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(Γ, Γ')-FREE BORDISMS, CHARACTERISTIC NUMBERS AND STATIONARY POINT SETS

S. S. KHARE* (Shillong)

Introduction

C. N. Lee and Wasserman [7] developed the notion of characteristic numbers for G-manifolds and proved their G-bordism invariance. In [2] we defined characteristic numbers for an unoriented singular G-bordism and proved their invariance with regard to singular G-bordism. The case of oriented singular G-bordism is considered in [3] and [4]. One of our primary aims in this paper is to develop these notions for (Γ, Γ')-free singular bordisms, $\Gamma' \subset \Gamma$ being families of subgroups in a finite group G. (For definition see [6]). In [2] we tackled this problem for some special pairs of families (for so called "almost adjacent" pairs). In an effort to consider more general pairs of families, we get an analogue of Stong's result [6, Proposition 2] for finite abelian groups in §3. In this section we prove that if (M^n, θ) is a G-manifold with stationary point free induced action of the subgroup G_2, then (M^n, θ) is a G-boundary, G a finite abelian group. Lastly in §4 this analogue has been used to show that if the fixed point set F of G_2 in M^n is nonempty and if F has an equivariant trivial normal bundle in M^n, then (M^n, θ) is a G-boundary.

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Characteristic numbers for an almost adjacent pair (Γ, Γ')

Let G be a finite group and X be a G-space. Let h^* be an equivariant cohomology theory and h_* be the associated equivariant homology theory [1] given by $h^* = H^*A$ and $h_* = H_*A$, where A is a functor from the category of G-spaces and G-maps to the category of topological spaces and continuous maps, H^* is the singular cohomology theory and H_* is the associated singular homology theory. Let

$$\langle , \rangle \colon h^*(X;\, G) \underset{H^*(pt.)}{\otimes} h_*(X;\, G) \to H_*(pt.)$$

be the Kronecker pairing.

Suppose for each compact G-manifold W there exists a class $[W, \partial W] \in$ $\in h_*(W, \partial W\colon G)$ such that

a) $[W_1 \cup W_2, \partial W_1 \cup \partial W_2] = [W_1, \partial W_1] + [W_2, \partial W_2]$

and

b) $\partial_*[W, \partial W] = [\partial W]$.

Such an element $[W, \partial W] \in h_*(W, \partial W;\, G]$ is called a *topological class* of W.

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Following Stong [5], a family Γ in G, is a collection Γ of subgroups of G such that (i) $H\in\Gamma$ implies that all the subgroups of H also belong to Γ (ii) $H\in\Gamma$ implies that $gHg^{-1}\in\Gamma$, $\forall g\in G$. Let $\Gamma'\subset\Gamma$ be families in G such that each member of $\Gamma-\Gamma'$ is maximal in G. Such a pair (Γ, Γ') is called a pair of almost adjacent families. For any subgroup H of G, let $K=\frac{N}{H}$, N being the normalizer of H in G. Let $F_H(X)$ be the set of $x\in X$ such that $hx=x$, $\forall h\in H$. Consider the action of K on $F_H(X)$ by $(gH)x=gx$. Let EK be the total space of the universal K-bundle. For a pair (Γ, Γ') of almost adjacent families, consider the equivariant cohomology and equivariant homology

$$h^*(X;\,G) = \bigoplus_H H^*\big((EK\times F_H(X)/K;\ \mathbf{Z}_2\big)$$

$$h_*(X;\,G) = \bigoplus_H H_*\big((EK\times F_H(X)/K;\ \mathbf{Z}_2\big),$$

the summation is over the set of all representatives of the conjugacy classes of subgroups H in $\Gamma-\Gamma'$. Let X be a G-space and $[M^n, \partial M^n, \varphi, \theta, f]$ be an element of (Γ, Γ')-free bordism group $\varkappa_n(G;\Gamma,\Gamma')(X)$ [5]. Then

$$h_*(M^n;\,G) \approx \bigoplus_H \bigoplus_{k=0}^{n} H_*\big(F_H^k(M^n)/K;\ \mathbf{Z}_2\big),$$

where $F_H^k(M^n)$ is the union of k-dimensional submanifolds in $F_H(M^n)$. We define a topological class $[M, \partial M]$ of M^n in $h_*(M^n, \partial M^n; G)$ to be $\bigoplus_H \bigoplus_{k=0}^{n} \sigma_k^H$ where $\sigma_k^H\in H_k\big(F_H^k(M^n)/K;\mathbf{Z}_2\big)$ is the fundamental class of $F_H^k(M^n)/K$. Let $u\in h^*\big(X\times B(0, G)_n;\, G\big)$. Let $\tau_{M^n}\colon M^n\to B(0, G)_n$ be the tangent map.

