SOME PROBLEMS IN RING THEORY:

(VON NEUMANN) REGULARITY AND ANTI - REGULARITY IN MODULES AND RINGS

Ms. MAISNAM IBEMHAL DEVI

DEPARTMENT OF MATHEMATICS

NORTH - EASTERN HILL UNIVERSITY

Submitted in fulfilment of the requirements of the Degree of

DOCTOR OF PHILOSOPHY

To

NORTH - EASTERN HILL UNIVERSITY

SHILLONG

NOVEMBER 1992

TO

MY TEACHERS

WHO HAVE TAKEN THE TROUBLE OF

TEACHING ME

CERTIFICATE

I certify that the dissertation entitled "SOME PROBLEMS IN RING THEORY: (VON NEUMANN) REGULARITY AND ANTI-REGULARITY IN MODULES AND RINGS" submitted by Ms. Maisnam Ibemhal Devi in fulfilment of the requirements for the degree of Doctor of Philosophy is the outcome of a study undertaken by the candidate. I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented to for the award of a degree in any University before.

This dissertation may be placed before the examiners for evaluation and necessary formalities.

I certify that this dissertation is worthy of consideration by the examiners.

Mangesh B. Rege

Shillong,

November 1, 1992.

Mangesh B. Rege
Supervisor
Department of Mathematics
North-Eastern Hill University
SHILLONG.

ACKNOWLEDGEMENTS

This work was carried out under the guidance of Prof.M.B. Rege. I wish to express my sincere thanks to him for all the trouble he has taken during the research period and for the completion of this thesis.

My sincere thanks are also due to Prof. S.M. Bhatwadekar and Prof. Amit Roy for helpful suggestions.

My sincere regards are also due to Prof. M.B. Rege, Dr. H.K. Mukherjee, Dr. P. Saikia, Prof. P. Jothilingam, Prof. S.S. Khare, Prof. Vijay Kumar, Dr. B.K.Dev Sharma and Shri S.L. Marbaniang for the effort they had put in to give us proper foundation for research during M.Sc. courses.

The NEHU had provided me with NEHU fellowship for 4 years for my research work. My deep sense of gratitude is due to Prof. S.S. Khare, then Head of the Department of Mathematics, for the initiative he had taken and for following up all the official formalities. My sincere thanks are also due to all NEHU Officials who had made the necessary arrangements to provide with the NEHU fellowship.

Contd...2/-

Some portions of this thesis have been published in National Academy Science Letters jointly with Prof. M.B. Rege and Ms. I. Choudhuri. I am thankful for the valuable co-operation from them.

I also express my deep sense of gratitude to Dr. A.B. Chakraborty and Dr. N. J. Deb for valuable advice and encouragement, and to Mrs. S. Dutta for encouragement, companionship and moral support extended to me during the last seven years.

I am also indebted to my research colleague Ms. Sima Jajodia with whom I could spend some time during the preparation of her M.Phil. thesis. I felt I had gained a lot mathematically during this short period. My sincere thanks are also due to my colleagues Ms. Amita Mitra and Ms. Debadutta Bhattacharjee for their valuable co-operation.

I also express my sincere thanks to all the research fellows, M.Sc. students and office staff of Mathematics Department for their kind co-operation.

Finally, I express my sincere thanks to Mr. V.T. James for the effort he has taken and the valuable time he has spent for typing the thesis in an excellent manner.

Ms. Maisnam Ibemhal Devi

CONTENTS

INTRODUCTION

John von Neumann [vN:36]* initiated the study of rings which satisfy the following condition: for each element a of the ring there exists an element b such that aba = a. These rings, which later came to be known as (von Neumann) regular rings, arise naturally in the study of modular lattices. However, regular rings are interesting objects in their own right and have been studied at a great length during the last half-century. (See the bibliography, containing 270 items, in [G] .)

In this thesis we study (von Neumann) regularity and anti-regularity in modules and rings. Apart from Chapter 0 it consists of four chapters. Since there are varying definitions in the literature, we collect in Chapter 0 those we use; we also recall some basic results. The rest of the thesis divides naturally into two parts: Chapters 1 and 2 are devoted to a study of regularity in modules and rings; Chapters 3 and 4 to anti-regularity in modules and rings.

---

* The bibliography is divided into two parts: books and monographs; research papers. The presence of a two-digit number in the citation (indicating the year of publication)means that the reference is to a research paper.

The concept of a regular ring was extended to that of a regular module by a number of authors; see § 1E of Chapter 1. In this thesis the term regular module is used in the sense of Zelmanowitz [Z:72] . Throughout R denotes an associative ring with identity and modules are unitary. (Note, however, that this assumption is absent in [Z:72] .) Let M be a left R-module. Then $N = \mathrm{Hom}_R(M, R)$ is a right R-module in a natural manner. An element m of M is called <u>regular</u> (in M) if $(mf)m = m$ for some $f \in N$. A module M is called <u>regular</u> if each of its elements is regular. When $M = R$ we recover the usual definitions in the ring case.

There is a different definition of a regular module due to Elliger [E:71] . A natural element-wise extension of this definition is given in § 1A of Chapter 1. We denote the set of all regular elements of a module M by Reg(M) and the set of all regular elements in the sense of Elliger by eReg(M). Several results concerning the sets Reg(M) and eReg(M) are recorded in Chapter 1. We have , for example (see § 1D):

Result A: Let R be a left self-injective ring, D a left R-module, and B a submodule of D. Then $B \cap \mathrm{Reg}(D) = \mathrm{Reg}(B)$.

In §2 of Chapter 1 a module B is called absolutely regular if for each overmodule D of B, every element of B is regular in D. A ring R is left absolutely regular if the module ${}_RR$ is absolutely regular. Using Result A above we show (1.40) that every left self-injective, regular ring is left absolutely regular.

This fact raises the question of the existence of regular rings which are not absolutely regular. In §2 we construct commutative regular rings which are not absolutely regular.

In §3 of Chapter 1 we extend to modules a lemma due to McCoy which has been used in the study of regular rings. Among the applications of this extension is a technical Proposition (1.55) which yields an alternative proof of the following result, which is a crucial step in the proof of Theorem 2.8 of [Z:72] :

Corollary 1.56. Let $D = A \oplus B$. Then D is regular if and only if A and B are regular.

Central localizations of regular rings have been studied in [AFS:74] . In § 4 of Chapter 1 we study central localizations of regular modules. Among the results we prove are:

Corollary 1.60. Let ${}_RM$ be a module. Let T be a multiplicatively closed subset of the centre of R. Then the

$T^{-1}R$-module $T^{-1}M$ is regular. (This extends to modules a result (1.61) well-known for rings.)

Example 1.62 shows that even when M_v is a regular R_v-module for each maximal ideal v of the centre of R, the R-module M need not be regular.

Theorem 1.64 asserts that if M is a finitely presented left R-module and M_v is a regular R_v-module for each maximal ideal v of the centre of R then M is a regular R-module.

In Chapter 2 we are concerned with regularity in rings. We collect some basic results concerning one-sided regularity in §1. In §2 we call a ring right 2-finite if every right regular element- in the sense of Azumaya [A:54] (see 0.17)- is left regular. After noting that for each field K the matrix ring $M_n(K)$ is left and right 2-finite, it is pointed out that all 2-finite rings are directly finite (2.11). The rest of §2 is devoted to giving sufficient conditions for the 2-finiteness of a ring. The main result of §3 characterises normal, right 2-finite rings as those rings in which $a = a^2b$ implies $a = ba^2$.

In §4 of Chapter 2 we give sufficient conditions on a ring R which ensure that 'property (RC) holds', i.e., the subset Reg(R) is closed under multiplication. We first note that the (RC) property holds (trivially) in commutative rings and in regular rings. The result that the (RC) property holds in normal rings is deduced from a proposition (2.34) valid in modules.

In Chapters 3 and 4 we study anti-regularity in modules and rings. In the theory of generalised inverses a ring has been called anti-regular if for each non-zero element a there exists a non-zero element b such that bab = b. (Such an element b has been called a 2-inverse of a.) In this thesis we introduce a module-theoretic generalisation of this concept. Let M be a left R-module and N the right R-module $\mathrm{Hom}_R(M, R)$. An element m of M is anti-regular if there exists a non-zero element f of N such that $f(mf) = f$. A module M is anti-regular if each of its non-zero elements is anti-regular. When M = R we recover the definitions in the ring case.

Chapter 3 is devoted to a study of anti-regularity in modules. This is carried out in the setting of Morita contexts; we need only their basic properties and these are recalled in §0C. Several equivalent conditions for the anti-regularity of an element of a module are given in Theorem 3.5. We show next (Proposition 3.8(b)) that regular implies anti-regular (for non-zero elements, modules and rings). Thus the class of regular modules is contained in the class of anti-regular modules. This leads to the question: which properties of regular modules hold for anti-regular modules? In §3 we show that the class of anti-regular modules is closed under submodules and direct sums. Unlike the class of regular modules, this class is also closed under direct products. It is shown in § 4 that if M is an anti-regular module, then M is non-singular and Rad(M) = 0.

Endomorphism rings of anti-regular modules are studied in §5. It is shown that if M is an anti-regular left R-module, then $S = \mathrm{End}({}_R M)$ is an anti-regular ring and M is an anti-regular right S-module. This extends several results of Nicholson [N:75] . In §6 we give conditions under which the converse of Proposition 3.8(b) holds, i.e. anti-regular implies regular. A very general condition under which anti-regular implies regular is 'the module satisfies either the ascending or the descending condition on direct summands' (Theorem 3.37). It follows (3.38) that noetherian anti-regular rings are semi-simple. In Proposition 3.39 we prove that a ring R is semi-simple if and only if every (resp. cyclic , resp. indecomposable, resp.simple) left R-module is anti-regular.

While the chapter headings broadly indicate the themes of the respective chapters, a complete compartmentalisation has been neither possible nor desirable. A case in point is Chapter 4, which is mainly devoted to the study of anti-regularity in rings. In §4 of this chapter we consider, for a left R-module M and an idempotent e of R, the ring eRe and the left eRe-module eM. It was considered appropriate to collect together a number of results concerning the regularity and anti-regularity of eM as well as concerning regularity / anti-regularity of the ring eRe in this section.

The following are some of the results obtained in the rest of Chapter 4. An example of a commutative anti-regular ring R with a multiplicatively closed subset T such that $T^{-1}R$ is not anti-regular is given in §1. Reduced anti-regular rings have properties similar to those of reduced regular rings; this is shown in §2. Rings satisfying the condition 'every factor ring is anti-regular' are studied in §3; if such a ring is commutative, then it is necessarily regular. Formulae for the number of regular/ anti-regular elements in Z/nZ are obtained in § 5. We conclude with some open questions concerning similar formulae for certain familiar finite rings.

****

CHAPTER 0

PRELIMINARIES

In this chapter we record some basic notation, terminology, definitions and known results which will be used in later chapters.

A. Generalities

0.1. Rings and modules. By a ring we mean an associative ring with identity; all modules, ring homomorphisms and subrings are unitary. The letter R always denotes a ring; rings may also be denoted by R_i, R', R_0. A left R-module M is denoted as ${}_RM$ and a right R-module as M_R. Modules and ideals may also be denoted by A, B, D as well.

When rings without identity ('rngs') are considered that fact is explicitly mentioned.

Domain means a (possibly non-commutative) ring without divisors of zero.

The letters a,b,c,d denote elements of a ring (or a module). This usage is generally clear from the context and may not be made explicit. When a ring or a module or an element is nonzero that fact will also be usually clear from the context.

0.2. Submodules and homomorphisms. When there is no possibility of confusion we may omit a reference to the ring under consideration. Thus $B \leqslant D$ means B is a subgroup of D as well as (when the ring R is clear from the context) B is an R-submodule of D. By a module pair $(B, {}_RD)$ (or (B, D)) we mean B is an R-submodule of D. We write $B \overset{\oplus}{\leqslant} D$ for ' B is a direct summand of D '. A map $f:A \to B$ means a (group or R-module) homomorphism. Let D, M be left R-modules, $f:D \to M$ a map, and $d \in D$. The image of d under f is denoted by df. Generally, an effort is made to write homomorphisms of modules on the side opposite to the scalars. However, functional notation is used for ring homomorphisms, denoted by θ or ϕ .

0.3. Notation. The letter N denotes the set of natural numbers, Z the ring of integers, Q the field of rationals, H the division ring of real quaternions.

0.4. Notation. For each set I we let $M^{(I)}$ (resp. M^I) denote the direct sum (resp. the direct product) of copies of M indexed by I. Let R be a ring. Then R^N is actually a ring, the ring of sequences of elements of R.

0.5 Convention. Let P be a one-sided property of rings. We shall say R is P (or R has P, if appropriate) if R is left or right P. When a ring is left and right P that fact will be explicitly mentioned.

0.6. Annihilators. Let J be a subset of a ring R. By its left annihilator $l_R(J)$ we mean the set $\{ x \in R \mid xa = 0\}$ for each a in J . Thus $l_R(J)$ is a left ideal of R. Similarly the right annihilator $r_R(J)$ of J is a right ideal of R. When the ring is clear from the context we drop the letter R. We write $l(a)$ for $l(\{a\})$ and $r(a)$ for $r(\{a\})$. When the left and right annihilators of a subset coincide (for example, in a commutative ring or a reduced ring (0.10)) we may write $\mathrm{Ann}(J)$ for $l(J) = r(J)$.

0.7. Centralizers. For an element a of R, the centralizer of a in R, i.e. the set $\{x \in R \mid xa = ax\}$, is denoted by $C(a)$. The centre of R will be denoted by $C(R)$; C denotes a subring of $C(R)$, often $C(R)$ itself. The letter T is used for a multiplicatively closed subset of C.

0.8. Idempotents. We denote by $\bar{I}(R)$ (resp. $I(R)$, resp. $B(R)$) the set of all (resp. all non-zero, resp. all central) idempotents in R.

0.9. Normal rings. A ring is normal (also called abelian in the literature) if every idempotent is central (i.e. $\bar{I}(R) \subset C(R)$ or $\bar{I}(R) = B(R)$).

