Phase transition in a supersymmetric theory

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We study the thermodynamics of a globally supersymmetric theory in the presence of a net background of "charge" associated with a continuous symmetry. We solve exactly a prototypical model involving two chiral superfields in the weak-coupling limit and demonstrate the existence of a second-order phase transition associated with the spontaneous breaking of this symmetry. This is done by showing that one of the scalar fields in the supermultiplet develops a vacuum expectation value below a critical temperature. We also find that this phase transition is in the same universality class as the usual Bose gas condensation.

I. INTRODUCTION

There is currently a great deal of interest in theories with broken supersymmetry[1] (SUSY). Apart from their intrinsic beauty, these theories may play an important role in understanding of the gauge hierarchy problem.[2] Such theories have an underlying invariance under simultaneous transformation between boson and fermion fields; in such theories, bosons and fermions are treated on the same footing by putting them in the same supermultiplet having a common mass. In a realistic model based on SUSY, such a symmetry must be broken, because there are no bosons and fermions in nature which are degenerate in mass. Furthermore, if SUSY has relevance to actual physics, it must be incorporated into a grand unified theory.[3] It then becomes necessary to understand the finite-temperature behavior of a SUSY theory.[4] Such a study is important in describing the evolution of the early Universe if SUSY is a good symmetry at high energy.

In the case of ordinary symmetries, it is well known that if they are broken at $T=0$, they may be restored at a higher temperature.[5] This occurs at T_c, at which the effective scalar-boson mass term vanishes. This T_c is determined by the various coupling constants in the theory, and also by the presence of a conserved background "charge" (e.g., weak isospin and hypercharge, etc.).[5,6] It is also known that the effect of a background of charges carried by the fermions, such as baryon number, etc., is negligible on such phase transitions.[7]

In this paper, we study the critical behavior of a globally SUSY theory in the presence of a net background of charge.[8] Because of SUSY, the charge, in general, is carried by both bosons and fermions. To study the effect of a net background of conserved charge in a SUSY theory, we consider a prototypical globally SUSY model of two complex scalars A_+ and A_- and a Dirac fermion ψ. The Lagrangian of the model is[9]

$$\mathcal{L} = |\partial A_+|^2 - |s + MA_+ + gA_+^2|^2 + |\partial A_-|^2$$
$$- |(M + 2gA_+^*)A_-|^2 + \bar{\psi}(i\partial - M)\psi$$
$$- (2gA_+\bar{\psi}_-\psi_+ - gA_-^*\psi_+^T C^{-1}\psi_+ + \text{H.c.}) , \qquad (1)$$

where ψ_+ and ψ_- are the usual chiral projections of ψ and ψ_+^T stands for transpose with C, the charge-conjugation operator. Here M, g, and s are constants with dimensions 1, 0, and 2, respectively. The Lagrangian is invariant under the SUSY transformation

$$\delta A_+ = \bar{\epsilon}_-\psi ,$$

$$\delta\psi = -(M + 2gA_+^*)A_-\epsilon_+ - (s + MA_+ + gA_+^2)\epsilon_-$$
$$\qquad - i\partial A_+\epsilon_- - i\partial A_-\epsilon_+ ,$$

$$\delta A_- = \bar{\epsilon}_+\psi ,$$

up to a total space-time gradient. Here ϵ is, as usual, an infinitesimal anticommuting Majorana spinor. This Lagrangian is nothing but the component-form version of the most general renormalizable Lagrangian of two chiral superfields Φ_+ and Φ_- of opposite chirality (Φ_+ and Φ_-^* have the same chirality) from which the auxiliary fields have been eliminated through equations of motion. It is also invariant under the following global symmetry transformation (R invariance):

$$A_+ \rightarrow A_+, \quad \psi_+ \rightarrow e^{-i\alpha}\psi_+ ,$$
$$A_- \rightarrow e^{-2i\alpha}A_-, \quad \psi_- \rightarrow e^{-i\alpha}\psi_- . \qquad (2)$$

Thus the "fermionic charges" of A_+, A_-, and ψ are 0, -2, and -1, respectively.[1,9]

The summary of the rest of the paper is as follows. In Sec. II we solve the model exactly in the weak-coupling limit and show the existence of a phase transition for spatial dimensions $d > 2$. We show this by demonstrating that the vacuum expectation value (VEV) of A_- is nonzero for $T < T_c$. Thus the continuous symmetry (2) is broken below T_c. In Sec. III we study the critical behavior of the model in detail and show that it is in the same universality class as the ordinary Bose gas.[10] Our conclusions are in Sec. IV.

