VON NEUMANN REGULAR RINGS
AND
RELATED RINGS

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of the Degree of Master of Philosophy**

TO

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CERTIFICATE

I certify that the dissertation entitled "von Neumann regular rings and related rings " submitted by Ms. M. Ibemhal Devi in partial fulfilment of the requirements for the degree of Master of Philosophy is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

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INTRODUCTION

This dissertation is devoted to a survey of some investigations carried out in certain topics in Ring Theory. We are interested here in von Neumann regular rings and related rings. John von Neumann ([vN:36], [vN]) initiated the study of rings in which for each element a there exists b such that aba = a (called by him as regular rings) in 1936. These rings arise naturally in the study of modular lattices. However , regular rings are interesting objects in their own right and have been studied at great length in the past four decades.

Plenty of evidence can be adduced in support of this assertion. The bibliography of Prof. K.R. Goodearl's book, "Von Neumann Regular Rings" [GII], contains 270 items. Several dozen additional references can be found in Reviews in Ring Theory (See the Bibliography at the end of this dissertation.)

Thus it is impossible to survey more than a small fraction of the field of regular rings in a work of this size. In this dissertation we have restricted ourselves to a survey of some investigations on regular rings, strongly regular rings, weakly regular rings, V-rings, p-V-rings, V'-rings, p-V'-rings and SF-rings which were carried out by Armendariz, Fisher, Ming, Pillay, Ramamurthi, Rangaswamy, Raphael, Steinberg and others.

Each of these classes of rings is 'related' to the class of regular rings in the sense that its defining condiion is equivalent to regularity under the assumption of commutativity. So these classes turn out to be of independent interest only in the context of non-commutativity. Thus our rings are associative but usually non-commutative.

For carrying out a systematic study of these rings some 'general' classes of rings turn out to be of importance. Among these are: abelian rings, reduced rings, symmetric rings, duo rings, quasi-duo rings, quasi-simple rings and non-singular rings. Results about them are scattered through the literature. For the sake of uniformity of terminology, completeness and convenience of reference these rings have been studied in Chapter I. Knowledge of the following topics in Ring and Module Theory is assumed: artinian and noetherian modules and rings, semi-simple modules and rings, injective and projective modules, tensor product and flat modules, rings and modules of fractions. Most of this material can be found in Stenström's book [St.]; we shall usually follow his terminology.

We take up the main theme of the dissertation, a study of regular rings and related rings, in Chapter II. The topics covered there are described in some detail in the introduction to that chapter.

The Bibliography is divided into two parts: (I) books and monographs, and (II) research papers. A book may be cited as [X II] or as [XYZ]; here [X] or [XYZ] will denote the initial letter(s) of the name(s) of the author(s). A memoir may be cited, e.g., as [XYZ:53]; here '53 gives the year in which the paper was published. This will give an approximate idea of when the research was carried out.

CONTENTS.

CHAPTER I

SOME CLASSES OF RINGS

As mentioned in the introduction, this chapter is devoted to a survey of the basic properties of certain classes of rings. We need some of this material for a study of regular rings and other related rings to be carried out in the next chapter.

§1. Some notations and conventions

In this section we shall fix some general notations and conventions. We shall try to adhere to these throughout this dissertation.

1.1. CONVENTION. Unless otherwise mentioned, by a ring we shall mean an associative ring with identity; all modules, ring homomorphisms and subrings will be unitary.

1.2. CONVENTION. Let (P) be a property (of a ring or an element) which applies on the left as well as on the right. We shall say that a ring is (P) (or an element is (P)) if the ring is left or right (P) (the element is left or right (P)).

1.3. CONVENTION. Throughout, R will denote an associative ring with identity. By a module ${}_RM$ we mean a left R-module M and by a module M_R a right R-module M.

1.4. NOTATION. Let A and B be subgroups of an abelian group M. We shall use $A \leq B$ to denote that A is contained in B. If, moreover, M is a left R-module $A \leq_R M$ will mean that A is an R-submodule of M. In particular,

$A \leqslant_R R$ will mean that A is a left ideal of R.

1.5. NOTATION. Let $_RM$ be a module, $K \leqslant R$ and $N \leqslant M$. As usual, KN will denote the subgroup of (M, +) generated by all products of the form kn where k belongs to K and n belongs to N. Thus KN consists of all finite sums of the form $\sum k_i n_i$, $k_i \in K$, $n_i \in N$. We denote KK by K^2. If $K \leqslant R$, $L \leqslant R$ and $a \in R$, KaL, aKL etc. will have their usual meanings.

1.6. ABBREVIATIONS. We shall abbreviate finitely generated as f.g., directly finite as d.f., principal ideal ring as P.I.R.

Further notations and conventions will be introduced when required.

§2. Basic definitions.

Since there are varying definitions in the literature we indicate those to be used in this dissertation.

2.1. DEFINITION. A ring R is called abelian if every idempotent of R belongs to the centre of R.

2.2 DEFINITIONS. By a domain we mean a (possibly) non-commutative ring without divisors of zero. A ring is called a reduced ring if it has no non-zero nilpotent elements. It is called prime if for ideals A,B we have AB = 0 implies A = 0 or B = 0 (Ideal means two-sided ideal). It is called semiprime if it has no non-zero nilpotent ideals. A ring R is called semi-primitive if its Jacobson radical Rad R equals 0 . A ring R is called semi-simple if every left ideal of R is a direct summand of R. A ring R is called indecomposable if it cannot be expressed as $R_1 \times R_2$ where R_1, R_2 are non-zero rings.

2.3 DEFINITION. A ring R is called directly finite if xy = 1 implies yx = 1 for elements x,y of R.

Directly finite rings have also been called von Neumann finite by Herstein [H] and Peterson [P1: 75], Dedekind finite by Faith [FII : p.85], finite by Kaplansky [KI], fini by Renault [R 4: 73].

2.4 DEFINITIONS. A ring R is called left (right) duo if every left (right) ideal of R is an ideal. Following Convention 1.2 a ring is called duo if it is left or right duo.

2.5 DEFINITIONS. A ring R is called left (right) quasi-duo if every maximal left (right) ideal of R is an ideal.

2.6 DEFINITIONS. A ring R is called left symmetric if for a, b, c in R, abc = 0 implies acb = 0. It is called right symmetric if abc = 0 implies bac = 0

2.7 DEFINITIONS. A ring R is called local if R/ Rad R is a division ring; it is called semi-local if R/ Rad R is a semi-simple ring.

We give some easy examples of these concepts in the remarks below.

2.8 REMARK We have, trivially, commutative rings are abelian, directly finite, left and right duo and left and right symmetric.

2.9 REMARK. The only idempotents in a domain are 0 and 1. So domains are abelian. They are reduced, directly finite and left and right symmetric. The last remark applies to products of domains.

2.10 REMARK. Let R be a finite product of division rings. Then R is left and right duo, semisimple and semi-primitive.

2.11 REMARK. Other definitions will be introduced whenever needed (see, especially, sections 8 and 19).

§3. Abelian rings

In three propositions below, we give sufficient conditions (which are also trivially necessary) for an idempotent e to be central. As corollaries we obtain sufficient conditions for a ring to be abelian. These conditions (except in 3.2) are also necessary.

3.1 PROPOSITION. Let $e = e^2 \in R$. If e commutes with all nilpotent elements in a ring R, then e is central.

Proof. For any z in R, $(ez-eze)^2 = 0$. So $e(ez-eze) = (ez-eze)e = 0$ shows $ez = eze$. Similarly $(eze-ze)^2 = 0$ yields $ze = eze$. So $ez = ze$.

3.2 COROLLARY. If R is reduced then R is abelian.

3.3. PROPOSITION. If $eR = Re$, for an idempotent e, then e is central.

Proof. Let $x \in R$. Then $xe = ey$ for some y in R and $ex = ze$ for some z in R.

Therefore $exe = e^2y = ey = xe$ and $exe = ze^2 = ze = ex$ So $ex = exe = xe$ for all x in R. So e is central.

3.4 COROLLARY. If $eR = Re$ for each idempotent e, then R is abelian.

3.5 PROPOSITION. Let $e = e^2 \in R$. If e commutes with all (other) idempotents then e is central.

Proof. Let $n = ez-eze$. Then $n^2 = 0$, $en = n$ and $ne = 0$. Therefore $(e+n)^2 = e^2+en+ne+n^2 = e+n$. So $e+n$ is an idempotent. Therefore (by hypothesis) $e(e+n) = (e+n)e$ yields $n = en = ne = 0$. So $ez = eze$. Similarly we get $eze = ze$. Thus $ez = ze$ for all z in R.

3.6 COROLLARY. If all idempotents in R commute, then R is abelian.

3.7 PROPOSITION. Let R be abelian. Then R is directly finite.

Proof. Let $xy = 1$. Then $(yx)^2 = yxyx = y.1.x = yx$ shows that yx is an idempotent. So, as yx is central, we have, $yx.x = xyx = x$ implying $yx = yx.xy = xy = 1$.

3.8 REMARK. The converse of 3.7 is not valid. For each natural number n, $M_n(K)$ is a directly finite ring (K a field) : $AB = I$ for square matrices A,B implies $BA = I$. However, for $n \geqslant 2$, $M_n(K)$ is not abelian ring.

§4. Directly finite rings (d.f. rings)

See 2.3 for the definition of a d.f. ring.

4.1 REMARK. A subring of a d.f.ring is a d.f.ring.

4.2 PROPOSITION ([FII , p. 85]) The ring R is d.f. if and only if $R/\mathrm{Rad}\ R$ is d.f.

4.3 COROLLARY. Semilocal rings are d.f.

4.4 COROLLARY Local rings are d.f.

4.5. PROPOSITION. [O 1 : 71 Corollary3] Let R be a commutative ring and A an R-algebra which is a finitely generated R-module. Then A is a d.f. ring.

4.6 PROPOSITION. If R is a left noetherian ring then R is a d.f. ring. (This follows from the well known fact that a surjective endomorphism of a noetherian module is necessarily an isomorphism.).

4.7 PROPOSITION (Utumi [U : 65]) If R is left and right selfinjective, then R is a d.f. ring.

4.8 REMARK. As noted in 3.7, abelian rings are d.f. Hence, by 3.2, reduced rings are d.f. So are duo rings (Proposition 7.6 below). We prove below that quasi-duo rings are d.f. (7.9).

4.9 EXAMPLE. Let V be an infinite-dimensional vector space over a field K (say $[V:K] = \aleph_0$) Then in $R = \mathrm{End}_K(V)$ there exist elements f,g such that $gof = 1_V$ and $fog \neq 1_V$. So R is a prime ring which is not directly finite. (See §8 below).

§5. Some conditions on annihilators

5.1 NOTATION. Let S be a subset of a ring R . By its left annihilator $l(S)$ we mean the set $\{ x \in R \mid xa = 0$ for each a in S $\}$. Thus l(S) is a left ideal of R. Similarly the right annihilator S, $r(S) = \{ x \in R \mid ax = 0$ for each a in S $\}$, a right ideal of R. If S is the singleton set $\{a\}$, we write l(a) for l(S) and r(a) for r(S).

5.2 THE CONDITIONS

We shall consider the following conditions on annihilators ideals:

5 (A) For each $a \in R$, $l(a) = r(a)$.

5 (A_l) For each $a \in R$, $l(a) \leq r(a)$.

5 (A_r) For each $a \in R$, $r(a) \leq l(a)$.

5 (B_l) For each $a \in R$, $l(a)$ is an ideal of R.

5 (B_r) For each $a \in R$, $r(a)$ is an ideal of R.

5 (C_l) For each subset S of R, $l(S)$ is an ideal of R.

5 (C_r) For each subset S of R, $r(S)$ is an ideal of R.

We note below the implications that hold between these conditions.

5.3 <u>REMARK.</u> Suppose (A_r) holds and let x be an arbitrary element of R. Suppose $y \in l(x)$. Then $yx = 0$ and so $x \in r(y)$. By (A_r), $r(y) \leq l(y)$. Hence $x \in l(y)$ and so $xy = 0$. Thus $y \in r(x)$. So $l(x) \leq r(x)$ and thus (A_l) holds. By symmetry (A_l) implies (A_r). Thus (A_l), (A_r) and (A) are all equivalent.

5.4 <u>REMARK.</u> Suppose (B_r) holds. Let c be an arbitrary element of R and let $b \in l(c)$. Let $x \in R$. Then $bc = 0$ implies $c \in r(b)$ and so $xc \in r(b)$ (an ideal). So $bxc = 0$ showing that $bx \in l(c)$. Thus $l(c)$ is an ideal. So (B_l) holds. Conversely, (B_l) implies (B_r) and so (B_l) and (B_r) are equivalent. We can thus talk simply of condition (B) (or 5(B))

5.5 <u>REMARK</u> We have $l(S) = \bigcap_{a \in S} l(a)$ and $r(S) = \bigcap_{a \in S} r(a)$.

Therefore (B_l) is equivalent to (C_l) and (B_r) to (C_r). So, by (5.4), (C_l) and (C_r) are equivalent and equivalent to (B). Trivially (A) implies (B).

5.6 REMARK. If condition (B) holds, then R is abelian. For let $e = e^2 \in R$. Then $l(1-e) = Re$ and $r(1-e) = eR$ are ideals and so $Re = ReR = eR$. Hence by 3.4 R is abelian.

In the remarks below we record some properties of annihilators to be used later.

5.7 REMARKS. (I) Let $S \subset T$ be subsets of a ring R. Then, by definition, $l(T) \leq l(S)$ and $r(T) \leq r(S)$. Thus, taking annihilators is an inclusion-reversing operation.

(II) Let S be a subset of R. For each element a of S, we have $l(S) \cdot a = 0$ (by definition of $l(S)$) and so $a \in r\left[l(S)\right]$. So, we have,

$$S \subseteq r\left[l(S)\right] \quad \text{for all subsets } S \text{ of } R \tag{1}$$

Similarly, we get

$$S \subseteq l\left[r(S)\right] \quad \text{for all subset } S \text{ of } R \tag{2}$$

Now let J be a subset of R. Applying (1) to $r(J)$, we get

$$r(J) \leq r\left[l(r(J))\right] \tag{3}$$

Applying (2) to J we get

$$J \subset l\left[r(J)\right] . \tag{4}$$

Now (by Remark (I)) (4) yields

$$r(J) \geq r\left[l(r(J))\right] \tag{5}$$

Thus (3) and (5) imply,

$$r(J) = r\left[l(r(J))\right] . \tag{6}$$

Similarly, we get

$$l(J) = l\left[r\,(l(J))\right] \tag{7}$$

for all subsets J of R.

§6. Reduced and symmetric rings

The concept of a symmetric ring can be easily generalised to that of a symmetric module.

6.1 DEFINITION (Raphael [R 2 : 74]) A right R-module M is called symmetric if for elements r, s in R and m in M $m\,r\,s = 0$ implies $msr = 0$

6.2 **REMARKS.** (I) A ring R is right symmetric if and only if R_R is a symmetric module.

(II) A submodule of a symmetric module is symmetric.

(III) From (I) and (II) we get: if R is a right symmetric ring every right ideal of R is a symmetric module.

Next, we show that symmetry (as is appropriate) a left-right symmetric condition.

6.3 REMARKS. (I) Let R be a right symmetric ring. Then (taking $c = 1$ in the notation of 2.6) $ab = 0$ if and only if $ba = 0$. So we have $l(a) = r(a)$ for each a in R. Thus a right symmetric ring satisfies condition (A) of Section 5.

(III) By taking $a = 1$ in the definition of a left symmetric ring (2.6) we see that a left symmetric ring also satisfies the same condition 5(A).

6.4. PROPOSITION. If R is left symmetric, then R is right symmetric (and conversely).

Proof. Let $abc = 0$. Then as R is left symmetric we have $acb = 0$ i.e. $ac \in l(b)$. By condition 5(A), we have $ac \in r(b)$ i.e. $bac = 0$.

6.5. REMARKS (I) In view of proposition 6.4 and convention 1.2, a symmetric ring will be a left and right symmetric ring.

(II) As seen above (6.3) symmetric rings satisfy 5(A) and 5(B). Hence, by 5.6, they are abelian and so, by 3.7, directly finite.

6.6. PROPOSITION (Lambek (See [S3 : 73])) If R is reduced then R is symmetric.

Proof. We repeatedly use the fact that R is reduced. So let a,b,c in R such that $abc = 0$.

Then $(cab)^2 = cab\ cab = 0$ yields $cab = 0$;

$(abac)^2 = abacabac = 0$ yields $abac = 0$;

$(bacba)^2 = bacbabacba = 0$ yields $bacba = 0$;

$(cba)^2 = cbacba = 0$ yields $cba = 0$ and finally,

$(acb)^2 = acbacb = 0$ implies $acb = 0$

In the results below, we record some more properties of reduced rings.

6.7 PROPOSITION. Let R be a ring. Then R is reduced if and only if whenever a in R is such that $a^2 = 0$ we have $a = 0$

Proof. Only to prove "if" part. Let a in R such that $a^n = 0$. We assert $a = 0$. If not there exists an integer $m \geqslant 2$ such that $a^m = 0$ but $a^{m-1} \neq 0$. Taking $b = a^{m-1}$ we get $b^2 = 0$, $b \neq 0$, a contradiction.