0.10. Reduced rings. A ring is reduced if it has no non-zero nilpotent elements. In a reduced ring $l(x) = r(x)$ holds for each element x; see Lemma 5.1 of Chapter XII of [S].

0.11. Duo rings. By an ideal of a ring we mean a two-sided ideal. A ring is left duo if every left ideal is an ideal.

0.12. Semi-commutative rings. A ring is semi-commutative if it satisfies the following equivalent conditions.

(i) Whenever $a, b \in R$ satisfy $ab = 0$ we have $arb = 0$ for each $r \in R$.

(ii) For each $a \in R$, $l(a)$ is an ideal.

(iii) For each $a \in R$, $r(a)$ is an ideal.

0.13. Direct finiteness. A ring is directly finite if for elements a, b of the ring $ab = 1$ implies $ba = 1$. These rings have been called von Neumann finite by Herstein

([H] , p.33) and Peterson [P:75] , Dedekind finite by Faith ([FII] , p.85), finite by Kaplansky [KlI], fini by Renault [R:73] .

A ring is directly finite if and only if every right invertible element is left invertible; note that right and left can be interchanged here. The endomorphism ring of a vector space V is directly finite if and only if V is finite dimensional.

0.14. Chart. We have the following chart of implications.

$$\begin{tikzcd}
 & & \text{Domain} \arrow[d] & & \\
 & \text{Left Duo} \arrow[dr] & \text{Reduced} \arrow[d] & & \\
\text{Commutative} \arrow[ur] \arrow[dr] & & \text{Semi-commutative} \arrow[r] & \text{Normal} \arrow[r] & \text{Directly finite} \\
 & \text{Right Duo} \arrow[ur] & & &
\end{tikzcd}$$

All these implications are well-known and easy to prove. None of these implications is reversible. We shall mention only one example here: If H is the division ring of real quaternions, then H [x] is semi-commutative, but is neither left nor right duo. Examples of normal rings which are not semi-commutative are mentioned in 2.27 below.

0.15. Remarks. (i) All our left-sided concepts and results have right-sided counterparts. These are not recorded explicitly.

(ii) Generally, concepts (and terminological conventions) introduced in this thesis are given the paragraph-heading 'Definition'. Known concepts are not given this distinction.

(iii) While an effort has been made to avoid duplication of symbols, we have not been totally successful in this. For example, the symbol m has been often for an element of a module M; it has also been used for a natural number. It is hoped that such usages are too spatially separated to cause confusion.

B. Regularity

0.16 An element $a \in R$ is regular if there exists $b \in R$ such that $a = aba$. A ring is (von Neumann) regular if each of its elements is regular.

The terms in the next paragraph were introduced by Azumaya [A:54] .

0.17. An element $a \in R$ is left regular if there exists an element $b \in R$ such that $a = ba^2$; a is right regular if there exists b such that $a = a^2 b$. (Note that convention 0.5 applies to rings and not to elements. Thus a regular element is not the same thing as a left or right regular element.)

An element a is strongly regular if it is right regular as well as left regular.

0.18. In the presence of two or more rings we use terms like R-regular/ regular in R, R'-left regular etc.

0.19. Note that some authors (see, e.g., p. 283 of [S]) call an element a right regular if $r(a) = 0$, and regular if $r(a) = 0 = l(a)$. Also von Neumann regular rings are to be distinguished from the homologically regular rings of commutative algebra. Perhaps with this in mind some authors have called von Neumann regular rings by the name 'absolutely flat' rings; see e.g. Exercise 27 of Chapter 2 of [AM] .

0.20. We call d a semi-inverse of a if $a^2 d = a$, $d^2 a = d$ and $ad = da$; a is called a semi-unit [H:78] if it has a semi-inverse. If a is a semi-unit clearly it is regular and strongly regular. As recorded in [A:54] (see 2.6(b) below) strongly regular elements are semi-units; so the term 'semi-unit' is really redundant.

An application of the property of reduced rings mentioned in 0.10 yields the following result.

0.21. Proposition. Let R be a reduced ring. Then the following are equivalent for an element a of R:

(i) a is left regular; (ii) a is regular; (iii) a is right regular.

An application of 0.14 and 0.21 yields the next result. (See Proposition 12.3 of Chapter I of [S].)

0.22. Theorem. The following are equivalent for a ring R.

(i) Every element of R is left regular.
(ii) Every element of R is right regular.
(iii) Every element of R is strongly regular.
(iv) R is regular and left duo.
(v) R is regular and right duo.
(vi) R is regular and reduced.
(vii) R is regular and semi-commutative.
(viii) R is regular and normal.

0.23. A ring is <u>strongly regular</u> if it satisfies the conditions of Theorem 0.22.

0.24. Convention. Let Q be a property of rings. We say R is <u>completely Q</u> if every factor ring of R has property Q. (This condition clearly implies ' R is Q'.)

This terminology is used in the following remarks. Related questions are considered in §3 of Chapter 4.

0.25. Remark. Trivially, a regular ring is completely regular.

0.26. Remark. Left duo rings are completely left duo.

0.27. Remark. Duo rings are completely semi-commutative and completely normal. (This follows from 0.26 and 0.14 .)

0.28. Remark. Condition (i) of Theorem 0.22 passes down to factor rings. So strongly regular rings are completely strongly regular.

C. Morita contexts

For various treatments of general Morita contexts we refer to ([S] , p.112), ([JII] , §3.12) and [A:71]. Let M be a left R-module. The standard Morita context for M is denoted by (R,M,N,S); namely, $N = \mathrm{Hom}_R(M,R)$, $S = \mathrm{End}_R(M)$ and M,N have the usual bimodule structures ${}_R M_S$ and ${}_S N_R$. We have bimodule homomorphisms $(\ ,\ ) : M \otimes_S N \to R$, $[\ ,\ ] : N \otimes_R M \to S$ where $(m,f) = mf$ and $[f,m]$ is defined by $m'[f,m] = (m'f)m$ for $f \in N$, $m,m' \in M$. We also have $[f,m]g = f(mg)$ for $f,g \in N$, $m \in M$. Generalised associativity holds for meaningful strings of elements of R,M,N and S. For instance, we have the following formulae:

for $x,y \in M$, $g,h \in N$

0.29 $\quad [g,x][h,y] = [g(xh), y]$

0.30 $\quad x[g,y]h = (xg)(yh)$

Morita contexts can be defined for right modules as well. All related results have right-sided counterparts.

D. References

0.31. For the sake of completeness a number of definitions have been recalled below. See specially 1.10, 1.11 and 2.17.

0.32. For other undefined concepts we refer to the books mentioned in the bibliography. An encyclopaedic reference for regular rings is [G] .

CHAPTER 1

REGULARITY IN MODULES

In this chapter we shall study regularity in modules. Definitions of regular (elements in) modules in the sense of Zelmanowitz and Elliger are given in §1. Some basic results concerning these are also recorded in this section. In §2 McCoy's lemma for rings is extended to modules; this result is then applied to yield, inter alia, another proof of the fact that A, B regular modules implies $A \oplus B$ regular. We devote §3 to the construction of pairs of commutative rings R, U such that R is regular but R is not regular in ${}_R U$. Localizations of regular modules are studied in §4.

§ 1. Regularity and e-regularity.

In this section we define regular and e-regular elements and modules. Easy consequences of these definitions are derived and examples given. Many of these results may be known. However, we have not seen them recorded in the literature.

This section is divided into five parts. The heading of each will make its contents clear.

A. Basic definitions

1.1. An element m of a module ${}_RM$ is regular in M (when there is no possibility of confusion, regular) if there exists an element f in $\mathrm{Hom}_R(M, R)$ such that $(mf)m = m$.

1.2. A subset B of M is regular if each element of B is regular.

1.3. A module is regular if each of its elements is regular.

The following proposition can be extracted from the proof of Theorem 2.2 of [Z:72] .

1.4. Proposition. Let ${}_RM$ be a module and let $m \in M$. Then m is regular if and only if Rm is a projective direct summand of M.

Remarks concerning the proof. In view of the fundamental nature of this proposition, we point out the following:

"Only if" part. A *direct proof can be found in* the proof of Lemma 1.1 of [N:76] .

"If" part. Since we assume (in contrast to Zelmanowitz) that R has an identity element, a shorter proof can be given: Consider the map $h: R \to Rm$ defined by $rh = rm$. As Rm is projective h is split by a map

$g: Rm \to R$ satisfying $goh = 1_{Rm}$. As Rm is a direct summand of M, there exists an extension $f: M \to R$ of g. Now we have $m = (mg)h = (mf)h = (mf)m$, which proves the regularity of m. #

Elliger (§ 4, [E:71]) has defined regular modules differently; they have been called as e-regular by us. Below we give his definition, preceded by an element-wise definition.

1.5. An element m of a module ${}_RM$ is e-regular in M (or e-regular) if $Rm \leq^{\oplus} M$.

1.6. A subset B of M is e-regular if each element of B is e-regular.

1.7. A module is e-regular if each of its elements is e-regular.

1.8. Notation. For a module ${}_RD$, $\mathrm{Reg}({}_RD)$ (respectively, $\mathrm{eReg}({}_RD)$) will denote the set of all regular (respectively, e-regular) elements of D. When the ring is clear from the context we shall drop the letter R.

1.9. Remark. $\mathrm{Reg}(D) \subset \mathrm{eReg}(D)$; this follows from 1.4 and 1.5. Conditions on D which ensure that $\mathrm{eReg}(D) = \mathrm{Reg}(D)$ are given in 1.17. If D is an e-regular non-regular module clearly we have $\mathrm{eReg}(D) = D \neq \mathrm{Reg}(D)$. Examples of such modules are given in 1.18 and 1.24(iii).

1.10 . Shrikhande [S:73]$_2$ called a module <u>hereditary</u> if every submodule is projective and <u>semi-hereditary</u> if every finitely generated submodule is projective. A ring R is <u>left</u> <u>hereditary</u> (<u>semi-hereditary</u>) if ${}_RR$ is hereditary (semi-hereditary). Evans [E:72] called a module a <u>C.P.module</u> if each of its cyclic submodules is projective . Hill [H:85] called a module a <u>P.P.module</u> if it is a projective C.P. module. A ring R is a <u>left</u> <u>p.p.ring</u> if ${}_RR$ is a C.P.module (equivalently, a P.P.module). We have the implications:

$$\begin{tikzcd}
\text{hereditary} \arrow[r] & \boxed{\begin{array}{l}\text{projective}\\ \text{semi-hereditary}\end{array}} \arrow[r] \arrow[d] & \text{P.P.} \arrow[r] \arrow[d] & \text{projective} \\
\text{regular} \arrow[r] & \text{semi-hereditary} \arrow[r] & \text{C.P.} &
\end{tikzcd}$$

Of these, regular → semi-hereditary is a part of Theorem 2.2 of [Z:72] . The rest are obvious.

Torsion free modules over commutative domains are C.P.modules. Thus the Z-modules Q and Z^N are non-projective C.P. modules. The ring $R= Z/4Z$ is not a p.p.ring; so ${}_RR$ is a non-C.P. module which is projective.

1.11. We shall follow Bourbaki [BII8] in the use of <u>semi-simple.</u> A ring is <u>semi-primitive</u> if its Jacobson radical is zero; such rings have been called as <u>(Jacobson-)semi-simple</u> by some.

Some of the above terminology will be used in the following remarks; the proofs are straight-forward.

1.12. Remark (Elliger). Semi-simple modules are e-regular.

1.13. Remark. Semi-simple, projective modules are regular.

1.14. Remark. Let ${}_RD$ be a module such that $\mathrm{Hom}_R(D, R) = 0$. Then $\mathrm{Reg}(D) = 0$.

1.15. Remark. An element d of a module ${}_RD$ is regular if and only if it is e-regular and Rd is projective.

1.16. Remark. A module is regular if and only if it is e-regular and a C.P.module.

1.17. Remark. Assume that either (1) D is projective or (2) D is a C.P.module. Then $\mathrm{eReg}(D) = \mathrm{Reg}(D)$.

1.18. Remark. Let p be a prime. Since $\mathrm{Hom}_Z(Z/pZ, Z) = 0$ we have $\mathrm{Reg}({}_Z(Z/pZ)) = 0$ (by 1.14). In particular, the Z-module Z/pZ is non-regular. Since it is simple, it is e-regular(over Z as well as Z/pZ), by 1.12 .

B. The classical case

The following (known) facts connect the above concepts with the 'classical' case of regularity in a ring.

1.19. Remark. Let $M = {}_RR$; then $\mathrm{Hom}_R(M, R) = R$. Hence the condition in 1.1 reduces to: there exists b in R such that $aba = a$, recovering the definition (0.16) of a regular element in a ring. Since ${}_RR$ and

R_R are projective and since Definition 0.16 is left-right symmetric we have:

1.20. Proposition. The following conditions are equivalent for an element a of a ring R.

(1) a is regular in ${}_RR$.

(2) a is regular in R_R.

(3) a is e-regular in ${}_RR$.

(4) a is e-regular in R_R.

(5) There exists an element b in R such that $aba = a$.

1.21. Remark. It follows from 1.20 that $\mathrm{Reg}({}_RR) = \mathrm{Reg}(R_R) = \mathrm{eReg}({}_RR) = \mathrm{eReg}(R_R)$ is the set of all regular elements of R. This set will be denoted by Reg(R).

C. Change of ring

Let $\phi\colon R \to R'$ be a ring homomorphism, D a left R'-module. Then D has a natural R-module structure via ϕ.

1.22. Proposition. Suppose ϕ is onto. Then we have

1.22(*) $$\mathrm{eReg}({}_RD) = \mathrm{eReg}({}_{R'}D)$$

Proof. Let d be an element of D. Since ϕ is onto we have, (1) $Rd = R'd$ and (2) the equivalence of the conditions "Rd is an R-direct summand of D" and "R'd is an R'-direct summand of D". The desired conclusion follows. #

1.23. Corollary. Let R be a regular ring and let A be an ideal of R. Then R/A is e-regular as a left (as well as a right) R-module.

