqslant KA \leqslant K$. Therefore, $KA = f(KB) = 0$. So we get $K \leqslant A \cap l(A) = 0$. Hence f is an isomorphism.

2.5.17. REMARK. We shall show in § 2.7 below that the annihilator condition in 2.5.16 cannot be omitted.

2.5.18. PROPOSITION. Let R be a left duo ring. Then ${}_RR$ has property E_1.

<u>Proof</u>. Take A = R in proposition 2.5.16.

2.5.19. THEOREM. Let R be a left duo ring. Then every cyclic left R-module has property E_1.

Proof. Let R be a left duo ring and M = R/B be a cyclic left R-module. Let N = A/B be a sub-module of M and let $f:N \longrightarrow M$ be an R-epimorphism. Then B is an ideal of R, the ring R/B is left duo and f is R/B-linear. So by 2.5.18 f is R/B -isomorphism. So M has E_1.

It follows from the above results that if R is either left noetherian (2.5.1), left duo (2.5.18), directly finite and left p-injective or strongly π-regular (2.5.10 and 2.5.14) then ${}_RR$ has E_1. An example of a ring R such that ${}_RR$ is hopfian but does not have E_1 must therefore lie outside these classes of rings. Such an example will be furnished in 2.5.22 below.

Recall that a domain R is called a left Ore domain if for all non-zero a, b in R we have $Ra \cap Rb \neq 0$ [St., p.52]. Commutative domains are trivially Ore. Left noetherian domains are left Ore. (See [St.,p.53]) In the theorem below we shall show that for domains the three properties (I)R is left Ore, (II)${}_RR$ has E_1, and (III)${}_RR$ has E_3 are equivalent.

2.5.20. LEMMA. Suppose a is a right non-zero divisor in a ring R such that ${}_RR$ is an indecomposable module with property E_3. Then Ra is an essential left ideal of R.

Proof. Let $Ra \cap Rb = 0$. As a is a right non-zero-divisor the mapping defined by $f(xa+yb) = x$ is an R-epimorphism from Ra+Rb onto R. So by property E_3 we have $R \xrightarrow{\sim} Ra \oplus Rb$. By the indecomposability of ${}_RR$, we get Rb = 0.

2.5.21. THEOREM. Let R be a domain. Then the following conditions are equivalent:

(1) The ring R is a left Ore domain.

(2) ${}_RR$ has property E_1.

(3) ${}_RR$ has property E_3.

Proof. Let R be a left Ore domain. Let A be a left ideal of R and $f: A \longrightarrow R$ be an epimorphism. Now, f splits and so $A = Ra \oplus B$. Since R is left Ore, Ra is essential in R. Hence B = 0 and f is an isomorphism. This proves (1) implies (2). The implication (2) implies (3) is trivial and it follows from the lemma above that if R is a domain having property E_3 then Ra is essential in R for each non-zero element a of R. This completes the proof.

2.5.22. EXAMPLE. We show by an example that ${}_RR$ hopfian does not imply that ${}_RR$ has E_1 and that (unlike hopfianness) R_R has E_1 does not imply that ${}_RR$ has E_1. Let $f: K \longrightarrow K$ be an endomorphism of a field K which is not onto. The skew polynomial ring $R = K_f[X]$ is a right ore domain which is not left Ore [St., p.53]. Thus R is a d.f. ring. R_R has E_1 but ${}_RR$ does not have E_1, by the above theorem.

§ 2.6. The monomorphism properties

In this section we shall consider the monomorphism properties M_1, M_2, M_3. After dualising a result of Orzech and proving some results for duo rings we shall prove a theorem showing the equivalence of various conditions for a left p-injective ring.

First, we shall dualise 2.5.1.

2.6.1. PROPOSITION. Every artinian module has M_1.

Proof. Let U be artinian, W a submodule of U and $f: U \longrightarrow U/W$ be a monomorphism. Suppose that $\operatorname{Im}(f) = T_1/W$ with $W \leq T_1 \leq U$. Let $f(T_1) = T_2/W$, say, and continuing define $f(T_i) = T_{i+1}/W$ Thus we get a decreasing sequence of submodules containing W :

$$T_1 \geqslant T_2 \geqslant \ldots\ldots T_i \geqslant T_{i+1} \cdots$$

As U is artinian this series must terminate. So, let $T_n = T_{n+1}$. Then $f(T_{n-1}) = f(T_n)$. As f is a monomorphism, we get $T_{n-1} = T_n$. Continuing this argument we finally get $U = T_1$, i.e., f is an epimorphism.

It is possible to give a direct proof of the next result. However, we shall derive it as a corollary of 2.4.15.

