Classical and quantum spherical models of spin-glasses: A complete treatment of statistics and dynamics

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A generalization of the usual spherical model of spin-glasses is introduced, and its static and dynamic behavior is fully studied. The generalized model is given the name quantum spherical model to contrast it with the usual model which we call the classical spherical model. The results of the quantum model reduce to those of the classical model in appropriate limits. However, the quantum model shows some new and interesting features. The equilibrium entropy of spin-glasses which is negative at sufficiently low temperatures in the classical model is found positive in the quantum version, and it vanishes at zero temperature. The dynamic properties of quantum and classical spherical models are similar except in the infinite time limit when different equilibrium behavior is obtained as expected. The magnetization, order-parameter, and on-site correlations are found to decay exponentially above the transition temperature, and algebraically at and below it in the long-time limit. A closed-form expression for the dynamic susceptibility valid at all temperatures is also obtained. The algebraic decay predicted by our model may serve as a model form of behavior in the interpretation of experimental relaxation data on spin-glasses.

I. INTRODUCTION

In recent years there has been considerable progress in the theory of spin-glasses. In the most commonly studied model,[1–7] the system is described by N spins — each pair of spins interacting through a random exchange J_{ij} with average value zero and variance $N^{-1}\sigma^2$. The spins can be classical[1–4] or quantum.[6,7] Sherrington and Kirkpatrick[2] (SK) solved the static model for classical spins using the replica trick.[1] Later, Kosterlitz, Thouless, and Jones[3] (KTJ) solved it for classical spherical spins obtaining results similar to those of SK. These solutions, though apparently exact, have some unphysical properties. Most prominently, the entropy at low temperatures turns out to be negative. The existing solutions for quantum spins[6,7] have similar defect. On the other hand, phenomenological[4] and numerical[5] studies show that the model itself is physical and the solutions mentioned above are in error at low temperatures. There are some formal attempts[8] to correct the SK solution but there is no general consensus on these so far.

The dynamics of spin-glass systems with infinite-range random exchange interactions has also been studied recently by several authors.[5,9,10] Kinzel and Fischer[9] (KF) have employed the Glauber model for this purpose and have obtained results valid at and much above the spin-glass transition temperature T_Q. The KF calculation has been improved upon by Kirkpatrick and Sherrington[5] (KS) by taking into account the Onsager reaction field neglected by KF. These authors (KS) have also studied this model by Monte Carlo techniques. Ma and Rudnick[10] (MR) have considered a time-dependent Ginzburg-Landau model (TDGL) and have obtained results for $T \leq T_Q$.

The plan of this paper is as follows. Section II introduces our quantum spherical model of spin-glasses. In Sec. III its statics are solved exactly and a closed-form expression for the free energy is obtained. In Sec. IV the dynamics of the model are studied starting from a phenomenological Langevin-like equation. In Secs. III and IV various contacts are made with the earlier results in appropriate special cases. However, the analysis contained in these sections is more exhaustive, and many new results are obtained. Section V contains some concluding remarks.

II. MODEL

Let us first consider a model of spin-glasses characterized by the Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\sum_{i,j} J_{ij}S_iS_j + \mu\sum_i S_i^2 + \sum_i \frac{P_i^2}{2I} \quad . \tag{2.1}$$

Here S_i and P_i, $i = 1, 2, \ldots, N$, are canonically conjugate classical variables ranging from $-\infty$ to $+\infty$. Following usual nomenclature in the field, the variables S_i will be called spins. The interaction J_{ij} is a random variable with mean zero and variance $N^{-1}\sigma^2$. The first term in (2.1) gives the exchange energy of spins S_i. The second term, with an adjustable parameter μ, is introduced to satisfy a constraint in the model, viz.,

$$\sum_i \langle S_i^2 \rangle = N \quad , \tag{2.2}$$

where the angular brackets denote thermal average. The third term in (2.1) is a kinetic energy term with I as a moment-of-inertia-like parameter.

