ACCELERATION AND GRAVITATIONAL THERMODYNAMICS IN BRANE-GRAVITY BASED PHANTOM COSMOLOGY

BY

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DEPARTMENT OF MATHEMATICS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

To

NORTH-EASTERN HILL UNIVERSITY

SHILLONG - 793022

NOVEMBER, 2010

To

My Loving Mother Shrimati Sunanda Dutta

and

My Father (Late) Gopika Ranjan Dutta

CERTIFICATE

This is to certify that the thesis entitled "*ACCELERATION AND GRAVITATIONAL THERMODYNAMICS IN BRANE-GRAVITY BASED PHANTOM COSMOLOGY*" submitted by Jibitesh Dutta who got his name registered on 08.09.2008 for the award of Ph.D. degree in Mathematics of North-Eastern Hill University, is absolutely based upon his own work under the supervision of Dr. M. Ansari (Associate Professor), Department of Mathematics, North-Eastern Hill University, Shillong-793022 and Prof. Subentry Chakraborty (D. Sc.), Department of Mathematics, Jadavpur University, Calcutta-700032, and that neither this thesis nor any part of its has been submitted for any degree / diploma or any other academic award anywhere before.

We certify that the sources from which ideas borrowed, have been duly referred to.

This thesis may be placed before the examiners for evaluation and necessary formalities. We certify that this thesis is worthy of consideration for the Ph.D. degree.

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DECLARATION

I, Jibitesh Dutta, hereby declare that the subject matter of this thesis is the record of work done by me, that the contents of this thesis did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

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Acknowledgements

I express my deep gratitude to my supervisors Dr. M. Ansari and Prof. Subenoy Chakraborty for their guidance and support throughout my work. It was an enlighting experience for me to work under them. I further thank them for stimulating discussions and careful analysis of the problems.

I am deeply indebted to (L) Prof. S. K. Srivastava, Department of Mathematics, NEHU, with whom I have started the work. I am thankful to him for giving M. Phil. courses and for his initial guidance. He has co-authored two of my papers.

I express my profound gratitude to Prof. S. S. Khare, Prof. H. K. Mukerjee and Dr. A. K. Das, Department of Mathematics, NEHU, for their help and suggestions. They offered me full cooperation right from the very inception of my studies.

I am also thankful to all the teachers, research scholars and office staff of the Department of Mathematics, NEHU, for extending all possible help to me.

My sincere thanks to Dr. Ujjal Debnath, Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah, Ms. Nairwita Mazumder and Mr. Ritabrata Biswas, Department of Mathematics, Jadavpur University, Kolkata, for their continuous support and encouragement. Thanks to IUCAA (Pune, India) for their warm hospitality and facilities during my visits there, as lot of my work were *carried out there.*

I would also like to thank my colleagues of Department of Basic Sciences and Social Sciences, NEHU, where I am presently employed as an Assistant Professor, for their support.

Finally I am thankful to my parents, especially my mother, for giving me constant inspiration and providing me with opportunities to continue my studies.

Jibitesh Dutta

Preface

General Relativity, in spite of being the most successful gravitational theory has left some problems without answer. After the advent of General Relativity, a gravitational theory in a 4-dimensional space-time, Kaluza-Klein theory emerged as a higher dimensional theory. This is a 5-dimensional theory with metric tensor components depending on electromagnetic fields. It emerged from the desire to unify gravity and electromagnetism. In this theory, the extra dimensional space is very small and compact. Moreover, this theory has a problem that standard model particles are not observed in Kaluza-Klein frame work. In Ref. [20], it is proposed that the reason for not observing the standard model particles in Kaluza-Klein set up is confinement of these particles in 4-dimensional space-time, called 3-brane. This idea was the key ingredient of brane-gravity. Thus according to the theory of brane-gravity, major portion of gravity lies in higher-dimensional "bulk" space-time as well as very small portion of gravity and standard model particles are trapped to the 3-brane [86]. The 3-brane is identified with our observable universe. This kind of cosmic picture is inspired by the string-theory/M-theory.

One of the first example of this type was the Hořova and Witten set up [19] of 11- dimensional M-Theory compactified on $S^1/\mathbb{Z}_2$ orbifold. In this theory, two (1+9) dimensional branes are located at two ends of the orbifold. These two branes are endowed with the product manifold of

(1+3)-dimensional non compact and 6-dimensional compact manifold. When six extra dimensions are compactified on a very small scale close to the fundamental scale, their effect is realised on (1+3)-dimensional brane. Thus, Hořova-Witten set up provided an effective 5-dimensional model where the extra dimension can be large relative to the fundamental scale in contrast to Kaluza-Klein theory, where extra dimension is very small [2, 11, 18]. This solution was used by L. Randall and R. Sundrum in their seminal paper to solve the "hierarchy problem" by a *warped* or curved space-time showing that fundamental scale could be brought down from the Planck scale to 100 GeV by this approach. Thus, Randall-Sundrum approach brought the theory to scales below 100 GeV being the electroweak scale (so far results could be verified experimentally up to this scale only). In this model, extra-dimension is large having $(1+3)$-branes at its ends. These branes are $\mathbb{Z}_2$-symmetric (mirror symmetry) and have tension to counter the negative cosmological constant in the "bulk", which is AdS_5. The model, having *two* (1+3)-branes at the ends of the orbifold $S^1/\mathbb{Z}_2$, is known as RS-I model [22]. In RS-I model, one brane is hidden and other is visible.

In another paper, published in the same year, these authors proposed another brane-model as an alternative to compactification. In this model, there is only *one* $(1+3)$-brane at one end of the extra-dimension and the other end tends to infinity. This model is known as RS-II model [23]. In this thesis, we consider RS-II model which is very popular among cosmologist due to its simplicity and predictive power. Singularity problem of phantom cosmology is resolved easily using RS-II model of brane-gravity.

Another approach for brane-gravity was proposed by Dvali, Gabadaze and Porrati in the year 2000 [25, 26, 27, 28]. This is an induced gravity model. The main idea of the DGP model is the inclusion of a four dimensional Ricci scalar into the action. The model is then characterized by a cross over length scale

$$r_c = \frac{M_P^2}{2M_5^2},$$

such that gravity is 4-dimensional theory at scales $a \ll r_c$ where matter behaves as pressure less dust but gravity *leaks out* into the bulk at scales $a \gg r_c$ and matter approaches the behaviour of a cosmological constant.

Braneworld cosmology for these models is a strong candidate from the beginning of this century after *Wilkinson Microwave Anisotropy Probe* (WMAP) observations having conclusive evidence for cosmic acceleration beginning in the recent past. Being String theory inspired, braneworld models provide corrections to the General Relativity which is considered low energy limit of string theory. We find that novel cosmologies are obtained which potentially answers to some longstanding problems of modern cosmology, such as origin and nature of *Dark Energy* (DE). At the same time success of standard 4-dimensional cosmologies is preserved and in some cases the treatment in the framework of brane cosmology is even more satisfactory. The phantom dark energy, being a non-luminous cosmic fluid and having gravitational effect, is found more suitable source to cause late acceleration observed by scientist working in *WMAP* project.

In General Relativity based models, lot of works in phantom cosmology are available in the literature. But phantom cosmology, in the framework of both brane-gravity theories, needs proper attention as brane-corrections in Friedmann equations bring drastic changes in results obtained from General Relativity [97, 98, 99]. A part of this thesis is addressed to this area.

Apart from many developments in cosmology, thermodynamics in expanding universe is a very important issue which requires serious attention. In the past it has been the subject of many papers based on General Relativity [105, 106, 107, 108, 109, 110]. Recently the connection between gravity and thermodynamics has been extended to braneworld scenario [121, 122, 123, 124].

In this thesis, we investigate certain aspects of brane world modifications to cosmological dynamics. In particular, it is devoted to following two important aspects of Brane Cosmology.

- Acceleration and deceleration of the universe.
- Validity of Generalised second law of thermodynamics [106].

This thesis is a collection of six papers based on my research on the above mentioned two aspects of Brane Cosmology. Except last chapter, all the chapters are based on published papers in international journals.

It contains seven chapters followed by two appendixes where some detailed calculation are shown. First chapter gives a general introduction to the higher dimension, braneworld scenario, brane cosmology and accelerated phantom dominated universe. Chapter ends with a discussion on gravitational thermodynamics.

In 2nd chapter, we consider RS-II model of brane gravity and analyze phantom universe using a non-linear equation of state. Phantom fluid is known to violate the weak energy condition (WEC). It is found that this characteristic of phantom energy is affected drastically by the negative brane-tension λ of the RS-II model. It has been found that up to a certain value of energy density ρ satisfying $\rho/\lambda < 1$, weak energy condition is violated and universe super-accelerates. Moreover, on sufficient increase in phantom energy density, even strong energy condition (SEC) is not violated due to effect of brane corrections. As a consequence, it is found that the present model of the universe accelerate up to a finite time, explaining present acceleration but decelerates later on. Also expansion of the universe stops when $\rho = 2\lambda$. This is contrary to earlier results of phantom universe exhibiting acceleration only. Moreover, the model is free from big-rip problem. This chapter is based on [137].

In 3rd chapter, DGP model of brane-gravity is considered for the phantom universe using a nonlinear equation of state. Here, DGP model of brane-gravity is analyzed and compared with the standard General Relativity (GR) and Randall-Sundrum cases. It is found that in DGP model SEC is always violated and the universe accelerates only where as WEC is violated only for a special range of energy density. Chapter ends with an expression of the scale factor and analysis of its behaviour in the late universe. This chapter is based on [138].

In 4th chapter, cosmology of the late and future universe is obtained from $f(R)-$ gravity with non-linear curvature terms R^2 and R^3 (R being the Ricci scalar curvature). Here, curvature terms induce dark energy, dark matter and cosmological constant, which appear in the Friedmann equation for the late universe derived from $f(R)-$ gravitational equations. It has been observed that curvature-induced dark energy, obtained here, mimics phantom with the equation of state parameter $\omega = -5/4$. Moreover, Friedmann equation contains phantom DE term as $\rho_{\rm DE}[1-\rho_{\rm DE}/2\lambda]$. The correction term $-\rho_{\rm DE}^2/2\lambda$, with λ being the cosmic tension, is analogous to such a term in RS-II model Friedmann equation. Different phases of this model, including acceleration and deceleration during phantom phase have been investigated. This chapter is based on [144].

5th chapter deals with gravitational thermodynamics in DGP brane world. In particular, we study the validity of the generalised second law of gravitational thermodynamics (GSLT) of the universe bounded by the event horizon. Here, the radius of the event horizon is calculated by establishing a correspondence between holographic dark energy (HDE) and the effective energy density in the DGP brane world. It is shown that in the absence of cold dark matter (CDM), GSLT is always respected. In the presence of CDM, we investigate validity of GSLT in three different models of DGP braneworld. The result shows that, the relation between thermodynamics and gravity is not just accident in GR, but it has deep meaning which other theory of gravity also supports. This chapter is based on [152].

In 6th chapter, in contrast to 5th chapter, we study the validity of GSLT in both branches of DGP model, i.e. DGP(+) and DGP(-) model. Moreover, in this chapter matter in the universe is taken in the form of non-interacting two fluid system :- one component is the holographic dark energy and the other component is in the form of dust. Also, here we examine the validity of GSLT on both apparent and event horizons. At the apparent horizon, it is shown that GSLT is always respected regardless of specific form of DE. But in case of event horizon GSLT may breakdown in the future universe. This chapter is based on [162].

Finally in chapter 7, we extend the investigation of the validity of the GSLT to interacting holographic dark energy model in the DGP braneworld. The thermodynamics of interacting HDE model in General Relativity set up have been extensively studied in literature [165, 166]. In a recent paper [167], validity of GSLT has been studied on event horizon for interacting DE. Assuming first law of thermodynamics on the event horizon, they have found conditions for validity of GSLT in both cases when FRW universe is filled with interacting two fluid system:- one in the form of cold dark matter and the other is either HDE or new agegraphic DE. Here, we use this method of extracting entropy for interacting HDE in the DGP model. This chapter is based on [168].

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List of Symbols

$\mathbb{Z}$	set of integers
G	Newton's gravitational constant
M_P	$= 10^{19}$GeV is the Planck mass
M_5	5-dimensional fundamental Planck mass
H	Hubble parameter
a	scale factor
h	Planck's constant
c	speed of light
z	redshift
k_{B}	Boltzmann constant
$\hbar = h/2\pi$	reduced Planck Constant
$\mathbb{Z}_2$	$\{1, -1\}$, the multiplicative group of order 2
$GL(n)$	group of $n \times n$ non-singular complex matrices
$SO(3,1)$	special ortogonal group that preserves the metric in Minkowski space-time
$U(n)$	the unitary group, consisting of all unitary $n \times n$ complex matrices
$SU(n)$	the special unitary group, denotes the subgroup of $U(n)$ consisting of matrices that have determinant 1
$\dot{A}$	ordinary derivative of A with respect to cosmic time t
$\partial_\mu A$ or $A_{,\mu}$	partial derivative of A
$A^{\nu}_{;\mu}$	covariant derivative
$\Box$	$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}g^{ij}\frac{\partial}{\partial x^j}\right)$
$\simeq$	approximately equal to

Here, GeV is used as a fundamental unit and we have $1\text{GeV}^{-1} = 6.58 \times 10^{-25}$sec.

Chapter 1

Introduction

1.1 A brief overview of extra dimension

The idea that the world we live in has more dimension than we can see is not a new concept. This idea has long fascinated and puzzled physicists, mathematicians and writers of speculative fiction over the dimension of our universe [6, 7]. History of science shows that most of the major advancements, in physics, have taken place through attempts for *unification*. In the later part of nineteenth century, Maxwell unified *electricity* and *magnetism*, which explained the nature of *light*. Special theory of relativity came out of Maxwell's theory and basic principles of newtonian mechanics. The advent of Special Relativity [8] and Maxwells theory of electromagnetism led to Minkowski's suggestion [9] that we should understand physics geometrically in four-dimensional space-time rather than three-dimensional space. This led to the unification of space and time as space-time and migration from three to four dimensional world. The mixing of space and time is noticed only at very high speeds, through phenomenon such as length contraction and time dilation.

Space-time dimensions in addition to observed four were first introduced by Gunnar Nordström in 1914 [10] and became popular from the work of Kaluza and Klein in 1920s [11]. The early work on Nordström-Kaluza-Klein theories stemmed from the desire to unify gravity and electromagnetism. Kaluza [12] in 1919 (though his work was published in 1921) generalised Einstein's theory of gravity (described in 4-dimensional space-time) to 5-dimensional world, where fifth dimension introduces electromagnetic potential as component of the metric tensor. Later on, in 1926, Klein [13] used compactification of gauge group $U(1)$ and used the extra-dimension (fifth dimension) as circle being the 1-dimensional compact manifold. As a result, all field variables could be treated *periodic* in the fifth co-ordinate and could be expanded in Fourier series. Individual efforts, to unify gravity with electromagnetism made by Kaluza and Klein independently, led to a new theory called as Kaluza-Klein theory. This is a 5-dimensional theory, described in the space-time having topology $M^4 \otimes S^1$, where M^4 is the usual 4-dimensional space-time and S^1 is the circle (1-dimensional compact manifold) having radius too small to observe. The idea of extra dimension occurs again in the context of string theory, a modern attempt to unify general relativity and quantum field theory.

Quantum field theory (QFT) and general relativity (GR) are two fundamental theories of physics. QFT is successful in explaining physical phenomena at *micro-cosmos* level, i.e. down to distance 10^{-15} cm, being the Compton wavelength $\hbar/(m_0 c)$ of a hadron of rest-mass m_0 (hadrons are strongly interacting particles for example, proton, neutron, Σ-particles, various mesons etc.). On the other hand, GR explains physics at *macro-cosmos* level at distance much larger than hadron domain and gives a fascinating description of the universe in accordance with observations. These two fundamental theories have *no* compatibility as if *nature* has two minds working in their own way without bothering for each other. This is *not* an *ideal* situation. So, physics needs *unification* of these theories.

Physicists have been making concerted efforts to achieve this *goal*. Intense theoretical researches has shown that, quite likely, secret to this mystery lies in *gauge symmetry*. Due to courtesy of C. N. Yang and R. A. Mills [14, 15], it was known that *unity* and *symmetry* of basic laws could be understood using corresponding Lie groups and Lie algebra. Use of local Yang-Mill's theory, in quantum domain, was successful in killing divergences in QFT, which shows that *nature loves symmetry*.

There are *four* fundamental forces (known so far) in *nature*. These are electro-magnetism (obeying the abelian group $U(1)$), weak interaction (obeying gauge-symmetry given by the non-abelian group $SU(2)$), strong interaction (obeying gauge-symmetry given by the non-abelian group $SU(3)$) and GR (gravitation) having gauge-symmetry given by the group $GL(4)$ or $SO(3,1)$. But the main problem of unification of QFT and GR was still untouched. One of the most promising models of unification is string theory (for an introduction see [16]). String Theories remove the infinities that are present in a classical unification by describing particles as extended 1-dimensional strings rather than point particles. However the mathematics of string theory does not work in a 4-dimensional universe. These 1d strings live in a 10-dimensional space, or 11d for supergravity (which is a combined theory of gravity and supersymmetry).

There are two classes of strings: closed and open strings. Gravity is described by closed strings and matter is described by open strings.

Open and closed strings in the theory can predict only the existence of bosonic particles including graviton and tachyon but not fermionic particles. In order to include fermions, string theory needs to be combined with supersymmetry and this results in the new theory called superstring theory. There are 5 different types of superstring theories: type I, type IIA, type IIB, type $SO(32)$ heterotic, and type $E8 \otimes E8$ heterotic. All of these superstring theories required $9+1$-space-time dimensions in order to be consistent [17]. Later on, using the powerful duality-transformations, it was found that type I, type IIA, type IIB superstrings, $E_8 \otimes E_8$ heterotic strings, $SO(32)$ heterotic strings and 11-dimensional supergravity are limiting cases of a larger theory known as M-theory. Thus modern ideas of extra dimensions were introduced by string theory/M-theory.

1.1.1 Need for higher-dimensional theory : a heuristic view

So far experiments probe particles up to the energy scale $100\text{GeV} = 1\text{TeV}$ which is the electro-weak energy scale M_{w}. In a 4-dimensional gravity, gravitational constant $G = M_P^{-2}$ giving Planck scale $M_P = 10^{19}\text{GeV}$ as the *fundamental* scale. Now, a question arises "Why is the fundamental scale is 10^{17} times higher than the electro-weak scale M_{w} ?" This is the "hierarchy problem" of fundamental scale. Obviously, a natural solution to this problem is to raise the gravitational constant and as a result to lower down the fundamental scale from the Planck level. A possible approach, to do so, is to think of higher-dimensional gravity with the action

$$S_g^{(4+d)} = \frac{1}{16\pi G_{(4+d)}} \int d^4x d^d y \sqrt{|g^{(4+d)}|} R^{(4+d)}, \tag{1.1.1}$$

where space-time has topology $M^4 \otimes E^d$ (M^4 is the usual 4-dimensional non-compact manifold and E^d is extra compact manifold).

The Ricci scalar curvature is given by

$$R^{(4+d)} = R^{(4)} + R^{(d)}. \tag{1.1.2}$$

Connecting equations (1.1.1) and (1.1.2), we have

$$S_g^{(4+d)} = \frac{1}{16\pi G_{(4+d)}} \int d^4x d^d y \sqrt{|g^{(4+d)}|} [R^{(4)} + R^{(d)}], \tag{1.1.3}$$

On integrating over extra space, equation (1.1.3) yields

$$S_g^{(4)} = \frac{1}{16\pi G} \int d^4x \sqrt{|g^{(4)}|} [R^{(4)} + R^{(d)}], \tag{1.1.4}$$

where

$$G_4 = G = G_{(4+d)}/V_d \quad \text{with} \quad V_d = \int d^d y \sqrt{|g^{(d)}|}. \tag{1.1.5}$$

So,

$$G^{-1}_{(4+d)} = M^{(2+d)}_{(4+d)} = M_P^2/V_d < M_P^2, \tag{1.1.6}$$

for $V_d > 1$.

Adding matter to the gravitational action as

$$S_m^{(4+d)} = \int d^4x d^d y \sqrt{|g^{(4+d)|}} \mathcal{L}_m, \tag{1.1.7}$$

and using the condition $\delta(S_g^{(4+d)} + S_m^{(4+d)})/\delta g^{MN} = 0$, field equations are obtained as

$$R_{MN}^{(4+d)} - \frac{1}{2}g_{MN}^{(4+d)}R = -8\pi G_{(4+d)}T_{MN}^{(4+d)}, \tag{1.1.8}$$

where $g_{MN}^{(4+d)}$ and $R_{MN}^{(4+d)}(M, N = 0, 1, 2, 3, 4, \cdots, (4+d))$ are metric and Ricci tensor components in higher-dimensional space-time.

In the weak static limit, equation (1.1.8) yield Poisson's equation

$$\Box^{(4+d)}\phi^{(4+d)} = 4\pi G_{(4+d)}M/r^{(3+d)}. \tag{1.1.9}$$

On solving equation (1.1.9), it is obtained that

$$\phi^{(4+d)} \sim G_{(4+d)}/r^{(1+d)}. \tag{1.1.10}$$

If length scale of the universe is L, $V_d \sim L^d$.

Further,

$$\phi^{(4+d)} \gtrsim G_{(4+d)}/rL^d = G_4/r \sim \phi^{(4)}, \quad \text{for} \quad r < L$$

and

$$\phi^{(4+d)} \lesssim G_{(4+d)}/rL^d = G_4/r \sim \phi^{(4)}, \quad \text{for} \quad r \gtrsim L.$$

These observations show that higher-dimensional gravitational potential is higher than 4-dimensional gravitational potential below a certain scale and, above this scale, higher-dimensional gravitational potential is lower than 4-dimensional gravitational potential. It means that below a certain length scale, gravity is stronger in higher-dimensional space-time.

Experiments probe 4-dimensional gravity up to the length scale 0.1mm = 10^{15}TeV^{-1}. So,

$$L \lesssim 10^{15}\text{TeV}^{-1}. \tag{1.1.11}$$

Connecting equations (1.1.6) and (1.1.11), it is found that the fundamental scale is lowered down as

$$M_{(4+d)} = 10^{(34-15d)/(d+2)}\text{TeV}.$$

Above arguments explain why higher-dimensional gravity is not observed at low energy, i.e. at energy less than 10^{-15}TeV, but at energy higher than this scale gravity leaks heavily to higher-dimensions.

1.2 Braneworld Scenario

After point-like particles and 1-dimensional extended objects namely strings, it was natural to think of p-dimensional membrane (p-branes) being topological defects. For example when $p = 0$, we have zero-brane which is just a point. When $p = 1$, the object is a 1-dimensional line and is called one-brane or string and for $p = 2$ we get a membrane. Among p-branes, D-branes (p-branes obeying Dirichlet condition) are important. Just as propagating point sweeps out a curve - the world line in space-time, a Dp-brane (p-dimensional D-branes) sweeps out a $p + 1$ dimensional volume. They are special class of p-branes in superstring theories. These Dp-branes are surfaces where open strings must start and finish. Theory of D-branes provides a simple description of various non-perturbative objects required by string-duality. Using duality-transformations, it is found that type I, type IIA and type IIB superstrings are different limits of D-branes.

Superstring is described in a 10-dimensional space-time. Open strings, with Neumann boundary condition

$$n^a \partial_a X^\mu(\sigma, \tau) = 0, \mu = 0, 1, \cdots, p$$

[(σ, τ) are world-sheet co-ordinates] and Dirichlet conditions

$$X^\mu = 0 \quad \text{for} \quad \mu = (p+1), \cdots, 9$$

have ends on p-brane. The 10-dimensional space-time is called "bulk". The p-brane is the Dirichlet brane, which contains the standard particles and fields. Closed strings, formed by gluing two open strings end to end, move throughout the bulk and carry "gravitons". D-branes became important in giving new insight into quantum mechanics of black-holes and nature of space-time at the shortest distance.

During 1960-70, the idea that our (3 + 1)-dimensional world could be realized as a 3d surface in higher dimensional space was actively discussed in the context of general relativity. In this scenario our observable universe is a 3-brane (4-dimensional space-time) containing standard matter and fields. It is a hyperspace of higher-dimensional space-time, being the (4 + d)-dimensional bulk. In this brane-bulk scenario, all matter and gauge interactions are localised on a brane while gravity may propagate into whole space-time. This means that gravity is fundamentally a higher dimensional interaction and we only see the effective 4d theory on the brane. This approach paved a new way to get 4-dimensional space-time as an alternative to Kaluza-Klein type of compactification. We need a mechanism of compactification since all our daily experiences can perfectly be explained using just a four dimensional space-time.

It has further inspired a class of classical models of the universe, in which extra dimensions can be included in general relativity, and their possible implications for classical cosmology can be investigated phenomenologically without any dependence on a particular model of string theory. This is known as the braneworld scenario. It is one of the most important ideas in high energy physics and cosmology. In this connection, Hořova-Witten set-up has played an important role. A brief discussion, on it, is given as follows.

1.2.1 Hořova and Witten Set Up

In 1996, Hořova and Witten [19] proposed an example of this approach using 11-dimensional M-theory. They considered M-theory on the orbifold $R^{10} \otimes S^1/\mathbb{Z}_2$ (where R^{10} is the 10-dimensional space-time and $S^1/\mathbb{Z}_2$ is the orbifold, which is made by folding a circle S^1 on itself along its diameter). They argued that as M-theory on $R^{10} \otimes S^1$ reduces to type II superstrings, when circle shrinks to zero, M-theory on $R^{10} \otimes S^1/\mathbb{Z}_2$ reduces to $E_8 \otimes E_8$ heterotic strings. Thus, in this set-up, E_8 propagates at two ends of the orbifold $S^1/\mathbb{Z}_2$. In the low energy limit, E_8 breaks to $SU(3) \otimes SU(2) \otimes U(1)$. 9-branes at the ends of the orbifold contain these particles and fields of standard GUT (grand unified theory) model. Topologically, 9-branes are $R^4 \otimes C^6$ with R^4 being the non-compact 4-dimensional space-time and C^6 being compact 6-dimensional complex Calabi-Yau manifold. (Kähler manifold has hermitian and closed metric $\Omega = (\partial^2\phi/\partial Z^i \partial \bar{z}^j) dZ^i \wedge d\bar{z}^j$. If Ω is exact and determinant of $(I + \frac{i}{2\pi}\partial^2\phi/\partial Z^i\partial\bar{z}^j)$ vanishes, we have Calabi-Yau manifold). In case, compactification length scale is much less than the distance between two ends of the orbifold, compact six dimensions can be integrated out and we are left with the effective 5-dimensional theory with 5-dimensional bulk having topology given by $R^4 \otimes S^1/\mathbb{Z}_2$, where R^4 is the 3-brane. This result inspired many later braneworld models. In what follows, we discuss three distinct mechanisms by means of which the laws of 4d gravity can be obtained on the brane.

1.2.2 Arkani Hamed-Dimopoulos-Dvali (ADD) model

In 1998, Arkani Hamed, Dimopoulos and Dvali [20, 21] put forward the idea that a large volume for compact extra dimensions would lower down the fundamental scale from Planck scale giving solution to "hierarchy problem". In contrast to *Hořova-Witten* model, ADD model requires more than 1 equivalent extra dimensions. This model is very simple, where 3-brane is embedded in (4 + d)-dimensional space-time. Here d extra dimensions are compactified on a torus of size L. On $r \ll L$, gravity is obtained from the higher-dimensional bulk, but for $r > L$ usual 4-dimensional gravity is recovered. This set up is often referred as **Braneworlds with Compact Extra Dimensions.**

1.2.3 Randall-Sundrum (RS) models

In this section we describe another way of obtaining 4d gravity on a brane. It is based on a phenomenon of localisation of gravity discovered by Randall and Sundrum (RS) [22, 23] and set up is often referred as **Braneworlds with Warped Extra Dimensions**.

Hořova-Witten solution provided an effective 5-dimensional model where extra dimension could be large relative to the fundamental scale in contrast to Kaluza-Klein approach where extra dimension is very small. The pioneering work, in this direction, was done by L. Randall and R. Sundrum in 1999. According to RS approach, our 3-brane (observable universe) is identical to a domain wall in a 5-dimensional *anti de-Sitter* space-time (AdS_5). In their *first* paper [22], they proposed to solve the "hierarchy problem" by a *warped* or curved dimension in contrast to ADD model, where extra-dimensions are flat. In this model, extra-dimension is large having 3-branes at there ends. These branes are $\mathbb{Z}_2$-symmetric (mirror symmetry) and have tension to counter the negative cosmological constant in the "bulk", which is AdS_5. The model, having *two* 3-branes at the ends of the orbifold $S^1/\mathbb{Z}_2$, is known as RSI model.

In another paper, in the same year, these authors proposed another brane-model as an alternative to compactification . In this model, there is only *one* 3-brane at one end of the extra-dimension and the other end tends to infinity. This model is known as RSII model [23].

The "bulk" cosmological constant Λ_5 squeezes gravity to the 3-brane. In *gaussian* normal co-ordinates $x^M = (x^i, y)(M = 0, 1, 2, 3, 4)$, AdS_5 metric is given as

$$ds^2_{(5)} = e^{-2|y|/l}\eta_{ij}dx^i dx^j + dy^2. \tag{1.2.1}$$

where y is the co-ordinate of the extra dimension ($0 \le y \le L$) and η_{ij} denotes the Minkowski metric. Here,

$$\Lambda_5 = -\frac{6}{l^2}, \tag{1.2.2}$$

where l is the radius of curvature for AdS_5. Further distinction, in these models, are given below.

RSI model

In this model, two 3-branes are located at the ends $y = 0$ and $y = L$ of the orbifold $S^1/\mathbb{Z}_2$. The action of the RSI brane world model is

$$S_{\rm RS} = S_{\rm Gravity} + S_{\rm Mat\ brane} + S_{\rm Hid\ brane}, \qquad (1.2.3)$$

where

$$\begin{aligned} S_{\rm Gravity} &= \frac{M_5^2}{2}\int d^4x \int_{-L}^{L} dy\sqrt{|g^{(5)}|}(R^{(5)} + 2\Lambda_5), \\ S_{\rm Mat\ brane} &= \int d^4x\sqrt{-g_{\rm Mat\ brane}}\ \lambda_{\rm Mat\ brane}, \\ S_{\rm Hid\ brane} &= \int d^4x\sqrt{|g_{\rm Hid\ brane}|}\ \lambda_{\rm Hid\ brane}, \end{aligned}$$

and the subscripts, "Mat brane" and "Hid brane" denote our matter brane (which contains the standard model fields) and the hidden brane respectively. Here λ is the brane tension.

