On Wave Solutions of Kilmister and Newmann's Weakened Field Equations

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Abstract

Gravitational situations arising out of the introduction of the two functions A and B in the flat metric have been examined by Sharan (1965) and some cases have been investigated in presence of electromagnetic field and the cosmological term. In the present paper I have considered the weakened field equations originally given by Kilmister and Newman (1961), Pirani, Rund, Eddington and Rund (1967) in a space-time metric $ds^2 = -dx^2 - dy^2 - dz^2 - dt^2 + 2A\,dz\,dt + 2B\,dx\,dy$, when $A = A(z,t)$ and $B = B(x,y)$ and have established the existence of plane wave-like solutions. In this space-time the interesting case is that the field equations provide two differential equations that are integrable in terms of elliptic functions.

Key Words: electromagnetic field, weakened field equations, wave-like solutions, elliptic functions.

Introduction

Einstein's field equations in the theory of general relativity is given by

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = -kT_{ik} \,, \qquad (1.1)$$

where k is a coupling constant to be determined by comparison with experiments and T_{ik} is the energy momentum tensor in the presence of

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matter, R_{ik} is the Ricci tensor and R is the scalar curvature defined by

$$R = g^{ik} R_{ik}$$

The tensor G_{ik} enjoys the properties

$$(a)\ G_{ik} = G_{ki} \quad and \quad (b)\ G^{i}_{k;i} = \left(g^{hi} G_{hk}\right)_{;i} = 0 ,$$

where a semicolon (;) followed by an index denotes covariant differentiation. In vacuum T_{ik} vanishes and (1.1) reduces to

$$R_{ik} = 0 \qquad (1.2)$$

Eddington, Buchdahl, Kilmister and Newman, Rund and Du Plessis have suggested vacuum field equations alternative to (1.2) which are weaker than (1.2) in the sense that they admit a class of solutions for which (1.2) holds, and have called such field equations "Weakened Field Equations". Thompson [1] has made a study of these equations and has concluded that the weakened field equations are too weak. D. Lovelock [2], [3] has solved a set of five weakened field equations, namely

$$J_{ijk} \equiv R^{a}_{ijk;a} = 0 \qquad (1.3)$$

$$G_{jk} \equiv (-g)^{\frac{1}{4}} \Big[g^{ih} R_{kj;ih} - g^{ih} R_{ij;kh} + \frac{1}{6} R_{;kj} - \frac{1}{6} g_{jk} g^{ih} R_{;ih} - R^{ih} C_{jhik} + \frac{R}{6} g^{ih} C_{jhik} \Big] = 0 \qquad (1.4)$$

with properties

$$(a)\ G_{jk} = G_{kj} \quad and \quad (b) G^{j}_{k;j} = 0,$$

$$E^{hk} \equiv (-g)^{\frac{1}{4}} \Big[g^{hj} g^{ki} \{ 2R_{jlim} R^{ml} + g^{ml} R_{ij;lm} - R_{;ij} \} - \frac{1}{2} g^{hk} \left(R^{l}_{m} R^{m}_{l} - g^{lm} R_{;lm} \right) \Big] = 0 \,. \qquad (1.5)$$

with properties

$(a)\ E^{hk} = E^{kh}\ and\ (b)\ E^{hk}_{;k} = 0,$

$$\epsilon^{rs} \equiv (-g)^{\frac{1}{2}} \left[(g^{rs}g^{tu} - \frac{1}{2}g^{rt}g^{su} - \frac{1}{2}g^{ru}g^{st})R_{;ut} + R\left(R^{sr} - \frac{1}{4}g^{sr}R\right)\right] = 0\ , \qquad (1.6)$$

with properties

$(a)\ \epsilon^{rs} = \epsilon^{sr}\ \ and\ (b)\ \ \epsilon^{rs}_{;r} = 0,$

and

$$H^{ij}_{k} \equiv R^{ij}_{;k} = 0. \qquad (1.7)$$

Here C_{jhik} is the Weyl curvature given by

$$C_{jhik} = R_{jhik} - \frac{1}{2}(R_{jk}g_{hi} - R_{hi}g_{jk} - R_{ji}g_{hk} + R_{hk}g_{ji}) + \frac{R}{6}(g_{ij}g_{hk} - g_{hi}g_{jk}) \qquad (1.8)$$