II. MODEL AND SOLUTION

From (1) it is clear that we are dealing with a system of particles A_+, A_-, and ψ (and their antiparticles) which can be mutually interconverted by various "reactions" allowed by the Lagrangian. The individual numbers of various particles [given by $N(A_+)$, $\overline{N}(A_+)$, etc. where the bar refers to antiparticles] are not conserved in equilibrium but the net fermionic charge N, corresponding to the global continuous symmetry (2), given by

$$N = -2[N(A_-) - \overline{N}(A_-)] - [N(\psi) - \overline{N}(\psi)] , \qquad (3)$$

is conserved. If the interactions are completely neglected, then the individual particle numbers will be conserved. We take an intermediate weak-coupling approach where all interactions are neglected but the various reactions are kept so that the constraint (3) is satisfied. Assuming that an equilibrium is reached with respect to all the "reactions" allowed by the Lagrangian (1), the partition function is given by[11]

$$Z(\mu,T) = \sum_{N=0}^{\infty} \left[e^{\beta\mu Q} \sum_{\left\{ \sum_E n_E = N \right\}} e^{-\beta \mathcal{H}} \right] ,$$

where

$$Q \equiv (-\tfrac{1}{2}N) = \sum_{E(A_-)} [n_E(A_-) - \overline{n}_E(A_-)]$$
$$+ 2 \times \tfrac{1}{2} \times \sum_{E(\psi)} [n_E(\psi) - \overline{n}_E(\psi)] \qquad (4)$$

and

$$\mathcal{H} = \sum_{E(A_+)} E(A_+)[n_E(A_+) + \overline{n}_E(A_+) + 1]$$
$$+ \sum_{E(A_-)} E(A_-)[n_E(A_-) + \overline{n}_E(A_-) + 1]$$
$$+ 2 \sum_{E(\psi)} E(\psi)[n_E(\psi) + \overline{n}_E(\psi) - 1] , \qquad (5)$$

and μ is the chemical potential,[8] corresponding to the constraint (4) and $\beta = 1/T$, as usual. Here $n_E(A_+)$, etc., denote the number of particles in the single-particle states labeled by $E(A_+)$, etc. The factors of 2 in (4) and (5) come from the spin sum for the Dirac fermion. Notice

the cancellation of the zero-point energy in (5) due to the fact that

$$E(A_+) = E(A_-) = E(\psi) , \qquad (6)$$

due to SUSY. Using the plane-wave representation for the free-particle states and standard methods,[11] we get in the thermodynamic limit

$$Z(\mu,T) = Z_B(0,T)Z_B(\mu,T)Z_F(\tfrac{1}{2}\mu,\tfrac{1}{2}T) , \qquad (7)$$

where

$$\ln Z_{B,F}(\mu,T) = \frac{V}{(2\pi)^d} \int d\mathbf{k} \sum_{r=1}^{\infty} r^{-1} \epsilon_r e^{\beta\mu r}$$
$$\times e^{-\beta r (k^2 + M^2)^{1/2}} , \qquad (8)$$

$$\epsilon_r = 1 \text{(bosons)}, \quad \epsilon_r = (-1)^{r+1} \text{ (fermions) .} \qquad (9)$$

