A STUDY OF SOME $p-$ADIC SERIES

ABSTRACT

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To

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Thesis

This thesis can be broadly divided into two parts. The first part deals with the question of finding a series in $\mathbb{Q}$ which converges to different preassigned $p-$adic numbers whereas the second part deals with infinite $p-$adic series involving factorials, namely series of the following three types:

- $$\sum_{n=0}^{\infty} n^k(n+j)!$$
- $$\sum_{n=0}^{\infty} p^{n!}$$
- $$\sum_{n=0}^{\infty} \left(\frac{n!}{n!^2+1}\right)^{n!^2}$$

In what follows we briefly describe about the thesis chapter wise as well as broadly outline main results of each chapter.

In Chapter 1 we introduce $p-$adic norms, convergence with respect to those norms as well as with respect to the usual norm. We also define *algebraic element* over a general field $\mathbb{F}$. However in the later chapters we restrict the study of algebraic element over the field $\mathbb{Q}$ only.

Burger and Struppeck [2] have obtained a series in $\mathbb{Q}$ converging to preassigned $p-$adic numbers as well as real number with respect to respective norms. In Chapter 2 we have a computationally efficient algorithm as well as an elementary proof for the existence of a series converging to given $p-$adic numbers as well as real numbers. Using this efficient algorithm we have computed first few terms of a series converging to 0 with respect to $p-$adic norms as well as converging to e with respect to the usual norm.

In Chapter 3 the question of the rationality of the series of the type $\sum_{n=0}^{\infty} n^k(n+j)!$ for fixed positive integers k, j has been studied. Murty and Sumner [4] have proved that the question of rationality of the sum of the $p-$adically convergent series $\sum_{n=0}^{\infty} n^k(n+j)!$ is linked to the question of "non-vanishing of Uppulari-Carpenter numbers." We explore the work of Murty and Sumner. Also it has been found that *Stirling numbers, Kurepa's number* and certain polynomials (see section (3.3)) etc. could help in our study of the $p-$adic series. So we introduce and study these numbers as well as polynomials. We also study the approach taken by Dragovich [1] and obtain a simpler proof of some of the result of Murty and Sumner [4].

Using ideas similar to the well known proof for the transcendence of the sum $\sum_{n=0}^{\infty} 2^{-n!}$ (using Liouville's criteria) Suter [3] has obtained an elementary proof for the transcendence of the $p-$adic sum $\sum_{n=0}^{\infty} p^{n!}$. We begin fourth chapter by including the proof of transcendence of the $p-$adic sum $\sum_{n=0}^{\infty} p^{n!}$ as given by Suter. Using Liouville's theorem Burger and Struppeck [2] have shown that the series $\sum_{n=0}^{\infty} \left(\frac{n!}{n!^2+1}\right)^{n!^3}$ converges to transcendental number with respect to $p-$adic norm as well as the usual norm on $\mathbb{Q}$. In Chapter 4 we give a proof for transcendence of the sum $\sum_{n=0}^{\infty} \left(\frac{n!}{n!^2+1}\right)^{n!^2}$. As these results depend heavily upon Liouville's theorem, we include a proof of this theorem as well.

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CERTIFICATE

I certify that the dissertation entitled **“A STUDY OF SOME p–ADIC SERIES”** submitted by Mr. Deepak Subedi in partial fulfillment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources, from which ideas are borrowed, have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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DECLARATION

I, Deepak subedi, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

This dissertation is being submitted to the North-Eastern Hill University for the degree of Master of Philosophy in Mathematics.

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Preface

$p-$*adic numbers* were first studied systematically by Kurt Hensel (1861-1941) in the first decade of the twentieth century. Though these numbers appeared strange compared to the familiar real and complex numbers, Hensel's idea of completing the rationals with respect to the $p-$*adic distance* on the rationals (which he himself introduced), put these numbers on a firm mathematical footing. Though Hensel's ideas had their origin in Kummer's remarkable work on *cyclotomic fields,* it was only when Ostrowski(1918) showed that the $p-$adic numbers behave like the real numbers in a number of ways that the significance of the p-adic numbers became clear.

The realization that the field of $p-$adic numbers (for each rational prime p) is as natural as the field of real numbers has had a profound influence on the development of number theory as the growth of class number theory and adelic ideas testifies. One of the crowning achievements in this respect was the discovery of the $p-$*adic Riemann Zeta function* by Kubota and Leopoldt in the sixties through their interpretation of *Kummer's Congruences* of *Bernoulli's numbers.* In the seventies Iwasawa was able to find remarkable connection between cyclotomic fields and $p-$*adic L-functions*. In fact, one can say without any exaggeration, that the proof of *Fermat's Last Theorem* by Wiles, using *elliptic curves*, has been made possible by the extensive work on $p-$*adic L-functions* and $p-$*adic modular forms*, among others, carried out in the last forty years.

At another level, $p-$adic methods have been successfully used in deriving congruences involving *binomial coefficients*, *Bernoulli's numbers*, *Apery numbers*, *Stirling numbers* etc. modulo powers of primes. For example, the determination of certain *binomial coefficients* modulo p^2 by Chowla, Dwork and Evan [28] depended on the $p-$*adic Gamma function* and the Gross-

Koblitz formula for Gauss sum. Hahn and Lee [29] used similar techniques to prove generalisations of such congruences (this type of congruences go back to Gauss, Jacobi and Eisenstein). On the other hand, Washington [19] has used properties of $p-$adic L-functions to generalise classical congruences for sums of powers of integers mod p. Thus there is ever-increasing use of $p-$adic methods in the study of various aspects of elementary number theory too.

In the last few years, $p-$adic numbers and techniques have been used by several physicists to give new interpretation of some ideas from physics. Apart from application in *Quantum mechanics*, one can cite a new model of *space-time* which uses $p-$adic concepts. On the other hand, recently there has been some path-breaking work in *genetics* using $p-$adic methods.

In fact, $p-$adic methods, because of some inherent properties of p-adic numbers, are suitable for studying structures or phenomena in which tree-like formations or *hierarchical formations* are encountered. This explains some attempts to apply $p-$adic analysis in economics and social sciences.

As far as $p-$adic analysis is concerned, *$p-$adic functional analysis* has witnessed a tremendous growth recently. On the other hand, there has been a growing interest in understanding various aspects of series of rationals which converges $p-$adically for every prime p. This dissertation is basically a study of certain questions about some such series.

Broadly speaking, we study the following three topics in this dissertation.

(a) the development of an efficient procedure for computing a series of *rational numbers* which not only converges $p-$adically to any given *$p-$adic numbers* for each prime p, but also converges (with respect to the usual norm) to any given real number;

(b) the study of the rationality questions of certain series with *factorials* which converges $p-$adically for every prime p;

(c) the study of the question of the *transcendence* of certain series with factorials which converges $p-$adically (for each prime p) as well as with respect to *the usual norm.*

Now we give a brief description of the contents of each of the chapters of our dissertation.

The first chapter is introductory in nature. In this chapter we have introduced basic notions like *valuation of integers, $p-$adic norms, convergence with respect to $p-$adic norms, algebraic element over the field* $\mathbb{Q}$ etc.

Burger and Struppeck [11] have described a procedure for obtaining a series which converges not only to pre-assigned real number but also to pre-assigned $p-$adic numbers (with respect to every $p-$adic norm, p being a prime). In the second chapter we modify the proof of Burger and Struppeck as well as, obtain an algorithm through which we could compute a series in $\mathbb{Q}$ converging to pre-assigned number of the type mentioned above. This algorithm is much simpler to handle as compared to Burger's construction. This simplification is due to the fact that, the new algorithm does not involve the Chinese Remainder Theorem. We also include a few computational work based on the new algorithm at the last portion of this chapter.

In the third chapter we study series of the type $\sum_{n=0}^{\infty} n^k(n+j)!$ where k, j are non-negative integers. We also look at a few special cases. We have connected the general series $\sum_{n=0}^{\infty} n^k(n+j)!$ with the series $\sum_{n=0}^{\infty} n!$ using certain polynomials $a_k(j), b_k(j)$ (where j is an indeterminate). The polynomial $a_k(j)$ has the property that if $a_k(j)$ has an integral root j_0 (say) then $\sum_{n=0}^{\infty} n^k(n+j_0)!$ would be a fixed $p-$adic integer whatever be the value of the prime p.

Murty, Sumner [22] and Dragovich [3] have studied the series $\sum_{n=0}^{\infty} n^k(n+j)!$. In fact Murty and Sumner have explicitly expressed the sum $\sum_{n=0}^{\infty} n^k n!$ in terms of the sum $\sum_{n=0}^{\infty} n!$ using Stirling numbers. We study the series $\sum_{n=0}^{\infty} n^k n!$ using the technique developed by Murty, Sumner and Dragovich and a few other preliminary techniques. This study leads to a simplified proof of some of the results of Murty and Sumner. Apart from Stirling numbers (as will be clear from the results in second chapter) that *Kurepa's numbers, Bell numbers* etc. helps in concluding rationality of the sum $\sum_{n=0}^{\infty} n^k n!$.

We begin the fourth chapter by introducing a few preliminary as well as known results. One of these is "the series $\sum_{n=0}^{\infty} p^{n!}$ converges $p-$adically to a *transcendental number*". *Liouville's theorem* is an important tool for the next result of the fourth chapter. We include a proof of the Liouville's theorem both in the real case as well as in the $p-$adic case. Burger and Struppeck have used Liouville's theorem to show that the series $\sum_{n=0}^{\infty} \left(\frac{n!}{n!^2+1}\right)^{n!^3}$

converges to a transcendental number. The proof can be easily modified to show that the series $\sum_{n=0}^{\infty}\left(\frac{n!}{n!^2+1}\right)^{n!^k}$ converges to a transcendental number for every $k \geq 3$. In this chapter we show that the sum of the series $\sum_{n=0}^{\infty}\left(\frac{n!}{n!^2+1}\right)^{n!^2}$ is also a transcendental number.

Contents

Chapter 1

Basic concepts

1.1 Introduction

In this chapter we introduce some basic concepts mainly from $p-$adic analysis. We also include results which are needed for this survey, most of them without proofs. The proofs can be found in standard books on $p-$adic analysis such as Koblitz [25].

1.2 Normed fields

To be able to do analysis in a field, one introduces a metric space structure on the field through the concept of a norm. As usual, $\mathbb{R}$ denotes the field of real numbers while $\mathbb{R}^{\geq}$ denotes the set of nonnegative real numbers. Also, for any field $\mathbb{F}$, we shall denote by $0_{\mathbb{F}}$ and $1_{\mathbb{F}}$ as the additive and multiplicative identity of $\mathbb{F}$ respectively.

Definition 1.2.1. Let $\mathbb{F}$ be a field. A mapping $|.| : \mathbb{F} \to \mathbb{R}^{\geq}$ is a *norm* on $\mathbb{F}$ provided it satisfied the following:

(a) For every $x \in \mathbb{F}, |x| = 0$ if and only if $x = 0_{\mathbb{F}}$.

(b) For every $x, y \in \mathbb{F}$,
$$|xy| = |x||y|.$$

(c) For every $x, y \in \mathbb{F}$,
$$|x + y| \leq |x| + |y|.$$

(d) If, instead of condition (c), $|.|$ satisfies the stronger inequality

$$|x+y| \leq \max\{|x|, |y|\}$$

then the norm on $\mathbb{F}$ is said to be a *non-archmmedean norm.*

In fact, a non-archimedian norm on $\mathbb{F}$ satisfies the "*isosceles triangle property*" which says

$$|x+y| = \max\{|x|, |y|\} \tag{1.1}$$

for every $x, y \in \mathbb{F}$ with $|x| \neq |y|$.

Every field $\mathbb{F}$ has a *trivial norm* given by

$$|x| = \begin{cases} 0 & \text{if } x = 0_{\mathbb{F}} \in \mathbb{F} \\ 1 & \text{if } x \neq 0. \end{cases}$$

To take another example of a norm, if $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q}$, the field of rational numbers, then the *usual norm* on $\mathbb{F}$ is given by the familiar *absolute value.* Thus for every $x \in \mathbb{F}$ the value of the usual norm of x is the absolute value of x that is:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Coming back to the discussion of norms on an arbitrary field $\mathbb{F}$, observe that $|1.1| = |1||1|$. where 1 is the multiplicative identity of $\mathbb{F}$. Since the norm of x in $\mathbb{F}$ is zero if and only if $x = 0$, and in a field $\mathbb{F}, 1 \neq 0$, it follows that $|1| = 1$.

Also note that in a field $(-1).(-1) = 1$; taking norms of both sides of the equality and then taking the square roots of the equality of real numbers thus obtained, we see that $|-1| = \pm 1$. However, the norm of every element being nonnegative, we conclude that $|-1| = 1$. Thus,

$$|1| = |-1| = 1.$$

It also follows that for any $y \in \mathbb{F}$, $|-y| = |-1|.|y| = |y|$. Therefore, for a non-archimedean norm, for any $x, y \in \mathbb{F}$ we see that

$$|x \pm y| \leq \max\{|x|, |y|\}. \tag{1.2}$$

There is a simple criterion for determining the non-archimedeanness of the norm on a field. We introduce some notation before stating this criterion. Given a field $\mathbb{F}$, we' let $n = n1_{\mathbb{F}} = 1_{\mathbb{F}} + \cdots + 1_{\mathbb{F}}$ (by an abuse of notation), which is $1_{\mathbb{F}}$ added to itself n times. Thus, from now onwards, 1 will be the multiplicative identity $1_{\mathbb{F}}$ of $\mathbb{F}$ (Similarly, we shall let 0 stand for additive identity). Here is the criterion for checking whether a norm is non-archimedean or not.

Theorem 1.2.2. *A norm $|.|$ on a field $\mathbb{F}$ is a non-archimedean norm if and only if*

$$|n| \leq 1$$

for every positive integer n.

Proof. If $|.|$ is a non-archimedean norm, then for any positive integer n

$$|n| = |1 + 1 + 1 + \cdots + 1| \leq \max\{|1|, |1|, \cdots |1|\} = 1,$$

proving the theorem one way. To prove it in the other way, assume that $|n| \leq 1$ for every nonnegative integer n. In particular, for a positive integer n, the norm of the binomial coefficient cannot exceed 1. Therefore, for a positive integer m and any $x, y \in \mathbb{F}$, by the binomial expansion (which is valid in any commutative ring) of $(x + y)^m$ we have

$$\begin{aligned} |x + y|^m &= |(x + y)^m| \\ &\leq |\sum_{i=0}^{m} \binom{m}{i} x^i y^{m-i}| \\ &\leq \sum_{i=0}^{m} |x|^i |y|^{m-i}, \end{aligned}$$

where to derive the second inequality we have used both the multiplicative property and the triangle inequality satisfied by the norm. However, note that

$$|x|^i |y|^{m-i} \leq \max\{|x|^m, |y|^m\}$$

for any nonnegative integer $i \leq m$.

It, therefore, follows from the preceeding inequality that for any arbitrary positive integer m,

$$|x+y|^m \leq (m+1)\max\{|x|^m, |y|^m\}.$$

Taking the m^{th} roots of both sides of the above inequality of nonnegative real number, and then letting $m \to \infty$, we finally obtain the inequality

$$|x+y| \leq \max\{|x|, |y|\},$$

as from elementary calculus we know that $\lim_{m\to\infty}(m+1)^{1/m} = 1$.
The proof of this theorem is complete. $\square$

The preceding theorem immediately shows that the usual norm on $\mathbb{R}$ or on $\mathbb{Q}$ cannot be a non-archimedean norm. In the following section, we shall see that there is an infinite family of non-archimedean norms on $\mathbb{Q}$, the field of rational numbers. To see that these norms are essentially different, we need to introduce the concept of *equivalent norm* on a field. The following brief discussion leads to that concept.

Given a norm on a field $\mathbb{F}$ the norm induces a distance function on $\mathbb{F}$ as follows: for any x, y the "distance" $d(x,y)$ is given by

$$d(x,y) = |x-y|.$$

Using the defining property of the norm $|.|$, it can be easily shown that d is a metric on $\mathbb{F}$ and so it induces a topology on $\mathbb{F}$. Thus analytic concepts are available in $\mathbb{F}$ (but with respect to the norm). For example a sequence $\{x_n\}$ of elements of $\mathbb{F}$ is a *Cauchy sequence* with respect to the norm $|.|$ if given any real number $\epsilon > 0$ there is a positive integer M such that whenever $m, n \geq M$

$$|x_m - x_n| < \epsilon.$$

It is also clear that $\{x_n\}$ converges to $l \in \mathbb{F}$ with respect to $|.|$ if the sequence $|x_n - l|$ of real numbers converges to 0 with respect to the usual norm on $\mathbb{R}$. We say that two norms on a field $\mathbb{F}$ are *equivalent* if they induce the same topology or metric space structure on $\mathbb{F}$.

For practical purposes, the following proposition is more useful in determining equivalence of norms.

Proposition 1.2.3. *Two norms $|.|_1$ and $|.|_2$ on the field $\mathbb{F}$ are equivalent if a sequence in $\mathbb{F}$ is Cauchy with respect to $|.|_1$ if and only if it is Cauchy with respect to $|.|_2$.*

We give an alternate but equally useful criterion. Let $|.|_1$ and $|.|_2$ be two norms on a field $\mathbb{F}$. Then $|.|_1$ and $|.|_2$ are equivalent if and only if

$$\{x \in \mathbb{F} : |x|_1 \leq 1\} = \{x \in \mathbb{F} : |x|_2 \leq 1\}.$$

We define *complete metric space* as in any standard textbook on analysis. we say that a field $\mathbb{F}$ with a norm is complete (with respect to the norm) if every Cauchy sequence in $\mathbb{F}$ is a convergent sequence. It is well known that the field of rational numbers is not a complete metric space with respect to the usual absolute value norm. Similarly in the next section we shall introduce another norm on $\mathbb{Q}$ which is not complete.

There is a standard construction in analysis of the *completion* of a metric space. For example, the completion of $\mathbb{Q}$ equipped with the usual absolute value norm is the field of real numbers. In the case of a normed field we can elaborate this construction process as follows: given a field $\mathbb{F}$ with a norm $|.|$, we can construct a field $\mathbb{K}$ such that $\mathbb{F}$ is a subfield of $\mathbb{K}$ and such that the norm $|.|$ can be extended to a norm on $\mathbb{K}$; Moreover, $\mathbb{K}$ is complete with respect to the extended norm. We shall not go into the details here but shall develop these ideas in the next section.

1.3 The $p-$adic norm

Let p be a fixed prime in $\mathbb{Z}$. For any nonzero integer c, let $v_p(c)$ denote the highest power of p dividing c. Note $v_p(p) = v_p(-p) = 1$ whereas $v_p(q) = 0$ for any prime q distinct from p.

Next, given a rational number $x \in \mathbb{Q}, x \neq 0$, we can represent x as a unique fraction a/b where a and b are relatively prime integers and $b > 0$. The requirement that a and b are relatively prime ensures that such a fractional representation is a unique one. Thus, if

$$x = a/b,$$

where a and b are integers with $b > 0$ and $(a.b) = 1$, then (without loss of any ambiguity), we may define

$$v_p(x) = v_p(a) - v_p(b).$$

Clearly $v_p(x)$ is an integer.