The 5d Einstein equation in the bulk is

$$G_{\alpha\beta} = -\Lambda_5 g_{\alpha\beta}, \qquad (1.2.4)$$

assuming that there are no additional fields in the bulk, i.e. $T_{\alpha\beta} = 0$.

These branes have the $\mathbb{Z}_2$ symmetry in order to make the bulk compact and periodic:

$$y \leftrightarrow -y \quad and \quad L + y \leftrightarrow L - y. \qquad (1.2.5)$$

The metric given by (1.2.1) to be a valid solution of the Einstein's equation (1.2.4), the matter on the two 3-brane must have *equal* and *opposite* brane tension $\pm\lambda$ with

$$\lambda = 3M_P^2/4\pi l^2. \tag{1.2.6}$$

The positive tension brane must be located at $y = 0$ where the warp factor is greatest, and the negative tension brane at $y = L$ where the warp factor is lowest.

In this set-up, *positive* tension brane is "hidden" and *negative* tension brane is "visible". These are positive and negative cosmological constants in "hidden" and "visible" 3-branes respectively. In other words, these are positive and negative vacuum energy density in "invisible" and "visible" 3-branes respectively.

The reduced Planck mass measured on our brane is not the fundamental mass but connected to the fundamental reduced Planck mass (M_5) by

$$\begin{aligned} M_P^2 &= M_5^3 \int_{-L}^{L} e^{-2|y|/l} dy \\ &= 2M_5^3 \int_{0}^{L} e^{-2|y|/l} dy \\ &= M_5^3 l \Big[1 - e^{-2L/l}\Big]. \end{aligned} \tag{1.2.7}$$

This gives a possibility to solve the hierarchy problem.

<u>RSII model</u>

The RSII model does not intend to solve the hierarchy problem. In this model, there is only *one* 3-brane with *positive* tension and $L \to \infty$, so

$$M_P^2 = M_5^3 l. \tag{1.2.8}$$

The single brane is located at $y = 0$ with one symmetry identification $y \leftrightarrow -y$. The action of the hidden brane is dropped.

1.2.4 Dvali-Gabadaze-Porrati (DGP) model

In this section we consider the third known mechanism of obtaining 4d gravity on a brane. This mechanism is different from the previously discussed mechanisms since it allows the volume of the extra space to be infinite and the set up is referred as **Braneworlds with Infinite Volume Extra Dimensions**. This mechanism was proposed by Dvali, Gabadaze and Porrati [25] in the year 2000 that has low energy modifications through an induced gravity term. The merit of the DGP model is to explain late-time acceleration without dark energy and the ability to recover standard cosmology at early times [26]. The model does not intend to solve the hierarchy problem. The action of the DGP braneworld is

$$S_{\mathrm{DGP}} = \frac{M_5^3}{2}\int d^4x \int dy \sqrt{|g^{(5)}|}R^{(5)} + \frac{M_P^2}{2}\int d^4x\sqrt{-g}R + \int d^4x \mathcal{L}_M. \tag{1.2.9}$$

The main idea of the DGP model is the inclusion of a four dimensional Ricci scalar into the action. The induced gravity term (4d Ricci scalar term in the brane action) can be motivated by possible quantum effects of the interaction between matter on the brane and the bulk gravity (String theories with a ghost free Gauss-Bonnet (GB) term in the bulk give rise to induced gravity terms on the brane [30]).

On the 4-dimensional brane the action of gravity is proportional to M_P^2 whereas in the bulk it is proportional to the corresponding quantity in 5-dimensions. The model is then characterized by a cross over length scale

$$r_c = \frac{M_P^2}{2M_5^2},$$

such that gravity is 4-dimensional theory at scales $a \ll r_c$ where matter behaves as pressure less dust but gravity *leaks out* into the bulk at scales $a \gg r_c$ and matter approaches the behaviour of a cosmological constant [25, 26, 27]. Contrary to RS model, DGP model lives in Minkowski bulk and there is no brane tension. In this model there is only one brane as in RSII, without the bulk cosmological constant ($\Lambda_5 = 0$).

1.3 Brane Cosmology

Cosmology is a subject which deals with the physical structure of the universe at large scale and it describes different processes during its evolution. Nowadays it is an active field of physical thought and of exciting experimental results. Its main goal is to describe the evolution of our universe from some initial time to its present form. One of its outstanding successes is the precise and detailed description of the very early stages of the universe evolution. Various experimental results confirmed that inflation describes accurately these early stages of the evolution. Cosmology can also help to understand the large scale structure of our universe as it is viewed today. It can provide convincing arguments why our universe is accelerating and it can explain the anisotropies of the Cosmic Microwave Background data.

The mathematical description of Cosmology is provided by the Einstein equations. A basic ingredient of all cosmological models is the matter content of the theory. Matter enters Einstein equations through the energy momentum tensor.

The modern era of brane cosmology began with [31] in the context of large extra dimensions, made possible by the hypothesis that the standard model of particle physics is localized on a D-brane. This notion of branes into cosmology offered another novel approach to our understanding of the Universe and of its evolution. The main paper motivating interest in brane cosmology was Ref. [89] (BDL). Their work was inspired by string theory, and in particular the *Hořova - Witten* model [19]. It was proposed that our observable universe is a three dimensional surface (domain wall, brane) embedded in a higher dimensional space. For development of a physical theory, we need a model consistent with known facts of the system. In what follows, we discuss Friedmann equations which are obtained by solving Einstein's field equations to predict various phenomena in the universe.

Friedmann Equations

In GR set up the most successful model, having tremendous predictive power for cosmology, was obtained by Friedmann, in 1922 as well as by Robertson and Walker in 1930, which is called as Friedmann-Robertson-Walker (FRW) model . This is a non-static centrally symmetric homogeneous model obeying the cosmological principle. This model is so successful that the standard hot big-bang theory is based on it. Here we shall consider FRW branes. This is a general cosmological solution where the Hubble rate decreases as the energy density goes down. The FRW solution describes the uniform expansion of a homogenous and isotropic perfect fluid.

We shall review the Friedmann equations, that govern the evolution of the model, for the standard GR case before we consider the brane results.

1.3.1 Result of General Relativity

FRW model was derived by solving the Einstein field equations with the cosmic matter as a non-interacting tiny gas particle behaving like a perfect fluid . In this model the line element is obtained as

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1.3.1}$$

where $a(t)$ is the scale factor with cosmic time t ; r , θ , ϕ are polar co-ordinates of the 3-space and k is the curvature parameter having values

$$k = \begin{cases} +1 & \text{for closed model} \\ 0 & \text{for flat model} \\ -1 & \text{for open model} \end{cases}$$

The metric equation (1.3.1) is same for Einstein field equations with or without cosmological constant (Λ).

For the Einstein's field equations

$$R_{ij} - \frac{1}{2}g_{ij}R = -8\pi G T_{ij}, \tag{1.3.2}$$

(R_{ij} being Ricci tensor components, g_{ij} being metric tensor and R being Ricci scalar), the time-time component for the metric (1.3.1) can be written as

$$\begin{aligned} H^2 &= \left(\frac{\dot{a}}{a}\right)^2 \\ &= \frac{8\pi G\rho}{3} - \frac{k}{a^2}\ , \end{aligned} \tag{1.3.3}$$

where ρ is the total cosmic fluid energy density and $H = \frac{\dot{a}}{a}$ is the Hubble's rate of expansion. This is the Friedmann equation giving cosmic dynamics. Equation (1.3.3) is obtained using the energy momentum tensor for the perfect fluid having components

$$T^{ij} = (\rho + p)u^i u^j - pg^{ij} \qquad\qquad (i, j = 0, 1, 2, 3),$$

with p being the pressure and u^i is the 4-velocity components in co-moving co-ordinates being normalized as $u^i u_i = 1$.

The energy momentum tensor with components T^{ij} is conserved by virtue of the identities $T^{ij}_{;j} = 0$, where semi-colon (;) denotes covariant derivative leading to the continuity equation

$$\dot{\rho} + 3H(p + \rho) = 0. \tag{1.3.4}$$

Equations (1.3.3) and (1.3.4) yield Raychoudhuri's equation

$$\dot{H} = \frac{k}{a^2} - 4\pi G(\rho + p). \tag{1.3.5}$$

Equations (1.3.3) and (1.3.5) lead to another useful equation

$$\begin{aligned} \frac{\ddot{a}}{a} &= \dot{H} + H^2 \\ &= \frac{-4\pi G}{3}(\rho + 3p). \end{aligned} \tag{1.3.6}$$

Equations (1.3.3) and (1.3.5) are standard GR Friedmann equations. When we consider the braneworld we will require the Israel junction conditions (See appendix A). This is because we now live on a hypersurface so the Friedmann equations only apply to a restricted part of the space-time.

1.3.2 Result of RS brane model

Projecting the 5d curvature and imposing the Darmois-Israel junction conditions and $\mathbb{Z}_2$ symmetry on the brane, Shiromizu *et al.* [32] found the effective Einstein equation on the brane (for details see appendix A):

$$^{(4)}G_{ij} = 8\pi G(\Lambda_4 g_{ij} - \tau_{ij}) + (8\pi G_5)^2 \pi_{ij} - \hat{\mathcal{E}}_{ik}, \qquad (1.3.7a)$$

where

$$\Lambda_4 = -4\pi G_5 \left[\Lambda_5 + \frac{4}{3}\pi G_5 \lambda^2 \right], \qquad (1.3.7b)$$

$$G = \frac{4}{3}\pi G_5^2 \lambda, \qquad (1.3.7c)$$

$$\pi_{ij} = -\frac{1}{4}\tau_{ik}\tau_j^k + \frac{1}{12}\tau\tau_{ij} + \frac{1}{8}g_{ij}\tau_{kl}\tau^{kl} - \frac{1}{24}g_{ij}\tau^2. \qquad (1.3.7d)$$

The tensor π_{ij} is the high energy correction term which is quadratic in the energy momentum tensor and $\hat{\mathcal{E}}_{ik}$ is the part of 5-dimensional space-time Weyl tensor (see appendix A). Equation (1.3.7c) shows that G is proportional to λ (vacuum energy of the brane). In case $\lambda < 0$, $G < 0$. For example, in the case of visible brane of RSI model, $G < 0$. It may be noted that first two terms on the right hand side of (1.3.7a) are the standard GR contribution of the matter on the brane and cosmological constant and the other terms are unique to brane models.

The unperturbed cosmological braneworld is a Friedmann brane in a Schwarzschild-AdS$_5$ bulk [90] (where the expansion of the universe can be interpreted as motion of the brane in the bulk), satisfying the usual energy conservation equation [86], but a modified Friedmann equation (for details see appendix B):

$$H^2 + \frac{k}{a^2} = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1 + \frac{\rho}{2\lambda}\right) + \frac{\Lambda_4}{3} + \frac{\mathcal{C}}{a^4}. \tag{1.3.8}$$

Compared to the usual Friedmann equation (1.3.3), obtained from einsteinian gravity, equation (1.3.8) contains *two* more terms

$$(i)\frac{8\pi G}{3}\frac{\rho^2}{2\lambda} \quad \text{and} \quad (ii)\frac{\mathcal{C}}{a^4}.$$

In these brane-gravity correction terms, the second term emerges spontaneously on performing integration of the field equation (i.e. $\mathcal{C}$ is an integration constant). No source is known for it. It corresponds to radiation density term, hence, it is called *dark radiation density*. It is interesting to note that the *energy density of the brane enters quadratically* (i.e. the first term) on the right hand side of equation (1.3.8) in contrast with standard four-dimensional Friedmann equation.

1.3.3 Result of DGP brane model

The bulk field equations are of the same form as in the RS model. The 4d energy momentum tensor is modified, in comparison to the RS model, as it includes the contribution from the induced gravity term in the action (1.2.9). By using the same assumptions and conditions as in the RS case (a perfect fluid undergoing isotropic and homogenous expansion), we obtain the induced gravity Friedmann equation to be [26]:

$$H^2 + \frac{k}{a^2} = \left(\sqrt{\frac{8\pi G\rho}{3} + \frac{1}{4r_c^2}} + \epsilon \frac{1}{2r_c} \right)^2 \tag{1.3.9}$$

or equivalently

$$H^2 - \frac{k}{a^2} - \frac{\epsilon}{r_c}\sqrt{H^2 - \frac{k}{a^2}} = \frac{8\pi G\rho}{3}. \tag{1.3.10}$$

In above equation, $r_c = \frac{M_P^2}{2M_5^2}$ is the crossover scale which determines the transition from 4d to 5d behaviour and $\epsilon = \pm 1$.

For $\epsilon = 1$, we have standard DGP(+) model which is self accelerating model while for $\epsilon = -1$, we have DGP(-) model which does not self accelerate in late universe. We have in addition the usual equation of conservation for the energy momentum tensor given by (1.3.4). It may be noted that the standard GR cosmology is recovered from equation (1.3.9) whenever $8\pi G\rho/3$ is large compared to $1/r_c^2$, so that the early phase of the model is analogous to standard cosmology. In the early phase, equation (1.3.9) reduces, at leading order, to the standard $4d$ Friedmann equation given by equation (1.3.3). However the late time behaviour is generically different as shown in [26].

Cosmological implications of the DGP braneworld have been analyzed in [27, 28, 29]. From Supernovae data the value of $r_c \approx 0.94 H_0^{-1}$ [33]. In DGP model, brane tension is assumed to be zero or cancelled out with a brane cosmological constant. The model can be generalized to include nonzero Λ_5 and brane tension λ (see Refs. [34, 35, 36, 37]), but we do not consider this case here.

1.4 Accelerated phantom dominated universe

The relativistic cosmology has witnessed many breakthrough after Einstein proposed static model of the universe. After 1981, cosmologists believed that, in the very beginning, universe accelerated and it came out of this phase after a short period heralding the big-bang scenario initially driven by radiation and particles. Thus, it was thought that after exit from inflationary phase, universe expanded, but it was slowing down. In 1998-99, contrary to above belief, accelerated expansion was pointed out by two groups (one led by A.G.Riess and other by S.Perlmutter) [38] from observations of Type Ia supernova explosion (SN Ia). In 2003, *Wilkinson Microwave Anisotropy Probe* (WMAP) [39] data also confirmed this result. This revolutionary observations [39, 40, 41] challenged cosmologists to develop an appropriate cosmological model explaining acceleration in the late universe. Such a model requires a source of energy to derive acceleration in the late universe. Also, observations support homogeneous and flat model of the universe [42]. In such a model, Friedmann equation (1.3.6) show that cosmic dynamics can exhibit acceleration only when $\rho + 3p < 0$ [43]. It shows violation of the cosmic strong energy condition (SEC) and indicates dominance of exotic fluid in the late universe. Such a exotic fluid violating SEC is called *Dark Energy* (DE). According to earlier observational data confirmed by WMAP (in 2003) up to unprecedented accuracy, dark energy density comprises 73% of the energy density of the present universe. Amongst the matter content of the universe, baryonic matter amounts to only 4%. The rest of the matter (23%) is believed to be in the form of a non-luminous component of non-baryonic matter with a dust like equation of state ($p = 0$) known as Cold Dark Matter (CDM). Dark energy is distinguished from dark matter in the sense that its equation of state ($p < -\frac{\rho}{3}$) is different from dark matter.

The existence of DE is one of the most significant discoveries over the last decade [38]. However, the nature of this energy remains a mystery. Various models of dark energy have been proposed, such as a small positive cosmological constant, quintessence, K-essence, phantom, holographic dark energy, etc. (see Ref. [43] for recent reviews with fairly complete list of references of different dark energy models.) The simplest candidate is the cosmological constant Λ introduced by Einstein in 1917 and model is popularly known as LCDM (Lamda Cold dark matter). Here the DE is the vacuum energy density with equation of state (EOS) parameter $\omega(= p/\rho) = -1$. It gives a very good fit to the observational data [40], but with an unnaturally small and fine tuned value of Λ. Also, following the more accurate data a more dramatic result appears: the recent analysis of the Type Ia supernovas data indicates that the time varying dark energy gives a better fit than a cosmological constant.

The condition $\rho + 3p < 0$ implies $-1/3 > \omega > -1$. In 2002-2003, Caldwell found the case $\omega < -1$ better fit for the observed astrophysical data and advocated for this case, which violates the weak energy condition (WEC) too [44, 45]. This fluid, is known as phantom (He got this name from Part I of the Star Wars movie series - Phantom Menace). Presently all observations do not rule out the possibility of the existence of matter with $\omega < -1$. Even in a model in which the Newton constant is evolving with respect to redshift z, the best fit $\omega(z)$ crosses the phantom divide $\omega = -1$ [47]. Hence the phenomenological model for phantom dark energy should be considered seriously. The phantom model explains the present and future acceleration of the universe, but it is plagued with the problem of big-rip singularity (singularity in finite future time when energy density, pressure and the scale factor diverge).

In the dark energy universe, different types of singularities appear. The future singularities can be classified in the following ways :

(i) Type I (big-rip): For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$.
(ii) Type II (sudden): For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|p| \to \infty$.
(iii) Type III : For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$.
(iv) Type IV : For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to 0$ and higher derivatives of H diverge.

Here t_s , a_s and ρ_s are constants with $a_s \neq 0$. The type I, big-rip singularity emerges for the phantom like equation of state $\omega < -1$. The type II corresponds to the sudden future singularity at which a and ρ are finite but p diverges . Type III singularity appears for the model with $p = -\rho - A\rho^{\alpha}$ and type IV singularity appears in the model with $p = -\rho - \frac{AB\rho^{\alpha\beta}}{A\rho^{\alpha} + B\rho^{\beta}}$.

The big-rip problem in phantom model of the universe was a great challenge to cosmologists . So, it was natural to think for its avoidance.

In this direction, first attempt was made by Elizalde *et al.* [46]. These authors suggested that making quantum corrections in energy density and pressure near big-rip singularity, this problem can be avoided. Later on, around the same time, Srivastava [48] suggested that we can get rid of this problem without making quantum corrections if phantom dark energy behaves as a generalized chaplygin gas (GCG) and barotropic fluid simultaneously. Also in [49] using brane-gravity (BG), he obtained singularity free phantom driven cosmic acceleration.

Thus, phantom was another exotic matter suggested by Caldwell. Different sources of exotic matter violating SEC [50 - 58] and WEC were proposed in the recent past [20, 44, 45, 48, 59 - 71]. A number of models appeared in the literature where the dynamical nature of phantom energy was constructed by taking a kinetic term with a wrong sign so that it can give rise to the present acceleration of the universe [55, 72 - 74]. Phantom dark energy models have also been studied in Brans-Dicke theory [75] or in an interacting scenario where the phantom field is coupled to some other field [76, 77]. A comprehensive review of these contributions is available in [43]. As the exotic matter is not known, different scalar fields with special attention to quintessence, phantom, K-essence and tachyon, were tried upon to probe actual nature of the matter responsible for required DE [43, and references therein]. Apart from, field-theoretic models, some fluid dynamical models such as Chaplygin gas and generalized chaplygin gas models due to its super-symmetric generalization and negative pressure were also tried out [43, 78].

As no satisfactory model came up, it was thought that curvature, being the source of gravitation, could also be a source of DE. The Einstein-Hilbert term $R/16\pi G$ (R being the scalar curvature and G being the gravitational constant) yields the geometrical component in Einstein's gravitational equations, so it was realized to use non-linear terms of curvature replacing matter lagrangian in the gravitational action. Motivated by this idea, S. Capozziello *et al.* and S. M. Carroll *et al.* proposed non-linear terms of curvature R^{-n} with $n > 0$ as a possible source of DE [79]. Although this model explained late cosmic acceleration, it exhibited instability and failed to satisfy solar system constraints. Further, this idea was taken up by Nojiri and Odintsov and models for gravitational alternative of DE were proposed taking different forms of $f(R)$ other than R^{-n}. These improved models satisfied solar system constraints exhibiting late cosmic acceleration for small curvature and early inflation for large curvature [80]. Thus, in $f(R)-$ *dark energy models*, non-linear curvature terms were considered as an alternative for DE [81, for detailed review]. Recently, Amendola *et al.* have criticized $f(R)-$ *dark energy models*, specially the model with R^n and R^{-m} (where $n > 0$ and $m > 0$ are real numbers) [80] on the ground that these models are unable to produce matter in the late universe prior to the beginning of late acceleration [82]. This criticism is responded to in Ref. [83], where Nojiri and Odintsov have discussed dark matter in these type of models.

In the race to investigate a viable cosmological model, satisfying observational constraints and explaining present cosmic acceleration, brane-gravity was also drawn into service and brane-cosmology was developed. A review on brane-gravity and its various applications with special attention to cosmology is available in [85 - 88].

So, apart from general relativity (GR)-based models and $f(R)-$ models, brane-gravity (BG)-based cosmological models were also tried upon to explain acceleration in the late universe. In particular, RS-II model got much attention due to its simple and rich conceptual base [89 - 98]. In [99], it is found that RS-II model of brane-gravity yields a phantom model giving transient acceleration (where acceleration stops after sometime in future) and avoiding big-rip singularity.

The present density of dark energy is found to be $0.73\rho^0_{\rm cr}$ with $\rho^0_{\rm cr} = 2.5 \times 10^{-47}{\rm GeV}^4$ (present critical energy density) [38, 39]. If late universe is dominated by phantom, present phantom energy density is $0.73\rho^0_{\rm cr}$. As discussed above, Friedmann equation, obtained from RS-II model of brane-gravity, contains the energy term as $(8\pi G\rho/3)[1 - (\rho/2\lambda)]$, where $\lambda = 48\pi G/k_5^4 = 48\pi/M_P^2 k_5^4$ with $k_5^2 = 8\pi G_5 = 8\pi G l = 8\pi l/M_P^2$, with newtonian gravitational constant $G = M_P^{-2}$ in natural units, M_P being the Planck mass and l being length of the extra-dimension of the 5-dimensional bulk [86, 87, 88]. As an example, if we set $k_5^2 = 1{\rm GeV}^{-3}$ as taken in Ref. [87], $\lambda = 48\pi/M_P^2 = 6.03 \times 10^{10}\rho^0_{\rm cr}$ and $\rho/2\lambda \sim 10^{-10}$ in the present universe. This example shows that the brane-correction term, in the Friedmann equation, may not be effective in the present universe unless length of the extra-dimension is sufficiently small. But, energy density of the phantom fluid will increase with expansion of the universe due to EOS parameter $\omega < -1$ in this case. So, even if the correction term is not effective in the present universe, it will be effective in future phantom universe.

In the present universe, brane-corrections are not effective (as obtained above) in RS-II model. This situation will continue until ρ will grow sufficiently. During this period WEC is violated and phantom universe will super-accelerate. As far as $\rho << 2\lambda$, universe will super-accelerate in future and ρ will grow with $a(t)$. It is reasonable to believe $\rho/\lambda \gtrsim 1$ in future due to rapid increase in $a(t)$ caused by super-acceleration. Increase in ρ will still continue with growing $a(t)$. On further increase in ρ, brane-corrections will be effective and only SEC will be violated up to a certain value of phantom energy density.

As a consequence, acceleration of the phantom universe will become comparatively slow. It means that, in this situation, phantom universe will accelerate, but it will not super-accelerate. It is because, $\ddot{a}/a > 8\pi G\rho/3$ when WEC is violated and $0 < \ddot{a}/a < 8\pi G\rho/3$, when only SEC is violated. This is the intermediate state. When ρ will increase more, none of SEC and WEC will be violated due to strong effect of brane-corrections and phantom universe will decelerate. Acceleration and super-acceleration manifest anti-gravity effect of dark energy. So, it is found that brane-corrections, in RS-II model, counter anti-gravity effect of phantom dark energy. In the case of quintessence, ρ decreases with expansion of the universe due to $\omega > -1$, so brane-corrections can not be effective in RS-II model-based present and future quintessence universe.

In addition to the theoretical problems associated with "Phantom" scalar fields, classically they have the unphysical behaviour as ρ increases with expansion of the universe which leads to the instability of quantum vacuum. Models with $\omega < -1$ but without quantum instability are therefore interesting. An effective ω such that $\omega < -1$ can occur in modified gravity theories without any phantom matter that causes theoretical problems. There is a class of brane world models which exhibits effective phantom behaviour. These models are a variant of DGP brane world models [100, 101]. Since DGP(-) does not self accelerate, therefore it requires DE on the brane. It experiences 5d gravitational modifications to its dynamics which effectively screen DE. At late times, the dynamics deviates from GR, as gravity leaks off the 4d brane.

1.5 Universe as a thermodynamical system

In the last section we have seen universe is experiencing accelerated expansion driven by DE. Except knowing that DE has negative pressure, very little is known about its theoretical nature and origin. In this conceptual set up, one of the important questions concerns the thermodynamical behaviour of an accelerated expanding universe driven by DE. The first hint on the connection between general relativity and thermodynamics was given by Bekenstein in 1973 [103]. He outlined the laws of thermodynamics in the presence of black holes which turned out to be equivalent to the laws of black hole mechanics [104]. Study of gravitational thermodynamics in an accelerating universe has been a strong candidate and has been addressed to in many papers based on GR [105 - 110]. The reason of interest in this subject is two fold (i) it is natural to study thermodynamical aspect of accelerating universe and (ii) the astonishing result for phantom obtained in [111] which either has negative temperature or negative entropy. This is another problem of phantom cosmology like big-rip singularity for which some viable solution are proposed in Ref. [48]. The main problem of studying thermodynamics of the Universe is to define the entropy and temperature on the boundary of the universe. Generally the entropy and hence the temperature is taken from black hole physics but in other gravitational theories (such as $f(R)$ gravity) some correction terms may be needed. Motivated by the profound connection between black hole physics and thermodynamics, in recent times there has been some deep thinking on the relation between gravity and thermodynamics. In what follows, we give a brief idea of black hole thermodynamics.

In semi-classical quantum description of black hole physics, it is found that a black hole behaves as a black body and emits thermal radiation (known as Hawking radiation). The temperature (known as Hawking temperature) and the entropy are proportional to the surface gravity at the horizon and area of the horizon respectively [103, 104]. The temperature (Hawking), entropy and mass of the black hole satisfy the first law of thermodynamics [112]. As the temperature and entropy are determined by purely geometric quantities (namely surface gravity and horizon area respectively), i.e. characterized by the space-time geometry and hence by Einstein field equations, so it is natural to speculate some relationships between black hole thermodynamics and Einstein equations. A pioneer work in this respect was done by Jacobson who disclosed that Einstein's gravitational field equation can be derived from first law of thermodynamics ($:\delta Q = T\delta S$ connecting heat Q, entropy S, and temperature T) and the relation between horizon area and entropy for all local Rindler causal horizons [113]. For a general static spherically symmetric space-time, Padmanabhan [114] was able to derive the first law of thermodynamics on the horizon, starting from Einstein equations. In a nutshell Chakraborty *et al.* [115] have given the following nice equivalence.

Laws of thermodynamics

$\Longleftrightarrow$ Analogous laws of black hole dynamics (Semi classical analysis)

$\Longleftrightarrow$ Einstein field equations (gravity theory) (Classical treatment),

which perhaps shows the strongest evidence for a fundamental connection between quantum physics and gravity.

Subsequently, this equivalence between Einstein equation and thermodynamical laws has been generalized in the context of cosmology.

The Universe can be considered as a thermodynamical system. The cosmological horizon has an associated entropy that may be interpreted analogously to the entropy of the black hole [116]. An observer living in an accelerated FRW universe has a lack of information about the regions outside its event horizon. The area of the horizon, $A = 4\pi R_h^2$ (R_h being radius of the horizon) represents a measure of this lack of information. As in the case of the black hole horizon, entropy of the cosmological horizon is defined as $S_h = A/4$. The thermodynamics in de Sitters space-time was first investigated by Gibbons and Hawking in [116]. The thermodynamical study of the Universe has been extended to the quasi de Sitter space in [107, 117, 118]. In the usual standard big bang model a cosmological event horizon does not exist. However, in a general accelerating universe dominated by dark energy, with the equation of state $\omega \neq -1$, the cosmological event horizon separates from that of the apparent horizon. The event horizon and apparent horizon coincide in a spatially flat de Sitters space-time. There is a subtlety between the definitions of apparent horizon and of the event horizon of the universe. Their thermodynamical properties are different although dynamical difference between them is not large. When the apparent horizon and event horizon of the Universe are different, it was found by Wang *et al*. [107] that first law and second law of thermodynamics hold on apparent horizon, while they break down when event horizon is considered. According to them the first law may apply to variations between nearby states of thermodynamic equilibrium, while event horizon reflects the global features of space-time. Besides in a non static universe the usual definition of the thermodynamical quantities on the event horizon may not be as simple as in static space-time. The apparent horizon was singled out as the largest surface whose interior can be treated as a Bekenstein system, which satisfies the Bekenstein's entropy/mass bound $S \leq 2\pi R_A$ (R_A is the radius of the apparent horizon) and Bekenstein's entropy/area bound $S \leq A/4$. These Bekenstein's bounds are universal in nature and all gravitationally stable special regions with weak self gravity satisfy Bekenstein bounds. Outside the apparent horizon, the thermodynamic system is no longer a Bekenstein's system as Bekenstein entropy/area bound is violated. As event horizon is larger than the apparent horizon, so the universe bounded by the event horizon is not a Bekenstein system.

Some recent discussion on the connection between gravity and thermodynamics on various gravity theories can be found on [119, 120]. Recently this connection between gravity and thermodynamics has been extended to brane world scenario [121 - 124]. There are lot of works [125 - 132] in literature dealing with thermodynamics of the universe bounded by apparent horizon as it is a Bekenstein system in GR set up. But so far, due to the above complicated nature of event horizon, thermodynamics of the universe bounded by the event horizon is not addressed in brane world scenario. In this thesis we try to address this problem of gravitational thermodynamics.

Often using BG one obtains cosmological surprises. In [133], Sahni has mentioned many cosmological surprises. In [98], Srivastava has found that quantum gravity makes drastic changes in the course of future accelerated universe driven by phantom dark energy in RSII-model. In subsequent chapters, we shall see some new surprises from braneworld cosmology in addition to many surprises mentioned by Srivastava and Sahni.

