ARMENDARIZ MODULES AND RINGS
A BRIEF SURVEY

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CERTIFICATE

I certify that the dissertation entitled ‘ARMENDARIZ MODULES AND RINGS: A BRIEF SURVEY ‘ submitted by Ms Kongkamaya Rymbai in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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DECLARATION

I, Kongkamaya Rymbai, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

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PREFACE

Let D be an integral domain, and let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_j^n b_j x^j$ be polynomials with coefficients in D. Suppose that $f(x)g(x) = 0$. Using the fact that the polynomial ring $D[x]$ is also an integral domain, we deduce that either $f(x) = 0$ or $g(x) = 0$. It follows that for each coefficient a_i of $f(x)$ and b_j of $g(X)$ we have $a_i b_j = 0$.(Of course, the converse assertion, namely, $a_i b_j = 0$ for each i and each j implies $f(x)g(x) = 0$, always holds, i.e., over all rings.) It was shown by E.Armendariz [EPA] that reduced rings, i.e., rings with no nonzero nilpotent elements, have a similar property.In order to study additional classes of rings which satisfy a similar condition, the notion of an Armendariz ring was introduced by M.B.Rege and Sima Chhawchharia in [MBR & SC]. Thus an associative ring R is called an *Armendariz* ring if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0 \ \forall \ i, j$. Using Nagata's method of idealisation (trivial extensions) M.B.Rege and Sima Chhawchharia gave a number of examples of Armendariz rings and non-Armendariz rings. Further results on Armendariz rings were obtained by Anderson and Camillo in [DDA & VC].

In this dissertation we have attempted to carry out a brief survey of the work done in the area of Armendariz rings and related classes of rings and modules.

It is well-known that if R is a reduced ring then every idempotent in R is central, i.e., R is abelian. A result implicit in the proof of one of the theorems in [DDA & VC] is the fact that all Armendariz rings are abelian. Anderson and Camillo also studied (in the commutative situation) the relationship of these rings with the class of Gaussian rings, which had beeen studied earlier by Tsang, Gilmer and others. Basic results concerning Armendariz rings and related rings have been collected in the first chapter. ?

A ring R is called $semi-commutative$ if whenever $a, b \in R$ satisfy $ab = 0$ then $acb = 0$ for each $c \in R$; R is $reversible$ if whenever elements $a, b \in R$ satisfy $ab = 0$ we have $ba = 0$ It is easy to see that reduced rings are reversible, reversible rings are semi-commutative, and semi-commutative rings are abelian. In view of this, it is natural to ask for examples of Armendariz rings which are not semi-commutative. Such examples were furnished by Y.Lee, N.K.Kim and Agata Smoktunowicz in [YL& NKK & AS]. They also proved additional results about the relationships between Armendariz rings and semi-commutative rings.In [NKK & YL:2], Kim and Lee studied extensions of reversible rings and proved some results concerning their relationship with Armendariz rings. We survey results obtained by these authors in the third chapter.

A number of well-known ring theoretic properties have natural extensions to modules and it is often fruitful to study these extensions. In this spirit a study of semi-commutative modules and Armendariz modules was carried out by Buhphang and Rege in [AMB & MBR]. Their results have been surveyed in section 2.2 (devoted to Armendariz modules) and section 2.3 (semi-commutative modules).

After the initial results from [MBR & SC] and [DDA & VC], a comprehensive study of the notion of Armendariz ring was carried out during 1999 - 2005. To quote from the introduction to [TKL & YZ], ' The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring R and the annihilators of the polynomial ring $R[x]$ .. The reason behind these is the fact that Armendariz rings are precisely the rings R for which there is a natural bijection between the set of annihilators of R and the set of annihilators of $R[x]$'

After the class of Armendariz rings had been extensively studied (by M.B.Rege, S.Chhawchharia, D.D.Anderson,V.Camillo, Y.Lee, N.K.Kim, C. Huh, A.Smoktunowicz, T.-K.Lee, Y.Zhou and others - see the References), a number of authors introduced and studied analogues of the Armendariz property. Prominent among these analogues are the classes of weak Armendariz rings, quasi-Armendariz rings and Armendariz(PS) rings (studied by T.-K.Lee, T.L.Wong, Y.Hirano, A.M.Buhphang, M.B.Rege and others). These concepts have module analogues as well. Basic properties and examples of some of these analogues are recorded in the fourth chapter.

CONTENTS

Chapter 1

Basic definitions and results

In this chapter we record some basic definitions and notations used in this dissertation. We also recall some well-known results needed by us.For unexplained concepts and results we refer to the books by Anderson-Fuller [AF], Stenström [S] and Lambek [L].

There are two sections in this chapter dealing with basic concepts in (1) rings, and (2) modules.

1.1 Rings

By a *ring* we mean an associative ring with an identity element. Subrings and ring homomorphisms are unitary. The term 'rng' may occasionally be used to denote 'a ring which may not have an identity element'. (Integral) domains need not be commutative.

All our left-sided (resp., right-sided) concepts and results have right-sided (resp., left-sided) counterparts. They may not be stated explictly.

1.1.1 Notation. The letter R denotes a ring and D denotes a domain (in particular, a division ring).

1.1.2 Definition. An element a of a ring R is called an *idempotent* element if $a^2 = a$.

1.1.3 Notation. We let $I(R)$ denote the set of all idempotent elements of R.

1.1.4 Definition. An element a of a ring R is called a *nilpotent* element if $a^n = 0$ for some positive integer n.

1.1.5 Notation. We let $Nil(R)$ denote the set of all nilpotent elements of R.

1.1.6 Definition. A ring R is called a *reduced* ring if it has no non-zero nilpotent elements.

1.1.7 Proposition. *For a ring R, the following conditions are equivalent.*

(1) R is reduced. ✓

(2) Whenever $a \in R$ satisfies $a^2 = 0$, then $a = 0$. ✓

1.1.8 Notation. For a ring R we let $C(R)$ denote the centre of R.

1.1.9 Definition. A ring R is called an *abelian ring* if all idempotent elements of R are central i.e. $I(R) \subseteq C(R)$.

1.1.10 Definition. A ring R is called a (*vonNeumann*) *regular* ring if given $a \in R$, there exists $b \in R$ such that $aba = a$.

1.1.11 Examples. If D is a division ring, then D is regular. The endomorphism ring of a vector space is always regular.

1.1.12 Proposition. *Let R be a regular ring. Then the following statements are equivalent.*

(1) R is abelian. ✓

(2) R is reduced. ✓

Proof. (1) $\Rightarrow$ (2) Let $a \in R$ satisfy $a^2 = 0$. By hypothesis there exists $b \in R$ such that $aba = a$. This implies that $(ba)^2 = ba \in I(R)$. Since R is abelian we get $ba \in C(R)$, and this gives $a = aba = baa = ba^2 = 0$. Hence R is reduced.

(2) $\Rightarrow$ (1) Suppose R is reduced. Let $e \in I(R)$, then $(1 - e) \in I(R)$. Now we have $(1 - e)e = 0$. Since R is reduced and $[ex(1 - e)]^2 = 0$ we get $[ex(1 - e)] = 0$. This implies

$$ex = exe \ \forall \ x \in R \qquad (*)$$

Similarly we have $(1 - e)xe = 0$ i.e.

$$xe = exe \ \forall \ x \in R \qquad (**)$$

Therefore from (*) and (**) we get $ex = xe \ \forall \ x \in R$.Hence R is abelian. □

1.1.13 Notation. The letter $\mathbb{H}$ denotes the division ring of real quaternions, $\mathbb{Z}$ the ring of integers and $\mathbb{Q}$ the field of rational numbers.

1.2 Modules

1.2.1 Definition. Let R be a ring. By a left R-module we mean a set M equipped with a binary operation $+$ satisfying the conditions:

(1) $(M, +)$ is an abelian group.

(2) There exists a map $. : R \times M \to M$, known as the external b.o of R on M (or the scalar multiplication), such that the following hold.

(M1) $r.(m_1 + m_2) = r.m_1 + r.m_2$, $\forall\ r \in R, m_1, m_2 \in M$.

(M2) $(r_1 + r_2).m = r_1.m + r_2.m, \forall\ r_1, r_2 \in R, m \in M$.

(M3) $(rs).m = r.(s.m)$, $\forall\ r, s \in R, m \in M$.

(M4) $1.m = m$, $\forall\ m \in M$.

1.2.2 Notation. The notation ${}_RM$ shall often be used to denote the left R-module M. We shall also simplify $r.m$ (the notation for the scalar multiplication of $r \in R$ with $m \in M$) as rm.

Unless otherwise mentioned, by an R-module we mean a left R-module.

1.2.3 Remark. A right R-module M is similarly defined as an additive abelian group in which there exists a scalar multiplication $. : M \times R \to M$ satisfying

(M1)$'$$(m_1 + m_2).r = m_1.r + m_2.r\ \forall\ m_1, m_2 \in M, r \in R$.

(M2)$'$$m.(r_1 + r_2) = m.r_1 + m.r_2\ \forall\ m \in M, r_1, r_2 \in R$.

(M3)$'$$m.(rs) = (m.r).s\ \forall\ m \in M, r, s \in R$.

(M4)$'$$m.1 = m$, $\forall\ m \in M$.

1.2.4 Notation. M_R shall often denote the right R-module M. We shall again simplify $m.r$ (the notation for the scalar multiplication of $r \in R$ with $m \in M_R$) to mr.

1.2.5 Definition. A module M which is both a left and a right R-module and satisfies $(rm)r' = r(mr')\ \forall\, r, r' \in R$ and $m \in M$ is called a *left $R-$, right $R-$ bimodule* (abbreviated to *$R-$ bimodule*)

1.2.6 Definition. Let M,M' be left R-modules. A map $\theta : M \rightarrow M'$ is called an *$R-$ module homomorphism* or an *$R-$ homomorphism* or an *$R-$ linear map* if

(1) $\theta(m_1 + m_2) = \theta(m_1) + \theta(m_2)\ \forall\ m_1, m_2 \in M$.

(2) $\theta(rm) = r\theta(m),\ \forall\ r \in R, m \in M$.

1.2.7 Definition. An R-module P is *projective* if given an exact sequence $\beta : M \rightarrow N \rightarrow 0$ (β is onto) and an R-homomorphism $g : P \rightarrow N$ there exists an R-homomorphism $f : P \rightarrow M$ such that $\beta \circ f = g$.

1.2.8 Definition. A subset B of an R-module M is a *basis* for M if B generates M and B is linearly independent.

1.2.9 Definition. An R-module M is free if it has a basis.

1.2.10 Remarks. If M is a free R-module, then M is projective.

Let M be a free module over a ring R. Then M is isomorphic to $\bigoplus_I R$, for some indexing set I; the converse also holds.

A module P over a ring R is projective iff P is isomorphic to a direct summand of a free R-module.

1.2.11 Definition. Let D be a commutative domain and let M be a D-module. The set $T(M) = \{m \in M | \ \exists r \in D, r \neq 0 \ \textit{such that}\ rm = 0\}$ is a submodule of M called the *torsion submodule* of M.

1.2.12 Definition. Let D be a domain. A left D-module M is said to be torsion free if for $m \in M$ $m \neq 0$, and $r \in D$ $r \neq 0$ we have $rm \neq 0$.

1.2.13 Remark. If M is a free module over a domain D, then M is torsion free. For each module M over a commutative domain D the module $M/T(M)$ is torsion free .

Chapter 2

Basic results on Armendariz rings and Armendariz modules

The notion of an Armendariz ring was introduced by M.B.Rege and Sima Chhawchharia in [MBR & SC]. Further results on Armendariz rings were obtained by Anderson and Camillo in [DDA & VC]; they also studied (in the commutative situation) the relationship of these rings with the class of Gaussian rings, which had beeen studied earlier by Tsang, Gilmer and others.Armendariz modules were studied by A.M.Buhphang and M.B.Rege in [AMB & MBR].

After the initial results from [MBR & SC] and [DDA & VC], a comprehensive study of the notion of Armendariz ring was carried out during 1999 - 2005. To quote from the introduction to [TKL & YZ], ‘ The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring R and the annihilators of the polynomial ring $R[x]$.... The reason behind these is the fact that Armendariz rings are precisely the rings R for which there is a natural bijection between the set of annihilators of R and the set of annihilators of $R[x]$' (This last fact was noted by Hirano[YH].)

In this chapter we record basic properties of Armendariz rings, Armendariz modules and related rings and modules following various authors. Some results in this chapter are new.

2.1 Armendariz rings

This section is devoted to the study of the basic properties and examples of Armendariz rings.

2.1.1 Definition. Let R be a ring. Then R is called an *Armendariz* ring if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[X]$ satisfy $f(x)g(x) = 0$ then $a_i bj = 0 \ \forall \ i, j$

2.1.2 Examples. (a) Let D be an integral domain. Then D is an Armendariz ring

Proof : Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$. Since $D[x]$ is a domain, we have $f(x) = 0$ or $g(x) = 0$. Hence $a_i b_j = 0 \ \forall \ i, j$.

(b) Let R_1, R_2 be two Armendariz rings. Then $R_1 \times R_2$ is Armendariz

Proof : Let $f(x), g(x) \in R_1[x] \times R_2[x]$. Then $f(x) = (f_1(x), f_2(x)) = (\sum a_{i1} x^i, \sum a_{i2} x^i), g(x) = (g_1(x), g_2(x)) = (\sum b_{j1} x^j, \sum b_{j2} x^j)$ where $f_1(x), g_1(x) \in R_1[x], f_2(x), g_2(x) \in R_2[x]$.

Assume $f(x)g(x) = 0$. Then this implies that $f_1(x)g_1(x) = 0, f_2(x)g_2(x) = 0$. Since R_1, R_2 are Armendariz rings, we have $a_{i1} b_{j1} = 0, a_{i2} b_{j2} = 0 \ \forall \ i, j$. Therefore $(a_{i1}, a_{i2})(b_{j1}, b_{j2}) = 0$. Thus $R_1 \times R_2$ is Armendariz.

(c) Every direct product of Armendariz rings is Armendariz.

(d) Subrings (also, subrngs) of Armendariz rings are Armendariz.

Proof : Let R_0 be a subring of an Armendariz ring R and let $f(x) = \sum a_i x^i, g(x) = \sum b_j x^j \in R_0[x]$ satisfy $f(x)g(x) = 0$. Since $f(x), g(x) \in R[x]$ it implies that $a_i b_j = 0 \ \forall \ i, j$ (since R is Armendariz).

Therefore R_0 is also Armendariz.

(e) A direct sum of Armendariz rings is Armendariz (as a ring without identity)

This follows from (c) and (d)

(f) Let F be a field (or, more generally a nonzero ring). Take $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, then R is not an Armendariz ring.

Consider the two polynomials $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x \in R[x]$. Then we have $f(x)g(x) = 0$, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$.

(g) Let R be a nonzero ring. Then the $n \times n$ upper triangular matrix ring over R is not Armendariz for $n \geq 2$.

2.1.3 Proposition. *Let R be a reduced ring. For any $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ satisfying $f(x)g(x) = 0$ we have $a_i b_j = 0 \ \forall\ i, j$.*

Proof. Suppose $f(x)g(x) = 0$ (without loss of generality, we assume n greater than m). Then

$$a_0 b_0 = 0 \tag{0}$$

$$a_1 b_0 + a_0 b_1 = 0 \tag{1}$$

$$a_2 b_0 + a_1 b_1 + a_0 b_2 = 0 \tag{2}$$

$$a_{n-1} b_0 + a_{n-2} b_1 + \dots\dots + a_{n-m-1} b_m = 0 \tag{$n-1$}$$

$$a_n b_m = 0 \qquad (n+m)$$

Now $a_0b_0 = 0$ implies $(b_0a_0)^2 = 0$, and since R is reduced, we have $b_0a_0 = 0$. Similarly in general, whenever $a_ib_j = 0$, we have $b_ja_i = 0$ (since R is reduced). Multiplying equation (1) by b_0 (from the left), (1) reduces to $b_0a_1b_0 = 0$ Therefore $(a_1b_0)^2 = 0$ this implies that $a_1b_0 = 0$.
By continuing the same process, we have $a_ib_0 = 0 \ \forall \ i$.
Therefore the system of equations (1),(2),............(n-1) reduces to $a_0b_1 = 0, a_1b_1 + a_0b_2 = 0, \ldots\ldots\ldots, a_{n-2}b_1 + \ldots\ldots\ldots + a_{n-m-1}b_m = 0$.
Since $a_0b_1 = 0$ implies $b_1a_0 = 0$, therefore $a_1b_1 + a_0b_2 = 0$ yields $b_1a_1b_1 = 0$ which implies that $a_1b_1 = 0$.
Similarly we have $a_ib_1 = 0 \ \forall \ i$.
Finally after repeating the same process we get $a_ib_j = 0 \ \forall \ i, j$. □

Below we record some consequences of the above result.

2.1.4 Remarks. (a) Every reduced ring is Armendariz.
(b) If R is a Boolean ring, then R is Armendariz. This follows from the fact that a Boolean ring is a reduced ring.
(c) If R is a commutative regular ring, then R is Armendariz.
(d) An Armendariz ring need not be a regular ring. Consider $R = \mathbb{Z}$.

2.1.5 Proposition. *Let D be a commutative P.I.D. and A an ideal of D. Then D/A is Armendariz.*

Proof. Given D is a commutative P.I.D. and A an ideal of D, therefore $A = Da$ for some $a \in D$. We have only to look at the case $a \neq 0$.
Since $a \in D, a = p_1^{e_1} p_2^{e_2} \dots\dots\dots p_k^{e_k}$ where $p_i^{'}s$ are distinct prime elements in D.
Now $D/A = D/Dp_1^{e_1} \dots . p_k^{e_k} \cong D/Dp_1^{e_1} \times \dots\dots\dots\dots \times D/Dp_k^{e_k}$ (by the Chinese remainder theorem)
To show that D/A is Armendariz, it suffices to show that D/Dp^e is Armendariz for each $e \in \mathbb{N} \bigcup \{0\}$.
Let $\overline{f(x)} = \sum_{i=0}^{n} \overline{a_i} x^i$, $\overline{g(x)} = \sum_{j=0}^{m} \overline{b_j} x^j \in D/Dp^e[x]$ satisfy $\overline{0} = \overline{f(x)g(x)} = \overline{f(x)g(x)}$.
Now $\overline{f(x)g(x)} = \overline{0}$ implies that $p^e | f(x)g(x)$.
Case 1 $e = 1$.
In this case Dp is a maximal ideal of D and so D/Dp is a field. Therefore D/Dp is Armendariz.
Case 2 $e > 1$.
Now $p^e | f(x)g(x)$ implies $f(x) = p^{e_1} f_1(x)$, $g(x) = p^{e_2} f_2(x)$ where p does not divide the g.c.d of the coefficients of $f_l(x)$ for $l = 1, 2$. Now $p^e | f(x)g(x)$ implies that $e \leq (e_1 + e_2)$ which implies that $p^e | a_i b_j \ \forall \ i, j$ which finally implies that $\overline{a_i b_j} = \overline{0} \ \forall \ i, j$.
Hence D/A is Armendariz. □

2.1.6 Corollary. *For each integer n, $\mathbb{Z}_n$ is an Armendariz ring.*

Proof. This follow from Proposition (2.1.5) as $\mathbb{Z}$ is a P.I.D. □

2.1.7 Remarks. (a) If R is Armendariz and if A an ideal of R, then R/A need not be an Armendariz ring.
Consider $R = \boldsymbol{Z}[x], A = (4) + (x^2)$. Let $\overline{f(T)}, \overline{g(T)} \in R[T]/A[T]$, defined as

$\overline{f(T)} = \bar{2} + \bar{3}xT$ and $\overline{g(T)} = \bar{2} - \bar{3}xT$. Then we have
$\overline{f(T)g(T)} = 0$ but $(\bar{3}x)\bar{2} = \bar{6}x \neq 0$.
(b) The above example also shows that if D is a U.F.D.and if A is an ideal of R, then D/A need not be an Armendariz ring.

2.1.8 Definition. Let R be a commutative ring and let M be an R-module.Then the group $R \oplus M$ is a ring under multiplication defined by $(r, m).(s, n) = (rs, rn + sm)$ where $r, s \in R$ and $m, n \in M$. We denote this ring by $R(+)M$ or $T(R, M)$.We call $T(R, M)$ the *Nagata or trivial extension* of R by M.

2.1.9 Definition. Let R be a commutative ring and $h : R \to R$ be a ring homomorphism. Let M be an R-module. Then by $R(+)_h M$ we mean the group $R \oplus M$ with multiplication defined as $(r, m).(s, n) = (rs, h(r)n + sm)$.

2.1.10 Definition. Let A be an ideal of a ring R. Then $R \oplus (R/A)$ is a ring under multiplication defined as $(r, \bar{x})(s, \bar{y}) = (rs, \overline{ry + xs})$. We denote it by $R(+)R/A$

The Nagata (or trivial) extension terminology is used in the contex of t/ the constructions considered in 2.1.9 and 2.1.10 as well.

2.1.11 Remark. We have the identifications $(R/A)[x] \cong R[x]/A[x]$ and $(R(+)M[x]) \cong R[x](+)M[x]$

2.1.12 Proposition. *Let D be a domain, A an ideal of D. Suppose the ring D/A is Armendariz. Then $D(+)(D/A)$ is Armendariz.*

Proof. Let $f(x) = \sum(a_i, \overline{u_i}).x^i = (f_0(x), \overline{f_1(x)})$ and $g(x) = \sum(b_j, \overline{v_j}).x^j = (g_0(x), \overline{g_1}) \in \{D(+)(D/A)\}[x]$ satisfy $f(x).g(x) = 0$.

Now $f(x).g(x) = 0 \Rightarrow (f_0(x), \overline{f_1(x)}).(g_0(x), \overline{g_1(x)}) = 0 \Rightarrow$

$$f_0(x).g_0(x) = 0 \tag{1}$$

and

$$\overline{f_0(x).g_1(x) + g_0(x).f_1(x)} = 0 \tag{2}$$

Since $D[x]$ is a domain, we have $f_0(x) = 0$ or $g_0(x) = 0$.

Case(1) If $f_0(x) = 0$, then (2) becomes $\overline{g_0(x)f_1(x)} = 0$ in $(D/A)[x]$ this implies $\overline{b_j u_i} = 0 \; \forall \; i, j$ (since D/A is Armendariz).

and $f_0(x) = 0 \Rightarrow a_i = 0 \; \forall \; i$.

Therefore $(a_i, \overline{u_i})(b_j, \overline{v_j}) = 0 \; \forall \; i, j$.

Case(2) $g_0(x) = 0$.

The proof is similar to that in case (1).

Therefore in both cases we have $(a_i, \overline{u_i})(b_j, \overline{v_j}) = 0 \; \forall \; i, j$.

Hence $D(+)D/A$ is Armendariz. □

2.1.13 Proposition. *Suppose that there exist $u, v \in R$ such that $u^2 = 0 = v^2$ and $uv = vu \neq 0$. Then R is not an Armendariz ring.*

Proof. Consider $f(x) = u + vx, g(x) = u - vx \in R[x]$. By the given condition on u, v we have $f(x)g(x) = 0$ but $uv = vu \neq 0$. Hence R cannot be an Armendariz ring. □

2.1.14 Theorem. *A ring R is reduced iff the trivial extension $T(R, R) = R(+)R$ is Armendariz.*

Proof. ($\Rightarrow$) Suppose R is reduced

Let $f(x) = \sum(a_i, u_i).x^i = (f_0(x), f_1(x))$ and $g(x) = \sum(b_j, v_j).x^j = (g_0(x), g_1(x))$

belonging to $(R(+)R)[x]$ satisfy $f(x)g(x) = 0$. Then

$$f_0(x)g_0(x) = 0 \tag{1}$$

and

$$f_0(x)g_1(x) + f_1(x)g_0(x) = 0 \tag{2}$$

Since R is reduced, we get $g_0(x)f_0(x) = 0$.
Therefore $g_0(x)f_0(x)g_1(x)+g_0(x)f_1(x)g_0(x) = 0$ which implies $(f_1(x)g_0(x))^2 = 0$. This implies that

$$f_1(x)g_0(x) = 0 \tag{3}$$

Using (3) we get $f_0(x)g_1(x) = 0$. Since R is reduced, we have from (1), (2), (3) $a_ib_j = 0, u_ib_j = 0$ and $a_iv_j = 0 \;\forall\; i, j$. So $(a_i, u_i)(b_j, v_j) = 0 \;\forall\; i, j$
Therefore $R(+)R$ is Armendariz.
($\Leftarrow$) Assume $R(+)R$ is Armendariz.
Suppose, if possible, R is not reduced .Then there exists $0 \neq a \in R$ such that $a^2 = 0$.
Let $v = (0, 1)$ and $u = (a, 0) \in R(+)R$,then we have $v^2 = u^2 = 0$,but $vu = uv = (0, a) \neq 0$ (since $a \neq 0$).Hence $R(+)R$ is not Armendariz by the above proposition.A contradiction. Thus R is reduced. □

2.1.15 Remarks. (a) If R is Armendariz,then $R(+)R$ need not be an Armendariz.

Example 1. By Corollary (2.1.6) for each natural number n the ring $\mathbb{Z}_n$ is Armendariz. It is not reduced, if n is not square-free. It follows from the previous result that (for example) for the Armendariz ring $\mathbb{Z}_4$ the ring $\mathbb{Z}_4(+)\mathbb{Z}_4$ is not Armendariz.

Example 2. We give another example where we show non-Armendarizness by direct computation. This example is of independent interest because of its usefulness in other situations.

Let T_0 be a nonzero reduced ring and consider $R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in T_0\}$.

Then R is Armendariz as $R \cong T_0(+)T_0$.

Let $R_0 = \{\begin{pmatrix} A & B \\ 0 & A \end{pmatrix} : A, B \in R\}$ and let

$$f(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ & \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ & \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$$

and

$$g(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ & \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$$

Then we have $f(x)g(x) = 0$ with $f(x), g(x) \in R_0[x]$ but

$$\begin{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\\ \\ \begin{pmatrix}0&0\\0&0\end{pmatrix}&\begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}\begin{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}&\begin{pmatrix}1&0\\0&1\end{pmatrix}\\ \\ \begin{pmatrix}0&0\\0&0\end{pmatrix}&\begin{pmatrix}0&1\\0&0\end{pmatrix}\end{pmatrix}\neq 0\ .$$

Thus $R_0 = R(+)R$ is not Armendariz.

For a subset S of R, the notation $l_R(S)$ denotes the left annihilator of S in R.