Definition 1.3.1. Let p be a prime. Define $|.|$ on $\mathbb{Q}$ as follows: For $x \in \mathbb{Q}$,

$$|x|_p = \begin{cases} 0 & \text{if } x = 0; \\ p^{-v_p(x)} & \text{whenever } x \neq 0. \end{cases}$$

That $|.|_p$ is a norm on $\mathbb{Q}$ is a routine verification; we omit it. Further, note that for any nonzero integer n, $v_p(n)$ is a nonnegative integer, so that $p^{-v_p(n)} \leq 1$. It follows from Theorem (1.2.2) that $|.|_p$ is a non-archimedean norm on $\mathbb{Q}$.

Proposition 1.3.2. *Let p be a prime. Then* $|.|_p$ *on* $\mathbb{Q}$ *is a non-archimedian norm on* $\mathbb{Q}$.

$|.|_p$ is known as the *$p-$adic norm* on $\mathbb{Q}$. This p-adic norm provides an interesting metric space structure on $\mathbb{Q}$, which is radically different from the one induced by the usual absolute value norm. For example, $2-$adic distance between 0 and 2 is $1/2$ whereas $2-$adic distance between 0 and 25 is 1. Loosely speaking, the higher the power of a prime p dividing $a - b$ for two rationals a and b, the smaller the distance between a and b.

To take another example, note that the $p-$adic norm of p^n is p^{-n}. It follows that the sequence $\{p^n\}$ converges $p-$adically to 0. On the other hand, if q is a prime distinct from p, then it can be easily shown that $\{q^n\}$ is not even a Cauchy sequence with respect to the $p-$adic norm. Also observe that $\{p^n\}$ cannot be a Cauchy sequence in $\mathbb{Q}$ with respect to the usual norm of $\mathbb{Q}$. Thus we have outlined the proof of the following result.

Proposition 1.3.3. *The $p-$adic norm on* $\mathbb{Q}$ *is not equivalent to the $q-$adic norm if q is a prime distinct from p. It is also not equivalent to the usual norm on* $\mathbb{Q}$.

Though we shall not require it, we state the classical theorem of Ostrowski about the possible norms on $\mathbb{Q}$. At this point, we introduce a notation which simplifies certain statements about norms on $\mathbb{Q}$. We let $|.|_\infty$ denote the usual norm on $\mathbb{Q}$.

Theorem 1.3.4. *(Ostrowski)Any nontrivial norm on* $\mathbb{Q}$ *is equivalent to either* $|.|_p$ *for some prime p or for* $p = \infty$.

A basic result of real analysis states that $\mathbb{Q}$ is not complete with respect to the usual absolute value norm and that the completion of $\mathbb{Q}$, with respect to that norm, is the field $\mathbb{R}$ of real numbers. The norm on $\mathbb{Q}$ also extends to what we call as the usual norm on $\mathbb{R}$.

A similar situation occurs when we consider $\mathbb{Q}$ with respect to the $p-$adic norm for a fixed rational prime p. To show that $\mathbb{Q}$ is not complete with respect to the $p-$adic norm (for a fixed prime p), consider the sequence $\{S_n\}$ in $\mathbb{Q}$ given by

$$S_n = \sum_{i=0}^{n} p^{i!}.$$

Since for positive integers m, n with $m < n$, we have

$$\begin{aligned}
|S_n - S_m|_p &= |p^{(m+1)!} + \cdots + p^{n!}|_p \\
&\leq \max\{|p^{(m+1)!}|_p, \cdots, |p^{n!}|_p\} \\
&= \max\{p^{-(m+1)!}, \cdots, p^{-n!}\} \\
&= p^{-(m+1)!}.
\end{aligned}$$

It follows by choosing m large enough that we can make $|S_n - S_m|_p$ as small as we like. Thus S_n is a Cauchy sequence with respect to the $p-$adic norm but we shall show in a later section (see Theorem (3.2.2)) that $\{S_n\}$ cannot converge to a rational number. Therefore for every prime p, $\mathbb{Q}$ is not complete with respect to the $p-$adic norm.

1.4 $\mathbb{Q}_p$, the field of $p-$adic numbers

In this section we give a brief description of $\mathbb{Q}_p$, the completion of $\mathbb{Q}$ with respect to the $p-$adic norm (for a fixed but arbitrary prime p), focusing on some important features. We state several results without proofs; their proofs can be found in Koblitz [25].

The elements of $\mathbb{Q}_p$ are, technically speaking, equivalence classes of Cauchy sequences in $\mathbb{Q}$ with respect to the $p-$adic norm; two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ are *equivalent sequences*, if the sequence $\{x_n - y_n\}$ is a null sequence, i.e., $|x_n - y_n|_p \to 0$ (as a sequence of reals). Thus given any $\alpha \in \mathbb{Q}_p$, α has a representative Cauchy sequence, say $\{x_n\}$, in $\mathbb{Q}$. For example, $0 \in \mathbb{Q}_p$ is the equivalence class of any sequence in $\mathbb{Q}$ which converges to

$0 \in \mathbb{Q}$. In fact, we can identify $\mathbb{Q}$ as a subset of $\mathbb{Q}_p$ by identifying any $a \in \mathbb{Q}$ with the equivalence class of the constant sequence $\{a\}$ in $\mathbb{Q}$. By standard arguments used in the process of completion, one can show that $\mathbb{Q}_p$ is *the smallest complete field containing $\mathbb{Q}$ as a subfield.*

One introduces a norm in $\mathbb{Q}_p$, which we shall also call as the $p-$adic norm and denote it by $|.|_p$, as follows: Given a Cauchy sequence $\{x_n\}$ in $\mathbb{Q}$, with respect to the $p-$adic norm, the sequence $\{|x_n|_p\}$ is a Cauchy sequence of nonnegative reals, for $||x_n|_p - |x_m|_p| \leq |x_n - x_m|_p$. Thus, $\{|x_n|_p\}$ converges to a nonnegative real. Moreover, it is easy to see that if $\{x_n\}$ and $\{y_n\}$ are equivalent Cauchy sequences in $\mathbb{Q}$, then $\lim_{n\to\infty} |x_n|_p = \lim_{n\to\infty} |y_n|_p$ Therefore, for any $\alpha \in \mathbb{Q}_p$ with a representative Cauchy sequence $\{x_n\}$ in $\mathbb{Q}$, we can define, without any ambiguity, the p-adic norm of α as

$$|\alpha|_p = \lim_{n\to\infty} |x_n|_p. \tag{1.3}$$

The following proposition gives the important properties of $\mathbb{Q}_p$ with respect to $|.|_p$.

Proposition 1.4.1. *As defined in* (1.3), *the norm $|.|_p$ is a nonarchimedean norm on $\mathbb{Q}_p$ and extends the $p-$adic norm on $\mathbb{Q}$. Moreover, $\mathbb{Q}$ is a dense subfield of $\mathbb{Q}_p$ with respect to the metric induced by $|.|_p$ on $\mathbb{Q}_p$.*

Since the newly introduced norm on $\mathbb{Q}_p$ extends the $p-$adic norm on $\mathbb{Q}$, we are justified in denoting the new norm as $|.|_p$.

$\mathbb{Q}_p$ is called the field of *p-adic numbers*. Note that the value group of the p-adic norm on $\mathbb{Q}_p$ remains the same as the one of the p-adic norm on $\mathbb{Q}$. i.e.,

$$\{|\alpha|_p : \alpha \in \mathbb{Q}_p\} = \{|x|_p : x \in \mathbb{Q}\}.$$

The next result is a collection of some useful facts. Note that the ring $\mathbb{Z}$ of rational integers, being a subring of $\mathbb{Q}$, is also a subring of $\mathbb{Q}_p$.

Proposition 1.4.2. *(a) If p^n does not divide a where $a \in \mathbb{Z}$ then $|a|_p > p^{-n}$.*

(b) For $x \in \mathbb{Q}$, if $|x|_p \leq 1$ then $x = a/b$ for some $a, b \in \mathbb{Z}$ satisfying the condition p does not divide b.

(c) For $x \in \mathbb{Q}_p$ if $|x|_p \leq 1$ then $|1 + x|_p \leq 1$.

(d) For $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{p^n}$ then $|a-b|_p \leq p^{-n}$.

The unit closed disc in the metric space $\mathbb{Q}_p$ is denoted by $\mathbb{Z}_p$. Thus

$$\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1\}.$$

The elements of $\mathbb{Z}_p$ are called *p-adic integers*. The next proposition collects the basic facts about $\mathbb{Z}_p$.

Proposition 1.4.3. *(a) $\mathbb{Z}_p$ is an integral domain with*

$$P = \{\alpha \in \mathbb{Z}_p : |\alpha|_p < 1\}$$

as the unique maximal ideal. In fact, $\mathbb{Z}_p$ is a PID whose nontrivial ideals are precisely

$$P = p\mathbb{Z}_p \supseteq p^2\mathbb{Z}_p \supseteq \cdots \supseteq p^n\mathbb{Z}_p \supseteq \cdots$$

(b) $\mathbb{Z}_p$ is the topological closure of $\mathbb{Z}$ in $\mathbb{Q}_p$. Every element of $\mathbb{Z}_p$ is the limit of a sequence of rational integers; in fact we can choose terms of the sequence to be nonnegative rational integers. Conversely, every Cauchy sequence of rational integers has a limit in $\mathbb{Z}_p$.

The second part of the proceeding proposition follows from an important technical lemma which can be stated as follows.

Lemma 1.4.4. *For $\alpha \in \mathbb{Z}_p$ and a positive integer m, there is an integer $y_m \in \mathbb{Z}$ with $0 \leq y_m \leq p^m - 1$ such that*

$$|\alpha - y_m|_p \leq p^{-m}.$$

This approximation lemma, applied successively, allows us to express any $\alpha \in \mathbb{Z}_p$ as a special kind of series in $\mathbb{Q}_p$ (see next section for details of series in $\mathbb{Q}_p$). To be precise, given $\alpha \in \mathbb{Z}_p$, we can find nonnegative integers $a_i \leq p-1$ such that

$$\alpha = a_0 + a_1p + a_2p^2 + \cdots$$

This is not a formal power series: the series on the right actually converges to α in $\mathbb{Q}_p$. Suppose now that $\alpha \in \mathbb{Q}_p$ with $|\alpha|_p = p^m$. Then $p^m\alpha \in \mathbb{Z}_p$ and so the preceding series expansion leads to what is known as the *p-adic expansion* of elements of $\mathbb{Q}_p$. This expansion is similar to the decimal expansion of real numbers.

Proposition 1.4.5. *Any p-adic number α has a unique p-adic expansion*

$$\alpha = a_{-m}p^{-m} + \cdots + a_{-1}p^{-1} + a_0 + a_1p + a_2p^2 + \cdots$$

where each of the nonnegative integers a_i is less than p and m is a finite integer. Furthermore $|\alpha|_p = p^m$ and thus $\alpha \in \mathbb{Z}_p$ if and only if $a_i = 0$ for $i < 0$.

Considering the partial sums of the p-adic expansion of $\alpha \in \mathbb{Z}_p$, we see that α is represented by a Cauchy sequence of rational integers $\{b_n\}$ such that

$$(a) \quad 0 \leq b_n < p^{n+1},$$
$$(b) \quad b_n \equiv b_{n+1} \pmod{p^{n+1}}.$$

Another conclusion, we can draw from the preceding proposition is that for any integer $n \geq 0$,

$$p^n\mathbb{Z}_p = \cup_{a=0}^{p-1}(ap^n + p^{n+1})\mathbb{Z}_p.$$

Then a standard argument shows that $\mathbb{Z}_p$ *is a compact set* . Since $\mathbb{Z}_p$ is a neighbourhood of $0 \in \mathbb{Z}_p$ and $\alpha + \mathbb{Z}_p$ is a neighbourhood of any $\alpha \in \mathbb{Q}_p$, it follows that $\mathbb{Q}_p$ is *locally compact.* We end this section by discussing congruences in $\mathbb{Z}_p$. Taking a cue from Proposition (1.4.2) part (d), we can define, for any $\alpha, \beta \in \mathbb{Z}_p$,

$$\alpha \equiv \beta \pmod{p^n} \text{ if and only if } |\alpha - \beta|_p \leq p^{-n}. \tag{1.4}$$

Thus, congruences modulo powers of p can be introduced in $\mathbb{Z}_p$; it is clear that if α, β are actually in $\mathbb{Z}$, then the congruence $\alpha \equiv \beta \pmod{p^n}$ in $\mathbb{Z}_p$ implies that p^n divides $\alpha - \beta$ in $\mathbb{Z}$.

1.5 Convergence in $\mathbb{Q}_p$

As $\mathbb{Q}_p$ is a metric space with respect to the $p-$adic norm, the usual results about convergence of sequences and series in metric spaces also hold in $\mathbb{Q}_p$. We now discuss the additional features of p-adic convergence which arise out of the fact that the p-adic norm on $\mathbb{Q}_p$ is nonarchimedean norm. A word about terminology: when a series converges in $\mathbb{Q}_p$, we shall sometimes say that the series converges $p-$adically.

As in any metric space, a sequence $\{a_n\}$ in the field $\mathbb{Q}_p$ is said to be a *null sequence* if the sequence converges to 0 in $\mathbb{Q}_p$. Thus $\{a_n\}$ is a *null sequence* in $\mathbb{Q}_p$ if the sequence of real numbers $|a_n|_p$ converges to the real number 0 with respect to the usual norm on $\mathbb{R}$.

As the next two propositions show, the non-archimedeanness of the p–adic norm on $\mathbb{Q}_p$, results in some simpler criterion for convergence of sequences and series in $\mathbb{Q}_p$.

Theorem 1.5.1. *A sequence $\{a_n\}$ in $\mathbb{Q}_p$, is a Cauchy sequence (and hence a convergent sequence) if and only if the sequence $\{b_n\}$ is a null sequence where $b_n = a_n - a_{n-1}$.*

Proof. We need to prove only that if the sequence $\{b_n\}$ is a null sequence in $\mathbb{Q}_p$, then the sequence $\{a_n\}$ is a Cauchy sequence. Assuming b_n to be a null sequence, for a given real number $\epsilon > 0$ there exist a positive integer M such that

$$|a_n - a_{n-1}|_p < \epsilon$$

whenever $n > M$. Therefore, for any positive integers n and m such that $n > m > M$, by the *stronger triangle inequality* for the p-adic norm, we see that

$$\begin{aligned} |a_n - a_m|_p &< \max\{|a_n - a_{n-1}|_p, |a_{n-1} - a_{n-2}|_p, \cdots, |a_{m+1} - a_m|_p\} \\ &< \epsilon. \end{aligned}$$

□

The next theorem shows that checking the convergence of p-adic series is even simpler.

Theorem 1.5.2. *A series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$, converges in $\mathbb{Q}_p$ if and only if the sequence $\{a_n\}$ is a null sequence in $\mathbb{Q}_p$, in which case*

$$|\sum_{n=0}^{\infty} a_n|_p \leq \max_n |a_n|_p.$$

Proof. The series $\sum_{n=0}^{\infty} a_n$ converges if and only if the sequence of partial sums $S_n = \sum_{i=0}^{n} a_i$ converges. But $a_n = S_n - S_{n-1}$. It follows from Theorem (1.5.1) that a_n tends to 0 if and only if the series converges. For proving the inequality given in the theorem, assume that $\sum_{n=0}^{\infty} a_n$ converges. If $\sum_{n=0}^{\infty} a_n = 0$, there is nothing to prove. Otherwise there exist an integer M such that for every $n \geq M$,

$$|\sum_{n=0}^{\infty} a_n - S_n|_p < |\sum_{n=0}^{\infty} a_n|_p$$

which, because of the isosceles triangle property of the $p-$adic norm, implies that

$$|\sum_{n=0}^{\infty} a_n|_p = |S_M|_p.$$

On the other hand, by the strong triangle inequality for the p-adic norm it is clear that

$$|S_M|_p \leq \max \{|a_n|_p | 1 \leq n \leq M\} \leq \max_{n} |a_n|_p.$$

The preceding two expression completes the proof. ☐

We now present a couple of examples illustrating the use of Theorem (1.5.2). Since $|2^n|_2 = 2^{-n}$ whereas $|2^n|_p = 1$ for $p \neq 2$, it therefore follows from Theorem (1.5.2) that $\sum_{n=0}^{\infty} 2^n$ converges in $\mathbb{Q}_2$ but diverges in all other $\mathbb{Q}_p$.

One of the simplest example of a convergent series in $\mathbb{R}$ is provided by $\sum_{n=0}^{\infty} \frac{1}{n!}$. Again by the preceding theorem, this series diverges with respect to every $p-$adic norm. However, some other series involving factorials give us a rich supply of series which converge with respect to every p-adic norm. We begin with the simplest result of this kind.

Proposition 1.5.3. *For fixed nonnegative integers k and j, the series*

$$\sum_{n=0}^{\infty} n^k (n+j)!$$

converges $p-$adically for every prime p.

Proof.

$$|p^k!|_p = |p^{k-1}!|_p p^{-k} \leq p^{-k}.$$

For a given positive integer n, we can choose a positive integer k such that $p^k > n > p^{k-1}$. So, for a fixed nonnegative integer j it is clear that

$$\begin{aligned}|n^k(n+j)!|_p &= |n^k(p^k!)(p^k+1)\cdots(n+j)|_p \\ &\le |p^k!|_p \\ &\le p^{-k}.\end{aligned}$$

Thus we have

$$\lim_{n\to\infty} |n!|_p = 0.$$

Hence by Theorem (1.5.2) the result follows. ☐

Corollary 1.5.4. *The series*

$$\sum_{n=0}^{\infty} n!$$

converges $p-$adically.

We now discuss the convergence of a series with factorials that we study later in fourth chapter.

Proposition 1.5.5. *For a fixed positive integer j and a fixed nonnegative integer k, the series*

$$\sum_{n=0}^{\infty} \left(\frac{n!}{(n!)^j + 1}\right)^{(n!)^k}$$

converges p-adically for every prime p.

Proof. By the definition of $p-$adic norm, for every $n \ge p$, it is clear that (as p does not divide $n!^j + 1$)

$$|n!|_p = \left|\frac{n!}{(n!)^j + 1}\right|_p.$$

It follows that

$$\lim_{n\to\infty} \left|\frac{n!}{n!^j + 1}\right|_p = \lim_{n\to\infty} |n!|_p = 0.$$

However, for every integer $n \ge p$,

$$\left|\frac{n!}{n!^j + 1}\right|_p \le 1.$$

Therefore

$$\left|\left(\frac{n!}{(n!)^j+1}\right)^{(n!)^k}\right|_p \leq \left|\frac{n!}{n!^j+1}\right|_p.$$

Hence

$$\lim_{n\to\infty}\left|\left(\frac{n!}{n!^j+1}\right)^{n!^k}\right|_p = \lim_{n\to\infty}\left|\frac{n!}{n!^j+1}\right|_p = 0.$$

Thus by Theorem (1.5.2) the result follows. □

Corollary 1.5.6. *The series*

$$\sum_{n=0}^{\infty}\frac{n!}{n!^2+1}$$

converges in $\mathbb{Q}_p$ for every prime p.

For certain applications, we need a better estimate of $|p^k!|_p$. In the rest of this section, we derive such estimates. We begin by recalling the following well-known result.

Theorem 1.5.7. *For any nonnegative integer n, the following holds:*

$$v_p(n!) = \frac{n - S_p(n)}{p-1},$$

where $S_p(n)$ denotes the sum of the p-adic digits in the unique $p-$adic expansion of n.