Here M is the common mass of each particle (antiparticle) and V is the volume of the system ultimately going to infinity. The integral in (8) can be done by standard methods.[10] We get

$$\ln Z_{B,F}(\mu,T) = \frac{4\beta V}{\Lambda^{d+1}} \sum_{r=1}^{\infty} r^{-d'} \epsilon_r \cosh(\beta\mu r) K_{d'}(\beta M r) , \qquad (10)$$

with

$$\Lambda = (2\beta\pi/M)^{1/2}, \quad d' = \tfrac{1}{2}(d+1) ,$$

and $K_\nu(z)$ a standard Bessel function.[12] Therefore, $Z(\mu,T)$ can be obtained using (7). Since $\beta p V = \ln Z$, where p is the pressure of the system, one can get all the thermodynamic quantities. For example, we list the charge density $\rho = Q/V$ and the energy density $u = U/V$. We have

$$\rho = \left[\frac{\partial p}{\partial \mu} \right]_{\beta,V} = \rho_B(\mu,T) + \rho_F(\tfrac{1}{2}\mu,\tfrac{1}{2}T) , \qquad (11)$$

$$u = - \left[\frac{\partial}{\partial \beta}(\beta p) \right]_{\beta\mu,V} = u_B(\mu,T) + 2u_F(\tfrac{1}{2}\mu,\tfrac{1}{2}T) , \qquad (12)$$

where

$$\rho_{B,F}(\mu,T) = \frac{4\beta}{\Lambda^{d+1}} \sum_{r=1}^{\infty} r^{-(d'-1)} \epsilon_r \sinh(\beta\mu r) K_{d'}(\beta M r) , \qquad (13)$$

$$u_{B,F}(\mu,T) = \frac{4M\beta}{\Lambda^{d+1}} \sum_{r=1}^{\infty} \{ r^{-(d'-1)} \epsilon_r \cosh(\beta\mu r)[K_{d'+1}(\beta M r) - (\beta M r)^{-1} K_{d'}(\beta M r)] \} . \qquad (14)$$

By using the large-z form of $K_\nu(z)$, namely,[12]

$$K_\nu(z) = \left[\frac{\pi}{2z} \right]^{1/2} e^{-z}, \quad z \gg 1 , \qquad (15)$$

we see that the sum in (13) for $\rho_B(\mu,T)$ is divergent for $\mu = M$ for $d \leq 2$. So a nonzero T_c exists only for $d > 2$. By contrast, ρ_F is found to be convergent for all $T > 0$.

Therefore, the fermions do not affect the phase transition. For $d > 2$, the critical temperature T_c is given by

$$\rho = \rho_B(M,T_c) + \rho_F(\tfrac{1}{2}M,\tfrac{1}{2}T_c) . \qquad (16)$$

The numerical value of T_c is clearly affected by fermions. For $T \leq T_c$, as usual, μ sticks at M and (11) becomes

$$\rho - \rho_0(T) = \rho_B(M,T) + \rho_F(\tfrac{1}{2}M, \tfrac{1}{2}T) , \tag{17}$$

where $\rho_0(T)$ is the condensate density. To relate it to oth-

er quantities of interest, we note the usual equality between the VEV of a field and the square root of the ground-state particle number density.[13] In our case, the ground-state densities are given by

$$\rho_0(A_+) = V^{-1}(e^{\beta M} - 1)^{-1} ,$$

$$\rho_0(A_-) = V^{-1}(e^{\beta(M-\mu)} - 1)^{-1} , \quad \bar{\rho}_0(A_-) = V^{-1}(e^{\beta(M+\mu)} - 1)^{-1} , \tag{18}$$

$$\rho_0(\psi) = V^{-1}(e^{2\beta(M-\mu/2)} + 1)^{-1} , \quad \bar{\rho}_0(\psi) = V^{-1}(e^{2\beta(M+\mu/2)} + 1)^{-1} .$$

For $T \leq T_c$, all the ρ_0's vanish (in the thermodynamic limit) except $\rho_0(A_-)$ which is nonzero and is given by (17). There we can write

$$\rho_0(T) = \rho_0(A_-) \neq 0 \quad \text{for } 0 \leq T < T_c$$

$$= 0 \quad \text{for } T \geq T_c . \tag{19}$$

This signals the breakdown of the global symmetry (2) below T_c. The ground-state energy of the system is given by

$$u_g = M\rho_0(A_-) = M\rho_0(T) \neq 0 \tag{20}$$

for $T < T_c$ and is zero for $T \geq T_c$.

