The nearest correlation matrix problem: Solution by differential evolution method of global optimization

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I. Introduction: A product moment correlation matrix R of order m is a symmetric positive semi-definite matrix such that $r_{ij} = r_{ji} \in R$ lies between –1 and 1 (inclusive). Moreover, $r_{ii} = 1$. Each r_{ij} is the cosine of angle θ between two variates, say x_i and x_j; $i, j \in \{1, 2, ..., m\}$. Such matrices have many applications, particularly in marketing and financial economics as reflected in the works of Chesney and Scott (1989), Heston (1993), Schöbel and Zhu (1999), Tyagi and Das (1999), Xu and Evers (2003), etc. These matrices find frequent applications in risk management and option pricing (Rebonato and Jäckel, 1999). The need to forecast demand for a group of products in order to realize savings by properly managing inventories also requires the use of correlation matrices (Budden et al. 2007).

In many cases, due either to paucity of data/information or dynamic nature of the problem at hand, it is not possible to obtain a complete correlation matrix. Some elements of R are unknown. Then the problem is to obtain a valid complete product moment correlation matrix. Several methods have been suggested to complete a correlation matrix - that is to obtain a valid complete correlation matrix from an incomplete correlation matrix, some of whose elements are unknown. Works of Stanley and Wang (1969), Glass and Collins (1970), Johnson (1980), Olkin (1981), Barett et al. (1989), Helton et al. (1989), Grone et al. (1984), Barett et al. (1998), Laurent (2001), Kahl and Günther (2005) and Budden et al. (2007) etc are notable. Mishra (2007) proposed an algorithm (and provided a Fortran program) that applies the differential evolution method of global optimization to obtain a complete correlation matrix from an incomplete correlation matrix of an arbitrary order.

In many cases, however, the matrix available to the analyst/decision-maker is complete, but it is an invalid (not a positive semi-definite) correlation matrix. There could be many reasons that give rise to such invalid matrices (Mishra, 2004). In such cases, the problem is to obtain a positive semi-definite correlation matrix, R, which, in some sense, is closest to the given invalid matrix, Q. A number of methods have been developed to obtain such nearest correlation matrices. The works of Rebonato and Jäckel (1999), Higham (2002), Anjos et al. (2003), Pietersz and Groenen (2004), Grubisic and Pietersz (2004) and Mishra (2004) are some of them. Of these methods, Mishra (2004) minimizes the maximum (Chebyshev) norm while others minimize the Euclidean, Erhardt-Schmidt or Frobenius norm of the difference matrix Δ = Q-R, where Q is the invalid and R is the valid product moment correlation matrix.

II. The Objective of the Present Paper: Till date, the various methods proposed by different authors are based on majorization, hypersphere decomposition, semi-definite programming, geometric programming or Bergman/von Neumann divergence (Mishra, 2008). In this paper we propose to obtain the nearest valid correlation matrix by the differential evaluation method of global optimization.

III. The Method of Differential Evolution: The differential Evolution (DE) method of Storn and Price (1995) is perhaps the fastest evolutionary computational procedure yielding most accurate solutions to continuous global optimization problems. It consists of three basic steps: (i) generation of (large enough) population with individuals in the m-dimensional space, randomly distributed over the entire domain of the function in question and evaluation of the individuals of the so generated population by finding f(x), where x is the decision variable; (ii) replacement of this current population by a better fit new population, and (iii) repetition of this replacement until satisfactory results are obtained or the given criteria of termination are met.

The strength of DE lays on replacement of the current population by a new population that is better fit. Here the meaning of 'better' is in the Pareto improvement sense. A set S_a is better than another set S_b ***iff :*** (i) *no* $x_i \in S_a$ is inferior to the corresponding member of $x_i \in S_b$; ***and*** (ii) *at least one* member $x_k \in S_a$ is better than the corresponding member $x_k \in S_b$. Thus, every new population is an improvement over the earlier one. To accomplish this, the DE method generates a candidate individual to replace each current individual in the population. A crossover of the current individual and three other randomly selected individuals obtains the candidate individual from the current population. The crossover itself is probabilistic in nature. Further, if the candidate individual is better fit than the current individual, it takes the place of the current individual else the current individual passes into the next iteration (Mishra, 2006).

