# Generalization of Indirect Least Squares to Estimation of Structural Equations of a Multi-equation Linear Econometric Model

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**I. Introduction**: This paper has two objectives, (i) to generalize the method of Indirect Least Squares (ILS) and (ii) to show that such a generalization, call it the Generalized Indirect Least Squares (GILS), yields Two-Stage Least Squares (2-SLS) under the zero restriction on some structural coefficients characterizing over-identification.

**II. The Multi-equation Linear Econometric Model**: The equation system,

$$YA + XB + U = 0 \qquad \dots (1)$$

where, $Y_{n,m}$ = current endogenous variables;
$X_{n,k}$ = pre-determined (non-stochastic) variables;
$U_{n,m}$ = structural disturbances;
$A_{m,m}$ = structural coefficient matrix associated with Y, matrix A is assumed to be of full rank such that $A^{-1}$ exists, all the main diagonal elements of A are a-priori known to be –1, and other elements may be unknown or zero;
$B_{k,m}$ = coefficient matrix associated with X, some of the elements of B may be a-priori known to be zero and others are unknown;
m = number of current endogenous variables in the model;
k = number of pre-determined variables in the model;
n = number of observations (sample size);

is called a multi-equation linear econometric model.

**III. Problems in Estimation of A and B**: If n >k and X has a full column rank of k, it is possible to proceed to estimation of the coefficients A and B. However, estimation of A and B by Ordinary Least Squares (OLS) is prohibited due to the stochastic regressor problem. The current endogenous variables (Y) are random and correlated with the structural disturbances (U) as noted by Theil (p. 452). This fact defies the applicability of OLS (Intriligator, pp. 375-377).

To eliminate stochastic regressors from (1) we obtain the reduced form equations by post-multiplying (1) by $A^{-1}$. Thus,

$$YAA^{-1} + XBA^{-1} + UA^{-1} = 0, \text{ or } Y = X\Pi + V \qquad \dots (2)$$

where, $\Pi = -BA^{-1}$ and $V = -UA^{-1}$. The equation system (2) is amenable to estimation by OLS. We estimate $\Pi$ by P as

$$P = [X'X]^{-1}X'Y = X^{-g}Y = X^{+}Y \qquad \dots (3)$$

where $X^{+}$ is the Moore-Penrose inverse of X, which, due to X having a full rank of k, provides the least squares g-inverse of $X = X^{-g}$.

**IV. The Identification Problem**: Is it possible to obtain estimated A and B provided that only P is numerically known as in (3) ? Now, if all the columns of A and B can be obtained, then A and B also can be obtained completely. If a particular column of A and B (say $a_i$ and $b_i$) can be obtained from the relation

$$Pa_i = -b_i \qquad \ldots(4)$$

then equation i of (1) is identifiable. Since this is true of any equation in (1), and (4) holds for any particular column of A and B, we will drop the subscript i and rewrite (4) as

$$Pa = -b \qquad \ldots(5)$$

It is to be recalled that $P$, $a$ and $b$ are k x m, m x 1 and k x 1 matrices respectively. Moreover, the $i^{th}$ element of $a$ is $-1$ if (5) relates to equation i. It is obvious that (5) is in k equations and k+m-1 unknowns. So, as such, (5) cannot be solved for $a$ and $b$. Now, suppose, using a-priori information in (1) we obtain the values of some (say $k_2$) elements of $b$ and some other (say $m_2$-1, noting that the $i^{th}$ element is already known to be -1) elements of $a$. Then, out of k elements of $b$ only $k-k_2 = k_1$ elements are unknown. Similarly, out of m elements of $a$, now $m_2$ elements are known, while $m_1 = m-m_2$ elements are unknown. Thus we proceed to identify by restriction on the structural coefficient matrices, A and B.

Pre-multiplying (5) by a suitable permutation matrix $Q$ we may obtain

$$QPa = -Qb \qquad \ldots(6)$$

such that $Qb$ can be partitioned into two sub-matrices, first of which (say, $b_1$) has $k_1$ unknown elements and the second (say $b_2$) has $k_2$ known elements. Correspondingly, $QP$ can be partitioned into $P_1$ and $P_2$. Note that such a permutation only reshuffles the equations in (5) without any bearing on the solution. Thus, we may rewrite (6) in a partitioned form as

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} a = -\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}; Q^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b \qquad \ldots(7)$$

Note that $P_1$ is in $k_1$ rows and m columns, $P_2$ is in $k_2$ rows and m columns. Similarly, $b_1$ is a $k_1$ x 1 matrix while $b_2$ is a $k_2$ x 1 matrix.