Proof. Let $R' := R/A$, $D := R'$ and $\phi: R \to R'$ the canonical ring homomorphism. Now R' is a regular ring and is therefore (as seen in § 1B) e-regular as a left module over itself. Hence R' is e-regular as a left R-module. #

1.24. Examples. The following examples may help understand the concepts introduced here. They will show that when ϕ is not onto 1.22(*) may not hold.

(i) Let $R = Z$, $R' = Q$ and ϕ the canonical inclusion. Let $D = Q$. Then $\mathrm{eReg}({}_Z Q) = \{ 0 \}$ but $\mathrm{eReg}({}_Q Q) = Q$ shows that $\mathrm{eReg}({}_R D) \subsetneqq \mathrm{eReg}({}_{R'} D)$ in this situation.

(ii) Let $R = K$, a field and $R' = K[x]$ and ϕ the canonical inclusion. Let $D = K[x]$.

Then $\mathrm{eReg}({}_K K[x]) = K[x]$ and $\mathrm{eReg}({}_{K[x]} K[x]) = K$ shows that $\mathrm{eReg}({}_R D) \supsetneqq \mathrm{eReg}({}_{R'} D)$ in this situation.

(iii) Let K be a field, $U = K^N$, $A = K^{(N)}$; A is an ideal of the regular ring U and U/A is also a regular ring (by 0.25). It follows, by 1.23, that the U-module U/A is e-regular.

Now let $f:U/A \to U$ be U-linear and $I = (U/A)f$. Then $AI = 0$ implies that $I \subset \mathrm{Ann}({}_U A) = 0$. This shows that $\mathrm{Hom}_U(U/A, U) = 0$ and therefore, by 1.14, $\mathrm{Reg}({}_U U/A) = 0$; in particular, U/A is not regular as a U-module.

There are natural ring homomorphisms $K \to U$ and $U \to U/A$. Thus U is "sandwiched" between K and U/A. However, we have, using the fact that U/A is a K-vector space,

$$\mathrm{Reg}({}_K U/A) = U/A \supset \mathrm{Reg}({}_U U/A) = \{0\} \subset \mathrm{Reg}({}_{U/A} U/A) = U/A.$$

This example and 1.18 show that there is no regular analogue of 1.22.

1.25. Remark. Cheatham and Enochs [CE:81] have called a ring R (left) quasi-perfect if every finitely generated flat left R-module is projective . They have shown (Part(c) of Theorem 2) that over left quasi-perfect rings factor modules of regular modules are regular.

Elsewhere in that paper they have asserted that left coherent rings are left quasi-perfect. Notice that the ring U of Example 1.24(iii) is coherent, since it is regular. However the regular U-module U has U/A as a non-regular factor module. Therefore U cannot be quasi-perfect. This can also be seen directly (the U-module U/A is flat but non-projective) as was pointed out in another context by Rege (Remark 3.5 in [R:86]).

D. Module pairs

In what follows, $(B, {}_RD)$ denotes a module pair (0.2).

1.26. Remark. Let b be an element of B which is e-regular in D. Since Rb is a direct summand of D, it is also a direct summand of B. This shows $B \cap \mathrm{eReg}(D) \subset \mathrm{eReg}(B)$.

1.27. Remark. As in 1.26 we have $B \cap \mathrm{Reg}(D) \subset \mathrm{Reg}(B)$.

1.28. Remark. Over $R = Z$ we have $\mathrm{Reg}(Q) = \mathrm{eReg}(Q) = \{0\}$. and $\mathrm{Reg}(Z) = \mathrm{eReg}(Z) = \{0, 1, -1\}$. Thus strict inclusions can hold in 1.26 and 1.27

1.29. Remarks. (i) Suppose that D is an e-regular module. Then $D = \mathrm{eReg}(D)$ yields (using 1.26) that $B = \mathrm{eReg}(B)$. Thus we get Elliger's result (Proposition 4.5 of [E:71]) that submodules of e-regular modules are e-regular.

(ii) The regular analogue of this result due to Zelmanowitz ((1.4) of [Z:72]) can be similarly derived from 1.27.

Now we give a condition which ensures that $B \cap \mathrm{Reg}(D) = \mathrm{Reg}(B)$.

1.30. Definition. A module pair $(B, {}_RD)$ satisfies condition (C_1) if each R-homomorphism from B to R can be extended to a R-homomorphism from D to R

(equivalently, the natural map $\mathrm{Hom}_R(D, R) \to \mathrm{Hom}_R(B, R)$ is onto).

1.31. Remark. The pair $(B, {}_RD)$ satisfies (C_1) in each of the following cases:

(1) R is left self-injective, and

(2) (R is any ring and) B is a direct summand of D.

As an immediate consequence we have

1.32. Remark. Assume that (B, D) satisfies (C_1). Then $B \cap \mathrm{Reg}(D) = \mathrm{Reg}(B)$. Hence B is regular if and only if B is regular in D.

1.33. Remark (e-regular analogue of 1.32).

Assume that $B \leqslant^{\oplus} D$. Then $B \cap \mathrm{eReg}(D) = \mathrm{eReg}(B)$. Hence B is e-regular if and only if B is e-regular in D.

E. Other definitions

While we shall not study them in this thesis it should be noted that there are some other definitions of regular modules. Fieldhouse calls a module regular ("f-regular") if every submodule is pure [F:69] . Ware has studied regular projective ("w-regular") modules [W:71] . For projective modules the various classes (regular, e-regular, f-regular and w-regular) coincide; in particular each concept yields von Neumann regularity in the case of a ring. Note that for some authors, notably Nicholson [N:75] and Zelmanowitz [Z:72] , rings do not always have identity elements. Cheatham and Enochs [CE:81] have recorded some relationships between regular modules and f-regular modules. Hirano [H:81] has made a comparative study of regular modules and V-modules.

§2. Absolutely regular modules.

We shall not be interested in e-regularity in the rest of this thesis (except in a few remarks). Thus, hereafter, regularity means regularity in the sense of Zelmanowitz (1.1 - 1.4).

In this section we shall prove the existence of commutative rings R and U satisfying the following conditions: R is a regular subring of U, but R is not regular in $_RU$. The following definitions are suggested by this situation:

1.34. Definition. A module B is absolutely regular if for each overmodule D, B is regular in D.

1.35. Definition. A ring R is left absolutely regular if the module $_RR$ is absolutely regular. (In the commutative case we drop the adjective 'left'.)

We also introduce the following definition.

1.36. Definition. A module B has the <u>weak</u> <u>extension</u> <u>property</u> if for each overmodule D, the pair (B, D) satisfies (C_1). (See 1.30.)

1.37. Remark. It follows from 1.31 that ${}_RB$ has the weak extension property in each of the following cases.

(1) R is left self-injective and B arbitrary.

(2) (R is any ring and) B is injective .

1.38. Remark. Clearly, every absolutely regular module is regular.

1.39. Proposition. Every regular module with the weak extension property is absolutely regular.

Proof. Let B be a regular module with the weak extension property. Let D be an overmodule of B. Then, by assumption, (B, D) satisfies (C_1). This implies, by 1.32, that B is regular in D. Hence B is absolutely regular. #

1.40. Corollary. Every regular, left self-injective ring is left absolutely regular.

1.41. Corollary. Semi-simple rings (1.11) are left and right absolutely regular.

The rest of this section is devoted to the construction (for each field of characteristic zero) of a class of regular rings which are not absolutely regular. These results arose from a consideration of Example 4.4 of [Z.- H.:76].

1.42. Let K be a commutative ring, $U := K^N$, $A := K^{(N)}$ and R_0 a subring of U containing A. Clearly A is a large R_0-submodule of U. If $R_0 \neq U$, then R_0 cannot be a R_0-direct summand of U, and in particular cannot be self-injective. Therefore with a suitable choice of K and R_0 we can hope to get the examples we want. (See 1.40.)

Assume that K is a field of characteristic zero. (In fact, some results can be proved under the assumption that K is any infinite field.) Let $\underline{R}$ be the regular subring of U consisting of sequences $(a_i)_{i \in N}$ for which the set $\{ a_i \mid i \in N \}$ is finite.

1.43. Proposition. Let R be any subring of $\underline{R}$ containing A (so that U becomes an R-module). Let $f \in \mathrm{Hom}_R(U, R)$. Then $Uf \leqslant A$.

Proof. Suppose, if possible, $\underline{t}f \notin A$ for some $\underline{t} = (t_i)_{i \in N} \in U$ Assume $\underline{t}f = (a_i)_{i \in N}$. Then $J := \{i \in N \mid a_i \neq 0\}$ is an infinite set. Now define an element $\underline{u}$ of U as follows. If $i \in J$, write $u_i = it_i/a_i$, if $i \notin J$ write $u_i = 0$; finally set $\underline{u} = (u_i)_{i \in N}$.

We shall identify K with a subring of R via the map $a \to (a,a,a \ldots)$. Write $e^i = (\delta_{ij})_{j \in N}$, where δ_{ij} denotes Kronecker delta. Then $e^i \in A \subset R$. Let $i \in J$. Note that $e^i\underline{u} = (i/a_i)e^i\underline{t}$. Using the R-linearity of f, we have: the i th co-ordinate of $\underline{u}f$ equals

$$((e^i\underline{u})f)_i = i/a_i \ [(e^i\underline{t})f]_i = i/a_i(\underline{t}f)_i = i.$$

Since K is a field of characteristic zero, and J is an infinite set, this contradicts the requirement that $\underline{u}f \in R \leqslant \underline{R}$. #

1.44. Corollary. $\mathrm{Reg}({}_RU) = A$; therefore the module ${}_RU$ is not regular.

Proof. Let $t \in \mathrm{Reg}({}_RU)$. Then $t = (tf)t$ for some $f \in \mathrm{Hom}_R(U, R)$. Hence $t \in At \subset A$. Next let $a \in A$. Then there exists $a' \in A$ such that $aa'a = a$. Consider $f: U \to R$ defined by $tf = ta'$ for $t \in U$. Then $a = aa'a = (af)a$ shows that $a \in \mathrm{Reg}({}_RU)$.

1.45. Examples. Let K be a field of characteristic zero. Set

$$\mathcal{R}_K = \{ R \mid R \text{ is a regular subring of } \underline{R} \text{ containing } A \} .$$

Then any ring in $\mathcal{R}_K$ *is regular but not absolutely regular* (by 1.44). Examples of rings in $\mathcal{R}_K$ are: (1) $\underline{R}$ itself, and (2) the ring of sequences $\underline{a}$ in U such that all but finitely many entries of $\underline{a}$ are equal.

1.46. Remark. The ring U is regular and self-injective (see Corollary 5.2 of [SV :74] for a proof of self-injectivity). Therefore U is absolutely regular by 1.40. The examples in 1.45 show that regular subrings of absolutely regular rings need not be absolutely regular.

§3. McCoy's lemma for modules.

The following lemma due to McCoy has been used in the study of regular rings ([KI], pp. 111-115).

1.47. Lemma. Let R be a ring and a, d be elements of R. If $dad - d$ is regular, then d is regular.

If D is a left R-module, $\mathrm{Hom}_R(D, R)$ has the *natural structure of a right R-module; see §C of Chapter 0.*

This structure will be exploited to extend 1.47 to modules:

1.48. Proposition ("McCoy's lemma for modules"). Let $d \in {}_R D$ and $q \in \mathrm{Hom}_R(D, R)$ such that $(dq)d - d$ is regular (in D). Then d is regular (in D).

Proof. By hypothesis, there exists $g \in \mathrm{Hom}_R(D, R)$ such that $\underline{(dq)d -} d = [((dq)d - d)g]((dq)d - d)$.

On expanding and rearranging this equation we get $d = (dh)d$ where

$$h = q - g + q(dg) + g(dq) - q(dg)(dq) \in \mathrm{Hom}_R(D, R). \quad \#$$

1.49. Remark. On putting $D = R$ in Proposition 1.48 we get 1.47.

The rest of this section is devoted to some applications of 1.48.

1.50. Proposition. Let D, M be left R-modules and $k : D \to M$ be an R-homomorphism with kernel B. Assume that B is regular in D. If $d \in D$ is such that dk is regular in M, then d is regular in D.

Proof. Since dk is regular in M there exists $f \in \mathrm{Hom}_R(M, R)$ satisfying $((dk)f)dk = dk$. Write

$q: = kf \in \mathrm{Hom}_R(D, R)$. The element $(dq)d - d$ belongs to B and is therefore regular in D. By 1.48 the element d is regular in D. #

1.51. Corollary. Let $B \lessdot D$ and assume that B is regular in D. (This happens if B is regular and (B, D) satisfies (C_1).) If d in D is such that $\bar{d}$ is a regular element of the factor module D/B, then d is regular in D. If D/B is a regular module, then D is a regular module.

1.52. Corollary. If $B \lessdot D$, if B is absolutely regular, and if D/B is a regular module, then D is a regular module.

1.53. Definition. Let a be an element and B a submodule of a module D. The pair $\underline{(a, B) \text{ satisfies condition } (C_2)}$ $\underline{\text{(in } D)}$ if there exists $f:D \to R$ such that $(af)a = a$ and $Bf = 0$.

1.54. Remark. The seemingly artificial condition (C_2) applies in the following situation: Suppose that $D = A \oplus B$, and $a \in A$. Then the pair (a, B) satisfies (C_2) if and only if a is regular in A.

1.55. Proposition. If the pair (a, B) satisfies (C_2) in D, if B is regular in D, and if b is an element of B, then $a + b$ is regular in D.

Proof. By hypothesis, there exists $f:D \to R$ such that $(af)a = a$ and $Bf = 0$. Write

$$z: = [(a + b)f] (a + b) - (a + b).$$

Now $z = (af)b - b$ is an element of B and is therefore regular in D. It follows, by 1.48, that $a + b$ is a regular element of D. #

1.56. Corollary. Let $D = A \oplus B$. Then D is a regular module if and only if A and B are regular modules.

Proof. By 1.29 (ii) it is sufficient to prove the "if part". Let $a + b$ (with $a \in A$ and $b \in B$) be an element of D. By 1.54 the pair (a, B) satisfies (C_2). Hence by 1.55 $a + b$ is regular in D. #

1.57. Remark. The result proved in 1.56 is a crucial step in the proof of Theorem 2.8 of [Z:72] . However, the proof given here is different.