The Hamiltonian given by (2.1) without the kinetic energy term is called the spherical model of spin-glasses. It is not difficult to see that the inclusion of the kinetic energy term does not affect the properties of the model.[11] We shall therefore call (2.1) including the kinetic energy term as the classical spherical model. We call it classical in order to contrast it with another model in which the variables S_i, and P_i in (2.1) become operators and are required to obey the Heisenberg commutation relation

$$[S_i, P_{i'}] = i\hbar\delta_{ii'} \quad . \tag{2.3}$$

The model (2.1) with the additional requirement (2.3) will be called the quantum spherical model. It should be noted that the quantumness of the model is only in the commutation relation (2.3), and not in the discreteness or vector nature of S_i, P_i. This model has the virtue of being exactly solvable, and showing some interesting differences from the classical spherical model. It is convenient to rewrite H in a generalized Fourier space whose bases are the orthonormalized eigenfunctions of the large random matrix J_{ij}. The matrix J_{ij} can be diagonalized by the transformation,

$$\sum_i J_{ij}\phi_\lambda(i) = J_\lambda \phi_\lambda(j) \quad , \tag{2.4a}$$

$$\sum_i S_i \phi_\lambda(i) = S_\lambda \quad , \tag{2.4b}$$

$$\sum_i P_i \phi_\lambda(i) = P_\lambda \quad , \tag{2.4c}$$

with

$$[S_\lambda, P_{\lambda'}] = i\hbar\delta_{\lambda\lambda'} \quad . \tag{2.5}$$

The distribution of J_λ is given by[12]

$$P(J_\lambda) = \begin{cases} (N/2\pi\sigma^2)[(2\sigma)^2 - J_\lambda^2]^{1/2} & \text{for } J_\lambda^2 \le (2\sigma)^2 \\ 0, & \text{otherwise} \quad . \end{cases} \tag{2.6}$$

We obtain

$$\mathcal{H} = \frac{1}{2}\sum_\lambda (2\mu - J_\lambda) S_\lambda^2 + \sum_\lambda \frac{P_\lambda^2}{2I} \quad . \tag{2.7}$$

It is evident that the Hamiltonian is a sum of harmonic oscillators. This fact simplifies the analysis considerably.

III. STATICS

The Hamiltonian (2.7) diagonalizes to

$$\mathcal{H} = \sum_\lambda (n_\lambda + \tfrac{1}{2})\hbar w_\lambda, \quad n_\lambda = 0, 1, 2, \ldots \quad , \tag{3.1}$$

where,

$$w_\lambda = \left(\frac{(2\mu - J_\lambda)}{I} \right)^{1/2} \quad . \tag{3.2}$$

The partition function is given by

$$\begin{aligned} Z &= \sum_{\lambda, n_\lambda} e^{-\beta H} \\ &= \prod_\lambda \sum_{n_\lambda} \exp[-\beta(n_\lambda + \tfrac{1}{2})\hbar w_\lambda] \quad , \end{aligned} \tag{3.3}$$

where $\beta = (k_B T)^{-1}$. Summation over n_λ yields

$$Z = \prod_\lambda [2\sinh(\tfrac{1}{2}\beta\hbar w_\lambda)]^{-1} \quad . \tag{3.4}$$

The free energy per spin is

$$\begin{aligned} \mathcal{F} &= -N^{-1} k_B T \ln Z \\ &= N^{-1} k_B T \sum_\lambda \ln[2\sinh(\tfrac{1}{2}\beta\hbar w_\lambda)] \quad . \end{aligned} \tag{3.5}$$

The spherical constraint (2.2) becomes

$$\left.\frac{\partial \mathcal{F}}{\partial \mu}\right|_T = 1$$

or

$$N^{-1} \sum_\lambda \left(\frac{\hbar}{I w_\lambda} \right) \{ \tfrac{1}{2} + [\exp(\beta\hbar w_\lambda) - 1]^{-1} \} = 1 \quad . \tag{3.6}$$

To understand the nature of the phase transition the behavior of μ as a function of T has to be studied. From (3.6), we see that at high temperatures μ behaves as $\frac{1}{2}k_B T$ with quantum corrections, and it decreases as T decreases till it hits σ. Let T_Q be the temperature when μ equals σ. For $T < T_Q$, μ sticks at σ as usual to keep the free energy real. The transition temperature T_Q is determined by the equation,

$$\frac{4g}{3\pi} + \frac{8g}{\pi} \int_0^1 \kappa(1-\kappa^2)^{1/2} [\exp(2\beta_Q \sigma g \kappa) - 1]^{-1} d\kappa = 1 \quad , \tag{3.7}$$

where $g = (\hbar^2/I\sigma)^{1/2}$ and $\beta_Q = (k_B T)^{-1}$. In writing (3.7), the sum in (3.6) has been converted to an integral using (2.6). When g is greater than $3\pi/4$, Eq. (3.7) has no solution for positive T_Q, meaning that there is no phase transition. When g is equal to $3\pi/4$, $T_Q = 0$. For g less than $3\pi/4$, there is a finite T_Q which depends on $\hbar$, I, and σ.