Although the physical implications of these weakened field equations are yet not well established many authors have tried to find the solutions of these field equations in the hope that these may be useful in future. Kilmister & Newman [4] and Pirani, Rund, Eddington & Rund [5] originally suggested these field equations. Here we propose to solve the above field equations (1.3) to (1.7) in a space-time whose metric is in the form [6]

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2A dz dt - 2B dx dy, \qquad (1.9)$$

where $A \equiv A(z,t)$ and $B \equiv B(x,y)$

1. Calculation of Christoffel Symbols and Ricci Tensors

The determinant g of the metric (1.9) is found to be

$$g = -(1 + A^2)(1 - B^2) \qquad (2.1)$$

Since g is negative, $B^2 < 1$. The non–vanishing components of g^{ij} are

$$g^{11} = g^{22} = \frac{g^{12}}{B} = \frac{-1}{1-B^2}$$

$$-g^{33} = g^{44} = \frac{g^{34}}{A} = \frac{1}{1+A^2} \qquad (2.2)$$

and the surviving components of Christoffel symbols of second kind $\left\{ {k \atop i\ \ j} \right\}$ are

$$\left\{ {1 \atop 1\ \ 1} \right\} = \frac{-BB_1}{1-B^2}, \qquad \left\{ {3 \atop 3\ \ 3} \right\} = \frac{AA_3}{1+A^2},$$

$$\left\{ {2 \atop 1\ \ 1} \right\} = \frac{-B_1}{1-B^2}, \qquad \left\{ {4 \atop 3\ \ 3} \right\} = \frac{A_3}{1+A^2},$$

$$\left\{ {1 \atop 2\ \ 2} \right\} = \frac{-B_2}{1-B^2}, \qquad \left\{ {3 \atop 4\ \ 4} \right\} = \frac{-A_4}{1+A^2}, \qquad (2.3)$$

$$\left\{ {2 \atop 2\ \ 2} \right\} = \frac{-BB_2}{1-B^2}, \qquad \left\{ {4 \atop 4\ \ 4} \right\} = \frac{AA_4}{1+A^2},$$

where B_1 , B_2 , A_3 and A_4 stand for $\frac{\partial B}{\partial x}$, $\frac{\partial B}{\partial y}$, $\frac{\partial A}{\partial z}$ and $\frac{\partial A}{\partial t}$ respectively.

The non-vanishing independent components of R_{ij} are given by

$$R_{11} = R_{22} = \frac{-R_{12}}{B} = X\ ,$$

$$-R_{33} = R_{44} = \frac{R_{34}}{A} = Y, \qquad (2.4)$$

where

$$X \equiv \frac{B_{12}}{1-B^2} + \frac{BB_1B_2}{(1-B^2)^2}\ ,$$

$$Y \equiv \frac{A_{34}}{1+A^2} - \frac{AA_3A_4}{(1+A^2)^2}. \tag{2.5}$$

From (2.2) & (2.4) the scalar curvature R is given by

$$R = 2(-X+Y). \tag{2.6}$$

From (2.2) & (2.4) the non-vanishing components of R^{ij} are

$$R^{11} = R^{22} = \frac{R_{12}}{B} = \frac{X}{1-B^2},$$

$$-R^{33} = R^{44} = \frac{R_{34}}{A} = \frac{Y}{1+A^2}. \tag{2.7}$$

The non-vanishing components of curvature tensor $R_{ijlm}\left(= -R_{jilm} = -R_{ijml} = R_{lmij}\right)$ are given by

$$R_{1212} = B_{12} + \frac{BB_1B_2}{1-B^2},$$

$$R_{3434} = A_{34} - \frac{AA_3A_4}{1+A^2} \tag{2.8}$$

Using (2.4), (2.6) & (2.8), the non-vanishing components of Weyl curvature tensor C_{jhik} from (1.8) are computed as