2.1.16 Theorem. *Let R be an Armendariz ring and let I be an ideal of R. Then $R/l_R(I)$ is an Armendariz ring.*

Proof. Let $\overline{f(x)} = \overline{a_0} + \overline{a_1}x + ... + \overline{a_n}x^n$ and $\overline{g(x)} = \overline{b_0} + \overline{b_1}x + ... + \overline{b_m}x^m$ belong in $R[x]/l_R(I)[x]$ where $\overline{a_i} = a_i + l_R(I)$ and $\overline{b_j} = b_j + l_R(I)$ satisfy $\overline{f(x)g(x)} = 0$.

Then we have for $0 \leq k \leq n+m$, $\sum_{i+j=k} a_i b_j \in l_R(I)$ where $0 \leq i \leq n$ and $0 \leq j \leq m$ and since $l_R(I)$ is the left annihilator of I in R, we get for all $u \in I$, $\sum_{i+j=k} a_i b_j u = 0$. This yields

$(a_0 + a_1 x + ... + a_n x^n)(b_0 u + b_1 u x + ... + b_m u x^m) = 0 \in R[x]$.

Since R is Armendariz, we have $a_i b_j u = 0 \ \forall \ i, j$ and $\forall \ u \in I \Rightarrow a_i b_j \in l_R(I) \Rightarrow \overline{a_i b_j} = 0 \ \forall \ i, j$

Therefore $R/l_R(I)$ is an Armendariz ring. $\square$

2.1.17 Proposition. *Let R be a reduced ring. Then $T_1 = \{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} :$ $a, b, c, d \in R\}$ is an Armendariz ring*

Proof. Let $f(x) = \sum \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} x^i$ and $g(x) = \sum \begin{pmatrix} p_j & q_j & r_j \\ 0 & p_j & s_j \\ 0 & 0 & p_j \end{pmatrix} x^j$ $\in T_1[x]$ satisfy

$$f(x)g(x) = 0 \qquad (1)$$

Now $f(x)$ and $g(x)$ can be written as

$$f(x) = \begin{pmatrix} \sum a_i x^i & \sum b_i x^i & \sum c_i x^i \\ 0 & \sum a_i x^i & \sum d_i x^i \\ 0 & 0 & \sum a_i x^i \end{pmatrix} = \begin{pmatrix} f_0(x) & f_1(x) & f_2(x) \\ 0 & f_0(x) & f_3(x) \\ 0 & 0 & f_0(x) \end{pmatrix}$$

$$\text{and } g(x) = \begin{pmatrix} \sum p_j x^j & \sum q_j x^j & \sum r_j x^j \\ 0 & \sum p_j x^j & \sum s_j x^j \\ 0 & 0 & \sum p_j x^j \end{pmatrix} = \begin{pmatrix} g_0(x) & g_1(x) & g_2(x) \\ 0 & g_0(x) & g_3(x) \\ 0 & 0 & g_0(x) \end{pmatrix}$$

from (1) we have $f_0(x)g_0(x) = 0, f_0(x)g_1(x) + f_1(x)g_0(x) = 0$

$f_0(x)g_2(x) + f_1(x)g_3(x) + f_2(x)g_0(x) = 0$ and $f_0(x)g_3(x) + f_3(x)g_0(x) = 0$

Since R is reduced we have

$f_0(x)g_0(x) = 0, f_0(x)g_1(x) = 0, f_1(x)g_0(x) = 0, f_0(x)g_2(x) = 0,$

$f_1(x)g_3(x) = 0, f_2(x)g_0(x) = 0, f_0(x)g_3(x) = 0$ and $f_3(x)g_0(x) = 0$

Then $a_i p_j = 0, a_i q_j = 0, b_i p_j = 0, a_i r_j = 0, b_i s_j = 0,$

$c_i s_j = 0, a_i s_j = 0$ and $d_i p_j = 0 \ \forall \ i, j$

$$\text{Hence } \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \cdot \begin{pmatrix} p_j & q_j & r_j \\ 0 & p_j & s_j \\ 0 & 0 & p_j \end{pmatrix} = 0 \ \forall \ i, j$$

Therefore T_1 is an Armendariz ring. □

2.1.18 Remarks. (a) Let R be a non-zero ring and let

$$R_4 = \{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & a_{23} & a_{24} \\ 0 & 0 & a_{11} & a_{34} \\ 0 & 0 & 0 & a_{11} \end{pmatrix} : a_{ij} \in R \}$$

Consider two polynomials $f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$

and

$$g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x \in R_4[x]$$

Then $f(x)g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$

$+ \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$, but

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$ So R_4 is not Armendariz.

Similarly for $n \geq 5$ R_n (defined analagously) is not Armendariz.

2.1.19 Proposition. *Let R be an Armendariz ring. Then we have the following results:*

(1) If $ab = 0, ac^n b = 0$ for $a, b, c \in R$ and some integer $n \geq 1$, then $acb = 0$.

(2) Let $a, b, c \in R$ satisfy $ab = 0$ and $c^n \in C(R)$ for some integer $n \geq 1$, then $acb = 0$.

Proof. (1) Consider $f(x) = a(1 - cx)$ and $g(x) = (1 + cx + ... + c^{n-1}x^{n-1})b \in R[x]$. Then $f(x)g(x) = a(1 - cx)(1 + cx + ... + c^{n-1}x^{n-1})b = 0$ (since $ab = 0, ac^nb = 0$).

Since R is Armendariz, we get $acb = 0$.

(2) Given $ab = 0$, we get $ac^nb = 0$ (since $c^n \in C(R)$)

Therefore from (1) we get $acb = 0$. ☐

2.1.20 Corollary. *Armendariz rings are abelian.*

Proof. Assume that R is an Armendariz ring. Let $r \in R$ be arbitrary and let $e \in I(R)$.

Write $a = e, b = 1 - e$ and $c = er(1 - e)$. Then we have $ab = e(1 - e) = 0$ and $c^2 = er(1 - e)er(1 - e) = 0$. Therefore we get $ab = 0, ac^2b = 0$

Hence by Proposition (2.1.19(1)) we get $acb = 0$ i.e $eer(1 - e)(1 - e) = 0$ which implies $er(1 - e) = 0 \Rightarrow$

$$er = ere \qquad (1)$$

Next take $a_1 = 1 - e, b_1 = e, c_1 = (1 - e)re$. Then using the same process we get $a_1c_1b_1 = 0 \Rightarrow$

$$re = ere \qquad (2)$$

Therefore from (1) and (2) we have $er = re \; \forall r \in R$.

Hence R is abelian.

Alternative proof:

Consider $f(x) = (1-e) + (1-e)rex$ and $g(x) = e - (1-e)rex$. Then

$f(x)g(x) = [(1-e) + (1-e)rex][e - (1-e)rex] = 0$

Since R is Armendariz we get $(1-e)(1-e)re = 0 \Rightarrow (1-e)re = 0$

$$\Rightarrow re = ere \tag{1}$$

Let $f_1(x) = e + er(1-e)x, g_1(x) = (1-e) - er(1-e)x$

Then $f_1(x)g_1(x) = 0$. Since R is Armendariz we have $er(1-e) = 0$

$$\Rightarrow er = ere \tag{2}$$

From (1) and (2) we have $er = re \ \forall\ r \in R$

Therefore R is abelian. □

2.1.21 Remarks. (a) Armendariz rngs (rings without identity) need not be abelian

Consider $R = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$

then we have $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in I(R)$ but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A \neq A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Therefore R is not abelian.

(b) If R is abelian, then R need not be an Armendariz ring

Let $R = \mathbb{Z}_4(+)\mathbb{Z}_4$ then R is abelian but R is not Armendariz. Take

$f(x) = (\bar{2}, \bar{0}) + (\bar{2}, \bar{1})$ and $g(x) = (\bar{2}, \bar{0}) - (\bar{2}, \bar{1}) \in \{\mathbb{Z}_4(+)\mathbb{Z}_4\}[x]$

then $f(x)g(x) = 0$ but $(\bar{2}, \bar{1})(\bar{2}, \bar{0}) = (\bar{0}, \bar{2}) \neq 0$

2.1.22 Proposition. *Let R be an Armendariz ring. Then the ring R is directly finite, i.e., $yx = 1$ whenever $xy = 1$ for $x, y \in R$.*

Proof. This holds since Armendariz rings are abelian (by Corollary 2.1.20) and (as seen below) all abelian rings are directly finite:
Let $x, y \in R$ (an abelian ring) satisfy $xy = 1$. Since $xy = 1$, we have $(yx)^2 = (yx)(yx) = y(xy)x = yx$; hence $yx \in I(R)$ and therefore $yx \in C(R)$. Now $1 = (xy)^2 = (xy)(xy) = x(yx)y = (yx)xy = yx$. Therefore R is directly finite. □

2.1.23 Proposition. *Let R be an Armendariz ring and let $f_1, f_2, ...f_n \in R[x]$ satisfy $f_1 f_2 .....f_n = 0$. Then $a_1 a_2 ...a_n = 0$ where a_i is a coefficient of f_i.*

Proof. Assume $f_1 f_2 .....f_n = 0$. Then we have $f_1(f_2...f_n) = 0$. Since $f_1, (f_2 f_3 ...f_n) \in R[x]$ and R is Armendariz , we have $a_1 b = 0 \forall$ coefficient b of $f_2 f_3 ...f_n$. Thus we have $a_1(f_2 f_3 ...f_n) = 0$. Again $a_1 f_2, (f_3....f_n) \in R[x]$ and $(a_1 f_2)(f_3....f_n) = 0$, we have $a_1 a_2 c = 0$ $\forall$ coefficient c of $(f_3 f_4....f_n)$ and a_2 a coefficient of f_2. Therefore $a_1 a_2 f_3 ...f_n = 0$. By continuing in this way, we have $a_1 a_2 a_3 ...a_n = 0$. □

2.1.24 Proposition. *A ring R is Armendariz iff $R[x]$ is Armendariz.*

Proof. ($\Rightarrow$) Suppose R is an Armendariz ring. Consider two polynomials $f(T) = f_0 + f_1 T + ... + f_n T^n$ and $g(T) = g_0 + g_1 T + ... + g_m T^m \in R[x][T]$ where $f_i, g_j \in R[x]$. Assume

$$f(T)g(T) = 0 \qquad (1)$$

Let $l = deg(f_0) + deg(f_1) + ... + deg(f_n) + deg(g_0) + deg(g_1)..... + deg(g_m)$ where $deg(f_i)$ (etc.) means the x-degree of the polynomial f_i. Then we have $f(x^l) = f_0 + f_1 x^l + ... + f_n x^{ln}$ and $g(x^l) = g_0 + g_1 x^l + ... + g_m x^{lm}$. From $f(x^l)g(x^l) = 0$ (a consequence of (1)), since R is Armendariz and since l

is 'sufficiently large', it follows that $\forall\ i,j$ the product ab vanishes for each coefficient a of f_i and b of g_j. This yields $f_i g_j = 0\ \forall\ i,j$.

Hence $R[x]$ is Armendariz.

($\Leftarrow$) This is trivial as subrings of Armendariz rings are again Armendariz . □

2.1.25 Proposition. *Let R be an abelian ring. Then the following conditions are equivalent:*

(1) R is an Armendariz ring.

(2) eR and $(1-e)R$ are Armendariz for every idempotent element e of R.

(3) eR and $(1-e)R$ Armendariz for some idempotent e of R.

Proof. (1) $\Rightarrow$ (2).Suppose R is Armendariz. Let $f(x) = \sum a_i x^i$ and $g(x) = \sum b_j x^j \in eR[x]$ satisfy $f(x)g(x) = 0$. Since$f(x), g(x)$ also belong to $R[x]$ and R is Armendariz, we have $a_i b_j = 0\ \forall\ i,j$. Hence eR is Armendariz.Similarly $(1-e)R$ is Armendariz

(2) $\Rightarrow$ (3) is trivial

(3) $\Rightarrow$ (1) Since $e \in (I(R) \bigcap C(R))$, so we have $R \cong eR \times (1-e)R$ as rings; hence R is Armendariz. □

2.1.26 Remarks. (a) If R is Armendariz, then eRe is also Armendariz for each idempotent $e \in I(R)$.

(b) If eRe is Armendariz for any nontrivial non-identity idempotent e of R, then R need not be an Armendariz ring.

Consider $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. Then R is not Armendariz by Example (2.1.2(f)). The only nontrivial non-identity idempotents of R are

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. But we have $eRe \cong \mathbb{Z}_2$, so eRe is Armendariz for every nontrivial non-identity idempotent e in R

2.1.27 Proposition. *Let R be a regular ring. Then the following conditions are equivalent:*

(1) R is abelian

(2) R is Armendariz

Proof. (1) $\Rightarrow$ (2) Suppose R is abelian. Let $a \in R$ satisfy $a^2 = 0$. Since R is regular, $\exists\ b \in R$ such that

$$a = aba \tag{1}$$

$e = ab$, then $e \in I(R)$ and since R is abelian, ab commutes with every element of R .

So $a = (ab)a = a(ab) = a^2b = 0$ (since $a^2 = 0$) $\Rightarrow a = 0$.

This implies that R is reduced, hence R is Armendariz.

(2) $\Rightarrow$ (1) By Corollary (2.1.20) we have R is abelian. ☐

2.1.28 Proposition. *Let $n \geq 2$. Then R is reduced iff $R[x]/(x^n)$ is Armendariz.*

Proof. ($\Rightarrow$) Assume R is reduced. Let u denote $\overline{x}$ where $\overline{x} = x + (x^2)$ in $R[x]/(x^n)$. Then we have $R[x]/(x^n) = R[u] = R + Ru + ... + Ru^{n-1}$, since for $a_0 + a_1\overline{x} + ... + a_{n-1}\overline{x^{n-1}} \in R[x]/(x^n)$, we have

$a_0 + a_1\overline{x} + ... + a_{n-1}\overline{x^{n-1}} = a_0 + a_1u + ... + a_{n-1}u^{n-1} \in R[u]$ and

$R + Ru + ... + Ru^{n-1}$.

Similarly taking $b_0 + b_1u + ... + b_{n-1}u^{n-1} \in R + Ru + ... + Ru^{n-1} (\in R[u])$ we have $b_0 + b_1u + ... + b_{n-1}u^{n-1} = b_0 + b_1\overline{x} + ... + b_{n-1}\overline{x^{n-1}} \in R[x]/(x^n)$

Let $f = p_0 + p_1T + ... + p_mT^m$ and $g = q_0 + q_1T + ... + q_kT^k \in R[u][T]$

where $p_i = a_{0i}+a_{1i}u+...+a_{n-1}u^{n-1}$ and $q_j = b_{0j}+b_{1j}u+...+b_{n-1j}u^{n-1} \in R[u]$

Therefore $f = (a_{00} + a_{10}u + ... + a_{n-1}u^{n-1}) + (a_{01} + ... + a_{n-11}u^{n-1})T + .. + (a_{0m} + a_{1m}u + ... + a_{n-1}u^{n-1})T^m \Rightarrow$

$f = (a_{00} + a_{01}T + .. + a_{0m}T^m) + (a_{10} + a_{11}T + ... + a_{1m}T^m)u + .. + (a_{n-10} + a_{n-11}T + .. + a_{n-1m}T^m)u^{n-1} = f_0 + f_1u + .. + f_{n-1}u^{n-1}$

with $f_i = a_{i0} + a_{i1}T + ... + a_{im}T^m \in R[T]$

Similarly $g = g_0 + g_1u + .. + g_{n-1}u^{n-1}$ with $g_j = b_{j0} + .. + b_{jk}T^k$

$\in R[T]$

Assume $fg = 0$. Note that $f_ig_ju^{i+j} = 0 \; \forall \; i, j$ with $i + j \geq n$, so it suffices to check the cases of $i + j < n$. From $fg = 0$ we have

$f_0g_0 = 0, f_0g_1 + f_1g_0 = 0, f_0g_2 + f_1g_1 + f_2g_0 = 0$

.

.

$.f_0g_{n-1} + f_1g_{n-2} + ... + f_{n-1}g_0 = 0$

Since R is reduced, we get $f_ig_j = 0 \; \forall \; i, j = 0, ...n - 1$

Therefore $R[x]/(x^n)$ is Armendariz.

($\Leftarrow$) Suppose $R[x]/(x^n)$ is Armendariz. Let $r \in R$ satisfy

$r^n = 0$ for some positive integer n. Consider the two polynomials $f(T) = r - \overline{x}T$

and $g(T) = r^{n-1} + r^{n-2}\overline{x}T + ... + \overline{x^{n-1}}T^{n-1} \in \{R[x]/(x^n)\}[T]$. We have $f(T)g(T) = r^n - \overline{x^n}T^n = 0$. Since $R[x]/(x^n)$ is Armendariz we have $r\overline{x^{n-1}} = 0$ and this implies that $r = 0$.

Therefore R is reduced. ☐

2.1.29 Proposition. *Let I be an ideal of a ring R. Suppose R/I is an*

Armendariz ring. If I is reduced (as a ring without identity), then R is Armendariz.

Proof. Write $\overline{I} = I[x]$ and let $a, b \in R$. If $ab = 0$, then $(bIa)^2 = 0$. Since I is reduced and $bIa \subseteq I$, we get $bIa = 0$. Let $f(x) = \sum a_i x^i$ and $g(x) = \sum b_j x^j$ $\in R[x]$ satisfy $f(x)g(x) = 0$. Then $(f(x) + \overline{I})(g(x) + \overline{I}) = 0$ in $R[x]/I[x]$. Since R/I is Armendariz we get $a_i b_j \in I \; \forall \; i, j$. Now we proceed by induction on m with $m \geq 0$. If $m = 0$, then we are done and so we assume $m \geq 1$

Claim: $a_0 b_j = 0 \; \forall \; j \in \{0, 1, 2, ...n\}$

Assume $a_0 b_j \neq 0$ for some $j \in \{1, 2...n\}$. Let l be the smallest integer in $\{1, 2, ...n\}$ such that $a_0 b_l \neq 0$. Therefore we have $a_0 b_j = 0 \; \forall 0 \leq j \leq l-1$ and this gives $b_j I a_0 = 0$.....(1) (by the previous argument).

Hence $(a_{l-j} b_j)(a_0 b_l)^2 = a_{l-j} b_j (a_0 b_l)(a_0 b_l) \in a_{l-j} b_j I a_0 b_l$ (since $a_i b_j \in I \; \forall \; i, j$).

from (1) we get $a_{l-j}(b_j I a_0) b_l = 0$.

Therefore we have

$$a_{l-j} b_j (a_0 b_l)^2 = 0 \tag{2}$$

($j \in \{0, 1, 2, ...l-1\}$)

Now from $f(x)g(x) = 0$, we have the coefficient of x^l is $0 = a_0 b_l + a_1 b_{l-1} + ... + a_l b_0$

$$\Rightarrow 0 = a_0 b_l + \sum_{j=0}^{l-1} a_{l-j} b_j \tag{3}$$

Multiplying equation (2) (from the right) by $(a_0 b_l)^2$ we get

$0 = (a_0 b_l)(a_0 b_l)^2 + (\sum_{j=0}^{l-1} a_{l-j} b_j)(a_0 b_l)$

$\Rightarrow (a_0 b_l)^3 = 0)$ (using equation (2))

Since I is reduced and $a_0 b_l \in I$, we get $a_0 b_l = 0$ which is a contradiction to the assumption that $a_0b_l \neq 0$. Hence it follows that

$a_0b_l = 0 \forall j \in 0, 1, ..., n$.

Now let $f'(x) = \sum_{i=1}^{m} a_ix^i$, then $f'(x)g(x) = 0$

Take $f_1(x) = \sum_{i=1}^{m-1} a_ix^i$ but $xf_1(x) = f'(x)$ so we get $xf_1(x)g(x) = 0$ which implies that $f_1(x)g(x) = 0$. Since degree of $f_1(x)$ is $m - 1 < m$, so by induction hypothesis, we get $a_ib_j = 0 \ \forall \ i, j$

Hence R is Armendariz. $\square$

2.1.30 Remark. If R/I is Armendariz for every non-zero ideal I of R with I Armendariz, then R need not be an Armendariz ring. (Here we consider I as a rng, i.e., a ring which may not have an identity.)

Take $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field.

Let $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$
$\in R[x]$

Then we have $f(x)g(x) = 0$ but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$

So R is not Armendariz. Now we show that R/I and I are Armendariz for any non-zero ideal I of R.

We have $I_1 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $I_3 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ are the only three non-zero proper ideals of R.

Then $R/I_1 \cong F$ so R/I_1 is Armendariz .

Let $f(x) = \begin{pmatrix} \sum a_ix^i & \sum b_ix^i \\ 0 & 0 \end{pmatrix}$ and $g(x) = \begin{pmatrix} \sum a'_jx^j & \sum b'_jx^j \\ 0 & 0 \end{pmatrix} \in I_1[x]$ satisfy $f(x)g(x) = 0$ then we get

$(\sum a_i x^i)(\sum a_j' x^j) = 0$ and $(\sum a_i x^i)(b_j' x^j) = 0$

Since F is a field, we have $a_i a_j' = 0$ and $a_i b_j' = 0 \ \forall \ i, j$

Therefore $\begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_j' & b_j' \\ 0 & 0 \end{pmatrix} = 0.$

So I_1 is Armendariz.

We also have $R/I_2 \cong F$ so R/I_2 is Armendariz. Applying the same method as in the case of I_1 we get I_2 is Armendariz.

Finally $R/I_3 \cong F(+)F$, so R/I_3 is Armendariz. Also $I_3^2 = 0$ so I_3 is Armendariz.

2.1.31 Definition. Let R be a commutative ring with identity. For $f \in R[x]$, the *content* A_f of f is the ideal of R generated by the coefficients of f. The ring R is called a *Gaussian* ring if $A_{fg} = A_f A_g \ \forall \ f, g \in R[x]$.

2.1.32 Proposition. *If R is a Gaussian ring, then R is Armendariz.*

Proof. Let $f = \sum a_i x^i, g = \sum b_j x^j \in R[x]$ satisfy $fg = 0$. Clearly $A_{fg} = 0$. Since R is Gaussian we have $0 = A_{fg} = A_f A_g$ and this implies that $a_i b_j = 0$ $\forall \ i, j$ (since $a_i \in A_f$ and $b_j \in A_g$).

Therefore R is an Armendariz. □

2.1.33 Definition. Let D be a commutative integral domain with quotient field K. By a *fractional ideal* I of K we mean a non-zero D-submodule of K such that $dI \subset D$ for some non-zero $d \in D$. For a fractional ideal I of K by I^{-1} (the inverse of I) we mean the set of all $x \in K$ with $xI \subset D$; I^{-1} is again a fractional ideal. We say that I is *invertible* if $II^{-1} = D$.

2.1.34 Definition. A Prüfer domain is a commutative domain in which every non-zero finitely generated ideal is invertible.

2.1.35 Remark. The converse of Proposition 2.1.32 is false, i.e., if R is a commutative Armendariz ring then R need not be Gaussian. This follows from the theorem (due to Tsang)that a commutative integral domain is Gaussian iff it is Prüfer.(See [DDA & BGK, Theorem 1.3].) Hence the ring of polynomials in one (or more)indeterminate(s) over the ring of integers, being a non-Prüfer domain, is an Armendariz ring which is not Gaussian.

2.1.36 Proposition. *Let R be a commutative ring. If R is Gaussian, Then R/I is also Gaussian.*

Proof. Let $f(x) = \sum a_ix^i, g(x) = b_jx^j \in R[x]$, then by hypothesis we have

$$A_{fg} = A_fA_g \tag{$*$}$$

Now for $\overline{f(x)} = f(x) + I[x], \overline{g(x)} = g(x) + I[x] \in R[x]/I[x]$ we are to show $A_{\bar{f}\bar{g}} = A_{\bar{f}}A_{\bar{g}}$. Since $A_{\bar{f}\bar{g}} \subset A_{\bar{f}}A_{\bar{g}}$, therefore it is enough to show that $A_{\bar{f}}A_{\bar{g}} \subset A_{\bar{f}\bar{g}}$. We know that a generator of $A_{\bar{f}}A_{\bar{g}}$ look likes $\overline{a_i}\overline{b_j} = \overline{a_ib_j}$. Now $\overline{a_ib_j} \in A_{\bar{f}}A_{\bar{g}}$, which implies that $a_ib_j \in A_fA_g$. Then by (*) we get $a_ib_j \in A_{fg}$ and this gives $a_ib_j = \sum r_kc_k$ (for some $c_k \in A_{fg}$). This implies that $\overline{a_i}\overline{b_j} \in A_{\bar{f}\bar{g}}$. Hence $A_{\bar{f}}A_{\bar{g}} \subset A_{\bar{f}\bar{g}}$. Therefore $A_{\bar{f}\bar{g}} = A_{\bar{f}}A_{\bar{g}}$. Thus R/I is Gaussian. □

2.1.37 Proposition. *Let R be a commutative ring. Then R is Gaussian iff R/I is Armendariz for every ideal I of R.*

Proof. ($\Rightarrow$) Suppose R is Gaussian, then R/I is Gaussian (by Proposition (2.1.36)). Therefore by Proposition (2.1.32) we have R/I is Armendariz.

($\Leftarrow$) Suppose R/I is Armendariz for every ideal I of R . Let $f(x) = \sum a_ix^i, g(x) = \sum b_jx^j \in R[x]$. Since $A_{fg} \subset A_fA_g$, therefore it is enough to prove that

$$A_fA_g \subset A_{fg} \tag{$*$}$$

.For proving (*), it is sufficient to prove $A_fA_g/A_{fg} = 0$. A generator of A_fA_g/A_{fg} (as an R-module or ideal of R/A_{fg}) is of the form $\overline{a_ib_j}$. Now we have $\overline{f(x)g(x)} = 0 \in R/A_{fg}$, which implies that $\overline{a_ib_j} = 0 \ \forall \ i,j$ (since R/A_{fg} is Armendariz). Therefore $A_fA_g/A_{fg} = 0$ i.e., $A_fA_g = A_{fg}$. Hence R is Gaussian . ☐

2.1.38 Notation. Let R be a commutative ring. We denote the set of all prime ideals of R by $Spec(R)$ and the set of all maximal ideals of R by $Max(R)$.

2.1.39 Definition. Let R be a commutative ring and A be an ideal of R. Then A is *locally principal* if $\forall \ P \in Spec(R)$ the ideal A_P of R_P is principal.

For the proof of the following result we refer to [DDA & BGK, Theorem 1.1].