Proof. Let

$$n = a_0 + a_1p + a_2p^2 + a_3p^3 + \cdots + a_kp^k$$

be the unique $p-$adic expansion of a nonnegative integer n, where the p-adic digits a_i satisfy the condition $0 \leq a_i \leq p-1$. Then it is clear that

$$n - S_p(n) = a_1(p-1) + a_2(p^2-1) + \cdots + a_k(p^k-1).$$

Then a routine calculation shows that

$$\frac{n-S_p(n)}{p-1} = a_1 + a_2(p+1) + a_3(p^2+p+1) + \cdots + a_k(p^{k-1}+p^{k-2}+\cdots p+1).$$

For a nonnegative integer n, de Polignac's well-known formula (see [15]) for evaluating $v_p(n!)$ is given by

$$v_p(n!) = \sum_{i=1}^{k} [n/p^i].$$

Now, for $1 \leq j \leq k$, the p-adic expansion of n shows that

$$[n/p^j] = a_j + a_{j+1}p^1 + a_{j+2}p^2 + \cdots + a_k p^{k-j}.$$

It follows that

$$v_p(n!) = a_1 + a_2(1 + p) + a_3(1 + p + p^2) + \cdots + a_k(1 + p + \cdots + p^{k-1}).$$

The theorem follows. $\square$

Corollary 1.5.8. *For any positive integer n, we have*

$$|n!|_p = p^{-\frac{n - S_p(n)}{p-1}}.$$

In particular,

$$|(p^k)!|_p = p^{-\frac{p^k - 1}{p-1}}.$$

This corollary also allows us to evaluate the p-adic norm of the binomial coefficients.

Corollary 1.5.9. *For any positive integer n and integer r such that $0 \leq r \leq n$,*

$$|{}^nC_r|_p = p^{-\frac{S_p(n-r) + S_p(r) - S_p(n)}{p-1}}.$$

Choosing $n = p^k$ in Corollary (1.5.8), it is clear that

$$|{}^{p^k}C_r|_p = p^{-\frac{S_p(p^k - r) + S_p(r) - 1}{p-1}}$$

Using $S_p(p^k - r) + S_p(r) \geq 2$, we have $|{}^{p^k}C_r|_p < 1$. Hence the following corollary holds.

Corollary 1.5.10. *For positive inegers k and r, where $r < p^k$, ${}^{p^k}C_r$ is divisible by p.*

In the last chapter of this thesis, we shall examine whether certain $p-$adic numbers are algebraic (or transcendental) over $\mathbb{Q}$. In this section we state the basic results from field theory that we require.

1.6 Algebraic numbers

Let $\mathbb{K}$ be an extension of a field $\mathbb{F}$. An element $\alpha \in \mathbb{K}$ is called an *algebraic* element over $\mathbb{F}$ if α satisfies a nonzero polynomial (which forces the polynomial to be non-constant as well) over the field $\mathbb{F}$. If no such polynomial exists then $\alpha \in \mathbb{K}$ is *transcendental* over $\mathbb{F}$.

If $\alpha \in \mathbb{K}$ is *algebraic* over $\mathbb{F}$, then choose the *monic polynomial* of the least positive degree over $\mathbb{F}$ that α satisfies. It turns out that this is an *irreducible polynomial* and that α determines this polynomial uniquely. This unique monic, irreducible polynomial is called the *minimal polynomial* of α over $\mathbb{F}$.

If α is algebraic over $\mathbb{F}$ such that its minimal polynomial $m(x)$ has degree d, then we say that α *is algebraic of degree d.*

Let $\alpha_1, \alpha_2, \cdots, \alpha_d$ be the roots of $m(x)$ in some suitable extension of $\mathbb{F}$ (say in the *algebraic closure* or in the *splitting field* of $m(x)$ over $\mathbb{F}$). Then $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_d$ are called the conjugates of α over $\mathbb{F}$ and norm of α. $N_{\mathbb{K}/\mathbb{F}}(\alpha)$ can be given by

$$N_{\mathbb{K}/\mathbb{F}}(\alpha) = \left| \prod_{i=1}^{d} \alpha_i \right|^{1/d}$$

Recall that if a field $\mathbb{F}$ has *characteristic* 0 ($n1_{\mathbb{F}} \neq 0$ for every positive integer n), then the roots of any irreducible polynomial over $\mathbb{F}$ are distinct. Thus the minimal polynomial of $\alpha \in \mathbb{K}$ over $\mathbb{F}$ has distinct roots (in some suitably chosen extension of $\mathbb{F}$) or equivalently, the minimal polynomial can be factored into a product of distinct linear factors over such an extension.

Chapter 2

Series converging to given numbers

2.1 Introduction

Koblitz [25] posed the question as to whether or not there exists a series of rational numbers which converges with respect to every (non-equivalent norm) on $\mathbb{Q}$. This question was answered in the affirmative by Burger and Struppeck [11]. For a precise statement of Koblitz's questions, we need the following notation:

- p_k denotes the k^{th} rational prime; so that $p_1 = 2, p_2 = 3$ etc.
- $V_{\mathbb{Q}} = \{\infty, 2, 3, \ldots, p_k, \ldots\}$.

Koblitz's query actually leads to the following questions:

Question 1 : Does there exist an infinite series of non-zero rational numbers, such that for each $p \in V_{\mathbb{Q}}$, the series converges (with respect to $|.|_p$) in $\mathbb{Q}_p$?

Question 2 : Does there exist an infinite series of non-zero rational numbers so that the series converges to a rational number in $\mathbb{Q}_p$ for each $p \in V_{\mathbb{Q}}$?

Burger and Struppeck [11] have answered these questions by proving the following proposition.

Proposition 2.1.1. *Given α_p in $\mathbb{Q}_p$ for each $p \in V_{\mathbb{Q}}$, there exist a series $\sum_{n=0}^{\infty} a_n$ of rational numbers such that the following two condition are satisfied:*

(a) a_n is positive for any positive integer n;

(b) $\sum_{n=0}^{\infty} a_n$ converges to α_p for all values of $p \in V_{\mathbb{Q}}$.

We present two alternative proofs of Proposition (2.1.1) in this chapter. In section 2.2, we give a modified version of the procedure Burger and Struppeck [11] have outlined. Our version is simpler and avoids the ambiguity inherent in their procedure. In Section 2.3, we give an entirely different procedure for finding such a series. Compared to the first one, the second procedure is direct and avoids the Chinese Remainder Theorem.

2.2 The first proof

Let S_k denote the k^{th} partial sum of the required series $\sum_{n=0}^{\infty} a_n$, that is

$$S_k = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots + a_k,$$

where each a_k is a rational number, to be determined in such a way that the series $\sum_{n=0}^{\infty} a_n$ converges to α_p with respect to the p-adic norm for each $p \in V_{\mathbb{Q}}$.

We begin by setting

$$S_0 = a_0 = [\alpha_\infty] - 1. \tag{2.1}$$

Next, assuming that $k \geq 0$, we define the partial sum S_k inductively in such a manner that the following two inequalities will be satisfied:

$$\alpha_\infty - 1/2^{k-1} < S_k < \alpha_\infty - 1/2^k; \tag{2.2}$$

$$\|\alpha_{p_i} - S_k\|_{p_i} \leq p_i^{-k} \text{ for every } 1 \leq i \leq k. \tag{2.3}$$

For this inductive procedure, assume that $a_0, a_1, \ldots, a_{k-1}, a_k$ have been defined; we now describe a method to determine the rational number a_{k+1} such that the partial sum

$$S_{k+1} = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots + a_{k+1}$$

satisfies Inequalities (2.2) and (2.3) simultaneously. But first we fix some notations needed for our procedure. For any fixed nonnegative integer k and any positive integer $i \leq k+1$, let

$$\begin{aligned}
m_{k+1,i} &= \alpha_{p_i} - S_k, \\
\Delta_{k+1} &= \{i \mid 1 \leq i \leq k+1 \text{ and } |m_{k+1,i}|_{p_i} > 1\}, \\
M_{k+1} &= \begin{cases} \prod_{i \in \Delta_{k+1}} p_i^{k+1} |m_{k+1,i}|_{p_i} & \text{if } \Delta_{k+1} \neq \phi \\ \prod_{i=1}^{k+1} p_i^{k+1} & \text{if } \Delta_{k+1} = \phi, \end{cases} \\
R_k &= \alpha_\infty - S_k - \frac{1}{2^k}, \\
L_k &= \alpha_\infty - S_k - \frac{1}{2^{k-1}}.
\end{aligned}$$

Let us fix an integer z_{k+1} such that z_{k+1} satisfies the following two conditions:

$$z_{k+1}(R_{k+1} - L_{k+1}) > M_{k+1} \tag{2.4}$$

and

$$\text{g.c.d}(z_{k+1}, p_1 p_2 \cdots p_{k+1}) = 1. \tag{2.5}$$

Observe that $z_{k+1} = 2^{k+1} M_{k+1} + 1$ is one of the integer which satisfy the conditions given in Inequalities (2.4) and (2.5); to avoid ambiguity we fix $z_{k+1} = 2^{k+1} M_{k+1} + 1$. Next, we define q_{k+1} as follows:

$$q_{k+1} = \begin{cases} z_{k+1} \prod_{j \in \Delta_{k+1}} |m_{k+1,j}|_{p_j} & \text{if } \Delta_{k+1} \neq \phi \\ z_{k+1} & \text{if } \Delta_{k+1} = \phi. \end{cases}$$

It is clear that for any positive integer $i \leq k+1$,

$$|q_{k+1}|_{p_i} = \begin{cases} \dfrac{1}{|m_{k+1,i}|_{p_i}} & \text{if } \Delta_{k+1} \neq \phi \\ 1 & \text{if } \Delta_{k+1} = \phi. \end{cases} \tag{2.6}$$

We claim that for any fixed positive integer $i \leq k+1$,

$$q_{k+1} m_{k+1,i} \in \mathbb{Z}_{p_i}. \tag{2.7}$$

To prove our claim, we first consider the case when $\Delta_{k+1} \neq \phi$. Then $|m_{k+1,i}|_{p_i} > 1$; on the other hand Eqn.(2.6) shows, in this case, that

$$|q_{k+1}|_{p_i} = \frac{1}{|m_{k+1,i}|_{p_i}}.$$

It follows that

$$|q_{k+1}|_{p_i}|m_{k+1,i}|_{p_i} = 1.$$

In case $\Delta_{k+1} = \phi$, we have $|m_{k+1,i}|_{p_i} \le 1$ and $|q_{k+1}|_{p_i} = 1$. So, in this case

$$|q_{k+1}m_{k+1,i}|_{p_i} \le 1.$$

Thus, in both the cases, Eq.(2.7) holds proving our claim. Since $\mathbb{Z}$ is a dense subset of $\mathbb{Z}_{p_i}$, it follows that for the integer i, we can find a rational integer $d_{k+1,i}$ to approximate the p_i-adic integer $q_{k+1}m_{k+1,i}$ in the following manner:

$$|q_{k+1}m_{k+1,i} - d_{k+1,i}|_{p_i} \le \frac{|q_{k+1}|_{p_i}}{p_i^{k+1}}; \tag{2.8}$$

Observe that the right hand side of Inequality (2.8) is the p_i-adic norm of $\frac{p_i^{k+1}}{|q_{k+1}|_{p_i}}$ which itself is a positive integral power of p_i. After choosing the integer $d_{k+1,i}$ for each positive integer $i \le k+1$, we consider the following system of $k+1$ linear congruences in $\mathbb{Z}$ for $i = 1, 2, \ldots, k+1$:

$$x_{k+1} \equiv d_{k+1,i} \quad (\text{mod}\ \frac{p_i^{k+1}}{|q_{k+1}|_{p_i}}). \tag{2.9}$$

By the Chinese Remainder Theorem, there exists a solution of the system of congruences unique modulo M_{k+1}, as

$$M_{k+1} = \prod_{i=1}^{k+1} \frac{p_i^{k+1}}{|q_{k+1}|_{p_i}}.$$

Note that for any solution x_{k+1} of the system of congruences (2.9), the Inequalities (2.8) show that

$$|x_{k+1} - m_{k+1,i}q_{k+1}|_{p_i} \le \frac{|q_{k+1}|_{p_i}}{p_i^{k+1}}. \tag{2.10}$$

On the other hand, our choice of q_{k+1} implies that

$$q_{k+1}(R_{k+1} - L_{k+1}) > M_{k+1}.$$

Therefore one can choose a solution r_{k+1} of the preceding system of $k+1$ congruences, satisfying Inequality (2.9), such that it also satisfies

$$q_{k+1}L_{k+1} < r_{k+1} < q_{k+1}R_{k+1}. \tag{2.11}$$

Finally, we set

$$a_{k+1} = r_{k+1}/q_{k+1}.$$

Then, by the definition of $m_{k+1,i}$ and Inequality (2.10), we can conclude that

$$\begin{aligned} |\alpha_{p_i} - S_{k+1}|_{p_i} &= |\alpha_{p_i} - S_k - a_{k+1}|_{p_i} \\ &= \left| \frac{m_{k+1,i}q_{k+1} - r_{k+1}}{q_{k+1}} \right|_{p_i} \\ &\le \frac{1}{p_i^{k+1}} \end{aligned}$$

Also, by Inequality (2.11) and the definitions of a_{k+1}, L_k and R_k, we see that

$$\alpha_\infty - S_k - \frac{1}{2^k} < a_{k+1} < \alpha_\infty - S_k - \frac{1}{2^{k+1}},$$

that is,

$$\alpha_\infty - \frac{1}{2^k} < S_{k+1} < \alpha_\infty - \frac{1}{2^{k+1}}.$$

The preceding inequalities show that the partial sum S_{k+1} satisfies the Inequalities (2.2) and (2.3). So the proof of Proposition (2.1.1) is complete.

As an example of our procedure as outlined in the preceding proof, we determine the first few terms of a series of rational numbers which converges to 0 with respect to the p-adic norm for every rational prime p and at the same time converges to the well-known real number e with respect to the usual norm. Thus, if each of the S_k is a partial sum of the required series, then by Eq.(2.1),

$$S_0 = [e] - 1 = 1.$$

Following the inductive steps of our proof, we determine the next three terms; we collect the results of our calculations in the following table:

k	M_k	q_k	r_k	S_k
1	2	5	5	2
2	36	145	40	66/29
3	27000	216001	7446	15646500/6264029

We remark, without proof, that the procedure of determining the partial sums S_k that we had described in our proof, can be so modified that if S_k satisfies the inequality

$$\|\alpha_{p_i} - S_k\|_{p_i} \le p_i^{i-1-k} \text{ for every } 1 \le i \le k, \tag{2.12}$$

even then S_k converges to given p-adic numbers for each $p \in V_{\mathbb{Q}}$. As the preceding inequality is weaker than Inequality (2.3), it turns out that the sizes of the digits of integers involved in S_k are smaller so that it is possible for us to list more terms. For example, going back to the question of finding the partial sums of a series that converges to 0 with respect to the p-adic norm for each prime p and to e with respect to the usual norm, we can give the following table for the partial sums if the partial sums are chosen to satisfy Inequality (2.12):

k	M_k	q_k	r_k	S_k
1	2	5	5	2
2	12	49	22	120/49
3	360	2881	240	$357480/(49 \cdot 2881)$
4	75600	$49 \cdot 1209601$	3554280	$S_3 + 3554280/(49 \cdot 3704401)$

Of course, as before S_0 is given by

$$S_0 = [e] - 1 = 1.$$

2.3 The second proof

In this section, we present another proof of Proposition (2.1.1). This elementary proof does not involve the Chinese Remainder Theorem; we also derive a simplified algorithm for determining the partial sums of the required series which converges to given $\alpha_p \in \mathbb{Q}_p$ for primes $p \in V_{\mathbb{Q}}$ (see Section 2.1 for the relevant definitions). We need the following notation for our proof; k denotes a positive integer:

$\alpha_p(k)$: a rational number obtained from the p-adic expansion of α_p by summing the terms upto the k^{th} power of p. Observe that for a prime q different from p, $|\alpha_p(k)|_q \leq 1$.

π_k : the product $2 \cdot 3 \cdot 5 \cdot 7 \cdots p_k$ of the first k primes.

σ_k : $\sum_{i=1}^{k} (\pi_k / p_i)^k$,

Σ_k : $\sum_{i=1}^{k} \alpha_{p_i}(k).(\pi_k / p_i)^k$.

As in section (2.2), let S_k denote the k^{th} partial sum of the required series $\sum_{n=0}^{\infty} a_n$, that is

$$S_k = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots + a_k,$$

where each a_k is a rational number, to be determined in such a way that the series $\sum_{n=0}^{\infty} a_n$ converges to α_p with respect to the p-adic norm for each $p \in V_{\mathbb{Q}}$.

We construct the required series by setting

$$S_0 = a_0 = [\alpha_\infty] - 1$$

where $[x]$ denotes the greatest integer less than or equal to x. For $k \geq 1$, let

$$d_k = \left[\frac{2^{k+1}(\pi_k)^{k-1}}{\sigma_k}\right]\pi_k + \pi_k + 1, \tag{2.13}$$

$$c_k = \left[d_k \cdot \frac{(\alpha_\infty - 1/2^{k-1})\sigma_k - \Sigma_k}{\pi_k^k}\right] + 1. \tag{2.14}$$

Let S_k be the solution of the following linear equation

$$\{x - \alpha_2(k)\}\frac{1}{2^k} + \{x - \alpha_3(k)\}\frac{1}{3^k} + \cdots + \{x - \alpha_{p_k}(k)\}\frac{1}{p_k^k} = c_k/d_k, \tag{2.15}$$

for each positive integer k. We claim that as $k \to \infty$, S_k approaches α_p for every $p \in V_{\mathbb{Q}}$.

We establish this claim by proving the following proposition.

Proposition 2.3.1. *Let k be a fixed positive integer and S_k be the solution of Eq. (2.15). Then for every positive integer $j \leq k$, the following hold:*

(a) $|S_k - \alpha_{p_j}(k)|_{p_j} \leq \max\{p_j^{-k}, p_j^{-k}|\alpha_{p_j}|_{p_j}\}$.

(b) $\alpha_\infty - 1/2^{k-1} < S_k < \alpha_\infty - 1/2^k$.