Thermodynamic quantities like the entropy, and the specific heat can be obtained from (3.5) using

(3.7). The entropy below the transition temperature is particularly interesting. We obtain

$$S = N^{-1} k_B \sum_\lambda \{ \tfrac{1}{2}\beta\hbar w_\lambda \coth(\tfrac{1}{2}\beta\hbar w_\lambda) - \ln[2\sinh(\tfrac{1}{2}\beta\hbar w_\lambda)] \} \quad . \tag{3.8}$$

It is easy to check that the quantity in curly brackets, and therefore S, is always positive. For temperatures below the transition temperature, it is convenient to write the free energy and entropy in an integral form as

$$\mathfrak{F} = \frac{16 k_B T}{\pi A^4} \int_0^A (A^2-\kappa^2)^{1/2} \kappa^2 \ln(2\sinh\kappa)\, d\kappa \tag{3.9}$$

and

$$S = \frac{16 k_B}{\pi A^4} \int_0^A (A^2-\kappa^2)^{1/2} \kappa^2 [\kappa \coth\kappa - \ln(2\sinh\kappa)]\, d\kappa \quad , \tag{3.10}$$

where $A = \beta g \sigma$. The leading term in (3.10) as $T \to 0$, is

$$S = \frac{64}{9\pi} k_B D(2A) \quad , \tag{3.11}$$

where $D(2A)$ is the usual Debye function. Clearly as T goes to zero,

$$S = -\tfrac{8}{45} k_B \left(\frac{\pi k_B T}{g \sigma} \right)^3 + \cdots \quad . \tag{3.12}$$

The specific heat also goes like T^3 as $T \to 0$. It should be noted that this result comes essentially from using the semicircular law (2.6). It does not rule out that a model of spin-glasses based on short-range random interactions on a lattice would give a linear specific heat as observed in experiments.

To study the order parameter and susceptibilities we introduce into (2.1) a term $\sum_i H_i S_i$ where H_i is the magnetic field at the site i (in energy units). The problem now becomes that of coupled, shifted, harmonic oscillators, and can be solved similarly. The sole effect is to add a term

$$-(4N)^{-1} \sum_\lambda (\mu - \tfrac{1}{2} J_\lambda)^{-1} H_\lambda^2 \quad ,$$

where H_λ's are the components of H_i in the $\phi_\lambda(i)$ representation. The constraint equation (3.6) is changed to

$$N^{-1} \sum_\lambda \left(\frac{\hbar}{I w_\lambda} \right) \{ \tfrac{1}{2} + [\exp(\beta\hbar w_\lambda) - 1]^{-1} \} = 1 - Q^2 \quad , \tag{3.13}$$

where

$$Q^2 = N^{-1} \sum_\lambda \langle S_\lambda \rangle^2$$

is the order parameter, and

$$\langle S_\lambda \rangle = \frac{H_\lambda}{2\mu - J_\lambda} \quad .$$

The "shattered" and normal susceptibilities are, respectively,

$$\chi_\lambda = (2\mu - J_\lambda)^{-1} \quad ,$$

$$\chi = \sum_\lambda (2\mu - J_\lambda)^{-1} \quad .$$

For $T > T_Q$ in zero field, $\mu > \sigma$, so $\langle S_\lambda \rangle = 0$, which implies $Q = 0$. For $T \leq T_Q$, μ sticks at σ, as it does in the ordinary ferromagnetic case and in the classical case.[3] We find

$$\chi_{\lambda_{\max}} \sim \left(\frac{T - T_Q}{T_Q} \right)^{-2} \quad \text{for } T \gtrsim T_Q$$

$$\to \infty \quad \text{for } T \leq T_Q \quad ,$$

$$\chi \sim \sigma^{-1} \left[1 - \text{const} \left(\frac{T - T_Q}{T_Q} \right) \right] \quad \text{for } T \gtrsim T_Q$$

$$= \sigma^{-1} \quad \text{for } T \leq T_Q \quad .$$

Connections with the classical spherical model can be made by taking the limit $\hbar \to 0$. The critical temperature T_Q obtained from (3.7) becomes σ / k_B in agreement with the KTJ result. The quantum corrections to this value are of order $\hbar^2$, and tend to decrease T_Q. The leading term in entropy below T_Q, in the limit $\hbar \to 0$ becomes

$$S = -k_B \ln(g\sigma / k_B T) + \cdots \quad . \tag{3.14}$$

This also agrees with the KTJ result. Clearly as $T \to 0$ the entropy becomes negative, and diverges logarithmically. This is unphysical; however, it is a usual phenomenon in classical-spherical-model calculations.