2.1.40 Remark. Let R be a commutative ring and let $f(x) \in R[x]$. If A_f is locally principal then $A_{fg} = A_fA_g \ \forall \ g(x) \in R[x]$.

2.1.41 Definition. A commutative ring R is called an *arithmetical* ring if every finitely generated ideal of R is locally principal.

2.1.42 Proposition. *An arithmetical ring is Gaussian (and, therefore, Armendariz).*

Proof. Let R be an arithmetical ring and let $f(x) \in R[x]$. By hypothesis A_f (being finitely generated) is locally principal. Hence we get $A_fAg = A_{fg}$ $\forall \ g(x) \in R[x]$. Therefore R is Gaussian and by Proposition (2.1.32) R is Armendariz. ☐

2.1.43 Corollary. *Let R be a commutative principal ideal ring. Then R is Gaussian (and, therefore Armendariz).*

Proof. Let R be a commutative P.I.R. and let A be an ideal of R . Since R is a P.I.R., we have $A = Rx$ for some $x \in R$. Hence for every $P \in Spec(R)$, $A_P = R_P\frac{x}{1}$. Therefore R is arithmetical. Then by Proposition (2.1.42) R is Gaussian. Hence R is Armendariz. (This extends Proposition 2.1.5) $\square$

Commutative von Neumann regular rings and Dedekind domains are other examples of arithmetical (and, therefore, Gaussian) rings. (We already know that they are Armendariz since they are reduced.)

2.1.44 Remark. Some examples of Armendariz (PS) rings are given in Section 4.3 below. These are also automatically Armendariz rings.

2.2 Armendariz modules

This section is devoted to the study of the basic properties and examples of Armendariz modules. We follow [AMB & MBR] here.

2.2.1 Definition. Let R be a ring,and M an R-module. We call M an *Armendariz* module if for $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $g(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfying $f(x)g(x) = 0$, then we have $a_i m_j = 0 \ \forall \ i, j$.

2.2.2 Proposition. *Let D be a domain and M a torsion free D-module. Then M is Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in D[x]$ and $g(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)g(x) = 0$. Now M is torsion free implies $M[x]$ is a torsion free $D[x]$-module, so we get $f(x) = 0$ or $g(x) = 0$. Thus M is Armendariz. □

2.2.3 Corollary. *Let D be a domain and M is a free $D-$ module. Then M is Armendariz.*

Proof. This follow from Proposition (2.2.2) as free D-modules over domains are torsion free. □

2.2.4 Remarks. (a) A ring R is Armendariz iff the module ${}_RR$ is Armendariz
(b) Let R be a ring and M be a free R-module. Then M need not be an Armendariz module. Consider $R = M_2(\mathbb{Z}), M = R$. Clearly, ${}_RM$ is free but ${}_RM$ is not Armendariz

2.2.5 Proposition. *The class of Armendariz R-modules is closed under (1) direct products, (2) submodules and (3) direct sums.*

Proof. (1) Let $\{M_l\}_{l\in I}$ be a family of Armendariz R-modules. Let $f(x) = \sum_{i=0}^{k} a_i x^i \in R[x]$, $m_l(x) = \sum_{j=0}^{n} u_j x^j \in M_l[x]$. Suppose $f(x)(m_l(x))_{l\in I} = 0$. Then this implies that $(f(x)m_l(x))_{l\in I} = 0 \Rightarrow f(x)m_l(x) = 0$ for each $l \in I$. Since M_l are Armendariz for each $l \in I$, so $a_i u_j = 0 \; \forall \; i, j$. Hence $\prod M_l$ is an Armendariz R-module.

(2) Let $f(x) = \sum_{i=0}^{k} a_i x^i \in R[x]$ and $n(x) = \sum_{j=0}^{l} n_j x^j \in N[x]$ where $N \leq M$. Suppose $f(x)n(x) = 0$; then $a_i n_j = 0 \; \forall \; i, j$ since $n(x) \in M[x]$ and M is Armendariz. Hence submodules of an Armendariz modules are again Armendariz.

(3) Since a direct sum is a submodule of the corresponding direct product, therefore by (1) and (2) we get direct sums of Armendariz modules are again Armendariz. ☐

2.2.6 Proposition. *Let R be an Armendariz ring and M is a free R- module. Then M is Armendariz.*

Proof. Given R is Armendariz and M is a free R- module, therefore we have $M \cong \bigoplus R$ and since a direct sum of an Armendariz modules is Armendariz, we get that M is an Armendariz module. ☐

2.2.7 Proposition. *Let R be an Armendariz ring and M an R-projective module. Then M is an Armendariz module.*

Proof. Since M is R-projective module, therefore $M \cong$ a direct summand of a free R-module F. But F is free, and so Armendariz by (2.2.6). Hence M is Armendariz as submodules of Armendariz modules are Armendariz. ☐

2.2.8 Remarks. (a) If R is reduced and M is a free R-module, then M is Armendariz

(b) If R is a reduced ring and M is R-projective, then M is Armendariz.

(c) If D is a domain, and M is D-projective, then M is Armendariz.

(d) Let D be a commutative domain, M is a D-module. Then $M/T(M)$ is Armendariz. This follows from the result that $M/T(M)$ over a commutative domain is torsion free and hence by Proposition (2.2.2) $M/T(M)$ is Armendariz

2.2.9 Proposition. *Let R be a commutative ring and M an R-module. If $R(+)M$ is Armendariz, then M is Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $g(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$.Assume

$$f(x)g(x) = 0 \qquad (1)$$

Consider $(f(x), 0), (0, g(x)) \in \{R(+)M\}[x]$. Then $(f(x), 0)(0, g(x)) = (0, f(x)g(x)) = (0, 0)$ by (1). Since $R(+)M$ is Armendariz, we get $(a_i, 0)(0, m_j) = (0, 0) \Rightarrow (0, a_i m_j) = 0 \Rightarrow a_i m_j = 0$. Hence M is Armendariz. ☐

2.2.10 Remark. The converse of Proposition (2.2.9) is false, i.e., if R is a commutative ring, and M is an Armendariz R-module, then $R(+)M$ need not be an Armendariz ring. Consider $R = \mathbb{Z}_4, M = \mathbb{Z}_4$, but $\mathbb{Z}_4(+)\mathbb{Z}_4$ is not Armendariz.

2.2.11 Proposition. *Let D be an integral domain, M a D-bimodule and $R' = D(+)M$, the Nagata extension of D by M. The ring R' is Armendariz if and only if both ${}_DM$ and M_D are Armendariz.*

Proof. ($\Rightarrow$) Suppose R' is Armendariz. Let $f(x) = \sum_{i=0}^{m} a_i x^i \in D[x]$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in M[x]$ satisfy $f(x)g(x) = 0$

Then for $(f(x),0),(0,g(x)) \in \{D(+)M\}[x]$, we have

$(f(x),0)(0,g(x)) = (0,f(x)g(x)) = (0.0)$. Since R' is Armendariz we have $(a_i,0)(0,b_j) = (0,0)$ and this implies $(0,a_ib_j) = 0 \Rightarrow a_ib_j = 0$.

Therefore M is Armendariz as a left (and similarly , right) D-module.

$(\Leftarrow)$ Suppose M is Armendariz. Let $f(x) = (f_0(x), f_1(x)) = \sum(a_i,m_i)x^i)$ and $g(x) = (g_0(x), g_1(x)) = \sum(b_j,n_j)x^j) \in \{D(+)M\}[x]$ satisfy $f(x)g(x) = 0$. This implies $(f_0(x), f_1(x))(g_0(x), g_1(x)) = 0$ and this implies $f_0(x)g_0(x) = 0$ and $f_0(x)g_1(x) + f_1(x)g_0(x) = 0$.

Since D is a domain we get $f_0(x) = 0$ or $g_0(x) = 0$ (or both).

Case(1): Assume $f_0(x) = 0$.

Then we get $f_1(x)g_0(x) = 0$. Since M is an Armendariz $D-$module we have $u_ib_j = 0 \ \forall\ i,j$.

Therefore $(0,u_i)(b_j,v_j) = (0,u_ib_j) = 0$

Case(2):$g_0(x) = 0, f_0(x) \neq 0$. The proof is similar to that in case (1)

Case (3): $f_0(x) = 0 = g_0(x)$ is trivial.

Therefore $D(+)M$ is Armendariz. □

2.2.12 Corollary. *If D is a commutative domain and M is a torsion free D- module, then $D(+)M$ is an Armendariz ring.*

Proof. From Proposition (2.2.2) M is Armendariz. Therefore on applying Proposition (2.2.11) we have $D(+)M$ is Armendariz. □

2.2.13 Corollary. *Let D be a domain (not neccesarily commutative). Let $R' = D(+)(D/A)$, then R' is an Amendariz ring iff D/A is an Armendariz ring.*

Proof. The proof follows from Proposition (2.2.11) □

2.2.14 Corollary. *If K is a field and V is a vector space over K, then $K(+)V$ is a commutative Armendariz ring which is not reduced if $V \neq 0$*

Proof. Since V is a torsion free K-module, then by Corollary (2.2.12), $K(+)V$ is Armendariz.

$K(+)V$ is commutative, since for $(a,u),(b,v) \in K(+)V$ we have
$(a,u)(b,v) = (ab, av+bu) = (b,v)(a,u)$ (since K is a field $ab = ba$)
This gives $K(+)V$ is commutative.
But $K(+)V$ is not reduced when $V \neq 0$, since for each $(0,v) \in K(+)V$ we have $(0,v)^2 = (0,0)$. □

2.2.15 Remarks. (a) If D is a commutative domain, M is a free D-module then $D(+)M$ is Armendariz.
(b) If R is not a domain, M is a free R-module then $R(+)M$ need not be an Armendariz ring. Consider $R = \mathbb{Z}_4, M = \mathbb{Z}_4$, then $R(+)M = \mathbb{Z}_4(+)\mathbb{Z}_4$ which is not Armendariz
(c) If D is a commutative domain, M is a torsion module, then $D(+)M$ can be Armendariz. Take $R = \mathbb{Z}, M = \mathbb{Z}_4$ then $R(+)M = \mathbb{Z}(+)\mathbb{Z}_4$ is Armendariz.

2.2.16 Proposition. *Let D a domain ,$h : D \to D$ is a ring monomorphism and M is a D-bimodule which is torsion free on both the left and the right. Then $D(+)_hM$ is an Armendariz ring.*

Proof. Let $f(x) = \sum(a_i, m_i)x^i = (f_0(x), f_1(x)), g(x) = \sum(b_j, n_j)x^j = (g_0(x), g_1(x)) \in \{D(+)_hM\}[x]$
Suppose $f(x)g(x) = 0$ then $(f_0(x), f_1(x))(g_0(x), g_1(x)) = (0,0)$
$\Rightarrow f_0(x)g_0(x) = 0$ and

$$h(f_0(x))g_1(x) + g_0(x)f_1(x) = 0 \qquad (1)$$

Since D is a domain we get $f_0(x) = 0$ or $g_0(x) = 0$

Suppose $f_0(x) = 0 (f_1(x) \neq 0)$ then equation (1) becomes $g_0(x) f_1(x) = 0$.

Since M is torsion free on both left and right we get $g_0(x) = 0$

Therefore $f(x) = (0, f_1(x)), g(x) = (0, g_1(x))$ and this implies that $(0, m_i)(0, n_j) = 0$

Next assume $g_0(x) = 0$ then equation (1) reduces to $h(f_0(x)) g_1(x) = 0$

$\Rightarrow$

$$h(f_0(x)) = 0 \tag{2}$$

(since M is torsion free on both left and right)

Since h is a monomorphism we have from (2) $f_0(x) = 0$

Therefore $f(x)g(x) = (0, f_1(x))(0, g_1(x)) = (0, 0) \Rightarrow (0, m_i)(0, n_j) = 0$

Hence $D(+)_h M$ is Armendariz. ☐

2.2.17 Corollary. *Let D be a commutative domain, let $h : D \rightarrow D$ be a ring monomorphism and let M be a free D-module. Then $D(+)_h M$ is Armendariz.*

Proof. This follows from Proposition (2.2.16) since M is free it implies that M is torsion free. ☐

2.2.18 Corollary. *Let K be a field, $h : K \rightarrow K$ is a field monomorphism and V is a vector space over K. Then the ring $K(+)_h M$ is Armendariz.*

Proof. This follows from Corollary (2.2.17). ☐

2.2.19 Remark. If R is not a domain, M is a free R-module, $h : K \rightarrow K$ is a ring monomorphism, then $R(+)_h M$ need not be an Armendariz ring. Take $R = \mathbb{Z}_4, M = \mathbb{Z}_4$ and $h = I$ then $R(+)_h M = \mathbb{Z}_4(+)\mathbb{Z}_4$ is not Armendariz.

2.2.20 Proposition. *Let M be an R-bimodule Then $R(+)M$ is an Armendariz ring iff the following conditions are satisfied:*

(a) R is an Armendariz ring

(b) M is an Armendariz left and right R-module

(c) If $f(x)g(x) = 0 \in R[x]$, then $f(x)M[x] \bigcap M[x]g(x) = 0$

Proof. ($\Rightarrow$) Suppose $R(+)M$ is Armendariz.

(a) Since R is a subring of $R(+)M$ ($R \cong R(+)0 \subseteq R(+)M$), therefore R is Armendariz.

(b) Let $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ and $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ satisfy $f(x)m(x) = 0$. Then $(f(x), 0)(0, m(x)) = (0, f(x)m(x)) = (0, 0)$ Since the ring $R(+)M$ is Armendariz, we get $(0, a_i m_j) = (0, 0)\ \forall\ i, j$ and this implies $a_i m_j = 0\ \forall\ i, j$

Therefore M is Armendariz as a left R-module (and similarly as a right R-module).

(c) Let $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$.

Suppose $f(x)m(x) = m'(x)g(x) \neq 0$ for some $m(x), m'(x) \in M[x]$

Let $(f(x), 0) + (0, m'(x))y$ and $(g(x), 0) - (0, m(x))y \in \{R(+)M\}[x][y]$. Then we have

$\{(f(x), 0) + (0, m'(x))y\}\{(g(x), 0) - (0, m(x))y\} = (0, 0)$. But $(f(x), 0)(0, m(x)) \neq 0$ which contradicts the fact that $\{R(+)M\}[x]$ is Armendariz. Therefore we conclude that $f(x)M[x] \bigcap M[x]g(x) = 0$ if $f(x)g(x) = 0$.

($\Leftarrow$) Suppose that all the three conditions are satisfied.

Let $\alpha(x) = \sum_{i=0}^{n} (a_i, m_i)x^i = (f(x), m(x))$ and $\beta(x) = \sum_{j=0}^{k} (b_j, l_j)x^j = (g(x), l(x)) \in \{R(+)M\}[x]$ where $f(x), g(x) \in R[x]$ and $m(x), l(x) \in M[x]$. Suppose $\alpha(x)\beta(x) = 0$. Then we have $f(x)g(x) = 0$

and $f(x)l(x) + m(x)g(x) = 0.....(1)$
Now using (a) (since $f(x)g(x) = 0$), we get $a_ib_j = 0 \ \forall \ i, j$ and from equation (1) we have $f(x)l(x) = -m(x)g(x) \in f(x)M[x] \bigcap M[x]g(x) = 0$. Now this implies that $f(x)l(x) = m(x)g(x) = 0$. Now using condition (b), we get $a_il_j = 0$ and $m_ib_j = 0 \ \forall \ i, j$
Therefore $(a_i, m_i)(b_j, l_j) = 0 \ \forall \ i, j$.
Hence $R(+)M$ is Armendariz. $\square$

2.2.21 Corollary. *Let R be a reduced ring and let A be an ideal of R, such that R/A is reduced. Then $R(+)(R/A)$ is Armendariz.*

Proof. By Proposition (2.2.20), we only need to show that
$f(x)(R/A)[x] \bigcap (R/A)[x]g(x) = 0$ whenever $f(x)g(x) = 0$ in $R[x]$ since conditions (a) and (b) are satisfied.
Since $f(x)(R/A)[x] \bigcap (R/A)[x]g(x) = 0$ iff $(f(x)R[x] + A[x]) \bigcap (R[x]g(x) + A[x]) \subseteq A[x]$ therefore it is enough to show that $(f(x)R[x]+A[x]) \bigcap (R[x]g(x)+A[x]) \subseteq A[x]$.
So let $f(x)f'(x) + A[x] = g'(x)g(x) + A[x] \in (f(x)R[x] + A[x]) \bigcap (R[x]g(x) + A[x])...(1)$ for some $f'(x), g'(x) \in R[x]$
Since R is reduced, $f(x)g(x) = 0 \Rightarrow g(x)f(x) = 0$. Therefore multiplying equation (1) (from the right) by $f(x)$, we have
$f(x)f'(x)f(x) + A[x] = A[x]$ which implies that $f(x)f'(x) \in A[x]$ (since R/A is reduced).
Therefore, $(f(x)R[x] + A[x]) \bigcap (R[x]g(x) + A[x]) \subseteq A[x]$.
Hence $R(+)(R/A)$ is also Armendariz. $\square$

2.2.22 Remark. If R is Armendariz and R/A is also Armendariz for some ideal A of R, then $R(+)(R/A)$ need not be Armendariz.

Take $R = \mathbb{Z}/8\mathbb{Z}$ and $A = 0$ then $R = \mathbb{Z}/8\mathbb{Z}$ and $R/A = (\mathbb{Z}/8\mathbb{Z})/0 = \mathbb{Z}/8\mathbb{Z}$ are Armendariz but $\mathbb{Z}/8\mathbb{Z}(+)\mathbb{Z}/8\mathbb{Z}$ is not Armendariz.

2.2.23 Proposition. *Let $\theta : R \to A$ be a ring homomorphism and let M be an A-module. Regard M as a left R-module via θ. Then we have the following results*

(1) If ${}_AM$ is Armendariz, then ${}_RM$ is Armendariz.

(2) If θ is onto, then the converse of the statement in (1) holds.

(3) If A is an Armendariz ring, then A is Armendariz as a left R- module.

Proof. (1) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$. Then we have

$\theta(f(x))g(x) = 0$ $(rm = \theta(r)mr \in R, m \in M)$ $\Rightarrow \sum \theta(a_i)x^i \sum b_j x^j = 0$

$\Rightarrow \theta(a_i)b_j = 0 \ \forall\ i, j$ (since ${}_AM$ is Armendariz)

Now $a_i b_j = \theta(a_i)b_j = 0 \ \forall\ i, j$.

Therefore ${}_RM$ is Armendariz.

(2) Assume ${}_RM$ is Armendariz. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in A[x]$ and $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$. Since θ is onto, therefore for each $a_i \in A$, $\exists b_i \in R$ such that $\theta(b_i) = a_i$. Now $f(x)m(x) = 0 \Rightarrow \sum a_i x^i m(x) = 0 \Rightarrow \sum_{i=0}^{n} \theta(b_i)x^i \sum_{j=0}^{k} m_j x^j = 0 \Rightarrow \theta(\sum b_i x^i) \sum m_j x^j = 0$ $\Rightarrow \sum b_i x^i \sum m_j x^j = 0$ (by definition $r.m = \theta(r)m$) $\Rightarrow b_i m_j = 0 \forall i, j$ (since ${}_RM$ is Armendariz). Then $a_i m_j = \theta(b_i)m_j = b_i m_j = 0 \ \forall\ i, j$. Therefore ${}_AM$ is Armendariz.

(3) Since A is an Armendariz ring, then by Remark (2.2.4(a)) ${}_AA$ is an Armendariz module. Hence by part (1) ${}_RA$ is Armendariz as a left R-module.□ □

2.2.24 Corollary. *For any $n \in \mathbb{Z}$, $\mathbb{Z}_n$ is an Armendariz $\mathbb{Z}$-module.*

Proof. Let $\theta : \mathbb{Z} \to \mathbb{Z}_n$ be the ring homomorphism defined as $\theta(x) = x + n\mathbb{Z}$. Then θ is onto. Since $\mathbb{Z}_n$ over $\mathbb{Z}_n$ is Armendariz (by Corollary (2.1.6)), therefore by Proposition (2.2.23(1)) $\mathbb{Z}_n$ is an Armendariz $\mathbb{Z}$-module. □

2.2.25 Proposition. *The following conditions are equivalent*

(1) R is an Armendariz ring.

(2) Every torsionless R-module is Armendariz.

(3) Every submodule of a free R-module is Armendariz.

(4) There exists a faithful R-module which is Armendariz.

Proof. (1) $\Rightarrow$ (2). Let M be a torsionless R-module. Then $M \leq \prod R$ for some product of copies of R. Since $\prod R$ is Armendariz as an R-module, and a submodule of an Armendariz module is Armendariz, M is also Armendariz.
(2) $\Rightarrow$ (3) Let N be a submodule of a free R-module M, then $N \leq \bigoplus R$ (since M is free $\Rightarrow M \cong \bigoplus R$) which implies that N is a torsionless R-module. Hence by (2) N is Armendariz.
(3) $\Rightarrow$ (4) Take $M = R$, then R is a faithful R-module which is Armendariz.
(4) $\Rightarrow$ (1) Let M be a faithful R-module which is Armendariz. Then $R \leq \prod M$ for some product of copies of M. Since $\prod M$ is Armendariz as M is Armendariz, we have R is an Armendariz ring.□ □

2.2.26 Notation. If M is a left R-module we denote $R/ann(M)$ by $\overline{R}$ and the ring of endomorphisms $End_R(M)$ by $E(M)$.

2.2.27 Proposition. *The following conditions are equivalent*

(1) The left R-module M is Armendariz

(2) The left $\overline{R}$-module M is Armendariz.

Proof. (1) $\Rightarrow$ (2) Let ${}_RM$ is Armendariz. We have the natural onto ring homomorphism $\theta : R \to \overline{R}$ defined as $\theta(r) = r + ann(M) = \overline{r}$. Therefore by Proposition (2.2.23(2)) we have ${}_{\overline{R}}M$ is also Armendariz.
(2) $\Rightarrow$ (1) Let ${}_{\overline{R}}M$ be Armendariz. Then by Proposition (2.2.23(1)) ${}_RM$ is Armendariz. □

2.2.28 Remarks. (a) If the left $\overline{R}$-module M is Armendariz, then $\overline{R}$ is an Armendariz ring.
Since M is a faithful $\overline{R}$-module, therefore we have $\overline{R} \leq \prod M$ for some product of copies of M. Now M is Armendariz implies $\overline{R}$ is an Armendariz ring.
(b) Suppose the right $E(M)$-module M is Armendariz, then the ring $E(M)$ is Armendariz:
Since M is faithful as a right $E(M)$-module, by (4) $\Rightarrow$ (1) of Proposition (2.2.25), we get the result that $E(M)$ is an Armendariz ring.

2.2.29 Proposition. *A module M is Armendariz iff every finitely generated submodule of M is Armendariz.*

Proof. ($\Rightarrow$) Suppose that M is an Armendariz R-module. Then by Proposition (2.2.5(2)) every finitely generated submodule of M is Armendariz.
($\Leftarrow$) Suppose every finitely generated submodule of M is Armendariz. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$ in $M[x]$. Then $m(x) \in \{Rm_0 + Rm_1 + ... + Rm_k\}[x]$ and since every finitely generated submodule of M is Armendariz, we get $a_i m_j = 0 \ \forall \ i, j$. Hence M is Armendariz. □

2.2.30 Corollary. *Let M be a $\mathbb{Z}$-module. Then (1) M is Armendariz as a $\mathbb{Z}$-module and (2) $\mathbb{Z}(+)M$ is an Armendariz ring.*

Proof. Since every finitely generated submodule of M over $\mathbb{Z}$ is Armendariz, then by Proposition (2.2.29) M is Armendariz over $\mathbb{Z}$. (We use Corollary (2.2.24) and the structure theorem for finitely generated $\mathbb{Z}$-modules). Now on applying Proposition (2.2.11) we get $\mathbb{Z}(+)M$ is also Armendariz. $\square$

We record the following special case of the above result (its extension to arithmetical rings is recorded in Corollary (2.2.41) below) and mention that - when D is a commutative integral domain - there are interesting connections between the Armendarizness of the D-module K/D and the condition 'D is integrally closed'. These will be studied elsewhere.

2.2.31 Corollary. *$\mathbb{Q}/\mathbb{Z}$ is an Armendariz $\mathbb{Z}$-module.*

Proof. Since every finitely generated submodule of $\mathbb{Q}/\mathbb{Z}$ over $\mathbb{Z}$ is Armendariz, so by Proposition (2.2.29) $\mathbb{Q}/\mathbb{Z}$ is Armendariz over $\mathbb{Z}$. $\square$

2.2.32 Notation. Let M be an R-module, let $C(R)$ be the centre of R and let T be a multiplicatively closed subset of $C(R)$. Then $T^{-1}M$ is an $T^{-1}R$-module, where $T^{-1}M = \{\frac{m}{s} : m \in M, s \in T\}$ and $T^{-1}R = \{\frac{a}{t} : a \in R, t \in T\}$.

2.2.33 Proposition. *Let M be an R-module and $C = centre(R)$. Then the following conditions are equivalent.*

(1) M is Armendariz

(2) $T^{-1}M$ is Armendariz $T^{-1}R$-module for each multiplicatively closed subset T of C.

(3) M_P is an Armendariz R_P module for each $P \in Spec(C)$.

(4) M_Q is an Armendariz R_Q-module for each $Q \in Max(C)$.

Proof. (1) $\Rightarrow$ (2) Let $f(x) = \sum_{i=0}^{k} \frac{a_i}{t_i} x^i \in \{T^{-1}R\}[x]$ and $m(x) = \sum_{j=0}^{l} \frac{m_j}{s_j} x^j \in \{T^{-1}M\}[x]$ satisfy $f(x)m(x) = \frac{0}{1}$. Then this gives $(t_0 t_1 ... t_k) f(x) (s_0 s_1 ... s_l) m(x) = \frac{0}{1} \Rightarrow \exists\ u \in T$ such that $u(t_0 t_1 .. t_k) f(x)(s_0 s_1 .. s_l) m(x) = 0$. Since M is Armendariz R-module we get $u(t_0 t_1 .. t_{i-1} t_{i+1} .. t_k) a_i (s_0 s_1 .. s_{j-1} s_{j+1} .. s_l) m_j = 0\ \forall$ i, j, then this gives $\frac{(t_0 t_1 .. t_{i-1} t_{i+1} .. t_k) a_i (s_0 s_1 .. s_{j-1} s_{j+1} .. s_l)}{(t_0 t_1 .. t_k)(s_0 s_1 ... s_l)} = \frac{0}{1} \Rightarrow \frac{a_i}{t_i} \frac{m_j}{s_j} = \frac{0}{1}\ \forall\ i, j$. Therefore $T^{-1}M$ is Armendariz as an $T^{-1}R$-module.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$, then this implies $\frac{f(x)}{1} \frac{m(x)}{1} = \frac{0}{1}$ in $M_Q\ \forall\ Q \in Max(C)$. Since M_Q is Armendariz R_Q-module, we get $\frac{a_i}{1} \frac{m_j}{1} = \frac{0}{1}$ and this gives $a_i m_j = 0$ $\forall\ i, j$. Therefore M is Armendariz. □

2.2.34 Proposition. *Let D be a commutative domain and M a D-module. Then the module M is Armendariz iff its torsion submodule $T(M)$ is Armendariz.*

Proof. ($\Rightarrow$) Suppose M is Armendariz. Since $T(M)$ is a submodule of M, therefore $T(M)$ is also Armendariz.