Proof. We begin our proof by multiplving Eq.(2.15) by π_k^k and then rearranging the terms of the equations, so that we can rewrite Eq.(2.15) as

$$x\sum_{i=1}^{k}(\pi_k/p_i)^k - \sum_{i=1}^{k}\alpha_{p_i}(k).(\pi_k/p_i)^k = \frac{c_k}{d_k}\pi_k^k.$$

It is, therefore, clear that the solution S_k of Eq.(2.15) is given by

$$S_k = \frac{\frac{c_k}{d_k}(\pi_k)^k + \Sigma_k}{\sigma_k} = \frac{c_k(\pi_k)^k + d_k\Sigma_k}{d_k\sigma_k}. \tag{2.16}$$

Now it is easy to see, by the definitions of σ_k, d_k and π_k that both σ_k and d_k are relatively prime to π_k. Therefore, for any positive integer $i \leq k$, we have

$$|\sigma_k|_{p_i} = 1 \tag{2.17}$$

and

$$|d_k|_{p_i} = 1. \tag{2.18}$$

We also note that the definition of π_k gives

$$|\pi_k^k|_{p_i} = p_i^{-k}. \tag{2.19}$$

Now, going back to the definition of $\alpha_{p_i}(k)$, it is easily seen that $\alpha_{p_i}(k)$, in general, is a rational number with certain nonnegative power of p_i as its denominator. Thus, for any positive integer i such that $i \neq j$, one has

$$|\alpha_{p_i}(k)|_{p_j} \leq 1.$$

It follows, from the preceding equations, that

$$|{\pi_k}^k {p_i}^{-k} \alpha_{p_i}(k)|_{p_j} \leq p_j^{-k}.$$

Therefore, the relation

$$\left|\Sigma_k - \sigma_k\alpha_{p_i}(k)\right|_{p_j} = \left|\sum_{i=1}^{k}(\pi_k/p_i)^k(\alpha_{p_i}(k) - \alpha_{p_j}(k))\right|_{p_j},$$

implies that

$$|\Sigma_k - \sigma_k\alpha_{p_j}(k)|_{p_j} \leq \max\{p_j^{-k}, p_j^{-k}|\alpha_{p_j}(k)|_{p_i}\}. \tag{2.20}$$

On the other hand, one can conclude from Equalities (2.16),(2.17) and (2.18) that

$$\left|S_k - \alpha_{p_j}(k)\right|_{p_j} = |c_k(\pi_k)^k + d_k(\Sigma_k - \alpha_{p_j}(k)\sigma_k)|_{p_j}.$$

It, therefore, follows from Eqs.(2.19) and (2.20) that

$$|S_k - \alpha_{p_j}(k)|_{p_j} \leq \max \{p_j^{-k}, p_j^{-k}|\alpha_{p_j}(k)|_{p_j}\}.$$

We note that $|\alpha_{p_j}(k) - \alpha_{p_j}|_{p_j} < p_j^{-k}$. Thus

$$\begin{aligned}|S_k - \alpha_{p_j}|_{p_j} &\leq \max \{|S_k - \alpha_{p_j}(k)|_{p_j}, |\alpha_{p_j}(k) - \alpha_{p_j}|_{p_j}\} \\ &\leq \max \{p_j^{-k}, p_j^{-k}|\alpha_{p_j}(k)|_{p_j}\}.\end{aligned}$$

Finally, using the fact that p-adic norm of a p-adic number is p-adic norm of the sum of its first few terms in the $p$adic expansion, that is $|\alpha_{p_j}(k)|_{p_j} = |\alpha_{p_j}|_{p_j}$, we have

$$|S_k - \alpha_{p_j}|_{p_j} \leq \max \{p_j^{-k}, p_j^{-k}|\alpha_{p_j}|_{p_j}\}$$

which proves part (a) of the proposition. In order to prove part (b) of the proposition, we need to show that S_k satisfies the inequality

$$\alpha_\infty - 1/2^{k-1} < S_k < \alpha_\infty - 1/2^k.$$

However, by the definition of S_k given in Eq.(2.16),

$$S_k = \frac{\frac{c_k}{d_k}(\pi_k)^k + \Sigma_k}{\sigma_k} = \frac{c_k(\pi_k)^k + d_k\Sigma_k}{d_k\sigma_k}.$$

An easy calculation then shows that S_k satisfies the required inequality if the rational numbers c_k and d_k satisfy the following inequality

$$L(k) < c_k/d_k < R(k), \tag{2.21}$$

where

$$R(k) = \{(\alpha_\infty - 1/2^k)\sigma_k - \Sigma_k\}(\pi_k)^{\ k}$$

and

$$L(k) = \{(\alpha_\infty - 1/2^{k-1})\sigma_k - \Sigma_k\}(\pi_k)^{-k}.$$

Thus, part(b) of the proposition will be proved once we show that the rational numbers c_k and d_k, as defined by Eqns. (2.14) and (2.13), satisfy inequality (2.21). Since for any real number x, the greatest integer function $[x] > x - 1$, it follows from (2.13) that

$$d_k \frac{\sigma_k}{2^k \pi_k^k} > 2.$$

On the other hand, the definitions of $R(k)$ and $L(k)$ makes it clear that

$$R(k) - L(k) = \frac{\sigma_k}{2^k \pi_k^k},$$

which, combined with the preceding inequality, implies that

$$d_k(R(k) - L(k)) > 2.$$

Therefore, we see that

$$d_k R(k) > [d_k L(k)] + 2.$$

However, the definitions of c_k, given in Eqns. (2.14), readily implies that $c_k = [d_k L(k)] + 1$. It then follows from the preceding inequality that

$$R(k) > \frac{c_k}{d_k}. \tag{2.22}$$

Next note that the relation $c_k = [d_k L(k)] + 1$ also implies that

$$c_k > d_k L(k) - 1 + 1.$$

Therefore

$$\frac{c_k}{d_k} > L(k). \tag{2.23}$$

The Inequalities (2.22) and (2.23) show that c_k and d_k do satisfy inequality (2.21).Thus, the proof of the proposition is complete. □

2.4 Explicit computations with more terms

In [11], an explicit computation of a series which converges to the real number e with respect to the usual norm but which converges to the $p-$adic number 0 with respect to every $p-$adic norm has been given. However, the choice of terms of the series in that computation is ambiguous. We now determine such a series based on the procedure we have outlined in Section 2.2. Our computation of the series avoids ambiguity; also, it involves considerably less calculations.

Recall that the first term of the series is given by

$$a_0 = [\alpha_\infty] - 1 = [e] - 1 = 1$$

and using $\alpha_p = 0$ it is clear that $\Sigma_k = 0$.

k	σ_k	d_k	c_k	S_k
1	2	11	10	$2^2 \cdot 5/11$
2	13	7	6	$2^3 \cdot 3^3/91$
3	4591	31	14	$7 \cdot 2^4 \cdot 3^3 \cdot 5^3/142321$
4	149482321	211	43	$43 \cdot 2^4 \cdot 3^4 \cdot 5^4 \cdot 7^4/(149482321 \cdot 211)$

We have not gone any further because of technicalities in computation and also the size of the computation.

Note that, in our procedure, the numerator of the kth partial sum S_k involves atleast kth powers of each of the first k primes and the denominator is relatively prime to these primes . This ensures p-adic convergence. However, it is still possible to ensure p-adic convergence if lesser powers of primes are involved in S_k. It turns out that we can modify our procedure given in Section 2.2 to achieve that end. In what follows, we show that we can define the partial sums S_k in such a manner that it involves the product of the decreasing powers of the first k primes, but the corresponding series still converges to the prescribed numbers.

We begin by considering the equation

$$\{x_k - \alpha_2(k)\}\frac{1}{2^k} + \{x_k - \alpha_3(k)\}\frac{1}{3^{k-1}} + \cdots + \{x_k - \alpha_{p_k}(k)\}\frac{1}{p_k} = c_k/d_k,$$

which determines S_k in place of the fundamental Eq.(2.15) of Section 2.2 Then S_k is given by

$$S_k = \{(c_k/d_k)\pi(k) + \Sigma_k\}/\sigma_k$$

where

$$\pi(k) = 2^k 3^{k-1} 5^{k-2} \cdots p_k,$$

$$\sigma_k = \frac{\pi(k)}{2^k} + \frac{\pi(k)}{3^{k-1}} + \frac{\pi(k)}{5^{k-2}} + \cdots + \frac{\pi(k)}{p_k},$$

$$\Sigma_k = \frac{\pi(k)}{2^k}\alpha_2(k) + \frac{\pi(k)}{3^{k-1}}\alpha_3(k) + \frac{\pi(k)}{5^{k-2}}\alpha_5(k) + \cdots + \frac{\pi(k)}{p_k}\alpha_{p_k}(k),$$

$$d_k = [2^{k+1}/\sigma_k]\pi(k) + \pi(k) + 1,$$

$$c_k = [\frac{d_k(\alpha_\infty - 1/2^{k-1})\sigma_k - \Sigma_k}{\pi(k)}].$$

In this case we have the following table for $\pi(k), \sigma_k, c_k, S_k$:

k	π_k	σ_k	c_k	S_k
1	2	1	9	$\frac{3^2 \cdot 2}{11}$
2	12	7	32	$\frac{2^7 \cdot 3}{25 \cdot 7}$
3	360	157	388	$\frac{2^8 \cdot 3^4 \cdot 5}{157 \cdot 19^2}$
4	75600	21349	55364	$\frac{13841 \cdot 2^6 \cdot 3^3 \cdot 5^2 \cdot 7}{19 \cdot 23 \cdot 173 \cdot 37 \cdot 577}$

The values of S_5, S_6, S_7, S_8 and the values of the corresponding σ's etc. have been not included in the table because of their big size but we have computed these values and given below.

$$\pi(5) = 17463000$$
$$\sigma_5 = 47 \cdot 605329$$
$$c_5 = 13 \cdot 1427 \cdot 4073$$
$$S_5 = \frac{13 \cdot 1427 \cdot 4073 \cdot 2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11}{174636001 \cdot 47 \cdot 605329}$$

$$\pi(6) = 5244319080000$$
$$\sigma_6 = 573955156153$$
$$c_6 = 3^4 \cdot 41 \cdot 1453 \cdot 319607$$
$$S_6 = \frac{41 \cdot 1453 \cdot 319607 \cdot 2^6 \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13}{149 \cdot 32189 \cdot 1093441 \cdot 17 \cdot 33762068009}$$

$$\pi(7) = 2677277333530800000$$
$$\sigma_7 = 9299611 \cdot 21710673421$$
$$c_7 = 3 \cdot 13 \cdot 149 \cdot 93902705682067$$
$$S_7 = \frac{149 \cdot 93902705682067 \cdot 2^7 \cdot 3^7 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17}{927640943 \cdot 23 \cdot 125483209 \cdot 9299611 \cdot 21710673421}$$

$$\pi(8) = 2596876017927536545200000$$
$$\sigma_8 = 25914797 \cdot 6122948552647979 9$$
$$c_8 = 47 \cdot 503 \cdot 181922776566716745473$$
$$S_8 = \left(\frac{47 \cdot 503 \cdot 181922776566716745473 \cdot 2^8 \cdot 3^7 \cdot 5^6 \cdot 7^5}{41 \cdot 45469123373 \cdot 139299891 36157 \cdot 25914797}\right) \cdot \left(\frac{11^4 \cdot 13^3 \cdot 17^2 \cdot 19}{61229485526479799}\right)$$

S_8 has been recorded in two rows.

2.5 Some questions

Our investigations into series converging to preassigned numbers have led us to formulate the following questions:

Question 1: Can the algorithm given in Section 2.3 be further simplified?

Question 2: Can the proof given in this chapter be further modified so as to allow the series to tend to infinity for various $p \in V_{\mathbb{Q}}$ and to diverge (because the limit of the sequence of the partial sums does not exist) for other $p \in V_{\mathbb{Q}}$?

Question 3: Given $\alpha_p \in \mathbb{Q}_p$ for each $p \in V_{\mathbb{Q}}$, is it possible to find a sequence of nonzero $a_i \in \mathbb{Q}(i \geq 1)$ such that if

$$A_k = a_1 \cdot a_2 \cdots a_k$$

then A_k converges to α_p for every $p \in V_{\mathbb{Q}}$?

Chapter 3

A few series and Stirling numbers

3.1 Introduction

Quite frequently one comes across $p-$adically convergent series whose terms are rational integers. The sums of such series are in $\mathbb{Q}_p$. This chapter deals with a fascinating aspect of $p-$adic analysis, mainly the study of rationality of sums of such series. We begin by looking at the series $\sum_{n=0}^{\infty} n^k p^n$ which generalizes the familiar series $\sum_{n=0}^{\infty} p^n$. There are many series for which the sums of the series are not explicitly known one such example is the sum $\sum_{n=0}^{\infty} n^k(n+j)!$ where k, j are arbitrary but fixed integers. Dragovich has investigated series of the type $\sum_{n=0}^{\infty} n^k(n+j)!$. He has shown that if the sum of the series $\sum_{n=0}^{\infty} n!$ is a rational number then the sum cannot be the same rational number for every prime p. We have produced a different proof of this statement using Kurepa's left factorials.

Stirling numbers are inherently connected to factorials; in fact, $n!$ (etc.) can be expressed as the sum of Stirling numbers of the first kind. Since these numbers have been found to throw light on $\sum_{n=0}^{\infty} n^k n!$ we have included several results on these numbers as well.

Murty and Sumner [21] have shown that if $k \equiv 0$ or $2 \pmod 3$ then under the assumption that $\sum_{n=0}^{\infty} n!$ is an irrational number, $\sum_{n=0}^{\infty} n^k n!$ would also be an irrational number. By exploring the work of Dragovich we are able to produce a different and simpler proof of the result obtained by Murty and

Summer.

3.2 Series of the type $\sum_{n=0}^{\infty} a_n p^n$

The simplest example of a $p-$adically convergent series converging to rationals are given by the series of the type $\sum_{n=0}^{\infty} a_n p^n$ where each a_n is an integer. The standard convergent series, considered in the following proposition, may be called the *$p-$adic geometric series.*

Proposition 3.2.1. *$\sum_{n=0}^{\infty} p^n$ converges in $\mathbb{Q}_p$ to the rational number $(1-p)^{-1}$. However, for a prime q different from p the series does not converge in $\mathbb{Q}_q$.*

Proof. The well-known formula for the geometric progression allows us to express the partial sums of the series $\sum_{n=0}^{\infty} p^n$ as follows:

$$1+p+p^2+p^3+\cdots+p^n = \frac{p^{n+1}-1}{p-1}.$$

But $\left|\frac{p^{n+1}}{p-1}\right|_p = p^{-(n+1)}$, which tends to zero as n tends to infinity.

So $1+p+p^2+p^3+\cdots = -\frac{1}{p-1}$, a rational number.

For a prime q different from p, it has been shown in Chapter 1 that the series $1+p+p^2+p^3+\cdots$ does not converge $q-$adically; in fact, this conclusion follows from the fact that a series $\sum_{n=0}^{\infty} a_n$ converges $p-$adically if and only if $\lim_{n\to\infty}|a_n|_p = 0$. □

The preceding proposition, in fact, provides the $p-$adic expansion (see Chapter 1) of the rational number $(1-p)^{-1}$: the $p-$adic digits in this expansion are all equal to 1. We now present a general result which relates the pattern of the $p-$adic digits in the $p-$adic expansion of an element of $\mathbb{Q}_p$ to its rationality.

Theorem 3.2.2. *A $p-$adic number α is a rational number if and only if $p-$adic expansion of α is eventually periodic (periodic after a certain stage).*

Proof. Given α in $\mathbb{Q}_p$, let

$$\alpha = a_0 p^{-n} + a_1 p^{-n+1} + \cdots$$

be the unique $p-$adic expansion of α. Assume that α has a repeating $p-$adic expansion that is, after a certain term, in the expansion, say corresponds to p^k, the $p-$adic digits appear in a repeating block. Let

$$a_{k+1}, a_{k+2}, \cdots, a_{k+t}$$

be one such repeating $p-$adic block. In other words, the $p-$adic expansion of α can be given as

$$\begin{aligned} a_0p^{-n} + \cdots + a_kp^{-n+k} &+ (a_{k+1}p^{-n+k+1} + \cdots + a_{k+t}p^{-n+k+t}) \\ &+ (a_{k+1}p^{-n+k+t+1} + \cdots + a_{k+t}p^{-n+k+2t}) + \cdots \end{aligned}$$

Rearranging the terms of this $p-$adic expansion, we obtain

$$\begin{aligned} \alpha =& a_0p^{-n} + \cdots + a_kp^{-n+k} \\ &+ (a_{k+1}p^{-n+k+1} + \cdots + a_{k+t}p^{-n+k+t})(1 + p^t + p^{2t} + p^{3t} + \cdots). \end{aligned}$$

As $t \geq 1$, it is clear that

$$1 + p^t + p^{2t} + p^{3t} + \cdots = \frac{1}{1-p^t}.$$

It follows that

$$\alpha = a_0p^{-n} + \cdots + a_kp^{-n+k} + (a_{k+1}p^{-n+k+1} + \cdots + a_{k+t}p^{-n+k+t})/(1-p^t).$$

Clearly α is a rational number.

Conversely, assume that $\alpha = a/b$, $b > 0$, a rational number such that $(a, b) = 1$. Without loss of generality, we can assume that p does not divide b. By the division algorithm a/b can be expressed in the form

$$q + r/b$$

where $-b+1 \leq r \leq 0$ and $q \in \mathbb{Z}$. Now, simply multiplying and dividing r/b by $p^{\phi(b)} - 1$, we can express r/b as follows:

$$r/b = (-r)\frac{\left(\dfrac{p^{\phi(b)}-1}{r}\right)}{1-p^{\phi(b)}}.$$

Using the equality

$$\frac{1}{1-p^{\phi(b)}} = 1 + p^{\phi(b)} + p^{2\phi(b)} + p^{3\phi(b)} + \cdots$$

in $\mathbb{Q}_p$, it is clear that

$$\begin{aligned} r/b &= (-r)\left(\frac{p^{\phi(b)}-1}{b}\right)\{1 + p^{\phi(b)} + \cdots\} \\ &= (-r)\left(\frac{p^{\phi(b)}-1}{b}\right) + (-r)\left(\frac{p^{\phi(b)}-1}{b}\right)p^{\phi(b)} + \cdots \qquad (3.1) \end{aligned}$$

As $(b,p) = 1$, it follows from Fermat's theorem that

$$\frac{p^{\phi(b)}-1}{b}$$

is an integer Again by the assumption $-b+1 \leq r \leq 0, b > 0$ it is clear that

$$0 \leq (-r)\left(\frac{p^{\phi(b)}-1}{b}\right) < p^{\phi(b)}.$$

Hence its $p-$adic expansion is given by

$$(-r)\left(\frac{p^{\phi(b)}-1}{b}\right) = a_0 + a_1 p + \cdots + a_{\varphi(b)-1} p^{\phi(b)-1}$$

and substituting the value of $(-r)\left(\frac{p^{\phi(b)}-1}{b}\right)$ in (3.1), we have the $p-$adic expansion of r/b. Moreover, it is clear that the $p-$adic expansion of r/b is repeating.

Observe that for a $p-$adic integer β with its $p-$adic expansion

$$c_0 + c_1 p + c_2 p^2 + c_3 p^3 + \cdots ,$$

$p-$adic expansion of $-\beta$ is given by

$$p - c_0 + (p-1-c_1)p + (p-1-c_2)p^2 + (p-1-c_3)p^3 \cdots$$

and thus β has a repeating $p-$adic expansion if and only if $-\beta$ has a repeating $p-$adic expansion.

Thus if r/b is an integer then we have nothing to prove (as a positive integer is having a repeating $p-$adic expansion with 0 being repeating digit). Otherwise there exist an integer $t > 0$ such that $|q| < p^t$ and t^{th}term in the $p-$adic expansion of r/b is non-zero.

If q is negative then the sum of q with the first t term(s) in the $p-$adic expansion r/b is positive and less than p^{t+1} and we $p-$adically expand this sum (a positive integer). Thus it is easy to see that upon addition of q to the $p-$adic expansion of r/b, $p-$adic expansion remains unaffected from a term containing p^{t+1} onwards. Thus $q+r/b$ possess a repeating $p-$adic expansion.

If q is positive then as $-r/b$ is having a repeating $p-$adic expansion, we can show by a similar argument that $-q-r/b$ is having a repeating $p-$adic expansion and thus $q+r/b$ is having a repeating $p-$adic expansion.

In either case we have proved the required result. ☐

Corollary 3.2.3. *$p-$adically convergent series such as $\sum_{n=0}^{\infty} p^{n^2}$ or $\sum_{n=0}^{\infty} p^{n!}$ do not converge to a rational number.*

We now consider series of the type

$$\sum_{n=0}^{\infty} a_n p^n$$

.

where a_n need not be p-adic digits.

Proposition 3.2.4. *For a fixed nonnegative integer k, the series*

$$\sum_{n=0}^{\infty} n^k p^n$$

converges in $\mathbb{Q}_p$ to a rational number.