$$-C_{1313} = -C_{2323} = C_{1414} = C_{2424} = \frac{-X+Y}{6},$$

$$C_{1212} = \frac{(1-B^2)(5X+Y)}{3},$$

$$C_{3434} = \frac{(1+A^2)(5Y+X)}{3},$$

$$C_{1324} = C_{1423} = \frac{AB(X-Y)}{6},$$

$$C_{1314} = C_{2324} = \frac{A(-X+Y)}{6},$$

$$C_{1323} = -C_{1424} = \frac{B(-X+Y)}{6}.$$

2. Solutions of the Weakened Field Equations (1.3)

The curvature tensor R^a_{ijk} satisfies the Bianchi identity [7]

$$R^a_{ijk;l} + R^a_{ikl;j} + R^a_{ilj;k} = 0. \tag{3.1}$$

Let us put a = 1 and sum with respect to 'a' remembering that R^a_{ijk} is skew-symmetric in *j* and *k*, we find

$$R^a_{ijk;a} = R_{ij;k} - R_{ik;j}. \tag{3.2}$$

Therefore, the weakened field equation (1.3) reduces to

$$R_{ij;k} - R_{ik;j} = 0 \tag{3.3}$$

Using the values of $\left\{ {k \atop i\ \ j} \right\}$ and R_{ij} from (2.3) and (2.4) in (3.3), we obtain

$$\partial_2 X + B\partial_1 X = 0, \qquad \partial_1 X + B\partial_2 X = 0, \tag{3.4a}$$

$$\partial_4 Y + A\partial_3 Y = 0, \qquad \partial_3 Y - A\partial_4 Y = 0, \tag{3.4b}$$

Solving (3.4a) & (3.4b), we obtain

$$\partial_1 X = 0, \quad \partial_2 X = 0, \; \partial_3 Y = 0, \quad \partial_4 Y = 0, \tag{3.5}$$

which implies

$$\frac{B_{12}}{1-B^2} + \frac{BB_1B_2}{(1-B^2)^2} = k_1 \tag{3.6}$$

and

$$\frac{A_{34}}{1+A^2} + \frac{AA_3A_4}{(1+A^2)^2} = k_2 \, , \tag{3.7}$$

where k_1 and k_2 are constants.

Equations (3.6) and (3.7) are non-linear partial differential equations and their solutions are complicated. However, on taking $B = B(x+y), A = A\,(z-t)$ and substituting $\sin\psi$ for B and $\sinh\phi$ for A, the equations (3.6) and (3.7) reduce to

$$\frac{d\psi}{\sqrt{c_1 + 2k_1 \sin\psi}} = d\beta \tag{3.8}$$

and

$$\frac{d\phi}{\sqrt{c_2 - 2k_2 \sinh\phi}} = d\alpha \tag{3.9}$$

respectively, where c_1 and c_2 are constants of integration and $\beta = x+y, \alpha = z-t$.

Putting $tan\frac{\psi}{2} = t_1, \; tan\frac{\phi}{2} = t_2$, it is easy to see that the equations (3.8) and (3.9) take the form

$$d\beta = \frac{2dt_1}{\sqrt{c_1 t_1^4 + 4k_1 t_1^3 - 2c_1 t_1^2 + 4k_1 t_1 + C_1}} \tag{3.10}$$

and

$$d\alpha = \frac{2dt_2}{\sqrt{c_2 t_2^4 + 4k_2 t_2^3 - 2c_2 t_2^2 - 4k_2 t_2 + C_2}} \, . \tag{3.11}$$

The equations (3.10) and (3.11) are integrals of the form $\int F\,(t,u)dt$, where *F* denotes a rational function of t & u and where

$$u^2 = a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4$$

is a quartic or cubic function of *t* without repeated factor, which can be evaluated in terms of elliptic functions [8].

Thus the solution of the weakened field equation (1.3) consists of g_{ij} given by (1.9) where $B = B(x+y) and\ A = A\,(z-t)$ which are determined respectively from equations (3.10) and 3.11).