§4. Localizations of regular modules.

Central localizations of regular rings have been studied by Armendariz, Fisher and Steinberg [AFS:74]. In this section we shall study central localizations of regular modules.

1.58. Preliminaries. Let (as before) R be a ring, C a subring of $C(R)$ = Centre (R), M a left R-module, $N = \mathrm{Hom}_R(M, R)$ (a C-module) and m a fixed element of M. The set $mN = \{mf \mid f \in N\}$ has been of interest to commutative algebraists, who have often denoted it as $O_M(m)$ (see, e.g., (2.1) in [BR:82]). In view of the right R-module structure on N, mN is a right ideal of R. The set $(mN)m$ ($= mNm$) is a C-submodule of Rm.

A multiplicatively closed subset of C will be denoted by T. Then $T^{-1}C$ is a subring of Centre$(T^{-1}R)$, and $T^{-1}M$ is a left $T^{-1}R$-module. If $p \in \mathrm{Spec}(C)$, we shall use the usual notations C_p, R_p and M_p; similar notation will be used when $v \in \mathrm{Max}(C)$. For basic properties of these rings and modules of fractions we refer to [AM] and [JII].

1.59. Proposition. Let m be a regular element of ${}_RM$ and t an element of T. Then m/t is a regular element of ${}_{T^{-1}R}T^{-1}M$.

Proof. Let $m = (mf)m$ for some $f \in \mathrm{Hom}_R(M, R)$. Consider $tf/1 \in \mathrm{Hom}_{T^{-1}R}(T^{-1}M, T^{-1}R)$. (Note that $t \in T \subset C \subset C(R)$.)

Since $[(m/t)tf/1]\ m/t = (mf)m/t = m/t$, we get the desired result. ⫲

1.60. Corollary. Let ${}_RM$ be a regular module. Then $T^{-1}M$ is a regular $T^{-1}R$-module for each T; in particular, M_v is a regular R_v-module for each $v \in \mathrm{Max}(C)$.

1.61. Remark. On putting $M = R$ in 1.60 we get the well-known result that if R is a regular ring then so is $T^{-1}R$ for each multiplicatively closed subset T of the centre of R.

1.62. Example. Recall Example 1.24(iii). The ring U is regular but U/A is not a regular U-module. For each $v \in \mathrm{Max}(U)$, U_v is a field; therefore $(U/A)_v$ is a regular U_v-module (being a vector space over U_v). This example shows that even when M_v is a regular R_v-module for each $v \in \mathrm{Max}(C)$, M need not be a regular R-module.

A finiteness condition on M which ensures the regularity of M will be given in Theorem 1.64. Note that this theorem extends (a) → (c) of Theorem in [AFS:74], since ${}_RR$ is trivially finitely presented.

1.63.Remark. The following notation will be used in the proof of Theorem 1.64. The C-module (Cm + mNm)/mNm will be denoted by W(m). Clearly m is regular ↔ $Cm \leq mNm$ ↔ W(m) = 0. (Thus W(m) is a "test module" for the regularity of m in M.)

1.64. Theorem. Let M be a left R-module. Consider the following conditions.

(1) ${}_RM$ is regular

(2) ${}_{T^{-1}R}\,T^{-1}M$ is regular for each T.

(3) ${}_{R_p}M_p$ is regular for each $p \in \mathrm{Spec}(C)$.

(4) ${}_{R_v}M_v$ is regular for each $v \in \mathrm{Max}(C)$.

Then (1) ↔ (2) → (3) → (4). If, further, M is a finitely presented R-module then the four conditions are equivalent.

Proof. (1) implies (2) was seen in 1.60 and (2) implies (1) is trivial (choose $T = \{1\}$). The assertions (2) implies (3) and (3) implies (4) are also clear. We shall now prove (4) implies (1) under the assumption that ${}_R M$ is finitely presented.

It is well-known that when ${}_R M$ is finitely presented for any left R-module M' the $T^{-1}C$-modules $T^{-1}\mathrm{Hom}_R(M, M')$ and $\mathrm{Hom}_{T^{-1}R}(T^{-1}M, T^{-1}M')$ are canonically isomorphic. (See, e.g., Proposition 2.13", Chapter I of [L´] .) Now let $m \in M$ and $v \in \mathrm{Max}(C)$. Identifying $(\mathrm{Hom}_R(M, R))_v$ and $\mathrm{Hom}_{R_v}(M_v, R_v)$ as above, we have (using standard properties of localization):

$$(W(m))_v = (C_v m/1 + m/1\mathrm{Hom}_{R_v}(M_v, R_v)m/1)/(m/1\mathrm{Hom}_{R_v}(M_v, R_v)m/1)$$

Since M_v is a regular R_v-module by assumption, this yields $(W(m))_v = 0$ for each $v \in \mathrm{Max}\ (C)$.

It follows (by an application of an analogue of Proposition 3.8 of [AM] for non-commutative rings) that $W(m) = 0$. Therefore m is a regular element of M by 1.63. #

1.65. Remark. As pointed out in §1.1 E, there is a definition of regular modules ("f-regular modules") due to Fieldhouse. For these modules he has proved (Theorem 11.2 of [F:69]): Let R be a commutative ring and M an R-module. Then ${}_RM$ is f-regular $\leftrightarrow {}_{R_v}M$ is f-regular for each maximal ideal v of R.

CHAPTER 2

REGULARITY IN RINGS

In this chapter we shall study regularity and one-sided regularity (i.e., left and right regularity) for elements of a ring. Some basic results concerning one-sided regularity are collected in § 1; 2-finite rings are defined and studied in § 2; normal, 2-finite rings are studied in §3; sufficient conditions for Reg(R) to be closed under multiplication are given in §4.

§1. One-sided regularity in rings.

A main result of this section (Proposition 2.6(a)) is in [A:54] ; see also Proposition 1(1) in [H:78] . However, our proof, which uses the concept of the idealizer, is different and some intermediate results are used later.

The terminology of §B of Chapter 0 will be followed; i denotes a non-negative integer and m, n denote natural numbers. The letters a, b, c denote, as usual, elements of a ring. The notation 0.6 for annihilators will be followed.

2.1. Let a be right regular with $a^2b = a$. Then:

(a) $l(a^2) = l(a)$.

(b) For each i, $a^{i+1}b^i = a$ holds. (This is clear for $i = 1$ (and for $i = 0$). For $i \geq 2$ the equation $a^ib^{i-1} = a$ yields $a^{i+1}b^i = aa^ib^{i-1}b = a^2b = a$.)

(c) Suppose that $i+1 \leq n$. Then we have $a^nb^i = a^{n-i-1}a^{i+1}b^i = a^{n-i-1}a = a^{n-i}$, using Part (b).

(d) Substituting $i = m$ and $n = 2m$ in (c) we get $a^{2m}b^m = a^m$. Thus a^m is right regular for each m. #

Although it is possible to deduce the next result from Lemma 3 of [A:54] we give a direct proof.

2.2. Proposition. Let a be right regular and let a^m be left regular for some m. Then a is left regular.

Proof. Let $a^2b = a$ and $ca^{2m} = a^m$ for some b, c.

Then using (b) and (d) of 2.1 we get $a = a^mb^{m-1} = ca^{2m}b^{m-1} = ca^2\ a^{m-1} = (ca^{m-1})a^2$. This shows that a is left regular. #

Next we recall a well-known concept.

2.3. Let A be a right ideal of a ring R. Let $\tilde{I}(A)$ be the set of all elements t of R satisfying $tA \leq A$. Then

$\widetilde{I}(A)$ is a subring of R, called the idealizer of A (in R); $\widetilde{I}(A)$ contains A as an ideal, being the largest subring which does so.

2.4. Proposition. Suppose that a is right regular with $a^2b = a$. Assume further that $r(a^2) = r(a)$. then:

(a) $b \in \widetilde{I}(r(a))$, the idealizer of $r(a)$.

(b) $ab^na = ab^{n-1}$ for each n.

(c) $c = ab^2$ is a semi-inverse of a; in particular, a is left regular.

(d) $ab = ac$ is an idempotent.

Proof. (a) Since $a^2b.r(a) = a.r(a) = 0$ we have $b.r(a) \leqslant r(a^2) = r(a)$.

(b) Since $\widetilde{I}(r(a))$ is a subring of R, by Part (a) we get $b^{n-1} \in \widetilde{I}(r(a))$ for each n. Also $a^2b = a$ yields $a^2(ba - 1) = 0$ and so $ba - 1 \in r(a^2) = r(a)$. So we get $b^{n-1}(ba-1) \in r(a)$. This implies that $ab^na = ab^{n-1}$ for each n.

(c) Part (b) yields, on putting n = 1,2,3 successively,

(2.5) $aba = a,\ ab^2a = ab,\ ab^3a = ab^2$.

Next, using 2.5, we get,

$ac = a^2b^2 = ab = ab^2a = ca$, $c^2a = ab^2ab^2\,a = ab.b^2a$ $= ab^3a = ab^2 = c$, and $a^2c = a^3b^2 = a$ (using 2.1 (b)).

This shows that c is a semi-inverse of a. Therefore a is left regular.

(d) This follows from 2.5 and (c). #

The proof of Part (c) of the above proposition yields a different proof of (most of) the following proposition due to Azumaya; we record 2.6 since we need it later.

2.6. Proposition(Azumaya). (a) Let a be right regular. Then a is left regular if and only if $r(a^2) = r(a)$.

(b) An element a is strongly regular if and only if a is a semi-unit. In this case a has a unique semi-inverse z. Moreover $C(a) = C(z)$ (in the notation of 0.7).

2.7. Proposition. Let a be an element of a ring R, which is a subring of R'. Suppose that a is R-right regular and R'-left regular (see 0.18). Then a is R-left regular.

Proof. By hypothesis, there exists an element t of R' such that $a = ta^2$. Therefore $r_R(a) = r_R(a^2)$. Hence, by 2.6(a), a is R-left regular. #

§2. 2-finite rings.

Let K be a field and $M_n(K)$, the ring of $n \times n$ matrices over K. Let matrices A, B satisfy $A = A^2B$. It can be shown using linear algebra that there exists a matrix C such that $A = CA^2$. Thus in $M_n(K)$ every right regular element is left regular and the dual property also holds. Thus $M_n(K)$ is left and right 2-finite in the sense of the following definition. (These properties of $M_n(K)$ also follow as a special case of 2.19 or 2.21 below.)

2.8. Definition. A ring R is <u>right 2-finite</u> if every right regular element is left regular.

2.9. Remarks. (i) A ring R has been called <u>n-finite</u> in the literature if $M_n(R)$ is a directly finite ring; see [L:70]. It is hoped that our usage of the term "2-finite", in a different sense, will not cause confusion.

(ii) Although the term "2-finite" is not used by Azumaya [A:54] his Theorem 1 asserts that every ring of bounded index is left and right 2-finite. Apart from this result we have not come across any study of 2-finite rings in the literature.

2.10. Remark. Let a be right invertible and left regular. The equations $ca^2 = a$ and $ab = 1$ yield

$$1 = ab = ca^2b = ca.ab = ca$$

It follows that $c = b$ and hence a is invertible.

2.11. Proposition. Every 2-finite ring is directly finite. (See 0.13.)

Proof. Let R be right 2-finite. Suppose that $a, b \in R$ satisfy $ab = 1$. Then $a = a^2b$ shows that a is right regular. By hypothesis a is left regular. It follows by Remark 2.10 that a' must be invertible. So by 0.13 R is directly finite. #

2.12. Remark. We do not have an example of a ring which is directly finite but not 2-finite. The results of this section may be of some use for settling the question of the existence of such rings.

In the rest of this section we give sufficient conditions for the 2-finiteness of a ring. These results are analogues of known (or easy) results concerning the direct finiteness of a ring. However many of the proofs are different. In 2.13 - 2.15 we deduce the 2-finiteness of a ring from the 2-finiteness of certain rings related to it.

2.13. Proposition. Let R be a subring of a right 2-finite ring R'. Then R is right 2-finite.

Proof. Let a be R-right regular (and hence R'-right regular). As R' is right 2-finite, a is R'-left regular. Hence, by 2.7, a is R-left regular. #

2.14. Remark. Let $\{R_i\}_{i \in I}$ be a family of rings. Then their direct product ΠR_i is right 2-finite if and only if each R_i is right 2-finite. (This can be verified easily.)

2.15. Proposition. Let R be a ring with centre C. Suppose that for each maximal ideal v of C the localization R_v is a right 2-finite ring. Then R is right 2-finite.

Proof. Let R' be the direct product of the rings R_v. By 2.14 the ring R' is right 2-finite. There is a canonical injective ring homomorphism from R to R'. Hence, by 2.13 R is also right 2-finite. #

2.16. Remark. Note that 2.13 - 2.15 have easily *verifiable analogues for directly finite rings.*

2.17. Azumaya [A:54] called an element a of a ring right π-regular if some suitable power of a is right regular; R is right π-regular if so is every element of R; an element (or a ring) is strongly π-regular if it is left and right π-regular. Dischinger [D:76] proved that every left π-regular ring is right π-regular; Zöschinger [D:76] and Hirano [H:78] have given simplified proofs of this fact. Thus there is no distinction between left π-regularity, right π-regularity and strong π-regularity for rings. Clearly strongly regular rings (0.23) and artinian rings are strongly π-regular. Commutative π-regular rings have been characterised in Lemma 5.6 of Storrer [S:68] . They are precisely the zero-dimensional rings.

It is easy to see that strongly π-regular rings *are directly finite.*

2.18. Remark. Let R be a strongly π-regular ring. Let a be a right regular element of R. By the left π-regularity of R the element a^m is left regular for some m. Hence by 2.2 a is left regular. This shows that R is right (and, similarly, left) 2-finite. #

It is known that left (or right) noetherian rings are directly finite. In the following theorem we consider larger classes of rings.

2.19. Theorem. Let R be a ring which satisfies the ascending chain condition on right annulets (i.e. right ideals of the form r(J)).Then R is right 2-finite.