Proof. Note that as

$$|n^k p^n|_p \leq p^{-n},$$

the $p-$adic norms of $n^k p^n$ tend to zero as n tends to infinity. It follows that the series

$$\sum_{n=0}^{\infty} n^k p^n$$

.

converges p-adically.
Let

$$\sigma(k) = \sum_{n=0}^{\infty} n^k p^n$$

be the sum of the series in $\mathbb{Q}_p$. We prove the proposition by induction on k. It is clear that

$$\sigma(0) = \sum_{n=0}^{\infty} p^n = (1-p)^{-1}$$

is a rational number. We may assume that $k \geq 1$ and that $\sigma(j)$ is a rational number for $0 \leq j \leq k-1$. To evaluate $\sigma(k)$, we split $\sigma(k)$ as follows:

$$\sigma(k) = \sum_{n=0}^{p-1} n^k p^n + \sum_{n=p}^{\infty} n^k p^n .$$

Changing the index of second summation by setting $n - p = m$ on the right hand side of the above equation we obtain

$$\sigma(k) = \sum_{n=0}^{p-1} n^k p^n + \sum_{m=0}^{\infty} (m+p)^k p^{m+p}. \tag{3.2}$$

Expanding $(m+p)^k$ binomially and then rearranging the terms of the infinite series of the right hand side of Eq.(3.2), we can express the infinite series as

$$p^p \sum_{m=0}^{\infty} \{ \sum_{r=0}^{k} \binom{k}{r} m^{k-r} p^r \} p^m .$$

Observe now that the sum of the series $\sum_{m=0}^{\infty} m^{k-r} p^m$ is precisely $\sigma(k-r)$. Hence equality (3.2) can be put in the form

$$\sigma(k) = \sum_{n=0}^{p-1} n^k p^n + p^p \sum_{r=0}^{k} \binom{k}{r} \sigma(k-r) p^r .$$

We break the second sum on the right hand side into two parts to obtain

$$\sigma(k) = \sum_{n=0}^{p-1} n^k p^n + p^p \sigma(k) + p^p \sum_{r=1}^{k} \binom{k}{r} \sigma(k-r) p^r .$$

It therefore follows that

$$\sigma(k) = (1-p^p)^{-1}\{\sum_{n=0}^{p-1} n^k p^n + \sum_{r=1}^{k} \binom{k}{r} \sigma(k-r) p^{p+r}\}.$$

By the induction hypothesis, each of the sums

$$\sigma(0), \sigma(1), \sigma(2), \cdots, \sigma(k-1)$$

is also a rational number. Hence the preceding expression shows that $\sigma(k)$ is also a rational number, completing the proof. □

Corollary 3.2.5. *Let $f(x)$ be a polynomial with integral coefficients. Then the series $\sum_{n=0}^{\infty} f(n)p^n$ converges $p-$adically to a rational number.*

We conclude this section by posing two questions which we believe to be open to the best of our knowledge and illustrate the difficulty in dealing with the rationality question:

- Is $\sum_{n=1}^{\infty} n^n p^n$ a rational number?
- Is $\sum_{n=0}^{\infty} n! p^n$ a rational number?

Note that both these series are $p-$adically convergent.

3.3 Series of the type $\sum_{n=0}^{\infty} n^k(n+j)!$

In this section as well as in the next section, we investigate series with factorials such as $\sum_{n=0}^{\infty} n^k(n+j)!$ for fixed nonnegative integers k and j. As mentioned in the Introduction of this chapter, Dragovich [3] along with Murty and Sumner [21] have considered such series with factorials mainly with reference to the rationality of such sums.

We begin by recalling from Chapter 1 that for $n \geq p^m$, the $p-$adic norm

$$|n^k(n+j)!|_p \leq p^{-m}$$

so any series of the type $\sum_{n=0}^{\infty} n^k(n+j)!$ converges $p-$adically to some element of $\mathbb{Q}_p$. In particular, the sum of the series

$$\sum_{n=0}^{\infty} n! \tag{3.3}$$

is in $\mathbb{Q}_p$ for every prime p. We shall denote the sum as α; note that the $p-$adic sum α actually depends on the prime p. This series has attracted a lot of attention but very little is known about it. For example, as mentioned in the introduction of this chapter, the following question has been around for a long time non without a satisfactory answer:

- Is $\alpha \in \mathbb{Q}_p$, a rational number for any prime p?

In contrast, as we shall see shortly, the $p-$adically convergent series $\sum_{n=0}^{\infty} n(n+1)!$ converges to a fixed integer. In fact

$$\sum_{n=0}^{\infty} n(n+1)! = -1.$$

This peculiarity has prompted us to consider the $p-$adically convergent series $\sum_{n=0}^{\infty} n^k(n+j)!$. In this connection note that Murty and Sumner [21] have investigated $\sum_{n=0}^{\infty} n^k n!$ relating it to $\alpha = \sum_{n=0}^{\infty} n!$. We begin by considering the series $\sum_{n=0}^{\infty} n(n+j)!$ for a fixed nonnegative integer j. Observe that

$$\begin{aligned}(n+j+1)! &= (n+j+1).(n+j)! \\ &= n.(n+j)! + (j+1)(n+j)!\end{aligned}$$

However

$$\sum_{n=0}^{\infty}(n+j+1)! = \sum_{n=0}^{\infty} n! - (0! + 1! + 2! + \cdots + j!)$$

and

$$\sum_{n=0}^{\infty}(n+j)! = \sum_{n=0}^{\infty} n! - (0! + 1! + 2! + \cdots + (j-1))!$$

Kurepa's left factorial, $K(m)$, for a nonnegative integer m, is given by

$$\begin{aligned}K(m) &= 0 && \text{if } m = 0\\ &= 0! + 1! + 2! + \cdots + (m-1)! && \text{if } m \text{ is a positive integer.}\end{aligned}$$

It then follows that

$$\begin{aligned}\sum_{n=0}^{\infty} n(n+j)! &= \sum_{n=0}^{\infty}(n+j+1)! - (j+1)\sum_{n=0}^{\infty}(n+j)\\ &= \alpha - K(j+1) - (j+1)\{\alpha - K(j)\}\\ &= -j\alpha - K(j+1) + (j+1)K(j).\end{aligned}$$

For reasons which will be clear in a moment, we let

$$\begin{aligned} a_1(j) &= -j \\ b_1(j) &= -K(j+1) + (j+1)K(j). \end{aligned}$$

Therefore we can express $\sum_{n=0}^{\infty} n(n+1)!$ as

$$\sum_{n=0}^{\infty} n(n+j)! = a_1(j)\alpha + b_1(j). \tag{3.4}$$

We are now ready to present the main result of this section.

Proposition 3.3.1. *For integers $k \geq 1$ and $j \geq 0$, define $a_k(j)$ and $b_k(j)$ recursively as follows:*

$$\begin{aligned} a_1(j) &= -j \\ b_1(j) &= -K(j+1) + (j+1)K(j) \end{aligned}$$

and for $k \geq 2$

$$\begin{aligned} a_k(j) &= a_{k-1}(j+1) - (j+1)a_{k-1}(j) \\ b_k(j) &= b_{k-1}(j+1) - (j+1)b_{k-1}(j). \end{aligned}$$

Let $k \geq 1$ and $j \geq 0$ be fixed integers. Then there are integers $a_k(j)$ and $b_k(j)$ such that

$$\sum_{n=0}^{\infty} n^k(n+j)! = a_k(j)\alpha + b_k(j)$$

Proof. The proof is by induction on k. The case $k = 1$ has already been worked out. So we may assume $k \geq 2$ and that the proposition holds for k.

$$\sum_{n=0}^{\infty} n^k(n+j+1)! = \sum_{n=0}^{\infty} n^{k+1}(n+j)! + (j+1)\sum_{n=0}^{\infty} n^k(n+j)!$$

Thus using $\sum_{n=0}^{\infty} n^k(n+j)! = a_k(j)\alpha + b_k(j)$ we have

$$a_k(j+1) - (j+1)a_k(j) = a_{k+1}(j)$$

and

$$b_k(j+1) - (j+1)b_k(j) = b_{k+1}(j).$$

This completes the proof. □

Corollary 3.3.2. *If for a given pair k, j of nonnegative integers the equality $a_k(j) = 0$ holds then the series $\sum_{n=0}^{\infty} n^k(n+j)!$ converges to an integer.*

We give a table for the first few values of $a_k(j)$ for $k = 1, 2, 3, 4, 5, 6$

k	$a_k(j)$
1	$-j$
2	$j^2 - 1$
3	$-j^3 + 3j + 1$
4	$j^4 - 6j^2 - 4j + 2$
5	$-j^5 + 10j^3 + 10j^2 - 10j - 9$
6	$j^6 - 15j^4 - 20j^3 + 30j^2 + 54j + 9$

We observe that $a_1(0) = 0$. Thus, we have

$$\sum_{n=0}^{\infty} n \cdot n!$$

is a fixed integer. Similarly,

$$\sum_{n=0}^{\infty} n^2(n+1)!$$

is a fixed integer. The following is an open question.

- Given k, what are the values of j for which $a_k(j)$ vanishes.

3.4 On the rationality of $\sum_{n=0}^{\infty} n!$

In this section as well as in the following ones, we deal with the series $\sum_{n=0}^{\infty} n!$, mainly on the question of its rationality. As mentioned in the Introduction, Dragovich [4] proved the following result to partially answer a question raised by Koblitz [25] in 1970. We give two proofs the first one is an illustration of Dragovich's proof whereas in the second, we use a result about Kurepa's left factorial for a simple derivation.

Theorem 3.4.1. *If the $p-$adic sum of the series $\sum_{n=0}^{\infty} n!$ is a rational number, then it cannot be the same rational number in $\mathbb{Z}_p$ for every prime p.*

Proof. Let $\alpha = \sum_{n=0}^{\infty} n!$ in $\mathbb{Q}_p$ for a prime p. (Recall that α may assume different values for different primes p). Suppose, if possible, there is a rational number a/b, where a and b are integers, $b > 0$, $(a,b) = 1$ such that $\sum_{n=0}^{\infty} n!$ converges to a/b for every $p-$adic norm on $\mathbb{Q}$, $p \neq \infty$. Let

$$S_k = \sum_{n=0}^{k} n!$$

be the k^{th} partial sum of the series $\sum_{n=0}^{\infty} n!$. Since

$$\alpha - S_n = (n+1)!\{1 + (n+2) + (n+2).(n+3) + \cdots\}$$

and since for a given prime p, the quantity inside the braces is a p-adic integer, we have

$$|\alpha - S_n|_p \leq p^{-v_p(n+1)!}.$$

Multiplying this inequality by $|b|_p$, and observing that $|b|_p \leq 1$, we obtain

$$|b\alpha - bS_n|_p \leq p^{-v_p(n+1)!}$$

which gives the following congruence in $\mathbb{Z}_p$:

$$a \equiv bS_n \ (\mathrm{mod} p^{v_p(n+1)!}).$$

Recall from Chapter 1 that if two integers are congruent modulo p^k in $\mathbb{Z}_p$ then they are congruent in $\mathbb{Z}$. Therefore, the preceding congruence between bS_n and a is a congruence in $\mathbb{Z}$. In other words $p^{v_p(n+1)!}$ divides $bS_n - a$ for any positive integer n. Since p is an arbitrary prime and $(n+1)!$ can be expressed as a product of prime powers (the power being $v_p(n+1)!$) we see that $(n+1)!$ divides $bS_n - a$ in $\mathbb{Z}$ that is,

$$a \equiv bS_n(\mathrm{mod}(n+1)!). \tag{3.5}$$

We now claim that there exist some positive integer N such that

$$bS_n \leq (n+1)!$$

for every $n \geq N$. In order to establish this claim, we consider the inequality

$$bS_n \leq b(n!) + 2b(n-1)!, \tag{3.6}$$

using the fact that $S_n = S_{n-1} + n!$ and $b > 0$, it is easy to see that the preceding inequality is equivalent to

$$S_{n-1} \leq 2(n-1)!. \tag{3.7}$$

It can be easily shown by induction that the preceding inequality and hence (3.6) will hold for $n \geq 4$. Also for sufficiently large n. we see that

$$b(n!) + 2b(n-1)! \leq (n+1)!. \tag{3.8}$$

Inequalities (3.6) and (3.8) imply that

$$0 < bS_n < (n+1)!. \tag{3.9}$$

This proves our claim. Consider the case $a < 0$. Then $0 < bS_n - a < 2(n+1)!$ which along with Congruence (3.5) yields that

$$bS_n - a = (n+1)!$$

for all $n \geq N$. It follows from preceding equality that

$$bS_{n+1} - bS_n = (n+2)! - (n+1)!$$

for all $n \geq N + 1$ that is,

$$b(n+1)! = (n+1)(n+1)!.$$

Since b is a fixed integer, we get a contradiction. If $a \geq 0$ then a similar argument leads to a contradiction. This completes the proof. □

It can be mentioned here that Dragovich [3] proved a more general result, namely. that for an integer x, $\sum_{n=0}^{\infty} n!x^n$ cannot be same rational number in $\mathbb{Q}_p$ for all primes p. We prove the following two lemmas before giving an alternative proof for the result if $\sum_{n=0}^{\infty} n!$ is a rational number then it cannot be the same in all $\mathbb{Z}_p$. We need a couple of preliminary results for our proof of Dragovich's Theorem (3.4.1).

Lemma 3.4.2. *Let x be a nonzero rational number. Then $\Pi|x| = 1$ where the product runs over all norms on $\mathbb{Q}$.*

Proof. For $x = \pm 1$, the statement is trivially true. If $x \neq 0, \pm 1$ then x can be uniquely as the finite product

$$x = p_1^{k_1} p_2^{k_2} \cdots p_l^{k_l}$$

where each of the p_i is a distinct prime and k_i is a nonzero integer. By properties of norms, we have that

$$|x|_{p_i} = \begin{cases} p_i^{-k_i} & \text{if } p = p_i; \\ \pm x & \text{if } \ p = \infty; \\ 1 & \ \text{otherwise} \ . \end{cases}$$

Hence the lemma follows. □

Lemma 3.4.3. *Let r be a rational number. If $|r|_p = 1$ for every odd prime p and $|r|_2 = 1/2$, then $r = \pm 2$.*

Proof. We know that every rational number r can be expressed in the form a/b where $a, b \in \mathbb{Z}$ and $(a, b) = 1$. Since for any odd prime p, $|r|_p = 1$. It follows that $(a, p) = (b, p) = 1$. On the other hand $|r|_2 = 1/2$, this implies that $2|a$. $(b, 2) = 1$ and $(a/2, 2) = 1$.
Hence $a = \pm 2, b = \pm 1$. Thus $a/b = \pm 2$.

Alternatively, using Lemma (3.4.2) for a nonzero rational number r, we have $r \prod_{p \neq 2} |r|_p = \dfrac{\pm 1}{|r|_2} = \pm 2$. Since $|r|_p = 1$ for any every prime p, the lemma follows. □

Recall that for a nonnegative integer n, Kurepa's left factorial is given by

$$K(0) = 0,$$

$$K(n) = \sum_{k=0}^{n-1} k!, n \in \mathbb{N}$$

Clearly $K(n)$ is the $(n-1)th$ partial sum of the series $\sum_{n=0}^{\infty} n!$. Barsky and Benzaghou [9] have recently proved what was known as "*Kurepa's left factorial conjecture*." The following is the statement of the conjecture.

Theorem 3.4.4. *The following statements hold* *(a) For every* $n \geq 2$,

$$(K(n), n!) = 2.$$

(b) n *does not divide* $K(n)$ *for every* $n > 2$.

(c) p *does not divide* $K(p)$ *for every odd prime* p.

(It can be shown that (a), (b), (c) are equivalent).

Barsky and Benzaghou [9] proved Theorem (3.4.4) by showing that p does not divide $\beta_{p-1} - 1$ (for the definition of β_n the nth Bell number we refer to Definition (3.6.6). However, from Corollary (3.6.8) $K(n)$ is related to Bell numbers β_n through the following congruence:

$$K(p) \equiv \beta_{p-1} - 1 \pmod{p}$$

Now, for an odd prime p, using Theorem (1.5.2) it is clear that

$$|\alpha - K(p)|_p \leq 1/p.$$

Hence, we have $\alpha \equiv K(p)$ $(mod p)$, for every odd prime p. Thus $|\alpha|_p = 1$ for an odd prime p.
Let us assume that α is an arbitrary but fixed rational number independent of the norm, that is, $\alpha = \frac{a}{b}$ where $(a, b) = 1$ but

$$\begin{aligned} |\alpha|_2 &= |0! + 1! + 2! + 3! + \cdots|_2 \\ &= |10 + 4! + 5! + 6! + \cdots|_2 \\ &= 1/2|5 + 4!/2 + 5!/2 + 6!/2 + \cdots|_2 = 1/2. \end{aligned}$$

Thus by lemma (3.4.3), $\alpha = \pm 2$.
That is,

$$\text{either } \alpha - 2 = 0 \text{ Or, } \alpha + 2 = 0$$

Which gives either

$$+2! + 3! + 4! + 5! + \cdots \equiv 0 \pmod{16}$$

Or

$$2 + 0! + 1! + 2! + 3! + \cdots \equiv 0 \pmod{16}$$

Neither of these assertions holds.

3.5 $\sum_{n=0}^{\infty} n!$ and subfactorials.

In this section, we explore the relationship between the series $\sum_{n=0}^{\infty} n!$ and certain interesting sequences of rational numbers known as *subfactorials*.

Definition 3.5.1. For a positive integer n the *subfactorial*, $S(n)$ is defined as follows:

$$S(n) = n!\sum_{k=0}^{n}(-1)^k/k!.$$

$S(n)$ can also be characterized as the number of derangements of n elements which is also the number of *permutations* of n elements without any fixedpoints. We now list a few interesting properties of $S(n)$.

$$\begin{aligned} S(n) =& nS(n-1) + (-1)^n \\ S(n) =& (n-1)S(n-1) + S(n-2) \end{aligned}$$

The following proposition due to Bernd C. Kellner [6] relates subfactorials to Kurepa's left factorials.

Proposition 3.5.2. *For any integer $n \geq 2$,*

$$K(n) \equiv (-1)^{n-1}S(n-1) \pmod{n}$$

Proof. The proposition is trivial for $n = 2$. Consider an integer $n \geq 3$. The definition of $S(n-1)$ then gives us

$$(-1)^{n-1}S(n-1) = \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}(n-1-k)!. \tag{3.10}$$

Changing the index of summation, k by $n-1-k$ and noting that

$$\binom{n-1}{k} = \binom{n-1}{n-1-k}.$$

Eq(3.10) can be written as

$$(-1)^{n-1}S(n-1) = \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}k!.$$

Now, for $0 \leq k \leq n-1$, one has

$$\begin{aligned}(-1)^k \binom{n-1}{k} k! &= (-1)^k (n-1)\cdots(n-k) \\ &\equiv k! \pmod{n}.\end{aligned}$$

Therefore

$$\begin{aligned}(-1)^{n-1} S(n-1) &= \sum_{k=0}^{n-1} (-1)^k k! \binom{n-1}{k} \\ &\equiv \sum_{k=0}^{n} k! \pmod{n}\end{aligned}$$

which proves the proposition. □

The following proposition gives $\alpha = \sum_{n=0}^{\infty} n!$ as the limit of subfactorials.

Proposition 3.5.3.

$$\alpha = -\lim_{n\to\infty} S(2^n - 1)$$

considering convergence with respect to 2−adic norm.