($\Leftarrow$) Suppose $T(M)$ is an Armendariz D-module. Let $f(x) = \sum_{i=0}^{k} a_i x^i \in D[x], g(x) = \sum_{j=o}^{n} m_j x^j \in M[x]$ satisfy $f(x)g(x) = 0$. Then we have

$$a_0 m_0 = 0 \qquad (1)$$

$$a_0 m_1 + a_1 m_0 = 0 \qquad (2)$$

$$a_0 m_2 + a_1 m_1 + a_2 m_0 = 0 \qquad (3)$$

$$a_k m_n = 0 \qquad (k+n+1)$$

Assume $a_0 \neq 0$. Now multiplying equation (2) by a_0 we get $(a_0)^2 m_1 = 0$ (since $a_0 a_1 m_0 = a_1 a_0 m_0 = 0$ from equation (1)) which implies that $(a_0)^2$ annihilates both m_0 and m_1 . On using the same process we get $(a_0)^3 m_2 = 0$ (from equation (3)).By continuing we get $g(x) \in T(M)[x]$. Since $T(M)$ is an Armendariz D-module we have $a_i m_j = 0 \ \forall \ i, j$. Hence M is an Armendariz D-module. □

2.2.35 Proposition. *Let R be a commutative ring. The ring R is Gaussian iff every cyclic R-module is Armendariz.*

Proof. Suppose R is Gaussian. Let M be a cyclic R-module, then $M = Rm$ for some $m \in M$. Since the map $\theta : R \rightarrow Rm$ defined as $\theta(r) = rm$ is onto, therefore $R/Ker(\theta) \cong Rm$. Then by applying Proposition (2.1.37) we get M is Armendariz.

Let every cyclic R-module be Armendariz. Then in particular the ring R/I is Armendariz (since the cyclic R-module R/I is Armendariz iff the ring R/I is Armendariz). Therefore on applying Proposition (2.1.37) we get the ring R is Gaussian. □

2.2.36 Definition. A ring R is *left (right) Gaussian* if every cyclic left (right) R-module is Armendariz.

2.2.37 Definition. A ring R is *left (right) strongly Gaussian* if every left (right) R-module is Armendariz.

2.2.38 Remarks. (a) A ring R is called *completely Armendariz* if every homomorphic image of R is Armendariz.

(b) Left Gaussian rings are completely Armendariz.

Proof : Let R be a left Gaussian ring and I an ideal of R. Then the cyclic left R-module R/I is Armendariz. Hence R/I is an Armendariz ring.Therefore R is completely Armendariz.

(c) Left invariant, completely Armendariz rings are left Gaussian.

Let R be a left invariant and completely Armendariz ring. If M is a cyclic left R-module, then $M = Rm$ for some $m \in M$. Therefore $M \cong R/I$, for some left ideal I of R. Now R/I is a quotient ring of R (since I is also a right ideal). Since R is a completely Armendariz ring, and $M \cong R/I$, the R-module M is Armendariz.

(d) (i) The Gaussian property is not left-right symmetric and (ii) Completely Armendariz rings need not be left Gaussian. We show this by an example.

Let L be a field and let $K = L(Y, T)$ be the field of rational functions in two commuting indeterminates Y and T. Let h be the endomorphism of the field K defined via $h(Y) = Y^2$ and $h(T) = T^2$. The ring $R = K(+)_h K$ has the following properties.

(i) Apart from R and 0 the only right ideal of R is $0(+)K$, which is also a left ideal.Since the ring $R_0 = R/(0(+)K)$ is a field, R_0 is Armendariz as a right R-module. We know that R is an Armendariz ring. The ring R is thus a right Gaussian and completely Armendariz ring.

(ii) Let $W = h(K) = L(Y^2, T^2)$,a subfield of K. Then $B = 0(+)W$ is a left ideal of R. We now show that the cyclic left R-module R/B is not Armendariz,and thus the ring R is not left Gaussian.

Let $f(X) = (0,Y) + (0,T)X \in R[X]$, and $g(X) = \overline{(Y,0)} + \overline{(-T,0)}X \in (R/B)[X]$. Then we have $f(X)g(X) = 0$, but $(0,Y)(\overline{(-T,0)} \neq 0$ in R/B.

A proof of the next proposition can be found in [DDA & BGK, Remark 1.5]. (By *the content* A_g of $g(x) \in M[x]$ we mean the R-submodule of M generated by coefficients of $g(x)$.)

2.2.39 Proposition. *Let R be a commutative ring and let $f(x) \in R[x]$.The ideal A_f is locally principal iff for each R-module M and each $g(x) \in M[x]$ we have $A_f A_g = A_{fg}$.*

2.2.40 Proposition. *Let R be a commutative ring. The ring R is strongly Gaussian iff R is arithmetical.*

Proof. Let R be a strongly Gaussian ring and let I be a finitely generated ideal of R. So $I = A_f$ for some $f(x) \in R[x]$. Let M be an R-module and $g(x) \in M[x]$. Then $f(x)\overline{g(x)} = 0$ in $M[x]/A_{fg}[x]$ (where $\overline{g(x)} = g(x) + A_{fg}[x]$) which implies that $A_{f\bar{g}} = 0$. Since M/A_{fg} is Armendariz we have $A_f A_{\bar{g}} = 0$, i.e., $A_f A_g / A_{fg} = 0$ which implies that $A_f A_g = A_{fg}$. Therefore by Proposition (2.2.39) we get $I = A_f$ is locally principal.Thus R is arithmetical.

Let R be an arithmetical ring, let M be an R-module and let $f(x) \in R[x]$, $g(x) \in M[x]$ satisfy $f(x)g(x) = 0$. Then we get $A_{fg} = 0$ and Since A_f is locally principal, we have $A_f A_g = A_{fg} = 0$ (by Proposition (2.2.39)) which implies that M is Armendariz. Therefore R is strongly Gaussian. $\square$

2.2.41 Corollary. *If R is arithmetical, then every R-module is Armendariz. In particular if R is arithmetical, then K/R is Armendariz as an R-module where $K = T_0^{-1}R$ (T_0 is the set of a non-zero-divisors of R)*

Proof. This follows directly from Proposition (2.2.40) (since R strongly Gaussian implies every R-module is Armendariz). $\square$

2.2.42 Remark. Let M be a module over $\mathbb{Z}$, then M is Armendariz as $\mathbb{Z}$-module. (This special case of Corollary (2.2.41) has already been recorded in (2.2.30).)

Chapter 3

Semi-commutativity and reversibility

In this chapter we study the concepts of semi-commutative rings, semi-commutative modules and reversible rings. These classes are intimately connected with the classes of Armendariz rings and modules; they are also objects of independent interest. The class of Armendariz rings, as well as the class of semi-commutative rings, are both 'sandwiched' between the class of reduced rings and the class of abelian rings. A comparative study of Armendariz rings and semi-commutative rings was carried out by C.Huh, Y.Lee and Agata Smoktunowicz in [CH,YL & AS]. Semi-commutative modules were introduced and studied in [AMB & MBR]. The class of reversible rings, studied by Cohn (in [PMC]) and other authors, lies between the classes of reduced rings and semicommutative rings. In [NKK & YL:2], Kim and Lee studied extensions of reversible rings and proved some results concerning their relationship with Armendariz rings.

3.1 Semi-commutative rings

This section is devoted to the study of the basic properties and examples of semi-commutative rings.

3.1.1 Definition. A ring R is called *semi − commutative* if whenever $a, b \in R$ satisfy $ab = 0$, then $acb = 0$ for each $c \in R$.

3.1.2 Example. Any commutative ring is semi-commutative

3.1.3 Remark. Subrings of semi-commutative rings are semi-commutative.

3.1.4 Proposition. *A semi-commutative ring is abelian.*

Proof. Let R be a semi-commutative ring. Then for any $a, b \in R$ satisfying $ab = 0$ we have $acb = 0$ forall $c \in R$.
Let $a = e, b = (1 - e)$ where $e \in I(R)$. Then $ab = 0$.
Since R is semi-commutative we have $axb = 0 \ \forall \ x \in R$,
$\Rightarrow ex(1 - e) = 0 \Rightarrow$

$$ex = exe \tag{1}$$

Similarly we have $ba = 0$, and this gives $(1 - e)xe = 0 \Rightarrow$

$$xe = exe \tag{2}$$

From (1) and (2) we have $xe = ex \ \forall \ x \in R$.
Therefore R is abelian. □

3.1.5 Proposition. *Let R be a ring. Then the following statements are equivalent.*

(1) R is semi-commutative

(2) The right annihilator of each element of R is an ideal of R.

(3) The left annihilator of each element of R is an ideal of R.

(4) For $a, b \in R$, $ab = 0$ implies $aRb = 0$.

Proof. (1) $\Rightarrow$ (2). Let $a, r \in R$ and let $x \in r_R(a)$. Now $x \in r_R(a) \Rightarrow ax = 0$.Since R is semi-commutative we get $arx = 0$ and this gives $rx \in r_R(a)$; hence $r_R(a)$ is a left ideal of R. Therefore we have $r_R(a)$ is an ideal of R (since $r_R(a)$ is a right ideal of R).

(2) $\Rightarrow$ (3) Let $x \in l_R(a)$, then $xa = 0$,so for any $t \in R$, $txa = 0 \Rightarrow tx \in l_R(a)$. Therefore $l_R(a)$ is a left ideal of R. Now $x \in l_R(a) \Rightarrow xa = 0 \Rightarrow a \in r_R(x)$. Since $r_R(x)$ is an ideal of R we get for any $t \in R$, $ta \in r_R(x)$, i.e., $xta = 0 \Rightarrow xt \in l_R(a)$. Therefore $l_R(a)$ is an ideal of R.

(3) $\Rightarrow$ (4) Let $a, b \in R$ satisfy $ab = 0$. Then we have $a \in l_R(b)$

Since left annihilator of every subset of R is an ideal, therefore we have (for $r \in R$) $ar \in l_R(b) \Rightarrow arb = 0 \ \forall \ r \in R \Rightarrow aRb = 0$.

(4) $\Rightarrow$ (1) Let $a, b \in R$ satisfy $ab = 0$. Then by the given condition $aRb = 0$, we get $acb = 0 \ \forall \ c \in R$.

Therefore we have R is semi-commutative. □

The following result of M.B.Rege and S.Chhawchharia has led to several results of the same type.

3.1.6 Proposition. *Let R be an Armendariz ring. Then R is semi-commutative iff $R[x]$ is semicommutative.*

Proof. ($\Rightarrow$) Suppose R is a semi-commutative ring.

Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy

$$f(x)g(x) = 0 \tag{1}$$

Now Consider a polynomial $h(x) = \sum_{k=0}^{l} c_k x^k \in R[x]$. Since R is Armendariz, from (1) we get $a_i b_j = 0 \ \forall \ i, j$

By hypothesis R is semi-commutative. Therefore we have $a_i c b_j = 0 \ \forall \ c \in R$.

Then this gives $a_i c_k b_j = 0 \ \forall \ i, j, k$.

Therefore we have $f(x)h(x)g(x) = 0 \ \forall \ h(x) \in R[x]$.

Hence $R[x]$ is semi-commutative.

($\Leftarrow$) This follows from the fact that a subring of a semi-commutative ring is again semi-commutative. ☐

3.1.7 Remark. If R is semi-commutative. Then $R[x]$ need not be. Take the ring $R = (\mathbb{Z}_2 + A)/I$ where $\mathbb{Z}_2$ is the field of integers modulo 2 , $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ is a free algebra of polynomials with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$ and I is an ideal generated by $a_0b_0, a_1b_2 + a_2b_1, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_2b_2, a_0rb_0, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$ with $r \in A$. (The details of the proof are given in [CH,YL & AS, Example (2)]).

3.1.8 Proposition. *Let R be a regular ring. Then the following conditions are equivalent.*

✓ *(1) R is an Armendariz ring*

(2) R is a reduced ring

(3) R is semi-commutative

Proof. (1) $\Rightarrow$ (2) Suppose R is Armendariz, then R is abelian. Now R is abelian and regular (given) implies R is reduced.

(2) $\Rightarrow$ (3) Let $a, b \in R$ satisfy $ab = 0$, then $ba = 0$ (since R is reduced) $\Rightarrow$ $axb = 0 \ \forall \ x \in R$ (on using R is reduced)

Therefore R is semi-commutative.

(3) $\Rightarrow$ (1) Let R be a semi-commutative ring. Now R is both regular and semi-commutative, hence R is reduced. Therefore by Proposition (2.1.3) R is Armendariz. ☐

3.1.9 Remarks. (a) If R is Armendariz and semi-commutative, then R need not be regular. Consider $R = \mathbb{Z}_4$.

(b) Semi-commutative ring need not be Armendariz.Consider $R = \mathbb{Z}_4(+)\mathbb{Z}_4$.

(c) If R is Armendariz,then R need not be semi-commutative. Take the ring in [CH,YL & AS, Example 14]

3.1.10 Proposition. *Let I be an ideal of R and R/I is a semi-commutative ring. If I is reduced (as a ring without identity), then R is semi-commutative.*

Proof. Let $a, b \in R$ satisfy $ab = 0$. Then we get

$(a + I)(b + I) = ab + I = I \Rightarrow ab \in I$. Since R/I is semi-commutative we get $aRb \subseteq I$.

Now $(bIa)^2 = bIabIa = 0$ (since $ab = 0$)

Since $bIa \subseteq I$ and I is reduced, we have

$$bIa = 0 \tag{1}$$

So $((aRb)I)^2 = aRbIaRbI = aR(bIa)RbI = 0$ (from (1))

Since I is reduced and $aRbI \subseteq I$, we get $aRbI = 0$

Now as $aRb \subseteq I$ we have $(aRb)(aRb) \subseteq aRbI = 0$. Since I is reduced and $aRb \subseteq I$, we get $aRb = 0$.

Hence R is semi-commutative. $\square$

3.1.11 Remark. However even If R/I and I are semi-commutative for every non zero proper ideal I of R, even then R need not be semi-commutative.

Consider $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field. Then R is not semi-commutative, for if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ we have $AB = 0$ but for $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $ACB \neq 0$.

Now we show that R/I is semi-commutative for every nonzero proper ideal' I of R. The only nonzero proper ideals I of R are

$\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

case(1) Let $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, then $R/I \cong F$. Therefore R/I is semi-commutative. Let $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in I$ satisfy $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = 0$. Then $\begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix} = 0 \Rightarrow ac = 0 = ad$.Since F is a field, either $a = 0$ or $c = d = 0$. If $a = 0$ then

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = 0 \ \forall \ e, f \in F$$

If $c = d = 0$ we have $\begin{pmatrix} a & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$.

Hence I is semi-commutative.

case(2) Let $I = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then $R/I \cong F$. Hence R/I is semi-commutative.

Let $\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \in I$ satisfy

$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = 0$. Then $ad = 0 = bd$. Since F is a field, either $a = 0 = b$ or $d = 0$.

If $a = 0 = b$ then $\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & e \\ 0 & f \end{pmatrix} \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = 0$.

If $d = 0$, we get $\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & e \\ 0 & f \end{pmatrix} \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = 0$.

Therefore I is semi-commutative.

case(3) Let $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, then $R/I \cong F(+)F$. Hence R/I is semi-commutative.

Let $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in I$, then we have $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = 0$, so

$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = 0 \ \forall \ e \in F$.

Therefore I is semi-commutative.

3.1.12 Proposition. *Let R be a reduced ring. Then $R(+)R$ is a semi-commutative ring.*

Proof. Let (a, b), $(c, d) \in R(+)R$ satisfy $(a, b)(c, d) = 0$. Then this gives $(ac, ad + bc) = 0$ which implies

$$ac = 0 \tag{1}$$

and

$$ad + bc = 0 \tag{2}$$

Since R is reduced, we have $ca = 0$. Now multiplying (2) (from the right)by a we get $ada + bca = 0 \Rightarrow ada = 0 \Rightarrow ad = 0$.

Then equation (2) reduces to $bc = 0$. Since R is reduced we get for $r, s \in R$

$$arc = 0, ard = 0, dasc = 0 \tag{$*$}$$

So for any $(r, s) \in R(+)R$, $(a, b)(r, s)(c, d) = (arc, ard + asc + brc) = 0$ (from (*)).

Hence $R(+)R$ is a semi-commutative ring. ☐

3.1.13 Remarks. (a) If $R(+)R$ is a semi-commutative ring, then R need not be a reduced ring. Take $R = \mathbb{Z}_4$,then R is not reduced but $R(+)R = \mathbb{Z}_4(+)\mathbb{Z}_4$ is semi-commutative as $R(+)R$ is commutative.

(b) If R is a semi-commutative ring, then $R(+)R$ need not be a semi-commutative ring.

Let $S_0 = \mathbb{H}(+)\mathbb{H}$. Then S_0 is semi-commutative by Proposition (3.1.12). Let $R = S_0(+)S_0$. Then for $X = \begin{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{pmatrix}$,

$$Y = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in S_0(+)S_0$$ we have $XY = 0$ but for

$$Z = \begin{pmatrix} \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \end{pmatrix},$$

$$XZY = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Therefore $R = S_0(+)S_0$ is not semi-commutative.

3.1.14 Proposition. *Let R be a reduced ring. Then $S_1 = \{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R\}$ is a semi-commutative ring.*

Proof. Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S_1$ satisfy $AB = 0$.

Then we have

$$a_1a_2 = 0 \tag{1}$$

$$a_1b_2 + b_1a_2 = 0 \tag{2}$$

$$a_1c_2 + b_1d_2 + c_1a_2 = 0 \tag{1}$$

$$a_1d_2 + d_1a_2 = 0 \tag{4}$$

From (1) we have $a_2a_1 = 0$ (since R is reduced)
Multiplying equation (2) (from the left) by a_2, we get $a_2a_1b_2 + a_2b_1a_2 = 0$ $\Rightarrow a_2b_1a_2 = 0$ (since $a_2a_1 = 0$) $\Rightarrow a_2b_1 = 0$ (since R is reduced) $\Rightarrow b_1a_2 = 0$. Therefore from (2) we get $a_1b_2 = 0$.
Again multiplying equation (4) (from the left) by a_2 we get $a_2d_1a_2 = 0 \Rightarrow$ $a_2d_1 = 0$ which implies that $d_1a_2 = 0$
Hence equation (4) reduces to $a_1d_2 = 0 \Rightarrow d_2a_1 = 0$
Now multiplying equation (3) (from the left) by a_2 we get $a_2c_1a_2 = 0 \Rightarrow$ $c_1a_2 = 0$. Therefore equation (3) reduces to $a_1c_2 + b_1d_2 = 0$....(5). On multiplying equation (5) (from the right)by a_1 we get $a_1c_2a_1 = 0 \Rightarrow a_1c_2 = 0$. Hence from (5) we get $b_1d_2 = 0$. Since R is reduced,we get R is semi-commutative.So for $r, s, t, u \in R$ we have

$$a_1ra_2 = 0, a_1rb_2 = 0, a_1sa_2 = 0, b_1ra_2 = 0 \tag{$*$}$$

$$a_1rc_2 = 0, a_1sd_2 = 0, b_1rd_2 = 0 \tag{$**$}$$

$$a_1ta_2 = 0, b_1ua_2 = 0, c_1ra_2 = 0, a_1rd_2 = 0 \tag{$***$}$$

$$a_1ua_2 = 0, d_1ra_2 = 0 \tag{$****$}$$

Now for any $\begin{pmatrix} r & s & t \\ 0 & r & u \\ 0 & 0 & r \end{pmatrix} \in S_1$ we have

$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} r & s & t \\ 0 & r & u \\ 0 & 0 & r \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} = 0$ (0n using equations (*),(**),(***) and(****)).

Hence S_1 is a semi-commutative ring. □

3.1.15 Remark. Let $T_0 = \{\begin{pmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & g \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d, e, f, g \in R\}$ where R is a reduced ring. Then T_0 is not semi-commutative.

Take $A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T_0$. Then we have $AB = 0$ but for $C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T_0$ we have

$ACB = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$. Therefore T_0 is not semi-commutative.

3.1.16 Proposition. *Let R be a reduced ring and consider the ring* $T_0 = \{\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{11} & x_{12} & x_{24} \\ 0 & 0 & x_{11} & x_{12} \\ 0 & 0 & 0 & x_{11} \end{pmatrix} : x_{ij} \in R\}$. *Then T_0 is a semi-commutative ring.*

Proof. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & a_{12} & a_{24} \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{12} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{11} & b_{12} & b_{24} \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & 0 & b_{11} \end{pmatrix}$ $\in T_0$ satisfy $AB = 0$. Then we have

$$a_{11}b_{11} = 0 \tag{1}$$

$$a_{11}b_{12} + a_{12}b_{11} = 0 \tag{2}$$

$$a_{11}b_{13} + a_{12}b_{12} + a_{13}b_{11} = 0 \tag{3}$$

$$a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{12} + a_{14}b_{11} = 0 \tag{4}$$

$$a_{11}b_{24} + a_{12}b_{12} + a_{24}b_{11} = 0 \tag{5}$$

Multiplying equation (2) (from the right)by a_{11} and used the condition that R is reduced, we get

$$a_{11}b_{12} = 0, a_{12}b_{11} = 0 \tag{6}$$

On multiplying (3) (from the right) by a_{11} we get $a_{11}b_{13}a_{11} + a_{12}b_{12}a_{11} + a_{13}b_{11}a_{11} = 0 \Rightarrow a_{11}b_{13}a_{11} = 0$ (since $b_{12}a_{11} = 0 = b_{11}a_{11}$). Since R is reduced,we get $a_{11}b_{13} = 0$. Therefore (3) reduces to

$$a_{12}b_{12} + a_{13}b_{11} = 0 \tag{7}$$

Again by multiplying (7) by a_{11} (from the right) and used that R is reduced we get

$$a_{12}b_{12} = 0, a_{13}b_{11} = 0 \tag{8}$$

Similarly by continuing the same process we get

$$a_{11}b_{24} = 0, a_{12}b_{12} = 0, a_{24}b_{11} = 0 \tag{9}$$

$$a_{11}b_{14} = 0, a_{12}b_{24} = 0, a_{13}b_{12} = 0, a_{14}b_{11} = 0 \tag{10}$$

Now from equations (1),(6),(7),(8),(10) (using that R is semi-commutative as R is reduced)we get

$$a_{11}rb_{11} = 0, a_{11}rb_{12} = 0, a_{12}rb_{11} = 0 \tag{$*$}$$

$$a_{11}rb_{13} = 0, a_{11}sb_{12} = 0, a_{12}rb_{12} = 0, a_{11}tb_{11} = 0 \tag{$**$}$$

$$a_{12}sb_{11} = 0, a_{13}rb_{11} = 0, a_{11}rb_{14} = 0, a_{11}sb_{24} = 0 \tag{$***$}$$

$$a_{12}rb_{24} = 0, a_{11}tb_{12} = 0, a_{12}sb_{12} = 0, a_{13}rb_{12} = 0 \tag{$****$}$$

$$a_{11}ub_{11} = 0, a_{12}vb_{11} = 0, a_{13}sb_{11} = 0, a_{14}rb_{11} = 0 \tag{$*****$}$$

$$a_{11}rb_{24} = 0, a_{11}sb_{12} = 0, a_{12}sb_{12} = 0, a_{11}vb_{11} = 0 \tag{$******$}$$

$$a_{12}sb_{11} = 0, a_{24}rb_{11} = 0 \tag{$*******$}$$

$\forall\ r, s, t, u, v \in R$. Now for any $C = \begin{pmatrix} r & s & t & u \\ 0 & r & s & v \\ 0 & 0 & r & s \\ 0 & 0 & 0 & r \end{pmatrix} \in T_0$ we get

$$ACB = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & a_{12} & a_{24} \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{pmatrix} \begin{pmatrix} r & s & t & u \\ 0 & r & s & v \\ 0 & 0 & r & s \\ 0 & 0 & 0 & r \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{11} & b_{12} & b_{24} \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & 0 & b_{11} \end{pmatrix} =$$

0 (on using equations (*),(**),(***),(****),(*****),(******) and (*******)).

Hence T_0 is semi-commutative. □

3.1.17 Proposition. *Let R be a ring and let e be a central idempotent element of R. Then R is semi-commutative iff eR and $(1-e)R$ are semi-commutative.*

Proof. ($\Rightarrow$) Suppose R is semi-commutative. Let $a_1 = ea, b_1 = eb \in eR$ satisfy $a_1 b_1 = 0$, i.e., $eaeb = 0 \Rightarrow eaxeb = 0 \ \forall \ x \in R$ (since R is semi-commutative) $\Rightarrow eayeb = 0 \ \forall \ y \in eR$.

Therefore eR is semi-commutative.

Similarly $(1-e)R$ is semi-commutative.

($\Leftarrow$) Assume eR and $(1-e)R$ are semi-commutative. Therefore we have $R \cong eR \times (1-e)R$ (Since $e \in (I(R) \bigcap C(R))$). Hence R is also semi-commutative. □

3.1.18 Corollary. *If R is semi-commutative, then eRe and $(1-e)R(1-e)$ are semi-commutative.*

Proof. This follows from Proposition (3.1.17) as $(1-e)R(1-e) = R(1-e)$. □

3.1.19 Corollary. *R is semi-commutative iff eRe is semi-commutative for all $e \in I(R)$.*

Proof. ($\Rightarrow$) This follows from the fact that a subrng (as a ring without identity) of a semi-commutative ring is again semi-commutative.

($\Leftarrow$) Suppose eRe is semi-commutative for all $e \in I(R)$, then choosing $e = 1, R = 1R1$ is semi-commutative. □

3.1.20 Remark. If eRe is semi-commutative for any non-trivial non-identity idempotent element e of R, then R need not be semi-commutative.