IV. The Proposed Method to find the Nearest Correlation Matrix: The 'nearest correlation matrix problem' is cast into a constrained minimization problem of differential evolution procedure. The given matrix (of order m) is first checked for positive definiteness (whether eigenvalues, m in number, are all non-negative). If any eigenvalue is found negative, the DE procedure swings into action. A population of N individual candidate vectors of eigenvalues (say $\lambda_i;\ i=1,2,...,N$, each with m elements) is generated by using uniformly distributed non-negative random numbers. With the eigenvectors (say V_0) of the original matrix (Q) and the candidate vectors of eigenvalues ($\lambda_i;\ i=1,2,...,N$) the candidate correlation matrices ($R_i;\ i=1,2,...,N$) are constructed. Each of these candidate correlation matrices (R_i) is checked for unitary principal diagonals, positive semi-definiteness and the trace that must equal m (i.e. the order of the matrix). Large positive penalties are set if any of the restrictions are violated. A difference matrix $\Delta_i = Q - R_i$ is constructed. The appropriate norm of Δ_i is computed. The sum of the norm of Δ_i matrix and the positive penalty (if any) make the objective function that is considered for minimization. An optimum solution has a zero penalty and the minimal value of the appropriate norm. Three alternative specifications of norm may be used: absolute (1), Frobenius (2) or Chebyshev (coded as 99 or lager).

V. Some Examples: Let us take some examples from the extant literature. First, the Q matrix from Rebonato and Jäckel (1999, p.9) and the solutions obtained by them by two methods (hypersphere decomposition and spherical decomposition), as presented in table-1.1. Using our method we have obtained three different solutions (R matrices, nearest to Q of Rebonato and Jäckel) as presented in table-1.2. They have been obtained by minimization of three different norms (absolute, Frobenius and Chebyshev). Minimization of these norms has its own advantages and disadvantages relative to the norm chosen.

Table-1.1: Invalid Correlation Matrix from Rebonato & Jäckel and their Estimated Nearest Correlation Matrices								
Given Invalid Matrix (Q)			Nearest Matrix (R_H) obtained by Hypersphere decomposition			Nearest Matrix (R_S) obtained by Spherical decomposition		
1.00	0.90	0.70	1.00000	0.89458	0.69662	1.00000	0.89402	0.69632
0.90	1.00	0.30	0.89458	1.00000	0.30254	0.89402	1.00000	0.30010
0.70	0.30	1.00	0.69662	0.30254	1.00000	0.69632	0.30010	1.00000

Table-1.2: Valid Nearest Correlation Matrices obtained from Q of Rebonato & Jäckel by Differential Evolution based Method Proposed in this Paper								
Obtained by Minimizing the Absolute Norm			Obtained by Minimizing the Frobenius Norm			Obtained by Minimizing the Chebyshev Norm		
1.0000000	0.8939631	0.6956880	1.0000000	0.8945707	0.6965991	1.0000000	0.8951409	0.6974547
0.8939631	1.0000000	0.3000001	0.8945707	1.0000000	0.3025051	0.8951409	1.0000000	0.3048590
0.6956880	0.3000001	1.0000000	0.6965991	0.3025051	1.0000000	0.6974547	0.3048590	1.0000000
All figures rounded off at the seventh place after decimal.								

The second example is from Higham (2002). His Q matrix and the nearest correlation matrix estimated by him are presented in table-2.1.

Table-2.1: Invalid Correlation Matrix from Higham and his Estimated Nearest Correlation Matrix					
Given Invalid Matrix (Q) by Higham			Nearest Matrix (R) obtained by Higham		
1.00	1.00	0.00	1.00000	0.76069	0.15731
1.00	1.00	1.00	0.76069	1.00000	0.76069
0.00	1.00	1.00	0.15731	0.76069	1.00000

We have estimated nearest correlation matrices by minimizing the aforesaid three norms by the method proposed in this paper. The estimated matrices are presented in table-2.2.