Chapter 2

Non-Linear Equation Of State, Cosmic Acceleration And Deceleration During Phantom Dominance

2.1 Prelude

In this chapter, RS-II model of brane-gravity is considered for phantom universe using a non-linear equation of state. Phantom fluid is known to violate the WEC. As discussed in section 1.4 of chapter 1, $\omega = -1$ divides the cases violating SEC and WEC. It is known as phantom divide. It means that ideal EOS $p = -\rho$ needs a correction term being linear or non-linear function of ρ causing deviation from the ideal situation as $p = -\rho \pm f(\rho)$. Here, only phantom fluid is considered, so we take the negative sign yielding $\omega < -1$. Moreover, $f(\rho)$ in the proposed non-linear EOS implies dependence of ω on ρ. In some earlier investigations [134, 135, 136], these types of EOS were used considering time-dependent viscosity with correction terms dependent on ρ and $H = \dot{a}/a (\dot{a} = da/dt)$.

In a recent paper [49], EOS $p = -|\omega|\rho$ for phantom fluid with constant $\omega < -1$ has been considered in RS-II model based Friedmann equation and it is found that brane-gravity corrections suppress the phantom characteristic to violate WEC and SEC, when ρ increases sufficiently with expansion of the universe. As a consequence, this model expands with acceleration up to some finite time explaining present cosmic acceleration, but it decelerates later on. In Ref. [49], $f(\rho)$ is a linear function of ρ. So, it is natural to study the effect of brane-corrections taking non-linear $f(\rho)$ too. The aim of this chapter is to extend the work of Ref. [49] taking EOS $p = -\rho - f(\rho)$, with $f(\rho)$ being non-linear functional of ρ. We shall show that known characteristic of phantom energy to violate SEC is drastically affected by the negative brane-tension λ of the RS-II model. It is interesting to see that up to a certain value of energy density ρ satisfying $\rho/\lambda < 1$, WEC is violated and universe super-accelerates. But as ρ increases more, only SEC is violated and universe accelerates. When $1 < \rho/\lambda < 2$, even SEC is not violated and universe decelerates. Expansion of the universe stops, when $\rho = 2\lambda$. This is contrary to earlier results of phantom universe exhibiting acceleration only. This chapter is based on [137].

2.2 Effective equation of state

Observations support homogeneous and isotropic model of the late universe, given by the line-element [42]

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2] \tag{2.2.1}$$

where a(t) is the scale factor.

In general, the last term in equation (1.3.8) is non zero, one can consistently make the choice $\mathcal{C} = 0$ if the bulk space time is anti-de Sitter. In this chapter, we study RS-II model with negative tension ($\lambda < 0$). RS-II model with negative tension has been extensively studied in [24, 49].

In this space-time, RS-II model based Friedmann equation (1.3.8) reduces to

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left[1 - \frac{\rho}{2\lambda}\right], \tag{2.2.2}$$

with G, ρ and λ as defined in previous chapter.

As mentioned in previous chapter, in RS models also, conservation equation is given as

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.2.3}$$

Connecting equations (2.2.2) and (2.2.3), it is obtained that [97, 98]

$$\frac{\ddot{a}}{a} = -4\pi G(\rho + p)\left[1 - \frac{\rho}{\lambda}\right] + \frac{8\pi G}{3}\rho\left[1 - \frac{\rho}{2\lambda}\right]. \tag{2.2.4}$$

Here, the non-linear equation of state for phantom fluid is taken as

$$p = -\rho - f(\rho). \tag{2.2.5}$$

As $\rho + p < 0$, equation (2.2.3) yields $\dot{\rho} > 0$. It shows that phantom energy density will increase in future with growing $a(t)$. Moreover, equation (2.2.2) shows that $a(t)$ will be maximum when $\rho = 2\lambda$.

In GR-based theory, Friedmann equation is obtained as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[\rho + 3P]. \tag{2.2.6}$$

Comparing equations (2.2.4) and (2.2.6), *effective EOS* with brane gravity corrections is obtained as

$$P = -\rho - f\Big[1 - \frac{\rho}{\lambda}\Big] + \frac{\rho^2}{3\lambda}, \tag{2.2.7}$$

using equation (2.2.5).

Equation (2.2.7) yields

$$\rho + P = -f\Big[1 - \frac{\rho}{\lambda}\Big] + \frac{\rho^2}{3\lambda} \tag{2.2.8}$$

and

$$\rho + 3P = -2\rho - 3f\Big[1 - \frac{\rho}{\lambda}\Big] + \frac{\rho^2}{\lambda}. \tag{2.2.9}$$

It is interesting to see from equations (2.2.8) and (2.2.9) that

$$\rho + P = -f\Big[1 + \frac{\rho}{\lambda}\Big] - \frac{\rho^2}{3\lambda} < 0,$$

showing violation of WEC, and

$$\rho + 3P = -2\rho - 3f\Big[1 + \frac{\rho}{\lambda}\Big] - \frac{\rho^2}{\lambda} < 0,$$

showing violation of SEC, if $\lambda > 0$, which is the case of RS-I model. But, in the RS-II model being addressed here, we find certain situations when these cosmic conditions are not violated due to effect of brane-gravity corrections.

Here, the case of RS-II model is analyzed taking the following three cases yielding different non-linear EOS (2.2.5) for the phantom fluid:

$$\begin{aligned}(I)\;\; f(\rho) &= A\rho^{\alpha},\\ (II)\;\; f(\rho) &= \frac{A\rho^{\alpha}}{\sqrt{1-\frac{\rho}{2\lambda}}},\\ (III)\;\; f(\rho) &= \frac{A\rho^{\frac{1}{2}}\ln(\rho/\rho_0)}{\sqrt{1-\frac{\rho}{2\lambda}}},\end{aligned}$$

with α being a real number and A being a coupling constant having dimension $(\text{mass})^{4-4\alpha}$ in cases (I) and (II). Moreover, A has dimension $(\text{mass})^2$ in case (III).

$\underline{Case\ I : f(\rho) = A\rho^{\alpha}}$

In this case, equation (2.2.8) implies that

$$P = -\rho - A\rho^{\alpha}\Big[1-\frac{\rho}{\lambda}\Big] + \frac{\rho^2}{3\lambda}. \tag{2.2.10}$$

Equation (2.2.10) yields effective pressure $P < 0$ for

$$\rho < 3\lambda\Big[1 + A\rho^{(\alpha-1)}\Big\{1-\frac{\rho}{\lambda}\Big\}\Big]. \tag{2.2.11}$$

Further, it is found that

$$\rho + P = -A\rho^{\alpha}\Big[1-\frac{\rho}{\lambda}\Big] + \frac{\rho^2}{3\lambda} < 0$$

till

$$\rho_0 < \rho < 3\lambda A\rho^{(\alpha-1)}\Big[1-\frac{\rho}{\lambda}\Big], \tag{2.2.12}$$

with ρ_0 being the present energy density. This result shows that WEC will be violated till ρ satisfies the inequality (2.2.12). It will not be violated when

$$\rho > 3\lambda A\rho^{(\alpha-1)}\Big[1-\frac{\rho}{\lambda}\Big]. \tag{2.2.13}$$

This means that phantom fluid will behave effectively as phantom dark energy till ρ obeys the inequality (2.2.12). It will not behave effectively as phantom when ρ increases more and obeys the inequality (2.2.13).

Moreover, equation (2.2.9) shows that SEC will be violated till

$$\rho < \lambda\Big[2+3A\rho^{(\alpha-1)}\Big\{1-\frac{\rho}{\lambda}\Big\}\Big]. \tag{2.2.14}$$

This shows that when ρ will increase such that

$$3\lambda A\rho^{(\alpha-1)}\Big[1-\frac{\rho}{\lambda}\Big] < \rho < \lambda\Big[2+3A\rho^{(\alpha-1)}\Big\{1-\frac{\rho}{\lambda}\Big\}\Big], \tag{2.2.15}$$

only SEC will be violated. It shows that when ρ will satisfy the inequality (2.2.15), phantom characteristic to violate WEC will be suppressed by brane-gravity effects for negative brane tension and phantom fluid will behave effectively as quintessence. These results yield *effective phantom divide* at

$$\rho = \rho_{\rm phd} = 3\lambda A\rho_{\rm phd}^{(\alpha-1)}\Big[1-\frac{\rho_{\rm phd}}{\lambda}\Big]. \tag{2.2.16}$$

It is interesting to see that even SEC will not be violated when

$$\lambda\Big[2+3A\rho^{(\alpha-1)}\Big\{1-\frac{\rho}{\lambda}\Big\}\Big] < \rho < 2\lambda. \tag{2.2.17}$$

which implies that, during the range given by (2.2.17), dark energy characteristic to violate SEC and WEC will be suppressed completely by brane-corrections in RS-II model.

Thus, the above analysis shows that universe will accelerate till ρ satisfies the inequality (2.2.14) and it will decelerate during the range of ρ given by (2.2.17).

$$\underline{Case\ II : f(\rho) = A\rho^{\alpha}/\sqrt{1 - \frac{\rho}{2\lambda}}}$$

In this case, equation (2.2.8) implies that

$$P = -\rho - \frac{A\rho^{\alpha}}{\sqrt{1-\frac{\rho}{2\lambda}}}\left[1 - \frac{\rho}{\lambda}\right] + \frac{\rho^2}{3\lambda}. \tag{2.2.18}$$

This equation yields effective pressure $P < 0$ for

$$\rho < 3\lambda\left[1 + \frac{A\rho^{(\alpha-1)}}{\sqrt{1-\frac{\rho}{2\lambda}}}\left\{1 - \frac{\rho}{\lambda}\right\}\right]. \tag{2.2.19}$$

Further, it is found that

$$\rho + P = -\frac{A\rho^{\alpha}}{\sqrt{1-\frac{\rho}{2\lambda}}}\left[1 - \frac{\rho}{\lambda}\right] + \frac{\rho^2}{3\lambda} < 0$$

till

$$\rho_0 < \rho < 3\lambda\frac{A\rho^{\alpha}}{\sqrt{1-\frac{\rho}{2\lambda}}}\left[1 - \frac{\rho}{\lambda}\right]. \tag{2.2.20}$$

This result shows that WEC is violated till ρ satisfies the inequality (2.2.20). It will not be violated when

$$\rho > 3\lambda A\rho^{(\alpha-1)}\Big[1 - \frac{\rho}{\lambda}\Big]. \tag{2.2.21}$$

So, like the case I, in this case too, we find that phantom fluid will not behave effectively as phantom when ρ satisfies the inequality (2.2.21) due to brane-corrections.

Moreover, equation (2.2.9) shows that SEC will be violated till

$$\rho < \lambda\left[2 + \frac{3A\rho^{(\alpha-1)}}{\sqrt{1-\rho/2\lambda}}\Big\{1 - \frac{\rho}{\lambda}\Big\}\right], \tag{2.2.22}$$

but as ρ will increase with time such that

$$3\lambda\frac{A\rho^{\alpha}}{\sqrt{1-\frac{\rho}{2\lambda}}}\Big[1 - \frac{\rho}{\lambda}\Big] < \rho < \lambda\left[2 + \frac{3A\rho^{(\alpha-1)}}{\sqrt{1-\rho/2\lambda}}\Big\{1 - \frac{\rho}{\lambda}\Big\}\right], \tag{2.2.23}$$

only SEC will be violated. It shows that when ρ satisfies the inequality (2.2.23), phantom characteristic to violate WEC will be suppressed by brane-gravity effects for negative brane tension and phantom fluid will behave effectively like quintessence. These results suggest *effective phantom divide* at

$$\rho = \rho_{\rm phd} = 3\lambda A\rho_{\rm phd}^{(\alpha-1)}\Big[1 - \frac{\rho_{\rm phd}}{\lambda}\Big]. \tag{2.2.24}$$

It is interesting to see that even the SEC will not be violated for

$$\lambda\left[2 + \frac{3A\rho^{(\alpha-1)}}{\sqrt{1-\rho/2\lambda}}\Big\{1 - \frac{\rho}{\lambda}\Big\}\right] < \rho < 2\lambda, \tag{2.2.25}$$

which implies that, during the range (2.2.25), dark energy characteristic to violate SEC and WEC will be suppressed completely by brane-corrections in the RS-II model.

Thus, the above analysis shows that universe will accelerate till ρ satisfies the inequality (2.2.22) and it will decelerate during the range of ρ given by (2.2.25).

$$\underline{Case\ III : f(\rho) = A\rho^{1/2}\ln(\rho/\rho_0)/\sqrt{1-\frac{\rho}{2\lambda}}}$$

In this case, equation (2.2.8) implies that

$$P = -\rho - \frac{A\rho^{1/2}\ln(\rho/\rho_0)}{\sqrt{1-\frac{\rho}{2\lambda}}}\Big[1-\frac{\rho}{\lambda}\Big] + \frac{\rho^2}{3\lambda}. \tag{2.2.26}$$

This equation yields effective pressure $P < 0$ for

$$\rho < 3\lambda\left[1 + \frac{A\rho^{1/2}\ln(\rho/\rho_0)}{\sqrt{1-\frac{\rho}{2\lambda}}}\Big\{1-\frac{\rho}{\lambda}\Big\}\right]. \tag{2.2.27}$$

Further, it is found that

$$\rho + P = -\frac{A\rho^{1/2}\ln(\rho/\rho_0)}{\sqrt{1-\frac{\rho}{2\lambda}}}\Big[1-\frac{\rho}{\lambda}\Big] + \frac{\rho^2}{3\lambda} < 0$$

till

$$\rho_0 < \rho < 3\lambda\rho^{-1/2}\frac{A\ln(\rho/\rho_0)}{\sqrt{1-\frac{\rho}{2\lambda}}}\Big[1-\frac{\rho}{\lambda}\Big]. \tag{2.2.28}$$

This result shows that WEC will be violated till ρ satisfies the inequality (2.2.28). It will not be violated when

$$\rho > 3\lambda\frac{A\rho^{-1/2}\ln(\rho/\rho_0)}{\sqrt{1-\frac{\rho}{2\lambda}}}\Big[1-\frac{\rho}{\lambda}\Big]. \tag{2.2.29}$$

So, like the case I and II, in this case also, we find that phantom fluid will not behave effectively as phantom when ρ satisfies the inequality (2.2.29) due to brane-corrections.

Moreover, (2.2.9) shows that, in this case, SEC will be violated till

$$\rho < \lambda \left[2 + \frac{3A\rho^{-1/2}\ln(\rho/\rho_0)}{\sqrt{1-\rho/2\lambda}} \left\{ 1 - \frac{\rho}{\lambda} \right\} \right]. \tag{2.2.30}$$

It shows that as ρ increases with time such that

$$3\lambda \frac{A\rho^{-1/2}\ln(\rho/\rho_0)}{\sqrt{1-\frac{\rho}{2\lambda}}} \left[1 - \frac{\rho}{\lambda} \right] < \rho < \lambda \left[2 + \frac{3A\rho^{-1/2}\ln(\rho/\rho_0)}{\sqrt{1-\rho/2\lambda}} \left\{ 1 - \frac{\rho}{\lambda} \right\} \right], \tag{2.2.31}$$

only SEC will be violated. It shows that when ρ satisfies the inequality (2.2.31), phantom characteristic to violate WEC will be suppressed by brane-gravity effects for negative brane tension and phantom fluid will behave effectively as quintessence. These results suggest *effective phantom divide* at

$$\rho = \rho_{\rm phd}^{3/2} = 3\lambda \frac{A\ln(\rho_{\rm phd}/\rho_0)}{\sqrt{1-\frac{\rho_{\rm phd}}{2\lambda}}} \left[1 - \frac{\rho_{\rm phd}}{\lambda} \right]. \tag{2.2.32}$$

It is interesting to see that even SEC will not be violated for

$$\lambda \left[2 + \frac{3A\rho^{-1/2}\ln(\rho/\rho_0)}{\sqrt{1-\rho/2\lambda}} \left\{ 1 - \frac{\rho}{\lambda} \right\} \right] < \rho < 2\lambda. \tag{2.2.33}$$

which implies that, during the range (2.2.33), dark energy characteristic to violate SEC and WEC will be suppressed completely by brane-corrections in RS-II model.

Thus, the above analysis shows that universe will accelerate till ρ satisfies the inequality (2.2.38) and it will decelerate during the range of ρ given by (2.2.33).

2.3 Cosmic expansion with acceleration and deceleration

In the preceding section, we obtained different conditions for changes in the behaviour of phantom fluid dominating the RS-II model-based universe due to brane-gravity corrections. In what follows, we derive scale factor $a(t)$ solving Friedmann equation (2.2.2) and conservation equation (2.2.3). It helps to find the time period during which WEC and SEC will be violated and the time period during which these will not be violated.

$\underline{Case\ I : f(\rho) = A\rho^{\alpha}}$

In this case, connecting equations (2.2.3) and (2.2.5), we obtain

$$\dot{\rho} - 3A\frac{\dot{a}}{a}\rho^{\alpha} = 0. \tag{2.3.1}$$

It integrates to

$$\rho = \left[\rho_0^{1-\alpha} + 3A(1-\alpha)\ln\left(\frac{a}{a_0}\right)\right]^{\frac{1}{1-\alpha}}, \tag{2.3.2}$$

where $\rho_0 \leqslant \rho \leqslant 2\lambda$.

Equations (2.2.2), (2.2.3) and (2.2.5) yield

$$\dot{\rho} - 3A\sqrt{\frac{8\pi G}{3}}\rho^{\alpha+\frac{1}{2}}\sqrt{1-\frac{\rho}{2\lambda}} = 0, \tag{2.3.3}$$

where $\rho_0 \leqslant \rho \leqslant 2\lambda$.

Exact solution of this equation is obtained as

$$t = \frac{1}{A\sqrt{24\pi G}}\Big[t_0 + 2(2\lambda)^{(1/2)-\alpha}\Big\{\sqrt{1-\frac{\rho_0}{2\lambda}}2F_1\Big(\frac{1}{2},\frac{1}{2}+\alpha,\frac{3}{2},1-\frac{\rho_0}{2\lambda}\Big)$$

$$-\sqrt{1-\frac{\rho}{2\lambda}}2F_1\Big(\frac{1}{2},\frac{1}{2}+\alpha,\frac{3}{2},1-\frac{\rho}{2\lambda}\Big)\Big\}\Big], \tag{2.3.4}$$

where $2F_1(a,b,c,x)$ is the hypergeometric function. Further, using (2.3.2) in (2.3.4), we get a relation between time t and the scale factor $a(t)$.

As maximum value of ρ is 2λ, so phantom universe will expand up to time t_m given as

$$t_m = \frac{1}{A\sqrt{24\pi G}}\Big[t_0 + 2(2\lambda)^{(1/2)-\alpha}\sqrt{1-\frac{\rho_0}{2\lambda}}2F_1\Big(\frac{1}{2},\frac{1}{2}+\alpha,\frac{3}{2},1-\frac{\rho_0}{2\lambda}\Big)\Big] \tag{2.3.5}$$

with t_0 being the present time. Moreover, from equation (2.3.2), it is found that

$$3(1-\alpha)A\ln\left(\frac{a_m}{a_0}\right) = (2\lambda)^{1-\alpha} - \rho_0^{1-\alpha}, \tag{2.3.6}$$

where $a_m = a(t_m)$. This equation shows that if $\alpha \gtreqless 1, 2\lambda > \rho_0$ as $a_m > a_0$.

From equations (2.2.16) and (2.3.4), we obtain *effective phantom divide* at time

$$t = t_{\rm phd} = \frac{1}{A\sqrt{24\pi G}}\Big[t_0{+}2(2\lambda)^{(1/2)-\alpha}\Big\{\sqrt{1-\frac{\rho_0}{2\lambda}}2F_1\Big(\frac{1}{2},\frac{1}{2}{+}\alpha,\frac{3}{2},1{-}\frac{\rho_0}{2\lambda}\Big)$$

$$-\sqrt{1-\frac{\rho_{\rm phd}}{2\lambda}}2F_1\Big(\frac{1}{2},\frac{1}{2}+\alpha,\frac{3}{2},1-\frac{\rho_{\rm phd}}{2\lambda}\Big)\Big\}\Big]. \tag{2.3.7}$$

Inequalities (2.2.15) and (2.2.17) show that for ρ satisfying

$$\rho < \lambda\Big[2 + 3A\rho^{(\alpha-1)}\Big\{1 - \frac{\rho}{\lambda}\Big\}\Big], \tag{2.3.8}$$

SEC will be violated. This means that the universe will accelerate till ρ obeys (2.3.8). But as ρ grows more and satisfies

$$\rho > \lambda\Big[2 + 3A\rho^{(\alpha-1)}\Big\{1 - \frac{\rho}{\lambda}\Big\}\Big], \tag{2.3.9}$$

the SEC will not be violated. This means that the universe will decelerate when ρ obeys the inequality (2.3.9). It shows a transition from acceleration to deceleration at $\rho = \rho_{\rm tr}$ given by the equation

$$\rho_{\rm tr} = \lambda\Big[2 + 3A\rho_{\rm tr}^{(\alpha-1)}\Big\{1 - \frac{\rho_{\rm tr}}{\lambda}\Big\}\Big]. \tag{2.3.10}$$

Connecting equations (2.3.4) and (2.3.10), we find that this transition will take place at time

$$t = t_{\rm tr} = \frac{1}{A\sqrt{24\pi G}}\Big[t_0 + 2(2\lambda)^{(1/2)-\alpha}\Big\{\sqrt{1 - \frac{\rho_0}{2\lambda}}\,2F_1\Big(\frac{1}{2}, \frac{1}{2} + \alpha, \frac{3}{2}, 1 - \frac{\rho_0}{2\lambda}\Big)$$

$$-\sqrt{1 - \frac{\rho_{\rm tr}}{2\lambda}}\,2F_1\Big(\frac{1}{2}, \frac{1}{2} + \alpha, \frac{3}{2}, 1 - \frac{\rho_{\rm tr}}{2\lambda}\Big)\Big\}\Big]. \tag{2.3.11}$$

If $\alpha = 1$, equation (2.3.3) integrates to

$$\rho = \left[\frac{1}{2\lambda} + \left\{\sqrt{\frac{1}{\rho_0} - \frac{1}{2\lambda}} - A\sqrt{6\pi G}(t - t_0)\right\}^2\right]^{-1}, \tag{2.3.12}$$

where $\rho_0 < 2\lambda$ is the current energy density of the DE. It shows that phantom energy density will increase with time, which is consistent with results of GR based theory.

From equations (2.2.2) and (2.3.12), it is found that

$$H^2 = \frac{8\pi G}{3}\frac{\left\{\sqrt{\frac{1}{\rho_0}-\frac{1}{2\lambda}} - A\sqrt{6\pi G}(t-t_0)\right\}^2}{\left[\frac{1}{2\lambda}+\left\{\sqrt{\frac{1}{\rho_0}-\frac{1}{2\lambda}} - A\sqrt{6\pi G}(t-t_0)\right\}^2\right]^2}. \tag{2.3.13}$$

The solution of equation (2.3.13) is

$$a(t) = a_0\rho_0^{-\frac{1}{3}}\left[\frac{1}{2\lambda}+\left\{\sqrt{\frac{1}{\rho_0}-\frac{1}{2\lambda}} - A\sqrt{6\pi G}(t-t_0)\right\}^2\right]^{-\frac{1}{3}}. \tag{2.3.14}$$

Using $\rho = 2\lambda$ in equation (2.3.12), it is found that phantom era will end at time

$$t_e = t_0 + \frac{1}{A\sqrt{6\pi G}}\sqrt{\frac{1}{\rho_0}-\frac{1}{2\lambda}}. \tag{2.3.15}$$

In this case, equation (2.2.4) reduces to

$$\frac{\ddot{a}}{a} = \frac{4\pi G\rho}{3}\Big[3A\Big(1-\frac{\rho}{\lambda}\Big)+2\Big(1-\frac{\rho}{2\lambda}\Big)\Big], \tag{2.3.16}$$

which yields

$$\ddot{a} \gtreqless 0 \qquad \text{when} \qquad \frac{\rho}{\lambda} \lesseqgtr \frac{3A+2}{1+3A}. \tag{2.3.17}$$

Equation (2.3.12) and (2.3.16) show that $\ddot{a}=0$ when

$$\frac{3A+1}{(3A+2)\lambda} = \frac{1}{2\lambda}+\left\{\sqrt{\frac{1}{\rho_0}-\frac{1}{2\lambda}} - A\sqrt{6\pi G}(t_{vm}-t_0)\right\}^2. \tag{2.3.18}$$

This yields time for transition from acceleration to deceleration as

$$t_{vm} = t_0 + \frac{1}{A\sqrt{6\pi G}}\left[\sqrt{\frac{1}{\rho_0}-\frac{1}{2\lambda}} - \sqrt{\frac{3A}{2(3A+2)\lambda}}\right]. \tag{2.3.19}$$

From equation (2.3.12), it is also found that
$\ddot{a} < 0$ when $2\lambda \geqslant \rho > (3A+2)\lambda/3A+1$. This means that during the time interval

$$t_0+\frac{1}{A\sqrt{6\pi G}}\sqrt{\frac{1}{\rho_0}-\frac{1}{2\lambda}} \geqslant t > t_0+\frac{1}{A\sqrt{6\pi G}}\left[\sqrt{\frac{1}{\rho_0}-\frac{1}{2\lambda}} - \sqrt{\frac{3A}{2(3A+2)\lambda}}\right], \tag{2.3.20a}$$

phantom universe will decelerate in the case $\alpha = 1$. It gives deceleration period

$$\sqrt{\frac{M_P^2}{4(3A+2)\pi G\lambda}}, \tag{2.3.20b}$$

which depends on magnitude of negative brane tension. There is no deceleration if $\lambda \gg M_P^2$. So the role of brane tension is very crucial here [140].

$$\underline{Case\ II : f(\rho) = \frac{A\rho^\alpha}{\sqrt{1-\frac{\rho}{2\lambda}}}}$$

In this case, connecting equations (2.2.3) and (2.2.5), we obtain

$$\dot{\rho} - 3A\frac{\dot{a}}{a}\frac{\rho^\alpha}{\sqrt{1-\rho/2\lambda}} = 0, \tag{2.3.21}$$

which integrates to

$$3A\ln\left(\frac{a}{a_0}\right) = \frac{1}{3}2^{2-\alpha}\lambda^{1-\alpha}\Big[\left(1-\frac{\rho_0}{2\lambda}\right)^{3/2}2F_1\left(\frac{3}{2},\alpha,\frac{5}{2},1-\frac{\rho_0}{2\lambda}\right)$$

$$-\left(1-\frac{\rho}{2\lambda}\right)^{3/2}2F_1\left(\frac{3}{2},\alpha,\frac{5}{2},1-\frac{\rho}{2\lambda}\right)\Big], \tag{2.3.22}$$

where $\rho_0 \leqslant \rho \leqslant 2\lambda$.

Equations (2.2.2), (2.2.3) and (2.2.5) yield

$$\dot{\rho} - 3A\sqrt{\frac{8\pi G}{3}}\rho^{\alpha+\frac{1}{2}} = 0 \tag{2.3.23}$$

Exact solution of this equation is obtained as

$$\rho = \left[\rho_0^{(1-2\alpha)/2} + \frac{1}{2}(1-2\alpha)A\sqrt{6\pi G}(t-t_0)\right]^{2/(1-2\alpha)}. \tag{2.3.24}$$

Thus, equations (2.3.22) and (2.3.24) yield the scale factor $a(t)$ as a function of time t.

As in RS-II model, expansion stops at $\rho = 2\lambda$; here phantom driven universe will expand up to time t_m, given as

$$t_{\rm m} = t_0 + \frac{2}{A(1-2\alpha)\sqrt{6\pi G}}\Big[(2\lambda)^{(1-2\alpha)/2} - \rho_0^{(1-2\alpha)/2}\Big]. \qquad (2.3.25)$$

In this case, WEC will be violated till

$$t < t_0 + \frac{2}{A(1-2\alpha)\sqrt{6\pi G}}\Big[\rho^{(1-2\alpha)/2} - \rho_0^{(1-2\alpha)/2}\Big], \qquad (2.3.26a)$$

and ρ will satisfy the inequality (2.2.20). It is found that WEC will not be violated when

$$t > t_0 + \frac{2}{A(1-2\alpha)\sqrt{6\pi G}}\Big[\rho^{(1-2\alpha)/2} - \rho_0^{(1-2\alpha)/2}\Big], \qquad (2.3.26b)$$

and ρ will satisfy the inequality (2.21). So, the *effective phantom divide* is obtained at time

$$t_{\rm phd} = t_0 + \frac{2}{A(1-2\alpha)\sqrt{6\pi G}}\Big[\rho_{\rm phd}^{(1-2\alpha)/2} - \rho_0^{(1-2\alpha)/2}\Big], \qquad (2.3.26c)$$

where $\rho_{\rm phd}$ is given by equation (2.2.24).

The inequality (2.2.23) and (2.3.24) yield the time period during which SEC will be violated. This shows that phantom fluid will behave effectively as quintessence fluid due to brane-corrections in this case. So, the universe will accelerate till

$$t < t_{\rm ae} = t_0 + \frac{2}{A(1-2\alpha)\sqrt{6\pi G}}\Big[\rho_{\rm ae}^{(1-2\alpha)/2} - \rho_0^{(1-2\alpha)/2}\Big], \qquad (2.3.27a)$$

where

$$\rho = \rho_{\rm ae} = \lambda\left[2 + \frac{3A\rho^{(\alpha-1/2)}}{\sqrt{1-\rho_{\rm ae}/2\lambda}}\Big\{1 - \frac{\rho_{\rm ae}}{\lambda}\Big\}\right]. \qquad (2.3.27b)$$

Similarly, inequalities (2.2.25) and (2.3.27b), as well as equations (2.3.24) and (2.3.27a), yield the time period $t_{\rm ae} < t < t_m$, during which brane-corrections of RS-II model will be so effective that neither SEC nor WEC will be violated for phantom fluid. As a consequence, universe will decelerate during this period.

As in case I, we take $\alpha = 1$ as an example. In what follows, the above results are analyzed if $\alpha = 1$. Using equation (2.3.26c), the time for *effective phantom divide* will be obtained at

$$t_{\rm phd} = t_0 + \frac{2}{A\sqrt{6\pi G}}\Big[\rho_0^{-1/2} - \rho_{\rm phd}^{(1-2\alpha)/2}\Big], \tag{2.3.28a}$$

where

$$\rho_{\rm phd} = \frac{3\lambda A}{1+3A}. \tag{2.3.28b}$$

According to equation (2.3.25), universe will expand till

$$t_{\rm m} = t_0 + \frac{2}{A\sqrt{6\pi G}}\Big[\rho_0^{-1/2} - (2\lambda)^{-1/2}\Big]. \tag{2.3.29}$$

Equations (2.3.22) and (2.3.24) show that at time

$$t_{\rm br} = t_0 + \frac{2}{A\sqrt{6\pi G\rho_0}} \tag{2.3.30}$$

ρ is divergent and $a(t)$ is complex. It is an unphysical situation.