2.6.2. PROPOSITION. Let R be left duo. Then every cyclic left R-module has property M_2.

Proof. Let U be a cyclic left R-module. Then U = R/A where A is a left ideal and hence an ideal of R. It follows that U is quasi-projective over R. (see, e.g., 1.7.11 (IV) ≡ [RV : 72, Corollary to Lemma 4.1]). Let W be a submodule of U. Then W = B/A where B is a (left) ideal of R. Now each R-endomorphism of U = R/A is given by right multiplication by an element of R. Hence W is a fully invariant submodule of U. It now follows from 2.4.15 that U has property M_2.

2.6.3. COROLLARY. Suppose R is left duo and strongly π-regular. Then every cyclic left R-module has properties M_1 and M_2.

Proof. Let U be a cyclic left R-module, W a submodule of U and $f: U \longrightarrow U/W$ be a monomorphism. As U has property M_2 (by proposition), we have W = 0. By Proposition 2.2.22

U is cohopfian and so f is an isomorphism. Thus U has M_1 as well.

2.6.4. COROLLARY. Let R be a commutative zero-dimensional ring. Then every cyclic R-module has properties M_1 and M_2. (See 2.2.21)

Recall that if R is a regular ring then R is strongly regular if and only if R is left (or right) duo if and only if R is reduced ([St., p.40]).

2.6.5. COROLLARY. Let R be strongly regular. Then every cyclic left (or right) module has properties M_1 and M_2.

In the final result of this section we shall prove the equivalence of some E and M properties for a left p-injective ring.

2.6.6. THEOREM. Let R be a left p-injective ring. Then the following conditions are equivalent.

(I) ${}_RR$ has properties M_1 and M_2.

(II) ${}_RR$ is cohopfian.

(III) ${}_RR$ has properties E_1 and E_2.

(IV) ${}_RR$ is hopfian.

<u>Proof</u>. Since the E_1 property is stronger than hopfianness

and the M_1 property implies cohopfianness we get the implications (I) implies (II) and (III) implies (IV). (II) implies (III) holds for any ring by Proposition 2.5.9. Now, assume (IV) and let $f: R \longrightarrow R/A$ be a monomorphism, where A is a left ideal of R. Suppose $f(1) = \bar{a}$. As f is monic, it follows that $l(a) = 0$.(Here $l(a)$ denotes the left annihilator of a.) Since R is left p-injective, it follows (see, for example, theorem 1 in [IN:54]) that $aR = r(l(a)) = R$. As R is directly finite (2.2.12) we get $R = Ra$. Now let z be an element of A. Write $z = xa$. Then $f(x) = \overline{xa} = 0$ shows $x = 0$, and so $z = 0$. Thus $A = 0$ and ${}_RR$ has property M_2. As R is directly finite, left p-injective, by Proposition 2.2.10 the map f $(: R \longrightarrow R)$ must be an isomorphism. This completes the proof of the Theorem.

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§ 2.7. Rings with hopfian (cohopfian) ideals

In [H:86, Remark 7] Hiremath asks whether M hopfian and N a submodule of M implies that N is hopfian. We answer this question in the negative in 2.7.1. below.

2.7.1. REMARKS. (I). Let R be a ring and M a left R-module. Consider $S = R \oplus M$, made into a ring by defining $(a,m)(b,n) = (ab, an+bm)$; $(1,0)$ is the identity of this ring. Now M becomes an ideal of S through the embedding $j: m \longmapsto (0,m)$. The maps $f: R \longrightarrow S$, $f(a) = (a,0)$ and $h: S \longrightarrow R$, $h(a,m) = a$, are ring homomorphisms such that $h \circ f = 1_R$, the identity map of R. Clearly, S is directly finite if and only if R is.

Now any R-submodule U of M becomes an ideal of S via j. Moreover, for R-submodules U,V of M any R-linear map $g: V \longrightarrow U$ is also left and right S-linear. (Since S acts on U,V via the map h, for $(a,m) \in S$, $v \in V$ we have $g((a,m)v) = g(av) = ag(v) = (a,m)g(v)$.)

It follows that U has E_i as an R-module if and only if U has E_i as a left (or right) ideal.

(II). Consider the following special case of (I). Let M be a cohopfian R-module which is not hopfian, for example,

$M = \mathbb{Z}(p^{\infty})$ over the ring $\mathbb{Z}$. Then in the ring $S = \mathbb{Z} \oplus M$, M is a cohopfian ideal which is not hopfian. As ${}_S S$ is a hopfian module this answers Hiremath's query in the negative.