Table-2.2: Valid Nearest Correlation Matrices obtained from Q of Higham by Differential Evolution based Method Proposed in this Paper								
Obtained by Minimizing the Absolute Norm			Obtained by Minimizing the Frobenius Norm			Obtained by Minimizing the Chebyshev Norm		
1.0000000	0.7071068	0.0000000	1.0000000	0.7606931	0.1573081	1.0000000	0.7807764	0.2192235
0.7071068	1.0000000	0.7071068	0.7606931	1.0000000	0.7606931	0.7807764	1.0000000	0.7807764
0.6956880	0.7071068	1.0000000	0.1573081	0.7606931	1.0000000	0.2192235	0.7807764	1.0000000
All figures rounded off at the seventh place after decimal.								

Table-3.1: Al-Subaihi's Given Invalid and Estimated Valid Nearest Correlation Matrices by Differential Evolution based Method Proposed in this Paper									
Given Invalid Matrix (Q)					Estimated Matrix Obtained by Minimizing the Absolute Norm				
1.00	0.50	0.50	0.00	0.00	1.000000000	0.475565731	0.475565731	0.000002657	0.000002657
0.50	1.00	0.84	0.84	0.84	0.475565731	1.000000000	0.840024064	0.798952659	0.798952659
0.50	0.84	1.00	0.84	0.84	0.475565731	0.840024064	1.00000000	0.798952659	0.798952659
0.00	0.84	0.84	1.00	0.84	0.000002657	0.798952659	0.798952659	1.000000000	0.840012746
0.00	0.84	0.84	0.84	1.00	0.000002657	0.798952659	0.798952659	0.840012746	1.000000000

As the third example, we use the invalid correlation matrix (Q) reported in Al-Subaihi (2004). He has obtained the valid matrix, which is grossly inoptimal (Mishra, 2004) and hence we do not feel a necessity to present it here. From his matrix we obtain three valid matrices as presented in tables 3.1 and 3.2.

Obtained by Minimizing the Frobenius Norm					Obtained by Minimizing the Chebyshev Norm				
1.0000000	0.4772712	0.4772712	0.0198652	0.0198652	1.0000000	0.4776281	0.4776281	0.0181106	0.0181106
0.4772712	1.0000000	0.8647105	0.8185024	0.8185024	0.4776281	1.0000000	0.8623712	0.8176281	0.8176281
0.4772712	0.8647105	1.0000000	0.8185024	0.8185024	0.4776281	0.8623712	1.0000000	0.8176281	0.8176281
0.0198652	0.8185024	0.8185024	1.0000000	0.8587108	0.0181106	0.8176281	0.8176281	1.0000000	0.8623697
0.0198652	0.8185024	0.8185024	0.8587108	1.0000000	0.0181106	0.8176281	0.8176281	0.8623697	1.0000000

As the fourth example we have a 7x7 invalid matrix (Q) presented in table 4.1. The matrix is strongly non-positive semidefinite and its elements are inconsistent among themselves. Two of its eigenvalues are negative and substantially large. We present in tables 4.2 through 4.4 the estimated nearest correlation matrix obtained by minimizing absolute, Frobenius and Chebyshev norms. The results have been obtained by the method proposed in this paper.

Table-4.1: A 7x7 Invalid (highly inconsistent) Correlation Matrix (Q)						
1.0	0.3	0.8	0.9	0.6	0.5	0.8
0.3	1.0	0.7	-0.9	-0.8	0.5	0.6
0.8	0.7	1.0	0.3	0.5	0.6	0.9
0.9	-0.9	0.3	1.0	0.5	0.8	0.2
0.6	-0.8	0.5	0.5	1.0	0.9	0.9
0.5	0.5	0.6	0.8	0.9	1.0	-0.8
0.8	0.6	0.9	0.2	0.9	-0.8	1.0