Connecting equations (2.3.29) and (2.3.30), it is found that

$$t_{\rm m} = t_{\rm br} - \frac{1}{A\sqrt{3\pi G\lambda}}. \tag{2.3.31}$$

This shows that expansion of the universe will stop before encountering the unphysical situation occurring at time $t_{\rm br}$, given by equation (2.3.30).

$$\underline{Case\ III : f(\rho) = A\rho^{1/2}\ln(\rho/\rho_0)/\sqrt{1-\frac{\rho}{2\lambda}}}$$

In this case, using equations (2.2.2), (2.2.3) and (2.2.5), we get

$$\dot{\rho} - 3A\sqrt{\frac{8\pi G}{3}}\rho\ln\rho = 0, \tag{2.3.32}$$

which integrates to

$$\ln\left(\frac{\rho}{\rho_0}\right) = \sqrt{24\pi G}(t-t_0). \tag{2.3.33}$$

The *effective phantom divide* is obtained at time

$$t = t_{\mathrm{phd}} = t_0 + \frac{\rho_{\mathrm{phd}}}{A\rho_0\sqrt{24\pi G}}. \tag{2.3.34}$$

This shows that, at $t < t_{\mathrm{phd}}$, the phantom fluid will violate WEC, but, at $t > t_{\mathrm{phd}}$ WEC will not be violated.

The inequality (2.2.30) shows that even SEC will be violated till $\rho < \rho_{\mathrm{ae}}$ and it will not be violated when $\rho > \rho_{\mathrm{ae}}$, where

$$\rho_{\mathrm{ae}} = \lambda\left[2 + \frac{3A\rho_{\mathrm{ae}}^{-1/2}\ln(\rho_{\mathrm{ae}}/\rho_0)}{\sqrt{1-\rho_{\mathrm{ae}}/2\lambda}}\left\{1-\frac{\rho_{\mathrm{ae}}}{\lambda}\right\}\right]. \tag{2.3.35}$$

The phantom energy will acquire the value ρ_{ae} at time

$$t_{\mathrm{ae}} = t_0 + \frac{\rho_{\mathrm{phd}}}{A\rho_0\sqrt{24\pi G}}, \tag{2.3.36}$$

being obtained from equations (2.3.33) and (2.3.35).

As in the above cases, t_{m} is obtained as

$$t_{\mathrm{m}} = t_0 + \frac{2\lambda}{A\rho_0\sqrt{24\pi G}}. \tag{2.3.37}$$

The results (2.3.36) and (2.3.37) show that universe will accelerate till $t_0 \leq t < t_{\rm ae}$ and will decelerate for $t_{\rm ae} < t < t_{\rm m}$.

2.4 Conclusion

In this chapter, we analyze the behaviour of phantom fluid in RS-II model of brane-gravity having negative brane-tension λ. Three cases of non-linear equations of state for the phantom fluid are taken. It is found that, contrary to RS-I model, in RS-II model, brane-corrections make drastic changes in the behaviour of phantom fluid, which is characterized by violation of WEC and the accelerating universe ending up in big-rip singularity in most of the models. Interestingly, RS-II model based phantom cosmology is found different from the usual picture of phantom universe. Energy conservation for phantom fluid yields that phantom energy density increases as universe expands. Above results suggest that behaviour of phantom fluid will change in the future universe as energy density will grow with expansion. Here, the model of the future universe begins at time t_0, the present age of the universe, and it stops expanding when phantom energy density ρ grows to 2λ by the time $t_{\rm m}$. The above analysis shows that, during the period $t_0 \leq t < t_{\rm m}$, two transitions will take place. The first one will take place at $t_{\rm phd}$, the time of transition from *violation of WEC* to *non-violation of WEC* and *violation of SEC*. The second one will take place at $t_{\rm ae}$, the time of transition from *violation of SEC* to *non-violation of SEC*. These transitions are caused by brane-corrections due to negative brane-tension in RS-II model-based universe. As a consequence, it is found that the present model of the universe will accelerate during the period $t_0 \leq t < t_{\rm ae}$ and decelerate during the period $t_{\rm ae} < t < t_{\rm m}$. Moreover, the model is free from big-rip problem. Thus, it is found that the role of brane-tension is crucial. When it is negative, it causes drastic changes in the behaviour of phantom dark energy, but phantom fluid has usual behaviour when brane-tension is positive.

Chapter 3

Non-Linear Equation Of State and Effective Phantom Divide In DGP Model

3.1 Prelude

In the previous chapter, we have studied the effect of brane corrections taking non-linear EOS. It was shown that known characteristic of phantom fluid to violate SEC is drastically affected by the negative brane-tension λ of the RS-II model. Here DGP model of brane-gravity is analysed and compared with the standard general relativity and Randall-Sundrum cases using non-linear equation of state. It is found that in DGP model SEC is always violated and the universe accelerates only where as WEC is violated only for a special range of energy density. Finally, we derive an expression of the scale factor and analyse its behaviour in the late universe. A part of this chapter is based on [138].

3.2 Basic Equations

As in Chapter 1, we take line-element

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2], \tag{3.2.1}$$

where a(t) is the scale factor.

In this space-time the standard GR Friedmann equation (1.3.3) is given by

$$H^2 = \frac{\kappa^2}{3}\rho, \tag{3.2.2}$$

where $\kappa^2 = M_P^{-2} = 8\pi G$.

On the brane in both RS-II and DGP model the conservation equation (1.3.4) holds:

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{3.2.3}$$

In the space-time given by equation (3.2.1), the DGP Friedmann equations (1.3.9) and (1.3.10) are respectively given by

$$H^2 = \left(\sqrt{\frac{\kappa^2\rho}{3} + \frac{1}{4r_c^2}} + \epsilon\frac{1}{2r_c}\right)^2, \tag{3.2.4}$$

$$H^2 - \epsilon\frac{H}{r_c} = \frac{\kappa^2\rho}{3}, \tag{3.2.5}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, ρ is the cosmic fluid energy density and $r_c = \frac{M_P^2}{2M_5^2}$ is the crossover scale which determines the transition from 4d to 5d behavior and $\epsilon = \pm 1$.

Here we take DGP(-) model. Thus Friedmann equations (3.2.4) and (3.2.5) simplifies to

$$H^2 = \left(\sqrt{\frac{\kappa^2\rho}{3} + \frac{1}{4r_c^2}} - \frac{1}{2r_c}\right)^2 \tag{3.2.6}$$

and

$$H^2 + \frac{H}{r_c} = \frac{\kappa^2 \rho}{3}. \tag{3.2.7}$$

3.3 Effective Equation of state and Cosmic Expansion

As in the previous chapter, the nonlinear EOS for phantom fluid is taken as

$$p = -\rho - f(\rho), \tag{3.3.1}$$

where $f(\rho) = A\rho^\alpha$, i.e.

$$p = -\rho - A\rho^\alpha. \tag{3.3.2}$$

Connecting equations (3.2.3) and (3.3.2), we have

$$\rho = \rho_0 \left[1 + 3\tilde{A}(1-\alpha)\ln\frac{a}{a_0}\right]^{\frac{1}{1-\alpha}}, \tag{3.3.3}$$

where $\alpha \neq 1$ and $\tilde{A} = A\rho_0^{\alpha-1}$. This EOS is suitable for cosmological data and centered around cosmological constant EOS ($A = 0$). Parameters A and α measures deviation from cosmological constant EOS. The sign of the parameter A determines whether DE is phantom or quintessence regime.

At high energy (early universe), $\frac{1}{r_c}$ is small and can be neglected, therefore from equation (3.2.6), we obtain the standard GR Friedmann equation as

$$H^2 = \frac{\kappa^2}{3}\rho. \tag{3.3.4}$$

In the late universe, the extra dimension effect cannot be neglected and we shall use a approximation form of Friedmann equation [139] as follows:

The DGP Friedmann equation (3.2.6) can be written as

$$H^2 = \frac{1}{4r_c}\left[\sqrt{1+\frac{4\rho r_c^2}{3M_P^2}}-1\right]^2. \tag{3.3.5}$$

Expanding in terms of $\rho r_c^2/M_P^2 \ll 1$, we get at the lowest order

$$H \approx \frac{\kappa^2 r_c}{3}\rho. \tag{3.3.6}$$

In this case, using equations (3.2.3) and (3.3.2), we have

$$\dot{H} = \frac{\kappa^4 r_c^2 A}{3}\rho^{\alpha+1}. \tag{3.3.7}$$

Consequently

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = \frac{\kappa^4 r_c^2}{3}\left(A\rho^{\alpha+1}+\frac{\rho^2}{3}\right). \tag{3.3.8}$$

In terms of κ^2, GR based Friedmann equation (2.2.6) can be written as

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{\kappa^2}{3}[\rho+3P]. \tag{3.3.9}$$

Comparing equations (3.3.8) and (3.3.9), the effective EOS in this case is obtained as

$$P = -\rho - \frac{2}{3}\kappa^2 r_c^2 \rho^2\left(\rho^{\alpha-1}+\frac{1}{3}\right)+\frac{2}{3}\rho. \tag{3.3.10}$$

Equation (3.3.10) yields that

$$\rho + P = \frac{2}{3}\rho - \frac{2}{3}\kappa^2 r_c^2\left(A\rho^{\alpha+1}+\frac{\rho^2}{3}\right) \tag{3.3.11}$$

and

$$\rho + 3P = -2\kappa^2 r_c^2 \rho^2\left(A\rho^{\alpha-1}+\frac{1}{3}\right). \tag{3.3.12}$$

From equation (3.3.12) we see that, unlike RS-II model, SEC is always violated in this case. Further from equation (3.3.11)

$$\rho + P = \frac{2}{3}\rho - \frac{2}{3}\kappa^2 r_c^2 \left(A\rho^{\alpha+1} + \frac{\rho^2}{3} \right) < 0$$

till

$$\rho_0 < \rho < \kappa^2 r_c^2 \left(A\rho^{\alpha+1} + \frac{\rho^2}{3} \right), \tag{3.3.13}$$

with ρ_0 being the present energy density. In particular, if we choose $\alpha = 2$, then for violation of WEC, the energy density must be restricted as

$$\rho > \frac{-1 + 3\sqrt{D_{r_c}}}{6A}, \tag{3.3.14}$$

where $\quad D_{r_c} = \frac{1}{9} + \frac{4A}{\kappa^2 r_c^2}$.

The parameter $\tilde{A}$ in equation (3.3.3) determines the type of behaviour of the DE density with the expansion of the universe. The speed of change of the DE density is given by

$$\frac{d\rho}{da} = 3\tilde{A}\rho_0 \left[1 + 3\tilde{A}(1-\alpha)\ln\frac{a}{a_0} \right]^{\frac{1}{1-\alpha}-1}. \tag{3.3.15}$$

This shows that for $\tilde{A} > 0$, DE density grows, for $\tilde{A} = 0$, the dark energy is constant while for $\tilde{A} < 0$, it decreases with the expansion of the universe. Thus for $\tilde{A} > 0$, phantom energy increases with expansion and there will be a time when

$$\rho > \kappa^2 r_c^2 \left(A\rho^{\alpha+1} + \frac{\rho^2}{3} \right). \tag{3.3.16}$$

For $\alpha = 2$, in this case energy density is restricted as

$$\rho < \frac{-1 + 3\sqrt{D_{r_c}}}{6A}. \tag{3.3.17}$$

This shows that WEC will be violated till ρ satisfies the inequality (3.3.13). It will not be violated when ρ satisfies (3.3.16). It means that when ρ satisfies the inequality (3.3.16) the phantom character of violating WEC will be suppressed by brane-gravity effects and phantom fluid will behave effectively as quintessence. This yields the *effective phantom divide* at

$$\rho = \rho_{\rm phd} = \kappa^2 r_c^2 \left(\tilde{A}\rho_0^{1-\alpha}\rho_{\rm phd}^{\alpha+1} + \frac{\rho_{\rm phd}^2}{3} \right). \tag{3.3.18}$$

Connecting MFE (modified Friedmann equation) (3.3.6) and conservation equation (3.2.3), we have

$$t = t_0 - \frac{1}{\kappa^2 \alpha \tilde{A} r_c \rho_0^{1-\alpha}} \Big(\rho^{-\alpha} - \rho_0^{-\alpha} \Big). \tag{3.3.19}$$

Using equations (3.3.18) and (3.3.19), we get the effective phantom divide at time

$$t_{\rm phd} = t_0 - \frac{1}{\kappa^2 \alpha \tilde{A} r_c \rho_0^{1-\alpha}} \Big(\rho_{\rm phd}^{-\alpha} - \rho_0^{-\alpha} \Big). \tag{3.3.20}$$

It is to be noted that in GR in order to have phantom crossing EOS needs to have double value [136], i.e. $p = -\rho \pm f(\rho)$.

Equation (3.3.19) can be written as

$$\rho = \rho_0 \Big[1 - \kappa^2 \alpha \tilde{A} r_c \rho_0 (t - t_0) \Big]^{-\frac{1}{\alpha}}, \tag{3.3.21}$$

which can be rewritten as

$$\rho = \frac{3H_0^2\Omega_0}{\kappa^2} \left[1 - \frac{3\tilde{A}\alpha\Omega_0 H_0}{2c}(t - t_0) \right]^{-\frac{1}{\alpha}}, \tag{3.3.22}$$

where

$$\Omega_0 = \frac{\kappa^2 \rho_0}{3H_0^2} \qquad \text{and} \qquad \Omega_{r_c} = \frac{1}{4r_c^2 H_0^2}, \tag{3.3.23}$$

are the usual dimensionless density parameters.

In terms of these parameters, Friedmann equation (3.2.7) can be written as

$$\Omega_0 + 2\sqrt{\Omega_{r_c}} = 1. \tag{3.3.24}$$

Solving equations (3.3.6) and (3.3.25), the scale factor in this case is obtained as

$$\begin{aligned} a(t) = a_0 \exp\Bigg[&\frac{1}{6\tilde{A}(\alpha-1)\sqrt{\Omega_{r_c}}}\left(1 - \frac{3\tilde{A}\alpha\Omega_0 H_0(t-t_0)}{2\sqrt{\Omega_{r_c}}}\right)^{\frac{-1}{\alpha}} \\ &\times \left(3\tilde{A}\alpha\Omega_0 H_0(t-t_0) - 2\sqrt{\Omega_{r_c}}\right) + 2\sqrt{\Omega_{r_c}}\Bigg]. \end{aligned} \tag{3.3.25}$$

This is the expression of the scale factor at late stages of evolution when ρ is very small. Note that for $\alpha > 1$, $a(t)$ grows exponentially with time while for $\alpha \leqslant 1$, $a(t)$ becomes constant asymptotically, i.e. we shall have a static model of the universe. Thus for $\alpha > 1$, the late stage acceleration as demanded by the present day observation is obtained while for $\alpha \leqslant 1$, the universe expands to a (finite) maximum volume and then becomes static asymptotically.

3.4 Conclusion

In this chapter, we analyse and compare behaviour of phantom fluid in DGP model (normal branch) of brane-gravity with RS-II model having negative brane tension λ. Brane corrections make drastic changes to the behaviour of phantom fluid which is characterised by violation of

WEC and accelerating the universe ending up in big-rip singularity in most of the models.

In both DGP and RS-II model, the phantom characteristic of violating WEC is suppressed by brane-gravity effects. Moreover both models are free from big-rip problem. In RS-II model brane tension plays a crucial role. When it is negative, it causes drastic changes in the behaviour of dark energy, but phantom fluid has usual behaviour when brane tension is positive.

In DGP model SEC is always violated, consequently there is acceleration only. Contrary to this in RS-II model, acceleration is transient showing the non violation of even SEC. In RS-II model the future universe begins at time t_0, the present age of the universe, and it stops expanding when energy density grows to 2λ by time t_m. Contrary to this in DGP model there is no maximum limit of expansion. The chapter ends with an expression of the scale factor at late stages of evolution.

Chapter 4

Acceleration And Deceleration In Curvature Induced Phantom Model

4.1 Prelude

In this chapter, cosmology of the late and future universe is obtained from $f(R)-$ gravity with non-linear curvature terms R^2 and R^3 (R being the Ricci scalar curvature). It is different from $f(R)-$dark energy models where non-linear curvature terms are taken as gravitational alternative for dark energy. In the present model, neither linear nor non-linear curvature terms are taken as dark energy. Rather, dark energy terms are induced by curvature terms and appear in the Friedmann equation derived from $f(R)-$gravitational equations. This approach has an advantage over $f(R)-$dark energy models in three ways (i) results are consistent with WMAP observations, (ii) dark matter is produced from the gravitational sector and (iii) the universe expands as $\sim t^{2/3}$ during dominance of the curvature-induced dark matter, which is consistent with the standard cosmology.

Here, curvature terms induce dark energy, dark matter and cosmological constant, which appear in the Friedmann equation (FE) for the late universe, derived from $f(R)-$ gravitational equations. It is interesting to see that curvature-induced dark energy, obtained here, mimics phantom with the equation of state (EOS) parameter $\omega = -5/4$. Moreover, FE contains phantom DE term as $\rho_{\rm DE}[1 - \rho_{\rm DE}/2\lambda]$. The correction term $-\rho_{\rm DE}^2/2\lambda$, with λ being the cosmic tension [140, 141], is analogous to such a term in RS-II model FE (2.2.2) as well as loop quantum gravity correction [142]. Like Refs. [140, 141], here also, this term is obtained from $f(R)-$ *gravity.* Here, cosmic tension λ is evaluated to be $5.77\rho^0_{\rm cr}$ with $\rho^0_{\rm cr}$ being the present critical density of the universe. Further, it is shown that universe, derived by curvature-induced dark matter, decelerates up to time $0.59t_0$ (t_0 being the present age of the universe). At this epoch and small red-shift $z = 0.36$, transition from deceleration to acceleration takes place. Interestingly, it is noted that as phantom energy density increases, effect of the term $-\rho_{\rm DE}^2/2\lambda$ gradually increases. As a result, it is found that universe will super-accelerate (expansion with high acceleration) during the period $0.59t_0 < t < 2.42t_0$, it will accelerate (expansion with low acceleration) during the period $2.42t_0 < t < 3.44t_0$ and, universe will decelerate even during the phantom phase when $3.44t_0 < t < 3.87t_0$. Phantom-dominance will end when $\rho = 2\lambda = 11.54\rho^0_{\rm cr}$ at time $t = 3.87t_0$ and dark matter will re-dominate causing decelerated cosmic expansion. It is natural to think that, even during decelerating phase with deceleration (derived by dark matter), phantom DE will grow with expansion. This chapter is based on [144].

Natural units ($k_B = \hbar = c = 1$) are used here.

4.2 Phantom phase of the late and the future universe from $f(R)$-gravity

Here, the action for $f(R)-$ gravity is taken as

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} + \alpha R^2 + \beta R^3\right], \tag{4.2.1}$$

where α is a dimensionless coupling constant and β is a constant having dimension $(\text{mass})^{-2}$ (as R has mass dimension 2).

Action (4.2.1) yields field equations

$$\frac{1}{16\pi G}\left(R_{ij} - \frac{1}{2}g_{ij}R\right) + \alpha\left(2R_{;ij} - 2g_{ij}\Box R - \frac{1}{2}g_{ij}R^2 + 2RR_{ij}\right)$$

$$+\beta\left(3R^2_{;ij} - 3g_{ij}\Box R^2 - \frac{1}{2}g_{ij}R^3 + 3R^2R_{ij}\right) = 0 \tag{4.2.2a}$$

using the condition $\delta S/\delta g^{ij} = 0$. The operator $\Box$ is defined as

$$\Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}g^{ij}\frac{\partial}{\partial x^j}\right). \tag{4.2.2b}$$

Taking trace of equation (4.2.2a), it is found that

$$-\frac{R}{16\pi G} - 6(\alpha + 3\beta R)\Box R - 18\beta g^{ij}R_{;i}R_{;j} + \beta R^3 = 0. \tag{4.2.3}$$

In equation (4.2.3), $(\alpha + 3\beta R)$ emerges as a coefficient of $\Box R$ due to presence of terms αR^2 and βR^3 in the action (4.2.1). If $\alpha = 0$, effect of R^2 vanishes and effect of R^3 is switched off for $\beta = 0$. So, an *effective* scalar curvature $\tilde{R}$ is defined as

$$\gamma\tilde{R} = \alpha + 3\beta R, \tag{4.2.4}$$

where γ is a constant having dimension $(\text{mass})^{-2}$ being used for the dimensional corrections.

Connecting equations (4.2.3) and (4.2.4), it is found that

$$-\Box\tilde{R} - \frac{1}{\tilde{R}}g^{ij}\tilde{R}_{;i}\tilde{R}_{;j} = \frac{1}{6\gamma}\left[\frac{1}{16\pi G} - \frac{\alpha^2}{3\beta}\right] - \frac{1}{6\gamma}\left[\frac{1}{16\pi G} - \frac{\alpha^2}{9\beta}\right]\frac{\alpha}{\gamma\tilde{R}} - \frac{\tilde{R}}{54\beta}\left[\gamma\tilde{R} - 3\alpha\right]. \quad (4.2.5)$$

In the space-time, given by equation (2.2.1), the above equation (4.2.5) reduces to

$$-\ddot{\tilde{R}} - 3\frac{\dot{a}}{a}\dot{\tilde{R}} - \frac{\dot{\tilde{R}}^2}{\tilde{R}} = \frac{1}{6\gamma}\left[\frac{1}{16\pi G} - \frac{\alpha^2}{3\beta}\right] - \frac{1}{6\gamma}\left[\frac{1}{16\pi G} - \frac{\alpha^2}{9\beta}\right]\frac{\alpha}{\gamma\tilde{R}} - \frac{\tilde{R}}{54\beta}\left[\gamma\tilde{R} - 3\alpha\right], \quad (4.2.6)$$

due to spatial homogeneous flat model of the universe.

For $a(t)$, being the power-law function of t, $\tilde{R} \sim a^{-n}$. For example, $\tilde{R} \sim a^{-3}$ for matter-dominated model. So, there is no harm in taking

$$\tilde{R} = \frac{A}{a^n}, \quad (4.2.7)$$

where $n > 0$ is a real number and A is a constant with mass dimension 2.

Using equation (4.2.7) in (4.2.6), it is found that

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + (3-2n)\left(\frac{\dot{a}}{a}\right)^2 = \frac{Ca^n}{nA}\left[1 - \frac{Da^n}{C}\right] - \frac{\gamma A}{54n\beta}\left[\frac{1}{a^n} - \frac{3\alpha}{\gamma A}\right]. \quad (4.2.8)$$

In the late universe, $a(t)$ is large, so this equation reduces to

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + (3-2n)\left(\frac{\dot{a}}{a}\right)^2 = \frac{Ca^n}{nA}\left[1 - \frac{Da^n}{C}\right] + \frac{\alpha}{18n\beta}, \quad (4.2.9a)$$

where

$$C = \frac{1}{6\gamma}\left[\frac{1}{16\pi G} - \frac{\alpha^2}{3\beta}\right] \quad (4.2.9b)$$

and

$$D = \frac{1}{6\gamma}\left[\frac{1}{16\pi G} - \frac{\alpha^2}{9\beta}\right]. \tag{4.2.9c}$$

Equation (4.2.9a) can be re-written as

$$\ddot{a} + (2-2n)\frac{\dot{a}^2}{a} = \frac{Ca^{n+1}}{nA}\left[1 - \frac{Da^n}{C}\right] + \frac{\alpha}{18n\beta}, \tag{4.2.10}$$

which integrates to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{E}{a^{6-4n}} + \frac{2C}{nA}\left[\frac{a^n}{(6-3n)} - \frac{Da^{2n}}{C(6-2n)}\right] + \frac{\alpha}{9n\beta(6-4n)}, \tag{4.2.11a}$$

where E is an integration constant having dimension $(\text{mass})^2$.

Equation (4.2.11a) is the modified Friedmann equation (MFE) giving cosmic dynamics. Terms, on r.h.s. of this equation, are imprints of curvature. The first term, proportional to $a^{-(6-4n)}$ emerges spontaneously. It is interesting to see that this term corresponds to matter density if $n = 3/4$, i.e. for this value of n it reduces to Ea^{-3} and yields the density of non-baryonic matter being spontaneously induced by curvature. So, it is identified as dark matter density.

Thus, for $n = 3/4$, equation (4.2.11a) looks like

$$\left(\frac{\dot{a}}{a}\right)^2 = \left[\frac{E}{a^3} + \frac{4\alpha}{81\beta}\right] + \frac{32Ca^{3/4}}{45A}\left[1 - \frac{5Da^{3/4}}{6C}\right]. \tag{4.2.11b}$$

On the r.h.s. of equation (4.2.11b), there are terms proportional to $a^{3/4}$ and $a^{3/2}$. If the density term $\rho_{\rm DE} = 4Ca^{3/4}/15A\pi G$ is put in the conservation equation given by equation (1.3.4), we have

$$\dot{\rho}_{\rm DE} + 3H(\rho_{\rm DE} + p_{\rm DE}) = 0. \tag{4.2.11c}$$

With $p_{\rm DE} = \omega\rho_{\rm DE}$, we obtain using (4.2.11c)

$$\omega = -\frac{5}{4} < -1. \tag{4.2.11d}$$

This result shows that $\rho_{\rm DE} = 4Ca^{3/4}/15A\pi G$ behaves as phantom dark energy density being induced by $f(R)-$ gravity.

Now, equation (4.2.11b) is re-written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left[\frac{8\pi G}{3}\rho_{\rm DM} + \frac{4\alpha}{81\beta}\right] + \frac{32Ca^{3/4}}{45A}\left[1 - \frac{5Da^{3/4}}{6C}\right], \quad (4.2.11e)$$

with

$$\rho_{\rm DM} = \frac{3E}{8\pi Ga^3}. \quad (4.2.11f)$$

Using current value of $\rho_{\rm DM}$ as $0.23\rho^0_{\rm cr}$, (4.2.11f) yields

$$\rho_{\rm DM} = 0.23\rho^0_{\rm cr}\left(\frac{a_0}{a}\right)^3, \quad (4.2.12a)$$

where $a_0 = a(t_0), 3E/8\pi G = 0.23\rho^0_{\rm cr}a_0^3$ and

$$\rho^0_{\rm cr} = \frac{3H_0^2}{8\pi G},$$

with $H_0 = 100{\rm km/Mpcsec} = 2.32 \times 10^{-42}h{\rm GeV}$ being the current Hubble's rate of expansion and $h = 0.68$. The present age of the universe is estimated to be $t_0 \simeq 13.7{\rm Gyr} = 6.6 \times 10^{41}{\rm GeV}^{-1}$ [145]. So,

$$H_0^{-1} = 0.96t_0. \quad (4.2.12b)$$

Further, a_0 is normalized as

$$a_0 = 1. \quad (4.2.12c)$$

Connecting equations (4.2.12a) and (4.2.12c), it is found that

$$\rho_{\rm DM} = \frac{0.23\rho^0_{\rm cr}}{a^3}. \quad (4.2.12d)$$

WMAP [145] gives decoupling of matter from radiation at red-shift

$$z_{\rm d} = \frac{1}{a_{\rm d}} - 1 = 1089. \quad (4.2.13)$$

So, it is supposed that dark matter begins to dominate cosmic dynamics when $a > a_{\rm d}$.

Now, equation (4.2.11e) is re-written as

$$\Big(\frac{\dot{a}}{a}\Big)^2 = \frac{8\pi G}{3}\Big[\Big\{\rho_{\rm DM} + \frac{\alpha}{54\beta\pi G}\Big\} + \rho_{\rm DE}\Big\{1 - \frac{\rho_{\rm DE}}{2\lambda}\Big\}\Big], \qquad (4.2.14a)$$

where

$$\rho_{\rm DE} = \rho^0_{\rm DE} a^{3/4}, \qquad (4.2.14b)$$

with $\rho^0_{\rm DE} = 0.73\rho^0_{\rm cr} = 4C/15A\pi G$, using $a_0 = 1$ and

$$\lambda = \frac{3C\rho^0_{\rm DE}}{5D}, \qquad (4.2.14c)$$

where C and D are given by equations (4.2.9b) and (4.2.9c) respectively.

The FE (4.2.14a) is obtained from $f(R)-$ gravity. Comparing it with GR based FE

$$\Big(\frac{\dot{a}}{a}\Big)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3},$$

it is found that the constant term in equation (4.2.14a) behaves as cosmological constant

$$\Lambda = \frac{4\alpha}{27\beta} \qquad (4.2.14d)$$

with vacuum energy density

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = \frac{2\alpha}{27\beta\pi G}. \qquad (4.2.14e)$$

It is interesting to see that equation (4.2.14a) contains a term $-(\rho_{\rm DE})^2/2\lambda$ analogous to the brane-gravity correction to the FE for negative brane-tension [86] and modifications in FE due to loop-quantum effects [57]. Here λ is called *cosmic tension* [140, 146]. Equation (4.2.14c) shows that *cosmic tension* λ depends on coupling constants α and β in the gravitational action (4.2.1). Moreover, a positive cosmological constant, too, emerges from curvature.

From (4.2.12a) and (4.2.14b), it is found that $\rho_{\rm DM} \sim a^{-3}$ and $\rho_{\rm DE} \sim a^{3/4}$. So $\rho_{\rm DM}$ decreases and $\rho_{\rm DE}$ increases with expansion of the universe. So, it is natural to think for values of these to come closer and to be equal at a certain time t_*. At this particular time, we have

$$0.23{a_*}^{-3} = 0.73a_*^{3/4},$$

using (4.2.12a), (4.2.12c) and (4.2.14b) as well as $a_* = a(t_*)$. This equation yields

$$a_* = \left(\frac{23}{73}\right)^{4/15}. \qquad (4.2.15)$$

It shows that, for $a < a_* = (23/73)^{4/15}$, $\rho_{\rm DM} > \rho_{\rm DE} > \rho_{\rm DE}^2$. So, equation (4.2.14a) is approximated as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\rho_{\rm DM} + \frac{\alpha}{54\beta\pi G}\right]. \qquad (4.2.16)$$

Connecting equations (4.2.12a) and (4.2.16), it is found that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{0.23H_0^2}{a^3}\left[1 + \frac{a^3}{B}\right], \qquad (4.2.17a)$$

with

$$B = \frac{\rho_{\rm DM}^0}{\rho_\Lambda} = \frac{4.66\beta H_0^2}{\alpha}, \qquad (4.2.17b)$$

obtained using (4.2.14e).