6.8 **REMARK.** If R is a reduced ring, then $l(a) = r(a)$ for each a in R. This result, which follows from Proposition 6.6 and Remarks 6.3 can also be seen directly as follows: $ba = 0$ implies $(ab)^2 = abab = 0$ and hence, as R is reduced, we get $ab = 0$. Thus $l(a) \leqslant r(a)$ for each $a \in R$ implying, by Remark 5.3, $l(a) = r(a)$ for each a in R.

6.9 **PROPOSITION.** Let R be a reduced ring and a an element of R. Let $A = l(a) = r(a)$, an ideal of R. Then $S = R/A$ is a reduced ring and $\bar{a}$ is a nonzero divisor in S.

Proof. Let x be an element of R. Suppose $\bar{x}^2 = 0$. Then $x^2 \in l(a)$ implying $x.xa = 0$. As $l(xa) = r(xa)$ we get $xax = 0$ and so $(xa)^2 = 0$ in R. Hence $xa = 0$ and so $x \in l(a)$. Thus $\bar{x} = 0$. It follows, by Proposition 6.7, that S is a reduced ring.

Next let $\bar{a}\bar{x} = 0$ in S. Then $ax \in l(a)$ implies $axa = 0$ and so $xa = 0$. Thus $x \in l(a)$ and $\bar{x} = 0$. Similarly $\overline{xa} = 0$ implies $\bar{x} = 0$. Thus $\bar{a}$ is a left and right non-zero divisor in S.

Domains are prime, reduced rings. The converse also holds. More generally, we have:

6.10 PROPOSITION. Let R be a prime ring satisfying condition 5(B). Then R is a domain.

Proof. Let $a \neq 0$ in R. Then, as r(a) is an ideal of R, we have

$$RaRr(a) = Rar(a) = R.0 = 0.$$

Since $RaR \neq 0$ we have $r(a) = 0$. Thus R is a domain.

As special cases of this results we have:

6.11. COROLLARY. If R is a prime, symmetric ring then R is a domain.

6.12 COROLLARY. If R is a prime, reduced ring then R is a domain.

(The above result is well-known. See, for example, [St. Ch.XII, Exer. 15], [M2 : 71] [G 2 : 84])

In view of Remark 5.6 it is natural to ask the following

6.13. QUESTION. If R is a prime, abelian ring is it necessary a domain?

§ 7 Duo and quasi-duo rings.

We state below only the "left-sided" versions of a number of results valid on both sides.

7.1 REMARKS. A finite product of left duo rings is left duo. A finite product of left quasi-duo rings is left quasi-duo.

7.2 REMARK. A ring R is left duo if and only if for each element a of R, Ra is an ideal of R. This holds because every left ideal A is a sum of the principal left ideals generated by elements of A.

7.3 REMARK. Trivially left duo rings are left quasi-duo.

7.4 EXAMPLES: (I) The ring of 2 x 2 lower triangular matrices over a division ring is a left quasi-duo ring which is not left duo.

(II) If R is a local ring, Rad R is the unique maximal left ideal of R. It is also an ideal of R. Thus R is a left (and, similarly, right) quasi-duo ring. But local rings may not be left duo as shown in (III) below.

(III) Let K be a field and $h: K \dashrightarrow K$ a field monomorphism which is not onto (eg. $K = L(X)$, the function field in one variable over a field L and h the map which is identity on L and takes X to X^2) Let V be a vector space over K. Then $K \oplus V$ becomes a ring under the definition

$(a,u)\ (b,v) = (ab, h(a)v + bu)$.

Consider the maps $j: K \rightarrow K \oplus V$ given by $a \longmapsto (a,0)$ and the projection $p: K \oplus V \rightarrow K$. They are both ring homomorphisms. We can identify V with the ideal $0 \oplus V$ of R; then $V^2 = 0$ and V is the unique maximal left (right) ideal of R. So R is a local ring and so is left and right quasi-duo. Let $M = h(K)$, a subfield of K. If W is a M-subspace of V which is not a K-subspace, then

$\widetilde{W} = \{(0,w) \mid w \in W\}$ is a left ideal of R which is not an ideal. Thus R will not be left duo.

7.5 <u>REMARK.</u> A result often stated in literature (see e.g. [H :81,p.137]) is that left and right duo rings are abelian. However onesided duoness is a sufficient condition for a ring to the abelian: Suppose R is left duo, then it clearly satisfies condition (B_1) of §5. Therefore, by Remark 5.4, R satisfies condition (B) and so, by Remark 5.6, R is abelian.

So duo rings are abelian and so, by Proposition 3.7, they are directly finite. It is instructive to give a dirdct proof of this fact:

7.6 <u>PROPOSITION.</u> If R is duo, then R is directly finite.

<u>PROOF.</u> Let xy = 1. If R is left duo, then Rx is an ideal and so $1 = xy \in Rx$. So $1 = zx$ for some z in R. Then, as usual, $z = z.1 = zxy = 1.y = y$. So $yx = 1$.

7.7 <u>PROPOSITION.</u> [R3 : 86, Proposition 4.4] Let R be quasi-duo, Then R/Rad R is a subring of a product of division rings.

<u>PROOF:</u> We have Rad R = $\bigcap m_i$, where $\{m_i\}_{i \in I}$ is the family of maximal left ideals of R. The canonical map $R \longrightarrow \prod_{i \in I} R/m_i$ induces an injection

$$g: R/\text{Rad } R \longrightarrow \prod_{i \in I} R/m_i$$

If R is left quasi-duo, m_i are all ideals of R. Hence R/m_i are all division rings and g is a ring monomorphism. This proves the proposition.

7.8 COROLLARY. A semiprimitive quasi-duo ring is reduced.

The following extends Corollary 4.4 and Proposition 7.6.

7.9 PROPOSITION [CDR:88] If R is quasi-duo then R is directly finite.

PROOF: It follows from Proposition 7.7 that R/Rad R (being a subring of a directly finite ring) is directly finite. (This can also be seen as follows: R/Rad R is reduced, so abelian by 3.2 and hence directly finite by 3.7.) Hence by 4.2 R is directly finite.

§8. Complements.

In this section we shall collect some more definitions and results that will be needed later.

8.1. DEFINITION. A ring R is called quasi-simple if R and 0 are the only ideals of R.

8.2. REMARKS. (I) In 8.1 we follow Bourbaki's terminology [B II 8, §5, Exercise 5]. Many authors call such rings simple rings.

(II) Division rings are clearly quasi-simple. More generally, if D is a division ring and n any natural number then $M_n(D)$ is a quasi-simple ring.

(III) Let V be a vector space over a field K. Let $R = \mathrm{End}_K(V)$. If $n = \dim_K V$ is finite, then $R \xrightarrow{\sim} M_n(K)$ and so is quasi-simple by Remark (II). If n is infinite, then $A = \{ f \in R \mid \mathrm{Im}\, f \text{ is finite-dimensional} \}$ is an ideal of R. Since $A \neq R$, $A \neq 0$, R is not quasi-simple.

(IV) In the notation of Remark (III), let $[V:K] = \aleph_0$ $R = \mathrm{End}_K(V)$. Then A is the only ideal of R such that $A \neq 0$, $A \neq R$. Hence, clearly, R is a prime ring (2.2). Moreover, $S = R/A$ is quasi-simple, neither left nor right noetherian.

(V) Examples of quasi-simple domains which are not division rings are given in [J III, p.211] and [FI, pp. 361-362].

(VI) If R is quasi-simple ring, then $C = \text{Centre } R$ can be easily seen to be a field. We shall extend this result to indecomposable, weakly regular rings in 11.7.

Next we record some basic properties of singular and nonsingular modules. We shall refer to standard texts for proofs of some results.

8.3. *DEFINITION.* Let L be a submodule of a module M. Then L is called a *large* (essential) *submodule* of M if for every non-zero submodule N of M we have $L \cap N$ is nonzero. By a *large left ideal* of R we mean a large submodule of $_RR$.

8.4. *PROPOSITION.* Let $f: M_1 \rightarrow M_2$ be a homomorphism of R-modules. If L is large in M_2, then $f^{-1}(L)$ is a large submodule of M_1.

PROOF: See [GI, Proposition 1.1(c)]

8.5. *PROPOSITION.* Let $_RM$ be a module. Define

$$Z(M) = \{ m \in M \mid Lm = 0 \} \text{ for some large left ideal } L \text{ of } R$$

Then Z(M) is a submodule of M.

PROOF: We use 8.4 . See [GI, §ID]

8.6. *DEFINITIONS.* We call Z(M) *the singular submodule* of M. If $Z(M) = 0$, M is called *a non-singular module* ; if $Z(M) = M$, M is called *a singular module.* We call a ring R a *left non-singular ring* if $Z(_RR) = 0$; a *right non-singular ring* if $Z(R_R) = 0$.

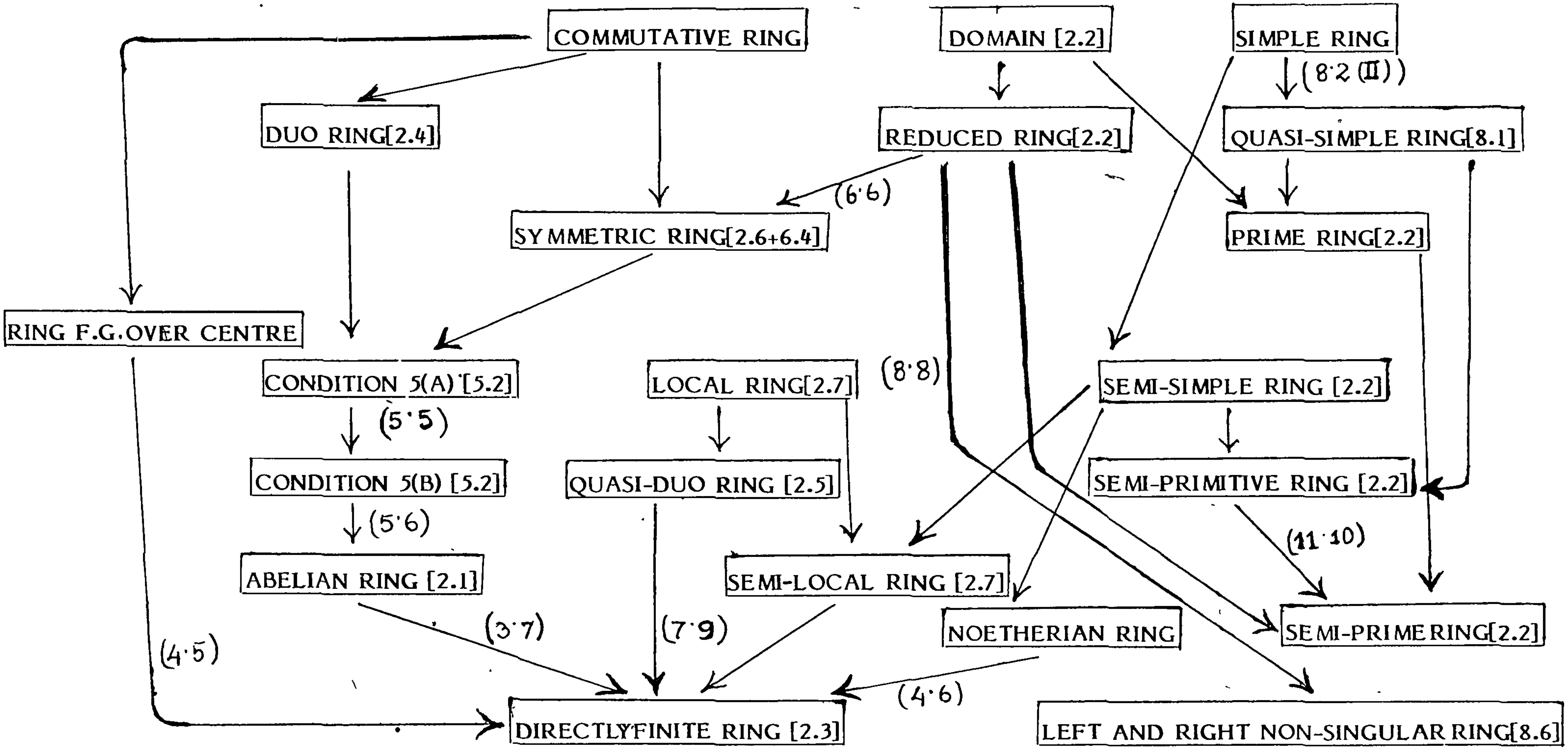
8.7. *REMARK.* Suppose $x \in Z(_RR)$. Then $Lx = 0$ for some large left ideal L. Hence $Lxr = 0$ implying $xr \in Z(_RR)$ for each $r \in R$. Thus $Z(_RR)$ is an ideal of R. However, there exists ring R for which $Z({}_RR) \neq Z(R_R)$. (See [GI, §1D, Exercise 1])

8.8 EXAMPLE. Let R be a reduced ring and let $x \in Z(R_R)$. Let $xL = 0$ for a large right ideal L of R. If $y \in L \cap xR$, then $y = xz \in L$. Hence $0 = xy = x^2z$ implying that $x^2 \in l(z)$. By 6.9, $x \in l(z)$ and so $y = 0$. Thus $L \cap xR = 0$ implying that $xR = 0$. Thus $x = 0$. Hence R is right (and left) non-singular.

8.9. REMARK. Let L be a large submodule of the left R-module M. For $m \in M$, the map $f:R \longrightarrow M$, defined by $f(r) = rm$ is R-linear. Hence by 8.4 $f^{-1}(L) = I$ is a large left ideal of R. Clearly, $Im \leq L$ showing that $I\bar{m} = 0$ in M/L.

Thus M/L is a singular left R-module. In particular, if L is a large left ideal of R, then R/L is a singular left R-module.

CHART FOR CHAPTER I



Note: (1) The reference in square brackets is to the paragraph where the concept is defined.
(2) The reference in round brackets is the paragraph where the implication is stated/proved.

CHAPTER II

REGULAR RINGS AND RELATED RINGS

IN THIS CHAPTER, we take up the main theme of the dissertation, namely, a study of regular rings. We shall also define and study the following related classes of rings: strongly regular rings, weakly regular rings, V-rings, p-V-rings, V'-rings, p-V'-rings, SF-rings. Each of these classes is 'related' to the class of regular rings in the sense that its defining condition is equivalent to regularity under the assumption of commutativity. (See 10.2, 11.4(II), 14.21, 15.4, 18.8 & 18.15). Hence a recurrent question will be: what are the other assumptions under which a particular defining condition is equivalent to regularity. This has been answered, for example, in 10.3, 11.20, 14.20, 15.9, 15.12 & 16.4.

It is well-known that a ring is semi-simple if and only if every module is projective if and only if every module is injective. The concept of injectivity generalises naturally to that of p-injectivity. (See §12 for definition.) It is known that a ring is regular if and only if every (one-sided) module is flat. (See 13.1) It can be shown that R is regular if and only if every (left) R-module is p-injective if and only if every right R-module is p-injective. (Theorem 12.13). These facts lead to a general study of the relationships between injectivity, p-injectivity and flatness. This was carried out by V. S. Ramamurthi in [R1:75] and is reproduced in §13 below. Left V-rings and regular rings are both contained in the class of left p-V-rings; these rings and V-rings are studied in §14. Rings over which simple left modules are flat, called left SF-rings, are studied in §15. In §16 we prove an "equivalence" theorem. In the last four sections we study central localisations, V'-rings, Kaplansky's question on the primitivity of prime, regular rings and polynomial rings over regular rings.

§9. Basic properties of regular rings

This section is devoted to the basic examples and properties of regular rings. For the proofs of many statements we shall refer to standard textbooks and monographs. In sections 11 and 14 a few properties of regular rings will be obtained in the context of larger classes of rings. Some properties will be obtained in element-wise situations

9.1 DEFINITIONS. Let R be a ring and a an element of R. An element b of R is called a 1-inverse of a if $aba = a$, and a is called a regular element (an R-regular element in case of ambiguity) if it has a 1-inverse in R. (Note that a number of authors use the term regular element for a non-zero-divisor. (See, e.g., [St: p.52].) Some others use this term for an invertible element).

9.2. DEFINITION. A ring R is called (von Neumann)regular if each element of R is regular.

9.3. EXAMPLES. (I) Division rings clearly are regular.

(II) Let $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_i$ is regular if and only if each R_i is regular. Hence it follows from (I) that if D_i are division rings the product $\prod_{i \in I} D_i$ is a regular ring.

(III) Let M_R be a semi-simple module. Then its endomorphism ring $S = \mathrm{End}_R(M)$ is a regular ring. (See [St., p.41] , [AF , Exercise 15.13])

(IV) We have, more generally ([St., Exercise I.50]) $\mathrm{End}_R(M)$ is regular if and only if for each $f \in \mathrm{End}_R(M)$ the submodules $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are direct summands of M.

(V) If R is a semi-simple ring, then it is regular.

(VI) Let V_D be a vector space over a division ring D. Then $S = \mathrm{End}_D(V)$ is a regular ring. ((V) and (VI) follow from (III)).