Consider $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. The non-trivial non-identity idempotent of R are

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $AB = 0$ but

$A \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0.$

Therefore R is not semi-commutative,but $eRe \cong \mathbb{Z}_2$ is semi-commutative.

3.1.21 Proposition. *Suppose that R is an Armendariz ring and let $C(R)$ be the center of R. If N is a right ideal of R such that $N \subseteq Nil(R)$, then $C(R) + N$ is both an Armendariz ring and a semi-commutative ring.*

Proof. Let $x, y \in C(R) + N$. Then $x = a + n, y = b + m$ for some $a, b \in C(R)$ and $n, m \in N$. Now $x - y = (a - b) + (n - m) \in C(R) + N$ (since $a, b \in Z(R)$ implies $a - b \in C(R)$ as $Z(R)$ is a subring of R and since N is a right ideal of R implies that for any $n, m \in N$ $n - m \in N$). Therefore $C(R) + N$ is a subring of R. Hence $C(R) + N$ is Armendariz as R is Armendariz.

Now we will show that $C(R) + N$ is semi-commutative.

Let $a+n, b+m \in C(R)+N$ satisfy $(a+n)(b+m)=0$, then for any $x \in C(R)$ we have

$$(a+n)x(b+m) = x(a+n)(b+m) = 0 \tag{1}$$

and for any $l \in N \ \exists$ some $k \in \mathbb{N}$ such that $l^k = 0$, we have $(a+n)l^k(b+m) = 0$ and by proposition (2.1.19) we have

$$(a+n)l(b+m) = 0 \tag{2}$$

Combining equations (1) and (2) we have $(a+n)(x+l)(b+m) = 0 \ \forall \ x+l \in C(R)+N$.

Therefore $C(R)+N$ is semi-commutative. ☐

3.1.22 Definition. A ring R is called *locally finite* if every finite subset of R generates a finite semi-group multiplicatively.

3.1.23 Proposition. *Let R be a locally finite Armendariz ring. Then R is a semi-commutative ring.*

Proof. Let $a, b \in R$ satisfy $ab = 0$. Since R is a locally finite ring, we have for every $r \in R \ \exists$ integers $m, l \geq 1$ such that $r^m = r^{m+l}$. Then $r^{m+2l} = r^{m+l}r^l = r^{m+l}$.

By induction we have $r^m = r^{m(l+1)}$. Put $h = l+1$, then with $h \geq 2$

$$(r^m)^h = r^m \tag{*}$$

Now $r^{(h-1)m} = r^{(h-2)m+m} = r^{(h-2)m}r^m = r^{(h-2)m}(r^m)^h$ from (*) which implies that $r^{(h-1)m} = r^{hm-2m+hm} = r^{2hm-2m} = r^{2(h-1)m} = (r^{(h-1)m})^2$.

Hence $r^{(h-1)m}$ is an idempotent element of R.

Since R is Armendariz, we have $r^{(h-1)m}$ is a central element of R .

Therefore $ar^{(h-1)m}b = r^{(h-1)m}ab = 0$ which implies that $arb = 0$ (by Proposition (2.1.19)

Hence R is semi-commutative. ☐

3.1.24 Proposition. *Let R be a ring and Ω be a multiplicatively closed subset of R consisting of central non-zero- divisors. Then R is semi-commutative iff $\Omega^{-1}R$ is semi-commutative.*

Proof. ($\Rightarrow$) Suppose R is semi-commutative.Let $\alpha = u^{-1}a$,$\beta = v^{-1}b \in \Omega^{-1}R$, where $u, v \in \Omega$ and $a, b \in R$ satisfy $\alpha\beta = 0$. This implies that $u^{-1}av^{-1}b = 0 \Rightarrow u^{-1}v^{-1}ab = 0$ (since $u, v \in C(R)$) which implies that $(uv)^{-1}ab = 0 \Rightarrow ab = 0$.

Since R is semi-commutative, we have $axb = 0 \ \forall \ x \in R$.

Let $\gamma = w^{-1}x$ where $w \in \Omega, x \in R$, then $\alpha\gamma\beta = u^{-1}aw^{-1}xv^{-1}b = (uwv)^{-1}axb = 0$ (since $axb = 0 \ \forall \ x \in R$)

Therefore $\Omega^{-1}R$ is semi-commutative.

($\Leftarrow$) This is trivial as a subring of semi-commutative ring is again semi-commutative. ☐

3.1.25 Remark. The map $\theta : R \to \Omega^{-1}R$ defined as $\theta(a) = a/1$ is $1-1$

Proof : Suppose $\theta(a) = 0$, then $a/1 = 0$ which implies that there exists $w \in \Omega$ such that $wa = 0$. Since w is a non-zero-divisor we get $a = 0$ .Hence θ is $1-1$.

3.1.26 Corollary. *Let R be a ring. Then $R[x]$ is semi-commutative iff $R[x; x^{-1}]$ is semi-commutative.*

Proof. ($\Rightarrow$) Assume $R[x]$ is semi-commutative. Consider $\Omega = \{1, x, x^2, ...\}$. Since $R[x; x^{-1}] = \Omega^{-1}R[x]$, by Proposition (3.1.24) $R[x; x^{-1}]$ is semi-commutative.

($\Leftarrow$) Since $R[x]$ is a subring of $R[x; x^{-1}]$, therefore $R[x]$ is semi-commutative.

□

3.1.27 Proposition. *Let R be a ring and let n be a positive integer. If R is reduced, then $R[x]/(x^n)$ is a semi-commutative ring, where (x^n) is the ideal generated by x^n.*

Proof. Let $S = R[x]/(x^n)$. If $n = 1$, then $S \cong R$. Hence S is semi-commutative (since R is reduced).
For $n \geq 2$ put $u = x + (x^n)$, then $S = R[u]$.
Let $A = a_0 + a_1u + ... + a_{n-1}u^{n-1}, B = b_0 + b_1u + ... + b_{n-1}u^{n-1} \in S$ satisfy $AB = 0$. Note that for $i + j \geq n$, $a_ib_ju^{i+j} = 0$, so it is sufficient to check for the case $i + j < n$.
From $AB = 0$, we have

$$a_0b_0 = 0 \tag{1}$$

$$a_0b_1 + a_1b_0 = 0 \tag{2}$$

$$a_0b_2 + a_1b_1 + a_2b_0 = 0 \tag{3}$$

$$a_0b_{n-2} + a_1b_{n-3} + ... + a_{n-3}b_1 + a_{n-2}b_0 = 0 \tag{$n-1$}$$

$$a_0b_{n-1} + a_1b_{n-2} + ... + a_{n-2}b_1 + a_{n-1}b_0 = 0 \qquad (n)$$

Since R is reduced, we have $a_ib_j = 0\ \forall\ i,j \Rightarrow a_icb_j = 0....(*)\ \forall\ c \in R$ and forall i,j (since R is reduced).

Let $C = c_0 + c_1u + ...c_{n-1}u^{n-1}$ be any element of S.

From (*) we have $a_ic_kb_j = 0\ \forall\ i,j,k$.

Therefore $ACB = 0$

Hence S is semi-commutative. □

3.1.28 Remarks. (a) If $R[x]/(x^n)$ is semi-commutative, then R need not be reduced. Take $R = \mathbb{Z}_4$, then $R[x]/(x^n)$ is semi-commutative as R is a commutative ring.

(b) If $R[x]/(x^2)$ is semi-commutative, then R is also semi-commutative.

(c) If R is semi-commutative, then $R[x]/(x^2)$ need not be semi-commutative. Take $R = \mathbb{H}(+)\mathbb{H}$, then R is semi-commutative as $\mathbb{H}$ is reduced.

Denote $\overline{x} = x + (x^2)$

Let $A = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\overline{x}$ and $B = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\overline{x} \in \{\mathbb{H}(+)\mathbb{H}\}[x]/(x^2)$. Then we have $AB = 0$.

Take $C = \begin{pmatrix} j & k \\ 0 & j \end{pmatrix} + \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ then we have

$$ACB = [\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\overline{x}][\begin{pmatrix} j & k \\ 0 & j \end{pmatrix} + \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\overline{x}][\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\overline{x}]$$

$$= [\begin{pmatrix} 0 & ij \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}\overline{x} + \begin{pmatrix} j & k \\ 0 & j \end{pmatrix}\overline{x}][\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\overline{x}]$$

$$= [\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} j & i+k \\ 0 & j \end{pmatrix} \overline{x}][\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overline{x}]$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \overline{x} + \begin{pmatrix} 0 & ji \\ 0 & 0 \end{pmatrix} \overline{x} + 0$$

$$= \begin{pmatrix} 0 & -k \\ 0 & 0 \end{pmatrix} \overline{x} + \begin{pmatrix} 0 & -k \\ 0 & 0 \end{pmatrix} \overline{x} \text{ (since } ji = -k \text{)}$$

$$= \begin{pmatrix} 0 & -2k \\ 0 & 0 \end{pmatrix} \overline{x} \neq 0.$$

3.2 Semi-commutative modules

This section is devoted to the study of the basic properties and examples of semicommutative modules.

3.2.1 Definition. Let M be a left R-module. Then M is *semi–commutative* if whenever $a \in R$ and $m \in M$ satisfy $am = 0$, then we have $acm = 0 \ \forall$ $c \in R$.

3.2.2 Examples. (a) If R is a commutative ring, then M is semi-commutative as an R-module.

(b) A ring R is semi-commutative iff the module ${}_RR$ is semi-commutative.

(c) Submodules of semi-commutative modules are semi-commutative.

3.2.3 Proposition. *Let D be a domain and let M be a torsion free D-module. Then M is semi-commutative.*

Proof. Let $a \in R$ and $m \in M$ satisfy $am = 0$. Since M is torsion free, we have either $a = 0$ or $m = 0$.

Therefore for any $c \in R$ we get $acm = 0$.

Hence M is semi-commutative. □

3.2.4 Proposition. *If $\{M_i\}$ is a family of semi-commutative modules over R; then we have the following results.*

(1) The direct product $\prod M_i$ is semi-commutative.

(2) Direct sum $\bigoplus M_i$ is semi-commutative.

Proof. (1) Let $a \in R$ and $(m_i)_{i \in I} \in \prod M_i$ satisfy $a(m_i)_{i \in I} = 0$. Then we have $am_i = 0 \ \forall \ i \in I$.

Since M_i is semi-commutative, we get $acm_i = 0 \; \forall \; c \in R$. Hence $ac(m_i)_{i \in I} = 0$ $\forall \; i \in I$.

Therefore $\prod M_i$ is semi-commutative.

(2) This follows from Example (3.2.2(c)) as the direct sum of a family of modules is a submodule of the direct product of the family. ☐

3.2.5 Proposition. *Let R be a semi-commutative ring and let M be a free R-module. Then M is semi-commutative.*

Proof. Since M is a free R-module, it implies that $M \cong \bigoplus R$. Now using the fact that a direct sum of copies of R is semi-commutative as R is semi-commutative, we get M is also semi-commutative over R. ☐

3.2.6 Proposition. *Let R be a semi-commutative ring and let M be a projective R-module. Then M is semi-commutative.*

Proof. Since M is projective R-module, therefore M is isomorphic to a direct summand of a free R-module F which is semi-commutative as $F \cong \bigoplus R$ (since $\bigoplus R$ is semi-commutative),and by Example (3.2.2(c)) we have M is a semi-commutative R-module. ☐

3.2.7 Remarks. (a) If M is a free R-module, then M need not be semi-commutative.

Take $R = M_2(\mathbb{Z}), M = R$, then ${}_RM$ is free but ${}_RM$ is not semi-commutative.

(b) If R is reduced and M is a free R-module, then M is semi-commutative.

(c) If R is reduced and M is R-projective module, then M is semi-commutative.

3.2.8 Proposition. *Let D be a (not necessarily commutative) integral domain and let M be a D-bimodule. Then the ring $D(+)M$ is semi-commutative iff M is a ' semi-commutative D-bimodule '.*

Proof. ($\Rightarrow$) Suppose $D(+)M$ is semi-commutative.

Let $a \in D$ and $m \in M$ satisfy $am = 0$.

Then we have $(a, 0)(0, m) = (0, am) = 0$. Since $D(+)M$ is semi-commutative, we get $\forall$ $(c, n) \in D(+)M$ $(a, 0)(c, n)(0, m) = (0, acm) = 0$ which implies that $acm = 0$.

Therefore M is semi-commutative .

($\Leftarrow$) Suppose M is semi-commutative as a D-bimodule.

Let $(a, m), (b, n) \in D(+)M$ satisfy $(a, m)(b, n) = 0$. Then we get $(ab, an + mb) = (0, 0)$,i.e.,

$$ab = 0 \tag{$*$}$$

and

$$an + mb = 0 \tag{$**$}$$

From $ab = 0$ we have either $a = 0$ or $b = 0$ (since D is a domain).

Case (1) Suppose $a = 0$ then equation (**)reduces to $mb = 0$. Since M is semi-commutative we have for any $c \in D$

$$mcb = 0 \tag{$***$}$$

Now for any $(c, l) \in D(+)M$ we have $(0, m)(c, l)(b, n) = (0, mc)(b, n) = (0, mcb) = 0$ (from equation (***)).

Case (2) Suppose $b = 0$ then equation (*) becomes

$$an = 0 \tag{1}$$

Then for any $c \in D$ we get

$$acn = 0 \tag{2}$$

(M is semi-commutative).
Let (c, l) be any element of $D(+)M$. Then we get $(a, m)(c, l)(0, n) = (0, acn) = 0$ (from equation (2)). Therefore in both cases we have $D(+)M$ is semi-commutative. ☐

3.2.9 Remark. If the ring $R(+)M$ is semi-commutative, then the ring R is semi-commutative and the modules ${}_RM$, M_R are semi-commutative.

3.2.10 Proposition. *Let M be an R-bimodule. Then $R(+)M$ is semi-commutative if the following conditions are satisfied:*
(1) R is semi-commutative
(2) M is semi-commutative.
(3) If $ab = 0$ in R, then $aM \bigcap Mb = 0$.

Proof. Let $(a, m), (b, n) \in R(+)M$ satisfy $(a, m)(b, n) = (0, 0)$.
Then we get $(ab, an + mb) = (0, 0)$ which implies that $ab = 0$ and $an + mb = 0$.
From equation $an + mb = 0$ we get $an = 0 = mb$ (since $aM \bigcap Mb = 0$).

Since R and M are semi-commutative we get

$$acb = 0 \tag{1}$$

and

$$acn = 0 = mcb \tag{2}$$

$\forall\ c \in R$. Now for any element $(c, l) \in R(+)M$, we have

$$(a, m)(c, l)(b, n) = (ac, al+mc)(b, n) = (acb, acn+alb+mcb) = (0, 0+alb+0) \tag{*}$$

(from (1) and (2)) Since $alb \in aM \bigcap Mb = 0$ so we have $alb = 0$. Therefore we have $(a, m)(c, l)(b, n) = (0, alb) = (0, 0)$.

Hence $R(+)M$ is semi-commutative. □

3.2.11 Remark. If R is a semi-commutative ring and M is semi-commutative as a left as well as right- module over R, then $R(+)M$ need not be semi-commutative.

Let $R = \mathbb{H}(+)\mathbb{H}, M = \mathbb{H}(+)\mathbb{H}$. Then $R(+)M$ is not semi-commutative by Remark (3.1.3(c)).

3.2.12 Proposition. *Let D be a domain, $h : D \to D$ be a ring monomorphism and let M be a D- bimodule which is torsion free on both left and right. Then $D(+)_h M$ is semi-commutative.*

Proof. Let $(a, m), (b, n) \in D(+)_h M$ ($m \neq 0, n \neq 0$) satisfy $(a, m)(b, n) = 0$. Then we have

$$ab = 0, h(a)n + mb = 0 \tag{1}$$

Since D is a domain we get $a = 0$ or $b = 0$.

Case (1) $a = 0$. Then equation (1) reduces to $mb = 0$. Since M is torsion free on the right, we get $b = 0$.

Therefore for any $(c, l) \in D(+)_h M$ we have

$(0, m)(c, l)(0, n) = (0, mc)(0, n) = (0, 0)$.

Case (2) $b = 0$.

Now $b = 0$, implies that equation (1) reduces to $h(a)n = 0$. Since M is torsion free on the left, we get $h(a) = 0$.

Now $h(a) = 0$ implies $a = 0$ (since h is a ring monomorphism)

Therefore for any $(c, l) \in D(+)_h M$ we have

$(0, m)(c, l)(0, n) = 0$.

Hence $D(+)_h M$ is semi-commutative. □

3.2.13 Remarks. (a) Let K be a field, $h : K \rightarrow K$ be a ring monomorphism, then $K(+)_h M$ is semi-commutative for every vector space V over K .

(b) If R is not a domain, M is a 'free R-bimodule' and $h : R \rightarrow R$ is a ring monomorphism, then $R(+)_h M$ need not be semi-commutative.

Take $R = \mathbb{H}(+)\mathbb{H}, M = R$ and $h = I : R \rightarrow R$, then by Remark (3.1.13(b)) $R(+)_h M$ is not semi-commutative.

3.2.14 Proposition. *Let R be a reduced ring and let A be an ideal of R such that R/A is reduced. Then $R(+)(R/A)$ is semi-commutative.*

Proof. Let $(a, \overline{r}), (b, \overline{s}) \in R(+)(R/A)$ satisfy $(a, \overline{r})(b, \overline{s}) = 0$.

Then we get

$$ab = 0 \tag{1}$$

and

$$\overline{as} + \overline{rb} = 0 \tag{2}$$

Now multiplying equation (2) (from the right) by a, we get $\overline{asa} = 0$. Since R/A is reduced we get $\overline{as} = 0$. Similarly $\overline{rb} = 0$.
On using the condition that R/A is semi-commutative we get

$$\overline{acs} = 0 \tag{3}$$

We also have

$$\overline{atb} = 0 \tag{4}$$

and

$$\overline{rcb} = 0 \tag{5}$$

$\forall\ c, t \in R$.
Therefore for any $(c, \bar{t}) \in R(+)(R/A)$ we have
$(a, \bar{r})(c, \bar{t})(b, \bar{s}) = (acb, \overline{acs + atb + rcb}) = (0, 0)$ (since $acb = 0\ \forall\ c \in R$ and from equations (3),(4) and (5)).
Hence $R(+)(R/A)$ is semicommutative. ☐

3.2.15 Proposition. *Let $\theta : R \to A$ be a ring homomorphism and let M be an A-module. Regard M as a left R-module via θ. Then we have*
(1) If ${}_AM$ is semi-commutative, then ${}_RM$ is semi-commutative.
(2) If θ is onto, then the converse of the statement in (1) hold.
(3) If A is a semi-commutative ring, then A is semi-commutative as a left R-module.

Proof. (1) Let $r \in R$, $m \in M$ with $rm = 0$.
Then $\theta(r)m = 0$ (since $rm = \theta(r)m$).
Therefore for any $s \in R$, $(rs)m = \theta(rs)m = \theta(r)\theta(s)m = 0$ since $\theta(r)m = 0$ in ${}_AM$ and ${}_AM$ is semi-commutative.

Hence ${}_RM$ is semi-commutative.

(2) Let ${}_RM$ be a semi-commutative module. Let $a \in A$, $m \in M$ satisfy $am = 0$.
Since θ is onto, so given any $y \in A$ $\exists$ $x \in R$ such that $\theta(x) = y$.
Therefore for $a \in A$ $\exists$ $r \in R$ such that $\theta(r) = a$.
Now $am = 0$ implies $\theta(r)m = 0$, i.e.,

$$rm = 0 \tag{1}$$

Let $b \in A$, then $\exists$ $s \in R$ such that $\theta(s) = b$.
Therefore we have $abm = \theta(r)\theta(s)m = \theta(rs)m = rsm = 0$ (from (1) and ${}_RM$ is semi-commutative)
Hence ${}_AM$ is semi-commutative.

(3) A is a semi-commutative ring, then ${}_AA$ is a semi-commutative module (by Example 3.2.2(b)). Therefore on applying result (1) we get ${}_RA$ is semi-commutative. □

3.2.16 Proposition. *The following conditions are equivalent:*
(1) R is a semi-commutative ring
(2) Every torsionless R-module is semi-commutative.
(3) Every submodule of a free R-module is semi-commutative.
(4) There exists a faithful R-module which is semi-commutative.

Proof. The proof is the same as in the case of Armendariz modules (Replace ‘ Armendariz ‘ by ‘semi-commutative ‘ in the proof of Proposition (2.2.25)) □

3.2.17 Notation. Recall that we denote $End_R(M)$ by $E(M)$ and $R/ann(M)$ by $\overline{R}$.

3.2.18 Proposition. *Let M be an R-module. Consider the following conditions*

(1) The left R-module M is semi-commutative.

(2) The left $\overline{R}$-module M is semi-commutative.

(3) $\overline{R}$ is a semi-commutative ring

(4) The right $E(M)$-module M is semi-commutative

(5) The ring $E(M)$ is semi-commutative.

Then we have

(1) iff (2)

(2) $\Rightarrow$ (3)

(4) $\Rightarrow$ (5)

Proof. (1) $\Rightarrow$ (2) Define a map $\theta : R \to \overline{R}$ as $\theta(r) = r + ann(M)$, then θ is clearly onto.

Hence by Proposition (3.2.15) we have ${}_{\overline{R}}M$ is semi-commutative.

(2) $\Rightarrow$ (1) By Proposition (3.2.15) ${}_RM$ is semi-commutative.

(2) $\Rightarrow$ (3) Since M is faithful as a left $\overline{R}$-module, so we have $\overline{R} \leq \bigoplus M$ for some direct sum of copies of M.

Hence $\overline{R}$ is a semi-commutative ring (since M is semi-commutative).

(4) $\Rightarrow$ (5) Since M is faithful as a right $E(M)$-module, on applying (4) $\Rightarrow$ (1) of Proposition (3.2.16) we get the ring $E(M)$ is a semi-commutative ring. ☐

3.2.19 Remark. If $\overline{R}$ is a semi-commutative (Armendariz) ring, then the left R-module M need not be semi-commutative (Armendariz): Consider the module given in [AMB & MBR,Example 2.12].

3.2.20 Proposition. *A module M is semi-commutative iff every cyclic submodule of M is semi-commutative.*

Proof. ($\Rightarrow$) This is trivial as submodules of semi-commutative modules are again semi-commutative.
($\Leftarrow$) Let $m \in M$, $a \in R$ satisfy $am = 0$. Since $m \in M$ implies that $m \in Rm$, therefore by hypothesis we have $acm = 0 \ \forall \ c \in R$.
Hence M is a semi-commutative R-module. □

3.2.21 Definition. A ring R is said to be *left invariant* if every left ideal of R is two sided.

3.2.22 Proposition. *If the cyclic left R-module R/J is semi-commutative, then J is an ideal of R.*

Proof. Let $x \in J$, $r \in R$. Then $x.\bar{1} = x(1 + J) = x + J = 0$ which implies that $rx\bar{1} = r(x + J) = r0 = 0$, and this give $rx \in J$.
Also since $x\bar{1} = 0$ and R/J is semi-commutative over R, then for any $r \in R$, $xr\bar{1} = 0$ which implies that $xr \in J$.
Therefore J is an ideal of R. □

3.2.23 Remark. The converse of Proposition (3.2.22) is not true. Consider the ring in [AMB & MBR, Example 2.12]

3.2.24 Proposition. *The following conditions are equivalent*

(1) R is left invariant

(2) Every left R-module is semi-commutative.

(3) Every cyclic left R-module is semi-commutative.

Proof. (1) $\Rightarrow$ (2) Let M be a left R-module and let $m \in M$, $a \in R$ satisfy $am = 0$. Since the left ideal Ra is two-sided, $aR \subseteq Ra$.

Then for any $c \in R$, $acm \in aRm \subseteq Ram = 0$.

Therefore M is a semi-commutative left R-module.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1)Let J be a left ideal of R. Since R/J is cyclic left R-module, by hypothesis the left R-module R/J is semi-commutative. Then on applying Proposition (3.2.22) we get R is left invariant. ☐

3.2.25 Proposition. *Let M be an R-module and let $C = C(R)$. Then the following conditions are equivalent.*

(1) M is semi-commutative

(2) $T^{-1}M$ is a semi-commutative $T^{-1}R$-module for each multiplicatively closed subset T of C.

(3) M_P is a semi-commutative R_P-module for each $P \in Spec(C)$.

(4) M_Q is a semi-commutative R_Q-module for each $Q \in Max(C)$.

Proof. (1) $\Rightarrow$ (2) Let $\frac{m}{s} \in T^{-1}M$, $\frac{a}{t} \in T^{-1}R$ satisfy $\frac{a}{t}\frac{m}{s} = \frac{0}{1}$. Then $\exists\ u \in T$ such that $uam = 0$ which implies that $(ua)m = 0$.

Since M is semi-commutative, we have for any $c \in R$

$(ua)cm = 0 \Rightarrow uacm = 0 \Rightarrow \frac{a}{t}\frac{c}{1}\frac{m}{s} = \frac{0}{1}$.

Therefore for any $\frac{c}{r} \in T^{-1}R$, we have $\frac{a}{t}\frac{c}{r}\frac{m}{s} = \frac{1}{r}\frac{a}{t}\frac{c}{1}\frac{m}{s} = 0$.

Therefore $T^{-1}M$ is a semi-commutative $T^{-1}R$-module.

(2) $\Rightarrow$ (3) is trivial

(3) $\Rightarrow$ (4) is trivial

(4) $\Rightarrow$ Let $a \in R$, $m \in M$ satisfy $am = 0$. Then this implies that $\frac{a}{1}\frac{m}{1} = \frac{0}{1} \in M_Q$ and since M_Q is semi-commutative $\forall$ $Q \in Max(C)$ we get $\frac{a}{1}\frac{c}{1}\frac{m}{1} = 0$ which implies that $acm = 0$.

Hence M is a semi-commutative R-module. $\square$

3.3 Reversible rings

This section is devoted to the study of the basic properties and examples of reversible rings. We have followed [NKK & YL:2] for some of this material.

3.3.1 Definition. A ring R is called a *reversible* ring if whenever $a, b \in R$ satisfy $ab = 0$ we have $ba = 0$

3.3.2 Examples. (a) Every commutative ring is reversible.

(b) Every reduced ring R is reversible.

Proof : Let $a, b \in R$ satisfy $ab = 0$. Then $(ba)^2 = baba = 0$. Since R is reduced we get $ba = 0$. Hence R is reversible.