Equation (4.2.17a) integrates to

$$a = B^{1/2}\sinh^{2/3}\left[\frac{3H_0\sqrt{0.23}}{2B^{3/2}}(t - t_{\rm d}) + \sinh^{-1}\left(\frac{a_{\rm d}}{B^{1/3}}\right)^{3/2}\right]. \qquad (4.2.18a)$$

This result is not consistent with the scale factor, obtained in the standard model of cosmology during matter dominance. So, to have a viable cosmology, it needs to be approximated as

$$a = a_{\rm d}\left[1 + \frac{3H_0\sqrt{0.23}}{2a_{\rm d}^{3/2}}(t - t_{\rm d})\right]^{2/3}, \qquad (4.2.18b)$$

which is possible till

$$\frac{3H_0\sqrt{0.23}}{2B^{3/2}}(t - t_{\rm d}) + \sinh^{-1}\left(\frac{a_{\rm d}}{B}\right)^{3/2} \lesssim 1 \tag{4.2.18c}$$

as $\sinh 1 = 1.18 \simeq 1$. Here $a_{\rm d}$ is given by (4.2.13), which is the scale factor at time $t = t_{\rm d} = 386{\rm kyr} = 2.8 \times 10^{-5} t_0$. Connecting equations (4.2.13), (4.2.17b), (4.2.18a) and (4.2.18c), it is found that

$$\frac{\alpha}{\beta H_0^2} \gtrsim 10^{-6}. \tag{4.2.18d}$$

Connecting equation (4.2.16) and $\alpha/\beta H_0^2 \simeq 10^{-6}$ from (4.2.18d), it is evaluated that

$$\rho_\Lambda = 2.19 \times 10^{-8} \rho^0_{\rm cr} = 5.48 \times 10^{-55}{\rm GeV}^4. \tag{4.2.18e}$$

The approximated form (4.2.17a) is obtained when $a << a_*$, but, as discussed above, $\rho_{\rm DM} \simeq \rho_{\rm DE}$ when $a \lesssim a_*$. So, in the narrow strip around $a = a_*$ for $a < a_*$, equation (4.2.14a) is approximated as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[2\rho_{\rm DM} + \frac{\alpha}{54\beta\pi G}\right], \tag{4.2.19}$$

using $\rho_{\rm DM} \simeq \rho_{\rm DE} > \rho^2_{\rm DE}$ in (4.2.14a).

Connecting (4.2.12a) and (4.2.19), it is found that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{0.46 H_0^2}{a^3}\left[1 + \left(\frac{a}{a_*}\right)^3\right], \tag{4.2.20}$$

with $a_*^3 = \alpha/167.67\beta H_0^2$. This is because at $a = a_*$, mater-dominated phase ends and dark energy dominance begins.

Equation (4.2.20) integrates to

$$a^{3/2} = a_*^{3/2} \sinh\left[\frac{3H_0\sqrt{0.46}}{2a_*^{3/2}}(t - t_{\rm d}) + \sinh^{-1}\left(\frac{a_{\rm d}}{a_*}\right)^{3/2}\right],$$

which is approximated as

$$a = a_{\rm d}\left[1 + \frac{3H_0\sqrt{0.46}}{2a_{\rm d}^{3/2}}(t - t_{\rm d})\right]^{2/3}, \tag{4.2.21}$$

as in the above case.

Using equation (4.2.21), it is found that

$$a_*^{3/2} \simeq a_{\rm d}^{3/2} + \frac{3H_0\sqrt{0.46}}{2}(t_* - t_{\rm d}).$$

This result yields

$$t_* \simeq 0.59t_0, \tag{4.2.22}$$

using values of a_* [from (4.2.15)], H_0^{-1} [from (4.2.12b)] and $t_{\rm d}$.

It has been discussed above that for $a > a_*, \rho_{\rm DM} < \rho_{\rm DE}$. In this case, equation (4.2.14a) is approximated to

$$\left(\frac{\dot{a}}{a}\right)^2 \simeq \frac{8\pi G}{3}\rho_{\rm DE}\left\{1 - \frac{\rho_{\rm DE}}{2\lambda}\right\}. \tag{4.2.23}$$

Thus, in the late universe, a phantom model is obtained from curvature without using an unknown scalar ϕ as a source of exotic matter. But this model contains a correction term $-4\pi G\rho_{\rm DE}^2/3\lambda$ due to curvature-induced cosmic tension λ being evaluated below.

Connecting equations (4.2.14b) and (4.2.23), it is found that

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = 0.73H_0^2a^{3/2}\left[a^{-3/4} - \frac{0.73\rho_{\rm cr}^0}{2\lambda}\right]. \tag{4.2.24}$$

Equation (4.2.24) integrates to

$$a(t) = \left[\frac{0.73\rho_{\rm cr}^0}{2\lambda} + \left\{\sqrt{1.26 - \frac{0.73\rho_{\rm cr}^0}{2\lambda}} - \frac{3}{8}H_0\sqrt{0.73}(t - t_*)\right\}^2\right]^{-4/3} \tag{4.2.25a}$$

as $a_*^{-3/4} = 1.26$. Equation (4.2.25a) shows that phantom model obtained here is singularity-free.

Also from equation (4.2.25a), it is found that

$$\ddot{a} = 0.27 H_0^2 a^{5/2} \left[\frac{1.7\rho_{\rm cr}^0}{\lambda} - \frac{11}{3} a^{-3/4} \right]. \qquad (4.2.25b)$$

This shows $\ddot{a} > 0$, when

$$\frac{1.7\rho_{\rm cr}^0}{\lambda} > \frac{11}{3} a^{-3/4}. \qquad (4.2.25c)$$

Further, equation (4.2.25a) yields

$$1 = a_0 = \left[\frac{0.73\rho_{\rm cr}^0}{2\lambda} + \left\{ \sqrt{1.26 - \frac{0.73\rho_{\rm cr}^0}{2\lambda}} - \frac{3}{8} H_0 \sqrt{0.73}(t_0 - t_*) \right\}^2 \right]^{-4/3}. \qquad (4.2.26)$$

Using (4.2.22) for t_* in (4.2.26), λ is evaluated as

$$\lambda = 5.77\rho_{\rm cr}^0. \qquad (4.2.27)$$

Equation (4.2.25a) exhibits accelerating universe when $t > t_*$. Thus, a transition from deceleration to acceleration takes place at

$$t = t_* = 0.59 t_0 \qquad (4.2.28)$$

and red-shift

$$z_* = \frac{1}{a_*} - 1 = \left(\frac{73}{23} \right)^{4/15} - 1 = 0.36, \qquad (4.2.29)$$

which is within the range $0.33 \leq z_* \leq 0.59$ given by 16 Type supernova observations [41]. Equation (4.2.25a) shows that universe expands till $\rho_{\rm DE}$ becomes equal to 2λ as it grows with expansion. It happens till $a(t)$ increases to $a_{\rm pe}$, satisfying

$${a_{\rm pe}}^{3/4} = \frac{2\lambda}{0.73\rho_{\rm cr}^0}. \qquad (4.2.30)$$

Thus, expansion (4.2.25a) stops at time

$$t_{\rm pe} = t_* + \frac{8}{3H_0\sqrt{0.73}}\sqrt{1.26 - \frac{0.73\rho^0_{\rm cr}}{2\lambda}}. \qquad (4.2.31)$$

4.3 Cosmic energy conditions as well as acceleration and deceleration during phantom era

In what follows, like RS-II model of 2nd chapter it is found that correction term $-4\pi G\rho^2_{\rm DE}/3\lambda$ effects the behavior of the phantom model drastically. In this section since DE dominates, therefore we take $\rho = \rho_{\rm DE}$ and $p = p_{\rm DE}$

From equations (4.2.11c) and (4.2.23), it is seen that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\Big[3(\rho + p)\Big[1 - \frac{\rho}{\lambda}\Big] - 2\rho\Big\{1 - \frac{\rho}{2\lambda}\Big\}\Big]. \qquad (4.3.1)$$

The correction term, in this equation, is caused due to curvature-induced cosmic tension λ in equation (4.2.23). This type of equation was obtained earlier in 2nd chapter in the context of RS-II model of brane-gravity.

Comparing equation (4.3.1) and analogous equation (2.2.6) of GR based theory, the effective pressure density P is given by

$$\rho + 3P = 3(\rho + p)\Big[1 - \frac{\rho}{\lambda}\Big] - 2\rho\Big[1 - \frac{\rho}{2\lambda}\Big]. \qquad (4.3.2)$$

Using (4.2.11d), equation (4.3.2) yields the effective pressure density in the curvature induced phantom model as

$$P = -\frac{5}{4}\rho + \frac{7\rho^2}{12\lambda}. \qquad (4.3.3)$$

Equation (4.3.3) yields

$$\rho + P = -\frac{\rho}{4} + \frac{7\rho^2}{12\lambda}. \qquad (4.3.4)$$

This equation shows that the WEC is violated when $\rho < 3\lambda/7 = 2.47\rho^0_{\rm cr}$, with λ given by (4.2.27). Moreover, $\rho + P = 0$ for $\rho = 2.47\rho^0_{\rm cr}$ and $\rho + P > 0$ for $\rho > (3/7)\lambda = 2.47\rho^0_{\rm cr}$.

From equation (4.2.14b), it is found that energy density ρ for phantom fluid increases with increasing scale factor $a(t)$. It is interesting to note from equation (4.3.4) that curvature-induced phantom fluid, obtained here, behaves effectively as phantom violating WEC till $\rho < 2.47\rho^0_{\rm cr}$, but phantom characteristic to violate WEC is suppressed by cosmic tension when ρ increases and obeys the inequality $\rho > 2.47\rho^0_{\rm cr}$.

Further, using (4.2.11d) in (4.3.2), it is also found that

$$\rho + 3P = -\frac{11\rho}{4} + \frac{7\rho^2}{4\lambda}. \tag{4.3.5}$$

Connecting equations (4.2.27) and (4.3.5), it is seen that SEC is violated when $\rho < 11\lambda/7 = 9.07\rho^0_{\rm cr}$. Also, it is found that $\rho + 3P = 0$ for $\rho = 9.07\rho^0_{\rm cr}$ and $\rho + 3P > 0$ for $\rho > 9.07\rho^0_{\rm cr}$.

Thus, it is seen that (i) WEC is violated for $\rho < 2.47\rho^0_{\rm cr}$, (ii) for $2.47\rho^0_{\rm cr} \leq \rho < 9.07\rho^0_{\rm cr}$ WEC is not violated, but SEC is violated and (iii) for $\rho > 9.07\rho^0_{\rm cr}$ neither of the two conditions is violated. Also it is interesting to note that these corrections cause *effective phantom divide* at

$$\rho = 2.47\rho^0_{\rm cr}. \tag{4.3.6}$$

Moreover, these results suggest that a transition from violation of SEC to non-violation of SEC will take place at

$$\rho = 9.07\rho^0_{\rm cr}. \tag{4.3.7}$$

Also, universe will super-accelerate till $\rho_* < \rho < 2.47\rho^0_{\rm cr}$, accelerate when $2.47\rho^0_{\rm cr} < \rho < 9.07\rho^0_{\rm cr}$ and decelerate when $9.07\rho^0_{\rm cr} < \rho < 11.54\rho^0_{\rm cr}$

as expansion of phantom phase of the universe will stop at $\rho = 11.54\rho^0_{\rm cr}$.

These results are also supported by equation (4.2.25b), as (4.2.25b) and (4.2.14b) yield

$$\ddot{a} = 0.27H_0^2 a^{5/2}\left[0.29 - \frac{2.68\rho^0_{\rm cr}}{\rho}\right]. \tag{4.3.8}$$

Connecting equations (4.2.14b), (4.2.25a) , (4.2.27) and (4.2.28a), it is found that

$$\rho = 0.73\rho^0_{\rm cr}[0.06 + \{1.094 - 0.32H_0(t - 0.59t_0)\}^2]^{-1}. \tag{4.3.9}$$

Equations (4.3.6) and (4.3.9) yield that *effective phantom divide* is obtained at time

$$t \simeq 2.42t_0. \tag{4.3.10}$$

Equations (4.3.8) and (4.3.10) yield that transition time for violation of SEC to non-violation of SEC will take place at

$$t \simeq 3.44t_0. \tag{4.3.11}$$

These results imply super-acceleration during the time interval $0.59t_0 < t < 2.42t_0$, acceleration during the time interval $2.42t_0 < t < 3.44t_0$ and deceleration during the time interval $3.44t_0 < t < 3.87t_0$. Expansion, driven by phantom, will stop at time $t = 3.87t_0$ as $\rho_{\rm DE}$ will acquire the value 2λ by this time.

When $t > 3.87t_0$, deceleration, driven by matter, will resume and Freidmann equation reduces to (4.2.17).

4.4 Re-appearance of matter-dominance and cosmic collapse

As mentioned above, DE terms are switched off in equation (4.2.14a) at $\rho_{\rm DE} = 2\lambda = 11.54\rho^0_{\rm cr}$ [obtained from (4.2.27)], $a = a_{\rm pe}$, given by equation (4.2.30) when $t = t_{\rm pe} = 3.87t_0$, given by equation (4.2.31). So, for $t > t_{\rm pe} = 3.87t_0$, equation (4.2.14a) will look like

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho^{\rm pe}_{\rm DM}\left(\frac{a_{\rm pe}}{a}\right)^3\left\{1 + \tilde{\Lambda}^{-1}\left(\frac{a}{a_{\rm pe}}\right)^3\right\}, \qquad (4.4.1a)$$

where

$$\rho^{\rm pe}_{\rm DM} = 3.68 \times 10^{-6}\rho^0_{\rm cr}, \qquad (4.4.1b)$$

$$\tilde{\Lambda} = \frac{\rho^{\rm pe}_{\rm DM}}{\rho_\Lambda} = \frac{243\pi G\beta\rho^{\rm pe}_{\rm DM}}{2\alpha} = 168, \qquad (4.4.1c)$$

which is evaluated using equations (4.2.18d), (4.2.30), (4.4.1b), and

$$\rho^{\rm pe}_{\rm DE}\left(\frac{a}{a_{\rm pe}}\right)^{3/4}\left\{1 - \left(\frac{a}{a_{\rm pe}}\right)^{3/4}\right\} = 0 \qquad (4.4.1d)$$

as

$$\rho^{\rm pe}_{\rm DE} = 2\lambda = 11.54\rho^0_{\rm cr}. \qquad (4.4.1e)$$

Equation (4.4.1a) integrates to

$$a(t) = \tilde{\Lambda}^{1/3}a_{\rm pe}\sinh^{2/3}\left[\sinh^{-1}\tilde{\Lambda}^{-1/2} + 2.22 \times 10^{-4}H_0(t - t_{\rm pe})\right], \quad (4.4.2a)$$

yielding

$$\frac{\ddot{a}}{a} = -\frac{2}{9}[2.22 \times 10^{-4}H_0]^2[\text{cosech}^2\theta(t) - 2], \qquad (4.4.2b)$$

with $\theta(t) = \sinh^{-1}\tilde{\Lambda}^{-1/2} + 2.22 \times 10^{-4}H_0(t - t_{\rm pe})$.

Equation (4.4.2b) shows decelerated expansion caused by matter-dominance till

$$\sinh\theta(t) < 1/\sqrt{2} = 0.707. \tag{4.4.2c}$$

The result (4.4.2a) is obtained when expansion is driven by the term

$$\rho^{\rm pe}_{\rm DM}\left(\frac{a_{\rm pe}}{a}\right)^3\left\{1+\tilde{\Lambda}^{-1}\left(\frac{a}{a_{\rm pe}}\right)^3\right\}$$

in the FE.

Moreover, though at $a = a_{\rm pe}$,

$$\rho^{\rm pe}_{\rm DE}\left(\frac{a}{a_{\rm pe}}\right)^{3/4}\left\{1-\left(\frac{a}{a_{\rm pe}}\right)^{3/4}\right\}$$

vanishes, it will be negative for $a > a_{\rm pe}$. So for $a > a_{\rm pe}$, FE (4.2.14a) is obtained as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\Big[\rho^{\rm pe}_{\rm DM}\Big(\frac{a_{\rm pe}}{a}\Big)^3\Big\{1+\tilde{\Lambda}^{-1}\Big(\frac{a}{a_{\rm pe}}\Big)^3\Big\} - \rho^{\rm pe}_{\rm DE}\left(\frac{a}{a_{\rm pe}}\right)^{3/4}\Big\{\Big(\frac{a}{a_{\rm pe}}\Big)^{3/4}-1\Big\}\Big]. \tag{4.4.3}$$

Obviously, the negative term in equation (4.4.3) will try to stop expansion and, on sufficient growth of $a(t)$ up to $a_{\rm m}$, expansion will reach its maximum such that $\dot{a}_{a=a_{\rm m}} = 0$ and $a_{\rm m}$ satisfies the equation

$$\rho^{\rm pe}_{\rm DM}\left\{\left(\frac{a_{\rm pe}}{a_{\rm m}}\right)^3+\tilde{\Lambda}^{-1}\right\} = 2\lambda\left(\frac{a_{\rm m}}{a_{\rm pe}}\right)^{3/4}\left\{\left(\frac{a_{\rm m}}{a_{\rm pe}}\right)^{3/4}-1\right\} \tag{4.4.4a}$$

being approximated as

$$\rho^{\rm pe}_{\rm DM}\Lambda^{-1} \simeq 2\lambda\left(\frac{a_{\rm m}}{a_{\rm pe}}\right)^{3/4}\left\{\left(\frac{a_{\rm m}}{a_{\rm pe}}\right)^{3/4}-1\right\}. \tag{4.4.4b}$$

Equation (4.4.4b) yields the solution

$$\left(\frac{a_{\rm m}}{a_{\rm pe}}\right) = \left\{\frac{1}{2}\left[1 + \sqrt{1 + 2\rho^{\rm pe}_{\rm DM}/\lambda}\right]\right\}^{4/3} = 1 + 4.25 \times 10^{-7}. \quad (4.4.4c)$$

The negative sign $(-)$ is ignored here as it yields $a_{\rm m} < a_{\rm pe}$, which is not possible. Using equation (4.4.4c), it is found that

$$\rho^{\rm m}_{\rm DM} = \rho^{\rm pe}_{\rm DM}\Big[1 - 1.28 \times 10^{-6}\Big]. \quad (4.4.4d)$$

Using (4.4.4c) in equation (4.4.2a), it is seen that the time $t = t_{\rm m}$ corresponding to $a = a_{\rm m}$ is given by

$$\begin{aligned}\Big[\sinh^{-1}\tilde{\Lambda}^{-1/2} &+ 2.22 \times 10^{-4} H_0(t_{\rm m} - t_{\rm pe})\Big] \\ &= \sinh^{-1}\left[\frac{1}{4\sqrt{\tilde{\Lambda}}}\left[1 + \sqrt{1 + 2\rho^{\rm pe}_{\rm DM}/\lambda}\right]^2\right] \\ &= \sinh^{-1}(0.18123) = 0.18123.\end{aligned}$$

(4.4.4e)

Equation (4.4.4e) confirms deceleration during time period $t_{\rm pe} < t < t_{\rm m}$ as it satisfies the condition (4.4.2c).

So, for $t_{\rm pe} < t < t_{\rm m}$, equation (4.4.2a) is obtained as

$$a(t) = a_{\rm pe}\Big[1 + 2.22 \times 10^{-4} H_0 \tilde{\Lambda}^{1/3} a_{\rm pe}^{-3/2}(t - t_{\rm pe})\Big]^{2/3}. \quad (4.4.5)$$

Connecting equations (4.2.12b), (4.4.1c) and (4.4.4e), $t_{\rm m}$ is evaluated as

$$t_{\rm m} - t_{\rm pe} = 5.32 \times 10^{-4} t_0 = 3.51 \times 10^{38}{\rm GeV}^{-1} = 694.4{\rm kyr}. \quad (4.4.6)$$

Further, it is interesting to note that the curve $a = a(t)$ will be continuous at $t = t_{\rm m}$, but the direction of tangent to this curve (pointed at $a = a_{\rm m}$) will change because it will attain its maximum at $t = t_{\rm m}$, yielding $\dot{a} < 0$ for $t > t_{\rm m}$, which used to be positive for $t < t_{\rm m}$. It means that universe will retrace back at $t = t_{\rm m}$ and will begin to contract. During the contraction phase, term proportional to a^{-3} will dominate over terms proportional to $a^{3/4}$ and (4.4.3) will yield

$$\frac{\dot{a}}{a} \simeq -\left[\frac{8\pi G}{3}\rho^{\rm m}_{\rm DM}\Big(\frac{a_{\rm m}}{a}\Big)^3\left\{1 + \tilde{\Lambda}^{-1}\Big(\frac{a}{a_{\rm m}}\Big)^3\right\}\right]^{1/2}. \tag{4.4.7}$$

On integrating equation (4.4.7), it is found that

$$a(t) = \tilde{\Lambda}^{1/3} a_{\rm m}\{\sinh[2.22\times 10^{-4}H_0](t_{\rm col} - t)\}^{2/3}, \tag{4.4.8a}$$

where

$$\begin{aligned} t_{\rm col} &= t_{\rm m} + [2.22\times 10^{-4}H_0]^{-1}\sinh^{-1}\tilde{\Lambda}^{-1/2} \\ &= t_{\rm m} + 3.33\times 10^2 t_0. \end{aligned} \tag{4.4.8b}$$

Equation (4.4.8a) shows that $a(t) = 0$ at $t = t_{\rm col}$ and the energy density of the universe

$$\begin{aligned} \rho &= \left[\rho^{\rm m}_{\rm DM}\Big(\frac{a_{\rm m}}{a}\Big)^3 + \rho^{\rm m}_{\rm DE}\Big(\frac{a}{a_{\rm pe}}\Big)^{3/4}\right] \\ &\simeq \rho^{\rm m}_{\rm DM}\Big(\frac{a_{\rm m}}{a}\Big)^3 \end{aligned} \tag{4.4.9}$$

will diverge. It means that universe will collapse at $t = t_{\rm col}$.

This result, suggested by the classical mechanics, is unphysical due to divergence of cosmic energy density. So, to have a viable physics around the epoch $t = t_{\rm col}$, the diverging component of the density of the universe, which is proportional to a^{-3}, needs to be finite. As this unphysical situation is predicted by classical mechanics, we have no other alternative other than to resort to quantum gravity. In [144], it is shown that cosmic collapse obtained from the classical mechanics can be avoided on making quantum gravity corrections relevant near collapse time due to extremely high energy density and large curvature analogous to the state of very early universe. This situation is analogous to very early universe, where quantum gravity effects are dominant. Earlier also, quantum gravity corrections were made in the equations of future universe under such circumstances to avoid finite time singularities [97, 98, 147].

4.5 Conclusion

Here, cosmology of the late and future universe is obtained from $f(R)-$ gravity, obtained by adding higher-order curvature terms R^2 and R^3 to the Einstein-Hilbert term linear in scalar curvature R. Here, problems of $f(R)-$ dark energy models, pointed out in [82], do not appear in $f(R)-$ gravity cosmology [84, 140, 141, 143, 146] as well as in the present model. Here, curvature scalar contributes to both geometrical and physical components of the theory. Thus, it plays dual role as a geometrical as well as physical fields, which was obtained earlier in [149].

Here, it is found that, in the late universe for the red-shift $z < 1089$, the dark matter term emerges spontaneously and phantom dark energy emerges as imprints of linear as well as non-linear terms of curvature. It is found that, during $0.36 < z < 1089$, dark matter dominated and universe expanded with deceleration as $t^{2/3}$. A transition from deceleration to acceleration took place at $z = 0.36$ and at time $t = 0.59t_0$ (t_0 being the present age of the universe). This transition is caused by dominance of curvature-induced phantom dark energy over curvature-induced dark matter. Dark energy gives anti-gravity effect and phantom dark energy exhibits this effect more strongly due to violation of WEC. So, phantom energy imposes a high jerk, causing super-acceleration.

The $f(R)-$ gravity inspired Friedmann equation, obtained here, contains two terms (i) $8\pi G\rho_{\rm DM}/3$ (with $\rho_{\rm DM}$ being the dark matter density) and (ii) $(8\pi G\rho_{\rm DE}/3)[1 - \rho_{\rm DE}/2\lambda]$ (with $\rho_{\rm DE}$ being the phantom dark energy density). Here $\lambda = 5.777\rho^0_{\rm cr}$ is called the cosmic tension, which is also curvature-induced. It is interesting to see that Friedmann equation, obtained here, contains a correction term $-4\pi G\rho^2_{\rm DE}/3\lambda$ analogous to such a term in Friedmann equations from RS-II model of brane-gravity [86] and loop quantum cosmology [57]. This correction is not effective in the present universe as $\rho^0_{\rm DE} << 2\lambda$, as well as till $\rho_{\rm DE} << 2\lambda$. But, as $\rho_{\rm DE}$ will increase in future with the growing scale factor $a(t)$, the effect of this term will increase. It is found that, on sufficient growth of $\rho_{\rm DE}$, the *effective EOS* does not violate WEC (which characterizes phantom), but violates SEC. On further increase in $\rho_{\rm DE}$, even SEC is not violated. Thus, at the beginning of the phantom phase, universe will super-accelerate during the period $0.59t_0 < t < 2.42t_0$; it will accelerate during the period $2.42t_0 < t < 3.44t_0$; and, during the period $3.44t_0 < t < 3.87t_0$, universe will decelerate even during the phantom phase. Phantom-dominance will end when $\rho = 2\lambda = 11.54\rho^0_{\rm cr}$ at time $t = 3.87t_0$. As a consequence, re-dominance of dark matter will begin giving decelerated expansion. But, as universe will expand, growth of $\rho_{\rm DE} \sim a^{3/4}$ will also continue giving $\rho_{\rm DE} > 2\lambda$. It causes the term $(8\pi G\rho_{\rm DE}/3)[1 - \rho_{\rm DE}/2\lambda]$ to switch over from positive to negative. The growth of this negative term will try to slow down expansion more rapidly. As a result, universe will reach a stage where the expansion will stop and the scale factor will acquire its maximum value in finite time $t_{\rm m} = 3.87t_0 + 694.4{\rm kyr}$. When $t > t_{\rm m}$, universe will bounce and contract. Results, obtained here, show that contraction of the universe will continue for sufficiently long period, $333t_0$, and universe will collapse at time $t_{\rm col} = 336.87t_0 + 694.4{\rm kyr}$, where energy density of the universe will diverge and scale factor will vanish. These results are obtained using prescriptions of the classical mechanics. In [144], it is probed further whether quantum gravity corrections can save the universe from the menace of collapse.

Chapter 5

Validity of Generalised Second Law in Holographic DGP Brane

5.1 Prelude

In this chapter, we study the validity of the generalised second law of gravitational thermodynamics (GSLT) of the universe bounded by the event horizon on the DGP braneworld. As mentioned in the first chapter, there is close connection between black hole physics and thermodynamics, therefore by extending generalised second law of black hole space times to cosmological settings, several authors have considered the interplay between ordinary entropy and the entropy associated to cosmological event horizons [105]. GSLT states that the time variation of the sum of the entropy of the horizon (S_h) and the entropy of the matter inside it (S_I) should be positive definite, i.e. $\dot{S}_{\rm tot} = \dot{S}_h + \dot{S}_I \geq 0$. In other words, according to the GSLT the entropy of the horizon plus its surroundings (in our case, the entropy in the volume enclosed by the horizon) cannot decrease. Hence, we must evaluate the total entropy to see its evolution during the present dark energy era. In recent years, a lot of attention has been paid to the GSLT in the accelerating universe driven by DE, besides the first law of thermodynamics. The

GSLT is as important as first law, governing the development of the nature [106, 107, 108, 109, 110].

An approach to the problem of DE arises from holographic principle which states that the number of degrees of freedom for a system within a finite region should be finite and is bounded by the area of its boundary. As in Ref. [150], one obtains holographic energy density as

$$\rho_\Lambda = 3c^2 M_P^2 L^{-2}$$

where L is an IR cut-off in units $M_P^2 = 1$. Li shows that [153], if we choose L as the radius of the event horizon we can get the correct equation of state and get the desired accelerating universe. The details of the physical basis of the holographic principle is available in the paper entitled "The holographic principle and the early universe" by Canfora and Vilasi (2005) [154].

As mentioned in first chapter, to examine the thermodynamical behaviour of a cosmological scenario, we consider universe as a thermodynamical system which has an associated entropy that may be interpreted analogously to the entropy of the black hole. However, it is not clear and is a matter of debate which horizon (apparent or event) must be used as a boundary of the universe to have consistent description.

In Ref. [151], it is shown that apparent horizon entropy extracted through connection between gravity and first law of thermodynamics satisfies the generalised second law of thermodynamics (GSLT) in DGP brane. But so far attempts to address these problems are made using apparent horizon only in braneworld scenario. So it is imperative to study GSLT using event horizon as the boundary of the universe in BG set up.

Here, the radius of the event horizon is calculated by establishing a correspondence between holographic dark energy (HDE) and the effective energy density in the DGP braneworld. We assume HDE density $\rho_\Lambda = 3c^2 M_P^2 L^{-2}$ still holds in the DGP model. It is shown that in the absence of cold dark matter (CDM), GSLT is always respected. In the presence of CDM, we investigate validity of GSLT in three different models of DGP braneworld. This chapter is based on [152].

5.2 Holographic dark energy in the DGP model

As in chapter 3, here we consider DGP(-) model in spatially flat FRW universe. The matter in the universe is taken in the form of non-interacting two fluid system:- one component is the general dark energy and the other component is in the form of dust (CDM), i.e. $\rho \equiv \rho_t = \rho_m + \rho_D$, where ρ_t is the total cosmic fluid energy density, ρ_m is the energy density of CDM and ρ_D is the energy density of DE.

Then the Friedmann equation (3.2.7) of chapter 3 can be rewritten as

$$H^2 = \frac{1}{3}(\rho_m + \rho_{\text{eff}}), \tag{5.2.1}$$

where ρ_{eff} is the effective energy density given by

$$\rho_{\text{eff}} = \rho_D - \frac{3H}{r_c}. \tag{5.2.2}$$

Here for simplicity we are using $\kappa^2 = 8\pi G = 1$.