(VII) Let E_R be a non-zero injective module (see 9.7 for references for injectivity) and $S = \mathrm{End}_R(E)$. Then S/Rad S is a regular ring. (This extends to quasi-injective modules. See [L, §4.4, Proposition 1 and Exercises 7 and 8] or [GII, Theorem 1.22].)

Many more examples of regular rings are given in standard text books, for example [GII] and [KII] .

9.4. PROPOSITION. Let R be a ring and A an ideal of R.
(I). If a is a regular element of R, then $\bar{a}$ is a regular element of R/A.

(II). If R is a regular ring, then so is the ring R/A.

Proof. Let b be an element of R such that aba = a. Then $\bar{a} = \overline{aba} = \bar{a}\bar{b}\bar{a}$ in R/A. This proves (I). Clearly, (II) follows from (I).

9.5 REMARKS (I). Let a be a regular element of a ring R. Let aba = a. Then e=ab, f=ba are idempotents and we have $Ra = Raba \leq Rf \leq Ra$ implying Ra = Rf and similarly aR = eR.

(II). It follows from (I) that if a is a regular element of an abelian ring R, then Ra and aR are ideals of R.

9.6. PROPOSITION. The following are equivalent on a ring R

(I) R is a regular ring.

(II), (II)' Every principal left(right) ideal is generated by an idempotent (equivalently, is a direct summand)

(III), (III)' Every f.g. left (right) ideal is generated by an idempotent (equivalently, is a direct summand).

PROOF: (I) $\rightarrow$ (II) and (I) $\rightarrow$ (II)' follow from Remark 9.5(I). (II) $\rightarrow$ (I). Let $Ra = Re$, $e = e^2$. Let $e = ba$, $a = xe$. Then $aba = ae = xe^2 = xe = a$, shows that a is a regular element. The proof of (II)' $\rightarrow$ (I) is similar. For (I) $\leftrightarrow$ (III) (and (I) $\leftrightarrow$ (III)') see [GII, Theorem 1.1] or [L, §3.5] or [R, Theorem 4.12].

9.7 REMARKS. We shall assume known the basic properties of projective and injective modules (see, for example, [St] , [AF] or [R] .) A ring will be called left hereditary if every left ideal is projective, left semi-hereditary if every f.g. left ideal is projective and a left p.p. ring if every principal left ideal is projective. It follows from 9.6 that regular rings are left (and right) semi-hereditary and left (and right) p.p. Let I be an infinite set and for each $i \in I$ let K_i be a field. Then, by Example 9.3 (II) , $R = \prod K_i$ is a regular ring, which is not semi-simple since the ideal $A = \bigoplus K_i$ is not a direct summand of R. Hence by a theorem of Osofsky [O2: 64] , there exists a cyclic R-module which is not injective. It is known (See either [S1 : 68, Lemma 7.2] (for a proof using a lemma of Snapper [S 5: 65]) or [SV:74 , 5.2]) that R is a self-injective ring. Thus the injective R-module R has non-injective quotients. Hence by ([R, Theorem 4.10]) R cannot be a hereditary ring. Thus regular rings need not be left or right hereditary.

Following Chase [C2 : 60] a ring is called <u>left coherent</u> ([<u>St;</u> p.43], [AF, §19] or [FI, 11.34]) if every finitely generated left ideal is finitely presented. Left noetherian rings are left coherent. A finitely generated projective module can be easily seen to be finitely presented. It follows that left semihereditary rings are left coherent. Therefore regular rings are left and right coherent. It follows by [St. , Proposition 13.3] or [FI, Chapter 11, Exercise 2] that if Ra and Rb are principal left ideals in a regular ring then $Ra \cap Rb$ is principal. A direct proof of this result can be found in [Sk., §2, Theorem 1] .

9.8. PROPOSITION. Suppose that 0 and 1 are the only idempotents of a ring R.

(I). If a is a non-zero regular element of R , then a is invertible.

(II). If R is a regular ring, then R is a division ring.

Proof. Let a = aba for an element b of R. Let e = ab.

Then $e^2 = e$ and $ea = aba = a \neq 0$ shows that $e \neq 0$. Hence, by hypothesis, $ab = e = 1$. Similarly, $ba = 1$. This proves (I). Clearly, (II) follows from (I).

9.9 COROLLARY. Let R be a regular, local ring. Then R is a division ring.

9.10. COROLLARY. Let R be a regular domain. Then R is a division ring.

9.11. PROPOSITION. Let R be a regular, left noetherian ring Then R is a semi-simple ring. (This gives a partial converse of 9.3(V)).

PROOF. Let A be a left ideal of R. As R is left noetherian, A is f.g. Hence, by 9.6, $A = Re$, for some $e = e^2$. Hence A is a direct summand of R, and so R is a semi-simple ring.

In the next two results, we assume a knowledge of the basic properties of localisations. This can be found for example, in [AM, Chapter 3] or [JII, Chapter 7] .

9.12. PROPOSITION. Let R be a ring and S a multiplicatively closed subset of the centre of R.

(I). If a is a regular element of R, then a/s is a regular element of $S^{-1}R$ for each $s \in S$.

(II). If R is regular, then so is $S^{-1}R$.

PROOF. Let $a = aba$ for some $b \in R$. As s is central, we have,

$$a/s \,.\, sb/1 \,.\, a/s = aba/s = a/s \text{ in } S^{-1}R.$$

This proves (I); (II) follows from (I).

9.13. PROPOSITION. Let R be a commutative ring. Then R is regular if and only if for each prime ideal p of R the localisation R_p is a field.

PROOF. By 9.12(II), R_p is a regular ring. Hence (being a commutative, regular, local ring) R_p is a field, by 9.9. Let $a \in R$. Now if $M = Ra/Ra^2$, then for each prime ideal p of R, $M_p = (R_p . a/1) / R_p (a/1)^2 = 0$ as R_p is a field. Hence $M = 0$. Thus $Ra = Ra^2$ and so $a = a^2b$ for some b in R. So R is a regular ring.

Central localisations will be studied further in section 17 where the ideas in 9.12 and 9.13 will be extended.

9.14. PROPOSITION. In a commutative regular ring R every prime ideal is maximal (i.e. R is zero-dimensional).

PROOF. Let p be a prime ideal of R. Then R/p is a commutative, regular, domain and so a field. So p is a maximal ideal.

§10. Strongly regular rings.

In this section we study regular rings which are also abelian. It will be shown that this condition is also equivalent to regular and reduced, to regular and duo and to a number of other conditions. (See Theorem 10.8 below).

10.1. DEFINITIONS. Let R be a ring and a an element of R. We say that a is left (respectively, right) strongly regular if there exists an element b of R such that $a = a^2b$ (respectively, $a = ba^2$). We call a ring R left (respectively, right) strongly regular if each element of R is left (respectively, right) strongly regular.

10.2. EXAMPLES. If R is a commutative ring then an element a is regular if and only if it is strongly regular. Division rings are strongly regular.

10.3. REMARK. In 10.5 below, we shall show that a ring R is left strongly regular if and only if it is right strongly regular. Thus this condition is left-right symmetric.

10.4. REMARK. If R is left strongly regular, then R is reduced, symmetric, abelian and d.f. We also have $l(a) = r(a)$ for each a in R.

PROOF. Suppose $a \in R$ such that $a^2 = 0$. Then as there exists $b \in R$ such that $a = a^2b$, we have $a = a^2b = 0.\ b = 0$ So, by Proposition 6.7, R is reduced. So R is symmetric(6.6) abelian (6.5(II)) and d.f. (5.7). The statement about annihilators follows from 6.8.

10.5. PROPOSITION. Suppose R is left strongly regular. Then R is regular and right strongly regular.

<u>PROOF.</u> Let $x \in R$ and $x = x^2y$. Hence $x\,(1-xy) = 0$ implying, by 10.4 $(1-xy)\,x = 0$. So $x = xyx$ showing that R is regular. Again, by 10.5, $x(1-yx) = 0$ yields $(1-yx)x = 0$ i.e. $x = yx^2$. Thus R is right strongly regular.

10.6 PROPOSITION. Let R be strongly regular. Then R is left (and right) duo.

<u>PROOF.</u> By Remarks 10.4. and 9.5(II) Ra is an ideal for each element a of R. Hence, by Remark 7.2, R is left duo. By symmetry (Proposition 10.5) R is right duo.

10.7. <u>REMARK.</u> It is easily seen that a factor ring of a strongly regular ring is strongly regular.

10.8. <u>THEOREM.</u> Let R be a regular ring. Then the following are equivalent.

(1) R is strongly regular

(2) R is reduced.

(3) R is abelian

(4) Every principal right ideal of R is generated by a central idempotent.

(5) Every right R-module is symmetric

(6) Every cyclic right R-module is symmetric

(7) R is a symmetric ring

(8) R is right duo

(9) R satisfies Condition 5(A) (See Remark 5.3)

(10) R satisfies Condition 5(B) (see Remark 5.4)

(Of course, the left-sided counterparts of (4), (5), (6) and (8) are also equivalent to (1))

PROOF. Note that the implications (2) $\longrightarrow$ (7) (6.6), (7) $\longrightarrow$ (9) (6.3), (9) $\longrightarrow$ (10), (10) $\longrightarrow$ (3) hold generally, that is without any regularity assumption on the ring R. (1) $\longrightarrow$ (2) is Remark 10.4. Assume (3) and let $a = aba$ Then $ab = e$ is central implies $a = ae = a^2b$. So (1) follows. Also we have, (1) $\longrightarrow$ (8) (10.5) and (8) $\longrightarrow$ (10) trivially. Hence (1), (2), (3), (7), (8), (9) and (10) are all equivalent. (3) $\longrightarrow$ (4). Let $a \in R$. Then $aR = eR$, where $e = e^2 \in$ Centre R, as R is regular and abelian. (4) $\longrightarrow$ (1). Let $a \in R$; let $aR = eR$, where $e \in$ Centre R. Then $e = ab$, $a = ea'$ gives $a = e^2a' = ea = ae = a^2b$.

Next we prove the equivalence (due to Raphael [R 2: 74]) of (1), (5), (6) and (7). The implications (5) $\longrightarrow$ (6) and (6) $\longrightarrow$ (7) are trivial and (1) $\longleftrightarrow$ (7) was seen above. Only to prove (1) $\longrightarrow$ (5): Let $m \in M$, r, $s \in R$ such that $mrs = 0$. Let I denote the annihilator of m. As R is right duo, I is an ideal of R. Now $rs \in I$ implies $\bar{r}.\ \bar{s} = 0$ in R/I, which is a strongly regular ring, by Remark 10.7. Hence, by 10.4, $\bar{s}.\bar{r}. = 0$, i.e. $sr \in I$. This implies msr = 0, showing that M is a symmetric R-module Thus (1), (4), (5), (6) and (7) are equivalent. This completes the proof of the theorem.

10.9. REMARK. We note the following extension of the implication (2) $\longrightarrow$ (8) due to Birkenmeier [B1: 83, Remark before Corollary 14]. Let A be a right (or left) ideal of a regular ring R. If A contains no non-zero nilpotent elements then A is an ideal.

10.10. LEMMA. Let a be a regular element of a ring R. Suppose aba = a and the idempotents e = ab and f = ba are both central. Then e = f.

PROOF. We have ae = ea = a and so

$$e = abab = afb = fab = fe = bae = ba = f.$$

The following result can be deduced from Lemma 10.10.

10.11. PROPOSITION (Raphael [R 2: 74]) The following are equivalent on a ring R.

(i) R is strongly regular

(ii) R is regular and if $a,b \in R$ with $a = aba$ then $ab = ba$.

10.12. TERMINOLOGY. We shall use the following terminology. By a <u>2-inverse</u> of an element a in a ring R we shall mean an element b such that b = bab. By a <u>1-2-inverse</u> of a we mean an element which is both a 1-inverse and a 2-inverse of a.

10.13. REMARK. If a has a 1-inverse, then it has a 1-2-inverse. Let $a = aba$. Then with $b_1 = bab$ we have $ab_1 a = a$ and $b_1ab_1 = bababab = bab = b_1$.

The following result is due to Savage [S4 : 80], who studied generalised inverses in regular rings.

10.14. PROPOSITION. Suppose b_1 and b_2 are two 1-2-inverses of an element a. Suppose ab_1 and b_1a are both central elemnts. Then $b_1 = b_2$.

PROOF. By Lemma 10.10, $ab_1 = b_1a = e_1$, say Let $e_2 = ab_2$ Then $e_1 = ab_1 = ab_2ab_1 = e_2e_1 = e_1e_2 = ab_1\ ab_2 = ab_2 = e_2$. Finally, $b_1 = b_1ab_1 = b_1\ e_2 = b_1ab_2 = e_1b_2 = e_2b_2 = b_2e_2 = b_2ab_2 = b_2$.

10.15. PROPOSITION. [St; Chapter 1, Exercise 47 (ii)] In a strongly regular ring every element has a unique 1-2-inverse

PROOF. Existence follows from 10.12 and uniqueness from 10.14.

§11. Weakly regular rings

In this section we shall study rings in which every left ideal is idempotent (and also the right-sided counterpart of this class). This class contains all regular rings and all quasi-simple rings.

11.1. DEFINITION. An element $a \in R$ is called a left (respectively, right) weakly regular element if $Ra = (Ra)^2$ (respectively $aR = (aR)^2$)

11.2. REMARKS: (1) An element $a \in R$ is <u>left weakly regular</u> if and only if $a \in (Ra)^2$.

(2) If a is a regular element (9.1) then a is left as well as right weakly regular.

(3) If $a \in$ Centre R, and a is weakly regular then $Ra = Ra\,Ra = aRa$ shows that a is regular.

(4) Let a be right strongly regular, so that $a = ba^2$ for some $b \in R$. Then $a = b.a.1.a \in RaRa$ show that a is <u>left</u> weakly regular. We shall show in Proposition 11.19(I) that the converse holds if Ra is an ideal of R.

11.3. DEFINITION. A ring R is <u>left weakly regular</u> if each element a of R is left weakly regular, i.e. $Ra = (Ra)^2$ for each $a \in R$. (See Proposition 11.9(I) below).

11.4 REMARKS. (I) By Remark 11.2(2) regular rings are left as well as right weakly regular.

(II) By Remark 11.2(3), commutative weakly regular rings are regular.

(III) Let R be a quasi-simple ring (8.1). Then for each non-zero element a of R, the nonzero ideal RaR coincides with R. Hence $RaRa = Ra$ and $aRaR = aR$. Thus R is left (and right) weakly regular.

(IV) Suppose R is left weakly regular i.e. <u>every</u> principal left ideal is idempotent. Then each left ideal A of R is also idempotent: for if $a \in A$, then $a \in Ra = (Ra)^2 \leq A^2$, so $A \leq A^2$; the reverse inclusion always holds, thus $A^2 = A$. Because of this left weakly regular rings have also been called as <u>left fully idempotent</u> rings in the literature.

In the following propositions (11.5, 11.9, 11.10, 11.11) we obtain some basic properties of weakly regular rings. In view of Remark 11.4(I) these are properties of regular rings as well.

11.5. PROPOSITION. (I). If x is a central regular element of a ring R and C = Centre R then x is C-regular (i.e. regular as an element of C).

(II) The centre of a weakly regular ring is regular.

<u>PROOF.</u> (I). Let $y \in R$ such that $x = x^2y = xyx$. Let $e = xy = yx$ Then $e^2 = e$, and $ex = x$

<u>Step 1.</u> $e \in C$: As $x \in C$, we have $ez = xyz = yzx = yzxyx = yxzyx = eze$ and so $ze = zyx = xzy = xyxzy = xyzxy = eze = ez$ for each $z \in R$. So $e \in C$.

<u>Step 2.</u> $ey \in C$: Next, we have, for $z \in R$, $eyz = yze = yzyx$ and $zey = ezy = yxzy = yzyx$, showing that $ey \in C$.

Finally, we have $x.ey.x = exyx = ex = x$ showing that $ey \in C$ is a 1-inverse of x.

(II) By 11.2(III) a central, left weakly regular element is regular. Hence this follows from (I).

11.6 REMARK. That the centre of a regular ring is regular is a classical result due to John von Neumann([**vN**: 36])

11.7. COROLLARY. Let R be an indecomposable, weakly regular ring. Then C = Centre R is a field.

<u>PROOF.</u> Since R is indecomposable 0 and 1 are the only idempotents of C, which is a regular ring by 11.5(II). Hence C is a field by 9.8 (II).

11.8. <u>COROLLARY.</u> The centre of a prime, regular ring is a field.

11.9. PROPOSITION. (I) Let x be a (left) weakly regular element. of a ring R such that $x \in \mathrm{Rad}\ R$. Then $x = 0$.

(II) A weakly regular ring is semi-primitive.

<u>PROOF.</u> Since $RxRx = Rx$, there exists $z \in RxR \subset \mathrm{Rad}\ R$ such that $x = zx$. Then, $(1-z)\ x = 0$ implying, as $1-z$ is a unit, that $x = 0$. This proves (I) . Clearly, (II) follows from (I) and its right-sided analogue.

11.10. <u>PROPOSITION.</u> A weakly regular ring is semi-prime.

<u>PROOF.</u> This follows from Remark 11.4 (IV): if every left ideal is idempotent then no nonzero ideal can be nilpotent. This also follows from 11.9 since semi-primitive rings are always semi-prime. (See, eg., [FI, 7.32C]).

