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COMPACT LIE GROUP ACTION AND EQUIVARIANT BORDISM

SHABD SHARAN KHARE¹

ABSTRACT. Let G be a compact Lie group and H a compact Lie subgroup of G contained in the center of G with H^m the maximal subgroup in the center, H being H -boundary. Let $p_r: H^m \rightarrow H$ be the projection onto the r th factor and H_r be the r th factor of H^m . Let $\{L_r\}$ be a family of subgroups of G such that $L_r \cap H_r$ is nontrivial. Consider a G -manifold M^n with $p_r(G_x \cap H^m)$ trivial or containing L_r , for every x in M^n . The main result of the paper is that if $\forall x \in M^n$, $p_r(G_x \cap H^m)$ is trivial at least for one r , then M^n is a G -boundary.

1. Introduction. This paper is a sequel to [3–5]. Conner and Floyd [1] proved that if \mathbf{Z}_2^k acts on a closed manifold M differentiably and without any fixed point, then M is a boundary. Stong [7] showed that if (M, θ) is a closed \mathbf{Z}_2^k -differential manifold without any stationary point, then (M, θ) is a \mathbf{Z}_2^k -boundary. In [3], we extended Stong's result for any finite abelian group of even order by proving the following. Let G be a finite abelian group of even order, (M, θ) a closed G -differential manifold and the elementary 2-group G_2 in G acts on M under θ without any stationary point. Then (M, θ) is a G -boundary. In [4], we initiated this problem for nonabelian groups S_3 and dihedral groups. In [5], we have extended the result of [3] for an arbitrary finite group with center of even order. One needs the elementary 2-group $G_2(C)$ of the center of G instead of G_2 .

In the present note, we consider the action of a compact Lie group and prove that the induced action of the central elementary H -subgroup of G determines G -bordism. This gives the results of [3–5] in particular cases.¹

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2. Preliminaries. Let H be a compact Lie group. If there exists an H -differential closed manifold (N, θ) such that the boundary $\partial N = H$ and the restriction of the action θ to H coincides with the operation in H , then we say that the compact Lie group H is H -boundary. Let G be a compact Lie group. By the *central elementary H -group* in G , we will mean the maximal subgroup $H^n (= H \times \cdots \times H, n \text{ times})$ contained in the center of G .

Consider a compact Lie group G with H^n the central elementary H -group in G , H being H -boundary. Let us fix a point h_0 of H . Let $p_r: H^n \rightarrow H$ denote the projection onto the r th factor, $1 \leq r \leq n$. Let H_r denote the subgroup of H^n

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with $p_i(H_r) = H$, if $i = r$ and $p_i(H_r) = h_0$, if $i \neq r$. Consider a family $\{L_r\}$ of subgroups L_r of G such that $L_r \cap H_r$ is a nontrivial subgroup of H_r , $1 \leq r \leq n$. By an $\{L_r\}$ -type action of G , we mean a differential action of G on a differential manifold such that for every x in M , $p_r(G_x \cap H^n)$ is either trivial or contains $L_r \forall r$, G_x being the isotropy group of x . A point x in M is said to be a *pseudo stationary point* if $p_r(G_x \cap H^n)$ is nontrivial $\forall r$, $1 \leq r \leq n$.

A family \mathcal{F} in G is a collection of subgroups of G such that if $K \in \mathcal{F}$, then all the subgroups of K and all conjugates of K are in \mathcal{F} . Let $\mathcal{F}' \subset \mathcal{F}$ be families in G such that there exists an H -boundary subgroup H of G satisfying the following conditions:

- (a) no nontrivial subgroup of H is contained in $K \forall K \in \mathcal{F} - \mathcal{F}'$,
- (b) the intersection I of all the members of $\mathcal{F} - \mathcal{F}'$ is in $\mathcal{F} - \mathcal{F}'$,
- (c) H is contained in the center.

We call such a pair $(\mathcal{F}, \mathcal{F}')$ of families an *admissible pair* in G with respect to $H \subset G$.

EXAMPLE 2.1. Consider a family $\{L_r\}$ of subgroups of G such that $L_r \cap H_r$ is a nontrivial subgroup of H_r . Let \mathcal{F}_r^L denote the family of all subgroups K of G for which $p_j(K \cap H^n)$ is trivial at least for one $j = 1, \dots, r$ and the nontrivial subgroups $p_j(K \cap H^n)$ of H_j contain L_j . Then $(\mathcal{F}_{r+1}^L, \mathcal{F}_r^L)$ is an admissible pair of families in G with respect to H_{r+1} , $0 \leq r \leq n$, \mathcal{F}_0^L being the empty family.