DEFINITION 2.1. We define the *u-characteristic number* of an (Γ, Γ')-free element $(M^n, \partial M^n, \varphi, \theta, f)$ by $\langle (f\times\tau_{M^n})^*(u),\ [M, \partial M]\rangle\in\mathbf{Z}_2$.

Regarding the bordism invariance, we establish

THEOREM 2.2. *$[M^n, \partial M^n, \varphi, \theta, f]\in\varkappa_n(G;\Gamma,\Gamma')(X)$ is zero if and only if all the u-characteristic numbers (corresponding to the theory h^*) of the (Γ, Γ')-free element $(M^n, \partial M^n, \varphi, \theta, f)$ are zero.*

PROOF. The G-equivariant map $f\colon M^n\to X$ gives K-equivariant map $f\colon F_H^k(M^n)\to F_H(X)$. Let

$$\nu_k^H\colon F_H^k(M^n)\to F_H\big(B(0, N)_{n-k}\big)$$

be the normal map. In fact the image of ν_k^H will be contained in $F'_H\big(B(0, N)_{n-k}\big)$ the union of path components of $p\in F_H(B(0, N)_{n-k})$ for which the fibre $(\gamma^{n-k})_p$ at p contains no trivial H-representation, γ^{n-k} being the universal real N-vector bundle. Let $\alpha_k^H\colon F_H^k(M^n)\to EK$ be the cover of the classifying map for K-bundle $F_H^k(M^n)\to\big(F_H^k(M^n)\big)/K$. Let f_k^H be the map obtained from $\alpha_k^H\times(f\times\nu_k^H)$ on passing to quotients. This gives the map

$$\eta\colon \varkappa_n(G;\,\Gamma,\Gamma')(X)\to \bigoplus_H \bigoplus_{k=0}^{n} \varkappa_k\big((EK\times\{F_H(X)\times F'_H(B(0, N)_{n-k})\})/K\big)$$

defined by

$$\eta([M^n, \partial M^n, \varphi, \theta, f]) = \bigoplus_H \bigoplus_{k=0}^{n} [F_H^k(M^n)/K, f_k^H].$$

We know that η is an isomorphism [5] and thus $[M^n, \partial M^n, \varphi, \theta, f]$ is zero if and only if $[F_H^k(M^n)/K, f_k^H]$ is zero, $\forall k$ and H. Next the group $h^*(X\times B(0, G)_n; G)$ is isomorphic to

$$\bigoplus_H \bigoplus_{k=0}^{n} [H^*((EK\times\{F_H(X)\times F'_H(B(0, N)_{n-k})\})/K; \mathbf{Z}_2)\otimes H^*(B0_k; \mathbf{Z}_2)].$$

Also $(f\times\tau_{M^n})^* = \bigoplus_H \bigoplus_{k=0}^{n} (f_k^H\times\tau_k^H)^*$ where $\tau_k^H\colon F_H^k(M^n)/K\to B0_k$ is the tangent map. Thus the u-characteristic number

$$\langle (f\times\tau_{M^n})*(u), [M, \partial M]\rangle = \langle \bigoplus_H \bigoplus_{k=0}^{n} (f_k^H\times\tau_k^H)*(u_k^H), \bigoplus_H \bigoplus_{k=0}^{n} \sigma_k^H\rangle$$

where u_k^H is given by $u = \bigoplus_H \bigoplus_{k=0}^{n} u_k^H$. This together with the fact that $[M^n, \partial M^n, \varphi, \theta, f]$ is zero if and only if $[F_H^k(M^n)/K, f_k^H]$ is zero gives the theorem.

An analogue of Stong's result and characteristic numbers for a more general pair of families

So far we confined ourselves to a pair of almost adjacent families. In an effort to get rid of almost adjacent families as much as possible, we come across an analogue of Stong's result [6, Proposition 2] for general groups. For this let G be a finite abelian group and Π be the family of all subgroups of G. Let $\Gamma'\subset\Gamma$ be families in G such that there exists an element a in G of order 2 such that

(1) $H\in\Gamma'\Rightarrow[H\cup\{a\}]\in\Gamma'$,

(2) $a\notin H$, $\forall H\in\Gamma-\Gamma'$,

(3) the intersection S of all the members of $\Gamma-\Gamma'$ is in $\Gamma-\Gamma'$. We call such a pair of families an *admissible pair* with respect to $a\in G$.