The behavior of the susceptibilities agrees with the KTJ classical results for $T \leq T_Q$. For $T > T_Q$, however, the quantum behavior agrees with the classical one only in the vicinity of T_Q.

IV. DYNAMICS

The Heisenberg equation of motion for S_λ is

$$\dot{S}_\lambda = (i\hbar)^{-1} [S_\lambda, \mathcal{H}] \quad . \tag{4.1}$$

The equation of motion for P_λ is written semiphenomenologically as

$$\dot{P}_\lambda = (i\hbar)^{-1} [P_\lambda, \mathcal{H}] - \beta\Gamma (i\hbar)^{-1} [S_\lambda, \mathcal{H}] + f_\lambda(t) \quad . \tag{4.2}$$

The first term on the right-hand side (rhs) is all we would have if the system were isolated. However, it is assumed to be in contact with a heat reservoir at temperature T. The second and third terms on the rhs represent the effect of this thermal contact. These terms can be understood as follows. The term $\dot{P}_\lambda$ on the left-hand side (lhs) is a force term according to Newton's second law. So the second, and third terms on the rhs have to be forcelike too. The second term is written as a general velocity-proportional viscous-force term. Γ is a phenomenological parameter and $(i\hbar)^{-1}[S_\lambda,H]$ is the velocity $\dot{S}_\lambda$. The term $f_\lambda(t)$ represents a random force arising from the thermal fluctuations. Its time average and correlation is given by the equations

$$\langle f_\lambda(t)\rangle = 0 \ , \tag{4.3}$$

$$\langle f_\lambda(t) f_{\lambda'}(t')\rangle = 2\Gamma\delta_{\lambda\lambda'}\delta(t-t') \ . \tag{4.4}$$

The quantity Γ on the rhs of (4.4) is the same Γ as in (4.2) by virtue of the Einstein relation. Equations (4.1) and (4.2) can be greatly simplified by using (2.1) and (2.3). All λ modes decouple, giving

$$\dot{S}_\lambda = P_\lambda / I$$

and

$$\dot{P}_\lambda = -(2\mu - J_\lambda)S_\lambda - \frac{\beta\Gamma}{I}P_\lambda + f_\lambda(t) \ .$$

The solution of the P_λ equation is

$$P_\lambda(t) = \int_{-\infty}^{t} [-(2\mu - J_\lambda)S_\lambda(t') + f_\lambda(t')] \times \exp[-\beta I^{-1}\Gamma(t-t')]\, dt' \ .$$

This solution is substituted in the S_λ equation and the terms that vary rapidly in the S_λ equation are neglected, i.e., $S_\lambda(t)$ is assumed nearly constant over time periods of the order $(k_B T/\Gamma)$. This is the usual procedure of elimination of fast modes. It yields

$$\dot{S}_\lambda(t) = -(k_B T/\Gamma)(2\mu - J_\lambda)S_\lambda(t) + f'_\lambda(t) \ , \tag{4.5}$$

where

$$f'_\lambda(t) = I^{-1}\int_{-\infty}^{t} dt' f_\lambda(t') \exp[-\Gamma(t-t')/k_B T I] \ .$$

It is easily verified,

$$\langle f'_\lambda(t)\rangle = 0 \tag{4.6}$$

and

$$\langle f'_\lambda(t) f'_{\lambda'}(t')\rangle = 2\Gamma'\delta_{\lambda\lambda'}\delta(t-t') \tag{4.7}$$

in the limit $\Gamma/k_B T I \to \infty$, where $\Gamma' = (k_B T)^2/\Gamma$. The limit $\Gamma/k_B T$ going to infinity should be understood to mean that we are interested in time periods much longer than $\Gamma/(k_B T)^2$. Equations (4.5)–(4.7) are the starting points of our analysis of dynamics. It is not difficult to see that the dynamical equations for the classical spherical model also have the same form as Eqs. (4.5)–(4.7). It is a property of the coupled harmonic oscillator nature of our model that its dynamics is the same in classical and quantum versions. Of course, the long-time equilibrium properties are distinct.