Proof. By proposition (3.5.2), we have

$$K(2^n) \equiv -S(2^n - 1) \pmod{2^n} \tag{3.11}$$

where $K(n)$ denotes the Kurepa's left factorial function.
Also

$$\alpha = \lim_{n\to\infty} K(n),$$

in $\mathbb{Q}_p$.
It follows from the congruence (3.11) that

$$\alpha = -\lim_{n\to\infty} S(2^n - 1)$$

in $\mathbb{Q}_2$. □

It is clear by a similar argument that

$$\alpha = -\lim_{n\to\infty} S(p^n - 1)$$

considering p−adic norm for any prime p.

3.6 Factorials and Stirling numbers

The Stirling number of first kind, $s(n,k)$ for positive integers n,k is defined as the number of ways to arrange n objects into k cycles and denoted as $s(n,k)$. It can also be characterized by the recurrance relation

$$s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1).$$

For analysing Stirling numbers further, we require the concept of falling factorial and rising factorial. For a real number x and a positive integer k falling factorial $x^{\underline{k}}$ and rising factorials $x^{\overline{k}}$ are defined as.

$$x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1).$$

$$x^{\overline{k}} = x(x+1)(x+2)\cdots(x+k-1).$$

Stirling number of the first kind can also be characterized as the coefficients of falling factorials that yield ordinary powers. Stirling numbers can generate factorials as given by the equation

$$\sum_{k=0}^{n} s(n,k) = n!. \tag{3.12}$$

Summing up this equation as n takes the values $1, 2, 3, \cdots, m$,we have

$$\sum_{n=1}^{m} n! = \sum_{n=1}^{m}\sum_{k=0}^{k=n} s(n,k).$$

Thus stirling numbers of the first kind are the building blocks for series with factorials.

However we are more interested in another kind of Stirling numbers called Stirling numbers of second kind and denoted as $S(n,k)$. Unless we explicitely mention by Stirling numbers we mean Stirling numbers of the second kind. Stirling numbers of second kind $S(n,k)$ can be defined in a number of ways. The following gives these numbers based on *partition*. A partion of a set $\{1, 2, 3, \cdots, n\}$ is a collection of non empty pairwise disjoint subsets of $\{1, 2, 3, \cdots, n\}$ called blocks whose union is equal to $\{1, 2, \cdots, n\}$. For example $\{1, 2\}, \{3, 4\}$ is a partition of $\{1, 2, 3, 4\}$ into two blocks. Stirling number of second kind, $S(n,k)$ is the number of ways in which we can partition a set with n elements into exactly k subsets. However in this dissertation we would be mainly regarding these numbers as the numbers generated by certain recurrence relation given below:

$$S(n+1,j) = S(n,j-1) + jS(n,j)$$

where n and j are positive integers and

$$S(1,1) = 1, S(1,0) = 0.$$

This recurrence relation is similar to the recurrence relation for the binomial coefficient

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$$

except for the coefficient j occurring in the first relation. The following proposition relates powers of x with falling factorials through Stirling numbers.

Proposition 3.6.1. *For any positive integer n*

$$x^n = \sum_{k=1}^{n} S(n,k) x^{\underline{k}}$$

where x stands for an indeterminate.

Proof. The proposition can be easily proved by induction on n. ☐

Now, noting that

$$(-x)^{\underline{n}} = (-1)^n x(x+1)(x+2)\cdots(x+n-1),$$

we obtain from Proposition (3.6.1) the following relation:

$$x^n = \sum_{k=1}^{k=n} S(n,k)(-1)^{n-k} x^{\overline{k}}.$$

This can be put in the form

$$x^{n-1} = \sum_{k=1}^{k=n} S(n,k)(-1)^{n-k} x^{\overline{k}}/x. \qquad (3.13)$$

Taking $x = 1, 2, 3, \cdots$ successively in (3.13), multiplying the resultant equations for $x = j$ by $j!$ and finally addding up, we get

$$\begin{aligned} &1^{n-1}1! + 2^{n-1}2! + \cdots \\ &= \textstyle\sum_{k=1}^{k=n} S(n.k)(-1)^{n-k}1^{\bar{k}}1!/1 + \sum_{k=1}^{k=n} S(n.k)(-1)^{n-k}2^{\bar{k}}2!/2 + \cdots \end{aligned} \tag{3.14}$$

The preceding series converges in $\mathbb{Q}_p$ because of the fact that the series $\sum_{n=0}^{\infty} n!$ converges in $\mathbb{Q}_p$ and the following observation:

$$j^{\bar{k}}\frac{j!}{j} = (j + k - 1)!.$$

and thus equality (3.14) is valid in $\mathbb{Q}_p$. Thus we have

$$1^{n-1}1! + 2^{n-1}2! + \cdots = \sum_{k=1}^{k=n} S(n,k)(-1)^{n-k}k! + \sum_{k=1}^{k=n} S(n,k)(-1)^{n-k}(k+1)! + \cdots$$

Assembling the coefficient of $n!, (n+1)!, (n+2)!, \cdots$ and observing that the coefficient of each of these factorials is the alternating sum

$$S(n,n) - S(n,n-1) + \cdots + (-1)^n S(n,1),$$

we have

$$\begin{aligned} &1^{n-1}1! + 2^{n-1}2! + \cdots \\ &= \{S(n,n) - S(n,n-1) + \cdots + (-1)^n S(n,1)\} \textstyle\sum_{i=n}^{\infty} i! + c_n \end{aligned} \tag{3.15}$$

where c_n is a constant depending on n. Just for notational convenience, we replace n by k to get the following equation

$$\begin{aligned} &1^{k-1}1! + 2^{k-1}2! + \cdots \\ &= \{S(k,k) - S(k,k-1) + \cdots + (-1)^k S(k,1)\} \textstyle\sum_{i=k}^{\infty} i! + c_k \end{aligned} \tag{3.16}$$

Let us set, using the notation of Murty and Sumner [21],

$$\sum_{n=0}^{\infty} n^k n! = a_k \alpha + b_k.$$

where a_k, b_k are unique integers and $\alpha = \sum_{n=0}^{\infty} n!$(as an element of $\mathbb{Q}_p$ Now, assume there is a prime p for which $\sum_{n=0}^{\infty} n^k n!$ is a $p-$adic irrational. Comparing coefficients of 1 and α in equation (3.16), we have

$$a_k = \sum_{i=1}^{k+1} (-1)^{k+1-i} S(k+1, i). \tag{3.17}$$

This relation is due to Murty and Sumner [21] and will be used repeatedly in what follows.

Lemma 3.6.2. *$\sum_{n=0}^{\infty} n^k n!$ can be the same rational number in all $\mathbb{Q}_p$ if and only if*

$$\sum_{i=1}^{k+1} (-1)^{k+1-i} S(k+1, i) = 0$$

Proof. Proof follows using the fact that α cannot be same rational number independent of prime p. □

The preceding lemma shows that Stirling numbers could play an important role in answering rationality of $p-$adic series.

Proposition 3.6.3. *For every pair n, k of positive integers*

$$S(n,k) = \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} j^n (-1)^{k-j}.$$

Proof. Denote the sum on the right hand side of the above equation by $\sigma(n,k)$. Since $\sigma(1,1) = S(1,1) = 1$ and $\sigma(1,0) = S(1,0) = 0$. It suffices to show that $\sigma(n,k)$ satisfies the recurrence relation for Stirling number of second kind.
Now, $\sigma(n+1, k+1)$ is given by

$$\sigma(n+1, k+1) = \sum_{j=0}^{k+1} \frac{1}{(k+1-j)!j!} j^{n+1} (-1)^{k+1-j}$$

which is the same as

$$\sigma(n+1, k+1) = \frac{(k+1)^{n+1}}{(k+1)!} + \sum_{j=0}^{k} \frac{-jj^n (-1)^{k-j}}{(k+1-j)!j!}. \tag{3.18}$$

Writing $-j$ as $k+1-j-(k+1)$, it is clear that

$$\frac{-j}{(k+1-j)!j!} = \frac{1}{(k-j)!j!} - \frac{k+1}{(k+1-j)!j!}.$$

Multipling this equation by $j^n(-1)^{k-j}$ and then summing up with respect to j it is clear that

$$\sum_{j=0}^{k} \frac{-jj^n(-1)^{k-j}}{(k+1-j)!j!} = \sum_{j=0}^{k} \frac{j^n(-1)^{k-j}}{(k-j)!j!} - \sum_{j=0}^{k}(k+1)\frac{j^n(-1)^{k-j}}{(k+1-j)!j!}.$$

Observe that one of the term on the right hand side is $\sigma(n,k)$ and so using (3.18) we get

$$\sigma(n+1,k+1) = \frac{(k+1)^{n+1}}{(k+1)!} - (k+1)\sum_{j=0}^{k} \frac{j^n(-1)^{k-j}}{(k+1-j)!j!} + \sigma(n,k).$$

The first two term on the right hand side of previous equality when added gives us

$$\frac{(k+1)^{n+1}}{(k+1)!} - (k+1)\sum_{j=0}^{k} \frac{j^n(-1)^{k-j}}{(k+1-j)!j!} = (k+1)\sum_{j=0}^{k+1} \frac{j^n(-1)^{k+1-j}}{(k+1-j)!j!}$$

which is nothing but $(k+1)\sigma(n,k+1)$.

Hence the result follows. □

Before proving next proposition we give a table for the first few values of $S(n,k)$. We assume $S(n,m)=0$ for $m>n$ and $S(0,0)=1$.

n	$S(n,0)$	$S(n,1)$	$S(n,2)$	$S(n,3)$	$S(n,4)$	$S(n,5)$	$S(n,6)$
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	1	3	1	0	0	0
4	0	1	7	6	1	0	0
5	0	1	15	25	10	1	0
6	0	1	31	90	65	15	1
7	0	1	63	301	350	140	21

Looking at this table, one can guess that if $2 \leq j \leq p-1$ then the Stirling numbers $S(p,j)$ are divisible by p for a prime p less than 8. The next proposition confirms our guess.

Proposition 3.6.4. *Let p be an odd prime and k be an integer such that $2 \leq k \leq p-1$ then*

$$S(p,k) \equiv 0 \pmod{p}.$$

Proof. By Proposition (3.6.3), we have

$$S(p,k) = \sum_{j=1}^{k} \frac{1}{k!} \binom{k}{j} j^p (-1)^{k-j}$$

On the other hand, by Fermat's Little theorem, for any integer j,

$$j^p \equiv j \pmod{p}.$$

It follows that

$$S(p,k)k! \equiv \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} j \pmod{p}.$$

Observe that

$$\sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} j = \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{k-j} = 0.$$

The required congruence follows from the fact that for $2 \leq k \leq p-1$, $(k!, p) = 1$. □

The following is a result stated by D. Barsky and B. Benzaghou [9].

Lemma 3.6.5. *For a prime p and any positive integer j such that $1 \leq j \leq p-1$*

$$S(p-1,j) \equiv (p-1-j)! \pmod{p}$$

Proof. We prove the congruence by induction on j. By Wilson's theorem, one has

$$(p-2)! \equiv 1 \pmod{p}.$$

Since $S(p-1,1) = 1$, the lemma holds for $j = 1$. Assume that the lemma holds for some $j \geq 1$. So using the congruence $S(p,k) \equiv 0 \pmod{p}$ for $2 \leq k \leq p-1$ in the following recurrence relation

$$S(p,j+1) = S(p-1,j) + (j+1)S(p-1,j+1), \tag{3.19}$$

we obtain

$$S(p-1,j) \equiv -(j+1)S(p-1,j+1). \tag{3.20}$$

Using our induction hypothesis, $S(p-1,j) \equiv (p-1-j)! \pmod{p}$, it is clear that

$$(p-1-j)! \equiv \{p-(j+1)\}S(p-1,j+1) \pmod{p} \tag{3.21}$$

Cancelling $p-j-1$ the induction argument is completed. □

We now introduce Bell numbers.

Definition 3.6.6. For any positive integer n the n^{th} Bell number, β_n is the total number of partitions of a set with n elements into disjoint subsets.

For example, $\beta_3 = 5$ because we can partition $\{1,2,3\}$ into subsets as follows:

$$\{1,2,3\};\{1,2\}\cup\{3\};\{1,3\}\cup\{2\};\{1\}\cup\{2,3\};\{1\}\cup\{2\}\cup\{3\}$$

The first few Bell numbers are as follows:

$$1, 2, 5, 15, 52, 203, 877, 4140, \cdots$$

The values of some more Bell numbers can be found at Sloane's Online Encyclopedia of integer sequences (see A000110). We now state some of the results without proof. The fundamental recurrence for Bell numbers is given by:

$$\beta_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k}$$

We are interested in Bell numbers as they are related to Stirling numbers.

Proposition 3.6.7. *Let n be a positive integer then $\beta_n = \sum_{k=0}^{n} S(n,k)$.*

Proof. The proof is clear using the definition of Bell number and the definition of Stirling numbers through partitions. ☐

Corollary 3.6.8. *Let p be a prime number then*

$$\beta_{p-1} - 1 \equiv \sum_{j=0}^{p-1} (p-1-j)! \pmod{p}.$$

Proof. Proof follows from Lemma (3.6.5) and the definition of β_{p-1}. ☐

3.7 Stirling numbers and their alternating sum

Recent works by Murty and Sumner [21] has shown the relevance of alternating sums of the Stirling numbers of the second kind with the question of rationality of $p-$adically convergent series with factorials. For a posititve integer n, we shall denote by w_n the following alternating sum:

$$w_n = \sum_{j=0}^{n} (-1)^j S(n, j).$$

It is clear from the definitions of a_n and w_n that

$$a_n = (-1)^{n+1} w_{n+1}.$$

The first few terms in the sequence $\{a_n\}$ are

$$1, -1, 0, 1, 1, -2, -9, -9, 50, 267, 413, -2180, -17731, -50533, \cdots$$

This sequence is the sequence $A000587$ of Sloane's On-Line Encyclopedia of Integer Sequences [24]. The integers of these sequences are also called Uppulari-Carpenter or Complementary Bell numbers with the generating function $exp(1-e^x)$. These numbers count the excess of the number of partitions of $\{1, \cdots, n\}$ into an even number blocks over the number of partitions into an odd number of blocks. It is interesting to note that this sequence appears in Example 5(2) of Section 8, Chapter 3 of Ramanujan's Second Notebook. By Eq. (3.17), it is clear that if

$$a_k = \sum_{i=1}^{k+1} (-1)^{k+1-i} S(k+1, i)$$

is nonzero then $\sum_{n=0}^{\infty} n^k n!$ is an irrational number p—adically whenever $\sum_{n=0}^{\infty} n!$ is an irrational number p—adically. Now we consider certain special values of k for which a_k is nonzero.

Proposition 3.7.1. *For a fixed prime p assume that $\alpha = \sum_{n=0}^{\infty} n!$ is a p-adic irrational number. Then for every natural number k,*

$$\sum_{n=0}^{\infty} n^{2^{2k}-1} n! \quad .$$

is also a p—adic irrational number.

Proof. Wannemacker [29] has proved that for positive integers n, k where $1 \leq k \leq 2^n$,

$$v_2(S(2^n, k)) = \sigma_2(k) - 1$$

where $v_2(n)$ denotes the highest power of 2 dividing n and $\sigma_2(k)$ denotes the number of 1 in the 2-adic representation of k. We observe that $\sigma_2(k) = 1$ if and only if $k = 2^j$ for some nonnegative integer j. Therefore

$$\begin{aligned} S(2^n, k) &\equiv 1 \pmod 2 \quad \text{if } k = 2^j \\ &\equiv 0 \pmod 2 \quad \text{otherwise.} \end{aligned}$$

It therefore, follows by putting $k = 2^{2m} - 1$ in (3.17) that

$$\begin{aligned} a_{2^{2m}-1} &\equiv \sum_{j=1}^{2^{2m}} S(2^{2m}, j). \pmod 2 \\ &\equiv 1 + 1 + \cdots 1 \pmod 2 \end{aligned}$$

where 1 added to itself $2m + 1$ times. Recall that

$$\sum_{n=0}^{\infty} n^{2^{2k}-1} n! = a_{2^{2k}-1}\alpha + b_{2^{2k}-1}.$$

Since the preceding congruence implies that $a_{2^{2k}-1} \neq 0$, it follows that $\sum n^{2^{2k}-1} n!$ is p-adic irrational number if α is assumed to be a p-adic irrational. □

The rationality questions for some other series, though similar to the ones we have considered in the preceding proposition, can be settled using different techniques as shown in the following.

Proposition 3.7.2. *Let p, q be two distinct primes. Assume that $\alpha = \sum_{n=0}^{\infty} n!$ is a $p-$adic irrational number. Then for any positive integer m, $\sum_{n=0}^{\infty} n^{q^m+1} n!$ is also a $p-$adic irrational number.*

Proof. Since

$$\begin{aligned}\sum_{n=0}^{\infty} n^k n! &= \sum_{n=0}^{\infty} (n+1)^k (n+1)! \\ &= \sum_{n=0}^{\infty} (n+1)^{k+1} n! \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{k+1} \binom{k+1}{r} n^r n!,\end{aligned}$$

it follows that

$$\sum_{n=0}^{\infty} n^{k+1} n! = -k \sum_{n=0}^{\infty} n^k n! - \sum_{n=0}^{\infty} \sum_{r=0}^{k-1} \binom{k+1}{r} n^r n!.$$

Hence the relation $\sum_{n=0}^{\infty} n^k n! = a_k \alpha + b_k$ yields that

$$a_{k+1}\alpha + b_{k+1} = -k(a_k \alpha + b_k) - \sum_{r=0}^{k-1} \binom{k+1}{r} (a_r \alpha + b_r).$$

Assuming α to be irrational for at least one prime and since $\alpha, 1$ are linearly independent over $\mathbb{Q}$ comparing rational and irrational part of the preceding equality, we obtain

$$a_{k+1} = -k a_k - \sum_{r=0}^{k-1} \binom{k+1}{r} a_r \tag{3 22}$$

and

$$b_{k+1} = -k b_k - \sum_{r=0}^{k-1} \binom{k+1}{r} b_r \tag{3.23}$$

Taking $k = p^m$ and observing $a_1 = 0$, $\binom{p^m+1}{r} \equiv 0 \pmod{p}$ for $r = 2, 3, 4, \cdots k-1$ and reducing (3.22) modulo p we have

$$a_{p^m+1} \equiv -1 \pmod{p}.$$

We conclude that a_{p^m+1} is a nonzero integer. Hence the proposition follows. □

Before proving next theorem on irrationality of the series $\sum_{n=0}^{\infty} n^k n!$ we have the folowing lemma proved by Murty and Sumner [21].