Proof. Let a be right regular. Consider the chain $r(a) \leqslant r(a^2) \leqslant \dots$. By the hypothesis on R this chain terminates and so $r(a^m) = r((a^m)^2)$ for some m. By 2.1(d), the element a^m is right regular. Hence by 2.6(a) a^m must be left regular. It follows, by an application of 2.2, that a must be left regular. #

2.20. Among rings satisfying the hypothesis of 2.19 are right noetherian rings, left artinian rings (their subrings) and domains.

2.21. Proposition. Let R be finitely generated as a module over its centre C. Then R is left and right 2-finite. (It is known that such rings are directly finite; see Corollary 3 of Orzech [O:71].)

Proof. Let $x_1, x_2, \ldots x_r$ be a set of generators of R as a module over C. Let a be a right regular element of R with $a^2b = a$. Let S_0 be a finite set of elements of C which appear as coefficients of the x_i when a, b, x_j+x_k and x_jx_k $(1 \leqslant j,k \leqslant r)$ are expressed as C-linear combinations of the x_i $(1 \leqslant i \leqslant r)$. Let C_0 be the subring of C generated by S_0 and let R_0 be the C_0-subalgebra of R spanned by the elements x_i. Then R_0 is left and right noetherian and $a, b \in R_0$. We are now through by Theorem 2.19.

Next we recall a result of Peterson ([P:75] , Proposition 2.3.)

2.22. Proposition. Let R be a ring integral over its centre C. Then R is directly finite.

By adapting Peterson's argument we prove:

2.23. Proposition. Let R be integral over its centre C. Then R is left and right 2-finite.

Proof. Assume that $a^2 b = a$. Write

$$b^{n+1} = \sum_{j=0}^{n} c_j b^j ,$$

with $c_j \in C$ for each j. Then, using 2.1(b), we get,

$$a = a^{n+2} b^{n+1} = \sum_{j=0}^{n} c_j a^{n+2} b^j = \sum_{j=0}^{n} c_j a^{n+2-j}$$

$$= \left(\sum_{j=0}^{n} c_j a^{n-j} \right) a^2$$

This shows that a is left regular. #

2.24. Remark. Although 2.21 can be deduced from 2.23 both proofs have been given since each appeared to be of independent interest.

§3. Normal, 2-finite rings.

The main result of this section characterises normal 2-finite rings. Notation 0.8 for sets of idempotents will be followed.

2.25. Theorem. The following conditions are equivalent for a ring R with centre C.

(1) R is normal and right 2-finite.

(2) If $a = a^2b$, then $ab \in C$.

(3) If $a = a^2b$, then $a = ba^2$.

(4) If $a = a^2b$, then ab and ba are central idempotents.

Proof. (1) → (2). Let $a = a^2b$ hold for elements a, b of R. Since R is right 2-finite, a is left regular; equivalently, $r(a) = r(a^2)$. By 2.4(d) $ab \in \bar{I}(R)$. As R is normal, $ab \in C$.

(2) → (3). Let $a = a^2b$. Then, by hypothesis, $ab \in C$ and therefore $a = aab = aba$. Hence $ba = (ba)^2.1$ implies (again invoking the hypothesis) $ba.1 \in C$. Therefore $a = aba = baa = ba^2$.

(3) → (1). Let $e = e^2$, $y \in R$. Write $n := (1 - e)ye$. Then $n^2 = 0$, $en = 0$ and $ne = n$. Hence $e^2(e + n) = e$, which implies that $(e + n)e^2 = e$. Therefore $n = 0$. It follows that $ye = eye$ for each idempotent e. Replacing

e by $(1 - e)$ in the last equation we get $y(1 - e) = (1 - e)y(1 - e)$ which yields $ey = eye$. Thus $ey = eye = ye$ for each $y \in R$, showing that $e \in C$. This proves that R is normal. It is trivially right 2-finite.

(2) → (4). This follows by the proof of (2) → (3); $a = aba$ yields the idempotency of ab.

(4) → (2) is trivial.

This completes the proof of the theorem.

In Proposition 2.26 it will be shown that semi-commutative rings are left and right 2-finite. It will then follow from 0.14 that all left or right duo rings are left and right 2-finite.

2.26. Proposition. Semi-commutative rings are (normal and) left and right 2-finite.

Proof. The normality of semi-commutative rings (0.14) will be used in the proof of this proposition. We shall verify Condition (2) of Theorem 2.25. Let $a = a^2b$, i.e. $a(1 - ab) = 0$. As R is semi-commutative

$ar(1 - ab) = 0$ for each $r \in R$. In particular, $ab(1 - ab) = 0$ showing that $ab \in \bar{I}(R) \subset C$, by the normality of R.

2.27. Examples. Normal, left and right 2-finite rings need not be semi-commutative. This is shown by the following examples.

(1) The ring of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a + d$, b and c are even integers is an example of a normal ring which is not semi-commutative. (See Example 5.5 in $[S:73]_1$.) As this ring is a subring of $M_2(Q)$ it is left and right 2-finite by 2.13.

(2) Let G be a finite group and $Z(G)$ its integral group ring. It is well-known (see, e.g., p.35 of [H]) that $Z(G)$ has no non-trivial idempotents; it is therefore a normal ring. As $Z(G)$ is finitely generated as a Z-module, it is left and right 2-finite, by 2.21. Since $Q(G)$ is the localization of $Z(G)$

with respect to the set of non-zero integers, it is easily seen that $Z(G)$ is semi-commutative if and only if $Q(G)$ is semi-commutative. By Maschke's theorem $Q(G)$ is semi-simple, and therefore, by Wedderburn's structure theorem, a finite product of matrix rings over division rings. It is easy to give examples of groups G such that $Q(G)$ is not reduced (equivalently, because of the above argument, not semi-commutative). For these groups $Z(G)$ is normal, left and right 2-finite, but not semi-commutative .

§ 4. Closure properties.

In this section we shall give sufficient conditions on a ring R which ensure that the subset $Reg(R)$ of all regular elements of R is closed under multiplication.

2.28. Definition. A ring R has *property (RC)* if the subset $Reg(R)$ is closed under multiplication.

2.29. Remark. If R is commutative then trivially R has property (RC): $a = aba$ and $c = cdc$ implies that $ac = aba\,cdc = ac\,bd\,ac$.

2.30. Remark. If R is regular then Reg(R) = R shows that R has property (RC).

2.31. Example. Let R' be a subring of a ring R. Then R has property (RC) does not imply R' has property (RC), as shown by the following example.

The ring $M_n(D)$, where D is a division ring, and $n \geqslant 2$ is regular and therefore has property (RC). Let $R' = UT_n(D)$, the upper triangular matrix ring, a subring of $M_n(D)$. The elements $E_{11} + E_{12}$ and E_{22} are idempotents and so belong to Reg(R').

But $(E_{11} + E_{12})E_{22} = E_{12} \notin$ Reg(R').

This example also shows the following:

(i) A left and right artinian ring need not have property (RC)

(ii) R/Rad(R) has property (RC) does not imply that R has property (RC). In the above example R'/Rad(R') is regular, being a finite product of copies of the division ring D.

2.32. Remark. Let $\{R_i\}_{i \in I}$ be a family of rings and let $R = \Pi R_i$. Then $\mathrm{Reg}(R) = \Pi \mathrm{Reg}(R_i)$, with natural identification. (This generalises the well-known result that R is regular if and only if each R_i is regular.)

2.33. Remark. It follows from 2.32 that the ring ΠR_i has property (RC) if and only if each R_i has property (RC).

The main theme of this chapter is 'regularity in rings'. However it seems natural to deduce a sufficient condition for the (RC) property to hold (2.36) from a proposition valid in modules.

2.34. Proposition. Let R be a ring with centre C and M a left R-module. Let $a \in R$ and $m \in M$. Assume that there is an element $b \in R$ such that $a = aba$ and $ba \in C$. Then we have:

(i) If m is e-regular, then am is e-regular.

(ii) If m is regular, then am is regular.

Proof. Write $e = ba$. Then $e^2 = e \in C$, and $Ra = Re$.

(i) By assumption, Rm is a direct summand of M. Since e is central, $Rm = Rem \oplus R(1 - e)m$. Thus $Ram = Rem$ is a direct summand of M. Hence am is e-regular in M.

(ii) Since m is regular there exists $f \in \mathrm{Hom}_R(M, R)$ such that $(mf)m = m$.

Now $am = (aba)(mf)m = (am)(fba)m$, as $ba \in C$.

Therefore, $am = (am)g(am)$ where $g = fb \in \mathrm{Hom}_R(M,R)$.

It follows that am is regular in M. #

2.35. Corollary. Let R be a normal ring and M a left R-module. If a is regular in R and m is e-regular (resp. regular) in M, then am is e-regular (resp. regular) in M.

2.36. Corollary. If R is a normal ring, then R has property (RC). (This extends 2.29 .)

2.37. Remarks.(i) Let $\mathrm{Reg}_l(R)$ (resp. $\mathrm{Reg}_r(R)$) denote the set of all left regular (resp. right regular) elements of R. Trivially, $\mathrm{Reg}_l(R)$ and $\mathrm{Reg}_r(R)$ are closed under multiplication in commutative rings and in strongly regular rings (0.22).

(ii) Let $St(R) = Reg_l(R) \cap Reg_r(R)$, the set of all strongly regular elements of R. Then remarks similar to 2.32 and 2.33 can be made about the sets $Reg_l(R)$, $Reg_r(R)$ and $St(R)$.

(iii) Let D be a division ring and $R = M_n(D)$, with $n \geqslant 2$. Let $A = E_{11} + E_{12}$ and $B = E_{22}$. Then A, B are idempotents and therefore belong to $St(R)$. However $AB = E_{12}$ is a non-zero nilpotent element; so AB is neither left regular nor right regular. #

In the final result of this section we give a sufficient condition for the strong regularity of a_1, a_2 to imply the strong regularity of a_1a_2. (As seen in 2.37 (iii) this does not always happen.)

2.38. Theorem. Let a_1, a_2 be strongly regular elements of a ring R. If a_1 and a_2 commute then a_1a_2 is strongly regular.

Proof. We shall use results of §1 of this chapter, especially 2.6(b). Let z_i be the semi-inverse of a_i for $i = 1, 2$. Then $C(a_i) = C(z_i)$ for $i = 1,2$.

Now $a_1 \varepsilon C(a_2)$ and $a_2 \varepsilon C(a_1)$ by hypothesis. So $a_1 \varepsilon C(z_2)$ and $a_2 \varepsilon C(z_1)$. Hence $z_1 \varepsilon C(a_2) = C(z_2)$. Thus the elements a_1, z_1, a_2, z_2 commute with each other. Therefore we have: $(a_1a_2)(z_1z_2) = (z_1z_2)(a_1a_2)$, $(a_1a_2)^2(z_1z_2) = a_1{}^2z_1 \, a_2{}^2z_2 = a_1a_2$ and finally, $(a_1a_2)\,(z_1z_2)^2 = a_1z_1{}^2 \, a_2z_2{}^2 = z_1z_2$.

These computations show that a_1a_2 is a strongly regular element of R. #

CHAPTER 3

ANTI-REGULARITY IN MODULES

In this chapter we shall study anti-regularity in modules. The definitions we need are collected in § 1. A theorem giving several equivalent conditions for the anti-regularity of an element of a module is proved in § 2. Basic properties and examples of anti-regular modules are recorded in § 3 and § 4. Endomorphism rings of these modules are studied in § 5. Conditions under which anti-regularity implies regularity are given in § 6.

The notation for standard Morita contexts (§0 C) and idempotents (0.8) will be followed.

§ 1. Basic definitions.

In this section we shall define anti-regularity for elements, modules and rings.

3.1. Definition. An element m of a left R-module M is anti-regular if there exists a non-zero $f \in N$ such that $f(mf) = f$.

3.2. Remark. It follows from 3.1 that any anti-regular element of a module is necessarily non-zero.

3.3. Definition. A module M is anti-regular if each non-zero element of M is anti-regular. (Thus in such modules the converse of 3.2 also holds.)

3.4. When $M = R$ we have $N = R$. Thus $a \in R$ is anti-regular as an element of ${}_RR$ if and only if there exists $b \in R$ such that $bab = b \neq 0$ (and if and only if a is anti-regular as an element of R_R). Thus we can talk about anti-regularity of a in R unambiguously. A ring R is anti-regular if each non-zero element of R is anti-regular. (This definition occurs in the literature of generalised inverses.)

§ 2. An equivalence theorem.

In this section we shall prove a theorem giving several equivalent conditions for the anti-regularity of an element of a module. We shall also show that non-zero regular elements are anti-regular; this will be deduced from an extension to modules of a result well-known for rings (Proposition 3.8).

3.5. Theorem. The following conditions are equivalent for an element m of a left R-module M.

(1) There exists $f \in N$ such that $mf \in I(R)$.

(2) There exists $f \in N$ such that $[f,m] \in I(S)$.

(3) There exist $f \in N$, $\alpha \in S$, such that $[f,m\alpha] = [f,m]\alpha \in I(S)$.

(4) There exist $r \in R$, $f \in N$ such that $(rm)f = r(mf) \in I(R)$.

(5) The element m is anti-regular in M.

(6) (See Proposition 1.4) Rm contains a non-zero, projective, direct summand of M.

Proof. (1) implies (2). Let $f \in N$ be such that $e = mf \in I(R)$. Then, applying 0.29, we get $[fe,m]^2 = [fe(mf)e,m] = [fe,m]$. Further, using 0.30, we get $m[fe,m]f = mf.e.mf = e \neq 0$. It follows that $[fe,m] \in I(S)$.

(2) implies (3) is clear; take $\alpha = 1_M$.

(3) implies (4). Suppose $f \in N$, $\alpha \in S$ are such that $[f, m]\alpha \in I(S)$. Let $g = \alpha f \in N$ and $r = mg = (m\alpha)f \in R$. Then $r^4 = m\alpha[f, m\alpha]^3 f$ $= m\alpha[f, m\alpha]f = r^2$ and (by an application of 0.29) $[fr^2, m\alpha] = [f, m\alpha]^3 = [f, m\alpha] \neq 0$, showing that $r^2 = r(mg) \in I(R)$.