3. \mathcal{F}_n^L -free action and G -bordism. The object of this section is to show that if (M, θ) is an \mathcal{F}_n^L -free closed G -manifold, then (M, θ) is G -boundary. Let $\mathfrak{N}_*(G; \mathcal{F}, \mathcal{F}')$ denote the $(\mathcal{F}, \mathcal{F}')$ -free G -bordism group for a pair $(\mathcal{F}, \mathcal{F}')$ of families in G . For a given family \mathcal{F} in G and a subgroup K of G , let \mathcal{F}_K denote the smallest family in G containing all the subgroups $[S \cup P]$, $S \in \mathcal{F}$ and P a subgroup of K .

THEOREM 3.1. *If $(\mathcal{F}, \mathcal{F}')$ is an admissible pair of families in G with respect to a subgroup H , which is H -boundary, then the homomorphism*

$$\mathfrak{N}_*(G; \mathcal{F}, \mathcal{F}') \rightarrow \mathfrak{N}_*(G; \mathcal{F}_H, \mathcal{F}'_H)$$

induced by the inclusion map $(\mathcal{F}, \mathcal{F}') \rightarrow (\mathcal{F}_H, \mathcal{F}'_H)$ is the zero homomorphism.

PROOF. Let $[M, \theta]$ be in $\mathfrak{N}_n(G; \mathcal{F}, \mathcal{F}')$. Let F denote the fixed points set of I in M , I being the intersection of all the members of $\mathcal{F} - \mathcal{F}'$. Since $\mathcal{F} - \mathcal{F}'$ is invariant under conjugation, I is normal in G so that the action θ induces an action on F , which we once again denote by θ . Let ν be the normal bundle of the imbedding of F in the interior of M and $D(\nu)$ be the disc bundle with the action θ^* of G on $D(\nu)$ induced by the real vector bundle maps covering the action θ on F . Since F is the fixed points set of I , no nontrivial subgroup of H is contained in $K \forall K \in \mathcal{F} - \mathcal{F}'$, no point of F will be fixed by the subgroup $[I \cup P]$, P being a nontrivial subgroup of H , so that H will act freely on F and hence on $D(\nu)$. Let $F' = F/H$ and $D'(\nu) = D(\nu)/H$. The actions θ and θ^* on F and on $D(\nu)$ induce actions θ' and θ'^* on F' and $D'(\nu)$ respectively, because H is contained in the center. Since H acts freely on F and $D(\nu)$, the quotient maps $\xi_1: F \rightarrow F'$ and $\xi_2: D(\nu) \rightarrow D'(\nu)$ are principal H -bundles. Since H is H -boundary, there exists an H -differential closed manifold (N, \emptyset) such that the boundary $\partial N = H$ and the restriction of the action \emptyset to H coincides with the operation in H . Consider the fibre bundles $\tilde{\xi}_1 = \xi_1[N]$ and $\tilde{\xi}_2 = \xi_2[N]$ associated to the principal H -bundles

ξ_1 and ξ_2 respectively. The total space E_1 of $\tilde{\xi}_1$ is given by $E_1 = (F \times N)/H$, where the action of H on $F \times N$ is given by $h(m, t) = (mh, h^{-1}t)$, $h \in H$ and $(m, t) \in F \times N$. Also the boundary ∂E_1 is diffeomorphic to $(F \times H)/H$. Let us take a fixed point \tilde{h} of H . Define a map $\eta: (F \times H)/H \rightarrow F$ as $\eta([m, h]) = m\tilde{h}$, where $h = \tilde{h}\tilde{h}$. Clearly η is a diffeomorphism. Let us define an action ψ_1 of G on E_1 as $g[m, t] = [mg, t]$. Then the diffeomorphism η preserves the H -action. Thus E_1 is a G -manifold with ∂E_1 being equivariantly diffeomorphic to F . Similarly the total space E_2 of $\tilde{\xi}_2$ is $(D(\nu) \times N)/H$, where the action of H on $D(\nu) \times N$ is given by $h(m, t) = (mh, h^{-1}t)$. Consider the action ψ_2 of G on E_2 as $g[m, t] = [mg, t]$. Let $\alpha: E_2 \rightarrow E_1$ be the map induced from $\nu': D'(\nu) \rightarrow F'$ by going to the fibre bundles; one has the commutative diagram:

$$\begin{array}{ccc} \xi_2(N): & E_2 & \rightarrow & D'(\nu) \\ & \downarrow \alpha & & \downarrow \nu' \\ \xi_1(N): & E_1 & \rightarrow & F' \end{array}$$