EXAMPLE 3.1. Let G be a finite abelian group of even order given by $G^2\times H$, where H is a finite group of odd order and $G^2 = \bigtimes_{i=1}^{r} (\mathbf{Z}_2 i)^{n_i}$. Let $\Gamma_k = \{U\times V\colon V$ is a subgroup of H and U is a subgroup of G^2 not containing $\mathbf{Z}_2^k = [t_1, \dots, t_k]$, $1\leqq k\leqq\gamma_r$ where $t_1, \dots, t_{\gamma_r}$ generate $(\mathbf{Z}_2)^{\gamma_r}$, $\gamma_r = \sum_{i=1}^{r} n_i\}$. It is simple to see that (Γ_{k+1}, Γ_k) is an admissible pair with respect to t_{k+1}. Here by Γ_{γ_r+1} we mean the family of all subgroups of G.

THEOREM 3.2. *If (Γ, Γ') is an admissible pair of families in G with respect to a, then a (Γ, Γ')-free element in $\varkappa_*(G; \Gamma, \Gamma')$ is zero as an element of $\varkappa_*(G; \Pi, \Gamma')$.*

PROOF. It is enough to show that the homomorphism $i_*\colon \varkappa_*(G; \Gamma, \Gamma')\to\varkappa_*(G; \Pi, \Gamma')$ induced from the inclusion $i\colon (\Gamma, \Gamma')\to(\Pi, \Gamma')$ is the constant homomorphism. Let $[M, \theta]$ be an element of $\varkappa_*(G; \Gamma, \Gamma')$. Let F be the closed submanifold of M consisting of all points of M fixed by S, S being the intersection of all the members of $\Gamma-\Gamma'$. Let ν be the normal bundle of the imbedding of F in the interior of M and let $D(\nu)$ be its disc bundle with the action θ^* of G on $D(\nu)$ induced by the real vector bundle maps covering the action θ on F. Since F is the fixed point set of S, $a\notin H$, $\forall H\in\Gamma-\Gamma'$ and no point of F is fixed by the subgroup $[S\cup\{a\}]$ generated by $S\cup\{a\}, a$ will act freely on F and hence on $D(\nu)$. Let $F'=F/[a]$ and $D'(\nu)=D(\nu)/[a]$. Since G is abelian, the actions θ and θ^* on F and $D(\nu)$ induce actions θ' and $\theta^{*\prime}$ on F' and $D'(\nu)$, respectively. Consider the quotient maps $q_1\colon D(\nu)\to D'(\nu)$ and $q_2\colon F\to F'$ which are equivariant and double covers over $D'(\nu)$ and F', respectively. Let C_1 and C_2 be the mapping cylinders of q_1 and q_2 and ψ_1^* and ψ_2^* be the induced actions on C_1 and C_2, respectively. We have the following commutative diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\gamma} & D'(\nu) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\nu'} \\ C_2 & \xrightarrow{\beta} & F' \end{array}$$

where $\alpha\colon C_1\to C_2$ is the map induced from $\nu'\colon D'(\nu)\to F'$ by going to mapping cylinder. It is simple to see that $\partial C_2\approx F$ and the action ψ_1^* on $\alpha^{-1}(\partial C_2)$ is isomorphic to the action θ^* on $D(\nu)$. Consider $W=M\times[0,1]\cup C_1/\sim$ where $\sim$ is the equivalence relation obtained by identifying $D(\nu)\times\{1\}$ with $\alpha^{-1}(\partial C_2)$. Let the action Θ of G on W be given by $\Theta|M^n\times I=\theta\times 1$ and $\Theta|C_1=\psi_1^*$. Take V to be

$$(\partial M\times I)\cup(M\times\{1\}-(D(\nu)\times\{1\})^0)\cup(\partial C_1-(\alpha^{-1}(\partial C_2))^0)$$

where ^{0} is the interior operator. Clearly V is (Γ',Γ')-free and ∂W is isomorphic to $M\cup V$ by identifying ∂V with ∂M. This shows that $[M,\theta]$ is zero in $\varkappa_*(G;\Pi,\Gamma')$.