3.3.3 Proposition. *Let R be a reversible ring. Then R is semi-commutative.*

Proof. Let $a, b \in R$ satisfy $ab = 0$. Then by hypothesis we get $ba = 0$ and $bax = 0 \ \forall \ x \in R$. Since R is reversible we have $axb = 0 \ \forall \ x \in R$. Therefore whenever $ab = 0$, we have $axb = 0 \ \forall \ x \in R$. Hence R is semi-commutative. $\square$

3.3.4 Remarks. (a) If R is semi-commutative, then R need not be reversible.

Let R be a reduced ring. Consider $T_1 = \{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \}$.

Then by Proposition (3.1.14) we know that T_1 is semi-commutative .

Now let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in T_1$.Then $AB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\neq 0$ but $BA = 0$. Therefore T_1 is not reversible.

(b) If R' is reversible, then R' need not be Armendariz. Take $R' = \mathbb{Z}_4(+)\mathbb{Z}_4$, then this ring is reversible (since it is commutative) but R' is not Armendariz.

(c) If R is Armendariz, then R need not be reversible. Consider $T_1 = \{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R$ (reduced ring) $\}$. Then by Proposition (2.1.17) T_1 is Armendariz but by Remark (3.3.4(a)) T_1 is not reversible.

3.3.5 Proposition. *Reversible rings are abelian.*

Proof. This follow from the facts that every reversible ring is semi-commutative and semi-commutative rings are abelian. □

3.3.6 Proposition. *Let R be a reduced ring. Then $R(+)R$ is a reversible ring .*

Proof. Let $(a,b),(c,d) \in R(+)R$ satisfy $(a,b)(c,d) = 0$. Then we have

$$ac = 0 \tag{1}$$

and

$$ad + bc = 0 \tag{2}$$

Since R is reduced, from (1) we have $ca = 0$. Now multiplying (2) (from the left) by c, we have $cad + cbc = 0$ which implies that $cbc = 0$ (since $ca = 0$) and this gives $bc = 0$ (since R is reduced). Hence equation (2) reduces to $ad = 0$. Since R is reduced we get $cb = 0$,$da = 0$ and this gives $(c,d)(a,b) = 0$.

Therefore $R(+)R$ is reversible. □

3.3.7 Remarks. (a) If R is reversible, then $R(+)R$ need not be reversible. This follows from Remark (3.1.13(b)).

(b) If $R(+)R$ is reversible, then R need not be reduced. Take $R = Z_4$, then $\mathbb{Z}_4(+)\mathbb{Z}_4$ is semi-commutative (since it is commutative).

In the following proposition $PQ = \{ab : a \in P, b \in Q\}$ where $P, Q \subseteq R$.

3.3.8 Proposition. *For a ring R the following conditions are equivalent:*

(1) R is reversible

(2) $r_R(P) = l_R(P)$ for each $P \subseteq R$

(3) For each $a \in R, l_R(a) = r_R(a)$

(4) $PQ = 0$ implies $QP = 0$ for any two non-empty subsets P, Q of R.

Proof. (1) ⇒ (2) Assume R is reversible. Let $x \in l_R(P)$, then $Px = 0$. Since R is reversible we have $xP = 0$, which implies that $x \in r_R(P)$. Hence

$$l_R(P) \subseteq r_R(P) \qquad (*)$$

Now for $y \in r_R(P)$, $yP = 0$ and this gives $Py = 0$ (since R is reversible), which implies that $y \in l_R(P)$. Then we get

$$r_R(P) \subseteq l_R(P) \qquad (**)$$

Therefore from (*) and (**) we have $r_R(P) = l_R(P)$.

(2) ⇒ (3) is trivial (take $P = \{y\}$)

(3) ⇒ (4) Assume for each $a \in R$, $l_R(a) = r_R(a)$. Let P, Q be two non-empty subsets of R satisfy $PQ = 0$. Then we have $ab = 0$ for $a \in P$, $b \in Q$. By hypothesis we have $ba = 0 \; \forall \; a \in P, b \in Q$ .Hence $PQ = 0$

(4) ⇒ (1) Take $P = \{x\}, Q = \{y\}$ for $x, y \in R$.Then whenever $xy = 0$ we

have $yx = 0$ (by hypothesis).

Therefore R is reversible. ☐

3.3.9 Proposition. *The class of reversible rings is closed under (1) subrings and (2) direct products.*

Proof. (1) Let R be a reversible ring and S be a subring of R. Let $a, b \in S$ satisfy $ab = 0$. Since a, b are in R, we have $ba = 0$ (by hypothesis). Hence S is reversible .

(2) Let $\{R_i\}_{i \in I}$ be a family of reversible rings and let $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_I R_i$ satisfy $(x_i)_{i \in I}(y_i)_{i \in I} = 0$. This gives $(x_i y_i) = 0$ which implies that $x_i y_i = 0 \; \forall \; i \in I$. Hence $y_i x_i = 0 \; \forall \, i \in I$ (since R_i is reversible). Therefore $(y_i x_i)_{i \in I} = 0$ which gives $(y_i)_{i \in I}(x_i)_{i \in I} = 0$. Thus $\prod_I R_i$ is reversible. ☐

3.3.10 Proposition. *Suppose that R/I is a reversible ring for some ideal of a ring R. If I is reduced, then R is reversible.*

Proof. Let $a, b \in R$ satisfy $ab = 0$. Then $(a + I)(b + I) = ab + I = I$ (since $ab = 0$). Since R/I is reversible we have $ba \in I$. Now $(ba)^2 = baba = 0$ and since I is reduced we get $ba = 0$.

Therefore R is reversible . ☐

3.3.11 Remark. If R/I is reversible for some non-zero ideal I of R and the ideal I is also reversible (as a rng), then R need not be reversible. This can be seen as follows.

Let D be a division ring and consider $R = \{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in D\}$.

By Remark (3.3.4(a)) R is not reversible .

We have $I_1 = \begin{pmatrix} 0 & D & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, I_3 = \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}$ and $I_4 = \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are the only non-zero proper ideals of R. We have I_1 is not reversible, since for

$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in I_1$ $AB = 0$ but $BA \neq 0$.

We now show that I_2, I_3, I_4 are reversible. Let $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & c & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I_2$ satisfy $AB = 0$. Then $BA = 0$. Therefore I_2 is reversible.

Let $A = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \in I_3$ satisfy $AB = 0$. Then this gives $BA = 0$.

Therefore I_3 is reversible.

Similarly for any $A, B \in I_4$ if $AB = 0$, then $BA = 0$. Hence I_4 is reversible.

Now we shall show that for $j = 2, 3, 4$, R/I_j is reversible.

Case (1) $j = 2$ We have $R/I_2 = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} + I_2 \}$. Therefore for any

$\alpha = \begin{pmatrix} x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 0 \end{pmatrix} + I_2$ and

$\beta = \begin{pmatrix} t & 0 & 0 \\ 0 & t & z \\ 0 & 0 & 0 \end{pmatrix} + I_2 \in R/I_2$ satisfying $\alpha\beta = 0$ we have

$$xt = 0 \tag{1}$$

,

$$xz + yt = 0 \tag{2}$$

From (1) we have $tx = 0$ (since D is a reduced ring). Multiplying (2) from the left by t we get $tyt = 0$ which implies $ty = 0$. Therefore (2) reduces to $xz = 0$, which implies that $zx = 0$. Hence $\beta\alpha = 0$.

Thus R/I_2 is reversible.

Case (2) $j = 3$.

$R/I_3 = \{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + I_3 \}$. Let $A = \begin{pmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} + I_3$ and

$B = \begin{pmatrix} t & z & 0 \\ 0 & t & 0 \\ 0 & 0 & z \end{pmatrix} + I_3$ satisfy $AB = 0$. Then $xt = 0, xz + yt = 0$, on using the condition that D is reduced we get $tx = 0, zx = 0, ty = 0$. Therefore $BA = 0$. Hence R/I_3 is reversible.

Case (3) $j = 4$

We have $R/I_4 = \begin{pmatrix} a & b & 0 \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix}$. Let $A = \begin{pmatrix} x & y & 0 \\ 0 & x & z \\ 0 & 0 & x \end{pmatrix}$ and $B = \begin{pmatrix} r & s & 0 \\ 0 & r & t \\ 0 & 0 & r \end{pmatrix} \in R/I_4$ satisfy $AB = 0$. Then we have $xr = 0, xs+yr = 0, xt + zr = 0$. Therefore on using that D is reduced we get $rx = 0, sx = 0, ry = 0, tx = 0, rz = 0$. Hence R/I_4 is reversible.

3.3.12 Proposition. *Let R be a ring. Then eR and $(1-e)R$ are reversible for some central idempotent e of R iff R is reversible.*

Proof. Suppose eR and $(1-e)R$ are reversible for some central idempotent e of R. Then we have $R \cong eR \times (1-e)R$; hence R is reversible.
Conversely let $ea, eb \in eR$ satisfy $eaeb = 0$ which implies $eab = 0$. By hypothesis we have $ebea = 0$ which gives eR is reversible. Similarly $(1-e)R$ is reversible. □

3.3.13 Remarks. (a) Let R be an abelian ring. Then eR and $(1-e)R$ are reversible for some idempotent e of R iff R is reversible.
(b) If R is reversible, then eRe is reversible for every idempotent $e \in R$. This follows from the fact that R is reversible implies that R is abelian, then $eRe = eR$. Therefore by Proposition (3.3.12) eRe is reversible.
(c) However if eRe is reversible for any nontrivial non-identity idempotent element e of R, then R need not be reversible.
Consider $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$.Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. We have $AB = 0$ but $BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$. Therefore R is not reversible.

Notice that the only non-trivial non-identity idempotents of R are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and we have $eRe \cong \mathbb{Z}_2$. Therefore eRe is reversible.

3.3.14 Proposition. *Let R be a ring and Δ be a multiplicatively closed subset of R consisting of central non-zero-divisors of R. Then R is reversible iff $\Delta^{-1}R$ is reversible.*

Proof. Suppose R is reversible. Let $\alpha = u^{-1}a, \beta = v^{-1}b \in \Delta^{-1}R$ satisfy $\alpha\beta = 0$. Then $u^{-1}av^{-1}b = 0$ which implies $u^{-1}v^{-1}ab = 0$ (since Δ is contained in centre of R) i.e. $(uv)^{-1}ab = 0$ and so $ab = 0$. Thus we have $ba = 0$ (since R is reversible) which gives $v^{-1}u^{-1}ba = 0$, i.e., $v^{-1}bu^{-1}a = 0$, hence $\beta\alpha = 0$. Therefore $\Delta^{-1}R$ is reversible.

The converse is trivial (since a subring of a reversible ring is reversible). □

3.3.15 Corollary. *Let R be a ring. Then $R[x]$ is reversible iff $R[x, x^{-1}]$ is reversible.*

Proof. Suppose $R[x]$ is reversible. Let $\Omega = \{1, x, x^2, ...\}$ then clearly Ω is a multiplicatively closed subset of $R[x]$. Since $R[x, x^{-1}] = \Omega^{-1}R[x]$, it follows that $R[x, x^{-1}]$ is reversible by Proposition (3.3.14) .

Conversely, if $R[x, x^{-1}]$ is reversible, then $R[x]$ is reversible . □

3.3.16 Proposition. *Let R be an Armendariz ring. Then the following conditions are equivalent:*

(1) R is reversible.

(2) R[x] is reversible.

(3) $R[x, x^{-1}]$ is reversible.

Proof. (1) ⇒ (2) Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$. Since R is Armendariz, we have $a_i b_j = 0 \ \forall \ i, j$. By hypothesis we get $b_j a_i = 0 \ \forall \ i, j$, which implies that $g(x)f(x) = 0$. Therefore $R[x]$ is reversible.

(2) ⇒ (3) By Corollary (3.3.15)

(3) ⇒ (1) Since subrings of reversible rings are reversible, therefore R is reversible. □

3.3.17 Remarks. (a) If a ring R is reversible, then $R[x]$ need not be reversible. Take the ring in [NKK & YL:2, Example 2.1]

3.3.18 Definition. A ring R is called a *symmetric* ring if $rst = 0$ implies $rts = 0$ for $r, s, t \in R$.

3.3.19 Proposition. *Let R be a semi-prime ring. The following statements are equivalent*

(1) R is a reduced ring.

(2) R is symmetric ring.

(3) R is a reversible ring.

(4) R is a semi-commutative ring.

Proof. (1) ⇒ (2) Let R be a reduced ring and let $r, s, t \in R$ satisfy $rst = 0$.(Since $r(st) = 0$ we have $(st)r = 0$ and $(rs)t = 0$ implies $t(rs) = 0$). Now $rst = 0$ further implies $r(sts) = 0 \Rightarrow (sts)r = 0 \Rightarrow s(tsrt) = 0 \Rightarrow (tsrt)s = 0$

$\Rightarrow (tsrt)sr = 0 \Rightarrow (tsr)^2 = 0 \Rightarrow tsr = 0$ since R is reduced. From the condition $(ts)r = 0$ we get $rts = 0$. Therefore R is symmetric.

(2) $\Rightarrow$ (3) Let R be a symmetric ring and let $a, b \in R$ satisfy $ab = 0$. Now $ab = 0$ gives $1ab = 0$. By hypothesis we get $1ba = 0$, i.e., $ba = 0$. Thus R is reversible.

(3) $\Rightarrow$ (4) This follows from Proposition (3.3.5).

(4) $\Rightarrow$ (1) Let R be a semi-commutative ring and let $a \in R$ satisfy $a^2 = 0$. Now $a^2 = 0$ gives $aRa = 0$ (since R is semi-commutative)which implies that $(Ra)^2 = 0$. By the hypothesis ' R is semiprime ' we have $Ra = 0$, i.e., $a = 0$. Hence R is reduced. □

3.3.20 Proposition. *Let R be a reduced ring and n be a positive integer. Then the ring $R[x]/(x^n)$ is reversible.*

Proof. Let $H = R[x]/(x^n)$. If $n = 1$, then $H \cong R$. Hence H is reversible. When $n = 2$, $H = R[x]/(x^2)$, put $u = x + (x^2)$, then $H = R[u]$. Let $f(u) = a_0 + a_1 u, g(u) = b_0 + b_1 u \in R[u]$ satisfy $f(u)g(u) = 0$. Then we have $a_0 b_0 = 0, a_0 b_1 + a_1 b_0 = 0$. Since R is reduced we have $b_0 a_0 = 0, b_0 a_1 = 0, b_1 a_0 = 0$ which implies that $g(u)f(u) = 0$. Hence H is reversible.

For $n \geq 3$ put $u = x + (x^n)$. Let $A = a_0 + a_1 u + a_2 u^2 + ... + a_{n-1} u^{n-1}$ and $B = b_0 + b_1 u + ... + b_{n-1} u_{n-1} \in H$ satisfy

$$AB = 0 \qquad (*)$$

Now $a_i b_j u^{i+j} = 0 \; \forall \; i, j$ with $i + j \geq n$, therefore it is enough to prove for the case $i + j < n$.

From (*) we have

$$a_0 b_0 = 0 \qquad (1)$$

$$a_0b_1 + a_1b_0 = 0 \tag{2}$$

$$a_0b_2 + a_1b_1 + a_2b_0 = 0 \tag{3}$$

$$a_0b_{n-1} + a_1b_{n-2} + \ldots\ldots\ldots + a_{n-1}b_0 = 0 \tag{n}$$

Since R is reduced we get $a_ib_j = 0\ \forall\ i, j$. Hence $b_ja_i = 0\ \forall\ i, j$ which gives $g(u)f(u) = 0$. Therefore $H = R[x]/(x^n)$ is reversible. □

3.3.21 Remarks. (a) If $R[x]/(x^n)$ is reversible, then R need not be reduced.See the example in Remark (3.1.28(a))

(b) If $R[x]/(x^n)$ is reversible, then R is reversible.

(c) If R is reversible, then $R[x]/(x^n)$ need not be reversible.See the example considered in Remark (3.1.28(c)).

3.3.22 Definition. Let S be a commutative ring. By an S-algebra we mean a ring R such that $\forall\ t \in S$, $r \in R$, ta is defined and this definition makes R into an S-module satisfying ′ if $t \in S$ and $\alpha, \beta \in R$, then $(t\alpha)\beta = \alpha(t\beta) = t(\alpha\beta)$ ′.

3.3.23 Definition. Let R be an algebra over a commutative ring S. Then *theDorroh extension* of R by S is the ring $R \times S$ with operations $(r, s) + (t, u) = (r + t, s + u)$ and $(r, s)(t, u) = (rt + st + ur, su)$, where $r, t \in R$ and $s, u \in S$.

3.3.24 Proposition. *(1) Let R be a symmetric ring and let I be an ideal of R that is an annihilator in R. Then R/I is a reversible ring.*

(2) Let R be an algebra over a commutative ring S, and let D_0 be the Dorroh

extension of R by S. If R is reversible and S is a domain, then D_0 is also reversible.

(3) Let R be a commutative domain and let h be an injective endomorphism of R. Then $R(+)_hR$ is reversible.

Proof. (1) Let $J \subseteq R$ and put $I = r_R(J)$. Let $\overline{a}, \overline{b} \in R/I$ satisfy $\overline{a}\overline{b} = 0$. Now $\overline{ab} = 0$ implies that $ab \in I$, i.e., $Jab = 0$. Since R is symmetric we have $Jba = 0$,which gives $ba \in I$, i.e., $\overline{b}\overline{a} = 0$ and thus R/I is reversible.

(2) Let $(r,s),(t,u) \in D_0$ satisfy $(r,s)(t,u) = 0$. Now from $(r,s)(t,u) = 0$ we have

$$rt + ur + st = 0 \tag{$*$}$$

and

$$su = 0 \tag{$**$}$$

Since S is a domain, we have from (**) $s = 0$ or $u = 0$. suppose $s = 0$, then equation (*) reduces to $0 = rt + ur = r(t+u)$; but R is reversible so we have

$$0 = (t+u)r = tr + ur \tag{1}$$

and this gives $(t,u)(r,s) = (tr + st + ur, us) = (tr + ur, 0) = 0$ (from (1)); similarly when $u = 0$. Therefore D_0 is reversible.

(3) Let $(r,s),(u,v) \in R(+)_hR$ satisfy $(r,s)(u,v) = 0$. Then we have

$$ru = 0, h(r)v + us = 0 \tag{1$*$}$$

which implies $r = 0$ or $u = 0$ (since R is a domain).

Case (1) Let $r = 0$ then equation (1*) reduces to $us = 0$, i.e., $u = 0$ or $s = 0$; hence we get $h(u)s = 0$ and therefore $(u,v)(r,s) = (ur, h(u)s + rv) = 0$.

Case (2) Let $u = 0$. Then we get $h(r)v = 0$, which implies that $h(r) = 0$ or $v = 0$ (since R is a domain). If $h(r) = 0$, we get $r = 0$ (since h is injective) and therefore $(u, v)(r, s) = (ur, h(u)s + rv) = 0$. Hence $R(+)_h R$ is reversible. $\square$

3.3.25 Remarks. Let R be a commutative reduced ring and let h be an injective endomorphism of R. Then the Nagata extension of R by R and h need not be reversible.

We take $R = D \times D$ where D is a commutative domain of characteristic zero. Then R is a commutative reduced ring which is not a domain. Let $h : R \to R$ be defined as $h(r, s) = (s, r)$. Then h is an automorphism of R. Now for $((0,1),(1,0)),((1,0),(0,1)) \in R(+)_h R$ we have $((0,1),(0,1))((1,0),(0,1)) = (0, h((0,1))(0,1)+(1,0)(0,1) = 0$ but $((1,0),(0,1))((0,1),(0,1)) = (0, h((1,0))(0,1)+(0,1)(0,1)) = (0,(0,2)) \neq 0$.

Chapter 4

Some analogues of the Armendariz property

After the class of Armendariz rings had been extensively studied (by M.B.Rege, S.Chhawchharia, D.D.Anderson,V.Camillo, Y.Lee, N.K.Kim, C.Huh, A.Smoktunowicz, T.-K.Lee, Y.Zhou and others - see the References), a number of authors intoduced and studied analogues of the Armendariz property. Prominent among these analogues are the classes of weak Armendariz rings, quasi-Armendariz rings and Armendariz(PS) rings (studied by T.-K.Lee, T.L.Wong, Y.Hirano, A.M.Buhphang, M.B.Rege and others). These concepts have module analogues as well.Basic properties and examples of some of these analogues are recorded in various sections of this chapter.

4.1 Weak Armendariz rings

This section is devoted to the study of the basic properties and examples of weak Armendariz rings. This class of rings was introduced and studied by Lee and Wong in [TKL & TLW].

4.1.1 Definition. A ring R is called a *weak Armendariz ring* if whenever two linear polynomials $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x$ satisfy $(a_0 + a_1x)(b_0 + b_1x) = 0 \in R[x]$, then we have $a_ib_j = 0 \ \forall\ i, j = 0, 1$.

4.1.2 Examples. (a) Every Armendariz ring is weak Armendariz
(b)A subring of a weak Armendariz ring is again weak Armendariz
(c) The converse of (a) is not true. Take the ring $R = \mathbb{Z}_3[x, y]/(x^3, x^2y^2, y^3)$ where $\mathbb{Z}_3$ denote the Galois field of three elements, $\mathbb{Z}_3[x, y]$ denotes the polynomial ring over $\mathbb{Z}_3$ and (x^3, x^2y^2, y^3) is the ideal of R generated by x^3, x^2y^2 and y^3. (See the proof in [TKL & TLW, Example 3.2].)
(d) $M_n(R)$ is not weak Armendariz when $R \neq 0$ and $n \geq 2$.

4.1.3 Proposition. *A ring R is reduced iff $R(+)R$ is weak Armendariz.*

Proof. This follows from the proof of Theorem (2.1.14). ☐

4.1.4 Remark. If R is weak Armendariz, then $R(+)R$ need not be. See the examples studied in Remark (2.1.15).

4.1.5 Theorem. *Let R be a ring and $n \geq 2$ a natural number. Then $R[x]/(x^n)$ is weak Armendariz iff R is reduced.*

Proof. ($\Rightarrow$) Suppose $R[x]/(x^n)$ is weak Armendariz.Let $r \in R$ satisfy $r^2 = 0$. Consider two polynomials $f(T) = r + \overline{x}^{n-1}T, g(T) = r - \overline{x}^{n-1} \in$

$R[x][T]/(x^n)[T]$, then we have $f(T)g(T) = (r + \overline{x}^{n-1}T)(r - \overline{x}^{n-1}T) = r^2 - r\overline{x}^{n-1}T + r\overline{x}^{n-1}T - \overline{x}^{2n-2}T^2 = 0$ (since $r^2 = 0$ and for $n \geq 2$ we have $2n - 2 \geq n$; hence $\overline{x}^{2n-2} = 0$). By hypothesis we have $r\overline{x}^{n-1} = 0$ which implies that $r = 0$. Therefore R is reduced.

($\Leftarrow$) This follows from Proposition (2.1.28) □

4.1.6 Proposition. *Let D be a domain and A an ideal of D. Suppose D/A is weak Armendariz. Then $D(+)(D/A)$ is weak Armendariz.*

Proof. This follows from Proposition (2.1.12) □

4.1.7 Proposition. *Let R be a ring. Then R is weak Armendariz iff eR and $(1 - e)R$ are weak Armendariz for some central idempotent e of R.*

Proof. The proof is similar to the proof of Proposition (2.1.25) □

4.1.8 Proposition. *Suppose R/I is weak Armendariz for some ideal I of R. If I is reduced (as a ring without identity), then R is weak Armendariz*

Proof. Let $f(x) = a_0 + a_1x, g(x) = b_0 + b_1x \in R[x]$ satisfy

$$f(x)g(x) = 0 \tag{1}$$

Since R/I is weak Armendariz we have $a_ib_j \in I \; \forall \; i, j = 0, 1$. From (1) we have

$$a_0b_0 = 0 \tag{2}$$

$$a_0b_1 + a_1b_0 = 0 \tag{3}$$

$$a_1b_1 = 0 \tag{4}$$

Claim: $a_0b_1 = 0$. Since $a_0b_0 = 0$ implies $(b_0Ia_0)^2 = 0$ therefore by hypothesis we have $b_0Ia_0 = 0$ (since $b_0Ia_0 \subseteq I$). Hence $(a_1b_0)(a_0b_1)^2 = a_1b_0(a_0b_1)a_0b_1 \in a_1(b_0Ia_0)b_1 = 0$ (since $b_0Ia_0 = 0$) which implies that

$$(a_1b_0)(a_0b_1)^2 = 0 \qquad (*)$$

. Now multiplying equation (3) by $(a_0b_1)^2$ (from the right) we get $(a_0b_1)(a_0b_1)^2 + (a_1b_0)(a_0b_1)^2 = 0$ which implies that $(a_0b_1)^3 = 0$ (from (*)). Since I is reduced and $a_0b_1 \in I$, we therefore conclude that $a_0b_1 = 0$; hence $a_1b_0 = 0$. Thus R is weak Armendariz. □

4.1.9 Remark. If R/I is weak Armendariz for any non-zero proper ideal I of R and I is also weak Armendariz as a rng, then R need not be a weak Armendariz ring. Consider the example studied in Remark (2.1.30).

4.1.10 Remarks. (a) If R is weak Armendariz, then eRe is also weak Armendariz for every idempotent $e \in R$.
(b) If eRe is weak Armendariz for every non-identity idempotent e of R, then R need not be weak Armendariz. Consider the example studied in Remark (2.1.26(b)

4.1.11 Proposition. *Let R be a ring. Consider the following statements.*
(1) The ring R is a weak Armendariz ring
(2) Let $a, b, c \in R$ be such that $ac = 0 = b^2$. Then $abc = 0$.
(3) The ring R is abelian.
(4) The ring R is directly finite, i.e., $yx = 1$ whenever $xy = 1$ for $x.y \in R$. Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Rightarrow$ (2) Let $f(x) = a - abx, g(x) = c + bcx \in R[x]$ with $ac = 0 = b^2$. Then $f(x)g(x) = (a - abx)(c + bcx) = ac + abcx - abcx - ab^2cx^2 = 0$ and since R is weak Armendariz, we have $abc = 0$.

(2) $\Rightarrow$ (3) Let $e \in I(R)$ and $a \in R$, then $(1 - e)e = 0 = ((1 - e)ae)^2$. Hence by (2) we have $(1 - e)((1 - e)ae)e = 0$ which implies that $(1 - e)ae = 0$, i.e.,

$$ae = eae \tag{$*$}$$

Similarly $e(1 - e) = 0 = (ea(1 - e))^2$ implies $ea(1 - e) = 0$ i.e.

$$ea = eae \tag{$**$}$$

Therefore from (*) and(**) we have $ea = ae \ \forall \ a \in R$. Hence R is abelian.