Lemma 3.7.3. *If* $k \equiv 2 \pmod 3$ *then* β_k *is even, otherwise* β_k *is odd.*

Proof. The proof is based on induction $\beta_0 = 1$ and $\beta_1 = 1$ and $\beta_2 = 2$, so that lemma holds for $k = 0, 1, 2$. Suppose the lemma is true for $j \leq k$. Recall the recursion relation for β_k

$$\beta_{k+1} = \sum_{j=0}^{k} \binom{k}{j} \beta_j.$$

By the induction hypothesis, if $j \equiv 2 (\mathrm{mod} 3)$ then β_j is even, otherwise β_j is odd. Hence the recurrence formula becomes

$$\sum_{j \equiv 1,2 (\bmod 3)} \binom{k}{j} \equiv \sum_{j \equiv 0 (\bmod 3)} \binom{k}{j} + \sum_{j \equiv 1 (\bmod 3)} \binom{k}{j} (\mathrm{mod} 2).$$

Let ζ be a non-real cube root of unity. Adding binomial expansion of $(1+rx)^k$ as r takes the values $1, \zeta, \zeta^2$ we get

$$\sum_{j=0}^{j=k} \binom{k}{j} x^j (1 + \zeta^j + \zeta^2 j) = (1+x)^k + (1+\zeta x)^k + (1+\zeta^2 x)^k.$$

Let $x = 1$ then

$$\sum_{j=0}^{j=k} \binom{k}{j} (1 + \zeta^j + \zeta^{2j}) = 2^k + (1+\zeta)^k + (1+\zeta^2)^k.$$

Using $\zeta^3 = 1$ and $1 + \zeta + \zeta^2 = 0$. It can be easily shown that

$$1 + \zeta^j + \zeta^{2j} = \begin{cases} 3 & \text{if } j \equiv 0(\text{mod} 3) \\ 0 & \text{otherwise.} \end{cases}$$

$$\zeta^{2k} = \begin{cases} 1 & \text{if } k \equiv 0(\text{mod} 3) \\ \zeta & \text{if } k \equiv 2(\text{mod} 3) \\ \zeta^2 & \text{if } k \equiv 1(\text{mod} 3) \end{cases} \tag{3.24}$$

Now for the other sum. Consider

$$\sum_{j=0}^{k} \binom{k}{j} x^j = (1 + x)^k$$

$$\sum_{j=0}^{k} \binom{k}{j} \zeta^{j-1} x^j = \zeta^2 (1 + \zeta x)^k$$

$$\sum_{j=0}^{k} \binom{k}{j} \zeta^{2j-2} x^j = \zeta (1 + \zeta^2 x)^k$$

Adding these together gives

$$\sum_{j=0}^{j=k} \binom{k}{j} x^j (1 + \zeta^{j-1} + \zeta^{2j-2}) = (1 + x)^k + \zeta^2 (1 + \zeta x)^k + \zeta (1 + \zeta^2 x)^k.$$

Let $x = 1$

$$\sum_{j=0}^{k} \binom{k}{j} (1 + \zeta^{j-1} + \zeta^{2j-2}) = 2^k + \zeta^2 (1 + \zeta)^k + \zeta (1 + \zeta^2)^k.$$

Now

$$1 + \zeta^{j-1} + \zeta^{2j-2} = \begin{cases} 3 & \text{if } j - 1 \equiv 0(\text{mod} 3) \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{j=0}^{j=k} \binom{k}{j} (1 + \zeta^{j-1} + \zeta^{2j-2}) = 3 \sum_{j \equiv 1(\bmod 3)} {}^{k}C_j = 2^k + \zeta^2 (1 + \zeta)^k + \zeta (1 + \zeta^2)^k.$$

Using $1+\zeta+\zeta^2=0$ repeatedly, we get

$$3\sum_{j\equiv 1(\bmod 3)} {}^{k}C_j = 2^k+\zeta^2(1+\zeta)^k+\zeta(1+\zeta^2)^k$$
$$= 2^k+(-1)^k\{\zeta^{2k+2}+\zeta^{k+1}.\}$$

But ζ^2 is a conjugate of ζ and thus

$$3\sum_{j\equiv 1(\bmod 3)} {}^{k}C_j = 2^k+2(-1)^k(\text{ real part of }\zeta^{k+1}). \qquad (3.25)$$

Let us note that

$$\zeta^{k+1}=\begin{cases} 1 & \text{if} \quad k+1\equiv 0(\bmod 3)\\ \zeta & \text{if} \quad k+1\equiv 1(\bmod 2)\\ \zeta^2 & \text{if} \quad k+1\equiv 2(\bmod 3)\end{cases} \qquad (3.26)$$

combining the information in equation , (3.24), (3.26) ,we see that if

$$k\equiv 1(\bmod 3)$$

then

$$\beta_{k+1}\equiv 0(\bmod 2),$$

and if

$$k\equiv 2,3(\bmod 3)$$

then

$$\beta_{k+1}\equiv 1(\bmod 2)$$

proving the required lemma. □

Murty and Sumner [21] uses the above lemma to prove the following theorem on alternating sums of Stirling numbers.

Theorem 3.7.4. *Let k be a positive integer such that $k\equiv 0$ or 2 (mod 3) then $a_k\neq 0$, and $\sum_{n=0}^{\infty} n^k n!$ is a $p-$adic irrational provided α is also a $p-$adic irrational.*

Proof. We have by (3.17) and definition of Bell numbers based on Stirling numbers

$$u_k \equiv \sum_{j=1}^{k+1} S(k+1,j) \equiv \beta_{k+1}(\mathrm{mod}2) \tag{3.27}$$

Hence the theorem follows. ☐

However, we have another simpler proof of Theorem (3.7.4) as shown below. Consider the series $\sum_{n=0}^{\infty} n^k(n-1)n!$. By simple calculation, we have

$$\sum_{n=0}^{\infty} n^k(n-1)n! = \sum_{n=0}^{\infty}(n+2)^k(n+1)(n+2)! = \sum_{n=0}^{\infty}(n+2)^{k+1}(n+1)^2 n! \tag{3.28}$$

Recalling the notation $\sum_{n=0}^{\infty} n^k n! = a_k\alpha + b_k$ and expanding $(n+2)^{k+1}(n+1)^2$ as powers of n and then comparing the coefficient of α (assuming α to be irrational), we get

$$a_{k+1} - a_k = \sum_{r=0}^{k+1} 2^{k+1-r}(a_{r+2} + 2a_{r+1} + a_r) \tag{3.29}$$

which upon reduction modulo 2 gives

$$a_k \equiv a_{k+3}(\mathrm{mod}2) \tag{3.30}$$

Now

$$a_1 = 0,$$
$$a_2 = S(3,3) - S(3,2) + S(3,1) = -1,$$
$$a_3 = S(4,4) - S(4,3) + S(4,2) - S(4,1) = 1$$

Thus we have

$$a_k \equiv \begin{cases} -1(\mathrm{mod}2) & \text{whenever } k \equiv 2(\mathrm{mod}3) \\ 1 \ (\mathrm{mod}2) & \text{whenever } k \equiv 0(\mathrm{mod}3) \\ 0 \ (\mathrm{mod}2) & \text{whenever } k \equiv 1(\mathrm{mod}3) \end{cases}$$

Hence the theorem follows.

Other congruences on alternating sum of Stirling numbers have also been obtained by Wannemacker like

$$\begin{aligned} a_n &\equiv a_{n+12} \,(\mathrm{mod}\, 4) \\ a_n &\equiv a_{n+48} \,(\mathrm{mod}\, 8) \end{aligned} \tag{3.31}$$

and in general

$$a_n \equiv a_{(n+3\;4^{h-1})} \,(\,\mathrm{mod} 2^h\,) \tag{3.32}$$

However, proof of this general congruence result (as given by Wannemacker) depends on concepts like companion matrix, Jacobson canonical form, etc. Using equation (3.29) we could possibly have an elementary proof of the general result namely congruence equation (3.31). Murty and Sumner [21] have obtained an expression for alternating sum of stirling numbers as e times sum of an infinite series with usual norm. The following lemma occurs in [21].

Lemma 3.7.5. $(-1)^k u_k = e\sum_{n=0}^{\infty} \dfrac{n^{k+1}}{n!}(-1)^n$

Proof. By equation (3.17), we have

$$\begin{aligned} (-1)^k u_k &= \sum_{j\geq 1}(-1)^j S(k+1,j) \\ &= \sum_{j\geq 1}\frac{(-1)^j}{j!}\sum_{0\leq r\leq j}(-1)^r\; {}^jC_r(j-r)^{k+1} \\ &= \sum_{r\geq 0}\sum_{j\geq r}\frac{(-1)^j}{r!(j-r)!}(j-r)^{k+1} \\ &= \sum_{r\geq 0}\frac{1}{r!}\sum_{j\geq r}\frac{(-1)^{j-r}}{(j-r)!}(j-r)^{k+1} \\ &= \sum_{r\geq 0}\frac{1}{r!}\sum_{n=0}^{\infty}\frac{n^{k+1}}{n!}(-1)^n \\ &= e\sum_{n=0}^{\infty}\frac{n^{k+1}}{n!}(-1)^n. \end{aligned}$$

□

3.8 Some questions

The material covered in this section throws up several questions whose answers, as far as we know, are not known. It is clear that the series $\sum_{n=0}^{\infty} n^k n!$ converges in every $\mathbb{Q}_p$ for every prime p.

Question 1 : Does there exist a $k \in \mathbb{N}$, $k > 1$ such that $\sum_{n=0}^{\infty} n^k n!$ is a rational number in every $\mathbb{Q}_p$? We can generalise the previous question and ask the question,

Question 2 : What are the integers k, j such that $\sum_{n=0}^{\infty} n^k (n+j)!$ is a fixed rational number?

Recall that $a_k(j)$ are defined inductively on k as follows for $k \geq 2$

$$a_k(j) = a_{k-1}(j+1) - (j+1)a_{k-1}(j)$$

and

$$\begin{aligned} a_1(j) &= -j \\ b_1(j) &= -K(j+1) + (j+1)K(j) \end{aligned}$$

We have the following question

Question 3 : For a fixed positive integer k, what are the roots of the equation $a_k(j) = 0$?

Recall that Kurepa's left factorials, $K(m)$ are defined by

$$\begin{aligned} K(m) &= 0 & &\text{if } m = 0 \\ &= 0! + 1! + 2! + \cdots + (m-1)! & &\text{if } m \text{ is a positive integer.} \end{aligned}$$

Barsky has shown that the remainder left out when $K(p)$ is divided by p is a nonzero quantity. We can further investigate this result and ask the question;

Question 4 : What is the least positive remainder when $K(p)$ is divided by p?

Chapter 4

Transcendental numbers in $\mathbb{R}$ or $\mathbb{Q}_p$

4.1 Introduction

It is often easier to decide whether a $p-$adic series converges or not as compared to deciding whether a real series converges or not. Thus the problem of convergence of a $p-$adic series is easier to handle but it is not easy to decide to what kind of number (rational or algebraic or transcendental) the $p-$adic series converges. For example, it is not known whether the series $\sum_{n=0}^{\infty} \frac{n!}{n!^2 + 1}$ converges $p-$adically to a rational number even for a single prime p. Burger and Struppeck [11] have shown that the series $\sum_{n=0}^{\infty} \left\{ \frac{n!}{n!^2 + 1} \right\}^{n!^3}$, converging in $\mathbb{Q}_p$ for every prime p as well as for $p = \infty$, is a transcendental number.

In this chapter, we show, using Liouville's theorem, that the sum $\sum_{n=0}^{\infty} \left\{ \frac{n!}{n!^2 + 1} \right\}^{n!^2}$ is transcendental number. We have not seen this recorded in the literature. We begin by treating the question of transcendence of certain simple series by elementary means.

4.2 The series $\sum_{n=o}^{\infty} p^{n!}$

Liouville, one of the pioneers for investigating transcendence of real numbers, had shown that the real number to which the series $\sum_{n=0}^{\infty}(-1)^n 2^{-n!}$ converges is transcendental over $\mathbb{Q}$. Suter in [13] gives a $p-$adic version of the proof given by Liouville to establish the following proposition.

Proposition 4.2.1. *For every prime p, series $\sum_{n=0}^{\infty} p^{n!}$ converges in $\mathbb{Q}_p$ to a transcendental number.*

Proof. Using the fact that $p-$adic norm of the terms of the series $\sum_{n=0}^{\infty} p^{n!}$ are tending to zero it is easy to see that the series $\sum_{n=0}^{\infty} p^{n!}$ converges in $\mathbb{Q}_p$. Now,

let

$$\beta = \sum_{n=0}^{\infty} p^{n!}$$

and $\beta_k = \sum_{n=0}^{k} p^{n!}$ be the k^{th} partial sum of the series $\sum_{n=0}^{\infty} p^{n!}$.

We prove that β is transcendental over $\mathbb{Q}$ by the method of contradiction. So assume that β is algebraic over $\mathbb{Q}$. We note that β is not a rational number as there is no repetition in the $p-$adic expansion of β. Clearly n the degree of β must be greater than 1.

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

be an irreducible polynomial over $\mathbb{Z}$ satisfied by β.

Write

$$q(x) = \frac{f(x) - f(\beta_k)}{x - \beta_k}$$

which can be shown to be a polynomial over $\mathbb{Z}$ using the fact that if i is a positive integer then $\frac{x^i - \beta_k^i}{x - \beta_k}$ is a polynomial over $\mathbb{Z}$. Clearly

$$|q(\beta)|_p \leq 1.$$

Also it is easy to see that

$$|f(\beta) - f(\beta_k)|_p = |\beta - \beta_k|_p |q(\beta)|_p. \tag{4.1}$$

Choose a positive integer M such that

$$p^M \geq \max \left\{ \sum_{i=0}^{n} |a_i|, \sum_{\substack{i=0 \\ i \neq j}}^{n} |a_i a_j|, \cdots, |a_1 a_2 a_3 \cdots a_n| \right\}. \tag{4.2}$$

where $|.|$ is the usual norm on $\mathbb{Q}$. Note that the highest power of p appearing in $f(\beta_k)$ is $p^{k!n}$, so we may write

$$f(\beta_k) = a_n p^{k!n} + b_1 p^w + \cdots + a_0.$$

Here b_1 is a sum of the $a_{i's}$ and p^w is the next highest power in the expanded form of $f(\beta_k)$. It follows by the properties of the usual norm that

$$f(\beta_k) \leq |f(\beta_k)| \leq p^{M+k!n} + p^{M+w} + \cdots + p^M \leq \sum_{i=0}^{k!n+M} p^{i!}.$$

Therefore,

$$f(\beta_k) \leq (p^{k!n+1+M} - 1)/(p-1) \leq p^{k!n+1+M}.$$

Hence $p^{-(k!n+1+M)} \leq |f(\alpha) - f(\alpha_k)|_p \leq |\beta - \beta_k|_p \leq p^{-(k+1)!}$. However as k approaches infinity $p^{-(k+1)!}$ tends to zero much faster than $p^{-(k!n+1+M)}$. Thus the preceding inequality is not valid for sufficiently large k. $\square$

4.3 A proof of Liouville's theorem

In this section we prove Liouville's theorem both in the p-adic case as well as in the real case.

Theorem 4.3.1. *Let p be either a rational prime or $p = \infty$. If $\alpha \in \mathbb{Q}_p$ is algebraic of degree d over $\mathbb{Q}$, then there exist a positive constant c such that for every rational number $r/s \neq \alpha$ with $(r, s) = 1$ and $s > 0$, the following holds:*

$$|\alpha - r/s|_p h(r/s)^d \geq c,$$

where

$$h(r/s) = max\{|r|, |s|\}.$$

Note that the definition of $h(r/s)$ involves the usual norms of the integers r and s and so $h(r/s)$ is a positive integer.

Proof. We assume first that α is a p-adic algebraic number of degree d. In other words, α is in $\mathbb{Q}_p$ and satisfies a monic irreducible polynomial of degree d over $\mathbb{Q}$. Consider the case when $d = 1$. Then α must be a rational number. The proof of the theorem in this case follows the one given by Calegari [12]. We begin by letting $\alpha = a/b$. Choose any rational number r/s such that $r/s \neq \alpha$ with $(r, s) = 1$. If the integers as and br are incongruent modulo p, then p does not divide $(as - br)$ so $|as - br|_p = 1$. Thus

$$\begin{aligned} |\alpha - r/s|_p h(r/s) &= \frac{|as - br|_p}{|bs|_p} h(r/s) \\ &= \frac{h(r/s)}{|bs|_p} \\ &\geq \frac{1}{|bs|_p}. \end{aligned}$$

As the p-adic norm of the integer bs is always less than or equal to 1, the theorem is verified in this case. So we can assume that as, br are congruent modulo p. Since
$\alpha = a/b \neq r/s$, it follows that $as - br$ is a non-zero integer and for any nonzero integer n,

$$|n|_p \geq \frac{1}{|n|}. \tag{4.3}$$

So

$$|as - br|_p \geq \frac{1}{|as - br|}.$$

Using the Inequality (4.3) we then see that

$$\begin{aligned} |\alpha - r/s|_p h(r/s) &= \frac{|as - br|_p}{|bs|_p} h(r/s) \\ &\geq \frac{1}{|as - br||bs|_p} h(r/s). \end{aligned} \tag{4.4}$$

However

$$|as - br| \leq h(r/s)(|a| + |b|).$$

So using the fact that for a non-zero integer s, $|bs|_p = |b|_p|s|_p \leq |b|_p$, it follows that

$$|as - br||bs|_p \leq h(r/s)(|a| + |b|)|b|_p.$$

The required inequality of the theorem then follows from Inequality (4.4). Hence the proof follows in this case. To complete the proof of Liouville's theorem in the p-adic case, we next assume that α is a p-adic algebraic number of degree $d > 1$. Let $\alpha = \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_d$ be all the conjugates of α over $\mathbb{Q}$. Therefore, by the elementary theory of field extensions if

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{d-1}x^{d-1} + x^d \tag{4.5}$$

is the minimal polynomial of α over $\mathbb{Q}$, then

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_d). \tag{4.6}$$

For an arbitrary rational number r/s where r, s are relatively prime integers, it is clear that

$$f(r/s) = (r/s - \alpha_1)(r/s - \alpha_2) \cdots (r/s - \alpha_d)$$

and so

$$s^d f(r/s) = (r - s\alpha_1)(r - s\alpha_2) \cdots (r - s\alpha_d).$$

That is,

$$\frac{s^d f(r/s)}{(r - s\alpha_2) \cdots (r - s\alpha_d)} = (r/s - \alpha)s. \tag{4.7}$$

Now, it is clear that if l is the lcm of the denominators of $a_0, a_1, \cdots, a_{d-1}$ then $ls^d f(r/s) \in \mathbb{Z}$. Next note that if n is a nonzero integer then $|n|\,|n|_p \geq 1$. Since degree d of α is greater than 1, it follows that

$$|ls^d f(r/s)|_p \geq \frac{1}{|ls^d f(r/s)|}. \tag{4.8}$$

Note that for any nonnegative integer $k \leq d$,

$$|r|^k |s|^{d-k} \leq \{h(r/s)\}^d.$$

By using Eq.(4.5) and then the triangle inequality for the quantity in the right hand side of Inequality (4.8), we have

$$\begin{aligned} |ls^d f(r/s)|_p &\geq \frac{\{l(|a_0| + |a_1| + |a_2| + \cdots + |a_{d-1}| + 1)\}^{-1}}{\{h(r/s)\}^d} \\ &= C\{h(r/s)\}^{-d} \end{aligned} \tag{4.9}$$

Where $C = \{l(|a_0|+|a_1|+|a_2|+\cdots+|a_{d-1} + 1|)\}^{-1}$ is a quantity independent of r/s. Using Inequalities (4.7), (4.9) and the non-archimedean property of $p-$adic norm, we have

$$|ls|_p|r/s-\alpha|_p \geq \frac{C_1}{\{h(r/s)\}^d \Pi_{j=2}^{j=d} max\{|r|_p, |s|_p|\alpha_j|_p\}}$$
$$\geq \frac{C_1}{\{h(r/s)\}^d \Pi_{j=2}^{j=d} max\{1, |\alpha_j|_p\}}$$

as $|n|_p \leq 1$ for $n \in \mathbb{Z}$. Now for a given α its conjugates are fixed and thus

$$\Pi_{j=2}^{j=d} max\{1, |\alpha_j|_p\}$$

is a constant. Hence the theorem follows in the $p-$adic case.