(4) implies (5). Let $r \in R$, $f \in N$ such that $r(mf) \in I(R)$. Let $g = fr(mf)r \in N$. We have $r(mg)(mf)$ $= (r(mf))^3 = r(mf) \neq 0$ and therefore $g \neq 0$. Further, we have, $g(mg) = f(r(mf))^3 r = fr(mf)r = g$, showing that m is anti-regular in M.

(5) implies (6). Let $m \in M$, $f \in N$ be such that $f(mf) = f \neq 0$. Then $(mf)(mf) = mf \neq 0$, clearly. Denoting mf by e, we have $e \in I(R)$ and $f = fe$. For $x \in M$, $xf = (xf)e$ shows that $Mf = Re = R(mf) = (Rm)f$. Now let h denote the restriction of f to Rm. As Re is R-projective the R-epimorphism h: $Rm \to Re$ is split

by an R-monomorphism $u: Re \to Rm$. Clearly, u splits $f:M \to Re$ as well. So $(Re)u$ is a non-zero, projective, direct summand of M contained in Rm.

(6) implies (1). Let W be a non-zero, projective, direct summand of M contained in Rm. Let $k:M \to W$ be the projection map. As $W \leqslant Rm$ and k is identitity on W, we have $W = (Rm)k = R(mk)$. As W is non-zero and projective there exists $e \in I(R)$ and an R-isomorphism $q:W \to Re$ with $(mk)q = e$. Letting $f = kq \in N$ we have $mf = e \in I(R)$.

This completes the proof of the theorem. #

3.6. Remark. An element m of M is called unimodular if there exists f in N such that $mf = 1$. Clearly, unimodular elements, which are of interest in certain problems involving projective modules, are regular as well as anti-regular.

In the special case when $M = R$ (implying that $N = R$, $S = R$) the above theorem yields the following (essentially well-known) result. (See, for example, Exercise 13 on p. 75 of [BII8] or Lemma 1.1 of [N:75]. In fact, our anti-regular rings are precisely the semi-primitive Zorn rings of Bourbaki and the semi-primitive I_0-rings (with identity) of Nicholson.)

3.7.Proposition. The following conditions are equivalent for an element a of a ring R.

(1) There exists $b \in R$ such that $ab \in I(R)$.

(2) There exists $b \in R$ such that $ba \in I(R)$.

(3) There exist $r, s \in R$ such that $ras \in I(R)$.

(4) The element a is anti-regular in R.

In the following proposition a result well-known for regular rings will be extended to regular elements in modules. (See Exercise 47(i) of Chapter I of [S].)

3.8. Proposition. (a) An element m of a module M is regular if and only if there exists $g \in N$ such that $(mg)m = m$ and $g(mg) = g$.

(b) Regular implies anti-regular (for non-zero elements, modules, rings).

Proof. (a) "If" part is trivial. So let m be regular and let $f \in N$ be such that $(mf)m = m$. Set $g = f(mf)$. Then $mg = (mf)(mf) = mf$ yields $(mg)m = (mf)m = m$ and $g(mg) = f(mf)^2 = f(mf) = g$.

(b) First assume that m is a non-zero regular element of M. Then, in the notation of part (a), $(mg)m = m \neq 0$ implies that $g \neq 0$. The equality $g(mg) = g$ further implies that m is anti-regular. The implications for modules and rings follow. #

§ 3. Basic properties and examples.

In this section we shall record some basic properties of anti-regular modules. It will be shown that the class of anti-regular modules over a given ring is closed under submodules, direct products (and therefore) direct sums. Some examples of anti-regular rings and modules are also recorded in this section.

We begin by showing that each pre-image (under a module homomorphism) of an anti-regular element is necessarily anti-regular. (Notice that in contrast to 1.50 no assumption on the kernel is required for this.)

3.9. Proposition. Let D, M be left R-modules and $k:D \to M$ be an R-homomorphism. Let d be an element of D such that dk is anti-regular in M. Then d is anti-regular in D.

Proof. By 3.5 there exists $f \in \mathrm{Hom}_R(M, R)$ satisfying $(dk)f \in I(R)$. Write $q: = kf \in \mathrm{Hom}_R(D, R)$. Then $dq = (dk)f \in I(R)$. It follows, by applying 3.5 again, that d is an anti-regular element of D. #

3.10. Corollary. Let D be a submodule of a module M. Let $d \in D$ be anti-regular as an element of M. Then d is anti-regular as an element of D.

Proof. Apply 3.9 to the inclusion $D \to M$. #

The following analogue of a result of Zelmanowitz (see 1.29(ii)) is now immediate.

3.11. Proposition. Let D be a submodule of a module M. If M is anti-regular then so is D.

3.12. The class of regular modules is closed under arbitrary direct sums but not under arbitrary direct products.

(See 3.14(b).) In 3.13 we prove that the class of anti-regular modules is closed under both direct sums and direct products.

3.13. Proposition. Let $\{M_i\}_{i \in I}$ be an arbitrary family of left R-modules. Let P_0 denote the direct product and S_0 the direct sum of this family. The following conditions are equivalent.

(1) Each M_i is anti-regular.

(2) The module P_0 is anti-regular.

(3) The module S_0 is anti-regular.

Proof. The assertions (2) implies (3) and (3) implies (1) hold by 3.11. Now assume (1) and let d be a non-zero element of P_0. For $i \in I$ let p_i denote the projection map $P_0 \to M_i$. Then for some $j \in I$ the element dp_j is non-zero and belongs to M_j which is an anti-regular module. By 3.9 the element d is anti-regular in P_0. Thus (1) implies (2).

3.14. Examples. (a) Anti-regular rings. For the sake of completeness we record the following known examples of anti-regular non-regular rings: the ring of continuous functions defined on the space of rationals, the ring of bounded sequences of real numbers, the ring of sequences of rational numbers which eventually take a constant integral value, and a non-commutative modification of this example due to Nicholson (Example 1.9 of [N:75]). It is trivial that a direct product of rings R_i is anti-regular if and only if each R_i is anti-regular.

(b) Anti-regular modules. All regular modules (3.8(b)) and all one-sided ideals in anti-regular rings (3.11) are anti-regular modules. If R is an anti-regular non-regular ring and Z the ring of integers the ideal $R\times 0$ in $R' = R\times Z$ is an anti-regular R'-module which does not belong to either of these classes. Let R be a commutative regular ring which is not self-injective. (Examples of such rings were given in 1.45.) By Remark (iv) on p. 349 of [Z:72] there exists an indexing set I such that R^I is not a regular module over R. However, by 3.13, R^I is an anti-regular module over R.

§ 4. The radical and the singular submodule. As observed by Zelmanowitz ((2.5) and (2.7) of [Z:72]), if M is a regular module then $\mathrm{Rad}(M) = 0$ and $Z(M) = 0$. This result is extended to anti-regular modules in 3.16.

3.15. Theorem. Let m be an anti-regular element of M. Then m cannot belong either to $\mathrm{Rad}(M)$ or to $Z(M)$.

Proof. Let m be an anti-regular element of a left R-module M. By Theorem 3.5 there exists $f \in \mathrm{Hom}_R(M,R)$ such that $mf \in I(R)$. It follows that mf cannot belong either to $\mathrm{Rad}(R)$ or to $Z({}_RR)$. Recall that if A,B are left R-modules and $g\colon A \to B$ an R-homomorphism then $\mathrm{Rad}(A)g \leq \mathrm{Rad}(B)$ and $Z(A)g \leq Z(B)$. Applying this observation to $f\colon M \to R$ we deduce that m cannot belong either to $\mathrm{Rad}(M)$ or to $Z(M)$. #

3.16. Corollary. Let M be an anti-regular module. Then (a) $\mathrm{Rad}(M) = 0$ and (b) $Z(M) = 0$.

3.17. Corollary Let R be an anti-regular ring. Then R

is (a) semi-primitive and (b) left and right non-singular. (See Renault's Remark about semi-primitive Zorn rings on p. 251 of [R:73] .)

3.18. Remark. Nicholson [N:75] calls a ring R an <u>I_0- ring</u> if every element a not belonging to Rad(R) is anti-regular. Therefore, by Theorem 3.15, if R is an I_0- ring then $Z({}_RR) \leqslant \mathrm{Rad}(R)$ and $Z(R_R) \leqslant \mathrm{Rad}(R)$.

§ 5. Endomorphism rings.

In this section we shall study endomorphism rings of anti-regular modules. The letter B denotes $\mathrm{End}(M_S) = \mathrm{Biend}({}_RM)$, the biendomorphism ring of ${}_RM$.

While it is possible to give a direct proof of the main theorem of this section, it seems worth-while to deduce it from some "elementwise" results. These will be recorded in 3.19 - 3.21.

3.19. Proposition. Let U be a ring and M a right U-module. Suppose that $m \in M$, $u \in U$ are such that mu

s an anti-regular element of M. Then u is an anti-egular element of U.

Proof. By Theorem 3.5 there exists $g \in \mathrm{Hom}_U(M,U)$ such that $g(mu) \in I(U)$, i.e., $g(m)u \in I(U)$. It follows by Proposition 3.7 that u is an anti-regular element of U. #

3 . 20. Proposition. Consider the following conditions for an element m of M.

(1) The element m is anti-regular in ${}_R M$.

(2) The element m is anti-regular in M_S.

(3) The element m is anti-regular in ${}_B M$.

Then: (a) (1) implies (2), and

(b) (2) and (3) are equivalent.

Proof. (a) By Theorem 3.5 there exists an element $f \in \mathrm{Hom}_R(M,R)$ such that $[f,m] \in I(S)$. Write $F(x) = [f,x]$ for $x \in M$. Then $F \in \mathrm{Hom}_S(M_S,S)$ and $F(m) = [f,m] \in I(S)$. Now apply (the right-sided version of) Theorem 3.5 to the element m in the right S-module M.

(b) It is well-known that the natural development that leads from R to $S = \mathrm{End}({}_RM)$ to $B = \mathrm{Biend}({}_RM)$ stabilizes i.e. $S = \mathrm{End}({}_BM) = \mathrm{Biend}(M_S)$. Therefore the equivalence of (2) and (3) holds by Part (a).

3.21. Proposition. Let $m \in M$, $\alpha \in S$ be such that $m\alpha$ is an anti-regular element of ${}_RM$. Then α is an anti-regular element of S.

Proof. By Proposition 3.20 $m\alpha$ is an anti-regular element of M_S. Hence, by Proposition 3.19, α is anti-regular in S.

3.22. Theorem. Let ${}_RM$ be an anti-regular module.

Then:

(a) The rings $S = \mathrm{End}({}_RM)$ and $B = \mathrm{Biend}({}_RM)$ are anti-regular.

(b) M is anti-regular as a right S-module (and as a left B-module), Proof. Let α be a non-zero element of S. Then there exists $m \in M$ such that $m\alpha$ is non-zero and therefore anti-regular in ${}_RM$. It follows by Proposition 3.21 that α is an anti-regular element of S, proving that S is an anti-regular ring. By Proposition 3.20 the module M_S is anti-regular. The above argument can therefore be repeated to yield the anti-regularity of the ring B and the left B-module M.

3.23. Remark. W. Zimmermann showed that if ${}_RM$ is a regular module then M_S is also regular. (See Bemerkung 3.6 on p. 33 of the dissertation of Birge Zimmermann-Huisgen [Z.- H.: 74] ; see also the proof of Theorem 3.3 of [Z.- H.: 75], a more accessible reference.) Theorem 3.22(b) includes the anti-regular analogue of this result.

3.24. Remark. While (anti-)regularity in ${}_RM$ implies (anti-) regularity in M_S the converse assertions are false. Let $R = Z$, $M = Q$. Then $S = Q = B$. Thus M is regular and anti-regular as an S-module. However (since $N = 0$) no non-zero element of ${}_RM$ can be either regular (as seen in 1.14) or anti-regular.

3.25. Remark. Recall that a left R-module M is called semi-prime if for each non-zero element m of M there exists an element f of N such that $(mf)m \neq 0$.

A ring R is semi-prime if it has no non-zero nilpotent (left, right or two-sided) ideals, equivalently, if the module $_RR$ is semi-prime. Now let m be a non-zero element of an anti-regular module $_RM$. By Theorem 3.5 there exists $f \in N$ such that $[(mf)m]f = (mf)(mf) = mf \neq 0$ implying that $(mf)m \neq 0$. Thus anti-regular implies semi-prime (for modules as well as rings). Therefore Theorem 3.22(a) above is stronger than observation (3.2) of [Z:72] . Note, however, that Zelmanowitz' argument actually shows that the endomorphism ring of a semi-prime module is semi-prime.

Even for rings without identity the classes of anti-regular "rngs" and of semi-primitive I_0 - rings [N:75] coincide. Therefore Part(a) of Theorem 3.22 extends several results of Nicholson ([N:75] , (5.8) - (5.11)).

3.26. Remark. It is well-known that the centre of a regular ring is regular. Example 1.9 of [N:75] shows that the centre of an anti-regular ring need not even be anti-regular. Thus all the anti-regular, non-regular rings mentioned in 3.14(a) have non-regular centres. We shall now apply the results of this section to furnish examples of anti-regular, non-regular rings with regular centres.

There exists a left ideal M in a regular ring R such that $S = \mathrm{End}_R(M)$ is not a regular ring (Example 3.1 of [Z:72]). In a preceding remark Zelmanowitz attributes to Cukerman and Ware the result that if M is an infinitely generated free module over a regular non-artinian ring R, then S is not a regular ring. However in each of these cases the module M is regular (by (1.4) and (2.8) of [Z:72]). Since M is an anti-regular module over R (by Proposition 3.8(b)) S is an anti-regular ring (by Theorem 3.22(a) above). Thus in each case S is an anti-regular, non-regular ring with regular centre (by Theorem 3.4 of [Z:72]).

§ 6. Indecomposability.

It was proved in Proposition 3.8(b) that regular implies anti-regular. In this section we shall give conditions under which the converse holds. Thus it will be shown that under conditions like ${}_RR$ indecomposable or M indecomposable, anti-regular implies regular.

We begin with a terminological convention.