(3) $\Rightarrow$ (4) See the proof of Proposition (2.1.22). ☐

4.1.12 Proposition. *Let R be a semiprime ring with the property that whenever $a, b, c \in R$ satisfy $ac = 0 = b^2$ we have $abc = 0$. Then the ring R is weak Armendariz.*

Proof. Consider $f(x) = a + bx, g(x) = c + dx \in R[x]$ with $f(x)g(x) = 0$. Then

$$ac = 0 = bd \tag{1}$$

and

$$ad + bc = 0 \tag{2}$$

For any $r \in R$ we have $(cra)^2 = 0$ (since $ac = 0$). Now we have $bd = 0 = (cra)^2$ and so by hypothesis we get $b(cra)d = 0$ i.e.

$$bcrad = 0 \tag{3}$$

On multiplying equation (2) by bcr (from the left) we get $bcrad + bcrbc = 0$ which implies that $bcrbc = 0 \ \forall \ r \in R$ (from (3)); hence we get $bcRbc = 0$ i.e. $(Rbc)^2 = 0$. By hypothesis we have $Rbc = 0$ which implies that $bc = 0$; hence $ad = 0$. Thus R is weak Armendariz. ☐

4.1.13 Remark. If R is a prime ring with the property that whenever $a, b, c \in R$ satisfy $ac = 0 = b^2$ we have $abc = 0$. Then R is weak Armendariz.

4.1.14 Proposition. *Let R be a weak Armendariz ring and let $a, b, c \in R$ satisfy $ac = 0 = ab^{2^s}c = 0$ for some integer $s \geq 1$, then we have $abc = 0$.*

Proof. Let $f(x) = a - (ab^{2^{s-1}})x, g(x) = c - (b^{2^{s-1}}c)x \in R[x]$. Then we have $f(x)g(x) = ac + ab^{2^{s-1}}cx - ab^{2^{s-1}}cx - ab^{2^s}cx^2 = 0$ (since by hypothesis we have $ac = 0$ and $ab^{2^s}c = 0$). As R is weak Armendariz we deduce $ab^{2^{s-1}}c = 0$. Continuing this way, we get $abc = 0$. ☐

4.1.15 Remark. Let R be a weak Armendariz ring. Suppose $ac = 0 = b^n$ for $a, b, c \in R$ and some integer $n \geq 1$, then $abc = 0$.

Choose a positive integer s such that $2^s \geq n$, then we have $ab^{2^s}c = 0$. Therefore by the above proposition we get $abc = 0$.

4.1.16 Proposition. *Let R be a weak Armendariz ring and let I be an ideal of R. Then $R/l_R(I)$ is a weak Armendariz.*

Proof. The proof is similar to that of Proposition (2.1.16). ☐

4.1.17 Proposition. *Let R be a regular ring. Then the following statements are equivalent*

(1) R is Armendariz

(2) R is reduced

(3) R is semi-commutative

(4) R is weak Armendariz

(5) R is reversible.

Proof. (The equivalence of (1), (2) and (3) has already been shown in Proposition 3.1.8.)

The implication (1) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (2). Suppose R is weak Armendariz, then R is abelian by (4.1.11). Hence R is reduced (being abelian and regular), by Proposition (1.1.12).

(2) $\Rightarrow$ (5). This follows from the fact that every reduced ring is reversible.

(5) $\Rightarrow$ (3). This is recorded in (3.3.3).

Note that for (2) $\Rightarrow$ (5) $\Rightarrow$ (3) we do not need the regularity assumption. $\square$

4.2 Weak Armendariz modules

This section is devoted to the study of the basic properties and examples of weak Armendariz modules. This class of modules was introduced and studied in [MBR & AMB].

4.2.1 Definition. An R module M is called a *weak Armendariz module* over R if whenever $f(x) = a_0 + a_1x \in R[x]$ and $m(x) = m_0 + m_1x \in M[x]$ satisfy $f(x)m(x) = 0$ we have $a_1m_0 = 0$.

4.2.2 Remarks. (a) Every Armendariz module is weak Armendariz.

(b) A module ${}_RR$ is weak Armendariz iff R is a weak Armendariz ring

(c) The class of weak Armendariz R-modules is closed under submodules, direct products and direct sums

(d) A weak Armendariz module over R need not be an Armendariz module.Take the ring R in [TKL & TLW, Example 3.2] (see Remark (4.2.2(b)) above) and $M = R$.

(e) An R-bimodule is called weakly Armendariz if it is weakly Armendariz on the left as well as on the right.

4.2.3 Definition. Let M be a left R-module. We call M *semiprime* if given $0 \neq m \in M$, there exists $f : M \to R$ such that $(mf)m \neq 0$.

4.2.4 Definition. Let R be a ring and let M be a left R-module. An element m of M is called *regular* if there exists $f \in Hom_R(M, R)$ with $(mf)m = m$. The module M is *regular* if every element of M is regular.

4.2.5 Definition. Let R be a ring and let M be a left R-module. Then M is called an *anti-regular* module if for every non-zero $m \in M$ there exists a non-zero $f \in Hom_R(M, R)$ such that $f(mf) = f$.

4.2.6 Remarks. (a) Every regular module is semiprime

(b) Every anti-regular module is semiprime.

4.2.7 Definition. An R-module$M$ is called *cyclically semiprime* if every cyclic submodule of M is semiprime.

4.2.8 Theorem. *Let M be a cyclically semiprime module which satisfies the property that whenever $a, b \in R, m \in M$ satisfy $am = 0$ and $b^2 = 0$ we have $abm = 0$. Then M is a weak Armendariz module.*

Proof. Let $f(x) = a_0 + a_1x \in R[x]$ and $g(x) = m_0 + m_1x \in M[x]$ satisfy $f(x)g(x) = 0$. Then we have

$$a_0m_0 = 0 \tag{1}$$

$$a_0m_1 + a_1m_0 = 0 \tag{2}$$

$$a_1m_1 = 0 \tag{3}$$

Suppose $a_0m_1 \neq 0$. Then by hypothesis there exists $\theta \in Hom_R(Rm_1, R)$ such that $((a_0m_1)\theta)a_0m_1 \neq 0$. From equation (2) we have $a_0m_1 = -a_1m_0$, then $0 \neq -((a_0m_1)\theta)a_1m_0$ implies $0 \neq ((a_0m_1)\theta)a_1m_0$, i.e., $0 \neq a_0(m_1\theta)a_1m_0$. Write $b = (m_1\theta)a_1$, then we get $b^2 = (m_1\theta)a_1(m_1\theta)a_1 = (m_1\theta)(a_1m_1\theta)a_1 = 0$ (from (3)). By hypothesis and From (1) and $b^2 = 0$ ($b = (m_1\theta)a_1$) we have $a_0bm_0 = 0$ i.e. $a_0(m_1\theta)a_1m_0 = 0$ which contradicts that $a_0(m_1\theta)a_1m_0 \neq 0$; hence our assumption that $a_0m_1 \neq 0$ is wrong. So we get $a_0m_1 = 0 = a_1m_0$. Thus M is a weak Armendariz module. □

4.2.9 Proposition. *Let M be an R- bimodule. Then $R(+)M$ is a weak Armendariz ring if the following conditions hold:*

(1) R is a weak Armendariz ring

(2) M is a weak Armendariz bimodule

(3) If $f(x) = a_0 + a_1x, g(x) = b_0 + b_1x \in R[x]$ satisfy $f(x)g(x) = 0$ then we have $(f(x)M(x)) \bigcap (M[x]g(x)) = 0$

Proof. Suppose that all the three conditions are satisfied.Let $(f(x), m(x)) = (\sum_{i=0}^{1} a_ix^i, \sum_{i=0}^{1} m_ix^i), (g(x), l(x)) = (\sum_{j=0}^{1} b_jx^j, \sum_{j=0}^{1} l_jx^j) \in \{R(+)M\}[x]$ satisfy $(f(x), m(x))(g(x), l(x)) = 0$.Then we have $f(x)g(x) = 0$ and

$$f(x)l(x) + m(x)g(x) = 0 \qquad (*)$$

By condition (1) we have $a_ib_j = 0 \ \forall \ i, j = 0, 1$.From equation (*) we have $f(x)l(x) = -m(x)g(x) \in f(x)M[x] \bigcap M[x]g(x) = 0$ (using condition (3) which implies that $f(x)l(x) = -m(x)g(x) = 0$). On using condition (2) we get $a_il_j = 0$ and $m_ib_j = 0 \ \forall \ i, j = 0, 1$. Thus $R(+)M$ is a weak Armendariz ring. ☐

4.2.10 Remark. If R is weak Armendariz and R/A is a weak Armendariz ring for some ideal A of R, then $R(+)(R/A)$ need not be weak Armendariz. See the example studied in Remark (2.2.22)

4.2.11 Proposition. *Let D be an integral domain. Then $D(+)M$ is a weak Armendariz ring iff M is a weak Armendariz D-bimodule.*

Proof. The proof is similar to that of Proposition (2.2.11) ☐

4.2.12 Proposition. *If $\theta : R \to A$ is a ring homomorphism and M is an A-module, regarding M as a left R-module via θ , we have the following results:*

(1) If ${}_AM$ is weak Armendariz, then ${}_RM$ is weak Armendariz.

(2) If θ is onto, then the converse of the statement in (1) hold.

(3) If A is a weak Armendariz ring, then A is weak Armendariz as a left R-module.

Proof. The proof is similar to that of Proposition (2.2.23) ☐

4.2.13 Proposition. *The following statements are equivalent*

(1) R is a weak Armendariz ring.

(2) Every torsionless R-module is weak Armendariz

(3) Every submodule of a free R-module is weak Armendariz.

(4) There exists a faithful R-module which is weak Armendariz.

Proof. The proof is similar to that of Proposition (2.2.25) ☐

4.2.14 Proposition. *A module M is weak Armendariz iff every submodule of M generated by at most two elements is weak Armendariz.*

Proof. The proof is similar to that of Proposition (2.2.29) ☐

4.2.15 Proposition. *Let M be an R-module and let $C = C(R)$. Then the following statements are equivalent.*

(1) M is weak Armendariz.

(2) $T^{-1}M$ is weak Armendariz $T^{-1}R$-module for each multiplicatively closed subset T of C.

(3) M_P is a weak Armendariz R_P-module for each $P \in Spec(C)$.

(4) M_Q is a weak Armendariz R_Q-module for each $Q \in Max(C)$.

Proof. The proof is similar to that of Proposition (2.2.33) ☐

4.2.16 Proposition. *Let D be a commutative domain and M a D-module. Then the module M is weak Armendariz iff its torsion submodule $T(M)$ is weak Armendariz.*

Proof. The proof is similar to that of Proposition (2.2.34) $\square$

4.3 Armendariz (PS) rings

This section is devoted to the study of the basic properties and examples of Armendariz(PS) rings.This class of rings was introduced and studied in [MBR & AMB].

4.3.1 Definition. A ring R is called an *Armendariz (PS) ring* if whenever $f(x) = \sum a_i x^i, g(x) = \sum b_j x^j \in R[[x]]$ satisfy $f(x)g(x) = 0$ we have $a_i b_j = 0$ $\forall\ i, j$.

4.3.2 Examples. (a) If R is a domain, then R is an Armendariz (PS) ring.
(b) Direct product of two Armendariz (PS) rings R_1, R_2 is again Armendariz (PS).
(c) Any direct product of Armendariz (PS) rings is Armendariz (PS).
(d) Subrings of Armendariz (PS) rings are Armendariz (PS).

4.3.3 Remark. If R is an Armendariz (PS) ring, then R is Armendariz.

4.3.4 Proposition. *Let R be a reduced ring. Whenever $f(x) = \sum a_i x^i, g(x) = \sum b_j x^j = 0 \in R[[x]]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0\ \forall\ i, j$.*

Proof. The proof is similar to that of Proposition (2.1.3) ☐

4.3.5 Remark. Every reduced ring is Armendariz (PS).

4.3.6 Proposition. *For a ring R, the trivial extension $T(R, R)$ is Armendariz (PS) iff R is reduced.*

Proof. The proof is similar to that of Proposition (2.1.14). ☐

4.3.7 Proposition. *Let R be a reduced ring. Then the ring*

$$T_1 = \{ \begin{pmatrix} a_1 & a_2 & a & b \\ 0 & a_1 & a_2 & c \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix} | a_1, a_2, a, b, c \in R\}$$ *is Armendariz (PS).*

Proof. Let $f(x) = \begin{pmatrix} f_1(x) & f_2(x) & f_3(x) & f_4(x) \\ 0 & f_1(x) & f_2(x) & f_5(x) \\ 0 & 0 & f_1(x) & f_2(x) \\ 0 & 0 & 0 & f_1(x) \end{pmatrix}$

$$= \begin{pmatrix} \sum_{i=0}^{\infty} a_{1i}x^i & \sum_{i=0}^{\infty} a_{2i}x^i & \sum_{i=0}^{\infty} a_{3i}x^i & \sum_{i=0}^{\infty} a_{4i}x^i \\ 0 & \sum_{i=0}^{\infty} a_{1i}x^i & \sum_{i=0}^{\infty} a_{2i}x^i & \sum_{i=0}^{\infty} a_{5i}x^i \\ 0 & 0 & \sum_{i=0}^{\infty} a_{1i}x^i & \sum_{i=0}^{\infty} a_{2i}x^i \\ 0 & 0 & 0 & \sum_{i=0}^{\infty} a_{1i}x^i \end{pmatrix}$$

and $g(x) = \begin{pmatrix} g_1(x) & g_2(x) & g_3(x) & g_4(x) \\ 0 & g_1(x) & g_2(x) & g_5(x) \\ 0 & 0 & g_1(x) & g_2(x) \\ 0 & 0 & 0 & g_1(x) \end{pmatrix}$

$= \begin{pmatrix} \sum_{j=0}^{\infty} b_{1j}x^j & \sum_{j=0}^{\infty} b_{2j}x^j & \sum_{j=0}^{\infty} b_{3j}x^j & \sum_{j=0}^{\infty} b_{4j}x^j \\ 0 & \sum_{j=0}^{\infty} b_{1j}x^j & \sum_{j=0}^{\infty} b_{2j}^{j} & \sum_{j=0}^{\infty} b_{5j}x^j \\ 0 & 0 & \sum_{j=0}^{j} b_{1j}x^j & \sum_{j=0}^{\infty} b_{2j}x^j \\ 0 & 0 & 0 & \sum_{j=0}^{\infty} b_{1j}x^j \end{pmatrix} \in T_1[[x]]$ satisfy

$$f(x)g(x) = 0 \tag{1}$$

From (1) we have

$$f_1(x)g_1(x) = 0 \tag{2}$$

$$f_1(x)g_2(x) + f_2(x)g_1(x) = 0 \tag{3}$$

$$f_1(x)g_3(x) + f_2(x)g_2(x) + f_3(x)g_1(x) = 0 \tag{4}$$

$$f_1(x)g_4(x) + f_2(x)g_5(x) + f_3(x)g_2(x) + f_4(x)g_1(x) = 0 \tag{5}$$

$$f_1(x)g_5(x) + f_2(x)g_2(x) + f_5(x)g_1(x) = 0 \tag{6}$$

using the condition that R is reduced we get $f_1(x)g_2(x) = 0, f_2(x)g_1(x) = 0, f_1(x)g_3(x) = 0, f_2(x)g_2(x) = 0, f_3(x)g_1(x) = 0, f_1(x)g_4(x) = 0, f_2(x)g_5(x) = 0, f_3(x)g_2(x) = 0, f_4(x)g_1(x) = 0, f_1(x)g_5(x) = 0, f_5(x)g_1(x) = 0$; hence $a_{1i}b_{1j} = 0, a_{1i}b_{2j} = 0, a_{2i}b_{1j} = 0, a_{1i}b_{3j} = 0, a_{2i}b_{2j} = 0, a_{3i}b_{1j} = 0, a_{1i}b_{4j} = 0, a_{2i}b_{5j} = 0, a_{3i}b_{2j} = 0, a_{4i}b_{1j} = 0, a_{1i}b_{5j} = 0, a_{5i}b_{1j} = 0 \ \forall\ i, j$, and these implies that

$$\begin{pmatrix} a_{1i} & a_{2i} & a_{3i} & a_{4i} \\ 0 & a_{1i} & a_{2i} & a_{5i} \\ 0 & 0 & a_{1i} & a_{2i} \\ 0 & 0 & 0 & a_{1i} \end{pmatrix} \begin{pmatrix} b_{1j} & b_{2j} & b_{3j} & b_{4j} \\ 0 & b_{1j} & b_{2j} & b_{5j} \\ 0 & 0 & b_{1j} & b_{2j} \\ 0 & 0 & 0 & b_{1j} \end{pmatrix} = 0\ .$$

Thus T_1 is Armendariz (PS). □

4.3.8 Remarks. (a) If R is reduced, then $R_0 = \{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in R\}$ is Armendariz (PS).

(b) If R is a reduced, then the ring $R_1 = \{ \begin{pmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & g \\ 0 & 0 & 0 & a \end{pmatrix} | a, b, c, d, e, f, g \in R\}$ is not Armendariz (PS). See the proof in Remark (2.1.18)

4.3.9 Proposition. *Suppose that the ring R is abelian. Then every idempotent of $R[[x]]$ ($R[x]$) is in R and $R[[x]]$ ($R[x]$) is abelian.*

Proof. Let $f(x) = \sum_{i=0}^{\infty} e_i x^i \in R[[x]]$. Assume $(f(x))^2 = f(x)$, then we have

$$e_0^2 = e_0 \tag{0}$$

$$e_0e_1 + e_1e_0 = e_1 \tag{1}$$

$$e_0e_2 + e_1e_1 + e_2e_0 = e_2 \tag{2}$$

$$e_0e_n + e_1e_{n-1} + ... + e_ne_0 = e_n \tag{n}$$

From equation (0) we have $e_0 \in I(R)$ so e_0 is central.
Now multiplying equation (1) (from the left)by e_0 we get
$e_0e_1 + e_0e_1e_0 = e_0e_1$ and this implies $e_0e_1e_0 = 0$ which implies $e_1e_0^2 = 0$ (since $e_0 \in C(R)$). This shows that $e_1e_0 = 0$
Hence equation (1) becomes $e_1 = 0$ (since $e_0e_1 = e_1e_0$)
Since $e_1 = 0$ equation (2) reduces to $e_0e_2 + e_2e_0 = e_2....(2)'$
Again multiplying equation $(2)'$ by e_0 (from the left) we get $e_0e_2e_0 = 0 \Rightarrow e_0e_2 = 0$. Hence $e_2 = 0$.
Assume that the result is true $\forall\ 1 \leq i \leq k$ (k is a positive integer) i.e $e_i = 0$ $\forall\ 1 \leq i \leq k$. Then the $(k+1)^{th}$ equation becomes $e_0e_{k+1} + e_{k+1}e_0 = e_{k+1}$. In multiplying by e_0 we get $e_{k+1} = 0$. Hence the result is true for any positive integer n, i.e., $e_0^2 = e_0$ and $e_i = 0\ \forall\ i$. This implies $f(x) = e_0 \in R$ and so $R[[x]]$ is abelian. ☐

4.3.10 Definition. A ring R is called a *Baer ring* if the right annihilator of every non-empty subset of R is generated by an idempotent.

4.3.11 Remark. The defining condition for a Baer ring is easily seen to be left-right symmetric.

4.3.12 Proposition. *For a ring R the following are equivalent*
(1) Every principal left ideal of R is projective.
(2) The left annihilator of each element of R is generated by some $e \in I(R)$.

Proof. (1) $\Rightarrow$ (2) For $x \in R$ consider the exact sequence
$0 \to l(x) \xrightarrow{j} R \xrightarrow{\theta} Rx \to 0$ where $\theta : R \to Rx$ is defined as $\theta(r) = rx$. Since Rx is projective, $\exists$ an R-homomorphism $\phi : Rx \to R$ such that $\theta o \phi$ is the identity map of Rx. Hence we have $l(x) = Re$ for some $e \in I(R)$. (We use the following theorem: "The left ideal A of a ring R is a direct summand of ${}_RR$ iff $\exists\ e \in I(R)$ such that $A = Re$ ")
(2) $\Rightarrow$ (1) Let $x \in R$. Then by hypothesis $l(x) = Re$. Consider the exact sequence (where $\theta : R \to Rx$ is defined by $\theta(r) = rx$)
$0 \to l(x) \xrightarrow{j} R \xrightarrow{\theta} Rx \to 0$.
We have $Rx \cong R/(ker\theta) = R/Re \cong R/l(1-e) \cong R(1-e)$
But we have $Re \bigoplus R(1-e) = R$ which implies that $R(1-e)$ is a direct summand of ${}_RR$. Hence $R(1-e)$ is projective as a left R-module. Therefore Rx is projective (since $Rx \cong R(1-e)$). □

4.3.13 Definition. A ring R is called a *left P.P. – ring* (also called as left p.p.-ring) if it satisfies the equivalent condition of Proposition (4.3.12)

4.3.14 Remarks. (a) A ring R is called a *P.P. – ring* if it is both a left and right P.P.-ring

(b) Baer rings are left and right P.P.

4.3.15 Proposition. *Let R be an Armendariz ring. Then R is a right P.P.-ring iff $R[x]$ is a right P.P.-ring.*

Proof. ($\Rightarrow$) Suppose R is a right P.P.-ring. Let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$. Since R is a right P.P.-ring $\exists\ e_i = e_i^2 \in R$ such that $r_R(a_i) = e_i R\ \forall\ 0 \leq i \leq m$ Now consider $e = e_0 e_1 e_2 ... e_m$ then by hypothesis we have $e^2 = e \in R$ and also $eR = \bigcap_{i=0}^{m} r_R(a_i)$. Since $e \in r_R(a_i) \forall i$ we have $f(x)e = a_0 e + a_1 ex + ... + a_m ex^m = 0 \Rightarrow$

$$eR[x] \ \subseteq r_{R[x]}(f(x)) \qquad (1)$$

Let $g(x) = \sum_{j=0}^{n} b_j x^j \in r_{R[x]}(f(x))$, then $f(x)g(x) = 0$ and by hypothesis R is Armendariz we get $a_i b_j = 0\ \forall\ i, j$. This implies that $b_j \in eR\ \forall j$.Hence $g(x) \in eR[x]$ and this imply

$$r_{R[x]}(f(x)) \ \subseteq eR[x] \qquad (2)$$

from (1) and (2) we get $eR[x] = r_{R[x]}(f(x))$. Thus $R[x]$ is a right P.P.- ring .
($\Leftarrow$) Suppose $R[x]$ is a right P.P.-ring. By Proposition (4.3.9) every idempotent of $R[x]$ is in R, so for any $a \in R, r_{R[x]}(a) = eR[x]$ for some idempotent $e \in R$.
Since $r_R(a) = r_{R[x]}(a) \bigcap R$, we get $r_R(a) = eR[x] \bigcap R = eR$.Therefore R is a right P.P.-ring. ☐

4.3.16 Proposition. *Let R be an Armendariz ring. Then R is a Baer ring iff $R[x]$ is a Baer ring.*

Proof. Suppose R is a Baer ring. Let V be a non-empty subset of $R[x]$ and let C_V denote the set of all coefficients of elements of V. Then $r_R(C_V) = eR$ for some $e \in R$. Since $e \in r_R(C_V) \Rightarrow e \in r_{R[x]}(V) \Rightarrow eR[x] \subseteq r_{R[x]}(V)$.....(1)
Let $g(x) = \sum_{j=0}^{n} b_j x^j \in r_{R[x]}(V)$. Then $Vg(x) = 0$ and so $f(x)g(x) = 0$ for any $f(x) = \sum_{i=0}^{m} a_i x^i \in V$. Since R is Armendariz, we get $a_i b_j = 0$ $\forall i, j$ which implies that $b_j \in r_R(C_V) = eR$. Therefore $b_j = ec_j$ for some $c_j \in R, J = 0, 1, ...n$ and this gives $g(x) = \sum_{j=0}^{n} ec_j x^j = e\sum_{j=0}^{n} c_j x^j \in eR[x]$ $\Rightarrow r_{R[x]}(V) \subseteq eR[x]$......(2)
Therefore from (1) and (2) we get $r_{R[x]}(V) = eR[x]$.
($\Rightarrow$) Suppose $R[x]$ is a Baer ring. By Proposition (4.3.9) every idempotent of $R[x]$ is in R, so for any $A \subset R, r_{R[x]}(A) = eR[x]$ for some $e \in R$. Since $r_R(A) = r_{R[x]}(A) \bigcap R$ we get $r_R(A) = eR[x] \bigcap R = eR$.
Hence R is a Baer ring ☐

4.3.17 Proposition. *Let R be an abelian ring.Then we have the following results*

(1) If $R[[x]]$ is a P.P.-ring, then R is a P.P.-ring.

(2) If $R[[x]]$ is a Baer ring, then R is a Baer ring

Proof. (1) Suppose $R[[x]]$ is a P.P.-ring and let $a \in R$. Then by hypothesis there exists $e \in I(R)$ such that $r_{R[[x]]}(a) = eR[[x]]$. Therefore $r_R(a) = eR$. Thus R is a P.P.-ring.
(2) The proof is similar to that of (1). ☐

4.3.18 Corollary. *Let R be an Armendariz ring. Then we have the following results*

(1) If $R[[x]]$ *is a P.P.-ring, then R is a P.P.-ring*

(2) If $R[[x]]$ *is a Baer ring, then R is a Baer ring.*

4.3.19 Remarks. The converse of Corollary (4.3.18(1)) does not hold. Take the ring R in [YL,NKK,& CYH, Example 1(1)]. Then R is a P.P.-ring (since R is a Boolean ring). Also R is Armendariz as it is reduced. But $R[[x]]$ is not a P.P.-ring by the argument in [YL & CH, Example 4].

4.4 Armendariz (PS) modules

This section is devoted to the study of the basic properties and examples of Armendariz(PS) modules. This class of modules was introduced and studied in [MBR & AMB].

4.4.1 Definition. A left R-module M is called an *Armendariz* (PS) module if whenever $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ satisfy $f(x)m(x) = 0$, we have $a_i m_j = 0 \; \forall \; i, j$.

4.4.2 Examples. (a) A ring R is an Armendariz(PS) ring iff ${}_R R$ is a Armendariz (PS) module.

(b) Submodules of Armendariz (PS) modules are Armendariz (PS).