Refer to §8 for basic results on nonsingular rings.

11.11 PROPOSITION. A left weakly regular ring R is right nonsingular.

PROOF. Let $a \in Z(R_R)$. Then there exists $z \in RaR$ such that $za = a$. As $Z(R_R)$ is an ideal we have $z \in Z(R_R)$. Hence $zL = 0$ for some large right ideal L of R Let $l \in L \cap aR$, then $l = at$ for some $t \in R$. Hence $l = at = zat = zl = 0$ implying $L \cap aR = 0$. As L is large we have $a = 0$, showing $Z(R_R) = 0$.

11.12 COROLLARY. A regular ring is left and right non-singular.

In 11.14, 11.21 and 11.22 below we give sufficient conditions for the equivalence of left and right weak regularity for a ring.

11.13. PROPOSITION. Suppose a is a left weakly regular element in a ring R satisfying the condition $l(a) \leq r(a)$. Then a is right weakly regular.

PROOF. By hypothesis, there exists $b \in RaR$ such that $a = ba$ i.e. $(1-b)a = 0$.
Now $l(a) \leq r(a)$ implies $a(1-b) = 0$ i.e., $a = ab \in aRaR$. So a is right weakly regular.

11.14. COROLLARY. [cf. [R1:73, Corollary 11]). Let R be a ring satisfying the condition that $l(a) = r(a)$ for each $a \in R$ (eg. a reduced ring (Remark 6.8)). Then R is left weakly regular if and only if R is right weakly regular.

The following proposition is due to Ramamurthi [R1: 73] Its proof is routine.

11.15. PROPOSITION. The following are equivalent for a ring R.

(1) R is left weakly regular.

(2) If I is a left ideal and K an ideal of R, then $I \cap K = KI$

(3) If I and J are left ideals of R, then $I \cap JR = JI$.

We shall follow the definitions of left zero divisor and left invertible in [AF, p.11]

11.16. PROPOSITION. Let R be a left weakly regular ring and x an element of R such that the ideal RxR does not equal R. Then x is a right zero divisor.

PROOF. Let $y \in RxR$ such that $x = yx$. Then $y \neq 1$ implies $1-y \neq 0$ and so $(1-y)x = 0$ shows that x is a right zero divisor.

11.17. COROLLARY. Let R be a domain. Then R is weakly regular if and only if R is quasi-simple.

11.18. COROLLARY. Let R be a right weakly regular ring. Then R is reduced if and only if R is a subdirect product of quasi-simple domains.

PROOF. See [R1 : 73]

Corollary 11.20 to the next proposition extends (8) $\longrightarrow$ (1) of Theorem 10.8.

11.19. PROPOSITION. Let a be a left weakly regular element of a ring R.

(I) If Ra is an ideal, then a is right strongly regular

(II) If aR is an ideal, then a is left strongly regular.

PROOF. (I). We have $aR \leqslant Ra$. Hence $a \in RaRa \leqslant RRaa = Ra^2$.

(II). We have $Ra \leqslant aR$. Hence $a \in RaRa \leqslant aRRa = aRa \leqslant aaR = a^2R$.

(See 1.4 for the notation $\leqslant$)

11.20 COROLLARY. A weakly regular, duo ring is strongly regular.

11.21. COROLLARY. Let R be duo. Then R is left weakly regular if and only if R is right weakly regular.

In § 16 we shall prove that quasi-duo, weakly regular rings are strongly regular, extending Corollary 11.20. Here we prove a preliminary result extending 11.21.

11.22. PROPOSITION. Let R be quasi-duo. Then R is left weakly regular if and only if it is right weakly regular.

PROOF: Assume, without loss of generality, that R is left quasi-duo. By 11.9, being a weakly regular ring, R is semi-primitive. Thus by 7.9, in either case, R is reduced. So, by 11.14, the equivalence of left and right weak regularity holds for R.

11.23. PROPOSITION: Let B be an idempotent left ideal of R and $g: B \longrightarrow R$ an R-epimorphism. Then BR = R. (Also, if ${}_RN$ is a module and $g:B \longrightarrow N$ R-epic then $BN = N$).

PROOF. As $B = B^2$ the left R-linearity of g yields $R = g(B) = g(B^2) = Bg(B) = BR$. (Similarly, we have, $N = BN$).

11.24. COROLLARY. Let x be a left weakly regular element of R such that RxR does not equal R. Then the left ideal Rx cannot be isomorphic to R.

11.25. REMARK. In [FI, Proposition 7.34] and [F: 72, Proposition 29] Faith has given longer proof of 11.24 (essentially) for a smaller class of rings, namely for left V-rings (see §14) and used it to determine the structure of left V-rings with certain additional conditions. Notice also that 11.16 can be deduced from 11.24.

11.26 NOTES. The concepts of a strongly regular element (10,1) and a weakly regular element (11.1) date back to at least 1950. They were apparently introduced by Bailey Brown and N.H. McCoy in [BM:50, §6]. However, they do not appear to have been studied later, the emphasis being on the study of strongly regular rings and weakly regular rings. The latter (defined as in 11.3, but without the assumption of an identity element) were studied in Ramamurthi's dissertation submitted to the Madurai University in 1970. His paper [R1:73] reproduces portions of this dissertation and contains our 11.5(II), 11.9(II), 11.11, 11.14(essentially) and 11.15-18. However the "element-wise" approach taken in 11.2(4), 11.5(I), 11.9(I), 11.13, 11.19 and 11.24 appears to be novel.

§12. On f-injective and p-injective modules and rings.

In this section we shall introduce two conditions weaker than injectivity - f-injectivity and p-injectivity - and characterise regular rings in terms of them.

Recall the following well-known criterion due to Baer.

12.1. THEOREM. A left R-module E is injective if and only if for every left ideal I of R every R-linear map $f:I \longrightarrow E$ can be extended to a R-linear map $g:R \longrightarrow E$.

PROOF. See [B:40] or [St., Chapter I, Proposition 5.4] or [L, §4.2, Lemma 1] or [AF, 18.3] or [R, Theorem 3.19] or [CE, Theorem 3.2 on page 8] □

We define f-injective and p-injective modules by weakening the condition in Baer's criterion. We shall state these definitions using the classical terminology of I-completeness due to Baer.

12.2. DEFINITION. Let I be a left ideal of R. A left R-module E is called I-complete of every R-homomorphism $f:I \longrightarrow E$ can be extended to a R-homomorphism $g: R \longrightarrow E$.

12.3. DEFINITIONS. A left R-module E is called f-injective (respectively, p-injective) if for every finitely generated (respectively, principal) left ideal I of R the module E is I-complete. A ring R will be called a left f-injective (respectively, left p-injective) ring if the module ${}_RR$ is f-injective (respectively, p-injective).

12.4. REMARK. Suppose the left ideal I is a direct summond of R. Then every left R-module E is I-complete; if $f:I\to E$ is R-linear and $I \oplus J = R$, then each $x \in R$ can be uniquely written as $x = i+j$ with $i \in I$ and $j \in J$, and the map $g: R\to E$ defined by $g(x) = f(i)$ extends f.

12,5 REMARK. By definition, every injective module is f-injective and every f-injective module is p-injective.

12.6. REMARK. V.S. Ramamurthi and K.M. Rangaswamy call a module E finitely injective if for every R-module M and every f.g. R-submodule N of M every R-homomorphism $f:N\to E$ can be extended to a R-homomorphism $g:M\to E$. Clearly, finitely injective implies f-injective in the above sense.

12.7. REMARK. Over a left noetherian ring f-injective modules are injective. Over a left principolideal ring p-injective modules are injective. Baer's criterion (12.1) yields these assertions.

12.8. REMARK. Direct products and direct factors (summands) of f-injective (respectively, p-injective) modules are f-injective (respectively, p-injective). This follows in the same way as the corresponding results for injective modules. (See [St. Lemma 6.4 of Chapter I] or [AF, Proposition 18.2] or [R, Theorem 3.14] or [L, Section 4.2, Proposition 2])

12.9. REMARK. By a result of H. Bass and Z. Papp R is left noetherian if and only if every direct sum of injective left R-modules is injective. ([R, Theorem 4.27], [AF, Proposition 18.13]). However, a direct sum of left f-injective (respectively, left p-injective) modules is left f-injective (respectively, left p-injective). This holds because when I is a f.g. left ideal of a ring R, the image of a R-homomorphism f: $I \longrightarrow \prod M_i$ is actually contained in a direct sum of finitely many M_i' s.

12.10 REMARK. Colby [C3 : 75] calls an f-injective module as $\mathcal{S}_0$ injective, Matlis [M4 : 85] as semi-injective. Damiano [D : 79] has defined a property called coflatness, which is more general than flatness. He has shown that a module is coflat if and only if it is f-injective.

12.11 REMARK. Recall that a ring R is called left self-injective if the module ${}_R R$ is injective. If S is principal ideal domain and A a non-zero ideal of S, then S/A is a self-injective ring [R, Theorem 4.28]. Left self-injective rings are clearly left f-injective and left p-injective.

The following theorem (12.13) was essentially proved by Ikeda and Nakayama [IN: 54] and reproved by Ming [M1: 74]

12.12. LEMMA. Let I be a left ideal of R. If I is principal and p-injective (or f.g. and f-injective) then I is a direct summand of R.

PROOF. Consider the identity map $1_I : I \longrightarrow I$. By hypothesis, there exists $g : R \longrightarrow I$ extending 1_I. So I is a direct summand of R.

12.13. THEOREM. The following conditions are equivalent for a ring R (of course, every left-sided statement has a right-sided counterpart).

(1) R is regular.

(2) (respectively(2′)) Every left R-module is f-injective (respectively, p-injective)

(3) ((3′)) Every cyclic left R-module is f-injective (p-injective)

(4) ((4′)) Every left ideal is f-injective(p-injective)

(5) ((5′)) Every principal left ideal is f-injective (p-injective)

PROOF. We shall prove the implications as follows:

$$\begin{tikzcd}
 & & (1) \arrow[dl] & \\
 & (2) \arrow[rr] \arrow[dl] \arrow[dd] & & (2') \arrow[dl] \arrow[dd] \\
(3) \arrow[rr] \arrow[dd] & & (3') \arrow[dd] & \\
 & (4) \arrow[rr] \arrow[dl] & & (4') \arrow[dl] \\
(5) \arrow[rr] & & (5') \arrow[uuuu, bend right] &
\end{tikzcd}$$

PROOF. (1) → (2). Let I be a f.g. left ideal of R. Then I is a direct summand of R, by 9.6. Hence every left R-module M is I-complete by 12.4. So M is f-injective. $(5)' \to (1)$ is the only other non-trivial implication. This follows by Lemma 12.12 and Proposition 9.6.

12.14. COROLLARY. If R is a regular ring then R is left and right f-injective(and p-injective).

12.15 REMARK. Recall that a module M is called quasi-injective if for every submodule N of M every R-homomorphism $f:N \to M$ extends to an endomorphism g of M. It is known [RV1:72] that if M is a quasi-injective module and N a submodule of M then N isomorphic to a direct summand of M implies that N is itself a direct summand of M. The following is an analogue of this result for f-injective (p-injective) rings.

12.16. PROPOSITION. Suppose R is a left f-injective (respectively, p-injective) ring. Suppose I is a f.g. (respectively, principal) left ideal of R which is isomorphic to a direct summand of R. Then I is a direct summand of R.

PROOF. Assume R is left f-injective. By 12.8, I is an f-injective left R-module. Hence, by 12.12, I is a direct summand of R. The proof in the p-injective case is similar.

See 9.7 for the terminology used below.

12.17. PROPOSITION (Damiano [D:79]) The following are equivalent for a ring R.

(1) R is regular

(2) R is left semi-hereditary and left f-injective

(3) R is left p.p. and left p-injective.

(As usual, left can be replaced by right in conditions (2) and (3))

PROOF. See Remarks 9.7 and Theorem 12.13 for (1)$\longrightarrow$(2); (2)$\longrightarrow$(3) is trivial. (3) $\longrightarrow$(1). Consider for a in R the R-epimorphism f:R$\longrightarrow$Ra defined by f(x)=xa. As Ra is projective f splits and Ra is isomorphic to a direct summand of R. So by Proposition 12.16, Ra is a direct summand of R. So R is a regular ring.

12.18. REMARK. The above result is stated in [AI : 82] . In [AI: 82] Ahsan and Ibrahim have also proved two theorems (3.4, 3.8) characterising regular rings. However, many results of section 3.8 of [AI:82] are included in Theorem 12.13 above and do not really require Azumaya's R-projectivity or flatness.

Regular rings are f-injective and p-injective. Although these names were used later, the study of f-injective and p-injective rings was initiated by Ikeda and Nakayama in [IN :54] . We shall show that their characterisation of left p-injective rings [IN:54, Theorem I(I)] can be obtained as a corollary to a proposition giving a necessary and sufficient condition for R to be Ra-complete for an element a of R. (The classical paper [IN:54] led to a number of further investigations involving annihilators and chain conditions. See for example, [F: 66], [RV2:77] and [Z:76].)

12.19. PROPOSITION. Let a be an element of a ring R. Then the following are equivalent.

(I) The module R^R is Ra-complete

(II) $aR = r[l(a)]$

(III) $aR = r(J)$ for some subset J of R

(i.e. aR is a *right annulet* in the terminology of Faith [F : 66])

Proof. (I) $\to$ (II). Assume R^R is Ra-complete. Since, by 5.7, $aR \leq r[l(a)]$ always, only to prove the reverse inclusion. So let $x \in r[l(a)]$, i.e. $l(a)x = 0$ and so $l(a) \leq l(x)$. Therefore, the map $f: Ra \to R$ given by $f(ra) = rx$ is easily seen to be well-defined. By the Ra-completeness of R, this map is given by right multiplication by an element y of R. Hence $x = f(a) = ay$ shows $x \in aR$.

The implication (II) $\to$ (III) is trivial. Now assume (III). Let $f: Ra \to R$ be R-linear, and let $y = f(a)$. Now $Ja = 0$ implies $Jy = Jf(a) = f(Ja) = f(0) = 0$. Hence $y \in r(J) = aR$, i.e. $y = az$ for some $z \in R$. Thus f is given by $f(xa) = xy = xaz$ and so the linear map $R \to R$ given by right multiplication by z extends f. This proves (I). (Note (III) $\to$ (II) can be proved directly: if $aR = r(J)$, then $r[l(a)] = r[l(aR)] = r[l(r(J))] = r(J) = aR$, by using 5.7.)

12.20 COROLLARY. The following are equivalent for a ring R.

(I) R is left p-injective

(II) $aR = r[l(a)]$ for each $a \in R$

(III) Every principal right ideal is a right annulet.

We record some related results. (See references cited for proofs).

E.A. Rutter, Jr. [R 5:75] also studied p-injective rings, calling them rings with the principal extension property. He rediscovered 12.20 and proved the next result.

12.21. PROPOSITION. Let R be a left p-injective ring with ascending chain condition on annihilator left ideals. Then

(a) R is right artinian

(b) R is left artinian if and only if the left socle of R is f.g. as a left ideal.

12.22. PROPOSITION ([IN:54, Theorem 1(ii)] or [St., Chapter XIV, Proposition 2.1]) Following are equivalent on a ring R.

(I) R is left f-injective

(II) R satisfies (i) $r(I_1 \cap I_2) = r(I_1) + r(I_2)$ for all f.g. left ideals. I_1, I_2.

and (ii) $r[l(a)] = aR$ for every $a \in R$.

12.23 PROPOSITION. ([IN:54, theorem 1(iii)] or [St, Chapter XIV, Proposition 2.2]) Let R be left self-injective.

Then: (i) $r(I_1 \cap I_2) = r(I_1)+r(I_2)$ for all left ideals I_1, I_2

(ii) $r[l(J)] = J$ for every f.g. right ideal .

12.24. COROLLARY. Let R be left noetherian. Then R is left self-injective if and only if

(i) $r(I_1 \cap I_2) = r(I_1)+r(I_2)$ for all left ideals I_1 and I_2.

(ii) $r[l(J)] = J$ for every f.g. right ideal J.

These results lead to the study of rings defined below.

12.25. DEFINITION. A right and left artinian ring R is called a quasi-Frobenius ring (or QF-ring) if it satisfies $r[l(A)] = A$, $l[r(B)] = B$ for all right ideals A and left ideals B.

12.26 PROPOSITION. If R is left or right noetherian and is left or right self-injective then R is a QF-ring.

PROOF. See [St., Chapter XIV, Proposition 3.4]

There is a vast literature on QF-rings. It will take us too far afield to discuss them here.

§13. Flatness and injectivity.

For basic results concerning tensor products and flatness we refer to standard texts (eg. [St., Chapter I], [AF, §19], [R, §2 and §3])

We begin with the following well-known characterisation of regular rings.

13.1 THEOREM. Let R be a ring. Then the following are equivalent.

(1) R is regular

(2) Every left R-module is flat.

(3) Every cyclic left R-module is flat.

PROOF. See [St., Chapter I, §12], [R, Theorem 4.24], [GII, Corollary 1.13], [L, § 5.4, Proposition 4] or [AF, Exercise 19.16].

