SOME PROBLEMS IN RING THEORY: (VON NEUMANN) REGULARITY AND ANTI - REGULARITY IN MODULES AND RINGS

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Submitted in fulfilment of the requirements of the Degree of

DOCTOR OF PHILOSOPHY

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CERTIFICATE

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I certify that this dissertation is worthy of consideration by the examiners.

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ACKNOWLEDGEMENTS

This work was carried out under the guidance of Prof.M.B. Rege. I wish to express my sincere thanks to him for all the trouble he has taken during the research period and for the completion of this thesis.

My sincere thanks are also due to Prof. S.M. Bhatwadekar and Prof. Amit Roy for helpful suggestions.

My sincere regards are also due to Prof. M.B. Rege, Dr. H.K. Mukherjee, Dr. P. Saikia, Prof. P. Jothilingam, Prof. S.S. Khare, Prof. Vijay Kumar, Dr. B.K.Dev Sharma and Shri S.L. Marbaniang for the effort they had put in to give us proper foundation for research during M.Sc. courses.

The NEHU had provided me with NEHU fellowship for 4 years for my research work. My deep sense of gratitude is due to Prof. S.S. Khare, then Head of the Department of Mathematics, for the initiative he had taken and for following up all the official formalities. My sincere thanks are also due to all NEHU Officials who had made the necessary arrangements to provide with the NEHU fellowship.

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Some portions of this thesis have been published in National Academy Science Letters jointly with Prof. M.B. Rege and Ms. I. Choudhuri. I am thankful for the valuable co-operation from them.

I also express my deep sense of gratitude to Dr. A.B. Chakraborty and Dr. N. J. Deb for valuable advice and encouragement, and to Mrs. S. Dutta for encouragement, companionship and moral support extended to me during the last seven years.

I am also indebted to my research colleague Ms. Sima Jajodia with whom I could spend some time during the preparation of her M.Phil. thesis. I felt I had gained a lot mathematically during this short period. My sincere thanks are also due to my colleagues Ms. Amita Mitra and Ms. Debadutta Bhattacharjee for their valuable co-operation.

I also express my sincere thanks to all the research fellows, M.Sc. students and office staff of Mathematics Department for their kind co-operation.

Finally, I express my sincere thanks to Mr. V.T. James for the effort he has taken and the valuable time he has spent for typing the thesis in an excellent manner.

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CONTENTS

INTRODUCTION

John von Neumann [vN:36]* initiated the study of rings which satisfy the following condition: for each element a of the ring there exists an element b such that aba = a. These rings, which later came to be known as (von Neumann) regular rings, arise naturally in the study of modular lattices. However, regular rings are interesting objects in their own right and have been studied at a great length during the last half-century. (See the bibliography, containing 270 items, in [G] .)

In this thesis we study (von Neumann) regularity and anti-regularity in modules and rings. Apart from Chapter 0 it consists of four chapters. Since there are varying definitions in the literature, we collect in Chapter 0 those we use; we also recall some basic results. The rest of the thesis divides naturally into two parts: Chapters 1 and 2 are devoted to a study of regularity in modules and rings; Chapters 3 and 4 to anti-regularity in modules and rings.

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* The bibliography is divided into two parts: books and monographs; research papers. The presence of a two-digit number in the citation (indicating the year of publication)means that the reference is to a research paper.

The concept of a regular ring was extended to that of a regular module by a number of authors; see § 1E of Chapter 1. In this thesis the term regular module is used in the sense of Zelmanowitz [Z:72] . Throughout R denotes an associative ring with identity and modules are unitary. (Note, however, that this assumption is absent in [Z:72] .) Let M be a left R-module. Then $N = \mathrm{Hom}_R(M, R)$ is a right R-module in a natural manner. An element m of M is called <u>regular</u> (in M) if $(mf)m = m$ for some $f \in N$. A module M is called <u>regular</u> if each of its elements is regular. When M = R we recover the usual definitions in the ring case.

There is a different definition of a regular module due to Elliger [E:71] . A natural element-wise extension of this definition is given in § 1A of Chapter 1. We denote the set of all regular elements of a module M by Reg(M) and the set of all regular elements in the sense of Elliger by eReg(M). Several results concerning the sets Reg(M) and eReg(M) are recorded in Chapter 1. We have , for example (see § 1D):

Result A: Let R be a left self-injective ring, D a left R-module, and B a submodule of D. Then $B \cap \mathrm{Reg}(D) = \mathrm{Reg}(B)$.

In §2 of Chapter 1 a module B is called absolutely regular if for each overmodule D of B, every element of B is regular in D. A ring R is left absolutely regular if the module ${}_RR$ is absolutely regular. Using Result A above we show (1.40) that every left self-injective, regular ring is left absolutely regular.

This fact raises the question of the existence of regular rings which are not absolutely regular. In §2 we construct commutative regular rings which are not absolutely regular.