Premultiplying $a$ by a suitable permutation matrix, S, and post-multiplying $QP$ by $S^{-1} = S'$ in equation (7) such that

$$Sa = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}; S^{-1}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a \qquad \ldots(7')$$

where, $a_1$ contains $m_1$ unknown elements and $a_2$ contains $m_2$ known elements. We may now rewrite (7) as

$$QPS^{-1}Sa = -Qb \qquad \ldots(8)$$

In partitioned form, we rewrite (8) as

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \ldots(9)$$

whence,

$$\begin{aligned} P_{11}a_1 + P_{12}a_2 &= -b_1 \\ P_{21}a_1 + P_{22}a_2 &= -b_2 \end{aligned} \qquad \ldots(10)$$

Now, since $a_2$ and $b_2$ are known, we get,

$$P_{21}a_1 = -[b_2 + P_{22}a_2] \qquad \ldots(11)$$

Note that $P_{21}$ is a $k_2$ x $m_1$ matrix giving rise to $k_2$ equations in $m_1$ unknowns as in (11).

**V. Indirect Least Squares and Exact Identification**: If $P_{21}$ in (11) is a square matrix and its rank is $m_1$ such that $[P_{21}]^{-1}$ exists, then,

$$a_1 = -P_{21}^{-1}[b_2 + P_{22}a_2] \qquad \ldots(12)$$

Once $a_1$ is known, substituting it into the second equation of (10), the elements of $b_1$ can be known. Solving (12) for $a_1$ and obtaining $b_1$ from (10) as mentioned, is called Indirect Least Squares due to MA Girshik (Hood and Koopmans, p. 140). This estimator exists iff $P_{21}^{-1}$ in (12) exists. Thus, the applicability of ILS requires that the equation concerned in (1) is just (or exactly) identified (Intriligator, p. 358). In other words, if $k_2 = m_1$ and rank$(P_{21}) = m_1$, then ILS estimator exists.

**VI. Generalization of Indirect Least Squares**: If $P_{21}$ is a rectangular (or square) matrix in $k_2$ rows and $m_1$ columns, and rank$(P_{21}) = r \le m_1$, then the Moore-Penrose inverse of $P_{21}$, That is, $P_{21}^{+}$ exists and is unique (Theil, pp. 268-273, Krishnamurthy & Sen, p. 183). Using this generalized inverse of $P_{21}$ we obtain from (10)

$$\begin{aligned} \hat{a}_1 &= -P_{21}^{+}[b_2 + P_{22}a_2] \\ \hat{b}_1 &= -[P_{11}\hat{a}_1 + P_{12}a_2] \end{aligned} \qquad \ldots(13)$$

This generalization and the consequent technique in (13) may be called GILS. The ILS is only a special case of GILS when $k_2=m_1=r$ and therefore, $P_{21}^{+} = P_{21}^{-1}$. However, if $k_2 \ge m_1$, and $r = m_1$ then $P_{21}^{+}$ yields the Least Squares g-inverse, $P_{21}^{-}$ or $P_{21}^{-g}$ (Theil, p. 271).

**VII. GILS in another Form**: It can be shown that (9) may be rewritten as

$$\begin{bmatrix} P_{11} & I \\ P_{21} & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = -\begin{bmatrix} P_{12}a_2 \\ b_2 + P_{22}a_2 \end{bmatrix} \qquad \ldots(14)$$

which may be rewritten as

$$\begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix}\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = -\begin{bmatrix} P_{12}a_2 \\ b_2 + P_{22}a_2 \end{bmatrix} \qquad \ldots(14')$$

From (14') we ob tain

$$\begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix}^{+}\begin{bmatrix} P_{12}a_2 \\ b_2 + P_{22}a_2 \end{bmatrix} = -\begin{bmatrix} \hat{b}_1 \\ \hat{a}_1 \end{bmatrix} \qquad \ldots(15)$$

It can be proved (as in the appendix) that

$$\begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix}^+ = \begin{bmatrix} I & -P_{11}P_{21}^+ \\ 0 & P_{21}^+ \end{bmatrix}$$

whence (15) may be rewritten as

$$\begin{bmatrix} I & -P_{11}P_{21}^+ \\ 0 & P_{21}^+ \end{bmatrix} \begin{bmatrix} P_{12}a_2 \\ b_2 + P_{22}a_2 \end{bmatrix} = -\begin{bmatrix} \hat{b}_1 \\ \hat{a}_1 \end{bmatrix} \qquad \ldots(16)$$

Therefore,

$$\hat{a}_1 = -P_{21}^+[b_2 + P_{22}a_2]$$

$$\hat{b}_1 = -[P_{11}\hat{a}_1 + P_{12}a_2]$$

which is the same as in (13).