THEOREM 3.3. *Let Γ be a family in a finite abelian group G such that there exists an element a in G of order 2 with $[a]\notin\Gamma$, $[a]$ being the subgroup of G generated by a. Then the homomorphism $i_*\colon \varkappa_*(G;\Gamma)\to\varkappa_*(G;\Pi)$ induced grom the inclusion $i\colon k\to\Pi$ is the zero homomorphism.*

PROOF. Let $[M,\theta]\in\varkappa_*(G;\Gamma)$. Since $[a]\notin\Gamma$, a will act freely on M and therefore the quotient map $q\colon M\to M/[a]$ will be a double cover over $M/[a]$. Let C be the mapping cylinder of the double cover with the induced action ψ of G on C. Clearly the boundary ∂C is isomorphic to M with $\psi|\partial C=\theta$. Consider $W=M\times[0,1]\cup C/\sim$ where $\sim$ is the equivalence relation obtained by identifying $M\times\{1\}$ with ∂C. Let the action Θ of G on W be given by $\Theta|M\times 1=\theta\times 1$ and $\Theta|C=\psi$. Clearly $\partial(W,\Theta)=(M,\theta)$. This shows that $[M,\theta]$ is zero in $\varkappa_*(G;\Pi)$.

Let G be a finite abelian group and $\mathbf{P}$ be the family of all proper subgroups of G. Suppose Γ' is another family in G such that there exists a chain of families $\Gamma'=\Gamma_1\subset\ldots\subset\Gamma_{r+1}=\mathbf{P}$ with (Γ_{i+1},Γ_i) being an admissible pair of families with respect to an element a_i, $i=1,\ldots,r$. By repeated application of Theorem 3.2 one concludes that the homomorphism $j_*\colon\varkappa_*(G;\Pi,\Gamma')\to\varkappa_*(G;\Pi,\mathbf{P})$ induced by the inclusion $j\colon(\Pi,\Gamma')\to(\Pi,\mathbf{P})$ is a monomorphism. Therefore we can give characteristic numbers for an (Π,Γ')-free element, since $(\Pi,\mathbf{P})$ is an almost adjacent pair. In this case we define equivariant homology and cohomology as $h_*(X;G)=H_*(F_G(X);\mathbf{Z}_2)$ and $h^*(X,G)=H^*(F_G(X);\mathbf{Z}_2)$, for a G-space X, $F_G(X)$ being the fixed points set of X under G. Thus corresponding to the equivariant homology and cohomology defined as above, using Theorems 2.2 and 3.2, one gets the following

THEOREM 3.4. *An (Π, Γ')-free element in $\varkappa_*(G; \Pi, \Gamma')$ is zero if and only if all the characteristic numbers are zero.*

Special cases. We will consider two cases $G=\mathbf{Z}_2^k$ and $G=\bigtimes_{i=1}^{r}(\mathbf{Z}_2 i)^{n_i}\times H$, H being any finite abelian group of odd order and each element of H commuting with each element of $\bigtimes_{i=1}^{r}(\mathbf{Z}_2 i)^{n_i}$.

Case I: $G=\mathbf{Z}_2^k=[t_1, \ldots, t_k]$. Let $\Gamma_i=\{U: U$ is a subgroup of $\mathbf{Z}_2^k$ not containing $\mathbf{Z}_2^i=[t_1, \ldots, t_i]$, $1\leqq i\leqq k\}$. It is easy to see that (Γ_{i+1}, Γ_i) is an admissible pair with respect to t_{i+1} and $\Gamma\subset\Gamma_2\subset\ldots\subset\Gamma_k=\mathbf{P}$. Therefore by repeated applications of Theorem 3.2 one infers that the homomorphism

$$j_{1*}\colon \varkappa_*(\mathbf{Z}_2^k;\ \Pi, \Gamma_1)\to\varkappa_*(\mathbf{Z}_2^k;\ \Pi, \mathbf{P})$$

induced from the inclusion $j_1\colon (\Pi, \Gamma_1)\to(\Pi, \mathbf{P})$ is an injection. Also $[t_1]\notin\Gamma_1$, therefore by Theorem 3.3 the homomorphism

$$j_{2*}\colon \varkappa_*(\mathbf{Z}_2^k;\ \Pi)\to\varkappa_*(\mathbf{Z}_2^k;\ \Pi, \Gamma_1)$$