As the two component matter system is non-interacting, so they satisfy energy conservation equation (1.3.4) separately, i.e.

$$\dot{\rho}_m + 3H\rho_m = 0, \tag{5.2.3}$$

$$\dot{\rho}_D + 3H(\rho_D + p_D) = 0. \tag{5.2.4}$$

where p_D is the thermodynamic pressure of DE. Also we have conservation equation for effective energy density [155]

$$\dot{\rho}_{\rm eff} + 3H(1 + \omega_{\rm eff})\rho_{\rm eff} = 0, \tag{5.2.5}$$

where $\omega_{\rm eff} = p_{\rm eff}/\rho_{\rm eff}$.

Equations (5.2.1) and (5.2.5) describe the equivalent GR model.

Our choice for HDE density is

$$\rho_\Lambda = \frac{3c^2}{R_E^2}, \tag{5.2.6}$$

where c is a constant and R_E is radius of the future event horizon, given by

$$R_E = a\int_t^\infty \frac{dt}{a} = a\int_a^\infty \frac{da}{Ha^2} = \frac{c}{\sqrt{\Omega_\Lambda}H}, \tag{5.2.7}$$

where $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} = \dfrac{c^2}{R_E^2 H^2}$ is the dimensionless DE.

As in Ref. [155], if we now establish the correspondence between the HDE density and the effective energy density in the DGP braneworld, we then have

$$\frac{3c^2}{R_E^2} = \rho_D - \frac{3H}{r_c}. \tag{5.2.8}$$

Differentiating this equation w.r.t cosmic time, we have

$$\frac{-6c^2}{R_E^3}\dot{R}_E = \dot{\rho}_D - \frac{3\dot{H}}{r_c}. \tag{5.2.9}$$

Also from equation (5.2.2), we have

$$\dot{\rho}_{\text{eff}} = \dot{\rho}_D - \frac{3\dot{H}}{r_c}. \tag{5.2.10}$$

Using equations (5.2.5), (5.2.9) and (5.2.10), we have

$$\dot{R}_E = \frac{R_E^3}{2c^2}(1 + \omega_{\text{eff}})H\rho_{\text{eff}}. \tag{5.2.11}$$

5.3 Gravitational thermodynamics

In this section, we examine the validity of GSLT on 3-DGP brane. Let us consider a region of FRW universe enveloped by the event horizon and assume that the region bounded by event horizon act as a thermal system with boundary defined by the event horizon and is filled with a perfect fluid of energy density ρ_t and pressure p_t.

We take as in Ref. [156]

$$p_t = \omega_{\text{eff}}\rho_{\text{eff}} \qquad \text{and} \qquad \rho_t = \rho_m + \rho_{\text{eff}}. \tag{5.3.1}$$

In a recent paper [157], validity of GSLT has been studied in Einstein's gravity and Gauss Bonnet gravity by assuming first law of thermodynamics and conditions for validity of GSLT have been obtained. Gravity on the brane does not obey Einstein theory, therefore usual area formula for the black hole entropy may not hold on the brane. So, we extract the entropy of the event horizon by assuming the first law of thermodynamics on event horizon.

The amount of energy crossing the event horizon in time dt has the expression

$$-dE = 4\pi R_E^3 H(\rho_t + p_t)dt. \tag{5.3.2}$$

So assuming the first law of thermodynamics, we have

$$\dot{S}_E = \frac{4\pi R_E^3 H}{T_E}\Big[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}\Big], \tag{5.3.3}$$

where S_E and T_E are the entropy and temperature of the event horizon respectively. Using Gibb's equation [106, 107],

$$T_E dS_I = dE_I + p_t dV,$$

we obtain the variation of the entropy of the fluid inside the event horizon as

$$\dot{S}_I = \frac{1}{T_E}\Big[V\dot{\rho}_t + (\rho_t + p_t)\dot{V}\Big], \tag{5.3.4}$$

where S_I and E_I are the entropy and energy of the matter distribution inside the event horizon. Here, we assume as in Refs. [158, 159, 160], the temperature of the source inside the event horizon is in equilibrium with the temperature associated with the horizon.

Connecting equations (5.2.3) and (5.2.5), we get

$$\dot{\rho}_t = -3H[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}]. \tag{5.3.5}$$

So starting with $E_I = \frac{4}{3}\pi R_E^3 \rho_t$, $V = \frac{4}{3}\pi R_E^3$ and using equations (5.3.4) and (5.3.5), and after some simplification one gets

$$\dot{S}_I = \frac{4\pi R_E^2}{T_E}\Big[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}\Big]\Big[\dot{R}_E - HR_E\Big]. \tag{5.3.6}$$

Adding equations (5.3.3) and (5.3.6), one gets the resulting change of entropy as

$$\dot{S}_{\rm tot} = \dot{S}_E + \dot{S}_I = \frac{4\pi R_E^2}{T_E}\Big[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}\Big]\dot{R}_E. \tag{5.3.7}$$

Substituting the value of $\dot{R}_E$ from equation (5.2.11) in (5.3.7), we get

$$\dot{S}_{\rm tot} = \frac{2\pi R_E^5 H}{T_E}\Big[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}\Big](1+\omega_{\rm eff})\rho_{\rm eff}. \tag{5.3.8}$$

If $\rho_m = 0$, i.e. in the absence of CDM, we have

$$\dot{S}_{\rm tot} = \frac{2\pi R_E^5 H}{T_E}(1+\omega_{\rm eff})^2\rho_{\rm eff}^2 \geq 0. \tag{5.3.9}$$

and GSLT is always respected.

When $\rho_m \neq 0$, we investigate the validity of GSLT in three concrete models of DGP braneworld.

5.3.1 Lambda-DGP (LDGP) model

The simplest DGP(-) model has cosmological constant Λ as dark energy, i.e. $\rho_D = \Lambda$ and following Lue and Starkman [100] it is called LDGP model. The observational constraints of this model have been considered by Lazkoz *et al.* [102]. This model can lead to phantom like acceleration of the late universe, but without the need of any phantom matter. In this model, we have following equation

$$H^2 + \frac{H}{r_c} = \frac{1}{3}(\rho_m + \Lambda), \tag{5.3.10}$$

and the conservation equation given by equation (5.2.6).

Using equations (5.3.10) and (5.2.6), we have

$$\dot{H} = \frac{-\rho_m}{2}\left[1 - \frac{1}{\sqrt{1+4r_c^2(\rho_m+\Lambda)/3}}\right]. \tag{5.3.11}$$

Using equations (5.2.2) and (5.2.5), we have

$$1+\omega_{\rm eff}=\frac{\dot{H}}{r_c H\rho_{\rm eff}}. \tag{5.3.12}$$

Since $\dot{H}<0$ by equation (5.3.11), we have effective phantom behaviour, $\omega_{\rm eff}<-1$, as long as $\rho_{\rm eff}>0$, i.e. for $H<r_c\Lambda/3$.

Thus in this case GSLT will be valid if $\rho_m+(1+\omega_{\rm eff})\rho_{\rm eff}<0$, i.e.

$$\frac{\rho_m}{\rho_{\rm eff}}<|1+\omega_{\rm eff}|. \tag{5.3.13}$$

5.3.2 Quintessence-DGP (QDGP) model

LDGP model was generalised by Chimento *et al.* [156], where $\rho_D=\rho_q$ to QDGP model. Here ρ_q is the quintessence dark energy with $\omega_q=$ *constant*(≥ -1). The interesting fact is that in GR with dark energy component, phantom crossing cannot occur with a single field [156] (Unless non minimal coupling between dark matter and dark energy is allowed [161]). In QDGP model, crossing is possible without resorting to multiple fields.

In this model, the effective energy density is given by

$$\rho_{\rm eff}=\rho_q-\frac{3H}{r_c}. \tag{5.3.14}$$

Since ω_q is constant, the conservation equations gives

$$\rho_m=\rho_{m_0}(1+z)^3, \qquad \rho_q=\rho_{q_0}(1+z)^{3(1+\omega_q)}, \tag{5.3.15}$$

where $z=\frac{1}{a}-1$ is the red shift, ρ_{m_0} and ρ_{q_0} are respectively the present energy densities of matter and quintessence DE.

As in Ref. [156], after little calculation, we have

$$E(z)\equiv\frac{H}{H_0}=\sqrt{\Omega_{r_c}+\Omega_m(1+z)^3+\Omega_q(1+z)^{3(1+\omega_q)}}-\sqrt{\Omega_{r_c}}, \tag{5.3.16}$$

$$\frac{\dot{E}}{H_0} = -\frac{3E(1+z)^3[\Omega_m + (1+\omega_q)\Omega_q(1+z)^{3\omega_q}]}{2(E+\sqrt{\Omega_{r_c}})}, \tag{5.3.17}$$

where

$$\Omega_m = \frac{\rho_{m_0}}{3H_0^2}, \qquad \Omega_q = \frac{\rho_{q_0}}{3H_0^2}, \qquad \Omega_{r_c} = \frac{1}{4r_c^2H_0^2}, \tag{5.3.18}$$

are the usual dimensionless density parameters. The equation (5.3.17) shows that $\dot{E}$ and hence $\dot{H} < 0, \forall\ z$, as in LDGP case.

After little computation, one can have

$$1+\omega_{\text{eff}}(z) = \frac{(1+\omega_q)\Omega_q E(z)(1+z)^{3(1+\omega_q)} - \sqrt{\Omega_{r_c}}\Omega_m(1+z)^3}{[E(z)+\sqrt{\Omega_{r_c}}][\omega_q(1+z)^{3(1+\omega_q)} - 2\sqrt{\Omega_{r_c}}E(z)]}. \tag{5.3.19}$$

The total equation of state parameter is defined as

$$\omega_t = \frac{p_t}{\rho_t} = \frac{\omega_{\text{eff}}\rho_{\text{eff}}}{\rho_m + \rho_{\text{eff}}}. \tag{5.3.20}$$

From equations (5.3.17) and (5.3.19), it follows that

$$1+\omega_t = \frac{\Omega_m(1+z)^3 + (1+\omega_q)\Omega_q(1+z)^{3(1+\omega_q)}}{E(z)[\Omega_{r_c} + E(z)]}. \tag{5.3.21}$$

Connecting equations (5.3.19), (5.3.20) and (5.3.21), we have $\omega_{\text{eff}} \geq -1$ provided $\rho_{\text{eff}} > 0$.

Thus in this case GSLT is valid provided $\rho_{\text{eff}} > 0$.

5.3.3 Chaplygin-DGP (CDGP) model

In this model the dark energy component on the brane is taken as generalised chaplygin gas (GCG), i.e. $\rho_D = \rho_{ch} = -\frac{A}{\rho_{ch}^{\alpha}}$, where ρ_{ch} is the energy density of GCG. So following [101], it is called Chaplygin DGP (CDGP) model. Unlike QDGP model, in this model the effective phantom behaviour remains valid at all red shifts for some choices of the free parameters of the model. It is based on the fact that, GCG has characteristics of interpolating between a CDM at early times and cosmological constant at late times.

In this model, the effective energy density is given by

$$\rho_{\text{eff}} = \rho_{ch} - \frac{3H}{r_c}. \tag{5.3.22}$$

Using the conservation equation (5.2.4) for ρ_{ch}, we have

$$\rho_{ch} = \rho_{ch_0}[A_s + (1 - A_s)(1 + z)^{3(1+\alpha)}]^{\frac{1}{1+\alpha}}, \tag{5.3.23}$$

where $A_s = A/\rho_{ch0}^{1+\alpha}$ and ρ_{ch0} is the present energy density of Chaplygin gas. Like previous case, we have

$$\frac{H}{H_0} = \sqrt{\Omega_{r_c} + \Omega_m(1 + z)^3 + \Omega_{ch}[A_s + (1 - A_s)(1 + z)^{3(1+\alpha)}]^{\frac{1}{1+\alpha}}} - \sqrt{\Omega_{r_c}}, \tag{5.3.24}$$

$$\frac{\dot{H}}{H_0^2} = -\frac{3H(1 + z)^3}{2(H + H_0\sqrt{\Omega_{r_c}})} \left\{\Omega_m + \frac{(1 - A_s)\Omega_{ch}(1 + z)^{3\alpha}}{[A_s + (1 - A_s)(1 + z)^{3(1+\alpha)}]^{\frac{\alpha}{1+\alpha}}}\right\}, \tag{5.3.25}$$

where $\Omega_{ch} = \dfrac{\rho_{ch_0}}{3H_0^2}$ is usual dimensionless density parameter. Here, we assume $0 < A_s < 1$ and $1 + \alpha > 0$. Clearly $\dot{H} < 0$, $\forall\ z$.

It can be shown [101], that effective energy density does not vanish and is always positive if

$$4A_s(1 - A_s) > \left(\frac{4\Omega_{r_c}\Omega_m}{\Omega_{ch}^2}\right)^{1+\alpha}, \tag{5.3.26}$$

and

$$\dot{\rho}_{\text{eff}} = \frac{9HH_0^2(1 + z)^3}{H + H_0\sqrt{\Omega_{r_c}}}\chi, \tag{5.3.27}$$

where

$$\chi = \Omega_m H_0 \sqrt{\Omega_{r_c}} - H(1-A_s)\Omega_{ch}(1+z)^{3\alpha}[A_s + (1-A_s)(1+z)^{3(1+\alpha)}]^{-\frac{\alpha}{1+\alpha}}. \tag{5.3.28}$$

At high red-shifts ($z \gg 1$), the quantity $\dot{\rho}_{\text{eff}}$ is non positive as

$$\chi \sim -H_0(1-A_s)^{\frac{1}{1+\alpha}}\Omega_{ch}(1+z)^{\frac{3}{2}}\sqrt{\Omega_m + \Omega_{ch}(1-A_s)^{\frac{1}{1+\alpha}}}. \tag{5.3.29}$$

Reversely when $z \approx -1$, $\dot{\rho}_{\text{eff}}$ is non negative as $\chi \sim \Omega_m H_0 \sqrt{\Omega_{r_c}}$ for $\alpha > 0$. Also

$$1 + \omega_{\text{eff}} = -\frac{\dot{\rho}_{\text{eff}}}{3H\rho_{\text{eff}}}. \tag{5.3.30}$$

Subject to equation (5.3.26), from equation (5.3.27), $\rho_{\text{eff}} > 0$ and the sign of $1 + \omega_{\text{eff}}$ is determined by $\dot{\rho}_{\text{eff}}$

Clearly $1+\omega_{\text{eff}}$ is positive at ($z \gg 1$). Thus GSLT is valid for ($z \gg 1$) subject to equation (5.3.26), but may break in future universe.

5.4 Conclusion

In this chapter, we analyse the validity GSLT of the universe bounded by the event horizon that is built on the DGP braneworld. Here, we have substantially extended the work of [124] to event horizon. We consider universe as a thermodynamical system and filled with two non-interacting fluids. One component is taken as CDM and other as general dark energy. In brane, the usual area formula for the black hole entropy may not hold, so we have not used the black hole entropy as the entropy of the event horizon. Instead of that, we assume the validity of first law of thermodynamics and calculate the entropy. It is shown that in the absence of CDM, GSLT is always valid. This shows that, the relation between thermodynamics and gravity is not just accident in GR, but it has deep meaning which other theory of gravity also supports.

Chapter 6

Holographic Dark Energy And Validity Of The Generalised Second Law Of Thermodynamics In The DGP Braneworld

6.1 Prelude

In the previous chapter, we considered universe as a thermodynamical system and filled with two non-interacting fluids and studied the validity of the GSLT in the DGP braneworld. One component was CDM and other was general dark energy. In this chapter, the matter in the universe is taken in the form of non-interacting two fluid system- one component is the holographic dark energy and the other component is in the form of dust (CDM). Also in contrast to the previous chapter, here we study the validity of GSLT in both the branches of the DGP model, i.e. DGP(+) and DGP(-) models.

The universe is chosen to be homogeneous and isotropic and the validity of the first law has been assumed here. As mentioned in the previous chapter, in Ref. [157] validity of GSLT has been studied in Einstein's gravity and Gauss Bonnet gravity by assuming first law of thermodynamics and conditions for validity of GSLT have been obtained. Here we extend the work of [157] in DGP model of BG. Moreover in [157], the boundary of the universe was taken as event horizon, but here we examine the GSLT on both apparent horizon and event horizon. At the apparent horizon it is shown that GSLT is always respected regardless of specific form of DE. But in case of event horizon, GSLT may breakdown in the future universe. This chapter is based on [162].

6.2 Equation of State in the DGP model

In flat, homogeneous and isotropic brane the Friedmann equations (1.3.9) and (1.3.10) respectively reduces to

$$H^2 = \left(\sqrt{\frac{\rho_t}{3} + \frac{1}{4r_c^2}} + \epsilon\frac{1}{2r_c}\right)^2 \tag{6.2.1}$$

and

$$H^2 - \epsilon\frac{H}{r_c} = \frac{\rho_t}{3}, \tag{6.2.2}$$

where ρ_t is the total cosmic fluid energy density as defined in the previous chapter.

Here the effective energy density is given by

$$\rho_{\text{eff}} = \rho_D + \epsilon\frac{3H}{r_c}. \tag{6.2.3}$$

For simplicity here also we are using $8\pi G = 1$.

Here we choose ρ_D as HDE, therefore from equation (5.2.4) of previous chapter, we have

$$\rho_D = \frac{3c^2}{R_E^2}. \tag{6.2.4}$$

The Friedmann equation (6.2.2) can be rewritten as

$$\frac{H}{H_0} = \sqrt{\Omega_m + \Omega_D + \Omega_{r_c}} + \epsilon\sqrt{\Omega_{r_c}}, \tag{6.2.5}$$

where

$$\Omega_m = \frac{\rho_m}{3H_0^2}, \qquad \Omega_D = \frac{\rho_D}{3H_0^2}, \qquad \Omega_{r_c} = \frac{1}{4r_c^2 H_0^2}, \tag{6.2.6}$$

are the dimensionless density parameters and H_0 is Hubble parameter at redshift $z = 0$.

It may be noted that here, we define the fractional energy densities with H_0 rather than H, by which these fractions are often defined in the literature.

Substituting the value of ρ_D from (5.2.4) in equation (5.2.7) [163], and then differentiating the resulting equation w.r.t. red shift $z = 1/a - 1$, we get the evolution of Ω_D as

$$\frac{d\Omega_D}{dz} = \frac{2\Omega_D^{3/2}}{c(1+z)}\left(\frac{c}{\sqrt{\Omega_D}} - \frac{1}{\sqrt{\Omega_m + \Omega_D + \Omega_{r_c}} + \epsilon\sqrt{\Omega_{r_c}}}\right). \tag{6.2.7}$$

From equation (6.2.2), setting $z = 0$ we get the initial condition of the above differential equation as

$$\Omega_D(0) = 1 - 2\epsilon\sqrt{\Omega_{r_c}} - \Omega_m(0). \tag{6.2.8}$$

We shall solve the above differential equation later on numerically to analyse GSLT.

6.2.1 Equation of State of HDE

From conservation equation (5.2.4) of HDE, we get

$$\omega_D = -1 + (1+z)\frac{1}{3\omega_D}\frac{d\Omega_D}{dz}. \tag{6.2.9}$$

Eliminating $\frac{d\Omega_D}{dz}$ from equations (6.2.7) and (6.2.9), we get

$$\omega_D = -\frac{1}{3} - \frac{2}{3c}\frac{\sqrt{\Omega_D}}{\sqrt{\Omega_m + \Omega_D + \Omega_{r_c}} + \epsilon\sqrt{\Omega_{r_c}}}. \tag{6.2.10}$$

We note that, as in GR based theory here also $\omega_D < -1/3$.

6.2.2 Equation of State of effective dark energy

Defining $\Omega_{\text{eff}} = \frac{\rho_{\text{eff}}}{3H_0^2}$, from equation (5.2.1) we can write

$$\left(\frac{H}{H_0}\right)^2 = \Omega_m + \Omega_{\text{eff}}. \tag{6.2.11}$$

Squaring equation (6.2.5), we have

$$\left(\frac{H}{H_0}\right)^2 = \Omega_m + \Omega_D + 2\Omega_{r_c} + 2\epsilon\sqrt{\Omega_{r_c}}\sqrt{\Omega_m + \Omega_D + \Omega_{r_c}}. \tag{6.2.12}$$

Comparing equations (6.2.11) and (6.2.12), we also find that

$$\Omega_{\text{eff}} = \Omega_D + 2\Omega_{r_c} + 2\epsilon\sqrt{\Omega_{r_c}}\sqrt{\Omega_m + \Omega_D + \Omega_{r_c}}. \tag{6.2.13}$$

Now from conservation equation (5.2.5), we have

$$\Omega_{\text{eff}} = \Omega_{\text{eff}}^{(0)} \exp\left(3\int_0^z \frac{1+\omega_{\text{eff}}(z')}{1+z'}dz'\right). \tag{6.2.14}$$

Equating equations (6.2.12) and (6.2.13) and then taking derivative on both sides w.r.t. z, we get

$$1 + \omega_{\text{eff}} = \frac{1}{3\Omega_{\text{eff}}}\left[\epsilon\sqrt{\Omega_{r_c}}\frac{3\Omega_m + (1+z)\frac{d\Omega_D}{dz}}{\sqrt{\Omega_m + \Omega_D + \Omega_{r_c}}} + (1+z)\frac{d\Omega_D}{dz}\right]. \tag{6.2.15}$$

Now the above equation can be solved numerically with the help of differential equation (6.2.7) and initial condition (6.2.8). The behavior of ω_{eff} w.r.t. z is shown in Figs. 6.1 and 6.3.

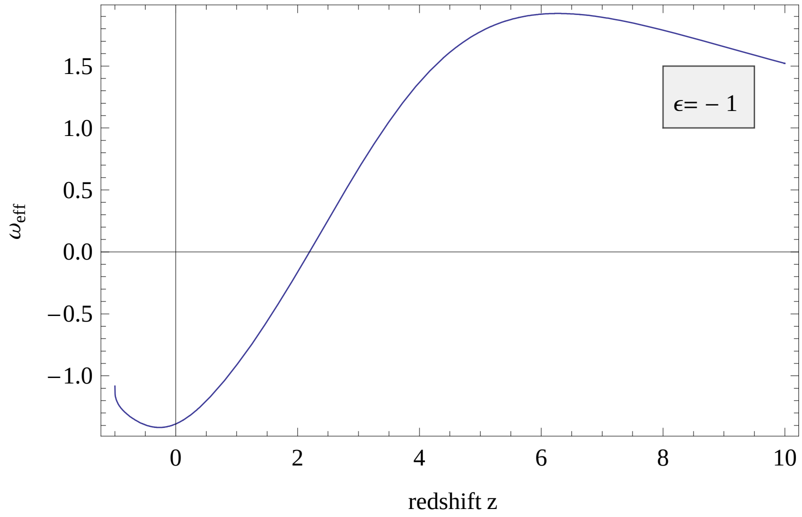

Figure 6.1: This figure shows the variation of ω_{eff} w.r.t. z for $\epsilon = -1$

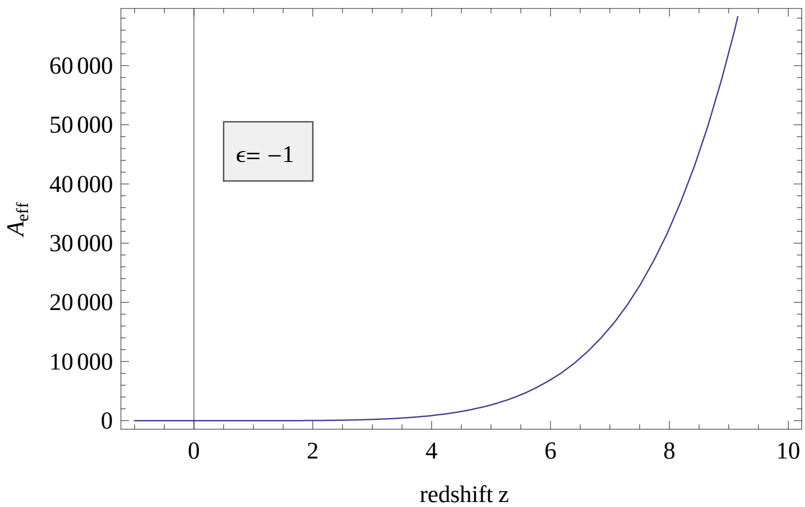

Figure 6.2: This figure shows the variation of A_{eff} w.r.t. z for $\epsilon = 1$

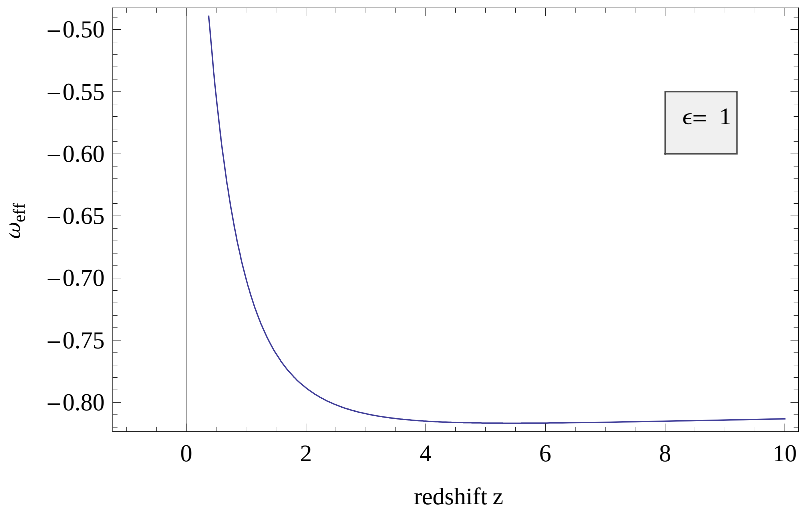

Figure 6.3: This figure shows the variation of ω_{eff} w.r.t. z for $\epsilon = -1$

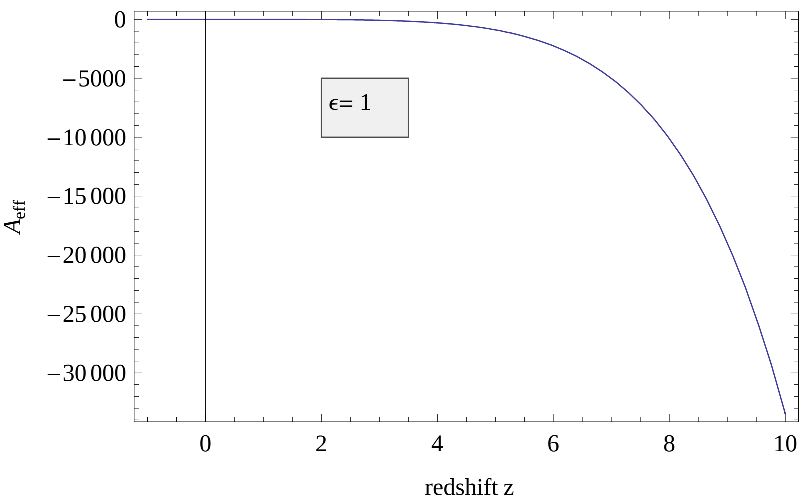

Figure 6.4: This figure shows the variation of A_{eff} w.r.t. z for $\epsilon = 1$

In all the figures the current density parameters used in the plots are $\Omega_{m0} = 0.3$ and $\Omega_{r_c} = 0.12$.

6.3 The Generalised Second Law of thermodynamics

In this section we examine the validity of GSLT on 3-DGP brane. By adapting the procedure of section 3 of previous chapter, and using equation (5.3.7), here the resulting change in the entropy is given by

$$\dot{S}_{\rm tot} = \dot{S}_h + \dot{S}_I = \frac{4\pi R_h^2}{T_h}\Big[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}\Big]\dot{R}_h. \tag{6.3.1}$$

Here the quantities S_h, R_h and T_h are the entropy, radius and temperature of the horizon respectively.

6.3.1 Universe bounded by apparent horizon

Here we study the validity of GSLT on apparent horizon. The apparent horizon for flat space is defined as

$$R_A = \frac{1}{H}. \tag{6.3.2}$$

In terms of apparent horizon, Friedmann equation (5.2.1) can be written as

$$\frac{1}{R_A^2} = \frac{1}{3}(\rho_m + \rho_{\rm eff}). \tag{6.3.3}$$

If we take the derivative of equation (6.3.3) w.r.t. cosmic time, then we get

$$\dot{R}_A = \frac{HR_A^3}{2}\left[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}\right]. \tag{6.3.4}$$

Substituting the above value of $\dot{R}_A$ in equation (6.3.1), we get

$$\dot{S}_{\rm tot} = \dot{S}_A + \dot{S}_I = \frac{2\pi R_A^5 H}{T_A}\left[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}\right]^2 \geq 0. \tag{6.3.5}$$

It may be noted that the result is true irrespective of specific form of DE. Thus it supports earlier investigations in GR based different DE models like the generalized chaplygin gas [164], the HDE [106, 107, 108, 109, 165] etc.