13.2. THEOREM (Villamayor) (See Chase [C 2: 60])
Let $0 \to K \to F \to B \to 0$ be an exact sequence of R-modules, where F is free, $K \leq F$ and $K \to F$ the inclusion map. Then the following are equivalent.

(i) B is flat

(ii) For every $v \in K$, there is a map
$h: F \to K$ with $h(v) = v$

(iii) For every $v_1 \dots v_n \in K$, there is a map
$h: F \to K$ with $h(v_i) = v_i$ for each $i, i = 1,2,3,\dots n$

PROOF. See [R, Theorem 3.39] or [FI, 11.27] for a proof. See also [St., Corollary 11.4 in Chapter I]

13.3. COROLLARY (Chase [C2 : 60]) Let K be a left ideal of a ring R. Then the following are equivalent.

(i) R/K is flat (as a left R-module)

(ii) For each a in K there is some b in K such that $ab = a$ (i.e. $a \in aK$).

PROOF. (i) $\to$ (ii) Consider the exact sequence

$$0 \to K \to R \to R/K \to 0$$

Let $a \in K$. By Theorem 13.2 there exists $h: R \to K$ such that $a = h(a) = ah(1) = ab$ where $b = h(1) \in K$.

(ii) $\to$ (i) Again apply Theorem 13.2

13.4. COROLLARY. Suppose the ring R satisfies condition 5(A) on annihilators. Let K be an ideal of R. Then R/K is left R-flat if and only if R/K is right R-flat.

PROOF. ($\to$) Let $a \in K$. Then by 13.3 there exists b in K such that $a = ab$. So by condition 5(A) $a = ba$. Hence by 13.3 again R/K is right R-flat.

For a left R-module M, let $M^* = \mathrm{Hom}_Z(M, Q/Z)$. M^* becomes a right R-module under the following definition : for $f \in M^*$, $r \in R$, fr is defined by $(fr)(m) = f(rm)$.

We have the following theorem due to Lambek.

13.5 THEOREM. A module ${}_RM$ is flat if and only if M^*_R is an injective module.

PROOF. see [St., Chapter I, 10.5]
[R, Theorem 3.35] or [FI, Theorem 5.60]
or [L, §5.3] or [AF, Lemma 19.14].

Let A be an ideal of a ring R. The next theorem gives an "injectivity criterion" for the flatness of the left R-module R/A. We include a proof for the sake of completeness. (Note that Theorem 13.6 has been attributed to Azumaya [A :73] by Yousif [Y:86, p.141] In [Y:86, Proposition 3.2] Yousif has proved that the two conditions in Theorem 13.6 are equivalent to the condition obtained by replacing 'injective' by 'p-injective' in (ii) (cf. 13.7 - 13.11 below).)

13.6. THEOREM. Let A be an ideal of R. Then the following are equivalent.

(i) R/A is flat as a left R-module

(ii) Every injective right R/A-module is injective as an R-module.

PROOF. (i) $\rightarrow$ (ii). Let $h:R \rightarrow R/A$ be the canonical quotient map. Let E be an injective right R/A-module. Let B be a right ideal of R and $f:B \rightarrow E$ an R-homomorphism. Consider the induced map $\bar{f}: B+A/A \rightarrow E$ defined by $\bar{f}(\bar{b}) = f(b)$. This is well-defined: if $\bar{x}=0$ i.e. $x \in B \cap A$, as R/A is left R-flat there exists $y \in A$ such that $x = xy$ (by 13.3) Hence as $EA = 0$ we get $f(x) = xf(y) \in EA = 0$. Hence as E is R/A-injective this extends to a homomorphism $k:R/A \rightarrow E$ and so $koh : R \rightarrow E$ extends f. So by Baer's criterion E is injective over R.

(ii) $\rightarrow$ (i) Consider the left R-module R/A and the right R-module $(R/A)^* = \mathrm{Hom}_Z(R/A, Q/Z)$. Now R/A is left R/A-flat, and so $(R/A)^*$ is an injective right R/A-module, by Lambek's theorem (13.5). Hence, by hypothesis $(R/A)^*$ is injective as a right R-module. So with another application of Lambek's theorem we conclude that R/A is flat as a left R-module.

Next, we shall prove two propositions, one giving a necessary condition and anothen giving a sufficient condition for the flatness of R/A as a left R-module, where A is an ideal of R.

13.7. PROPOSITION (Lemma (1.2) in [R1 : 75])Let A be an ideal of R. Assume R/A is left R-flat. Suppose B is a right ideal of R such that $B \leq A$ or $B + A = R$. Then R/A is B-complete (as a right R-module).

PROOF. Let $f: B \rightarrow R/A$ be (right) R-linear. We note first that if $x \in B \cap A$ then $f(x) = 0$: by 13.3, there exists $y \in A$ such that $x = xy$; hence $f(x) = f(x)y = 0$, as A annihilates R/A.

Case 1. $B \leq A$, then $B \cap A = B$ and so $f = 0$ and trivially has an extension $R \rightarrow R/A$.

Case 2. $B + A = R$. Then $1 = b+a$, for some $b \in B$, $a \in A$. So, for x in B, $x = bx + ax$ with bx in B and hence $ax = x - bx \in B \cap A$. So $f(ax) = 0$ implying $f(x) = f(bx) = f(b)x$. Hence f is given by left multiplication by f(b) and extends to a R-homomorphism $R \rightarrow R/A$.

13.8 REMARK. Note that the following are equivalent for an ideal A of R: (i) A is maximal as a left ideal (ii) R/A is a division ring (iii) A is maximal as a right ideal.

13.9. PROPOSITION. [R1 : 75] Suppose the ideal A satisfies the equivalent conditions of 13.8. If R/A is p-injective as a right R-module, then R/A is left R-flat.

PROOF. Let a be an element of A. We show that $a \in aA$ and so by Corollary 13.3, R/A will be left R-flat.

Consider the R-epimorphism $f: R/A \to aR/aA$ defined by $f(r+A) = ar + aA$.
If $f=0$, then $aR = aA$ and so $a \in aA$.
If $f \neq 0$, then R/A is a simple right R-module and hence f is an isomorphism. So aR/aA is p-injective as a right R-module. Hence the natural quotient map $h: aR \to aR/aA$ is given by left multiplication by an element $ar + aA$ of aR/aA. Thus, we have, in aR/aA, $\bar{a} = h(a) = \overline{ar}.\,a$
$= \overline{ara} = 0$ as $ara \in aA$. So $a \in aA$.

13.10. THEOREM. Let A be an ideal of R such that R/A is a division ring. Then the following are equivalent.

(i) R/A is left R-flat.
(ii) R/A is right R-injective.
(iii) R/A is p-injective as a right R-module.

PROOF. (ii) implies (iii) is trivial and (iii) implies (1) is Proposition 13.9.

(i) $\rightarrow$ (iii) First proof. Note that $S = R/A$ is a division ring and so injective as a right S-module. Applying Theorem 13.6 we get R/A is right R-injective.

Second proof. Let B be a right ideal of R. Then by the maximality of A as a right ideal, we have $B \leq A$ or $B + A = R$. Hence by proposition 13.7 R/A is B-complete. The desired conclusion follows.

13.11. COROLLARY. Let R be a commutative ring and M a simple R-module. Then the following are equivalent.

(i) M is flat
(ii) M is injective
(iii) M is p-injective.

13.12. REMARK. We do not know of a direct proof (i.e. one not involving flatness) of (iii) $\rightarrow$ (ii) in Corollary 13.11.

We shall now consider the question: if R is a commutative ring and A an ideal of R, is the flatness of R/A as an R-module a necessary or a sufficient condition for the injectivity of R/A on an R-module? The answer is negative in each case.

13.13. EXAMPLE. Let R be a commutative ring which is not selfinjective (eg. $R = Z$). Take $A = 0$. Then R/A is R-flat but not R-injective.

13.14. REMARK If R is a commutative selfinjective hereditary non-regular ring - we do not know of an explicit example of such a ring - then there would exist an ideal A such that R/A is not R-flat. However, by the argument of 9.7 R/A is R-injective. (An explicit example of a commutative ring R and an ideal A such that R/A is R-injective but not R-flat is given in [R1:75, §2]).

Next we apply the results of this section to weakly regular rings. First we prove a generalisation of [R1:75, Proposition 3.1(ii)] .

13.15 PROPOSITION. Let A be an ideal of a ring R. Assume that each element of A is left weakly regular. Then R/A is right R-flat.

PROOF. Let $a \in A$. Then, by hypothesis, $a = xa$, for some $x \in RaR \subset A$. Hence, by 13.3, R/A is right R-flat.

13.16. COROLLARY. Let R be a ring, A an ideal of R satisfying the equivalent conditions of 13.8. Suppose that each element of A is left weakly regular. Then R/A is left R-injective.

PROOF. Apply 13.15 and (the left-right dual of) 13.10

13.17. COROLLARY (Proposition 3.1(iii) of [R1:75])

Let R be a left weakly regular ring and A an ideal of R satisfying the equivalent conditions of 13.8. Then R/A is left R-injective.

13.18 PROPOSITION. (cf. Proposition 3.1(ii) of [R1:75])
A ring R is left weakly regular if and only if for each ideal A of R , R/A is a flat right R-module.

PROOF. "Only if" follows from Proposition 13.15. For the converse, let x be an element of R. Then R/RxR is right R-flat. Hence by 13.3, there exists $y \in RxR$ such that $yx = x$ So $x \in RxRx$.

§14. V-rings and related rings.

It is well-known and easy to show that a ring R is semi-simple if and only if every left R-module (or every cyclic left R-module) is projective and that R is semi-simple if and only if every left R-module is injective. A theorem of Osofsky ([O 2:64] or [O2: 68]) asserts that a ring is semi-simple if and only if every cyclic left (or right) R-module is injective. It is natural to ask what happens if in addition to the class of cyclic R-modules we consider a smaller class, that of simple R-modules, and in addition to injectivity and projectivity we consider p-injectivity and flatness.

Some results are well-known : it is a standard exercise that R is semi-simple if and only if every simple R-module is projective. (See, for example , [V:76,Theorem 2.6] or [MV:77, p. 566].) We have already mentioned (13.1) that R is regular if and only if every left R-module (cyclic left R-module) is flat. We have also seen (12.13) that "flat" can be replaced by "p-injective" here. Kaplansky proved that a commutative ring R is regular if and only if every simple R-module is injective (The usual reference for this result is [RZ:59] .) Around 1967, Villamayor considered rings R such that every simple left R-module is injective. Faith [F III] called such rings as left V-rings after Villamayor. In [MV:73] Michler and Villamayor studied the structure of V-rings. In [F1 : 74] Fisher gave an expository account of the connections between regular rings and V-rings. Ming([M1 : 74] and [M1 : 80]) and others have studied p-V-rings: A left p-V-ring is a ring R such that simple left R-modules are p-injective. Ramamurthi [R1 : 75] initiated the study of rings R such that simple left R-modules are flat, and called them as <u>left SF-rings.</u>

In this section we shall survey some research work done in the field of V-rings and p-V-rings. SF-rings will be considered in a later section.

14.1. <u>DEFINITION.</u> A ring R is a <u>left V-ring</u> if every simple left R-module is injective. (These rings have also been called as left co-semisimple rings in [AF, Exercises 13.10 and 18.23].)

14.2. DEFINITION A ring R is a left p-V-ring (respectively left f-V-ring) if every simple left R-module is p-injective (respectively, f-injective)

14.3. REMARK. Note that left p-V-rings are also called left SPI-rings in the literature. ([R1 : 75, Section 3])

14.4. NOTATION. For a left ideal L of R, L^* will denote the intersection of all maximal left ideals containing L (of course, $R^* = R$).

The following fundamental theorem gives a characterisation of left V-rings.

14.5. THEOREM. [MV : 73, Theorem 2.1]. The following conditions are equivalent for a ring R.

(1) R is a left V-ring.

(2) For each left R-module M, Rad M = 0

(3) For each cyclic left R-module M, Rad M = 0.

(4) For each left ideal I of R, we have $I^* = I$.

PROOF. See [MV : 73] ; note that a module M such that Rad M = 0 has been called as semi-simple there. See also [FI, 7.32A] and [SV:74, Proposition 2.2]

14.6. PROPOSITION. Let x be an element of a ring R. Assume that every simple quotient of Rx is Rx-complete. Then x is left weakly regular.

PROOF. Let, if possible, $Rx \neq (Rx)^2$. Then $M = Rx/(Rx)^2$ is a non-zero f.g. left R-module and hence has a maximal submodule N. Now $N = B/(Rx)^2$ for some left ideal B of R such that $(Rx)^2 \leq B \leq Rx$. Hence $M/N \cong Rx/B$ is simple and so Rx-complete, by hypothesis. So the natural quotient map $h: Rx \to Rx/B$ is given by $h(rx) = r\bar{xz} = \overline{rxz}$ for some element $z \in Rx$.

Hence $\bar{x} = \overline{xz} = 0$ in Rx/B since $xz \in xRx \leq (Rx)^2 \leq B$. Hence $x \in B$ and $M/N = 0$, a contradiction. So $Rx = (Rx)^2$.

14.7. COROLLARY. (See [M1 : 74, Lemma] or [R1 : 75 , Proposition 3.1]) Let R be a left p-V-ring. Then R is left weakly regular.

14.8. COROLLARY. Let R be a p-V-ring. Then R is semi-primitive and so semi-prime.

PROOF. This is a consequence of 11.9, 11.10 and 14.7.

14.9. REMARK. In [M1: 74, Lemma 1(ii)] Ming proves that if R is a left p-V-ring then every non-zero left ideal of R contains a maximal left subideal. However, this is a property of rings belonging to a much larger class, namely, of semi-primitive rings (Of course, it is also a property of left noetherian rings). In fact, analogous results hold for modules also. We prove all this in a sequence of results below.

14.10. PROPOSITION. Let M be a module and N a submodule of M such that $N \not\leq \text{Rad } M$. Then N has a maximal submodule.

PROOF: Since $N \not\leq \text{Rad } M$, there exists a maximal submodule Q of M such that $N \not\leq Q$. Consider the restriction g to N of the canonical quotient map $M \to M/Q$. As $N \not\leq Q$, $g \neq 0$; hence, as M/Q is simple g is onto. Hence $N/\text{Ker}\, g \cong M/Q$ is simple and so N has a maximal submodule.

14.11. COROLLARY. Let M be a module such that $\text{Rad } M = 0$. Then every nonzero submodule of M has a maximal submodule.

14.12. COROLLARY. Let R be a semi-primitive ring. Then every non-zero left (respectively, right) ideal of R contains a maximal left (respectively, right) subideal.

14.13. EXAMPLE. The following is an example of a ring which does not have the property mentioned in 14.12. Let $R = Z \oplus Q$ be made into a (commutative) ring by defining $(a,x)\ (b,y) = (ab,\ ay+xb)$. Then subideals of the ideal $A = 0 \oplus Q$ of R coincide with Z-submodules of Q. Since ${}_ZQ$ does not have a maximal submodule A cannot have a maximal subideal.

14.14. REMARK. In Example 1.7 of [V : 76] Varadarajan considers $R = Z\ ([X_n])_{n \in N}$, the polynomial ring in a countable number of indeterminates over Z, the ring of integers. Let I be the ideal of polynomials with constant term zero. He constructs an R-epimorphism $I \to Q$, where Q

is regarded as an R-module via the onto ring homomorphism $R \to Z$ taking a polynomial to its constant term. It is asserted next that since Q has no non-zero f.g. quotient (as a Z-module or as an R-module) I has no non-zero f.g. quotient module. Of course, this argument is incorrect. In fact, as seen above, since Rad R = 0, every nonzero ideal of R does have a simple quotient module.

Corollary 14.12 suggests the introduction of a number of related properties. So let us adopt the following

14.15 <u>TERMINOLOGY.</u> Let R be a ring. We shall say

(I) R has property (P_1) (respectively, (P_2)) if every non-zero left R-module (respectively, left ideal) has a simple quotient.

(2) R has property (P_3) ((P_4)) if every nonzero left R-module (left ideal) has a nonzero injective quotient.

(3) R has property (P_5) ((P_6)) if every nonzero left R-module (left ideal) has a nonzero p-injective quotient.

14.16. <u>REMARKS.</u> (I) Rings with property (P_1) have been variously called as (left) B-rings [FII , p.155] or max rings [V : 76, §4] in the literature.

(II) Trivially, (P_{2n-1}) implies (P_{2n}) for n = 1,2,3. Also $(P_3) \Rightarrow (P_5)$ and $(P_4) \Rightarrow (P_6)$

(III) Every left noetherian and (by 14.12) every semi-primitive ring has (P_2) but as ${}_{Z}Q$ has no simple quotient Z does not have (P_1).

(IV) Suppose R has (P_3) and let S be a simple left R-module. Then S has a nonzero injective quotient T and so $S \xrightarrow{\sim} T$ (as S is simple). Hence S is injective and R is a left V-ring.

(V) As in (IV) we can show that if R has (P_5) it is a left p-V-ring.

(VI) Suppose R is a left V-ring. Then by Theorem 14.5 for every ${}_{R}M$ we have $\operatorname{Rad} M = 0$. Hence M has a maximal submodule and so a simple quotient, i.e. an injective (non-zero) quotient. Thus R has (P_3).