III. CRITICAL BEHAVIOR

To study the critical behavior of the system, we have to study (11) and (13) near $\mu = M$. For $\mu \simeq M$, following Ref. 10, we get

$$\rho_B(\mu,T) = \rho_B(M,T) - \frac{\Gamma\left[\dfrac{4-d}{2}\right]}{(d-2)\pi^{d/2}2^{d-2}}$$

$$\times \left[\frac{M}{\beta}\right](M^2 - \mu^2)^{(d-2)/2}$$

$$+ O(M^2 - \mu^2), \quad 2 < d < 4 , \tag{21}$$

$$\rho_F(\mu,T) = \rho_F(M,T) + O(M^2 - \mu^2) . \tag{22}$$

Expanding these near T_c, and substituting in (11) and (13) we get

$$M^2 - \mu^2 \simeq C_1 t^{2/(d-2)}, \quad t = \frac{T - T_c}{T_c} > 0, \quad 2 < d < 4 , \tag{23}$$

$$C_1 = \left[\frac{2^{d-2}(d-2)\pi^{d/2}}{M\Gamma\left[\dfrac{4-d}{2}\right]}\right]^{2/(d-2)} [\rho_B'(T_c) + \rho_F'(T_c)] .$$

For $t \lesssim 0$, we get, similarly,

$$\rho_0(T) = |t| T_c [\rho_B'(T_c) + \rho_F'(T_c)] . \tag{24}$$

In (23) and (24) the primes indicate derivative with respect to temperature. Now comparing (23) and (24) with Eqs. (26) and (41), respectively, of Ref. 10, we see that all of the exponents of the system will be the same as those of the usual Bose gas. Therefore, the system is in the same universality class as the usual Bose gas. (Note that in Ref. 10, it was shown that the relativistic Bose gas is in the same universality class as the usual nonrelativistic Bose gas.)

Being in the same universality class, however, does not mean that the two systems are identical. Quantities like the critical temperature and various amplitudes can still be quite different in the two systems. In these quantities, the presence of fermion in the system will be felt. As an example, we describe the system in the extreme relativistic (ER) limit. In this limit, $\rho \ll M^d$ and using the low-z expansion of $K_\nu(z)$,[12]

$$K_\nu(z) = \frac{2^{\nu-1}\Gamma(\nu)}{z^\nu}, \quad z \ll 1 , \tag{25}$$

we get, from (7), (10), (11), and (13), to leading order (for $d = 3$),

$$p = \frac{2\pi^2 T^4}{45} + \frac{M\mu T^2}{3} - \frac{7}{24}M^2 T^2$$

$$+ \frac{\mu^2 T^2}{96} + \frac{T}{6\pi}(M^2 - \mu^2)^{3/2} + \cdots , \tag{26}$$

$$\rho = \frac{1}{3}MT^2 + \frac{\mu T^2}{48} - \frac{T\mu}{2\pi}(M^2 - \mu^2)^{1/2} + \cdots . \tag{27}$$

These equations may be compared with Eqs. (12) and (13) of Ref. 14 for the ER limit of the usual Bose gas. These give

$$T_c = \left[\frac{48}{17}\right]^{1/2} \left[\frac{\rho}{M}\right]^{1/2} , \tag{28}$$

as compared with

$$T_c = \sqrt{3} \left[\frac{\rho}{M}\right]^{1/2} , \tag{29}$$

for the usual Bose gas.[14] The influence of the fermions is seen to lower the critical temperature for the same overall density, as might have been anticipated. Also by studying (26) and (27) one can obtain the jump in the specific-heat derivative:

$$\left[\frac{\partial C_V}{\partial T}\right]_{T_c+} - \left[\frac{\partial C_V}{\partial T}\right]_{T_c-} = -\frac{289\pi^2}{72}\left[\frac{\rho}{M}\right] . \qquad (30)$$