6.3.2 Universe bounded by event horizon

It is well known that in a spatial flat de Sitter universe the cosmological event horizon given by equation (5.2.7) and the apparent horizon coincide and $\dot{R}_E = 0$ [106, 107, 108, 109]. Therefore for de Sitter space from equation (6.3.1), we see that $\dot{S}_{\text{tot}} = 0$, which correspond to reversible adiabatic expansion. In this case

$$R_h = R_E.$$

Differentiating Friedmann equation (5.2.1) w.r.t. cosmic time, we get

$$\dot{H} = -\frac{1}{2}\left[\rho_m + (1+\omega_{\text{eff}})\rho_{\text{eff}}\right]. \tag{6.3.6}$$

Also from equation (6.2.3), we have

$$\dot{\rho}_{\text{eff}} = \dot{\rho}_D + \epsilon\frac{3\dot{H}}{r_c}. \tag{6.3.7}$$

Connecting this equation with conservation equation of effective energy density and equation (6.2.4), we get

$$\dot{R}_E = \frac{R_E^3}{2c^2}\Big[H(1+\omega_{\text{eff}})\rho_{\text{eff}} - \frac{\epsilon}{2r_c}(\rho_m + (1+\omega_{\text{eff}})\rho_{\text{eff}})\Big]. \tag{6.3.8}$$

Connecting equations (6.3.1) and (6.3.8), we get

$$\dot{S}_{\rm tot} = \frac{2\pi R_E^5}{c^2 T_E}\Big[H(1+\omega_{\rm eff})\rho_{\rm eff} - \frac{\epsilon}{2r_c}(\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff})\Big]\Big[\rho_m + (1+\omega_{\rm eff})\rho_{\rm eff}\Big]. \tag{6.3.9}$$

In terms of density parameters this equation can be rewritten as

$$\dot{S}_{\rm tot} = \frac{18\pi H_0^5 R_E^5}{c^2 T_E} A_{\rm eff}(z), \tag{6.3.10}$$

where $A_{\rm eff}(z)$ is defined as

$$A_{\rm eff}(z) = \Big[\Omega_m + (1+\omega_{\rm eff})\Omega_{\rm eff}\Big]\Big[(\sqrt{\Omega_m + \Omega_D + \Omega_{r_c}} + \epsilon\sqrt{\Omega_{r_c}})(1+\omega_{\rm eff})\Omega_{\rm eff} - \epsilon\sqrt{\Omega_{r_c}}(\Omega_m + (1+\omega_{\rm eff})\Omega_{\rm eff})\Big]. \tag{6.3.11}$$

Now $\dot{S}_{\rm tot} \geq 0$ if $A_{\rm eff}(z) \geq 0$. After solving equations (6.2.7) and (6.2.8) together with equations (6.2.13) and (6.2.15), the evolution $A_{\rm eff}(z)$ is plotted in Figs. 6.2 and 6.4.

The above results lead to the following conclusions :

I. *For* $\epsilon = -1$, the GSLT will be valid on the event horizon if $(1 + \omega_{\rm eff}) > 0$, i.e. effective DE is not of the phantom nature. Then universe as a thermodynamical system with two non-interacting fluid components always obey the GSLT.

Also for late time universe or DE dominated universe $\Omega_m \to 0$ and in that case using equation (6.2.5), we have

$$A_{\rm eff}(z) = (1+\omega_{\rm eff})^2\Omega_{\rm eff}^2\left[\frac{H}{H_0} + \sqrt{\Omega_{r_c}}\right] \geq 0.$$

Thus for DE dominated universe, GSLT is always valid in this case. Further one may note that, in this case the entropy of the event horizon also increases with time while variation of the matter entropy with time is not positive definite, but the sum of the entropies increases with the evolution of the universe. Also the radius of the event horizon increases with time.

II. *For* $\epsilon = 1,$ we see that $A_{\rm eff}(z) \geq 0$ if

$$\sqrt{\Omega_{r_c}} \leq \left[\frac{\sqrt{\Omega_m + \Omega_D + \Omega_{r_c}} + \sqrt{\Omega_{r_c}}}{\Omega_m + (1+\omega_{\rm eff})\Omega_{\rm eff}}\right](1+\omega_{\rm eff})\Omega_{\rm eff}.$$

Thus in this case GSLT is valid subject to the above inequality.

For late time universe or DE dominated universe $\Omega_m \rightarrow 0$ and using equation (6.2.11), we have

$$A_{\rm eff}(z) = (1+\omega_{\rm eff})^2\Omega_{\rm eff}^2\left[\frac{H}{H_0} - \sqrt{\Omega_{r_c}}\right].$$

Thus for DE dominated universe GSLT is valid in this case if $H \geq H_0\sqrt{\Omega_{r_c}}$.

6.4 Conclusion

In this chapter, we have examined the validity of generalised second law of thermodynamics for universe as a thermodynamical system bounded by apparent horizon or event horizon in the DGP braneworld scenario. Assuming the validity of the first law of thermodynamics, the GSLT is always satisfied on the apparent horizon irrespective of the equation of state for non interacting holographic dark energy and choice of ϵ $(=\pm 1)$. For validity of GSLT on the event horizon, the choice of ϵ as well as the equation of state for holographic dark energy is crucial. Figures 6.1 and 6.3 show the variation of ω_{eff} as a function of redshift z for $\epsilon = -1$ and $+1$ respectively. For $\epsilon = -1$, the effective fluid may have phantom nature around $z = 0$, while for $\epsilon = +1$ the effective fluid can not have phantom behaviour throughout the evolution of the universe. Figures 6.2 and 6.4 show the validity of GSLT for $\epsilon = -1$ and $+1$ respectively. For $\epsilon = -1$, although A_{eff} is a decreasing function but it is positive throughout the evolution of the universe and hence GSLT is satisfied on the event horizon. However, for $\epsilon = +1$, although A_{eff} is an increasing function (decreasing function of z) but it is negative throughout and hence there is a violation of GSLT.

Moreover, from observational data the estimate of the parameters are the following: $\Omega_{m0} = 0.3$, $\Omega_{r_c} = 0.12$. Taking $c = 1.2$, for $\epsilon = -1$, using equations (6.2.7), (6.2.8) and (6.2.14), we get $\omega_{\text{eff}} = -1.38803$ at $z = 0$ and for $\epsilon = 1$ the value of $\omega_{\text{eff}} = -0.157402$ at $z = 0$. Thus for $\epsilon = -1$, we have effective phantom behaviour and for $\epsilon = 1$, we have effective quintessence behaviour in the present universe.

Chapter 7

Generalised Second Law Of Thermodynamics For Interacting Dark Energy In The DGP Braneworld

7.1 Prelude

In this chapter, we extend the investigation of the validity of the generalised second law of thermodynamics (GSLT) to interacting holographic dark energy (HDE) model in the DGP braneworld. The thermodynamics of interacting HDE model in GR set up have been extensively studied in literature [165, 166]. In a recent paper [167], validity of GSLT has been studied on event horizon for interacting DE. Assuming first law of thermodynamics on the event horizon, they have found conditions for validity of GSLT in both cases when FRW universe is filled with interacting two fluid system:- one in the form of cold dark matter and the other is either HDE or new agegraphic DE. Here, we use this method of extracting entropy for interacting HDE in DGP model. This chapter is based on [168].

7.2 Interacting Holographic dark energy in DGP model

The Friedmann equation (6.2.2) can be written in following effective Einstein form

$$\dot{H} = -\frac{1}{2}\left\{\rho_m + (\rho_{\text{eff}} + p_{\text{eff}})\right\}, \tag{7.2.1}$$

where ρ_{eff} is the effective energy density given by equation (6.2.3) and p_{eff} is the effective pressure given by

$$p_{\text{eff}} = p_D - \epsilon\frac{3H}{r_c} - \epsilon\frac{\dot{H}}{r_c H}. \tag{7.2.2}$$

The individual conservation equation for effective DE and CDM are respectively given by

$$\dot{\rho}_{\text{eff}} + 3H(1+\omega_{\text{eff}})\rho_{\text{eff}} = -Q \tag{7.2.3}$$

and

$$\dot{\rho}_m + 3H\rho_m = Q, \tag{7.2.4}$$

where $Q = \Gamma\rho_D$ is called interaction term [167] and the decay rate Γ corresponds to conversion of dark energy to dust (CDM). Following [167], if we define

$$\omega_{\text{eff}}^{(i)} = \omega_{\text{eff}} + \frac{\Gamma}{3H} - \epsilon\frac{\Gamma}{r_c\rho_{\text{eff}}} \qquad \text{and} \qquad \omega_m^{(i)} = -\frac{\Gamma u}{3H}, \tag{7.2.5}$$

then the above conservation equations can be written in non-interacting form as

$$\dot{\rho}_{\rm eff} + 3H(1+\omega_{\rm eff}^{(i)})\rho_{\rm eff} = 0 \qquad (7.2.6)$$

and

$$\dot{\rho}_m + 3H(1+\omega_m^{(i)})\rho_m = 0, \qquad (7.2.7)$$

where $u = \frac{\rho_D}{\rho_m}$ is the ratio of energy densities.

Also, using equation (6.2.3) in equation (7.2.6), the actual energy conservation for DE is

$$\dot{\rho}_D + 3H(1+\omega_D^{(i)})\rho_D = 0, \qquad (7.2.8)$$

where

$$\omega_D^{(i)} = \omega_D + \frac{\Gamma}{3H}. \qquad (7.2.9)$$

Combining equations (7.2.6) and (7.2.7), we get

$$\dot{\rho}_t + 3H(\rho_t + p_t) = 0, \qquad (7.2.10)$$

where

$$\rho_t = \rho_m + \rho_{\rm eff} \quad \text{and} \quad p_t = p_m + p_{\rm eff} = \omega_m^{(i)}\rho_m + \omega_{\rm eff}^{(i)}\rho_{\rm eff}. \qquad (7.2.11)$$

7.3 The Generalised Second Law of thermodynamics

In this section, we examine the validity of GSLT on 3-DGP brane. As in chapter 5, we consider a region of FRW universe enveloped by the horizon and assume that the region bounded by the horizon act as a thermal system with boundary defined by the horizon and is filled with a perfect fluid of energy density ρ_t and pressure p_t given by equation (7.2.11).

The amount of energy crossing the horizon in time dt has the expression

$$-dE = 4\pi R_h^3 H(\rho_t + p_t)dt, \tag{7.3.1}$$

where R_h is the radius of the horizon.

So from the first law of thermodynamics, we have

$$\dot{S}_h = \frac{4\pi R_h^3 H}{T_h}\left[(1+\omega_m^{(i)})\rho_m + (1+\omega_{\text{eff}}^{(i)})\rho_{\text{eff}}\right], \tag{7.3.2}$$

where S_h and T_h are the entropy and temperature of the horizon respectively.

Using Gibb's equation,

$$T_h dS_I = dE_I + p_t dV,$$

we obtain the variation of the entropy of the fluid inside the horizon as

$$T_h dS_I = d(\rho_m + \rho_D)V + (\rho_m + \rho_D + p_D)dV, \tag{7.3.3}$$

where S_I and E_I are the entropy and energy of the matter distribution inside the horizon. Like previous chapters, here also we assume the temperature of the source inside the horizon is in equilibrium with the temperature associated with the horizon.

So, starting with $E_I = \frac{4}{3}\pi R_h^3 \rho_t$, $V = \frac{4}{3}\pi R_h^3$ and using equations (7.2.7), (7.2.8) and (7.3.3), and after some simplification one gets

$$\dot{S}_I = -\frac{4\pi R_h^3 H}{T_h}\left[\rho_m(1+\omega_m^{(i)}) + \rho_D(1+\omega_D^{(i)})\rho_{\text{eff}}\right] + \frac{4\pi R_h^2}{T_h}\dot{R}_h\left[\rho_m + \rho_D(1+\omega_D)\right]. \tag{7.3.4}$$

Adding equations (7.3.2) and (7.3.4), one gets the resulting change of entropy

$$\dot{S}_{\text{tot}} = \dot{S}_h + \dot{S}_I = \frac{4\pi R_h^2}{T_h}\Big[-\frac{\epsilon \dot{H} R_h}{r_c} + \{\rho_m + \rho_D(1+\omega_D)\}\dot{R}_h\Big]. \quad (7.3.5)$$

We shall now examine the validity of GSLT, i.e.

$$\dot{S}_{\text{tot}} = \dot{S}_h + \dot{S}_I \geq 0,$$

for apparent and event horizon respectively.

7.3.1 Universe bounded by apparent horizon

Here

$$R_h = R_A = \frac{1}{H}. \quad (7.3.6)$$

So,

$$\dot{R}_A = -\frac{\dot{H}}{H^2} = \frac{1}{2H^2}\left\{\rho_m + (\rho_{\text{eff}} + p_{\text{eff}})\right\}. \quad (7.3.7)$$

Hence equation (7.3.5) simplifies to

$$\dot{S}_{\text{tot}} = \frac{4\pi R_A^2}{T_A}\left[\epsilon\frac{R_A}{2r_c} + \frac{1}{2H^2}\{\rho_m + \rho_D(1+\omega_D)\}\right]\Big\{\rho_m + \rho_{\text{eff}}(1+\omega_{\text{eff}})\Big\}. \quad (7.3.8)$$

7.3.2 Universe bounded by event horizon

In this case

$$R_h = R_E.$$

Here due to holographic nature of the DE, the energy density of the holographic matter can be written as

$$\rho_D = \frac{3c^2}{R_E^2}. \tag{7.3.9}$$

So, using the conservation equation (7.2.8), the time variation of the event horizon can be written as [157]

$$\dot{R}_E = \frac{3}{2} H R_E (1 + \omega_D^{(i)}). \tag{7.3.10}$$

Hence equation (7.3.5) now becomes

$$\dot{S}_{\text{tot}} = \frac{4\pi R_E^2}{T_E} \Big[\frac{\epsilon R_E}{2r_c} \Big\{ \rho_m + \rho_{\text{eff}}(1 + \omega_{\text{eff}}) \Big\} +$$

$$\frac{3}{2} H R_E \left(1 + \omega_D + \frac{\Gamma}{3H} \right) \Big\{ \rho_m + \rho_D (1 + \omega_D) \Big\} \Big]. \tag{7.3.11}$$

7.4 Conclusion

The following conclusion we can draw from the variation of the total entropy given by equations (7.3.8) or (7.3.11):

(a) The entropy variation does not depend on the interaction in the case of apparent horizon while in the case of event horizon there is dependence on interaction through $\dot{R}_E$ due to the choice of the energy density for HDE given by equation (7.3.9).

(b) If we put $r_c \to \infty$ (this happens at early epoch), i.e. neglect the brane effect then GSLT will always be satisfied for apparent horizon while GSLT will not be trivial for event horizon as it depends on the interaction term Γ.

(c) In deriving GSLT we do not need any specific choice for entropy and temperature at the horizon.

Appendix A

Gravitational field equations on 3-branes

As mentioned in the Chapter 1, in the braneworld scenario obtained from 5-dimensional gravity, our 4-dimensional world M^4 (3-brane) is a hypersurface with geometry described by the metric tensor components $g_{ij}(x)$ $(i, j = 0, 1, 2, 3)$ and co-ordinates x^i. M^4 is embedded in 5-dimensional space-time, namely "bulk" with metric tensor components $a_{\alpha\beta}(y)(\alpha, \beta = 0, 1, 2, 3, 4)$ and co-ordinates $y^\alpha = (x^i, y)$. If N^α are components of the unit vector normal to M^4, then

$$g_{\alpha\beta} = a_{\alpha\beta} + N_\alpha N_\beta \tag{A.1a}$$

with

$$N^\alpha g_{\alpha\beta} = 0 \tag{A.1b}$$

and

$$g^{\alpha\beta} = a^{\alpha\beta} - N^\alpha N^\beta. \tag{A.1c}$$

Here, $N^\alpha N_\alpha = 1$ with $N^i = (0, 0, 0, 0)$ and $N^4 = (0, 0, 0, 0, 1)$.

Now Gauss's equation is given by

$$R_{pijk} = (\Omega_{ik}\Omega_{jp} - \Omega_{ij}\Omega_{kp}) + \bar{R}_{\epsilon\beta\gamma\delta} y^\epsilon_{,p} y^\beta_{,i} y^\gamma_{,j} y^\delta_{,k}, \tag{A.2}$$

and Codazzi equation is given by

$$\begin{aligned}\Omega_{ij;k} - \Omega_{ik;j} &= -a_{\alpha\epsilon}N^{\epsilon}\bar{R}^{\alpha}_{\beta\gamma\delta}y^{\beta}_{,i}y^{\gamma}_{,j}y^{\delta}_{,k} \\ &= \bar{R}_{\beta\epsilon\gamma\delta}y^{\beta}_{,i}y^{\gamma}_{,j}y^{\delta}_{,k}N^{\epsilon}.\end{aligned} \tag{A.3}$$

where *extrinsic* curvature components are given as

$$\begin{aligned}\Omega_{ij} &= -a_{\alpha\beta}(\bar{D}_j N^{\alpha})y^{\beta}_{,i} \\ &= -(\bar{D}_j N_{\alpha})y^{\alpha}_{,i} = -N_{\alpha;\beta}y^{\alpha}_{,i}y^{\beta}_{,j}\end{aligned}$$

,

$$\bar{D}_i N^{\alpha} = -\Omega_{ij}g^{jl}y^{\alpha}_{,l}. \tag{A.4a}$$

and

$$\Omega = \Omega^i_i \tag{A.4b}$$

is the trace of Ω_{ij}.

Multiplying the *Gauss equation* (A.2) by g^{pl}, we have

$$\begin{aligned}R^l_{ijk} &= g^{pl}[\Omega_{ik}\Omega_{jp} - \Omega_{ij}\Omega_{kp}] + \bar{R}_{\epsilon\beta\gamma\delta}y^{\epsilon}_{,p}y^{\beta}_{,i}y^{\gamma}_{,j}y^{\delta}_{,k}g^{pl} \\ &= [\Omega_{ik}\Omega^l_j - \Omega_{ij}\Omega^l_k] + \bar{R}_{\epsilon\beta\gamma\delta}y^{\epsilon}_{,p}y^{\beta}_{,i}y^{\gamma}_{,j}y^{\delta}_{,k}g^{pl}.\end{aligned} \tag{A.5}$$

Multiplying the *Codazzi equation* (A.3) by g^{ik}, we have

$$\Omega^i_{j;i} - \Omega^i_{i;j} = \bar{R}_{\beta\epsilon\gamma\delta}y^{\beta}_{,i}y^{\gamma}_{,j}y^{\delta}_{,k}N^{\epsilon}g^{ik}, \tag{A.6}$$

Contracting l with j in equation (A.5), it is obtained that

$$R_{ik} = [\Omega\Omega_{ik} - \Omega_{il}\Omega^l_k] + \bar{R}^{\omega}_{\beta\gamma\delta}y^{\beta}_{,i}y^{\delta}_{,k}\delta^{\gamma}_{\omega}. \tag{A.7}$$

From equation (A.1a)

$$\delta^{\gamma}_{\omega} = a^{\gamma\alpha}a_{\alpha\omega} - N^{\gamma}N_{\omega}. \qquad (A.8)$$

Connecting equations (A.7) and (A.1a)

$$R_{ik} = [\Omega\Omega_{ik} - \Omega_{il}\Omega^{l}_{k}] + [\bar{R}_{\beta\delta} - \bar{R}^{\omega}_{\beta\gamma\delta}N^{\gamma}N_{\omega}]y^{\beta}_{,i}y^{\delta}_{,k} \qquad (A.9)$$

Multiplying this equation by g^{ik}, we have

$$\begin{aligned} R &= [\Omega^2 - \Omega^{ij}\Omega_{ij}] + [\bar{R}_{\beta\delta} - \bar{R}^{\omega}_{\beta\gamma\delta}N^{\gamma}N_{\omega}]y^{\beta}_{,i}y^{\delta}_{,k}g^{ik} \\ &= [\Omega^2 - \Omega^{ij}\Omega_{ij}] + [\bar{R}_{\beta\delta} - \bar{R}^{\omega}_{\beta\gamma\delta}N^{\gamma}N_{\omega}]a^{\beta\delta} \\ &= [\Omega^2 - \Omega^{ij}\Omega_{ij}] + [\bar{R}_{\beta\delta}a^{\beta\delta} - \bar{R}^{\omega}_{\beta\gamma\delta}N^{\gamma}N_{\omega}(g^{\beta\delta} + N^{\beta}N^{\delta}) \\ &= [\Omega^2 - \Omega^{ij}\Omega_{ij}] + [\bar{R} - 2\bar{R}_{\epsilon\gamma}N^{\epsilon}N^{\gamma}]. \end{aligned} \qquad (A.10)$$

This equation is obtained using equations (A.1c), (A.7),

$$\bar{R}_{\epsilon\beta\gamma\delta}g^{\beta\delta} = \bar{R}_{\epsilon i\gamma j}g^{ij} = \bar{R}_{\epsilon\gamma}$$

and

$$\bar{R}_{\epsilon\beta\gamma\delta}N^{\beta}N^{\delta} = \bar{R}_{\epsilon 4\gamma 4}N^4N^4 = \bar{R}_{\epsilon\gamma}.$$

Equations (A.9) and (A.10) readily give

$$\begin{aligned} G_{ik} &= R_{ik} - \frac{1}{2}g_{ik}R \\ &= [\Omega\Omega_{ik} - \Omega_{il}\Omega^{l}_{k}] + [\bar{R}_{\beta\delta} - \bar{R}^{\omega}_{\beta\gamma\delta}N^{\gamma}N_{\omega}]y^{\beta}_{,i}y^{\delta}_{,k} \\ &\quad -\frac{1}{2}g_{ik}[\Omega^2 - \Omega^{ij}\Omega_{ij}] - \frac{1}{2}g_{ik}[\bar{R} - 2\bar{R}_{\epsilon\gamma}N^{\epsilon}N^{\gamma}] \\ &= [\bar{R}_{\beta\delta} - \frac{1}{2}a_{\beta\delta}\bar{R}]y^{\beta}_{,i}y^{\delta}_{,k} + [\Omega\Omega_{ik} - \Omega_{il}\Omega^{l}_{k}]\frac{1}{2}g_{ik}[\Omega^2 - \Omega^{ij}\Omega_{ij}] - \mathcal{E}_{ik}, \end{aligned} \qquad (A.11)$$

where $g_{ik} = a_{\beta\delta} y^{\beta}_{,i} y^{\delta}_{,k}$ and

$$\mathcal{E}_{ik} = \bar{R}^{\omega}_{\beta\gamma\delta} N^{\gamma} N_{\omega} y^{\beta}_{,i} y^{\delta}_{,k}. \qquad (A.12)$$

In general, 5-dimensional Einstein's equations are given as

$$\bar{G}_{\alpha\beta} = \bar{R}_{\alpha\beta} - \frac{1}{2} a_{\alpha\beta} \bar{R} = -8\pi G_5 \bar{T}_{\alpha\beta}. \qquad (A.13)$$

The decomposition of $\bar{R}_{\alpha\beta\gamma\delta}$ is given as

$$\begin{aligned} \bar{R}_{\alpha\beta\gamma\delta} &= \frac{1}{3}(a_{\alpha\gamma}\bar{R}_{\beta\delta} - a_{\alpha\delta}\bar{R}_{\beta\gamma} - a_{\beta\gamma}\bar{R}_{\alpha\delta} + a_{\beta\delta}\bar{R}_{\alpha\gamma}) \\ &\quad -\frac{1}{12}(a_{\alpha\gamma}a_{\beta\delta} - a_{\alpha\delta}a_{\beta\gamma})\bar{R} + \bar{C}_{\alpha\beta\gamma\delta} \end{aligned} \qquad \text{(A.14)}$$

Connecting equations (A.12) and (A.14) as well as using $N_{\delta} y^{\delta}_{,i} = N_i = 0$ and $N^{\alpha} N_{\alpha} = 1$, we obtain

$$\begin{aligned} \mathcal{E}_{ik} &= [\frac{1}{3} a^{\alpha\omega}(a_{\alpha\gamma}\bar{R}_{\beta\delta} - a_{\alpha\delta}\bar{R}_{\beta\gamma} - a_{\beta\gamma}\bar{R}_{\alpha\delta} + a_{\beta\delta}\bar{R}_{\alpha\gamma}) \\ &\quad -\frac{1}{12} a^{\alpha\omega}(a_{\alpha\gamma}a_{\beta\delta} - a_{\alpha\delta}a_{\beta\gamma}) + \bar{C}^{\omega}_{\beta\gamma\delta}] N^{\gamma} N_{\omega} y^{\beta}_{,i} y^{\delta}_{,k} \\ &= [\frac{1}{3}(N^{\omega}N_{\omega}\bar{R}_{\beta\delta} - N^{\gamma}N_{\delta}\bar{R}_{\beta\gamma} - N^{\alpha}N_{\beta}\bar{R}_{\alpha\delta} + N^{\alpha}N^{\gamma}a_{\beta\delta}\bar{R}_{\alpha\gamma}) \\ &\quad -\frac{1}{12}(N^{\omega}N_{\omega}a_{\beta\delta} - N^{\gamma}N_{\delta}a_{\alpha\delta}a_{\beta\gamma}) + \bar{C}^{\omega}_{\beta\gamma\delta}N^{\gamma}N_{\omega}] y^{\beta}_{,i} y^{\delta}_{,k} \\ &= \frac{1}{3}(\bar{R}_{\beta\delta} - \frac{1}{2}a_{\beta\delta}) y^{\beta}_{,i} y^{\delta}_{,k} + \frac{1}{3} N^{\alpha}N^{\gamma}\bar{R}_{\alpha\gamma} g_{ik} + \frac{1}{12} a_{\beta\delta}\bar{R} y^{\beta}_{,i} y^{\delta}_{,k} + \bar{\mathcal{E}}_{ik} \end{aligned} \qquad \text{(A.15a)}$$

with

$$\hat{\mathcal{E}}_{ik} = \bar{C}^{\omega}_{\beta\gamma\delta} N^{\gamma} N_{\omega} y^{\beta}_{,i} y^{\delta}_{,k}. \qquad (A.15b)$$

Trace of equation (A.13) is obtained as

$$\bar{R} = \frac{16\pi G_5}{3} \bar{T}. \qquad (A.16)$$

So, connecting equation (A.13) in equation (A.16), we get

$$\bar{R}_{\alpha\beta} = -8\pi G_5(\bar{T}_{\alpha\beta} - \frac{1}{3}a_{\alpha\beta}\bar{T}). \qquad (A.17)$$

Using equations (A.15) - (A.17) in equation (A.11), we have

$$\begin{aligned} G_{ik} &= \frac{2}{3}(\bar{R}_{\beta\delta} - \frac{1}{2}a_{\beta\delta}\bar{R})y^{\beta}_{,i}y^{\delta}_{,k} + \frac{2}{3}\bar{R}_{\beta\delta}N^{\beta}N^{\delta}g_{ik} - \frac{1}{12}g_{ik}\bar{R} \\ &\quad +[\Omega\Omega_{ik} - \Omega_{il}\Omega^{l}_{k}] - \frac{1}{2}g_{ik}[\Omega^2 - \Omega^{ij}\Omega_{ij}] - \hat{\mathcal{E}}_{ik} \\ &= -\frac{16\pi G_5}{3}[\bar{T}_{\beta\delta}y^{\beta}_{,i}y^{\delta}_{,k} + (\bar{T}_{\alpha\beta} - \frac{1}{3}a_{\alpha\beta}\bar{T})g_{ik}N^{\alpha}N^{\beta} + \\ &\quad [\Omega\Omega_{ik} - \Omega_{il}\Omega^{l}_{k}] - \frac{1}{2}g_{ik}[\Omega^2 - \Omega^{ij}\Omega_{ij}] - \hat{\mathcal{E}}_{ik} \\ &= -\frac{16\pi G_5}{3}[\bar{T}_{\beta\delta}y^{\beta}_{,i}y^{\delta}_{,k} + (\bar{T}_{\alpha\beta}N^{\alpha}N^{\beta} - \frac{1}{4}\bar{T})g_{ik}] + \\ &\quad [\Omega\Omega_{ik} - \Omega_{il}\Omega^{l}_{k}] - \frac{1}{2}g_{ik}[\Omega^2 - \Omega^{ij}\Omega_{ij}] - \hat{\mathcal{E}}_{ik}\frac{1}{12}g_{ik}\bar{T}, \end{aligned}$$

(A.18a)

where

$$\hat{\mathcal{E}}_{ik} = \bar{C}_{\alpha\beta\gamma\delta}N^{\gamma}N_{\omega}y^{\beta}_{,i}y^{\delta}_{,k}. \qquad (A.18b)$$

It is to be noted that

$$\begin{aligned} g^{ik}\hat{\mathcal{E}}_{ik} &= \bar{C}^{\omega}_{\beta\gamma\delta}N^{\gamma}N_{\omega}a^{\beta\delta} \\ &= \frac{1}{2}[\bar{C}^{\omega}_{\beta\gamma\delta} - \bar{C}^{\omega}_{\beta\delta\gamma}]N^{\gamma}N_{\omega}a^{\beta\delta} \\ &= \frac{1}{2}[\bar{C}^{4}_{\beta 4\delta} - \bar{C}^{4}_{\beta\delta 4}]a^{\beta\delta} \\ &= \frac{1}{2}[\bar{C}_{\beta\delta} - \bar{C}_{\beta\delta}]a^{\beta\delta} \\ &= 0 \end{aligned}$$

(A.18c)

Connecting equations (A.6) and (A.17), it is obtained that

$$\begin{aligned} \Omega^i_{j;i} - \Omega^i_{i;j} &= \bar{R}_{\alpha\beta} N^\alpha y^\beta_{,j} \\ &= -8\pi G_5 (\bar{T}_{\alpha\beta} - \frac{1}{3} a_{\alpha\beta} \bar{T}) N^\alpha y^\beta_{,j} \\ &= -8\pi G_5 \bar{T}_{\alpha\beta} N^\alpha y^\beta_{,j}, \end{aligned} \tag{A.19}$$

as $a_{\alpha\beta} N^\alpha y^\beta_{,j} = N_\beta y^\beta_{,j} = N_j = 0$.

Israel-Dermois junction condition

So far, we have made an attempt to obtain *Gauss-Codazzi* equations for the 4-dimensional hypersurface of a 5-dimensional space-time. In what follows, we derive conditions at points where 3-branes are located in the extra-space. The 5th dimension, being an orbifold $S^1/\mathbb{Z}_2$ has two end points $y = 0$ and $y = L$. Without any loss of generality, we can obtain these conditions at one of these points. Here, $y = 0$ is chosen for this purpose.

At $y = 0$, the 4-dimensional hypersurface (3-brane) Σ separates the "bulk" into two regions Σ^+ (right-handed $y > 0$) and Σ^- (left-handed $y < 0$). In principle, these separate regions across the hypersurface Σ, may have two co-ordinate systems. So, to have a covariance, we can use tensor analysis, which provides to have *invariance* under co-ordinate transformations.

The unit normal vector, with components N^α, is tangential to geodesics intersecting the hypersurface orthogonally in a local neighbourhood around the point $y = 0$ consisting of small portions of the regions Σ^+

and Σ^-. So,

$$N^\alpha = \frac{dy^\alpha}{ds}$$

with s being the affine parameter along these geodesics and y^α being co-ordinates in the local neighbourhood. As the point $(x^i, 0)$ lies on the "bulk", coordinates are *smooth* across the hypersurface (3-brane).