(VII) If R is a left p-V-ring such that R has (P_1) then every nonzero left R-module has simple and hence (non-zero) p-injective quotient. So R has (P_5). We do not know if every left p-V-ring has (P_1) (i.e. is a max ring)

(VIII) Let R be regular and M a non-zero left R-module. Then by 12.13 M is p-injective and so M itself is a nonzero p-injective quotient of M. So regular rings have (P_5).

(IX) By (VI), (VII) and (VIII) above a left p-V-ring R which does not have (P_5) will have to be a ring R such that R is neither regular nor V- nor a max ring.

(X) As seen above (14.8) if R is a left p-V-ring, R is semiprimitive and so every nonzero left ideal has a nonzero simple and so p-injective quotient. So R has (P_6). Thus, by (V), we can assert that the following implications hold: R has $(P_5) \rightarrow R$ is a left p-V-ring $\rightarrow R$ has (P_6). It is natural to ask whether either of these implications can be reversed.

14.17. REMARK. A left V-ring is by definition a left p-V-ring and so left V-rings have the properties of left p-V-rings mentioned in 14.7 and 14.8. Because of 14.7 they also have properties of left weakly regular rings mentioned in §11.

The rest of this section will be devoted to various properties of V-rings and p-V-rings. A few of them will be proved and the rest stated without proof.

14.18. PROPOSITION. (cf. Theorem 3.2 of [MV:73]). The following conditions are equivalent for a ring R.

(1) R is semi-simple

(2) If ${}_RM$ is such that Rad M = 0, then M is injective.

(3) If ${}_RM$ is a cyclic module such that Rad M = 0, then M is injective.

PROOF. (1)→(2) and (2)→(3) are trivial Assume (3). Since simple modules are cyclic and have zero radical, condition (3) implies that R is a left V-ring.

So by Theorem 14.5 Rad $M = 0$ for each M. This, together with the hypothesis, implies that every cyclic left R-module is injective. So by Osofsky's theorem [O2 : 64] R is semi-simple.

14.19. REMARK. It is asserted in [MV:73, §3] that the above theorem answers a question of Cateforis and Sandomierski: [CS : 69] : If semisimple left R-modules are injective, is R necessarily semi-simple? However, it should be noted that in [CS : 69] semi-simple is used in our sense and not in the sense of the Jacobson radical being zero.

Next we explore the connection between V-rings and strongly regular rings. The following theorem was proved independently by Brown [B3: 73] and by Sarath and Varadarajan [SV : 74]. Our proof uses the characterisation (Theorem 14.5) of V-rings due to Michler and Villamayor and so differs from both these proofs.

14.20. THEOREM. The following are equivalent for a ring R.

(1) R is a strongly regular ring.

(2) R is a left duo, left V-ring.

(3) R is a left quasi-duo, left V-ring.

(Of course, left can be replaced throughout by right in (2) as well as (3)).

PROOF. (1) $\rightarrow$ (2). Let M be a cyclic left R-module. Then $M \xrightarrow{\sim} R/I$ where I is a left ideal of R. Since R is left duo (10.5) I is an ideal of R and R/I is a regular ring. Now the radical of the left R-module R/I equals the radical of R/I as a ring and hence vanishes by 11.9(II). (See 11.4(I).) So Rad $M = 0$ which implies, by 14.5, that R is a left V-ring. (3) $\rightarrow$ (1). Let I be a left ideal of R. Then, by 14.5, $I = I^*$. As R is left quasi-duo, I (being an intersection of maximal left ideals of R) is an ideal of R. Thus R is left duo. As R is also left weakly regular R is clearly strongly regular. (See Corollary 11.20)

14.21. COROLLARY. Let R be commutative. Then R is a V-ring if and only if it is regular.

14.22 EXAMPLES. We shall now record examples which show that the regularity of R is neither a necessary nor a sufficient condition for R to be a V-ring.

(I) Let W be an infinite dimensional vector space over a field K, and $R = \mathrm{End}_K(W)$. Regard W as a left R-module by defining $f.x = f(x)$ for any $f \in R$ and $x \in W$. By 9.3(6) R is a regular ring. If $x, y \in W$ and $x \neq 0$, then there exists $f \in R$ such that $y = f(x)$; this is a well-known property of vector spaces. It follows that W is a simple left R-module. It can be shown that W is not injective over R. (See [SV: 74, Proposition 2.4] for a detailed proof.)

So R is not a left V-ring. Thus R regular does not imply R left V-ring.

It will not be out of place to mention that the above result (W is not R-injective) is a special case of the following theorem of Sandomierski [S2 : 70, Theorem 1] : Let R be a ring, M_R a module such that $M = \bigoplus_{i \in I} M_i$, where each M_i is non-zero and I is an infinite indexing set. Let $S = \mathrm{End}(M_R)$. Then M is not injective as a left S-module.

(II) Cozzens [C4: 70] settled the question of the reverse implication in the ngative . He obtained examples of domains which are V-rings but not division rings. Hence by 9.10 these rings cannot be regular. These rings have been studied in detail in [FI] (pp. 358-362).

(III) Since left V-ring implies left p-V-ring and regular ring implies left p-V-ring (12.13) the example in (I) is an example of a left p-V-ring which is not a left V-ring. Similarly the examples in (II) are p-V-rings which are not regular.

§15. SF-rings.

Recall that a ring R is called a right SF-ring if every simple right R-module is flat. If R is regular then every R-module is flat, and so R is left and right SF. Ramamurthi initiated the study of SF-rings in [R1 : 75] and of the question whether an SF-ring is necessarily regular. Although a number of partial results have been obtained the question is still open. In this section we record some properties of SF-rings. The results 15.1 - 15.5 are essentially in [R1 : 75] .

15.1. <u>PROPOSITION.</u> Let R be a right SF-ring and x an element of R such that $K = l(x)$ is an ideal. Then x is regular.

<u>PROOF.</u> By hypothesis, $xR+K$ is a right ideal of R. Suppose, if possible, $xR+K \neq R$. Let M be a maximal right ideal containing $xR+K$. Then R/M is R-flat, and so by 13.3 $x = yx$ for some y in M. So $1-y \in K \leq M$ and $y \in M$ implies $1 \in M$, a contradiction! So $xR+K = R$ and hence $1 = xz+k$ for some z in R and k in K. Hence $x = (xz+k)x = xzx$, showing that x is regular.

15.2. <u>COROLLARY.</u> Let R be a SF-ring satisfying condition 5(B) (See Remark 5.4) Then R is strongly regular.

Proof. Apply Proposition 15.1 and Theorem 10.8

15.3. <u>COROLLARY</u>. A reduced SF-ring is (strongly) regular.

PROOF. A reduced ring satisfies condition 5(A) and hence 5(B). Now apply 15.2.

15.4. COROLLARY. A commutative SF-ring is regular.

15.5. PROPOSITION. The centre of an SF-ring is regular.

PROOF. Let R be a right SF-ring and let $x \in C$, the centre of R. Then $l(x)$ is an ideal of R, and so (by 15.1) there exists y in R such that $x = xyx$. It follows, by Proposition 11.5(I), that x is C-regular. Thus C is a regular ring.

The results 15.6 - 15.12 are from [R3 : 86].

15.6. PROPOSITION. A factor ring of a left SF-ring is a left SF-ring.

PROOF. Let $f:R \to T$ be an onto homomorphism of rings, and let S be a simple left T-module. Regard S as a left R-module via f. As f is onto S is a simple, and therefore flat, left R-module. Therefore $T \otimes_R S \cong S$ is T-flat.

15.7. PROPOSITION. Let R be a left SF-ring and M a maximal left ideal of R. Then M is flat.

PROOF. Consider the exact sequence

$$0 \to M \to R \to R/M \to 0 \qquad (I).$$

Taking character modules we get an exact sequence

$$0 \to (R/M)^* \to (R)^* \to (M)^* \to 0. \qquad (II)$$

(See [L, Chapter 5] or [R, Lemma 3.34])

As R/M and R are flat left R-modules, by Lambek's theorem (13.5), $(R/M)^*$ and R^* are injective right R-modules. Hence (II) splits, and being a direct summand of R^*, M^* is injective. Another application of Lambek's theorem yields that M is flat.

15.8. REMARK. A ring R is a left SF-ring if and only if every semi-simple left R-module is flat. This holds because a direct sum of flat modules is flat.

15.9. PROPOSITION. A semi-local SF-ring is semi-simple. (and therefore regular).

PROOF. Let R be a semi-local, left SF-ring. Then R/Rad R is a semi-simple, and therefore flat (15.8) left R-module. Let $a \in \text{Rad } R$. By 13.3 there exists $b \in \text{Rad } R$ such that $a = ab$. As $b \in \text{Rad } R$, $1 - b$ is a unit and hence $a = 0$. Thus Rad R = 0 and R is a semi-simple ring.

Following [CE:81] we introduce a definition.

15.10. DEFINITION. A ring R is called left quasi-perfect if every finitely generated flat left R-module is projective.

15.11. REMARKS. (I). It is known that integral domains, local rings and semi-perfect rings (see [AF, §27] for definition and basic properties) are left and right quasi-perfect.

(II). Let R be a left noetherian ring. Then every f.g. left R-module is finitely presented. Now Theorem 13.2 can be used to show that finitely presented flat modules are always projective. (See [R, Corollary 3.40]). Hence it follows that R is left quasi-perfect.

(III) It is claimed in [CE : 81] that left coherent rings are left quasi-perfect. However, as seen in 9.7, commutative regular rings are coherent. Let $R = \prod_{i=1}^{\infty} K_i$, where K_i are fields. Let $A = \bigoplus_{i=1}^{\infty} K_i$, an ideal of R. Then the cyclic R-module R/A is non-projective)(since A is not a direct summand of R) but is R-flat, by Theorem 13.1. Hence R cannot be quasi-perfect.

15.12. <u>PROPOSITION</u> If R is a left quasi-perfect, left SF-ring, then R is semi-simple (and therefore regular).

PROOF. Let S be a simple left R-module. Then as R is left SF, S is flat. As R is quasi-perfect and S is f.g. and flat over R, S is projective. Hence R is semi-simple. (See the references cited in the Introduction of § 14).

Finally, we record without proof some results of Ming. First, a definition.

15.13. DEFINITION. A ring R is called an ELT (MELT) ring if every essential (maximal essential) left ideal is an ideal.

15.14. PROPOSITION [M1:80] If R is a semi-prime, MELT, right SF-ring satisfying a polynomial identity,. then R is a regular, left and right V-ring.

15.15. PROPOSITION. [M1:80] A prime, MELT, right SF-ring is primitive.

15.16. PROPOSITION. [M1:80] Suppose R is a semi-prime, MELT, right SF-ring such that every primitive factor ring is regular. Then R is regular.

15.17. PROPOSITION. [M1: 80] Suppose every essential one-sided ideal of R is an ideal, then the following are equivalent :

(a) R is regular (b) R is a semi-prime, right SF-ring (c)R is a right SF, left p-injective ring.

15.18. PROPOSITION. [M1:81, Theorem 2.8] The following conditions are equivalent.

(1) R is left self-injective and regular.

(2) Every left annihilator ideal of R is quasi-injective and R is a right SF-ring.

There are many other results about SF-rings in [M1:81]. However a number of definitions will be required for stating them.

§ 16. An equivalence theorem

In this section we follow [R3:86] to prove the equivalence of a number of conditions introduced in this chapter under the assumption that R is a quasi-duo ring. For convenience, the proof of the nontrivial implications in Theorem 16.4 is broken into Propositions 16.1 to 16.3. In these propositions R denotes a left quasi-duo ring.

16.1 PROPOSITION. If R is left weakly regular, then R is a left V-ring.

PROOF. Let S be a simple left R-module. Then $S \cong R/M$ where M is a maximal left ideal and so an ideal of R. Let $x \in M$. As R is left weakly regular $Rx = RxRx$ and so there exists $y \in RxR \subseteq M$ (an ideal!) such that $yx=x$. Thus by 13.3'# R/M is a flat right R-module and so by 13.6' $S \cong R/M$ is an injective left R-module. Thus R is a left V-ring.

16.2. PROPOSITION. If R is a right SF-ring, then R is a left V-ring.

PROOF. If M is a maximal left ideal (and so an ideal) of R then R/M is a simple, and therefore flat, right R-module. By 13.6' R/M is an injective left R-module.

16.3 PROPOSITION. If R is a left SF-ring, then R is a right V-ring.

PROOF. Let A = R/Rad R. By 15.6 A is a left SF-ring. As R is a left quasi-duo ring, by 8.9 A is a reduced ring. By 10.8 A is strongly regular and hence, by 10.6 A is right quasi-duo. This implies that R also is right quasi-duo. So R is a right V-ring by 16.2'

The notation m.n' denotes the left-right dual of the result m.n .

16.4. THEOREM. Let R be a quasi-duo ring. Then the following conditions are equivalent.

(1) R is a left V-ring.
(2) R is a left p-V-ring.
(3) R is a left weakly regular ring.
(4) R is a left SF-ring.
(5) R is a right V-ring
(6) R is a right p-V-ring
(7) R is a right weakly regular ring.
(8) R is a right SF-ring
(9) R is a regular ring.
(10) R is a strongly regular ring.

PROOF. Assume R is left quasi-duo. (The proof in the right quasi-duo case follows by symmetry.)

Note that the implications (10)→(9) (10.5), (9)→(2)(§ 14), (2)→(3) (14.7) and (1)→(2) (trivial) always hold, that is without the assumption that R is quasi-duo.
Next for a left quasi-duo ring (3)→(1) by 16.1 and (1)→(10) by 14.20, and (3)↔(7) by 11.22. This proves the equivalence of (1), (2), (3), (7), (9) and (10). Now (8)→(1) by 16.2 and (9)→(8) by 13.1'; hence we get the equivalence of (8) and (9). Since (9)→(4) (13.1) and (5)→(6)→(7) always hold and (4)→(5) by 16.3, we have the equivalence of (4), (5), (6) and (7). This completes the proof of the theorem.

§17. Central Localisations

It was shown in 9.13 that a commutative ring R is regular if and only if the loffalisation of R at each maximal ideal is a field. In this section we shall study central localisations.

Throughout this section C will denote the centre of a ring R and S will denote a multiplicatively closed subset (containing 1) of C. Then $S^{-1}C$ is a commutative ring, and the C-module $S^{-1}R$ can be made into a C-algebra as usual. If Q is a left R-module $S^{-1}Q$ becomes a left $S^{-1}R$-module. We have the canonical maps $h_R: R \to S^{-1}R$ defined by $h_R(r) = r/1$ and $h_Q: Q \to S^{-1}Q$, $h_Q(q) = q/1$; h_R is a ring homomorphism and h_Q is a R-linear map. If m is a prime ideal of C, then $S = C - m$ is a multiplicatively closed subset of C. In this case, we denote $S^{-1}C$, $S^{-1}R$ and $S^{-1}Q$ by C_m R_m and Q_m respectively and call them central localisations (of C, R, Q respectively) at m. We refer to [AM] or [J] for the basic properties of localisations.

17.1 PROPOSITION. Let R be a ring with centre C. Let Q be a left R-module. Then: (1) For any S consisting entirely of regular elements, the map $h_Q: Q \longrightarrow S^{-1}Q$ is onto.

(2) Assume C is regular, and let $S = C - m$ where m is a maximal (prime (9.14)) ideal of C. Then

Ker h = mQ. Hence the sequence

$$0 \longrightarrow mQ \longrightarrow Q \longrightarrow Q_m \longrightarrow 0$$ is exact.

(3) Under the hypotheses of (2), $mR \cap C = m$

PROOF. (1) Let $q \in Q$ and $c \in S \subseteq C$. Then there exists $d \in C$ such that $c = c^2d$. Hence $q/c = dq/1 = h(dq)$ in $S^{-1}Q$. Thus h is onto.

(2) Let $x \in m$, $q \in Q$. Then $xyx = x$ implies $(1-xy)x = 0$ with $1-xy \notin m$. Hence $(1-xy)xq = 0$ showing $h(xq) = xq/1 = 0$ in Q_m. Thus $mQ \leq \text{Ker } h$.

On the other hand, let $q \in \text{Ker } h$, so that $q/1 = 0$ in Q_m. So there exists $x \in C \setminus m$ such that $xq = 0$ in Q. Let $xyx = x$ for some $y \in C$. Then $(1-xy)x = 0$ implies $1-xy \in m$. So $q = (1-xy)q \in mQ$. Thus $\ker h \leq mQ$. This shows $\ker h = mQ$.

(3) Now, clearly, $mR \cap C = \text{kernel } (C \to C_m)$ As C_m is a field (by (9.13)), $mR \cap C$ is a maximal ideal of C, and $mR \cap C \supset m$. Hence $mR \cap C = m$.

17.2 PROPOSITION. Let a be an element of a ring R. Let C be the centre of R. Then the following conditions are equivalent.

(I) a is a left weakly regular element of R

(II) For a multiplicatively closed subset S of C and $u \in S$, a/u is a left weakly regular element of $S^{-1}R$

(III) $a/1$ is a left weakly regular element of R_m for each maximal ideal m of C.

PROOF. (I) $\Rightarrow$ (II) let $a \in RaRa$. Then $a = \sum x_i a y_i a$ for some $x_i, y_i \in R$.