In the following we choose $\Gamma' = 1$ for convenience. Equation (4.5) can be easily solved to give

$$\langle S_\lambda(t)\rangle = \langle S_\lambda(\infty)\rangle + a_\lambda \exp[-\beta(2\mu - J_\lambda)t] \ , \tag{4.8}$$

where $a_\lambda = \langle S_\lambda(0)\rangle - \langle S_\lambda(\infty)\rangle$ and μ is given by

$$\kappa \equiv 2\beta(\mu - \sigma) = \begin{cases} (1-\beta\sigma)^2 & \text{for } \beta < \sigma^{-1} \\ 0, & \text{otherwise} \ . \end{cases} \tag{4.9}$$

For the correlation one obtains

$$\langle S_\lambda(0) S_\lambda(t)\rangle = \langle S_\lambda(0) S_\lambda(\infty)\rangle + b_\lambda \exp[-\beta(2\mu - J_\lambda)t] \ , \tag{4.10}$$

where

$$b_\lambda = \langle S_\lambda^2(0)\rangle - \langle S_\lambda(0) S_\lambda(\infty)\rangle \ .$$

It should be noted that while $\langle S_\lambda(0)\rangle$ in (4.8) is an arbitrary initial value, the quantity $\langle S_\lambda^2(0)\rangle$ in (4.10) is not. The value of $\langle S_\lambda^2(0)\rangle$ is related uniquely to the equilibrium susceptibility of the system via the fluctuation-dissipation theorem.[13] We now present our results.

1. *Magnetization*

We define $M(t) = \sum_i \langle S_i(t)\rangle$. Clearly, for the interactions given by (2.6), $M(\infty) = 0$. However, if the system is given a nonzero initial magnetization $M(0)$, it relaxes as follows:

$$M(t) = M(0)(\beta\sigma t)^{-1} I_1(2\beta\sigma t)\exp(-2\beta\mu t) \ ,$$

where I_1 denotes a modified Bessel function in the standard notation. For long times it gives

$$M(t) \sim M(0)(\beta\sigma t)^{-3/2}\exp(-\kappa t) \ .$$

In view of (4.9), $M(t)$ decays exponentially above $k_B T_Q = \sigma$, and as $t^{-3/2}$ at and below it.

2. *Order parameter*

An Edwards-Anderson-like[1] order parameter may be defined as $Q^2(t) = \sum_i \langle S_i\rangle^2$. In contrast with the magnetization, $Q^2(\infty)$ is $1-(T/T_Q)$ for $T < T_Q$ and zero otherwise. We get

$$Q^2(t) = Q^2(\infty) + [Q^2(0) - Q^2(\infty)] \times (2\beta\sigma t)^{-1} I_1(4\beta\sigma t)\exp(-4\beta\mu t) \ .$$

Again for long times $Q^2(t)$ relaxes to its equilibrium value exponentially above T_Q, and as $t^{-3/2}$ at and below it.

3. On-site correla‹ ›ns

The expression $G(t)=\sum_i \langle S_i(0)S_i(t)\rangle$ gives the on-site unequal time correlation. In calculating this quantity, an account has to be taken of the fluctuation dissipation theorem which dictates that

$$\langle S_\lambda^2(0)\rangle=\beta^{-1}\chi_\lambda(w=0)=[\beta(2\mu-J_\lambda)]^{-1}\ , \quad (4.11)$$

where $\chi_\lambda(w)$ is the dynamic shattered susceptibility of the system. We obtain

$$G(t)=G(\infty)+\sum_{\lambda\neq\lambda_m}[\beta(2\mu-J_\lambda)]^{-1}\times\exp[-\beta(2\mu-J_\lambda)t]\ , \quad (4.12)$$

where $G(\infty)=\beta^{-1}\chi_{\lambda_m}(0)$. For $T\leq T_Q$, the integral in (4.12) yields

$$G(t)=G(\infty)+N(\beta\sigma)^{-1}[I_0(2\beta\sigma t)+I_1(2\beta\sigma t)]\times\exp(-2\beta\sigma t)\ . \quad (4.13)$$

For $T>T_Q$, the sum in (4.12) can be evaluated in the long-time limit and gives

$$G(t)=G(\infty)+N(\beta\sigma)^{-3/2}\times[(\pi t)^{-1/2}\exp(-\kappa t)-\sqrt{\kappa}\,\mathrm{erf}(\sqrt{\kappa t})]\ . \quad (4.14)$$

For the long-time decay, we obtain from (4.13) and (4.14), respectively, $G(t)\sim t^{-1/2}$ at and below T_Q and $G(t)\sim t^{-3/2}\exp(-\kappa t)$ above T_Q. This behavior agrees with the calculations of MR, KS, and KF in the appropriate cases. One should note that the general result (4.14) has a crossover behavior. As one approaches T_Q from above, one has to wait a longer and longer time $t\sim\kappa^{-1}\sim(T-T_Q)^{-2}$ to see the exponential decays characteristic of the paramagnetic phase.