IV. CONCLUDING REMARKS

We have studied the thermodynamics of a globally supersymmetric theory in the presence of a net background of charge density. This charge is associated with a continuous global symmetry and, by virtue of SUSY, is carried by both bosons and fermions in the theory. We have solved a prototypical model exactly in the weak-coupling limit and shown that there is a second-order phase transition at $T_c \neq 0$, associated with the breaking of this global symmetry. This has been shown by demonstrating that

A_- develops a nonzero VEV below T_c. Thus, in SUSY models a conserved charge carried by fermions can be broken due to the condensation of bosons which must appear in the model due to SUSY. Although the phase transition is found to be in the same universality class as the usual Bose gas, there are important differences between the two. It is possible that if the model is solved with interactions fully taken into account, the differences from the Bose condensation may be brought more strongly.

ACKNOWLEDGMENT

[1]See, e.g., P. Fayet and S. Ferrara, Phys. Rep. **32C**, 249 (1977); P. Van Nieuwenhuizen, *ibid.* **68**, 189 (1981); J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University, Princeton, 1983).

[2]S. Weinberg, Phys. Lett. **62B**, 111 (1976); E. Gildener and S. Weinberg, Phys. Rev. D **13**, 333 (1976); E. Witten, Nucl. Phys. **B185**, 513 (1981).

[3]N. Sakai, Z. Phys. C **11**, 153 (1981); S. Dimopoulos and H. Georgi, Nucl. Phys. **B193**, 150 (1981).

[4]A. Das and M. Kaku, Phys. Rev. D **18**, 4540 (1978); L. Girardello, M. T. Grisaru, and P. Salomonson, Nucl. Phys. **B178**, 331 (1981); L. Van Hove, *ibid.* **B207**, 15 (1982); T. E. Clark and S. T. Love, *ibid.* **B217**, 349 (1983); M. B. Paranjape, A. Taormina, and L. C. R. Wijewardhana, Phys. Rev. Lett. **50**, 1350 (1983); J. Kapusta, S. Pratt, and V. Visnjic, Phys. Rev. D **28**, 3093 (1983); H. Aoyama and D. Boyanovsky, *ibid.* **30**, 1356 (1984); H. Matsumoto, M. Nakahara, Y. Nakano, and H. Umezawa, *ibid.* **29**, 2838 (1984).

[5]L. Dolan and R. Jakiw, Phys. Rev. D **9**, 3320 (1974); S. Weinberg, *ibid.* **9**, 3357 (1974).

[6]H. E. Haber and H. A. Weldon, Phys. Rev. D **25**, 502 (1982).

[7]B. J. Harrington and A. Yildiz, Phys. Rev. Lett. **33**, 324 (1974).

[8]P. N. Pandita and S. Singh, Phys. Rev. Lett. **50**, 1550 (1983). There were, unfortunately, some numerical errors in this paper which we have taken care of in the present paper, which, in any case, supercedes it. In both of these papers, we are considering a chemical potential μ associated with a conserved ordinary charge, and not a supercharge. There may be some problems in assigning a μ to a supercharge as pointed out by Paranjape *et al.* (Ref. 4). See also, in this connection, Kapusta *et al.* (Ref. 4). We take $\mu > 0$, without loss of generality.

[9]A. Salam and J. Strathdee, Phys. Rev. D **11**, 1521 (1975); Fort. Phys. **26**, 57 (1978).

[10]S. Singh and P. N. Pandita, Phys. Rev. A **28**, 1752 (1983).

[11]See, e.g., R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1972).

[12]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).

[13]N. N. Bogoliubov, Physica (Utrecht) Suppl. **26**, S1 (1960); J. D. Gunton and M. J. Buckingham, Phys. Rev. **166**, 152 (1968).

[14]H. E. Haber and H. A. Weldon, Phys. Rev. Lett. **46**, 1497 (1981).