Table-4.2: Estimated Correlation Matrix from Q in Table-4.1 through Minimization of Absolute Norm by Differential Evolution based Method Proposed in this Paper						
1.00000000	0.18737939	0.80000084	0.71327964	0.60000064	0.48595575	0.68101990
0.18737939	1.00000000	0.52901125	-0.49820429	-0.33088472	-0.01500142	0.40163850
0.80000084	0.52901125	1.00000000	0.30000009	0.54025823	0.45169718	0.76823665
0.71327964	-0.49820429	0.30000009	1.00000000	0.65453273	0.57264487	0.12263872
0.60000064	-0.33088472	0.54025823	0.65453273	1.00000000	0.59499803	0.47626442
0.48595575	-0.01500142	0.45169718	0.57264487	0.59499803	1.00000000	-0.05895502
0.68101990	0.40163850	0.76823665	0.12263872	0.47626442	-0.05895502	1.00000000

Table-4.3: Estimated Correlation Matrix from Q in Table-4.1 through Minimization of Frobenius Norm by Differential Evolution based Method Proposed in this Paper						
1.000000000	0.183618795	0.779746208	0.637889805	0.608249605	0.470975692	0.661543723
0.183618795	1.000000000	0.575256419	-0.543424780	-0.368307991	0.011512632	0.378861926
0.779746208	0.575256419	1.000000000	0.252021806	0.483198915	0.432142698	0.744340179
0.637889805	-0.543424780	0.252021806	1.000000000	0.709454952	0.577536122	0.073478552
0.608249605	-0.368307991	0.483198915	0.709454952	1.000000000	0.508116306	0.441058474
0.470975692	0.0115126322	0.432142698	0.577536122	0.508116306	1.000000000	-0.166939808
0.661543723	0.378861926	0.744340179	0.073478552	0.441058474	-0.166939808	1.000000000

Table-4.4: Estimated Correlation Matrix from Q in Table-4.1 through Minimization of Chebyshev Norm by Differential Evolution based Method Proposed in this Paper						
1.000000000	0.168991286	0.508346490	0.664536356	0.228095124	0.272100847	0.490989003
0.168991286	1.000000000	0.454972416	-0.425396613	-0.344769657	0.156909146	0.163178645
0.508346490	0.454972416	1.000000000	0.176233796	0.365688647	0.378941489	0.529777126
0.664536356	-0.425396613	0.176233796	1.000000000	0.264433128	0.325393819	0.020062501
0.228095124	-0.344769657	0.365688647	0.264433128	1.000000000	0.425392097	0.425392221
0.272100847	0.156909146	0.378941489	0.325393819	0.425392097	1.000000000	-0.325391995

0.490989003	0.163178645	0.529777126	0.020062501	0.425392221	-0.325391995	1.000000000

VI. Application of the Proposed Method to Completing a Given Incomplete Matrix: Earlier we have mentioned that in many cases, due either to paucity of data/information or dynamic nature of the problem at hand, it is not possible to obtain a complete correlation matrix. Some elements of R are unknown. Then the problem is to fill in the holes (cells occupied by the unknown elements of the incomplete correlation matrix). This problem has no unique solution and, generally, for every unknown element of the matrix there exists a bewildering large range ($r^H - r_L$) within which any value would make the 'completed' correlation matrix positive semi-definite. One may, therefore, generate a large list of valid correlation matrices (Mishra, 2007), which poses a difficult problem of choosing a particular matrix from among them. However, if the holes may be filed in by some rough guesstimates (guessed estimates) made by the analyst (and the matrix obtained through such subjective completion procedure turns out to be an invalid matrix, Q), one might try to obtain the nearest valid matrix from Q.

The difference between the 'complete the correlation matrix problem' and the one (the nearest correlation matrix problem) described in the earlier sections is that in the 'complete the correlation matrix problem' the known elements are considered as parameters and the unknown elements might be adjusted so that finally the completed correlation matrix is valid. This view results into the multiplicity of solutions of the problem. On the other hand, in the pure 'nearest correlation matrix' problem all the elements (barring the unitary elements in the principal diagonal) are subject to adjustments. However, if one is ready to allow for some adjustments in known elements (barring the unitary elements in the principal diagonal) too, the 'complete the correlation matrix' problem can be converted into the 'nearest correlation matrix' problem.