induced from the inclusion $j_2\colon (\Pi, \Phi)\to(\Pi, \Gamma_1)$ is a monomorphism. Thus $j_*\colon \varkappa_*(\mathbf{Z}_2^k; \Pi)\to\varkappa_*(\mathbf{Z}_2^k; \Pi, \mathbf{P})$ is a monomorphism, $j\colon (\Pi, \varphi)\to(\Pi, \mathbf{P})$. Let us define the equivariant homology and cohomology as follows: $h_*(X, \mathbf{Z}_2^k)=H_*(F_{\mathbf{Z}_2^k}(X);\ \mathbf{Z}_2)$ and $h^*(X;\ \mathbf{Z}_2^k)=H^*(F_{\mathbf{Z}_2^k}(X);\ \mathbf{Z}_2)$, where H_* and H^* are singular homology and cohomology, respectively. Using the monomorhism of

$$j_*\colon \varkappa_*(\mathbf{Z}_2^k;\ \Pi)\to\varkappa_*(\mathbf{Z}_2^k;\ \Pi, \mathbf{P})$$

and Theorem 2.2 for the almost adjacent pair $(\Pi, \mathbf{P})$, one immediately gets

THEOREM 3.5. *A Π-free element in $\varkappa_*(G; \Pi)$ is zero if and only if all its characteristic numbers (corresponding to the theories h^* and h_* defined as above) are zero.*

Case II: G is a finite abelian group of even order given by $G^2\times H$ where H is a finite group of odd order and G^2 is the 2-group $\bigtimes_{i=1}^{r}(\mathbf{Z}_2 i)^{n_i}$. Let $\mathbf{P}$ be the family of all subgroups of G of the type $U\times V$ where V is a subgroup of H and U is a subgroup of G^2 not containing $G_2=(\mathbf{Z}_2)^{\gamma_r}$, $\gamma_r=\sum_{i=1}^{r} n_i$. Let $(\mathbf{Z}_2)^{\gamma_r}$ be generated by $\{t_k\}$, $1\leqq k\leqq\gamma_r$.

THEOREM 3.6. *The following sequence is exact:*

$$0\to\varkappa_*(G;\ \Pi)\xrightarrow{j_*}\varkappa_*(G;\ \Pi, \mathbf{P})\xrightarrow{\partial_*}\varkappa_*(G;\ \mathbf{P})\to 0.$$

PROOF. By Example 3.1, (Γ_{k+1}, Γ_k) is an admissible pair with respect to t_{k+1}, $1\leqq k\leqq\gamma_r$. Therefore by repeated application of Theorem 3.2, one infers that the homomorphism

$$(j_1)_*: \varkappa_*(G; \Pi, \Gamma_1) \to \varkappa_*(G; \Pi, \mathbf{P})$$

is a monomorphism, j_1: $(\Pi, \Gamma_1) \to (\Pi, \mathbf{P})$. Since $[t_1] \notin \Gamma_1$, using Theorem 3.3 we get $(j_2)_*$: $\varkappa_*(G; \Pi) \to \varkappa_*(G; \Pi, \Gamma_1)$ to be monomorphism, j_2: $(\Pi, \varphi) \to (\Pi, \Gamma_1)$. Thus the inclusion j: $(\Pi, \varphi) \to (\Pi, \mathbf{P})$ induces monomorphism j_*. This completes the poof of the Theorem.

COROLLARY 3.7. *Let (M^n, θ) be a G-manifold and the induced action of the subgroup G_2 be stationary point free. Then (M^n, θ) bounds as a G-manifold.*

The stationary point set $F_{G_2}(M^n)$

As in Case II of §3, let $G = G^2 \times H$ be a finite abelian group of even order where H is an odd order group and $G^2 = \underset{i=1}{\overset{r}{\times}} (\mathbf{Z}_2 i)^{n_i}$. Let us denote the subgroup $(\mathbf{Z}_2)^{\gamma_r}$ by by G_2, $\gamma_r = \sum_{i=1}^{r} n_i$. Let $\mathbf{R}$ be the field of real numbers and $GL(\mathbf{R}, j)$ be the set of all isomorphisms of the vector space $\mathbf{R}^j$ onto itself, $j = 1, 2$. Any irreducible real representation of G will be either one dimensional or two dimensional. Let ϱ^j: $G \to GL(\mathbf{R}, j)$ be any nontrivial irreducible real j-dimensional representation of G, $j = 1, 2$.