At this stage, we draw *Heaviside function* into service to have distribution of a physical quantity into regions Σ^+ and Σ^-. This function is defined as

$$\theta(s) = \begin{cases} 1 & \text{for } s > 0, \\ 0 & \text{for } s < 0 \\ \text{indeterminate} & \text{for } s = 0. \end{cases} \tag{A.20a}$$

obeying properties

$$\theta(s)^2 = \theta(s), \tag{A.20b}$$

$$\theta(s)\theta(-s) = 0, \tag{A.20c}$$

$$\frac{d\theta(s)}{ds} = \delta(s). \tag{A.20d}$$

Also it is right to mention that $\theta(s)\delta(s)$ is *not* defined. Here $\delta(s)$ is the Dirac delta function, which is *divergent* at $s = 0$.

Using $\theta(s)$, metric tensor components $a_{\alpha\beta}$ (for the "bulk") are written as

$$a_{\alpha\beta} = \theta(s)a^+_{\alpha\beta} + \theta(-s)a^-_{\alpha\beta} \tag{A.21}$$

with $a^\pm_{\alpha\beta}$ being the metric tensor components for regions Σ^+ and Σ^- respectively. As 4-dimensional geometry will remain the same on the hypersurface, we have

$$g^+_{ij} = g^-_{ij}. \tag{A.22a}$$

This is the *first Israel condition*, which is re-written as

$$[g_{ij}] = 0 \qquad (A.22b)$$

with the bracket defined as $[X] = X^+ - X^-$.

As co-ordinates x^i and $y^\alpha = (x^i, y)$ are smooth, we have $[y^\alpha_{,i}] = 0$. So,

$$[a_{\alpha\beta}]y^\alpha_{,i}y^\beta_{,j} = [g_{ij}] = 0,$$

which yields

$$[a_{\alpha\beta}] = 0. \qquad (A.23)$$

The covariant derivative of $a_{\alpha\beta}$, given by equation (A.21), is obtained as

$$a_{\alpha\beta;\gamma} = \theta(s)a^+_{\alpha\beta;\gamma} + \theta(-s)a^-_{\alpha\beta;\gamma} + \delta(s)[a^+_{\alpha\beta} - a^-_{\alpha\beta}]N_\gamma, \qquad (A.24)$$

where

$$d\theta(-s)/ds = -d\theta(-s)/d(-s) = -\delta(-s) = -\delta(s)$$

and $N_\gamma = ds/dy^\gamma$ such that $N_\gamma N^\beta = \delta^\beta_\gamma$.

Equations (A.23) and (A.24) yield

$$a_{\alpha\beta;\gamma} = \theta(s)a^+_{\alpha\beta;\gamma} + \theta(-s)a^-_{\alpha\beta;\gamma}, \qquad (A.23)$$

Equation (A.25) can be re-written as

$$\begin{aligned} a_{\alpha\beta,\gamma} - \bar{\Gamma}^\delta_{\alpha\gamma}a_{\beta\delta} - \bar{\Gamma}^\delta_{\beta\gamma}a_{\alpha\delta} &= \theta(s)[a^+_{\alpha\beta,\gamma} - \bar{\Gamma}^{\delta+}_{\alpha\gamma}a^+_{\beta\delta} - \bar{\Gamma}^{\delta+}_{\beta\gamma}a^+_{\alpha\delta}] \\ &\quad +\theta(-s)[a^-_{\alpha\beta,\gamma} - \bar{\Gamma}^{\delta-}_{\alpha\gamma}a^-_{\beta\delta} - \bar{\Gamma}^{\delta-}_{\beta\gamma}a^-_{\alpha\delta}] \\ &= [\theta(s)a^+_{\alpha\beta,\gamma} + \theta(-s)a^-_{\alpha\beta,\gamma}] - [\theta(s)\bar{\Gamma}^{\delta+}_{\alpha\gamma} \\ &\quad +\theta(-s)\bar{\Gamma}^{\delta-}_{\alpha\gamma}]a_{\beta\delta} - [\theta(s)\bar{\Gamma}^{\delta+}_{\alpha\gamma} + \theta(-s)\bar{\Gamma}^{\delta-}_{\alpha\gamma}]a_{\beta\delta}]a_{\alpha\delta}. \end{aligned}$$

As $a^+_{\alpha\beta} = a^-_{\alpha\beta} = a_{\alpha\beta}$ and

$$\bar{\Gamma}^\delta_{\alpha\gamma} = \theta(s)\bar{\Gamma}^{\delta+}_{\alpha\gamma} + \theta(-s)\bar{\Gamma}^{\delta-}_{\alpha\gamma},$$

we have

$$a_{\alpha\beta,\gamma} = \theta(s)a^{+}_{\alpha\beta,\gamma} + \theta(-s)a^{-}_{\alpha\beta,\gamma} \qquad (A.26)$$

From equation (A.21), it is obtained that

$$a^{-}_{\alpha\beta} = \theta(-s)a_{\alpha\beta}$$

and using property (A.20b) of $\theta(s)$. Now, we have

$$\begin{aligned}\bar{\Gamma}^{\delta+}_{\alpha\gamma} &= \frac{1}{2}a^{\delta\beta+}[a^{+}_{\alpha\beta,\gamma} + a^{+}_{\beta\gamma,\alpha} - a^{+}_{\alpha\gamma,\beta}] \\ &= \frac{1}{2}a^{\delta\beta}\theta(s)[a_{\alpha\beta,\gamma} + a_{\beta\gamma,\alpha} - a_{\alpha\gamma,\beta}] \\ &\quad + \frac{1}{2}\delta(s)a^{\delta\beta}[a_{\alpha\beta}N_{\gamma} + a_{\beta\gamma}N_{\alpha} - a_{\alpha\gamma}N_{\beta}] \\ &= \theta(s)\bar{\Gamma}^{\delta}_{\alpha\gamma} + \frac{1}{2}\delta(s)a^{\delta\beta}[a_{\alpha\beta}N_{\gamma} + a_{\beta\gamma}N_{\alpha} - a_{\alpha\gamma}N_{\beta}]\end{aligned} \qquad \text{(A.27a)}$$

using equation (A.23) and $a^{+}_{\alpha\beta} = \theta(s)a_{\alpha\beta}$ from equations (A.20b,c) and (A.21).

Similarly, we have

$$\bar{\Gamma}^{\delta-}_{\alpha\gamma} = \theta(-s)\bar{\Gamma}^{\delta}_{\alpha\gamma} - \frac{1}{2}\delta(s)a^{\delta\beta}[a_{\alpha\beta}N_{\gamma} + a_{\beta\gamma}N_{\alpha} - a_{\alpha\gamma}N_{\beta}]. \qquad (A.27b)$$

From equations (A.27a) and (A.27b)

$$[\bar{\Gamma}^{\delta}_{\alpha\gamma}] = [\theta(s) - \theta(-s)]\bar{\Gamma}^{\delta}_{\alpha\gamma}$$

$$+\delta(s)a^{\delta\beta}[a_{\alpha\beta}N_{\gamma} + a_{\beta\gamma}N_{\alpha} - a_{\alpha\gamma}N_{\beta}] \neq 0. \qquad (A.28)$$

Thus, from equations (A.26) and (A.28)

$$\bar{\Gamma}^{\delta}_{\alpha\gamma,\beta} = \theta(s)\bar{\Gamma}^{\delta+}_{\alpha\gamma,\beta} + \theta(-s)\bar{\Gamma}^{\delta-}_{\alpha\gamma,\beta} + \delta(s)[\bar{\Gamma}^{\delta}_{\alpha\gamma}]N_{\beta}. \qquad (A.29)$$

Using these affine connections in the definition for Riemann curvature tensor, we have

$$\bar{R}^{\alpha}_{\beta\gamma\delta} = \theta(s)\bar{R}^{\alpha+}_{\beta\gamma\delta} + \theta(-s)\bar{R}^{\alpha-}_{\beta\gamma\delta} + \delta(s)\{[\bar{\Gamma}^{\alpha}_{\beta\gamma}]N_{\delta} - [\bar{\Gamma}^{\alpha}_{\beta\delta}]N_{\gamma}\}. \qquad (A.30)$$

Contracting α with γ, we obtain

$$\bar{R}_{\beta\delta} = \theta(s)\bar{R}^{+}_{\beta\delta} + \theta(-s)\bar{R}^{-}_{\beta\delta} + \delta(s)\{[\bar{\Gamma}^{\alpha}_{\beta\alpha}]N_{\delta} - [\bar{\Gamma}^{\alpha}_{\beta\delta}]N_{\alpha}\}. \qquad (A.31)$$

Further

$$\bar{R} = a^{\beta\delta}\bar{R}_{\beta\delta} = \theta(s)\bar{R}^{+} + \theta(-s)\bar{R}^{-}$$
$$+\delta(s)a^{\beta\delta}\{[\bar{\Gamma}^{\alpha}_{\beta\alpha}]N_{\delta} - [\bar{\Gamma}^{\alpha}_{\beta\delta}]N_{\alpha}\}. \qquad (A.32)$$

Using equations (A.31) and (A.32), Einstein tensor components are obtained as

$$\bar{G}_{\beta\delta} = \theta(s)\bar{G}^{+}_{\beta\delta} + \theta(-s)\bar{G}^{-}_{\beta\delta} - 8\pi G_5\delta(s)S_{\beta\delta}. \qquad (A.33a)$$

Getting $\bar{G}_{\beta\delta}$ from equations (A.31) and (A.32) and integrating the resulting equation with respect to y from $-\epsilon$ to ϵ and using the $\mathbb{Z}_2$-symmetry, it is obtained that

$$-8\pi G_5 S_{\beta\delta} = \{[\bar{\Gamma}^{\alpha}_{\beta\alpha}]N_{\delta} - [\bar{\Gamma}^{\alpha}_{\beta\delta}]N_{\alpha}\} - \frac{1}{2}a_{\beta\delta}\{[\bar{\Gamma}^{\alpha}_{\epsilon\alpha}]N^{\epsilon} - [\bar{\Gamma}^{\alpha}_{\epsilon\gamma}]a^{\epsilon\gamma}N_{\alpha}\} \qquad (A.33b)$$

From the "bulk" gravitational action

$$S_{\rm bulk} = \frac{1}{16\pi G_5}\int d^4x dy\sqrt{|a|}[\frac{\bar{R}}{16\pi G_5} - 2\Lambda_5] \qquad (A.34)$$

yields field equations

$$\bar{G}_{\beta\delta} = -8\pi G_5\Lambda_5 a_{\beta\delta}, \qquad (A.35)$$

where Λ_5 is the cosmological constant in the "bulk". So,

$$\bar{G}^{+}_{\beta\delta} = -8\pi G_5\Lambda_5 a^{+}_{\beta\delta}$$

and

$$\bar{G}^{-}_{\beta\delta} = -8\pi G_5 \Lambda_5 a^{-}_{\beta\delta}.$$

Using these results in equation (A.33a) and comparing it with equation (A.13), it is obtained that

$$\bar{T}_{\beta\delta} = -\Lambda_5 a_{\beta\delta} + \delta(s) S_{\beta\delta}. \qquad (A.36a)$$

It shows that at $s = 0$ (which corresponds to $y = 0$), energy momentum tensor components are given by equation (A.36a) and for $s \neq 0$

$$\bar{T}_{\beta\delta} = -\Lambda_5 a_{\beta\delta}. \qquad (A.36b)$$

It means that, according to classical theory, $\delta(s)$ enforces matter and fields to live in the 3-brane. As $S_{\beta\delta}$ are energy-momentum tensor components in the 3-brane, we have

$$S_{\beta\delta} = -\lambda g_{\beta\delta} + \tau_{\beta\delta} \qquad (A.37a)$$

with $N^{\beta} S_{\beta\delta} = 0$, λ being the vacuum energy (cosmological constant) in the 3-brane and $\tau_{\beta\delta}$ are energy-momentum tensor components of standard matter and fields in the 3-brane. Thus, from equation (A.37a)

$$S_{ij} = -\lambda g_{ij} + \tau_{ij} \qquad (A.37b)$$

As $N^{\alpha} = (0, 0, 0, 0, 1)$ are components of a unit vector perpendicular to the 3-brane. So,

$$N_{\alpha;\beta} = -\bar{\Gamma}^{\gamma}_{\alpha\beta} N_{\gamma} \qquad (A.38a)$$

and

$$N^{\alpha}_{;\beta} = \bar{\Gamma}^{\gamma\beta} N^{\gamma}. \qquad (A.38b)$$

These equations imply that

$$[N_{\alpha;\beta}] N^{\gamma} = -[\bar{\Gamma}^{\gamma}_{\alpha\beta}] \qquad (A.39a)$$

and

$$[N^{\alpha}_{;\alpha}]N_{\gamma} = \bar{\Gamma}^{\alpha}_{\gamma\alpha}. \qquad (A.39b)$$

Connecting equations (A.31b), (A.37a) and (A.37b), we have

$$\begin{aligned} -8\pi G_5 S_{\beta\delta} &= [N^{\alpha}_{;\alpha}]N_{\beta}N_{\delta} + [N_{\beta,\delta}]N^{\alpha}N_{\alpha} \\ &\quad -\frac{1}{2}a_{\beta\delta}([N^{\alpha}_{;\alpha}]N^{\epsilon}N_{\epsilon} + [N_{\epsilon;\gamma}]a^{\epsilon\gamma}). \end{aligned} \qquad \text{(A.40)}$$

So,

$$\begin{aligned} -8\pi G_5 S_{\beta\delta} y^{\beta}_{,i} y^{\delta}_{,j} &= [N^{\alpha}_{;\alpha}]N_{\beta}N_{\delta} y^{\beta}_{,i} y^{\delta}_{,j} + [N_{\beta;\delta}] y^{\beta}_{,i} y^{\delta}_{,j} \\ &\quad -\frac{1}{2}a_{\beta\delta} y^{\beta}_{,i} y^{\delta}_{,j}([N^{\alpha}_{;\alpha}]N^{\epsilon}N_{\epsilon} + [N_{\epsilon;\gamma}]a^{\epsilon\gamma}) \end{aligned} \qquad \text{(A.41a)}$$

as $N^{\alpha}N_{\alpha} = 1$. This equation is re-written as

$$-8\pi G_5 S_{ij} = [N^{\alpha}_{;\alpha}]N_i N_j + [N_{\beta;\delta}] y^{\beta}_{,i} y^{\delta}_{,j} - g_{ij}[N_{\epsilon;\gamma}]a^{\epsilon\gamma}. \qquad (A.41b)$$

Connecting equations (A.4a), (A.4b) and (A.41b), we obtain

$$-8\pi G_5 S_{ij} = -[\Omega_{ij}] + g_{ij}[\Omega]. \qquad (A.42a)$$

Trace of this equation is obtained as

$$-8\pi G_5 S = 3[\Omega]. \qquad (A.42b)$$

From equations (A.42a) and (A.42b)

$$[\Omega_{ij}] = 8\pi G_5 \{S_{ij} - \frac{1}{3}g_{ij}S\}.$$

This equation yields the second Israel junction condition, which can be re-written as

$$\Omega^{+}_{ij} = -\Omega^{-}_{ij} = 4\pi G_5 \{S_{ij} - \frac{1}{3}g_{ij}S\}$$

$$= 4\pi G_5\{\tau_{ij} - \frac{1}{3}g_{ij}(\tau - \lambda)\} \qquad (A.43)$$

using equation (A.37b). Interestingly, this equation gives *extrinsic curvature* in terms of energy-momentum of the matter in 3-brane.

Gravitational field equations

Due to symmetry in regions Σ^+ and Σ^-, quantities can be evaluated on the either side of $y = 0$. Thus, using equations (A.36b) and (A.43) in equation (A.18a), the field equations are obtained as

$$G_{ij} = 8\pi G(\Lambda_4 g_{ij} - \tau_{ij}) + (8\pi G_5)^2\pi_{ij} - \hat{\mathcal{E}}_{ik}, \qquad (A.44a)$$

where

$$\Lambda_4 = -4\pi G_5\Big[\Lambda_5 + \frac{4}{3}\pi G_5\lambda^2\Big], \qquad (A.44b)$$

$$G = \frac{4}{3}\pi G_5^2\lambda \qquad (A.44c),$$

$$\pi_{ij} = -\frac{1}{4}\tau_{ik}\tau^k_j + \frac{1}{12}\tau\tau_{ij} + \frac{1}{8}g_{ij}\tau_{kl}\tau^{kl} - \frac{1}{24}g_{ij}\tau^2 \qquad (A.44d)$$

and $\hat{\mathcal{E}}_{ik}$ is the part of 5-dimensional space-time Weyl tensor in equation(A.16b). Equation(A.44c) shows that G is proportional to λ (vacuum energy of the brane). In case $\lambda < 0$, $G < 0$. For example, in the case of visible brane of RSI model, $G < 0$.

Appendix B

Friedmann equation in the braneworld

The standard model of cosmology is based on Friedmann-Robertson-Walker space-time. So, it is natural to choose the *bulk* space-time in such a way that FRW model is recovered from it. In 2000, P.Binétry, C.Deffayet, U.Ellwanger and D.Langlois considered a 5-dimensional *bulk* space-time enveloping warped FRW type space-time as a 3-brane with ansatz

$$ds_5^2 = n^2(\tau, y)d\tau^2 - S^2(\tau, y)\gamma_{ab}dx^a dx^b - R^2(\tau, y)dy^2, \qquad (B.1)$$

where τ is the time and $\gamma_{ab}(a, b = 1, 2, 3)$ are components of the metric tensor of maximally symmetric 3-space parameterized by k. Here metric tensor components are obtained as

$$a_{00} = n^2(\tau, y), a_{ab} = -S^2(\tau, y)\gamma_{ab}, a_{44} = -R^2(\tau, y) \qquad (B.2a)$$

and

$$\gamma_{ab}dx^a dx^b = \frac{dr^2}{(1 - kr^2)} + r^2(d\theta^2 + sin^2\theta d\phi^2). \qquad (B.2b)$$

In this case, the 5-dimensional Einstein's field equations (A.13) are obtained as

$$\hat{G}_0^0 = 3\Big\{\frac{\dot{S}}{n^2 S}\Big(\frac{\dot{S}}{S}+\frac{\dot{R}}{R}\Big)+\frac{1}{R^2}\Big[\frac{S''}{S}+\frac{S'}{S}\Big(\frac{S'}{S}-\frac{R'}{R}\Big)\Big]+\frac{k}{R^2}\Big\} = -8\pi G_5 \hat{T}_0^0, \tag{B.3}$$

$$\hat{G}_b^a = \frac{1}{R^2}\delta_b^a\Big\{\frac{S'}{S}\Big(\frac{S'}{S}+2\frac{n'}{n}\Big)-\frac{R'}{R}\Big(2\frac{S'}{S}+\frac{n'}{n}\Big)+2\frac{S''}{S}+\frac{n''}{n}\Big\}$$

$$+\frac{1}{n^2}\delta_b^a\Big\{\frac{\dot{S}}{S}\Big(-\frac{\dot{S}}{S}+2\frac{\dot{n}}{n}\Big)-2\frac{\ddot{S}}{S}+\frac{\dot{R}}{R}\Big(-2\frac{\dot{S}}{S}+\frac{\dot{n}}{n}\Big)-2\frac{\ddot{R}}{R}\Big\}-k\delta_b^a = -8\pi G_5\hat{T}_b^a, \tag{B.4}$$

$$\hat{G}_0^0 = \frac{3}{n^2}\Big(\frac{n'}{n}\frac{\dot{S}}{S}+\frac{S'}{S}\frac{\dot{R}}{R}-\frac{\dot{S}'}{S}\Big) = -8\pi G_5\hat{T}_4^0, \tag{B.5}$$

$$\hat{G}_4^4 = 3\Big\{\frac{S'}{R^2 S}\Big(\frac{S'}{S}+\frac{n'}{n}\Big)-\frac{1}{n^2}\Big[\frac{\dot{S}}{S}\Big(\frac{\dot{S}}{S}-\frac{\dot{n}}{n}\Big)+\frac{\ddot{S}}{S}\Big]-\frac{k}{R^2} = -8\pi G_5\hat{T}_4^4, \tag{B.6}$$

where dot denotes derivative with respect to τ and prime denotes derivative with respect to y. Here

$$\hat{T}_{\alpha\beta} = T_{\alpha\beta}^{(bulk)} + \delta(y)R^{-1}S_{\alpha\beta}, \tag{B.7}$$

where $N^\alpha S_{\alpha\beta} = 0$. It means that non-zero components of $S_{\alpha\beta}$ are given by

$$S_{ij} = \lambda g_{ij} + T_{ij} \tag{B.8}$$

being energy-momentum tensor in the 3-brane. Moreover, T_{ij} are given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}. \tag{B.9a}$$

with

$$p = \omega\rho. \tag{B.9b}$$

In the space-time (B.1), extrinsic curvature components in 3-brane, are given by

$$\begin{aligned}[\Omega_{00}] &= -[N_{\alpha;\beta}]y^{\alpha}_{,0}y^{\beta}_{,0} \\ &= -[N_{\alpha,\beta}]y^{\alpha}_{,0}y^{\beta}_{,0} + \Gamma^{\gamma}_{\alpha\beta}N_{\gamma}y^{\alpha}_{,0}y^{\beta}_{,0} \\ &= \Gamma^{\gamma}_{00}N_{\gamma} \\ &= \frac{1}{2}a^{\gamma\delta}[a_{0\delta,0} + a_{0\delta,0} - a_{00,\delta}]N_{\gamma} \\ &= \frac{n(\tau,0)n'(\tau,0)}{R^2(\tau,0)}N_4 \\ &= \frac{n(\tau,0)n'(\tau,0)}{R(\tau,0)}\end{aligned} \quad \text{(B.10a)}$$

as $N^{\alpha}N_{\alpha} = -1$ and $N^{\alpha} = (0,0,0,0,1/R)$. Similarly

$$[\Omega_{ab}] = -\frac{S(\tau,0)S'(\tau,0)}{R(\tau,0)}. \quad (B.10b)$$

The $[A] = A^{+} - A^{-}$ is the jump function across the brane at $y = 0$. So, using equation (A.43), *junction conditions* are given by

$$[\Omega_{00}] = 8\pi G_5[S_{00} - \frac{1}{3}g_{00}S] \quad (B.11a)$$

and

$$[\Omega_{ab}] = 8\pi G_5[S_{ab} - \frac{1}{3}g_{ab}S] \quad (B.11b)$$

From equation (B.8), trace of S_{ij} is obtained as

$$S = \rho + 4\lambda - 3p. \quad (B.11c)$$

Connecting equations (B.10a), (B.11a) and (B.11c), it is obtained that

$$\frac{n'(\tau,0)}{n(\tau,0)R(\tau,0)} = \frac{8\pi G_5}{3}[3p + 2\rho - \lambda] \quad (B.12)$$

as $u_0^2 = n^2$ and $g_{00} = n^2$. Similarly, using $g_{ab} = -S^2(\tau, 0)$ and connecting equations (B.10b), (B.11b) and (B.11c), it follows that

$$\frac{S'(\tau,0)}{S(\tau,0)R(\tau,0)} = -\frac{8\pi G_5}{3}[\rho + \lambda] \qquad (B.13)$$

As $\hat{T}_4^0 = 0$, equation (B.5) yields

$$\frac{n'}{n}\frac{\dot{S}}{S} + \frac{S'}{S}\frac{\dot{R}}{R} - \frac{\dot{S}'}{S} = 0. \qquad (B.14)$$

From the condition (B.13),

$$\frac{\dot{S}'}{S} = \frac{\dot{S}}{S}\frac{S'}{S} - \frac{8\pi G_5}{3}\dot{\rho}R(\tau,0) - \frac{8\pi G_5}{3}(\rho+\lambda)\dot{R}(\tau,0). \qquad (B.15)$$

Using equations (B.12), (B.13), (B.14) and (B.15), we get

$$\dot{\rho} + 3\frac{\dot{S}(\tau,0)}{S(\tau,0)}(\rho + p) = 0, \qquad (B.16)$$

which is the usual conservation equation remaining unchanged for brane-world also.

To recover FRW type space-time at $y = 0$, we should have the synchronized time (cosmic clock synchronized at the beginning of the universe). In this case, we can redefine time as

$$dt = n(\tau, 0)d\tau. \qquad (B.17)$$

Using the synchronized time, equation (B.17) yields at $y = 0$

$$\frac{1}{R^2(t,0)}\Big(\frac{S'(t,0)}{S(t,0)}\Big)^2 - \Big[\Big(\frac{\dot{S}(t,0)}{S(t,0)}\Big)^2 + \frac{\ddot{S}(t,0)}{S(t,0)}\Big] - \frac{k}{R^2(t,0)} = -\frac{8\pi G_5}{3}\Lambda_5 \qquad (B.18)$$

as $T_4^{(bulk)4} = \Lambda_5, S_4^4 = 0$ and, in equation (B.17), $n(t,0) = 1$.

Integrating equation (B.18) with respect to time t

$$\Big(\frac{\dot{S}_0}{S_0}\Big)^2 = \frac{1}{2R_0^2}\Big(\frac{S_0'}{S_0}\Big)^2 + \frac{4\pi G_5}{3}\Lambda_5 + \frac{C}{S_0^4} - \frac{k}{S_0^2}, \qquad (B.19)$$

. Here C is an integration constant.

Using the condition (B.13), equation (B.19) can be rewritten as

$$\begin{aligned}\left(\frac{\dot{S}_0}{S_0}\right)^2 &= \frac{1}{2}\left(\frac{8\pi G_5}{3}\right)^2(\rho+\lambda)^2 + \frac{4\pi G_5}{3}\Lambda_5 + \frac{C}{S_0^4} - \frac{k}{S_0^2} \\ &= \frac{1}{2}\left(\frac{8\pi G_5}{3}\right)^2(\rho^2+2\lambda\rho) + \frac{1}{2}\left(\frac{8\pi G_5\lambda}{3}\right)^2 + \frac{4\pi G_5}{3}\Lambda_5 + \frac{C}{S_0^4} - \frac{k}{S_0^2} \\ &= \left(\frac{8\pi G_5}{3}\right)^2\lambda\rho\left(1+\frac{\rho}{2\lambda}\right) + \frac{1}{2}\left(\frac{8\pi G_5\lambda}{3}\right)^2 + \frac{4\pi G_5}{3}\Lambda_5 + \frac{C}{S_0^4} - \frac{k}{S_0^2}.\end{aligned} \tag{B.20}$$

Now Friedmann equation can be recovered from equation (B.19) by recognizing

$$\frac{8\pi G}{3} = \left(\frac{8\pi G_5}{3}\right)^2\lambda \tag{$B.21a$}$$

and

$$\frac{\Lambda_4}{3} = \frac{1}{2}\left(\frac{8\pi G_5\lambda}{3}\right)^2 + \frac{4\pi G_5}{3}\Lambda_5, \tag{$B.21b$}$$

where Λ_4 is the 4-dimensional cosmological constant in 3-brane.

Incorporating equations (B.21a) and (B.21b) in equation (B.20), we obtain

$$\left(\frac{\dot{S}_0}{S_0}\right)^2 = \frac{8\pi G}{3}\rho\left(1+\frac{\rho}{2\lambda}\right) + \frac{\Lambda_4}{3} + \frac{C}{S_0^4} - \frac{k}{S_0^2}. \tag{$B.22$}$$

For further simplification on the modified Friedmann equation (B.22), we take $S_0 = a(t)$. As a result, equation (B.22) looks like

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1+\frac{\rho}{2\lambda}\right) + \frac{\Lambda_4}{3} + \frac{C}{a^4} - \frac{k}{a^2}. \tag{$B.23$}$$

Some cosmological consequences

For $p = \omega\rho$, equation (B.16) yields

$$\rho = \rho(0)\left[\frac{a(0)}{a}\right]^q \tag{$B.24a$}$$

with

$$q = 3(1+\omega). \qquad (B.24b)$$

Case 1. When $\Lambda_4 \neq 0, C = 0, k = 0$

In this case, equation (B.23) integrates to

$$a(t) = a(0)\Big[-\frac{\kappa\rho(0)}{2\Lambda_4} + \rho(0)\sqrt{\frac{\kappa}{2\Lambda_4}\Big(\frac{1}{\lambda} - \frac{\kappa}{2\Lambda_4}\Big)}\{\sinh(q\sqrt{\Lambda/3}t)\cosh A$$

$$+\sinh A\cosh(q\sqrt{\Lambda/3}t)\}\Big]^{1/q}, \qquad (B.25a)$$

where

$$\sinh A = \frac{\Big[1 + \frac{\kappa}{2\Lambda_4}\rho(0)\Big]}{\rho(0)\sqrt{\frac{\kappa}{2\Lambda_4}\Big[\frac{1}{\lambda} - \frac{\kappa}{2\Lambda_4}\Big]}} \qquad (B.25b)$$

and

$$\kappa = 8\pi G. \qquad (B.26c)$$

When $\omega < -1, \quad q < 0$. In this case the solution (B.25a) exhibits the *big-rip* singularity at time $t = t_{\rm br}$ given by

$$\cosh(q\sqrt{\Lambda/3}t_{\rm br}) = \sqrt{\frac{\kappa}{2\Lambda_4}/\Big(\frac{1}{\lambda} - \frac{\kappa_N}{2\Lambda_4}\Big)}\mathrm{cosech}A + \sinh(q\sqrt{\Lambda/3}t_{\rm br})\coth A \qquad (B.26)$$

as $t \to t_{\rm br},\ a(t) \to \infty,\ \rho \to \infty$ and $p \to \infty$.

When brane-tension λ is given by $\lambda = \frac{2\Lambda_4}{\kappa}$,

$$\begin{aligned} a(t) &= a(0)\Big[-\frac{\rho(0)}{\lambda} + (1 + \rho(0)/\lambda)e^{q\sqrt{\Lambda/3t}}\Big]^{1/q} \\ &\simeq a(0)(1+\rho(0)/\lambda)^{1/q}e^{\sqrt{\Lambda_4/3t}}, \end{aligned} \qquad (B.27)$$

which is de Sitter expansion.

Case 2. When $\Lambda_4 = 0, C = 0, k = 0$

In this case, equation (B.23) integrates to

$$a(t) = a(0)\Big[-\frac{\rho(0)}{2\lambda} + \Big(1+\frac{\rho(0)}{\lambda}\Big)^{1/2} + \frac{q}{2}\sqrt{\kappa_N \rho(0)/3t}\Big]^{2/q}. \qquad (B.28)$$

It exhibits *big-rip* singularity at

$$t_{\rm br} = \left[-\frac{\rho(0)}{2\lambda} + \frac{\Big(1+\frac{\rho(0)}{\lambda}\Big)^{1/2}}{\frac{q}{2}\sqrt{\kappa\rho(0)/3}} \right]. \qquad (B.29)$$

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