3.27. Definition. R has no idempotents if R is a non-zero ring having no idempotents other than 0 and 1.

Recall that a non-zero module M is called indecomposable if it has no direct summands other than M and 0. The ring R has no idempotents if and only if ${}_RR$ (equivalently, R_R) is an indecomposable module.

Remarks 3.29 - 3.32 are about such rings. Remark 3.28 is well-known.

3.28. Remark. Suppose R is any ring and ${}_RM$ contains a unimodular element m, i.e. there exists $f \in N$ such that $mf = 1$. Then f is split by $g: R \to M$ defined by $rg = rm$. So Rm is a direct summand of M and Rm is isomorphic to R.

3.29. Remark. Suppose that R has no idempotents. Trivially an element a of R is anti-regular if and only if it is a unit. Therefore if R is also anti-regular then R is a division ring. (See Exercise 1(c) on p.75 of [BII8] .)

We extend Remark 3.29 to modules in 3.30 and 3.31 below.

3.30. Remark. Suppose that R has no idempotents. The element m in M is anti-regular if and only if it is unimodular. This holds by Theorem 3.5.

3.31. Remark. Suppose that R has no idempotents and M contains a non-zero anti-regular submodule M'. By 3.30 and 3.28 M' contains a copy of R. Now by 3.11 R is an anti-regular ring. Hence by 3.29 R is a division ring and M is a vector space over R. In particular M is a regular R-module.

3.32. Remark. Domains and local rings have no idempotents. If X is a connected topological space then the ring of continuous real-valued functions defined on X has no idempotents. It follows from 3.31 that if R is a ring belonging to one of these classes which is not a division ring then there are no non-zero regular or anti-regular R-modules. In particular, there are no non-zero regular or anti-regular abelian groups. #

Next we shall introduce a definition which is tailor made for the situation considered below.

3.33. Definition. A module M is <u>special</u> if each non-zero cyclic submodule of M is indecomposable.

3.34. Examples. Simple modules and torsion free modules over domains (in particular, vector spaces and free abelian groups) are special modules. For each prime p, $Z/(p^n)$ is special (as a Z-module and as a $Z/(p^n)$-module). If ${}_RR$ is a special module R has no idempotents.

3.35. Proposition. Let R be a ring, M a left R-module and m an anti-regular element of M.

(1) If M is indecomposable then M is projective and $M = Rm$.

(2) If M is indecomposable or special, then m is regular in M.

Proof. By Theorem 3.5 Rm contains a non-zero, projective submodule, say W, such that W is a direct summand of M, and hence of Rm as well.. Suppose that M is indecomposable. Then W = Rm = M shows that M is projective, proving (1). If M is special we have Rm = W. Thus in either case Rm is a projective direct summand of M. This yields, by 1.4, that m is regular in M. #

3.36. Proposition. Anti-regular special modules are regular and anti-regular indecomposable modules are simple, projective and regular.

Proof. This follows from. 3.35. In the indecomposable case the condition that every non-zero element is a generator implies the simplicity of the module. #

3.37. Theorem. Suppose that M is an anti-regular module satisfying either the ascending or the descending condition on direct summands (for example, M is artinian or noetherian). Then M is regular and of finite length. If, moreover, R is commutative then M is also injective.

Proof. If M is a non-zero module satisfying the given condition then by Proposition 10.14 in [AF] there exists a finite family $\{M_i\}_{i \in I}$ of indecomposable submodules such that $M = \bigoplus M_i$. As M is anti-regular each M_i is anti-regular. *Hence, by 3.36, each* M_i *is simple* and projective. Thus M is projective and semi-simple of finite length. Therefore, by 1.13, M is a regular module. The assertion for commutative R holds by Corollary 1.9 of [Z:72] #

3.38. Corollary. Noetherian anti-regular rings are semi-simple.

Chung and Luh [CL:76] proved that a ring R is semi-simple if and only if every left R-module is regular. We consider the anti-regular case in 3.39.

3.39. Proposition. The following conditions are equivalent for a ring R.

(1) R is semi-simple.

(2) Every left R-module is anti-regular.

(3) Every cyclic left R-module is anti-regular.

(4) Every indecomposable left R-module is anti-regular.

(5) Every simple left R-module is anti-regular.

Proof. The implications (1) implies (2), (2) implies (3), (2) implies (4), (3) implies (5) and (4) implies (5) are all trivial. Next assume (5) and let M be a simple left R-module. By 3.36 M will be projective, being indecomposable and anti-regular. It is well-known that if every simple left R-module is projective, then R is semi-simple. (For a proof see (3) → (1) of Theorem 2.6 of [V:76] .) #

CHAPTER 4

ANTI-REGULARITY IN RINGS

In this chapter we shall study anti-regularity in rings. Localizations of anti-regular rings are considered in §1. Reduced anti-regular rings have properties similar to those of reduced regular rings; this is seen in §2. The theme of §3 is a study of rings which satisfy the condition that each factor ring is anti-regular. In §4 we consider the ring eRe for an idempotent e and also the eRe-module eM. Cardinality results (for some finite rings like Z/nZ) are recorded in §5.

§ 1. Localization.

In this section we study localizations of anti-regular rings. We begin with some general results.

4.1. Proposition. Let u and a be elements of R.

(1) If a is anti-regular and u is left invertible, then ua is anti-regular.

(2) If a is left anti-regular and u is an invertible element of the centre of R, then ua is left anti-regular. (See Definition 4.5 below.)

Proof. We prove (1); the proof of (2) is similar.

Let $bab = b \neq 0$, and $vu = 1$. Then $b = bvu$ implies that $bv \neq 0$ and clearly $bv.ua.bv = bv \neq 0$. #

4.2. Proposition. Let $\phi: R \to R'$ be a one-to-one ring homomorphism and let a be anti-regular in R. Then $\phi(a)$ is anti-regular in R'.

Proof. The equation $bab = b \neq 0$ yields $\phi(b)\phi(a)\phi(b) = \phi(b) \neq 0$. #

4.3. Proposition. Let R be a ring, and T a multiplicatively closed subset of the centre of R. Assume that each element of T is a non-zero-divisor in R.

(1) If a is anti-regular in R and t is an element of T, then a/t is anti-regular in $T^{-1}R$.

(2) If R is an anti-regular ring then so is $T^{-1}R$.

Proof. (1) The hypothesis on T implies that the natural ring homomorphism $R \to T^{-1}R$ is one-to-one. By 4.2, $a/1$ is anti-regular in $T^{-1}R$. As $1/t$ is invertible in $T^{-1}R$, by 4.1(1) a/t is anti-regular in $T^{-1}R$.

4.4. Example. Let R denote the ring of all sequences of rational numbers which eventually take a constant integral value (this ring is also denoted as $B(Q, Z)$).

Thus

$R = \{(a_1, a_2, \ldots a_n, a, a, \ldots a, \ldots) \mid a_i \varepsilon Q$ for $1 \leqslant i \leqslant n$

and $a \varepsilon Z\}$

It is easily seen that R is an anti-regular ring. Moreover, we have:

(i) $\Theta: R \to Z$ defined by $\Theta(a_1, a_2, \ldots a, a, \ldots) = a$ is an onto ring homomorphism.

(ii) For $n \varepsilon N$ consider the sequence $e_n = (e_{nj})$ where $e_{nj} = 0$ for $1 \leqslant j \leqslant n-1$ and $e_{nj} = 1$ for $j \geqslant n$. Let $T = \{e_n \mid n \varepsilon N\}$. Then T is a multiplicatively closed subset of R. Since $\Theta(e_n) = 1$ for each $n \varepsilon N$ there is an induced ring homomorphism $\emptyset: T^{-1}R \to Z$. It is easily verified that $\emptyset$ is an isomorphism. Thus the ring $T^{-1}R$ is not anti-regular. Note that this is in contrast with the situation in regular rings (see Remark 1.61 above). We remark that (i) is a known fact, but (ii) has not appeared in the literature.

§2. Strongly anti-regular rings.

The main theorem of this section is an anti-regular analogue of a portion of 0.22.

By analogy with Azumaya's terminology (0.17) we introduce the following definition.

4.5. Definition. An element a of a ring R is <u>left anti-regular</u> if there exists a non-zero element b of R such that $b = b^2a$.

4.6. Remark. Let a be left invertible with $ba = 1$. Then $bab = b$ and $b^2a = b$ shows that a is anti-regular and left anti-regular.

4.7. Remark. Let a be left anti-regular with $b^2a = b \neq 0$. This yields $0 \neq b = b^2a = bb^2aa = b^3a^2 = \ldots = b^{n+1}a^n$ for each natural number n. Hence a cannot be nilpotent. #

Next we shall prove an anti-regular analogue of 0.21.

4.8. Proposition. Let R be a reduced ring and a an element of R. *Then the following conditions are equivalent.*

(1) a is left anti-regular.

(2) a is anti-regular

(3) a is right anti-regular.

Proof. (1) → (2) Since a is left anti-regular there exists $b \in R$, $b \neq 0$ such that $b^2a = b$, i.e. $b(ba - 1) = 0$ holds.

By 0. 10, $(ba - 1)b = 0$ holds. Hence a is anti-regular.

(2) → (1) and (2) ↔ (3) can be proved similarly. #

4.9. Theorem. The following conditions are equivalent for a ring R.

(1) Each non-zero element of R is left anti-regular.

(2) R is reduced and anti-regular.

(3) R is normal and anti-regular

(4) Each non-zero element of R is right anti-regular.

Proof. (1) → (2). R is reduced follows from 4.7 ; now apply 4.8.

(2) → (3) holds since all reduced rings are normal.

(3) → (1). Let a be a non-zero element of R. Then there exists a non-zero element b such that $bab = b$. Therefore ba (being an idempotent) belongs to the centre of R. So $b^2a = b$.

The equivalence of (2), (3) and (4) holds by symmetry; see 4.8. #

The following definition is suggested by the similarity between the theorem above and a portion of 0.22.

4.10. Definition. A ring R is <u>strongly anti-regular</u> if it satisfies the equivalent conditions of Theorem 4.9.

4.11. Corollary. If R is a duo, anti-regular ring, then R is strongly anti-regular.

Proof. By 0.14 R is normal. #

4.12. Remarks. (i) Since regular rings are anti-regular (3.8(b)) it follows from Theorem 4.9 that strongly regular rings are strongly anti-regular.

(ii) We do not know whether the converse of 4.11 holds.

§3. Completely anti-regular rings.

In this section we study completely anti-regular rings and related rings; the term "completely" is used in the sense of 0.24 .

4.13. Abbreviations. Strongly anti-regular (S.A. -R.), completely anti-regular (C.A.-R.), completely strongly anti-regular (C.S.A.-R.).

4.14. Remark. In contrast to 0.25 even commutative anti-regular rings need not be C.A.-R.; see Example 4.4.

4.15. Chart of implications.

$$\begin{tikzcd}
\text{Commutative regular} \arrow[r, "(1)"] \arrow[d, "(7)"] & \text{Commutative C.A.-R.} \arrow[r, "(4)"] \arrow[d, "(8)"] & \text{Commutative anti-regular} \arrow[d, "(9)"] \\
\text{Strongly regular} \arrow[r, "(2)"] \arrow[dd, "(10)"] & \text{C.S.A.-R.} \arrow[r, "(5)"] \arrow[d, "(11)"] & \text{S.A.-R.} \arrow[dd, "(13)"] \\
 & \text{Reduced C.A.-R.} \arrow[d, "(12)"] & \\
\text{Regular} \arrow[r, "(3)"] & \text{C.A.-R.} \arrow[r, "(6)"] & \text{Anti-regular}
\end{tikzcd}$$

These implications are either known (in the case of regular rings) or have been essentially proved earlier in this thesis or are obvious. The details are given in 4.16 below.

4.16. Remarks. (i) The implications (7) and (10) (which follow from 0.23) and their non-reversibility (see (iii) below) are standard facts in the theory of regular rings; we record them here for the sake of completeness.

(ii) The implications (1) and (3) follow from 0.25 and 3.8(b); (2) follows from 0.28 and 4.12(i); (4), (5) and (6) are trivial (see 0.24); (9) holds by 4.9 and (8) follows from (9); (11) and (13) hold by 4.9.

(iii) The implications (4), (5) and (6) are not reversible; see 4.4. The division ring of real quaternions shows the non-reversibility of (7), (8) and (9). If K is a field and $n \geqslant 2$, the regular ring $M_n(K)$ is not reduced. So (10), (12) and (13) are not reversible. The reversibility of (1), (2), (3) and (11) is discussed in 4.21 and 4.23.

4.17. For convenience, we recall some material from §1 of [FS:74]. The three conditions which follow have been considered there (the names (P) and (P') have been given here):

Condition (*): the union of every chain of semiprime ideals is semiprime.

(P): each prime factor ring of R is a domain.

(P'): each prime factor ring of R is a division ring.

Commutative rings trivially satisfy (P). If R is commutative, R satisfies (P') if and only if R is zero-dimensional. We have (P') → (P) (trivial) and (P) → (*) (see [FS:74]). The following result is a consequence of Theorem 1.1 of [FS:74] .

4.18. Proposition. If R is a semiprime ring satisfying (P'), then R is regular.

Since anti-regular rings are semiprime (3.25) the following fact is obvious.

4.19. Remark. C.A.-R. rings satisfy condition (*).

4.20. Proposition. If each prime factor ring of R is S.A.-R., then R satisfies (P').

Proof. Let R' be a prime factor ring of R. Then R' is a prime reduced anti-regular ring. It is well-known -see, e.g., Exercise 15 on p. 261 of [S] - that a prime reduced ring is necessarily a domain. Hence R' is a division ring, by 3.29. #

4.21. Corollary. Let R be a C.S.A.-R. ring; then R is strongly regular. (It follows that the implications (2) and (1) are reversible; further, the term C.S.A.-R. is redundant.)

Proof. Since R is S.A.-R., R is reduced. By 4.20 R satisfies (P'). Hence 4.18 yields regularity of R.

4.22. Corollary. If R is a duo C.A.-R. ring, then R is strongly regular.

Proof. This follows from 0.26 and 4.11.