THEOREM 4.1. Ker ϱ^j *contains a subgroup of G isomorphic to* $(\mathbf{Z}_2)^{\gamma_r - 1}$.

PROOF. It is simple to see that the image (ϱ^j/G_2) is either the trivial subgroup or is the subgroup consisting of just two elements, namely the identity element and the isomorphism θ: $\mathbf{R}^j \to \mathbf{R}^j$ given by $\theta(x) = (-1)x$, for every $x \in \mathbf{R}^j$. Therefore Ker (ϱ^j/G_2) is either G_2 itself or is isomorphic to $(\mathbf{Z}_2)^{\gamma_r - 1}$.

Let (M^n, θ) be a closed G-manifold and $F = F_{G_2}(M^n)$ be the stationary point set of M^n under the subgroup G_2. Consider the decomposition $F = \bigcup_{l=0}^{n} F^l$ where F^l is the l-dimensional component. Let $D(v_l)$ be the normal disc bundle of F^l in M^n with the induced action θ_l. Let Π be the family of all subgroups of G and $\mathbf{P}$ be the family of subgroups of G of the type $U \times V$ where V is a subgroup of H and U is a subgroup of G^2 not containing G_2.

DEFINITION 4.2. F is said to have an equivariant *trivial normal bundle* in M^n if G/G_2 acts trivially on F and there exists some positive dimensional real G-representations (W_l, φ_l) such that in $\varkappa_*(G, \Pi, \mathbf{P})$ $[D(v_l), \theta_l] = [F^l][D(W_l), \varphi_l]$ where $D(W_l)$ is the unit disc of W_l.

Given a real representation ϱ: $G \to GL(\mathbf{R}, j)$ of G one gets a j-dimensional vector space $\mathbf{R}^j$ over $\mathbf{R}$ with the action ψ: $G \times \mathbf{R}^j \to \mathbf{R}^j$ given by $\psi(g, x) = \varrho(g)(x)$. We say $(\mathbf{R}^j, \psi)$ a representation space of G or by abuse of language, a representation of G. Let $\{(V_k, \psi_k)\}_{1 \leq k \leq m}$ be the finite set of all nonisomorphic nontrivial irreducible real representations of G. Let $\mathbf{Z}^+$ be the set of nonnegative integers. Given any map f: $\{1, \ldots, m\} \to \mathbf{Z}^+$ one has a real representation $\big(V(f), \psi(f)\big)$ of G given by $V(f) = \bigoplus_{k=1}^{m} (V_k, \psi_k)^{f(k)}$ where $(V_k, \psi_k)^{f(k)}$ is the direct sum of $f(k)$ copies of (V_k, ψ_k). Let us denote the unit disc and unit sphere of $V(f)$ by $D(f)$ and $S(f)$, respectively.

THEOREM 4.3. *If F has an equivariant trivial normal bundle in (M^n, θ) then it is the boundary of some manifold and (M^n, θ) itself is the boundary of some G-manifold (N, Θ).*

PROOF. Since F has an equivariant trivial normal bundle in M^n, we have $[D(v_l), \theta_l]=[F^l][D(W_l), \varphi_l]$ for some positive dimensional real representations (W_l, φ_l) of G. Also $(W_l, \varphi_l)=(V(f_l), \psi(f_l))$ for some map f_l: $\{1, \ldots, m\}\to\mathbf{Z}^+$. Therefore $[D(v_l), \theta_l]=[F^l][D(f_l), \psi(f_l)]$. Let j_*: $\varkappa_*(G; \Pi)\to\varkappa_*(G; \Pi, \mathbf{P})$ be the homomorphism induced by the inclusion j: $(\Pi, \Phi)\to(\Pi, \mathbf{P})$. We have

$$j_*[M^n, \theta] = \sum_{l=0}^{n}[D(v_l), \theta_l] = \sum_{l=0}^{n}[F^l][D(f_l), \psi(f_l)].$$

Therefore from Theorem 3.6, one gets

$$(\partial_* j_*)[M^n, \theta] = \partial_* \sum_{l=0}^{n}[F^l][D(f_l), \psi(f_l)] = \sum_{l=0}^{n}[F^l][S(f_l), \psi(f_l)] = 0$$

in $\varkappa_*(G; \mathbf{P})$. Therefore there exists a $\mathbf{P}$-free G-manifold (D, η) such that

$$(1) \qquad (\partial D, \eta) = \bigcup_{l=0}^{n}\left(F^l\times(S(f_l), \psi(f_l))\right)$$