Take for instance a (valid) 4x4 matrix given in table 5.1 (panel-1). All of its eigenvalues are positive (2.21709657, 0.88487947, 0.734710693, 0.163313225). Now, suppose, we obliterate four of its elements (two in the upper and corresponding two in the lower diagonals) so as to produce an incomplete matrix, given in panel-2 of table-5.1 and guesstimate the holes (obliterated elements) so as to obtain an invalid correlation matrix given in panel-3 of table-5.1.

Now we obtain three estimated valid correlation matrices (given in table-5.2); the first by un-weighted minimization of the maximum (Chebyshev) norm (panel-1); the second by weighted minimization of the maximum norm; and the third by minimization of the weighted absolute norm. Guesstimated elements are assigned zero weights while the known elements are assigned unity weights.

Table-5.1: Transformation of an Incomplete Correlation Matrix into a Nearest Correlation Matrix Problem

A Valid Matrix					Incomplete Matrix					Guestimated Invalid Matrix			
1.00	0.80	0.50	0.30		1.00		0.50	0.30		1.00	1.00	0.50	0.30
0.80	1.00	0.30	0.20			1.00	0.30			1.00	1.00	0.30	0.70
0.50	0.30	1.00	0.15		0.50	0.30	1.00	0.15		0.50	0.30	1.00	0.15
0.30	0.20	0.15	1.00		0.30		0.15	1.00		0.30	0.70	0.15	1.00

Table-5.2: Nearest Correlation Matrix obtained from Incomplete Correlation Matrix Completed by Guesstimates											
By un-weighted min max norm*				By weighted min max norm				By weighted Absolute norm			
1.000000	0.914983	0.448617	0.385017	1.000000	0.879790	0.474272	0.325728	1.000000	0.883317	0.456305	0.319123
0.914983	1.000000	0.354975	0.714983	0.879790	1.000000	0.299798	0.722238	0.883317	1.000000	0.300000	0.711802
0.448617	0.354975	1.000000	0.233779	0.474272	0.299798	1.000000	0.124272	0.456305	0.300000	1.000000	0.150000
0.385017	0.714983	0.233779	1.000000	0.325728	0.722238	0.124272	1.000000	0.319123	0.711802	0.150000	1.000000
Rounded off at the sixth place after decimal. (*) May not be positive definite due to rounding off error											

VII. Concluding Remarks**:** We may draw some conclusions from this exercise. First, the 'nearest correlation matrix problem may be solved satisfactorily by the evolutionary algorithm like the differential evolution method. Other methods such as the Particle Swarm method (the results not presented here) also may be used. It may be so, however, that the solution obtained by such methods might not be strictly optimal, correct up to many (usually greater than 5 or 6) places after the decimal point. In that sense, these methods give only near-optimal results that may be practically acceptable. Secondly, these methods are easily amenable to choice of the norm to minimize. Thirdly, the 'complete the correlation matrix problem' can be solved (in a limited sense) by these methods. Fourthly, one may easily opt for weighted norm or un-weighted norm minimization. Fifthly, minimization of absolute norm to obtain nearest correlation matrices appears to give better results.

Finally, as one may observe, the resulting valid (nearest) correlation matrices are often near-singular and thus they are on the borderline of semi-negativity. One finds difficulty in rounding off their elements even at 6^{th} or 7^{th} place onwards after decimal, without running the risk of making the rounded off matrix non-positive semidefinite. Such matrices are difficult to handle. Nevertheless, it is possible to obtain more robust positive definite valid correlation matrices by constraining the determinant (the product of eigenvalues) of the resulting correlation matrix to take on a value significantly larger than zero. But this can be done only at the cost of a compromise on the criterion of 'nearness.' The method proposed by us does it very well.

Note: The computer program (FORTRAN) for the method proposed in this paper may be obtained from the author on request (contact mishrasknehu@yahoo.com). There are two programs: the one that uses the Differential Evolution and the other that uses the Particle Swarm Optimization. These programs are highly user-friendly and do not need any special instructions. All necessary instructions are given in the program and they pop up while the program is running.

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