**VIII. GILS and 2-SLS in Over-identified Case**: Equation system (14) can be rewritten as

$$\begin{bmatrix} P_{\cdot 1} & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix} d^* = -\begin{bmatrix} P_{12}a_2 \\ b_2 + P_{22}a_2 \end{bmatrix} = -P_{\cdot 2}$$

where,

$$P_{\cdot 1} = \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix}; \quad d^* = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}; \quad P_{\cdot 2} = \begin{bmatrix} P_{12}a_2 \\ b_2 + P_{22}a_2 \end{bmatrix} \qquad \ldots(17)$$

We assume that the elements of $b_2$ are known a-priori to be zero (zero restriction on $b_2$). Further, $a_2$ is a column vector whose first element is known to be –1 (normalization condition) and other elements ore zero (zero restriction on $a_2$). Under these conditions, we get from (17),

$$\begin{bmatrix} P_{\cdot 1} & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix} d^* = -P_{\cdot 2} \qquad \ldots(18).$$

From Theil (p. 459) we know that for any $i^{th}$ equation of (1)

$$X \begin{bmatrix} P_{i1} & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix} = [\hat{Y}_i \quad X_i] \qquad \ldots(19.1)$$

and again from Theil (p. 453) we know that

$$P_{i2} = -[X'X]^{-1}X'y_i = -X^+y_i \qquad \ldots(19.2)$$

From (19.1) we obtain

$$[X'X]^{-1}[X'X] \begin{bmatrix} P_{i1} & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix} = [X'X]^{-1}X'[\hat{Y}_i \quad X_i] \qquad \ldots(20)$$

From (18), (19.2) and (20) it follows that

$$[X'X]^{-1}X'[\hat{Y}_i \quad X_i]d^* = [X'X]^{-1}X' y_i \qquad \ldots(21)$$

Writing $Z_i = [\hat{Y}_i \quad X_i]$, we rewrite (21) as

$$[X'X]^{-1}X'Z_id^* = [X'X]^{-1}X'y_i \text{ or simply, } Z_id^* = y_i \qquad \ldots(22)$$

Whence (Intriligator, p. 390)

$$[Z_i]^+ y_i = d; \quad (d \text{ is the estimator of } d^*), \text{ or}$$

$$[Z_i'Z_i]^+ Z_i' y_i = d \qquad \ldots(23)$$

Now, provided that $Z_i$ has a full rank and, therefore, $[Z_i'Z_i]^+ = [Z_i'Z_i]^{-1}$, then we obtain Two-Stage Least Squares estimator of $d^*$. Thus, in case of over-identification, GILS yields 2-SLS estimator.

## Appendix

It can be shown that

$$\begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix}^+ = \begin{bmatrix} I & -P_{11}P_{21}^+ \\ 0 & P_{21}^+ \end{bmatrix} \qquad \ldots(\text{A -1})$$

where $[.]^+$ is the Moore-Penrose inverse of [.]. We show that

$$\begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix} \begin{bmatrix} I & -P_{11}P_{21}^+ \\ 0 & P_{21}^+ \end{bmatrix} \begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix} = \begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix} \qquad \ldots(\text{A -2})$$

or

$$\begin{bmatrix} I & 0 \\ 0 & P_{21}P_{21}^+ \end{bmatrix} \begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix} = \begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix} \qquad \ldots(\text{A -3})$$

or

$$\begin{bmatrix} I & P_{11} \\ 0 & P_{21}P_{21}^+P_{21} \end{bmatrix} = \begin{bmatrix} I & P_{11} \\ 0 & P_{21} \end{bmatrix} \qquad \ldots(\text{A -4})$$

Now, since $P_{21}P_{21}^+P_{21} = P_{21}$ (Theil, pp. 269-270), (A-4) is equal to (A-2) and hence the proof. It goes without saying that since the Least Squares g-inverse is only a special case of Moore-Penrose inverse, the proof given above is true of g-inverse, $[.]^{-g}$, as well.

## References


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