Hence $$a/u = \sum \frac{x_i u}{1} \, a/u \, \frac{y_i}{1} \, a/u$$ shows that $a/u \in S^{-1}R.\ a/u.\ S^{-1}R\ a/u$.

Hence a/u is left weakly regular

(II) $\longrightarrow$ (III) is trivial

(III) $\longrightarrow$ (I). Let $I = Ra$, a left ideal of R. Since localisation is an exact functor (equivalently, C_m is a flat C-module) the sequence $0 \to (I^2)_m \to I_m \to (I/I^2)_m \to 0$ is exact. Now $I_m = R_m.\ a/1$. and $(I^2)_m = (I_m)^2 = (R_m\ a/1)$. Hence, by hypothesis, $(I/I^2)_m = 0$ for every maximal ideal m of C. So $I = I^2$ and so a is left weakly regular.

17.3. COROLLARY. Following are equivalent for a ring R.

(I) R is left weakly regular

(II) $S^{-1}R$ is left weakly regular for each S.

(III) R_m is left weakly regular for each $m \in \mathrm{Max}\ (C)$.

The following result is the regularity analogue of 17.2

17.4 PROPOSITION. Let $a \in R$. Let C = Centre R. Then the following are equivalent.

(I) The element a is a regular element of R.

(II) For each multiplicatively closed subset S of C, and for each $u \in S$, a/u is a regular element of $S^{-1}R$.

(III) For each $m \in \operatorname{Max}(C)$, $a/1$ is a regular element of R_m.

PROOF. (I) $\rightarrow$ (II) If $aba = a$ then $a/u . ub/1 . a/u = a/u$ in $S^{-1}R$ showing a/u is regular.

(II) $\rightarrow$ (III) Trivial.

(III) $\rightarrow$ (I) Since $a/1$ is regular in R_m there exists $b_m/s_m \in R_m$ such that $a/1 . b_m/s_m . a/1 = a/1$ in R_m. So there exists $t_m \in C-m$ such that $t_m (ab_m a) = t_m s_m a = u_m a$ where $u_m = t_m s_m \in C-m$. So as usual the ideal of C generated by u_m is the unit ideal. (In the rest of the proof we write t_j for t_{m_j}, b_j for b_{m_j} etc.)

Now there exists a natural number n and elements $\{v_j\}_{j=1}^{n}$ in C such that $\sum_{j=1}^{n} v_j u_j = 1$.

Hence $a = \sum_{j=1}^{n} v_j u_j a = a\left(\sum_{j=1}^{n} v_j t_j b_j\right) . a = ada$ for some $d \in R$. So a is a regular element.

17.5. REMARK. In exactly the same manner as that of Proposition 17.4 we can prove the equivalence of the following conditions.

(I) a is left strongly regular

(II) For each $u \in S \subset C$, a/u is left strongly regular in $S^{-1}R$.

(III) For each $m \in \mathrm{Max}\,(C)$, $a/1$ is left strongly regular in R_m.

17.6. THEOREM. The following statements are equivalent for a ring R with centre C:

(a) R is regular

(b) R_m is regular for each $m \in \mathrm{Max}\,(C)$.

(c) C is regular and R/mR is regular for each $m \in \mathrm{Max}\,(C)$.

PROOF. (a) $\Longleftrightarrow$ (b) follows from Proposition 17.4.

(c) $\longrightarrow$ (b). By Proposition 17.1(2), as C is regular, $R/mR \xrightarrow{\sim} R_m$ as rings. Hence R_m is regular.

(b) $\longrightarrow$ (c). As R_m is left weakly regular for each m, by 17.3, R is left weakly regular. So, by 11.5 (II), C is regular. Hence $R_m \xrightarrow{\sim} R/mR$, by Proposition 17.1(2), showing that R/mR is regular. (Of course, we can prove (a) $\longrightarrow$ (c) (more easily), instead of (b) $\longrightarrow$ (c).)

Alternative proof of (b) $\longrightarrow$ (a). Notice that the commutative result that flatness is a local property [AM, Proposition 3.10] extends to central localisstions. Now let Q be a left R-module. Then Q_m is a flat R_m-module (since R_m is regular) for each maximal ideal m of C. So Q is R-flat. Hence by 13.1 R is a regular ring.

In the next theorem, the property, "all localisations at maximal ideals are fields" is extended to" all central localisations at maximal ideals of the centre are division rings".

17.7. LEMMA. A ring R is reduced if and only if R_m is reduced for each $m \in \mathrm{Max}\ (C)$.

PROOF. Suppose R is reduced. Let $(a/s)^2 = 0$ in R_m. Then there exists $t \in C - m$ such that $ta^2 = 0$. So $(at)^2 = 0$ implies $at = 0$, and so $a/s = 0$. Thus R_m is reduced by 6.7. Conversely, suppose each R_m is reduced. Let $a^2 = 0$ $(a \in R)$. Then $(a/1)^2 = 0$ in R_m for each m. Hence $a/1 = 0$ in each R_m. So $a = 0$ and thus R is reduced.

17.8. THEOREM. The following conditions are equivalent for a ring R.

(1) R is strongly regular

(2) R_m is strongly regular for each $m \in \mathrm{Max}\ (C)$

(3) R_m is a division ring for each $m \in \mathrm{Max}\ (C)$.

PROOF. (1)$\Leftrightarrow$(2) (First Proof) By Theorem 10.8 a ring is strongly regular if and only if it is regular and reduced. It follows by Theorem 17.5 and Lemma 17.6 that R is strongly regular if and only if R_m is strongly regular for each $m \in \mathrm{Max}\ (C)$.

(1)$\Leftrightarrow$(2) (Second Proof) This also follows from Remark 17.5. Since (3)$\rightarrow$(2) is trivial we have to prove only (1)$\rightarrow$(3)

to complete the proof of the theorem. So let $x \in R$, $s \in C-m$ be such that x/s is a non-zero element of R_m. Let $xyx = x$. Now R is abelian by 10.4 and so $e = xy \in C$. Note that $e \notin m$: for $e \in m$ implies $1 - e \notin m$ and so $x/s = xe/s = 0/1$ in R_m (since $(1-e)(0.s-xe) = 0$) So we have $x/s. ys/e = xys/es = es/es = 1/1$ in R_m. Thus R_m is a division ring.

17.9. THEOREM. The following conditions are equivalent for a ring R.

(1) R is a left V-ring.

(2) For each multiplicatively closed subset S of C, the localisation $S^{-1}R$ is a left V-ring.

(3) For each $m \in \mathrm{Max}(C)$, R_m is a left V-ring.

PROOF. (1) $\longrightarrow$ (2). By 11.5(II) C is a regular ring. Hence, by 17.2, $S^{-1}R$ is a factor ring of R. Hence $S^{-1}R$ is also a left V-ring.

(2) $\longrightarrow$ (3) is trivial. We next prove (3) $\longrightarrow$ (1)

We shall prove that for each left R-module Q, $\mathrm{Rad}\ Q = 0$. The desired conclusion will follow from Theorem 14.5. So let $q \in \mathrm{Rad}\ Q$. Then under the map $h_Q : Q \rightarrow Q_m$, $h_Q(\mathrm{Rad}\ Q) \leq \mathrm{Rad}\ (Q_m)$. Hence $q/1 \in \mathrm{Rad}\ (Q_m) = 0$ (as R_m is a left V-ring) for each $m \in \mathrm{Max}(C)$. So $q = 0$.

Next we consider SF-rings. We can apply Proposition 17.1 here, since the centre of a left SF-ring is regular, by Proposition 15.5.

17.10. PROPOSITION. Consider the following conditions

(1) R is a left SF-ring.

(2) For each maximal ideal m of C the localisation R_m is a left SF-ring. Then (1) implies (2). If R is finitely generated as a C-algebra then (2) implies (1).

PROOF. Let R be a left SF-ring. Then, as C is regular, $R_m \cong R/mR$ is a factor ring of R. Hence by 15.6 R_m is a left SF-ring. Thus (1) implies (2).

Conversely, let R be a f.g. C-algebra. Then, for a multiplicatively closed subset S of C, Centre $S^{-1}R = S^{-1}C$; in particular, Centre $R_m = C_m$. Hence C_m is a regular ring (in fact, a field) for each m . So C is regular (by 9.13 or 17.6). Next, if S is a simple left R-module, $R_m = R/mR$ and $S_m = S/mS$ (by 17.1(2)) is either zero or a simple left R_m-module. Thus S_m is a flat left R_m-module for each m and S is therefore R-flat. Thus R is a left SF-ring.

As an application of the above proposition we shall show that left SF implies regular for rings which are f.g. (as modules) over their centres.

17.11. THEOREM. Let R be a ring which is finitely generated as a module over its centre C. If R is left SF, then R is regular.

PROOF. By 15.5 C is a regular ring. So C_m is a field and R_m is a finite-dimensional C_m-algebra. Hence R_m is a left Noetherian ring. By 17.10 R_m is a left SF-ring and hence, by 15.11(II) and 15.12, a regular ring. It follows by 17.6 that R is a regular ring.

17.12. PROBLEMS (I) In view of the fact that regular rings and left V-rings are both classes of p-V-rings, it is natural to ask the following question: Is it true that R is a left p-V-ring if and only if R_m is a left p-V-ring for each $m \in \mathrm{Max}\ (C)$?

(II) R left SF implies regular has been proved for a number of classes of rings above.(See 15.2, 15.9, 15.11(2), 17.11.) Each of these classes is a sub-class of the class of directly finite rings. It is therefore natural to expect the following result: if R is a directly finite SF-ring then R is a regular ring.

17.13. NOTES. In 17.1 we have extended [AFS:74, Lemma 1] to modules. Proposition 17.3 and Theorems 17.6 and 17.8 are in [AFS : 74]. However, our proofs are different; we have derived these results from element-wise results 17.2, 17.4 and 17.5. Our proof of (1)$\longrightarrow$(3) of Theorem 17.8 is also more direct. Theorem 17.9 is also in [AFS: 74]; our proof of (3) implies (1) uses 14.5 and so differs from the proof in [AFS : 74]. Proposition 17.10 and Theorem 17.11 occur in [R3 : 86].

§18. V'-rings.

In this section we shall consider the singular versions of the rings considered in §14. Alin and Armendariz [AA : 68] called a ring R a (right) T-ring if every non-zero R-module has a nonzero socle. They investigated (in [AA:68]) T-rings having the following property:

(*) Every singular, simple (right) R-module is injective.

In [RR:72] Ramamurthi and Rangaswamy called a ring R, a *generalised right V-ring*, or for short, *a right GV-ring* if every simple right R-module is injective or projective.

Rings satisfying condition (*) were called V'-rings by Tominaga. He introduced the following terminology in [T : 77] (See [M1: 80].)

18.1 *DEFINITIONS.* A ring R is a right V'-ring (f-V'-ring, p-V'-ring) if every simple, singular module is injective (f-injective, p-injective)

The following result was stated without proof in [AA:68] and [RR:72, §2]

18.2. *PROPOSITION.* A ring R is a right GV-ring if and only if R is a right V'-ring.

PROOF: "Only If" part. Let S be a simple, singular right R-module. We shall prove that S is not R-projective. Then

S must be R-injective.

Let x be a nonzero element of S. Consider the R-epimorphism $f: R \rightarrow S$ defined by $f(r) = xr$. Suppose, if possible, S is R-projective. Then f splits and so $R = K \oplus N$ where $K = \text{Ker} f$ and N is a minimal right ideal of R, since $N \cong S$. As $x \in S = Z(S)$, $xL = 0$ for some large right ideal L of R. So by the minimality of N, $N \cap L \neq 0$ implies $N \leq L$. So $xL = 0$. A contradiction. So S cannot be R-projective.

"If" part. Assume that R is a right V'-ring and suppose that S is a simple, non-injective right R-module. We shall prove S is R-projective. As S is simple non-injective, $Z(S) \neq S$, so $Z(S) = 0$. Hence if $x \in S$, $x \neq 0$, then for each large right ideal L of R, we have $xL \neq 0$, now $M = \text{ann}(x)$ is a maximal right ideal of R (since $R/M \cong xR = S$) such that $xM = 0$. Hence M is not large and hence must be a direct summand of R. So $M \oplus N = R$ implies $S \cong R/M \cong N$ is projective over R.

Alin and Armendariz proved the following result [AA:68, Theorem 1.1]

18.3 PROPOSITION. For a (right) T-ring R the following are equivalent .

(a) All singular simple (right) R-modules are injective.

(b) $Z(R_R) = 0$ and $\mathrm{Rad}(R/I) = 0$ for any large right ideal I of R.

Ramamurthi and Rangaswamy [RR:72] obtained the following generalisation of 18.3.

18.4. THEOREM. The following are equivalent for any ring R.

(1) (a) for every large right ideal I of R, we have $I = I^*$, and (b) $Z(R_R) \cap J(R) = 0$.

(2) R is a right V'-ring

(3) For each right R-module M with $Z(M)$ large in M we have $J(M) = 0$.

(4) (a) For any right R-module M, if N is large in M, we have $N = N^*$ and (b) $Z(M) \cap J(M) = 0$.

PROOF: See [RR:72, Theorem 3.3]

18.5 EXAMPLE. The following example shows that (1) (a) of Theorem 18.4 does not by itself imply that R is a right V'-ring. Let $R = Z/(p^2)$ (p a prime number), then R has only 3 ideals 0, pR and R. So (1)(a) holds, but R/pR is neither injective nor projective over R.

18.6. PROPOSITION If R is a right V'-ring in which every primitive idempotent is central, then R is a right V-ring.

PROOF. Let S be a simple, projective right R-module. Then $S \cong eR$, where e is a primitive idempotent. Let $I \leq R_R$ and $f: I \longrightarrow eR$ any nonzero R-homomorphism with kernel K. Then f is onto and $I = K \oplus T$, where $T \cong eR$. Since e is central eR is a fully invariant submodule of R. Hence $eR = T$. Clearly, $K \leq (1-e)R$. Define $g: R \longrightarrow eR$ by $g = 0$ on $(1-e)R$ and $g = f$ on eR. Then g extends f. Hence S is injective. Since by hypothesis, every simple right module is either projective or injective it follows that R must be a right V-ring.

18.7. COROLLARY. If R is an abelian right V'-ring, then R is a right V-ring.

18.8. THEOREM. If R is a commutative ring then the following conditions are equivalent

(1) R is a V'-ring

(2) R is a V-ring

(3) R is a regular ring

PROOF (1)$\longrightarrow$(2) by Proposition 18.6 and (2)$\longrightarrow$(1) is trivial. (2)$\longleftrightarrow$(3) has been seen earlier. (14.21).

In §4 of [RR: 72] the authors consider semi-prime V'-rings. They begin with the following lemma (cf. 18.16 below).

18.9 LEMMA. In any right V'-ring each large right ideal is idempotent.

They prove the following results:

18.10 PROPOSITION. Let R be a ring in which every large right ideal is two-sided. Then R is semi-prime right V'-ring if and only if R is right weakly regular.

18.11 PROPOSITION. Any prime, right V'-ring R is right weakly regular.

18.12. THEOREM. The following are equivalent for any ring R.

(1) R is a semi-prime, right noetherian right V'-ring.

(2) R is either a quasi-simple, right noetherian right V-ring with zero socle or is a semi-simple ring.

(3) Every semi-simple right R-module is injective.

In [M:76] Ming considers p-V'-rings. Most of his results are consequences of a lemma, which we shall state in a modified form.

18.13. LEMMA. Let b be an element of R. Assume that every simple, singular left R-module be Rb-complete. Then there exists a left ideal K such that R = (RbR+$\underline{l}$(b)) $\oplus$ K.

PROOF:- Denote the left ideal RbR+l(b) by J. By Zorn's lemma, there exists a left ideal K such that J $\oplus$ K is large in R. If J+K $\neq$ R, let L be a maximal left ideal containing J+K. Then R/L is simple, singular (8.9) and hence Rb-complete. Define

$g: Rb \rightarrow R/L$ by $g(rb) = r+L$ for all r in R.

Then g is a well-defined R-linear map (since $l(b) \leq L$) and since R/L is Rb-complete, there exists c in R such that $g(rb) = rb(c+L)$ for all r in R. In particular $1+L = g(b) = bc+L$. Since $bc \in RbR \leq L$ we get $1 \in L$, contradicting the maximality of L. Thus $(RbR + l(b)) \oplus K = R$.

18.14. COROLLARY. If R is a left p-V'-ring, then for each element b of R, there exists a left ideal K such that $R = (RbR + l(b)) \oplus K$.

18.15 THEOREM. The following conditions are equivalent.

(1) R is strongly regular.

(2) R is a left duo, left p-V'-ring.

PROOF: $(1) \rightarrow (2)$ by 14.20

$(2) \rightarrow (1)$. Let $b \in R$. By 18.13, $R = (RbR+l(b)) \oplus K$ for some left ideal K of R. As R is left duo, $R = (Rb+l(b)) \oplus K$ and $KRb \leq Rb \cap K = 0$ implies $K = 0$ and hence $R = Rb + l(b)$. So $1 = ab+d$, $a \in R$, $d \in l(b)$. So $b = ab^2$, showing R is strongly regular.