(c) Direct product of Armendariz (PS) modules is Armendariz (PS).

(d) A direct sum of Armendariz (PS) modules is Armendariz (PS).

4.4.3 Proposition. *Let D be a domain and M a torsion free D-module. Then M is Armendariz (PS).*

Proof. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in D[[x]]$ and $0 \neq m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ satisfy $f(x)m(x) = 0$. Now M is torsion free implies $M[[x]]$ is also torsion free over $D[[x]]$; hence $f(x) = 0$. Therefore M is Armendariz (PS). □

4.4.4 Corollary. *Let D be a domain and M be a free D module. Then M is Armendariz (PS).*

4.4.5 Remark. If M is a module over a commutative domain, then $M/T(M)$ is a Armendariz (PS) module.

4.4.6 Proposition. *Let R be an Armendariz (PS) ring and M a free R-module. Then M is Armendariz (PS).*

Proof. Since M is a free R-module, we have $M \cong \bigoplus R($ for some direct sum of copies of R); hence M is Armendariz (PS). □

4.4.7 Corollary. *Let R be an Armendariz (PS) ring and let M be an R-projective module. Then M is Armendariz (PS).*

4.4.8 Remarks. (a) If R is reduced and M a free R-module, then M is Armendariz(PS).
(b) If R is reduced and M a projective R-module, then M is Armendariz (PS).

4.4.9 Proposition. *Let D be a commutative domain and M be an D-module. Then the ring $D(+)M$ is Armendariz (PS) if and only if M is an Armendariz (PS) module.*

Proof. The proof is similar to that of Proposition (2.2.11) □

4.4.10 Corollary. *If D is a commutative integral domain and M is torsion free D-module, then $D(+)M$ is Armendariz (PS).*

4.4.11 Corollary. *If D is a commutative domain and M a free D-module, then $D(+)M$ is Armendariz (PS).*

4.4.12 Proposition. *Let D be a commutative domain, $h : D \to D$ is a ring monomorphism and M is a torsion free D-module. Then $D(+)_hM$ is an Armendariz (PS) ring.*

Proof. The proof is same as in the case of Armendariz rings. □

4.4.13 Corollary. *Let K be a field , $h : K \to K$ a field monomorphism and V a K-vector space. Then the ring $K(+)_h M$ is Armendariz (PS)*

4.4.14 Proposition. *Let M be an R-bimodule. Then $R(+)M$ is an Armendariz (PS) ring if the following conditions are satisfied*
(1) R is Armendariz (PS)
(2) M is a Armendariz (PS) left and right R-module.
(3) If $f(x)g(x) = 0$ in $R[[x]]$, then $f(x)M[[x]] \bigcap M[[x]]g(x) = 0$.

Proof. The proof is similar to that in the Armendariz case. □

4.4.15 Corollary. *Let R be a reduced ring and R/A is also reduced for some ideal A. Then $R(+)(R/A)$ is Armendariz (PS).*

4.4.16 Proposition. *If ${}_RM$ is Armendariz (PS), then we have the following*
(1) ${}_RM$ is semi-commutative.
(2) ${}_{R[[x]]}M[[x]]$ is semi-commutative.

Proof. (1) Let $a \in R$, $m \in M$ satisfy $am = 0$. Consider $f(x) = a - acx \in R[[x]]$ and $m(x) = m + cmx + c^2mx^2 + c^3mx^3 + ... \in M[[x]]$, then we have
$f(x)g(x) = (a - acx)(m + cmx + c^2mx^2 + c^3mx^3 + ..) = am + acmx + ... + (-)acmx - ... = 0$.Since ${}_RM$ is Armendariz (PS), we get $acm = 0 \; \forall \; c \in R$.Hence ${}_RM$ is semi-commutative.
(2) Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ satisfy $f(x)m(x) = 0$. Then by hypothesis we get $a_i m_j = 0 \; \forall \; i, j$; hence $a_i b_k m_j = 0 \; \forall \; i, j, k$ and $\forall \; b_k \in R$ since R is semi-commutative. Therefore $f(x)g(x)m(x) = 0 \; \forall \; g(x) \in R[[x]]$. Thus ${}_{R[[x]]}M[[x]]$ is semi-commutative. □

4.4.17 Remarks. (a) If R is an Armendariz (PS) ring, then R and $R[[x]]$ are semi-commutative.

(b) We know that if R is an Armendariz (PS) ring, then R is Armendariz but the converse is not true.Take the ring R in [CH,YL & AS, Example 14], then the ring is Armendariz but not semi-commutative; hence R cannot be Armendariz (PS).

(c) If M is an Armendariz module over R, then M need not be a Armendariz (PS) module. Take $M = R$ in remark (b).

4.5 A characterization of Armendariz rings

We devote this section to a characterization of Armendariz rings given by Hirano in [YH]. Alternate proofs (due to Hirano) of some results (concerning polynomial rings over Baer rings and P.P.-rings) proved earlier by Kim and Lee in [NKK & YL:1] are also recorded.
Recall that the left (right) annihilator of a subset U of the ring R is defined by $l_R(U) = \{a \in R | aU = 0\}$ ($r_R(U) = \{a \in R | Ua = 0\}$)

4.5.1 Notation. We denote the set $\{r_R(U) | U \subset R\}$ by Γ or $rAnn_R(2^R)$ and $\{r_R(V) | V \subset R[x]\}$ by Δ or $rAnn_{R[x]}(2^{R[x]})$.

4.5.2 Remarks. (a) For a polynomial $f(x) \in R[x]$, we let C_f denote the set of coefficients of $f(x)$ and for a subset V of $R[x]$, we let C_V denote the set $\bigcup_{f \in V} C_f$. Then $r_{R[x]}(V) \bigcap R = r_R(V) = r_R(C_V)$.
(b) Let $\psi : \Delta \to \Gamma$ be defined as $\psi(I) = I \bigcap R$ for each $I \in \Delta$. Then ψ is surjective.
(c) Let $U \subseteq R$. Then $r_{R[x]}(U) = r_R(U)R[x]$.
(d) Let $\phi : \Gamma \to \Delta$ be the map defined by $\phi(I) = IR[x]$ for every $I \in \Gamma$. Then ϕ is injective.

4.5.3 Proposition. *Let $f(x)$ and $g(x)$ be two elements of $R[x]$. Then $f(x)Rg(x) = 0$ if and only if $f(x)R[x]g(x) = 0$.*

Proof. Suppose $f(x)Rg(x) = 0$. Let $h(x) = \sum_{k=0}^{l} c_k x^k$ be any element of $R[x]$. Then we have $f(x)h(x)g(x) = f(x)(\sum_{k=0}^{l} c_k x^k)g(x) = \sum_{k=0}^{l} f(x)c_k g(x)x^k = 0$ (by assumption). Since $h(x)$ is an arbitrary element of $R[x]$ we therefore conclude that $f(x)R[x]g(x) = 0$.

The converse is trivial. $\square$

4.5.4 Proposition. *Let R be a ring. The following statements are equivalent*

(1) R is Armendariz.

(2) $rAnn_R(2^R) \to rAnn_{R[x]}(2^{R[x]})$ defined by $A \to AR[x]$ is bijective.

Proof. (1) $\Rightarrow$ (2) Suppose R is Armendariz and let $\phi : rAnn_R(2^R) \to rAnn_{R[x]}(2^{R[x]})$ defined by $\phi(I) = IR[x]$ for every $I \in rAnn_R(2^R)$. Then by Remark (4.5.2(d)) ϕ is injective. Let S be a subset of $R[x]$ and let $f(x) = \sum_{i=0}^{m} a_i x^i \in S$. Now we first show $r_{R[x]}(f) = r_{R[x]}(C_f) = r_R(C_f)R[x]$. Let $g(x) = \sum_{j=0}^{n} b_j x^j \in r_{R[x]}(f)$. Then $f(x)g(x) = 0$ implies $a_i b_j = 0 \ \forall\ i,j$ (since R is Armendariz) and this gives $a_i g(x) = 0 \ \forall\ i$; hence we get

$$r_{R[x]}(f) \ \subseteq r_{R[x]}(C_f) \tag{1}$$

Let $g'(x) \in r_{R[x]}(C_f)$. Now $C_f g'(x) = 0$ implies $a_i g'(x) = 0 \ \forall\ a_i \in C_f$ implies $f(x)g'(x) = 0$, i.e., $g'(x) \in r_{R[x]}(f)$; hence we have

$$r_{R[x]}(C_f) \ \subseteq r_{R[x]}(f) \tag{2}$$

Therefore from (1) and (2) we get

$$r_{R[x]}(f) = r_{R[x]}(C_f) \tag{3}$$

Let $g(x) \in r_{R[x]}(C_f)$, then this implies that $a_i g(x) = 0 \ \forall\ a_i \in C_f$ and this implies $a_i b_j = 0 \ \forall\ i,j$ i.e. $g(x) \in r_R(C_f)R[x]$; hence

$$r_{R[x]}(C_f) \ \subseteq r_R(C_f)R[x] \tag{4}$$

Let $g'(x) \in r_R(C_f)R[x]$, then $g'(x) = ag''(x)$ where $a \in r_R(C_f)$ and $g''(x) \in R[x]$. Therefore we have $C_f ag''(x) = 0$ and this implies that $g(x) \in r_{R[x]}(C_f)$; hence

$$r_R(C_f)R[x] \ \subseteq r_{R[x]}(C_f) \tag{5}$$

Therefore from (3) ,(4) and (5) we have $r_{R[x]}(f) = r_{R[x]}(C_f) = r_R(C_f)R[x]$. Hence $r_{R[x]}(S) = \bigcap_{f\in S} r_{R[x]}(f) = \bigcap r_{R[x]}(C_f) = \bigcap r_R(C_f)R[x] = r_R(C_S)R[x] = \phi(r_R(C_S))$. Therefore ϕ is surjective.

(2) $\Rightarrow$ (1) Suppose (2) holds and let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$, then we have $r_{R[x]}(f) = AR[x]$ for some right ideal A of R. Let $g(x) = \sum_{j=0}^{n} \in R[x]$ satisfy $f(x)g(x) = 0$. Then this gives $g(x) \in r_{R[x]}(f)$, i.e., $g(x) \in AR[x]$; hence this gives $g(x) = \sum_{l=1}^{\lambda} \beta_l w_l(x)$ where $w_l(x) \in R[x]$ and $\beta_l \in A$. Let c_{l0} be the constant term of $w_l(x)$, then $b_0 = \sum_{l=1}^{\lambda} \beta_l c_{l0} \in A$. Similarly we get $b_j \in A \ \forall$ j. But $A \subseteq r_{R[x]}(f)$, therefore we have $b_j \in r_{R[x]}(f) \ \forall \ j$ i.e. $a_i b_j = 0 \ \forall \ i, j$. Hence R is Armendariz. □

Notice that on using the above proposition we can get alternate proofs of Propositions (4.3.15) and (4.3.16) as follows.

4.5.5 Corollary. *Let R be an Armendariz ring. Then R is a Baer ring iff $R[x]$ is a Baer ring .*

Proof. Suppose R is a Baer ring. Let $S \subseteq R[x]$. Then by hypothesis we have $r_R[x](S) = r_R(C_S)R[x]$. Since R is a Baer ring we have $r_R(C_S) = eR$ for some $e \in I(R)$; hence $r_{R[x]}(S) = (eR)R[x]$ i.e. $r_{R[x]}(S) = eR[x]$. Therefore $R[x]$ is a Baer ring.

Conversely suppose that $R[x]$ is a Baer ring. Let $U \subseteq R$, then $r_R(U) =$

$r_{R[x]}(U) \bigcap R = eR[x] \bigcap R = eR$ (since $R[x]$ is a Baer ring). Hence R is a Baer ring.

$\square$

4.5.6 Corollary. *Let R be an Armendariz ring. Then R is a P.P- ring if and only if $R[x]$ is a P.P- ring.*

Proof. Suppose R is a P.P- ring and let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. By hypothesis we have $r_{R[x]}(f(x)) = r_R(C_f)R[x]$ and by assumption we get $r_R(C_f) = eR$ for some idempotent $e \in R$. Therefore $r_{R[x]}(f(x)) = (eR)R[x] = eR[x]$. Hence $R[x]$ is a P.P ring.

Conversely suppose $R[x]$ is a P.P- ring. Let $a \in R$, then $r_R(a) = r_{R[x]}(a) \bigcap R$ and since $R[x]$ is a P.P- ring, we therefore have $r_{R[x]}(a) = eR[x]$; hence $r_R(a) = eR[x] \bigcap R = eR$. Thus R is a P.P- ring $\square$

4.6 Quasi-Armendariz rings

We devote this section to a study of quasi-Armendariz rings. This class of rings was introduced and studied by Hirano in [YH].

4.6.1 Definition. A ring R is called a $quasi-Armendariz$ ring if whenever $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then we have $a_i R b_j = 0 \; \forall \; i, j$.

4.6.2 Examples. Every Armendariz ring is quasi-Armendariz.

4.6.3 Remark. For commutative rings, the Armendariz and quasi-Armendariz conditions are equivalent.

4.6.4 Proposition. *For a ring R, the following statements are equivalent.*
(1) R is a quasi-Armendariz ring
(2) $\phi : rAnn_R(id(R)) \to rAnn_{R[x]}(id(R[x])); A \to AR[x]$ is bijective (where $id(R)$ is the set of ideals of R).

Proof. (1) $\Rightarrow$ (2) Suppose R is quasi-Armendariz. Let B be an ideal of $R[x]$ and let $f(x) = \sum_{i=0}^{m} a_i x^i \in B$. Then $\forall \; h(x) \in R[x]$, $h(x)f(x) \in B$. Now for $g(x) = \sum_{j=0}^{n} b_j x^j \in r_{R[x]}(B)$, we have $g(x)h(x)f(x) = 0 \; \forall \; h(x) \in R[x]$, i.e., $g(x)R[x]f(x) = 0$ and since R is quasi-Armendariz we get $g(x)Ra_i = 0$ and this gives

$$g(x)RC_f = 0 \qquad (1)$$

If $g(x)RC_f = 0$, we get

$$g(x)R[x]f(x) = 0 \qquad (2)$$

Therefore from (1) and (2) we have $r_{R[x]}(B) = \bigcap_{f(x) \in B} r_{R[x]}(f(x)) = \bigcap r_{R[x]}(RC_f) = r_R(RC_B)R[x]$ and this implies $r_{R[x]}(B) = \phi(r_R(RC_B))$; hence ϕ is surjective and by Remark (4.5.2(d)) we get ϕ is bijective.

(2) $\Rightarrow$ (1) Suppose (2) holds. Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy

$$f(x)R[x]g(x) = 0 \qquad (*)$$

From (*) we have $f(x)R[x]g(x)R[x] = 0$ and this implies that $f(x) \in r_{R[x]}(R[x]g(x)R[x])$, but

$$r_{R[x]}(R[x]g(x)R[x]) = AR[x] \qquad (**)$$

for some ideal A of R (since ϕ is surjective)

Therefore by the same arguments as given in the proof of Proposition (4.5.4) we get $a_i \in AR[x]$ (since $a_i \in A$ and $A \subseteq AR[x]$) $\forall$ i. From (**) we get $a_i R[x]g(x)R[x] = 0$ and this implies $a_i R[X]g(x) = 0$ i.e. $a_i R b_j = 0$ $\forall$ i, j. Hence R is quasi-Armendariz. □

4.6.5 Remark. Let R be a ring. Then the following conditions are equivalent

(1) R is quasi-Armendariz.

(2) $rAnn_R(id(R)) \rightarrow rAnn_{R[x]}(id(R[x])); A \rightarrow AR[x]$ is bijective.

(3) $lAnn_R(id(R)) \rightarrow lAnn_{R[x]}(id(R[x])); B \rightarrow R[x]B$ is bijective.

4.6.6 Definition. A ring R is a *subdirect sum* of the family of rings $\{R_i\}_{i \in I}$ if there is an injective homomorphism $f : R \rightarrow \prod_{i \in I} R_i$ such that for each $j \in I$ the map $\pi_j \circ f : R \rightarrow R_j$ is a surjective homomorphism, where $\pi_j : \prod_{i \in I} R_i \rightarrow R_j$ is the j^{th} projection.

4.6.7 Proposition. *If R be a subdirect sum of quasi-Armendariz rings, then R is a quasi-Armendariz ring.*

Proof. Given that R is a subdirect sum of a quasi-Armendariz rings say $\{R_i\}_{i\in I}$, there exists an injective homomorphism $\phi : R \to \prod_{i\in I} R_i$ such that $\pi_j \circ \phi : R \to R_j$ is surjective ($\pi_j : \prod R_i \to R_j$ is a projection map). Now $\pi_i \circ \phi : R \to R_j$ is surjective implies $R/Ker(\pi_j \circ \phi) \cong R_j$; hence $R/Ker(\pi_j \circ \phi)$ is quasi-Armendariz.Take $I_k = Ker(\pi_k \circ \phi), k \in I$,then we have $\bigcap_{k\in I} I_k = 0$.Suppose that two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$. Then we have $f(x)R[x]g(x) + I_k[x] = I_k[x]$ for each $k \in I$ and this implies that $a_i R b_j \subseteq I_k \ \forall \ i, j$ since R/I_k is quasi-Armendariz for each $k \in I$ .Therefore $a_i R b_j \subseteq \bigcap_{k\in I} I_k = 0$ i.e. $a_i R b_j = 0 \ \forall \ i, j$.Thus R is a quasi-Armendariz ring. □

4.6.8 Definition. A ring R is called a *prime* ring if whenever $IJ = 0$ for some ideals I, J, then we have $I = 0$ or $J = 0$.

4.6.9 Proposition. *Prime rings are quasi-Armendariz rings.*

Proof. Let R be a prime ring and let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$. Then $R[x]f(x)R[x]R[x]g(x)R[x] = 0$ and by hypothesis we have $R[x]f(x)R[x] = 0$ or $R[x]g(x)R[x] = 0$. If $R[x]f(x)R[x] = 0$ we get $a_i = 0 \ \forall \ i$; hence $a_i R b_j = 0 \ \forall \ i, j$. Similarly if $R[x]g(x)R[x] = 0$ we have $a_i R b_j = 0 \ \forall \ i, j$. Thus R is quasi-Armendariz. □

4.6.10 Corollary. *Semiprime rings are quasi-Armendariz rings*

Proof. If R is a semiprime ring, then R is a subdirect sum of prime rings; hence by Propositions (4.6.7) and (4.6.9) R is a quasi-Armendariz ring. □

4.6.11 Remark. Let R be a ring and suppose eR and $(1 - e)R$ are quasi-Armendariz for some central idempotent e of R. Then R is a quasi-Armendariz ring.

4.6.12 Notation. We denote $\begin{pmatrix} 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & 1 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}$ (where the i^{th} row, j^{th} column entry is 1 and the entries are zero everywhere else) by e_{ij} for $i, j = 1, 2, ...n$.

4.6.13 Theorem. *Let R be a quasi-Armendariz ring and let R_1 be a subring of $M_n(R)$ such that $e_{ii}R_1e_{jj} \subseteq R_1$ $\forall$ $i, j \in \{1, 2, ..., n\}$. Then R_1 also is a quasi-Armendariz ring.*

Proof. Let $f(x) = \sum_{k=0}^{m} \alpha^k x^k, g(x) = \sum_{l=0}^{n} \beta^l x^l \in R_1[x]$ satisfy $f(x)R_1[x]g(x) = 0$. From Proposition (4.5.3) we get

$$f(x)R_1g(x) = 0 \tag{1}$$

Now for any $ce_{pq} \in e_{pp}R_1e_{qq}$ where $c \in R$, we have $ce_{pq} \in R_1$ (since $e_{pp}R_1e_{qq} \subseteq R_1$); hence from (1) we get

$$g(x)ce_{pq}f(x) = 0inM_n(R[x]) \tag{2}$$

Equating the $(i, j)^{th}$ component of (2), we have $\sum_{t=0}^{m+n}(\sum_{r+s=t} \alpha_{ip}^r c \beta_{qj}^s)x^t = 0$ and this implies that

$$(\sum_{r=0}^{m} \alpha_{ip}^r x^r)c(\sum_{s=0}^{n} \beta_{qj}^s x^s) = 0 \tag{3}$$

Since $A = \{c \in R | ce_{pq} \in e_{pp}R_1e_{qq}\}$ is an ideal of R, therefore from (3) we have $(\sum_{r=0}^{m} \alpha_{ip}^r x^r)Rc(\sum_{s=0}^{n} \beta_{qj}^s x^s) = 0$ and since R is quasi-Armendariz we get $\alpha_{ip}^r c\beta_{qj}^s = 0 \ \forall\ r, s$. By the condition given on R_1 we have every element of R_1 is a sum of such ce_{pq}, we therefore conclude that $\alpha^r R_1 \beta^s = 0 \ \forall\ r, s$. Hence R_1 is a quasi-Armendariz ring. $\square$

4.6.14 Proposition. *If R is quasi-Armendariz ring, then for any idempotent $e \in I(R)$ the ring eRe is a quasi-Armendariz ring*

Proof. Let $f(x) = \sum_{i=0}^n a_i x^i, g(x) = \sum_{j=0}^m b_j x^j \in eRe[x]$ satisfy

$$f(x)eRe[x]g(x) = 0 \qquad (1)$$

Then $f(x)eR[x]eg(x) = 0$ and since $f(x), g(x) \in eRe[x]$ we have $f(x)e = f(x)$ and $eg(x) = g(x)$. From (1) we have $f(x)R[x]g(x) = 0$ and since R is quasi-Armendariz we have $a_i R b_j = 0 \, \forall\, i, j$. Also since $a_i e = a_i$ and $eb_j = b_j$ for each i, j; hence we have $a_i eReb_j = 0 \ \forall\ i, j$. Thus eRe is a quasi-Armendariz. $\square$

4.6.15 Proposition. *If R is a quasi-Armendariz ring, then $M_n(R)$ is quasi-Armendariz.*

Proof. This follows from the fact that $M_n(R)$ is a subring of itself which satisfy the condition that $e_{ii} M_n(R) e_{jj} \subseteq M_n(R) \ \forall\ i, j \in \{1, 2, ..., n\}$ and by Theorem (4.6.13) $\square$

4.6.16 Notation. $T_n(R)$ denotes the ring of all $n \times n$ upper triangular matrices over the ring R.

4.6.17 Corollary. *If R is a quasi-Armendariz ring, then for any positive integer n, the ring $T_n(R)$ also quasi-Armendariz.*

Proof. Since $T_n(R)$ is a subring of $M_n(R)$ and $e_{pp} T_n(R) e_{qq} \subseteq T_n(R)$ we therefore have by Theorem (4.6.13) $T_n(R)$ is quasi-Armendariz ring. $\square$

4.6.18 Remarks. (a) Let R be a quasi-Armendariz ring. Then a subring S of R need not be a quasi-Armendariz ring.

Consider $R = \mathbb{Z}_4$, then by Proposition (4.6.15) $M_2(R)$ is quasi-Armendariz. Now $S = \{\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{0} & \bar{a} \end{pmatrix} |\bar{a}, \bar{b} \in \mathbb{Z}_4\}$ is a subring of $M_2(R)$, but $S \cong \mathbb{Z}_4(+)\mathbb{Z}_4$; hence S is not quasi-Armendariz (since in commutative rings the Armendariz and quasi-Armendariz conditions are equivalent).

(b) If R is a quasi-Armendariz ring, then R need not be an Armendariz ring. Take $R = M_n(F)$ where F is a field and $n \geq 2$, then by Proposition (4.6.15) R is quasi-Armendariz but R is not Armendariz.

The following analogue of Proposition (2.1.24) also holds.

4.6.19 Proposition. *Let R be a quasi-Armendariz ring. Then the polynomial ring $R[x]$ is also a quasi-Armendariz ring.*

Proof. Let $f(T) = f_0 + f_1T + f_2T^2 + ... + f_nT^n$ and $g(T) = g_0 + g_1T + ... + g_mT^m \in R[x][T]$ where $f_i, g_j \in R[x]$ satisfy $f(T)R[x][T]g(T) = 0$ .From Proposition (4.5.3) we have

$$f(T)R[x]g(T) = 0 \tag{1}$$

Let $k = deg(f_0)+....+deg(f_n)+deg(g_0)+.....+deg(g_m)$ where $deg(f_i), deg(g_j)$ means the x-degree of polynomials f_i and g_j. Now $f(x^k) = f_0 + f_1x^k + ... + f_nx^{kn}$ and $g(x^k) = g_0 + g_1x^k + ... + g_mx^{km}$. Therefore, following the proof in the Armendariz case (using $f(x^k)R[x]g(x^k) = 0$ in $R[x]$), we get $f_iR[x]g_j = 0$ $\forall\ i, j$. Thus $R[x]$ is also quasi-Armendariz. □

4.6.20 Proposition. *Let D be a domain, and D/A be a quasi-Armendariz ring for some ideal A of D. Then $D(+)(D/A)$ is quasi-Armendariz.*

Proof. Let $(f_1(x), \overline{f_2(x)}) = \sum_{i=0}^{n}(a_i, \overline{u_i})x^i$ and $(g_1(x), \overline{g_2(x)}) = \sum_{j=o}^{m}(b_j, \overline{v_j})x^j \in \{D(+)(D/A)\}[x]$ satisfy $(f_1(x), \overline{f_2(x)})(D[x], D[x]/A[x])(g_1(x), \overline{g_2(x)}) = 0$. Then from Proposition (4.5.3) we get $(f_1(x), \overline{f_2(x)})(D(+)(D/A))(g_1(x), \overline{g_2(x)}) = 0$ and this gives

$$f_1(x)Dg_1(x) = 0 \tag{1}$$

and

$$f_1(x)D\overline{g_2(x)} + f_1(x)D/Ag_1(x) + \overline{f_2(x)}Dg_1(x) = 0 \tag{2}$$

Since D is a domain, we therefore have from (1) either $f_1(x) = 0$ or $g_1(x) = 0$. When $f_1(x) = 0$, equation (2) reduces to $\overline{f_2(x)}Dg_1(x) = 0$ i.e. $\overline{f_2(x)Dg_1(x)} = 0$ and since D/A is quasi-Armendariz we have $\overline{u_i}Db_j = 0 \ \forall \ i, j$; hence $(a_i, \overline{u_i})(D(+)(D/A))(b_j, \overline{v_j}) = 0 \ \forall \ i, j$. Similarly when $g_1(x) = 0$. Thus we have $D(+)(D/A)$ is quasi-Armendariz. □

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