Next we consider α to be a real number. As in the $p-$adic case, we start the proof by assuming α to be an algebraic real number of degree 1. Clearly $\alpha = a/b$ for some relatively prime integers $a, b, b \neq 0$ then

$$\left|\frac{a}{b} - \frac{r}{s}\right| h(r/s) = \frac{|as-br|}{|bs|} h(r/s).$$

But $as - br$ is a nonzero integer hence $|as - br| \geq 1$. Thus

$$|\frac{a}{b} - \frac{r}{s}| h(r/s) \geq \frac{1}{|bs|} h(r/s) \geq \frac{1}{|b|}$$

Hence Liouville's theorem follows for an algebraic number $\alpha \in \mathbb{R}$, of degree 1. To complete the proof in the real case we assume α to be an algebraic real number of degree greater than 1. Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + x^d$$

be the minimal polynomial of α over $\mathbb{Q}$.

Now by the Lagrange's Mean Value Theorem, there exist ξ such that

$$-f(r/s) = f(\alpha) - f(r/s) \doteq (\alpha - r/s) f'(\xi) \qquad (4.10)$$

where either $r/s < \xi < \alpha$ or $\alpha < \xi < r/s$ If our choice of rational number r/s leads to $|\alpha - \xi| \geq 1$ then

$$|\alpha - r/s| \geq 1.$$

Thus

$$|\alpha - r/s|s^d \geq 1.$$

Now, one can assume that $|\alpha - \xi| < 1$. Since $f'(\xi) = f'(\xi - \alpha + \alpha)$, $f'(\xi)$ can be considered as a polynomial in variable $\alpha - \xi$. Thus $f'(\xi)$ can be bounded above by a number c depending on α only. It then follows from Eq.(4.10) that

$$|\alpha - r/s| = \frac{|-f(r/s)|}{|f'(\xi)|} > |-f(r/s)|c. \tag{4.11}$$

Note that f is an irreducible polynomial of degree greater than 1. Therefore, a calculation, similar to the derivation of Inequality (4.8) shows that

$$|ls^d f(r/s)| \geq 1.$$

Using (4.11), we thus obtain

$$|\alpha - r/s| \geq \frac{c'}{s^d}.$$

Hence the theorem holds in this case too. □

Liouville's theorem for the case degree of α being equal to 1 gives a criterion for rationality of α. The following lemma gives a criterion for irrationality of $\alpha \in \mathbb{Q}_p$.

Lemma 4.3.2. *Let $\{p_n/q_n\}$ be a sequence of rational numbers with $p_n, q_n \in \mathbb{Z}, q_n \neq 0$ and with the sequence $\{q_n\}$ unbounded and suppose there exists a $\delta > 0$ such that*

$$0 < \left|\alpha - \frac{p_n}{q_n}\right|_p \leq \frac{1}{(max\{|p_n|, |q_n|\})^{1+\delta}} \tag{4.12}$$

for sufficiently large n then α is an irrational number.

4.4 Liouville's number

A number α is said to be *Liouville's number* if there exists,

(a) an infinite sequence of distinct rational numbers in the lowest form given by

$$P_1/Q_1, P_2/Q_2, P_3/Q_3, \cdots$$

and

(b) an infinite sequences of positive numbers

$$\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \cdots\}$$

tending to infinity, such that

$$0 < |\alpha - P_n/Q_n| < H_n^{-\Lambda_n} \quad (n = 1, 2, 3, \cdots)$$

where $H_n = \max(|P_n|, |Q_n|)$.

Lemma 4.4.1. *Liouville's numbers are transcendental.*

Proof. Note that $|\alpha - P_n/Q_n|H_n^d < H_n^{d-\Lambda_n}$. However $H_n^{d-\Lambda_n}$ tends to zero as $n \to \infty$. The lemma follows by Liouville's theorem. □

However there are certain transcendental number which are not liouvilles number at all many of the well known constants like π, e etc. are not Liouvilles number. As an application of Liouville's theorem, we have

Example 4.4.2. $\sum_{n=0}^{\infty} k^{-n!}$ is a transcendental number for k in $\mathbb{Z} \setminus \{1, -1, 0\}$.

The proof will follow by using the partial sums of this series and showing that for a given d.

$$\lim_{n \to \infty} |\alpha - s_n|_p h(s_n)^d = 0$$

As another application of Liouvilles theorem we have a proof of irrationality of $\sum_{n=0}^{\infty} p^{n^2}$, $\sum_{n=0}^{\infty} p^{u_n}$ where u_n denotes the n^{th} term of the fibonacci sequence

Proposition 4.4.3. $\sum_{n=0}^{\infty} p^{n^2}$ *is an irrational number p−adically*

Proof. Let $\alpha = \sum_{n=0}^{\infty} p^{n^2}, Q_n = \sum_{i=0}^{n} p^{i^2}$
Then

$$\begin{aligned} |\alpha - Q_n|_p &= |p^{(n+1)^2} + p^{(n+2)^2} + p^{(n+3)^2} + \cdots|_p \\ &= p^{-(n+1)^2} \end{aligned}$$

Thus

$$\begin{aligned}|\alpha - Q_n|_p h(Q_n) &= p^{-(n+1)^2}\sum_{i=0}^{n} p^{i^2} \\ &\le p^{-(n+1)^2}(n+1)p^{n^2} \\ &= (n+1)p^{-2n-1}\end{aligned}$$

which tends to zero as n tends to infinity. Thus by Liouville's Theorem

$$\sum_{n=0}^{\infty} p^{n^2}$$

is an irrational number $p-$adically. ☐

Proposition 4.4.4. *Let u_n denotes the n^{th} Fibonacci number. Then the sum $\sum_{n=0}^{\infty} p^{u_n}$ is an irrational number $p-$adically.*

Proof. Let

$$\alpha = \sum_{n=0}^{\infty} p^{u_n}, Q_n = \sum_{i=0}^{n} p^{u_i}$$

Then

$$\begin{aligned}|\alpha - Q_i|_p &= |p^{u(i+1)} + p^{u(i+2)} + p^{u(i+3)} + \cdots|_p \\ &= p^{-u_{(i+1)}}\end{aligned}$$

Hence

$$|\alpha - Q_n|_p h(Q_n)$$

tends to zero as n ends to infinity. Thus by Liouville's Theorem $\sum_{n=0}^{\infty} p^{u_n}$ is an irrational number $p-$adically. ☐

4.5 Transcendence of the sum $\sum_{n=0}^{\infty}(\frac{n!}{n!^2+1})^{n!^2}$

In this section, we show that the $p-$adically convergent series $\sum_{n=0}^{\infty}(\frac{n!}{n!^2+1})^{n!^2}$(for any prime p as well as $p = \infty$) converges to a $p-$adic number transcendental over $\mathbb{Q}$. We begin by a preparatory results.

Lemma 4.5.1. *For any real number $q > 1$ and a positive integer k, there exists a nonnegative integer N such that for every $n > N$,*

$$q^n > n^k$$

Proof. By the well known formula for binomial expansion, we have

$$q^n = (q-1+1)^n = \sum_{j=0}^{n} \binom{n}{j} (q-1)^j.$$

Now for $q > 1$ each term of the preceding sum is positive, so for $n \geq 2k$

$$\begin{aligned} q^n &> {}^nC_{2k}(q-1)^{2k} \\ &= \frac{n(n-1)\cdots(n-2k+1)}{(2k)!}(q-1)^{2k}. \end{aligned}$$

Next we choose

$$n > 4k-2;$$

note that as $k \geq 1$, this choice of n is still possible under the previous condition $n > 2k$ then it is easy to see that for such n

$$(n-2k+1) > n/2$$

so that for any nonnegative integer j such that $j < 2k-1$ we have

$$(n-j) > n/2$$

Hence we may conclude that

$$n(n-1)\cdots(n-2k+1) > \frac{n^{2k}}{2^{2k}}.$$

Thus

$$q^n > \frac{n^{2k}}{2^{2k}} \cdot \frac{(q-1)^{2k}}{(2k)!}$$

Therefore if n is chosen large enough such that

$$n^k > \max\left\{(4k-2)^k, \frac{2^{2k}(2k)!}{(q-1)^{2k}}\right\}$$

then

$$\begin{aligned} q^n &> \frac{n^{2k}}{2^{2k}(2k)!}(q-1)^{2k} \\ &> n^k. \end{aligned}$$

Hence the desired result is obtained. □

Corollary 4.5.2. *Let $q > 1$ be a real number, n be a nonnegative integer then there exists N such that for every $n > N$,*

$$q^n > (n+1)^k$$

Proof. By the above result there exists an integer N such that $q^n > (n)^{2k}$ for every $n \geq N$ but for $n > 2, n^2 > n+1$ and thus $q^n > (n+1)^k$ for every $n \geq N > 2$. □

Lemma 4.5.3. *Let d be a fixed positive integer, $q > 1$ be a real number, we define*

$$P(n) = \{\Pi_{i=1}^{i=n}(i!^2+1)^{i!^2}\}^d$$

$$R_q(n) = (q^n)^{(n+1)!^2}$$

Then

$$\lim_{n\to\infty} \frac{P(n)}{R_q(n)} = 0.$$

Proof. It is clear from the definition of $R_q(n)$ that

$$\begin{aligned}\frac{R_q(n+1)}{R_q(n)} &= q^{\{(n+1)(n+2)^2-n\}(n+1)!^2} \\ &= q^{\{n^3+5n^2+7n+4\}(n+1)!^2}.\end{aligned}$$

Note that for every positive integer n,

$$n^3+5n^2+7n+4 \geq n(n^2+5n+7).$$

It follows that

$$\frac{R_q(n+1)}{R_q(n)} \geq (q^n)^{(n+1)!^2(n^2+5n+7)}.$$

Therefore Corollary (4.5.2) with $k = 8d$ together with the preceding inequality (for sufficiently large n) implies that

$$\frac{R_q(n+1)}{R_q(n)} \geq \{((n+1)^2)^{4d}\}^{(n+1)!^2(n^2+5n+7)}.$$

Observes that

$$n^2+5n+7 > n^2+5n, (n+1)^2 \geq 2(n+1)$$

Thus we have

$$\frac{R_q(n+1)}{R_q(n)} \geq \{(2(n+1)^2)^{2d}\}^{(n+1)!^2(n^2+5n)}$$

However

$$\{(n+1)^2\}^{n+5} > \{(n+1)!\}^2. ((n+1)!^2)^{2d} \geq ((n+1)!^2+1)^d.$$

This shows that

$$\begin{aligned}\frac{R_q(n+1)}{R_q(n)} &\geq 2\{((n+1)!^2+1)^d\}^{(n+1)!^2 n}\\ &= 2\frac{P(n+1)}{P(n)} .\end{aligned}$$

Thus

$$\frac{R_q(n+1)}{R_q(n)} \geq 2\frac{P(n+1)}{P(n)}$$

and

$$\frac{R_q(n+2)}{P(n+2)} \geq 2^2\frac{R_q(n)}{P(n)}$$

Thus for $n \geq N$

$$\frac{R_q(n)}{P(n)} \geq 2^{n-N}\frac{R_q(N)}{P(N)}$$

Thus for a fixed N as n approaches to infinity, we have

$$\lim_{n\to\infty}\frac{R_q(n)}{P(n)} = \infty$$

and hence the lemma is proved. □

We are in a position to prove our main theorem.

Theorem 4.5.4. *The sum of the convergent series*

$$\sum_{n=0}^{\infty}\left(\frac{n!}{n!^2+1}\right)^{n!^2}$$

with respect to every norm on $\mathbb{Q}$ *is a transcendental number.*

Proof. Let

$$\alpha = \sum_{n=0}^{\infty}\left(\frac{n!}{n!^2+1}\right)^{n!^2}, r_n = \sum_{i=0}^{n}\left(\frac{i!}{i!^2+1}\right)^{i!^2}.$$

Since for a positive integer i,

$$\frac{i!}{i!^2+1} \leq 1/i!,$$

it follows that

$$\left\{\frac{i!}{i!^2+1}\right\}^{i!^2} \leq 1/i!,$$

which gives

$$r_n \leq \sum_{i=0}^{\infty} 1/i! = e, \text{ the well known constant.} \tag{4.13}$$

Let

$$\Pi_n = \Pi_{i=0}^{i=n}(i!^2+1)^{i!^2}$$

then

$$r_n = \frac{\sum_{i=0}^{n} \frac{i!^{i!^2}\Pi_n}{(i!^2+1)^{i!^2}}}{\Pi_n}$$

Let $r_n = \frac{a_n}{b_n}$ where a_n, b_n are relatively prime positive integers then b_n divides Π_n so

$$1 \leq b_n \leq \Pi_{i=0}^{i=n}(i!^2+1)^{i!^2}$$

and

$$1 \leq a_n \leq e\Pi_{i=0}^{i=n}(i!^2+1)^{i!^2}$$

Recall for $a_n > 0, b_n > 0$ and $(a_n, b_n) = 1$, $h(r_n)$ is given by

$$h(r_n) = \max\{a_n, b_n\}.$$

Thus

$$h(r_n)^d \leq \{e\Pi_{i=0}^{i=n}(i!^2+1)^{i!^2}\}^d. \tag{4.14}$$

We know that

$$\lim_{k\to\infty} |\alpha - r_k|_p = 0.$$

Thus there exists an integer M_n such that for every $k \geq M_n$,

$$|\alpha - r_k|_p \leq |(n+1)!^{(n+1)!^2}|_p. \tag{4.15}$$

However, for $k > n$, the non-archimedean property of the $p-$adic norm yields

$$|r_k - r_n|_p = |(\frac{(n+1)!}{(n+1)!^2+1})^{(n+1)!^2} + \cdots + (\frac{(k+1)!}{(k+1)!^2+1})^{(k+1)!^2}|_p \tag{4.16}$$

Observe that for $n > p, j \in \mathbb{N}$,

$$|(n+j+1)!^2 + 1|_p = 1.$$

Hence Equality (4.16) reduces to

$$\begin{aligned} |r_k - r_n|_p = & \max\{|((n+1)!^{(n+1)!^2}|_p, \cdots, |((k+1)!)^{(k+1)!^2}|_p\} \\ & \leq |((n+1)!^{(n+1)!^2}|_p. \end{aligned} \tag{4.17}$$

Thus for $k \geq max\{n+1, M_n\}$, using equality (4.15) and (4.17), we have

$$|\alpha - r_n|_p \leq \max\{|\alpha - r_k|_p, |r_k - r_n|_p\} \leq |(n+1)!^{(n+1)!^2}|_p. \tag{4.18}$$

Hence using Inequalities (4.14),(4.18) we have

$$|\alpha - r_n|_p h(r_n)^d \leq |(n+1)!^{(n+1)!^2}|_p \{e\Pi_{i=1}^{i=n}(i!^2+1)^{i!^2}\}^d.$$

Using the formula for $|n!|_p$ from Chapter 1 it follows that

$$|\alpha - r_n|_p h(r_n)^d \leq p^{\{-\frac{n+1-S_p(n+1)}{p-1}\}^{(n+1)!^2}}\}e.\{\Pi_{i=1}^{i=n}(i!^2+1)^{i!^2}\}^d$$

For $k \in \mathbb{N}$, if we choose $n_k = p^k - 1$ then $-\frac{n_k+1-S_p(n_k+1)}{p-1} = -\frac{n_k}{p-1}$ so

$$\begin{aligned} |\alpha - r_{n_k}|_p h(r_{n_k})^d & \leq \{e.\Pi_{i=1}^{i=n_k}(i!^2+1)^{i!^2}\}^d/(q^{n_k})^{(n_k+1)!^2}\} \\ & = e^d P(n_k)/R(n_k) \end{aligned}$$

which tends to zero as $n_k \to \infty$ by Lemma (4.5.3).

□

Lemma 4.5.5. *For* $n, k \in \mathbb{N}$*,* $k > n$ *the following inequality holds in* $\mathbb{R}$

$$|\{\frac{(n+1)!}{(n+1)!^2+1}\}^{(n+1)!^2} + \cdots + \{\frac{(k+1)!}{(k+1)!^2+1}\}^{(k+1)!^2} + \cdots|$$

$$\leq 2\{\frac{(n+1)!}{(n+1)!^2+1}\}^{(n+1)!^2}$$

Proof. Using $2(n+1)!(n!^2+1) \leq (n)!(n+1)!^2+1$ (which is true for every $n \in \mathbb{N}$) we obtain

$$\frac{(n+1)!}{(n+1)!^2+1} \leq 1/2\frac{n!}{n!^2+1}$$

thus

$$|\{\frac{(n+1)!}{(n+1)!^2+1}\}^{(n+1)!^2} + \cdots| \leq \{(\frac{(n+1)!}{(n+1)!^2+1})^{(n+1)!^2}\}(1 + 1/2 + 1/4 + 1/8 + \cdots)$$

$$= 2\{\frac{(n+1)!}{(n+1)!^2+1}\}^{(n+1)!^2}.$$

□

Proposition 4.5.6. $\sum_{n=0}^{\infty}(\frac{n!}{n!^2+1})^{n!^2}$ *is a transcendental number when the convergence is considered with respect to the usual norm.*

Proof. Proceeding as in Theorem (4.5.4)and using previous Lemma (4.5.5), we have

$$|\alpha - r_n|h(r_n)^d \leq 2\{\frac{(n+1)!}{(n+1)!^2+1}\}^{(n+1)!^2}\{e\Pi_{i=0}^{i=n}(i!^2+1)^{i!^2}\}^d$$

$$\leq 2\{\frac{1}{(n+1)!}\}^{(n+1)!^2}\{e\Pi_{i=0}^{i=n}(i!^2+1)^{i!^2}\}^d$$

Again it can be shown that

$$\lim_{n\to\infty}\{\frac{1}{(n+1)!}\}^{(n+1)!^2}\{e\Pi_{i=0}^{i=n}(i!^2+1)^{i!^2}\}^d = 0.$$

□

4.6 Some questions

Burger and Struppeck [11] have shown that the power series

$$\sum_{n=0}^{\infty}\left\{\frac{n!}{n!^2+1}\right\}^{n!^3} q^n$$

converges to a transcendental numbers for every rational number $q \neq 0$ and with respect to every p–adic norm on $\mathbb{Q}$.

Question 1 : Does there exist an irrational number r such that the sum of the power series $\sum_{n=0}^{\infty}\left\{\frac{n!}{n!^2+1}\right\}^{n!^3} r^n$ in $\mathbb{R}$ is a rational number?

Recall the definition of Liouville's numbers given in section (4.4)

Question 2 : Is the real number

$$\sum_{n=0}^{\infty}\left\{\frac{n!}{n!^2+1}\right\}^{n!^3}$$

a Liouville's number?

It is easy to see by the standard comparision test that the series

$$\sum_{n=0}^{\infty}\frac{1}{n^{n!}}$$

converges to a real number.

Question 3 : Is the real number

$$\sum_{n=0}^{\infty}\frac{1}{n^{n!}}$$

a transcendental number?

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series whose domain of convergence is $\mathbb{R}$.

Question 4 : Does there exist a sequence of irrational numbers $\{a_n\}$ such that for a fixed but arbitrary $r \in \mathbb{Q}$, $f(r)$ is a transcendental number?

If we write the sum $\sum_{n=0}^{\infty}\left\{\frac{n!}{n!^2+1}\right\}^{n!^3}$ in the usual decimal expansion of a real number given by

$$a_0.a_1a_2\ldots$$

than is it true that a_γ never vanishes?

It is known that the series $\sum_{n=0}^{\infty}\frac{1}{n^2}$ converges to $\pi^2/6$ which is a transcendental number.

Question 5 : Does the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

converges to a transcendental number or even an irrational number?

The sum

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

is known as Catalan's constant.

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