4.23. Remarks,. (i) We do not know whether the implications (3) and (11) are reversible; trivially, the reversibility of (3) will imply the reversibility of (11).

(ii) The reversibility of (1) also follows from an application of Exercise 22 on p. 64 of [KIII] , which asserts that commutative, reduced, zero-dimensional rings are regular; see also Lemma 5.6 of [S:68] .

§ 4. Idempotents.

Let $e^2 = e \in R$ and M a left R-module. Then eRe becomes a ring with identity e and the subgroup eM of M becomes a left eRe-module. In 4.26 below we prove that if ${}_RM$ is regular (resp. anti-regular) then ${}_{eRe}eM$ is regular (resp. anti-regular). Suppose R is a regular ring. It is well-known that eRe is also a regular ring. Elementwise/anti-regular analogues of this result are proved in 4.28.

4.24. Lemma. Let $f:M \to R$ be a R-homomorphism. Then $\underline{f}:eM \to eRe$ defined by $(ex)\underline{f} = (ex)f.e$ is an eRe-homomorphism.

Proof. Easy verification.

4.25. Proposition. Let em be a regular (resp. anti-regular) element of M. Then em is a regular (resp. anti-regular) element of ${}_{eRe}eM$.

Proof. First assume that em is regular. Then there exists $f:M \to R$ such that $(em)f.em = em$. Consider defined as in Lemma 4.24. Then $(em)\underline{f}.em = (em)f.eem = em$.

Next assume that em is anti-regular. By Theorem 3.5 there exists $f:M \to R$ such that $(em)f \in I(R)$. Write $e' := (em)f$. Clearly $ee' =$

Again consider $\underline{f}$ defined as in Lemma 4.24 . Then $(em)\underline{f} = e'e$. Now $e'ee'e = e'^2e = e'e$ and $e'ee' = e' \neq 0$ shows that $e'e \neq 0$. Another application of Theorem 3.5 yields the anti-regularity of em in ${}_{eRe}eM$.

4.26. Corollary. If ${}_RM$ is regular (resp. anti-regular, then ${}_{eRe}eM$ is regular (resp. anti-regular).

4.27. Remark. It follows from 4.26 that if R is a regular (resp. anti-regular) ring, then eR is regular (resp. anti-regular) as a left eRe-module. These results are actually special cases of 3.23 and Theorem 3.22 applied to the right R-module eR, since $eRe \cong \mathrm{End}(eR_R)$.

4.28. Proposition. Let G be any one of the properties regular, left regular, anti-regular, left anti-regular. Let $a \in eRe$. If a has property G as an element of R then a has the same property as an element of eRe.

Proof. We have $a = ea = ae = eae$.

Case: a regular. From $a = aba$ we get $a = eaebeae = aebea$.

Case: a left regular. From $a = ba^2$ we get $a = ebea^2$.

Case: a anti-regular. Suppose that $bab = b \neq 0$. Then $bea\ ebe\ ab = b\ eae\ b\ eab = babab = bab = b \neq 0$ yields $ebe \neq 0$.

Finally, $ebeaebe = ebe$ shows that a is anti-regular in eRe.

Case: a left anti-regular. Let $b^2a = b \neq 0$. We have $be = b$ and therefore $b = bebea$, yielding $ebe \neq 0$. Finally, $ebeebea = eb^2a = eb = ebe$ shows that a is left anti-regular in eRe.

4.29. Corollary. If R is regular/strongly regular/anti-regular/strongly anti-regular, then eRe also has the same property.

4.30. Remark. Alternative proofs of parts of 4.28 are possible. After proving that R regular/anti-regular implies that eRe is regular/anti-regular, the other cases will follow by noting that R reduced implies *eRe reduced*.

§5. Cardinality questions.

In this concluding section of the thesis we consider some cardinality questions. We determine the number of regular (anti-regular) elements in the ring Z/nZ of residue classes modulo n. We end by asking similar questions about some other finite rings.

4.31. Notation. For the notation Reg(R) see 1.21. Let An(R) denote the set of all anti-regular elements of R.

Let $\rho(R) :=$ #Reg(R), $\alpha(R) :=$ #An(R) and $\nu(R) := \#(R \setminus An(R))$

We shall further denote $\rho(Z/nZ)$ BY $\rho(n)$, $\alpha(Z/nZ)$ by $\alpha(n)$ and $\nu(Z/nZ)$ by $\nu(n)$.

4.32. Remark. It follows from 2.32 that if

$$R = R_1 \times R_2 \times \ldots\ldots \times R_r, \text{ then } \rho(R) = \prod_{i=1}^{r} \rho(R_i).$$

4.33. Remark. Let R be a commutative artin local ring. Then the unique maximal ideal m is nilpotent. Thus $\alpha(R) = \#(R \setminus m)$ and $\rho(R) = \alpha(R)+1$ (since 0 is a regular element).

4.34. Remark. Let p be a prime. Since Z/p^tZ is a finite local ring for each natural number t, we have

$$\alpha(p^t) = p^t - p^{t-1},\ \nu(p^t) = p^{t-1} \text{ and } \rho(p^t) = p^t - p^{t-1} + 1$$

4.35. Remark. Let n be any natural number. Write $n = p_1^{t_1} \ldots p_r^{t_r}$ where p_i are distinct primes.

Note that $Z/nZ \simeq \prod_{i=1}^{r} Z/(p_i^{t_i})$ by the Chinese remainder theorem ([L"], p. 65). Hence, by an application of 4.32 and 4.34

we have

$$\rho(n) = \prod_{i=1}^{r} (p_i^{t_i} - p_i^{t_i - 1} + 1).$$

(While this formula may be known we have not seen it in the literature.)

4.36. Remark. Let $R = R_1 \times R_2 \times \ldots \times R_r$, where R_i are rings. Then $(x_1, \ldots x_r)$ is anti-regular in R if and only if x_i is anti-regular in R_i for some i. In other words, $(x_1, \ldots x_r)$ is not anti-regular in R if and only if x_i is not anti-regular in R_i for each i. This yields

$$\nu(R) = \prod_{i=1}^{r} \nu(R_i).$$

Let n be any natural number. Then the above argument implies (with notation as in 4.35) that

$$\nu(n) = \prod_{i=1}^{r} \nu(p_i^{t_i}) = n/p_1 p_2 \ldots p_r \text{ , using 4.34.}$$

Hence $\alpha(n) = n(1 - 1/p_1 p_2 \ldots p_r)$.

4.37. Remark. As seen in Remarks 4.35 and 4.36 ρ' and ν are multiplicative arithmetic functions in the sense used in number theory (see, e.g., § 6-3 of [A]).

4.38. Remark. While noetherian anti-regular rings are semi-simple (by 3.38) anti-regular elements in finite rings need not be regular. Thus in $Z/36Z$ there are 30 anti-regular elements (by 4.36) but only 20 non-zero regular elements (by 4.35). Indeed every anti-regular element in Z/nZ is regular if and only if n is square-free if and only if Z/nZ is a regular ring; this can be seen easily.

4.39. Remark. Let K be a finite field of cardinality q, $f(x)$ a polynomial of degree n and $R = K[x] / (f(x))$ (a ring of cardinality q^n). Using the fact that $K[x]$ is a UFD computations similar to those in 4.35 and 4.36 can be carried out for the number of regular/anti-regular elements in R. We omit the details.

4.40. Question. Let K be a finite field of q elements and $R = UT_n(K)$, the upper triangular matrix ring over K. Then R has $q^{n(n+1)/2}$ elements. R is neither regular nor anti-regular. It would be interesting to obtain explicit formulae for the number of regular/anti-regular elements of R. Similarly it would be interesting to obtain formulae for the number of left regular elements of $M_n(K)$. (Such formulae can be easily derived in the case when $n = 2$ or $n = 3$. The general case seems difficult.)

BIBLIOGRAPHY

BOOKS AND MONOGRAPHS

[A] G.E. Andrews, Number Theory, W.B. Saunders Co., Philadelphia, 1974. (Reprinted by: Hindustan Publishing Corp., Delhi, 1984)

[AF] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, Springer Verlag, New York, 1974.

[AM] M.F. Atiyah and I.G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, Massachusettes, 1969.

[BII8] N.Bourbaki, Éleménts de Mathématique , Livre II, Chapitre 8, Modules at Anneaux Semi-simples, Hermann et Cie, Paris, 1958.

[FI] C.Faith, Algebra: Rings, Modules and Categories, Springer-Verlag, New York, 1973.

[FII] C. Faith, Algebra II, Ring Theory, Springer-Verlag, New York, 1976.

[G] K.R. Goodearl, Von Neumann Regular Rings, Pitman, London, 1979.

[H] I.N. Herstein, Notes from a Ring Theory Conference, CBMS series, American Math.Society, Providence, 1971.

[JI] N. Jacobson, Basic Algebra, Vol.I, Hindustan Publishing Corporation, Delhi, 1984.

[JII] N.Jacobson, Basic Algebra, Vol.II, Hindustan Publishing Corporation, Delhi, 1984.

[KI] I. Kaplansky, Fields and Rings, Second Edition, The University of Chicago Press, Chicago, 1972.

[KII] I. Kaplansky, Rings of Operators, W.A. Benjamin, New York, 1968.

[KIII] I. Kaplansky, Commutative Rings, Revised Edition, University of Chicago Press, Chicago, 1974.

[L'] T.Y. Lam, Serre's Conjecture, Lecture Notes in Mathematics 635, Springer-Verlag, New York, 1978.

[L"] S. Lang, Algebra, Addison-Wesley, Reading, Massachusetts, 1970.

[S] B. Stenström, Rings of Quotients: An Introduction to the Methods of Ring Theory, Springer-Verlag, New York, 1975.

RESEARCH PAPERS

[A:54] G. Azumaya, Strongly π-regular rings, J. Fac.Sci. Hokkaido Univ., Ser.I 1(1954),34-39.

[A:71] S.A. Amitsur, Rings of quotients and Morita contexts, J. Algebra 17(1971), 273-298.

[AFS:74] E.P. Armendariz, J.W. Fisher and S.A. Steinberg, Central localizations of regular rings, Proc.Amer. Math. Soc. 46(1974), 315-321.

[BM:50] B.Brown and N.H.McCoy, Some theorems on groups with applications to ring theory, Trans. Amer. Math. Soc. 69(1950), 302-311.

[BR:82] S.M. Bhatwadekar and A. Roy, Stability theorems for overrings of polynomial rings, Invent. math. 68(1982), 117-127.

[CE:81] T. Cheatham and E.Enochs, Regular modules, Math. Japon. 26(1981), 9-12.

[CL:76] L. Chung and J. Luh, A characterization of semi-simple Artinian rings, Math. Japon. 21(1976), 227-228.

[D:76] F. Dischinger, Sur les anneaux fortement π-reguliers, C.R. Acad.Sci.Paris, Ser.A 283(1976), 571-573.

[E:71] S. Elliger, Interdirekte Summen von Moduln, J. Algebra 18(1971), 271-303.

[E:72] M.W. Evans, On commutative P.P. rings, Pacific J. Math. 41(1972), 687-697.

[F:69] D.J. Fieldhouse, Pure theories, Math.Ann. 184(1969), 1-18.

[FS:74] J.W. Fisher and R.L. Snider, On the von Neumann regularity of rings with regular prime factor rings, Pacific J. Math. 54(1974), 135-144.

[H:78] Y. Hirano, Some studies on strongly π-regular rings, Math. J. Okayama Univ. 20(1978), 141-149.

[H:81] Y. Hirano, Regular modules and V-modules, Hiroshima Math. J. 11(1981), 125-142.

[H:85] D.A. Hill, Projective modules with prime endomorphism rings, Arch.Math. 44(1985), 126-137.

[L:70] G.Losey, Are one-sided inverses two-sided inverses in a matrix ring over a group ring? Canad. Math.Bull. 13(1970), 475-479.

[N:75] W.K. Nicholson, I_0-rings, Trans. Amer. Math. Soc. 207(1975), 361-373.

[N:76] W.K. Nicholson, Semi-regular modules and rings, Canad.J. Math. 28(1976), 1105-1120.

[0:71] M. Orzech, Onto endomorphisms are isomorphisms, Amer. Math. Monthly 78(1971), 357-362.

[P:75] R.D. Peterson, One-sided inverses in rings, Canad. J. Math. 27(1975), 218-224.

[R:73] G. Renault, Annaux reguliers auto-injectifs a droite, Bull. Math. Soc. France 101(1973), 237-254.

R:86] M.B. Rege, On von Neumann regular rings and F-rings, Math.Japon. 31(1986), 927-936.

S:68] H.H. Storrer, Epimorphismen von kommutativen Ringen, Comm.Math. Helv. 43(1968), 378-401.

[S:73]$_1$ G. Shin, Prime ideals and sheaf representation of a pseudo-symmetric ring, Trans.Amer.Math.Soc. 184(1973), 43-60.

[S:73]$_2$ M.S. Shrikhande, On hereditary and cohereditary modules, Canad.J.Math. 25(1973), 892-896.

[SV:74] B.Sarath and K. Varadarajan, Injectivity of certain classes of modules, J. Pure Applied Algebra 5(1974), 293-305.

[V:76] K. Varadarajan, A study of certain radicals, J. Pure Applied Algebra 9(1976), 107-120.

[vN:36] J. von Neumann, On regular rings, Proc.Nat. Acad.Sci. U.S.A. 22(1936), 707-713.

[W:71] R. Ware, Endomorphism rings of projective modules, Trans. Amer. Math.Soc. 155(1971), 233-256.

[Z:72] J. Zelmanowitz, Regular modules, Trans.Amer. Math.Soc. 163(1972), 341-355.

[Z.-H.:74] B. Zimmermann(-Huisgen), Endomorphismenringe von Selbstgeneratoren, Dissertation, Ludwig-Maximilians-Universität, München, 1974.

[Z.-H.:75] B. Zimmermann-Huisgen, Endomorphism rings of self-generators, Pacific J. Math. 61(1975), 587-602.

[Z.-H.:76] B.Zimmermann-Huisgen, Pure submodules of direct products of free modules, Math.Ann. 224(1976), 233-245.