4. Dynamic susceptibility

The dynamic susceptibility $\chi(w)$ is obtained from the correlation function by the use of the fluctuation-dissipation theorem

$$\chi(w)=\chi'(w)+i\chi''(w)=\chi(0)-iw\beta\int_0^\infty dt\, e^{-iwt}\langle S_i(0)S_i(t)\rangle\ .$$

We obtain

$$\chi'(w)=(2\beta\sigma^2)^{-1}[2\beta\mu-(A+B)^{1/2}]\ , \quad (4.15a)$$

$$\chi''(w)=(2\beta\sigma^2)^{-1}[w-(A-B)^{1/2}]\ , \quad (4.15b)$$

where for $T>T_Q$,

$$\mu=(2\beta)^{-1}(1+\beta^2\sigma^2)\ ,$$
$$A=\tfrac{1}{2}\{[(1+\beta\sigma)^4+w^2][(1-\beta\sigma)^4+w^2]\}^{1/2}\ ,$$
$$B=\tfrac{1}{2}[(1-\beta^2\sigma^2)^2-w^2]\ ,$$

and for $T\leq T_Q$,

$$\mu=\sigma\ ,$$
$$A=\tfrac{1}{2}(w^4+16\beta^2\sigma^2w^2)^{1/2}\ ,$$
$$B=-\tfrac{1}{2}w^2\ .$$

In the high-frequency limit, Eqs. (4.15a) and (4.15b) give $\chi(w)\sim\beta(1+\beta^2\sigma^2)w^{-2}-i\beta w^{-1}$ for $T>T_Q$, and $\chi(w)\sim 2\beta^2\sigma w^{-2}-i\beta w^{-1}$ for $T\leq T_Q$. In the low-frequency limit, they give, for $T>T_Q$,

$$\chi(w)-\chi(0)\sim-\beta(1-\beta^2\sigma^2)^{-2}[w^2+iw(1-\beta^2\sigma^2)]\ ,$$

and for $T\leq T_Q$,

$$\chi(w)-\chi(0)\sim-(2\beta\sigma^2)^{-1}w^{1/2}[1+i(2\beta\sigma)^{1/2}]\ .$$

It can be easily seen that the low-frequency dynamic susceptibility reveals a crossover similar to the one mentioned earlier in connection with the on-site correlations.

V. CONCLUDING REMARKS

The analysis presented in this paper offers a complete description of spin-glass statics and dynamics at a mean-field level. It agrees with the previous calculations and also fills some gaps in these. The entropy in our quantum model is always positive, and vanishes as T^3 in the zero-temperature limit. This also predicts a T^3 specific heat at low temperatures. However, the observed specific heat in most spin-glass-like systems is linear in T in the low-temperature region. This may be due to the spherical constraint, the mean-field nature of the calculation, or the use of the semicircular law. It is possible that a model of short-range random interactions will show a linear specific heat. The prediction of algebraic decays of various dynamic quantities at and *below* the transition temperature had not been noted before. It, however, supports the point brought out by Thouless, Anderson, and Palmer that in the spin-glass system the fluctuations remain anomalously large even below the transition temperature. This may be related also to the experimental situation which is somewhat uncertain.[14,15] If the fluctuations are anomalous below the transition temperature, then of course the experiments will not show either a very sharp order-parameter transition, or clean results below the transition.

Finally, we wish to comment on the usefulness of the spherical constraint in studying spin-glasses. The fact that it makes the model solvable is obvious. The solution, however, does not suffer from some pathologies that the spherical constraint produces in the ferromagnetic case. In that case the constraint is too severe. Due to the translational invariance the overall constraint (2.2), imposes a constraint on each individual spin, namely $\langle S_i^2 \rangle = 1$ for each i. In the spin-glass case, however, this is not so. Only the system as a whole has to satisfy Eq. (2.2). This permits more freedom to the system, as compared with the ferromagnetic case, to arrange itself in the lowest free-energy state.

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