(III). Another special case of (I) is furnished by choosing R = K, a field and M an infinite-dimensional vector space. Then the ideal M in $S = K \oplus M$ is neither hopfian nor cohopfian. It follows that it does not have either E_1 or M_1. Notice that S is a local ring with a unique prime ideal M. As S is zero-dimensional every cyclic S-module is cohopfian, by 2.2.21 and 2.2.22. This example also shows that the annihilator condition in 2.5.16 cannot be omitted.

2.7.2.REMARK. Consider the question dual to that asked by Hiremath: are quotients of cohopfian modules necessarily cohopfian ?

The following example shows that the answer is negative. Let $R = \mathbb{Z}$ and $T = \prod \mathbb{Z}/p\mathbb{Z}$ where p varies over the set of all primes. It is known that $N = \oplus \mathbb{Z}/p\mathbb{Z}$ is the torsion sub-group of T and T/N is a torsion free and divisible group. Therefore, T/N is an infinite-dimensional vector space over $\mathbb{Q}$ and so is not cohopfian as a $\mathbb{Q}$-module and as a $\mathbb{Z}$-module. Now let $f: T \longrightarrow T$ be a $\mathbb{Z}$-monomorphism (or a $\mathbb{Z}$-epimorphism).

Consider $\mathbb{Z}/p\mathbb{Z}$ as a sub-group of T in a natural manner. It follows from the considerations of p-torsion that $f(\mathbb{Z}/p\mathbb{Z}) \subset \mathbb{Z}/p\mathbb{Z}$. for each prime p. Hence the restriction of f to each $\mathbb{Z}/p\mathbb{Z}$ is an isomorphism. So f itself is an isomorphism. So T is hopfian as well as cohopfian as a $\mathbb{Z}$-module.

Next,we give a sufficient condition for each left ideal of a ring R to be hopfian.

2.7.3. PROPOSITION. Let R be semi-prime, left duo (equivalently, reduced, left duo). Then every left ideal of R has property E_1 and so is hopfian.

Proof. Let A be a left ideal of R and let $T = A \cap l(A)$. Then $T^2 = 0$. As R is semi-prime, this implies that T = 0. So by Proposition 2.5.16 $_RA$ has property E_1.

2.7.4. REMARK. Neither semi-primeness (example in 2.7.1 (III)) nor left duo property (Example 2.5.22) can be omitted from the hypotheses of Proposition 2.7.3.

Our next aim is to show that if R is a left duo, left p-injective ring then every left ideal of R is cohopfian. We shall first introduce some terminology modifying Baer's completeness terminology. (See Section 1.1.)

2.7.5. DEFINITION. Let I be a left ideal of a ring R. We say a left R-module M is <u>I-pseudo-complete</u> if any R-monomorphism $I \longrightarrow M$ can be extended to a R-homomorphism $R \longrightarrow M$.

2.7.6. DEFINITION. Let S be a subset of a ring R. A left R-module M will be called <u>$\hat{S}$-pseudo-complete</u> if for each a in S, M is Ra-pseudo-complete.

2.7.7. PROPOSITION. Let A be an left ideal of R. Suppose that (I) every left ideal of R properly contained in A is an ideal of R and (II) ${}_RR$ is $\hat{A}$-pseudo-complete. Then ${}_RA$ is cohopfian.

<u>Proof</u>. Let $f: A \longrightarrow A$ be an R-monomorphism. Let, if possible, $a \in A$, $a \notin f(A)$. By hypothesis (I), f(A) is an ideal of R. Let $y = f(a)$. Let $h: Ra \longrightarrow Ry$ be the restriction of f to Ra. Then h is an isomorphism with inverse $k: Ry \longrightarrow Ra$, say. It follows by hypothesis (II) that k is given by right multiplication by an element z in R. So $a = k(y) = yz \in f(A)$, as f(A) is an ideal. This contradiction proves that f must be onto, i.e. an isomorphism.

2.7.8. COROLLARY. Let R be a left p-injective, left duo ring. Then each left ideal of R is cohopfian.

2.7.9. COROLLARY (Birkenmeier [B : 76]) Each ideal of a commutative self-injective ring is cohopfian.

For the definition of a strongly regular ring see 2.2.20.

2.7.10. THEOREM. Let R be a strongly regular ring. Then every left (right) ideal of R is hopfian and cohopfian.

Proof. It is known [St., p.40] that strongly regular rings are regular, reduced and left and right duo. So by Proposition 2.7.3 every left (right) ideal of R is hopfian. Regular rings are left and right p-injective. So by Proposition 2.7.8 every left (right) ideal of R is cohopfian as well.

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