Similarly in the other three cases when
$B = B(x-y), \quad A = A\,(z-t); \;\; B = B(x+y),$
$A = A\,(z+t) and\ B = B(x-y), A = A\,(z+t)$
equations (3.6) & (3.7) reduce respectively to

$$\frac{d\psi}{\sqrt{c_1 - 2k_1 \sin\psi}} = d\beta, \qquad \frac{d\phi}{\sqrt{c_2 - 2k_2 \sinh\phi}} = d\alpha \, , \tag{3.12}$$

$$\frac{d\psi}{\sqrt{c_1 + 2k_1 \sin\psi}} = d\beta, \qquad \frac{d\phi}{\sqrt{c_2 - 2k_2 \sinh\phi}} = d\alpha \, . \tag{3.13}$$

and

$$\frac{d\psi}{\sqrt{c_1 - 2k_1 \sin\psi}} = d\beta, \qquad \frac{d\phi}{\sqrt{c_2 + 2k_2 \sinh\phi}} = d\alpha \, . \tag{3.14}$$

Like equations (3.8) & (3.9), equations (3.12), (3.13) and (3.14) each can be again reduced to forms (3.10) & (3.11) and hence will be integrable in terms of elliptic functions. Consequently the solutions of (1.3) corresponding to these chosen forms of A & B can be obtained on the same lines as in the case when $B = B(x+y), A = A(z-t)$.

3. Solutions of the Weakened Field Equations (1.4), (1.5) and (1.6)

Theorem 4.1 A necessary and sufficient condition that g_{ij} given by (1.9) be a solution of weakened field equations (1.4), (1.5) & (1.6) is

$$(a)\quad (1-B^2)(X_{11}-X_{22}) + (BB_1X_1 + B_1X_2 - BB_2X_2 - B_2X_1) = 0,$$

$$(b)\quad (1-A^2)(Y_{33}-Y_{44}) - (AA_3Y_3 + A_3Y_4 + AA_4Y_4 - A_4Y_3) = 0, \quad (4.2)$$

where the suffixes 1, 2, 3, 4 denote partial differentiation with respect to x, y, z, t respectively and the values *X* and *Y* are given by (2.5).

Proof: Substituting the components of $g_{ij}, g^{ij}, R_{ij}, R^{ij}, \left\{ {k \atop i\ j} \right\}, R_{ijlm}$ and C_{jhik} from section 2 in the field equations (1.4), (1.5) & (1.6) and solving each field equation separately, we see that each field equation gives the same condition (4.2).

Conversely, if the relation (4.2) holds then the weakened field equations (1.4), (1.5) & (1.6) are identically satisfied. This proves the theorem.

4. Solutions of the Weakened Field Equations (1.7)

Using the components of R^{ij} from (2.7) in the field equation (1.7), we get

$$\frac{\partial_1 X}{1-B^2} - \frac{2BB_1X}{(1-B^2)^2} = 0\,, \quad (5.1)$$

$$\frac{\partial_2 X}{1-B^2} - \frac{2BB_2X}{(1-B^2)^2} = 0\,. \quad (5.2)$$

$$\frac{\partial_3 Y}{1+A^2} + \frac{2AA_3Y}{(1+A^2)^2} = 0\,, \quad (5.3)$$

$$\frac{\partial_4 Y}{1+A^2} + \frac{2AA_4Y}{(1+A^2)^2} = 0\,. \quad (5.4)$$

Solving (5.1) to (5.4), we obtain equation (3.5) of section 3, which gives

$$X \equiv \frac{B_{12}}{1-B^2} + \frac{BB_1B_2}{(1-B^2)^2} = k_3 \quad (5.5)$$

and

$$Y \equiv \frac{A_{34}}{1+A^2} - \frac{AA_3A_4}{(1+A^2)^2} = k_4 \quad (5.6)$$

where k_3 and k_4 are constants.

Equations (5.5) and (5.6) are similar to equations (3.6) & (3.7) except for the constants which are now k_3 and k_4 instead k_1 of and k_2

Consequently, the solutions of the field equations (1.7) can be obtained on the same lines as those of (1.3) obtained in section 3.

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