Also $\alpha^{-1}(\partial E_1)$ is diffeomorphic to $D(\nu)$ and the action ψ_2 on $\alpha^{-1}(\partial E_1)$ is isomorphic to the action θ^* on $D(\nu)$. Consider

$$W = (M \times [0, 1]) \cup E_2 / \sim$$

where \sim is the equivalence relation in W obtained by identifying $D(\nu) \times \{1\}$ with $\alpha^{-1}(\partial E_1)$. Let the action Φ of G on W be defined by $\Phi|M \times [0, 1] = \theta \times 1$ and $\Phi|E_2 = \psi_2$. Take V to be

$$(\partial M \times [0, 1]) \cup (M \times \{1\} - (D(\nu) \times \{1\})^\circ) \cup (\partial E_2 - (\alpha^{-1}(\partial E_1))^\circ),$$

where $^\circ$ denotes the interior operator. Since I is the intersection of all members of $\mathcal{F} - \mathcal{F}'$, V will be $(\mathcal{F}'_H, \mathcal{F}'_H)$ -free. Also W is $(\mathcal{F}_H, \mathcal{F}'_H)$ -free and ∂W is diffeomorphic to $M \cup V$ identifying ∂V with ∂M . This shows that $[M, \theta]$ is zero in $\mathfrak{N}_*(G; \mathcal{F}_H, \mathcal{F}'_H)$. \square

Let \mathfrak{A} denote the family of all subgroups of G . Following the notations of Example 2.1 and using Theorem 3.1 we get the following

COROLLARY 3.2. *For every $r, 0 \leq r \leq n$, the homomorphism $\mathfrak{N}_*(G; \mathcal{F}_{r+1}^L, \mathcal{F}_r^L) \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_r^L)$ induced from the inclusion map $(\mathcal{F}_{r+1}^L, \mathcal{F}_r^L) \rightarrow (\mathfrak{A}, \mathcal{F}_r^L)$ is the zero one.*

PROOF. Since $(\mathcal{F}_{r+1}^L, \mathcal{F}_r^L)$ is an admissible pair of families with respect to the subgroup H_{r+1} , $0 \leq r < n$, and $(\mathcal{F}_r^L)_{H_{r+1}} = \mathcal{F}_r^L$, Theorem 3.1 gives the corollary. \square

COROLLARY 3.3. *Let M be a closed G -manifold with $\{L_r\}$ -type of action for some family $\{L_r\}$ of subgroups of G such that $L_r \cap H_r$ is nontrivial. If M does not have any pseudo stationary point, then M is a G -boundary.*

PROOF. It is enough to show that the homomorphism $\mathfrak{N}_*(G; \mathcal{F}_n^L) \rightarrow \mathfrak{N}_*(G; \mathfrak{A})$ induced from the inclusion map $\mathcal{F}_n^L \rightarrow \mathfrak{A}$ is the zero one. By Corollary 3.2 and the exact bordism sequence for the triple $(\mathfrak{A}, \mathcal{F}_{r+1}^L, \mathcal{F}_r^L)$, one gets that $j_*: \mathfrak{N}(G; \mathfrak{A}, \mathcal{F}_r^L) \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_{r+1}^L)$ is a monomorphism, $j: (\mathfrak{A}, \mathcal{F}_r^L) \rightarrow (\mathfrak{A}, \mathcal{F}_{r+1}^L)$ is the inclusion map. Therefore the composite

$$\mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_0^L) \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_1^L) \rightarrow \dots \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_n^L)$$

is a monomorphism and hence by the bordism exact sequence of the triple $(\mathfrak{A}, \mathcal{F}_n^L, \mathcal{F}_0^L)$, one gets that $\mathfrak{N}_*(G; \mathcal{F}_n^L, \mathcal{F}_0^L) \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_0^L)$ is the zero homomorphism. This completes the proof, since \mathcal{F}_0^L is empty. \square

REMARK 3.4. Taking G to be a finite abelian group of even order and H to be \mathbf{Z}_2 , one gets Corollary 3.7 of [3]. Considering G to be a finite group with center of even order and H to be \mathbf{Z}_2 , one gets Corollary 3.3 of [5].

REMARK 3.5. The case $H = \mathbf{Z}_2$ can also be obtained in a simpler way using the technique used in the Appendix of [2]. The case $G = \text{finite group}$ and $H = \mathbf{Z}_2$, has been obtained by Kosniowski [6] using the concept of slices.

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