In §3 of Chapter 1 we extend to modules a lemma due to McCoy which has been used in the study of regular rings. Among the applications of this extension is a technical Proposition (1.55) which yields an alternative proof of the following result, which is a crucial step in the proof of Theorem 2.8 of [Z:72] :

Corollary 1.56. Let $D = A \oplus B$. Then D is regular if and only if A and B are regular.

Central localizations of regular rings have been studied in [AFS:74] . In § 4 of Chapter 1 we study central localizations of regular modules. Among the results we prove are:

Corollary 1.60. Let ${}_RM$ be a module. Let T be a multiplicatively closed subset of the centre of R. Then the

$T^{-1}R$-module $T^{-1}M$ is regular. (This extends to modules a result (1.61) well-known for rings.)

Example 1.62 shows that even when M_V is a regular R_V-module for each maximal ideal v of the centre of R, the R-module M need not be regular.

Theorem 1.64 asserts that if M is a finitely presented left R-module and M_V is a regular R_V-module for each maximal ideal v of the centre of R then M is a regular R-module.

In Chapter 2 we are concerned with regularity in rings. We collect some basic results concerning one-sided regularity in §1. In §2 we call a ring <u>right 2-finite</u> if every right regular element- in the sense of Azumaya [A:54] (see 0.17)- is left regular. After noting that for each field K the matrix ring $M_n(K)$ is left and right 2-finite, it is pointed out that all 2-finite rings are directly finite (2.11). The rest of §2 is devoted to giving sufficient conditions for the 2-finiteness of a ring. The main result of §3 characterises normal, right 2-finite rings as those rings in which $a = a^2b$ implies $a = ba^2$.

In §4 of Chapter 2 we give sufficient conditions on a ring R which ensure that '<u>property (RC) holds</u>', i.e., the subset Reg(R) is closed under multiplication. We first note that the (RC) property holds (trivially) in commutative rings and in regular rings. The result that the (RC) property holds in normal rings is deduced from a proposition (2.34) valid in modules.

In Chapters 3 and 4 we study anti-regularity in modules and rings. In the theory of generalised inverses a ring has been called anti-regular if for each non-zero element a there exists a non-zero element b such that bab = b. (Such an element b has been called a 2-inverse of a.) In this thesis we introduce a module-theoretic generalisation of this concept. Let M be a left R-module and N the right R-module $Hom_R(M, R)$. An element m of M is anti-regular if there exists a non-zero element f of N such that f(mf) = f. A module M is anti-regular if each of its non-zero elements is anti-regular. When M = R we recover the definitions in the ring case.

Chapter 3 is devoted to a study of anti-regularity in modules. This is carried out in the setting of Morita contexts; we need only their basic properties and these are recalled in §OC. Several equivalent conditions for the anti-regularity of an element of a module are given in Theorem 3.5. We show next (Proposition 3.8(b)) that regular implies anti-regular (for non-zero elements, modules and rings). Thus the class of regular modules is contained in the class of anti-regular modules. This leads to the question: which properties of regular modules hold for anti-regular modules? In §3 we show that the class of anti-regular modules is closed under submodules and direct sums. Unlike the class of regular modules, this class is also closed under direct products. It is shown in § 4 that if M is an anti-regular module, then M is non-singular and Rad(M) = 0.

Endomorphism rings of anti-regular modules are studied in §5. It is shown that if M is an anti-regular left R-module, then $S = End(_R M)$ is an anti-regular ring and M is an anti-regular right S-module. This extends several results of Nicholson [N:75] . In §6 we give conditions under which the converse of Proposition 3.8(b) holds, i.e. anti-regular implies regular. A very general condition under which anti-regular implies regular is 'the module satisfies either the ascending or the descending condition on direct summands' (Theorem 3.37). It follows (3.38) that noetherian anti-regular rings are semi-simple. In Proposition 3.39 we prove that a ring R is semi-simple if and only if every (resp. cyclic , resp. indecomposable, resp.simple) left R-module is anti-regular.

While the chapter headings broadly indicate the themes of the respective chapters, a complete compartmentalisation has been neither possible nor desirable. A case in point is Chapter 4, which is mainly devoted to the study of anti-regularity in rings. In §4 of this chapter we consider, for a left R-module M and an idempotent e of R, the ring eRe and the left eRe-module eM. It was considered appropriate to collect together a number of results concerning the regularity and anti-regularity of eM as well as concerning regularity / anti-regularity of the ring eRe in this section.

The following are some of the results obtained in the rest of Chapter 4. An example of a commutative anti-regular ring R with a multiplicatively closed subset T such that $T^{-1}R$ is not anti-regular is given in §1. Reduced anti-regular rings have properties similar to those of reduced regular rings; this is shown in §2. Rings satisfying the condition 'every factor ring is anti-regular' are studied in §3; if such a ring is commutative, then it is necessarily regular. Formulae for the number of regular/anti-regular elements in Z/nZ are obtained in § 5. We conclude with some open questions concerning similar formulae for certain familiar finite rings.

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