Since each (W_l, φ_l) is a positive dimensional real representation of G, there exists a member $k(l)$ in the set $\{1, \ldots, m\}$ such that $f_l(k(l))\neq 0$. By Theorem 4.1 there exists a subgroup $H_{k(l)}$ of G isomorphic to $(\mathbf{Z}_2)^{\gamma_r-1}$ fixing $V_{k(l)}$. Let us fix some β, $0\leq\beta\leq n$. Let $A_{k(\beta)}$ be the largest subset of $\{1, \ldots, m\}$ such that $H_{k(\beta)}$ fixes V_j, $j\in A_{k(\beta)}$. Let $A_{k(\beta)}$ be the disjoint union of $B_{k(\beta)}$ and $C_{k(\beta)}$ where $B_{k(\beta)}$ consists of all $j\in A_{k(\beta)}$ such that V_j is one dimensional and $C_{k(\beta)}$ consists of all $j\in A_{k(\beta)}$ such that V_j is two dimensional. Let

$$\sum_{j\in B_{k(\beta)}} f_l(j)+\sum_{j\in C_{k(\beta)}} 2f_l(j) = \Delta(l, \beta)\in\mathbf{Z}^+.$$

Since $f_\beta(k(\beta))\neq 0$ and $A_{k(\beta)}$ contains $k(\beta)$, $\Delta(\beta, \beta)$ cannot be zero. From (1) we get

$$(2) \qquad F_{H_{k(\beta)}}(\partial D, \eta) = F_{H_{k(\beta)}}\Big(\bigcup_{l=0}^{n}\left(F^l\times(S(f_l), \psi(f_l))\right)\Big).$$

Suppose $F_{H_{k(\beta)}}(D)=F^*$ and $\mathbf{Z}_{2,\beta}\approx\mathbf{Z}$ be the complement of $H_{k(\beta)}$ in $(\mathbf{Z}_2)^{\gamma_r}$. Then from (2) we have

$$(\partial F^*, \eta|\mathbf{Z}_{2,\beta}) = \bigcup_{l=0}^{n}\left(F^l\times(S^{\Delta(l,\beta)-1}, a)\right)$$

where a is the antipodal involution. Since D is $\mathbf{P}$-free, F^* will have stationary point free action of $\mathbf{Z}_{2,\beta}$. Therefore $[\partial F^*, \eta|\mathbf{Z}_{2,\beta}]$ is zero in $\varkappa_*(\mathbf{Z}_{2,\beta}, \Gamma_1)$ so that

$$(3) \qquad \sum_{l=0}^{n}[F^l][S^{\Delta(l,\beta)-1}, a] = 0$$

in $\varkappa_*(\mathbf{Z}_{2,\beta}, \Gamma_1)$ where Γ_1 is the family in $\mathbf{Z}_{2,\beta}$ consisting of only trivial subgroup of $\mathbf{Z}_{2,\beta}$. We know that $\varkappa_*(\mathbf{Z}_2, \Gamma_1)$ is the free $\varkappa_*$-module with generators $\{[S^n, a], n\in\mathbf{Z}^+\}$.

Therefore (3) gives $[F^\beta]=0$, since $\Delta(\beta,\beta)\neq 0$. By varying β, one concludes that $[F^\beta]=0$, $\beta=0, \dots, n$. Hence $[F]=0$ in $\varkappa_*$. Also

$$j_*[M^n, \theta] = \sum_{l=0}^{n} [F^l][D(f_l), \psi(f_l)] = 0.$$

By Theorem 3.6, j_* is a monomorphism and therefore one infers that $[M^n, \theta]$ is zero in $\varkappa_*(G; \Pi)$. This completes the proof of the Theorem.

Combining Corollary 3.7 and Theorem 4.3, one infers that the fixed point data of the subgroup G_2 determines the equivariant bordism class of a G-manifold (M^n, θ).

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DEPARTMENT OF MATHEMATICS
NORTH-EASTERN HILL UNIVERSITY
BIJNI CAMPUS, LAITUMKHRAH
SHILLONG — 793 003
MEGHALAYA, INDIA