The following result [M1 : 76, Proposition 3(iii)] extends 18.9

18.16 PROPOSITION. Let R be a left p-V'-ring. Then every large left ideal of R is idempotent.

PROOF. Let I be a large left ideal of R. Let $b \in I$ and let $IR + l(b) = J$, a large left ideal of R. If $J \neq R$, let L be a maximal left ideal of R containing J. Then R/L is p-injective.

Hence we get a contradiction as in Lemma 18.13. Therefore $R = IR + l(b)$ and $1 = u+d$, $u \in IR$, $d \in l(b)$ implies $b = ub \in IRI = I^2$. So $I = I^2$.

The next result extends 18.11

18.17. PROPOSITION. Let R be a semi-prime, left p-V'-ring. Then R is left weakly regular.

PROOF. Let, if possible, $Rb \neq (Rb)^2$. Since R is semi-prime, $(Rb)^2$ is a large submodule of Rb. Let $L/(Rb)^2$ be a maximal submodule of the f.g. nonzero R-module $Rb/(Rb)^2$. Then Rb/L is simple, singular by 8.9 and so p-injective. If $g: Rb \rightarrow Rb/L$ is the canonical map, then there exists $c \in R$ such that $g(ab) = ab(cb+L)$ for all $a \in R$. Then $b-bcb \in L$ implying $b \in L$. So $Rb = L$, a contradiction. This shows $Rb = (Rb)^2$.

§19. Kaplansky's question.

In this section we shall consider a condition weaker than quasi-simplicity and stronger than primeness, namely, that of primitivity.

19.1. DEFINITION. A ring R is called (left) primitive if there exists a simple, faithful, left R-module.

19.2. REMARKS. (I). Let R be a quasi-simple ring (with 1) and M a maximal left ideal of R. Let $S = R/M$. Then

$A = \mathrm{ann}_R(S)$ is an ideal of R such that $A \neq R$ (as $1.S = S \neq 0$.) Hence $A = 0$. So S is faithful, simple left R-module. Thus R is left, (and right) primitive.

(II). Let R be a left primitive ring. Then there exists a faithful, simple left R-module S. Let A,B be nonzero ideals of R. Then S faithful $\to BS \neq 0 \to BS = S \to ABS = AS \neq 0 \to AB \neq 0$. Thus R is a prime ring.

(III). Let R be a commutative primitive ring. Then if S is a simple, faithful R-module, $S \cong R/M$ where M is a maximal ideal of R. So $M = \mathrm{ann}_R(S) = 0$ implying R is a field.

(IV). For any non-zero vector space V_D (D a division ring) $R = \mathrm{End}\,(V_D)$ is a left primitive ring; for V itself is a simple, faithful left R-module. If V is infinite dimensional then R is not quasi-simple (See Section 8.)

(V). Let S be a simple, faithful left R-module and $x \in \mathrm{Rad}\, R$. Then $S \cong R/M$ where M is a maximal left ideal of R and $x\bar{t} = \overline{xt} = 0$ in R/M, since $xt \in \mathrm{Rad}\, R \leq M$. So $xS = 0$ implying $x = 0$. Thus $\mathrm{Rad}\, R = 0$. This shows that primitive rings are semi-primitive (as expected from the terminology).

(VI). It follows from the above remarks, and known facts from Chapter I that we have the following chart:

$$\begin{tikzcd}
 & \text{division ring} \arrow[r] \arrow[d] & \text{simple ring} \arrow[r] \arrow[d] & \text{semi-simple ring} \arrow[ddl, dashed] \\
 & \text{domain} \arrow[d] \arrow[dl] & \text{quasi-simple ring} \arrow[d] \arrow[dl] & \\
\text{reduced ring} \arrow[dr] & \text{prime ring} \arrow[d] & \text{primitive ring} \arrow[l] \arrow[d] & \\
 & \text{semi-prime ring} & \text{semi-primitive ring} \arrow[l] &
\end{tikzcd}$$

The rest of this section will be devoted to the following query of Kaplansky ([K III, p.2]).

19.3. QUESTION. Are prime, regular rings primitive?

19.4. ELEMENTARY CASES. In the following remarks, R denotes a prime, regular ring.

(I). If R is commutative, then R is a commutative, regular domain and hence a field. So R is primitive. This result will be extended in (III) and (IV) below.

(II). Let R be left noetherian. Then R is semi-simple(9.11 and prime. Hence $R \xrightarrow{\sim} M_n(D)$, where D is a division ring, by Wedderburn's structure theorem. Hence by 19.2 (IV), R is primitive.

(III). Suppose that R is a f.g. module over its centre C. Then C is a regular, domain and hence a field. So R is a finite-dimensional C-algebra and hence a left artinian ring, So by Case (II) R is primitive.

(IV). Suppose R is abelian. Then R prime, regular implies that for each $a \neq 0$, $Ra = Re$ coincides with R. (Otherwise $Re.R(1-e) = 0$ gives a contradiction.) So R is a division ring.

Hence R is primitive.

Next we report on some recent work on Kaplansky's question. Goodearl studied prime ideals in regular, right self-injective rings in [G1: 73]. He proved the following result.

19.5. PROPOSITION. [G1: 73, Corollary 16]. Let R be prime regular, right self-injective. Then R is left and right primitive.

Next we consider the results of Fisher and Snider [FS:74]. First, a definition (which does not occur there).

19.6. DEFINITION. A ring is called anti-regular if every nonzero element has a nonzero 2-inverse.(See 10.12 .)

19.7. REMARK. By 10.13 regular implies anti-regular.

19.8. REMARK. If R is a prime ring then the set $\underline{S}$ of nonzero ideals of R is a directed set since I,J in $\underline{S}$ implies $I \cap J$ in $\underline{S}$ (since $I \cap J$ contains $IJ \neq 0$).

19.9. DEFINITION. A ring R is said to have a countable co-final subset of ideals if there exists a countable subset C of the set $\underline{S}$ of all nonzero ideals of R such that for each A in $\underline{S}$ there is an I in C such that $I \leq A$.

19.10 THEOREM. Let R be a prime anti-regular ring. Suppose R has a countable co-final subset of ideals. Then R is left and right primitive.

PROOF. See [FS:74] .

19.11. COROLLARY. If R is a countable, prime, regular ring, then R is primitive.

PROOF: The principal ideals form a countable co-final subset.

Fisher and Snider have also proved a number of results giving sufficient conditions for the primitivity of prime, regular group rings.

19.12. REMARK. Domanov [D1:77] has shown that the answer to Kaplansky's question, in all its generality, is negative by giving an example of a prime, regular ring which is not primitive.

§20. Polynomial rings over regular rings

Let R be a ring and $S = R[X]$ the polynomial ring in one indeterminate X over R. The basic properties of $R[X]$ are well known. Namely, S is commutative if and only if R is; S is a domain if and only if R is; S is a P.I.D. (principal ideal domain) if and only if R is a field.

In [M3 : 73] McCarthy proved the following result.

20.1. PROPOSITION. If R is commutative regular, then $S = R[X]$ is semi-hereditary.

In [C1 :74] Camillo proved:

20.2. PROPOSITION. If $R[X]$ is a commutative, semi-hereditary ring, then R is regular.

Polynomial ring over non-commutative regular rings were considered in [R 3: 75] and [P2 : 80].

The following result was proved in [R3 : 75]. The proof uses proposition 20.1 and the theory of Azumaya algebras.

20.3. PROPOSITION. Let R be a ring which is f.g. as a module over its centre C. Then $R[X]$ is left and right semi-hereditary.

Pillay [P2 : 80] proved the following Theorem 20.4. We shall reproduce it with proof which is selfcontained.

20.4. THEOREM. The following are equivalent

(1) R is regular

(2) For each $a \in R$, $aS + XS$ is a projective right ideal of S.

(3) For each $a \in R$, $Sa + SX$ is a projective left ideal of S.

PROOF: (1) implies (2). Let $a \in R$. Then there exists b in R such that $a = aba$. Let $e = ab$, $f = e+(1-e)X$. Then we have the equations:

$$a = fa \text{ and } X = f(1-e+eX) = (1-e+eX)f \qquad \text{(I)}$$

It follows that $aS + XS = fS$.
Now let $g = g(X) \in R[X]$ be such that $fg = 0$. Using (I) we get $Xg = 0$. So f is a nonzero divisor in S. So $S \xrightarrow{\sim} f \cdot S$. So $aS+XS$ is free and so projective. This proves (1) implies (2).

(2) implies (1). Fix $a \in R$ and let $K = aS+XS$, a projective right ideal of S, by hypothesis. By the dual basis lemma for projective modules (see, e.g., [FI, p. 141]) there exist S-homomorphisms u, v from K into S, such that for every $k \in K$, $k = au(k) + Xv(k)$. In particular, $a - au(a) = Xv(a)$. As X is central in S and u is an S-homomorphism, $Xu(a) = u(a)X = u(X)a$ so that we have,

$$aX - au(X)a = X^2 v(a). \qquad \text{(II)}$$

Write $u(X) = u_0 + u_1 X + \dots + u_t X^t$ and equate the coefficients of X on both sides of (II). Then we get $a = au_1 a$. Hence R is regular.

The equivalence of (1) and (3) follows from the left-right symmetry of (1).

20.5 <u>COROLLARY</u>. If $R[X]$ is either left or right semi-hereditary, then R is a regular ring.

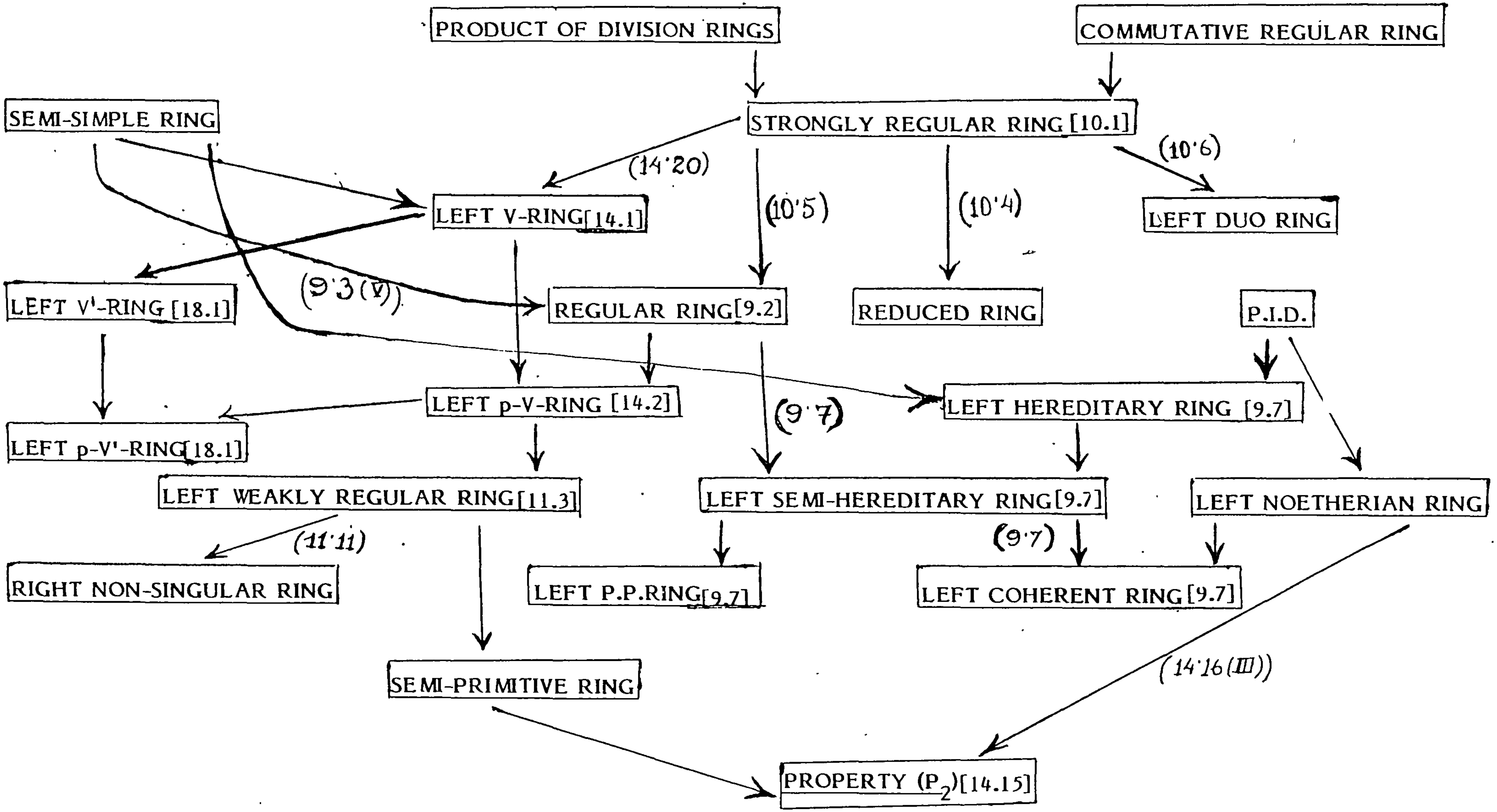
20.6 <u>REMARKS</u> (I) Corollary 20.5 also follows from a theorem of Jensen [J : 66] about weak global dimensions. This was noted in [R3 : 75 , Remark 3.13] and in [P2: 80] .

(II) The converse of Proposition 20.3 is clearly false. Let D be a division ring which is infinite dimensional over its centre K. Then $D[X]$ is a left and right principal ideal domain and hence a left and right semi-hereditary ring.

(III) As noted in 9.7 left semi-hereditary implies left coherent. Thus we can replace semi-hereditary by coherent in 20.1 and in 20.3. Coherence of polynomial rings and related problems were studied by Soublin in [S 6 : 70] and a number of other papers.

(IV) Professor Jensen mentions in his review of [P 2 : 80] (Mathematical Reviews, 81g : 16018) that Prof. Jøndrup has shown (unpublished) that there exists a regular ring R such that $R[X]$ is neither left nor right coherent and hence neither left nor right semi-hereditary. Hence Proposition 20.3 cannot be extended to arbitrary regular rings.

CHART FOR CHAPTER II



Notes: (1) The left-right dual of this chart is also clearly valid.
(2) The reference in square brackets is to the paragraph where the concept is defined. The reference in round brackets is the paragraph where the implication is stated/proved.

BIBLIOGRAPHY

The bibliography is divided into two parts: (I) books and monographs and (II) research papers. In each part the items are arranged alphabetically according to the initial letter of the name of the first author. However, in part (II), this rule is followed only to the extent of grouping together all references coming under a certain letter.

Many citations are followed by the review number in Mathematical Reviews (MR). This is followed by the "appearance number" in Reviews in Ring Theory (RRT) published by the American Mathematical Society, Providence, Rhode Island. We could not get hold of certain references because of inadequate library facilities. We know about them through MR or through references in other papers. These items are starred.

BOOKS AND MONOGRAPHS

[AF] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, Springer Verlag, New York, 1974 MR54 #5281 RRT 31.01.047

[AM] M.F. Atiyah and I.G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, Massachusettes, 1969.

[BII8] N. Bourbaki, Eléments de Mathematique, Livre II, Chapitre 8, Modules et Anneaux Semi-simples, Hermann et Cie, Paris, 1958.

[CE] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.

[FI] C. Faith, Algebra: Rings, Modules and Categories, Springer-Verlag, New York, 1973.

[FII] C. Faith, Algebra II, Ring Theory, Springer-Verlag, New York, 1976.

[FIII] C. Faith, Lectures on Injective Modules and Quotient Rings, Springer Lecture Notes No.49, Springer-Verlag, 1967.

[GI] K.R. Goodearl, Ring Theory: Non-singular Rings and Modules, Pure and Applied Mathematics Series, Vol. 33, Marcel Dekker, New York, 1976. MR55 #2970 RRT 31.02.051.

[GII] K.R. Goodearl, Von Neumann Regular Rings, Pitman. London, 1979.

[H] I.N.Herstein, Notes from a Ring Theory Conference, CBMS series, American Math. Society, Providence, 1971. MR 47 #1840 RRT 31.02.017

[JI] N. Jacobson, Basic Algebra, Vol. I, Hindustan Publishing Corporation, 1984.

[JII] N. Jacobson, Basic Algebra, Vol.II, Hindustan Publishing Corporation, 1984.

[JIII] N. Jacobson, Structure of Rings, Revised Edition, Amer. Math. Society, Providence, 1968. MR 36 #5158 RRT 31.01.019

[KI] I. Kaplansky, Rings with Operators, W.A. Benjamin, New York, 1968.

[KII] I. Kaplansky, Fields and Rings, Second Edition, The University of Chicago Press, Chicago, 1972.

[KIII] I. Kaplansky, Algesbraic and Analytic Aspects of Operator Algebras, CBMS series, American Math. Society, Providence, 1970.

[L] J. Lambek, Lectures on Rings and Modules, Blaisdell, , Waltham, 1966. MR 34 #5857 RRT 31.01.014

[R] J. J. Rotman, Notes on Homological Algebra, Van Nostrand Reinhold, New York., 1970.

[SK.] L.A. Skornyakov, Dedekindovi Strukturi s Dopolneniyem i Regulyarniye Koltsa, Gosudarstvennoye Izdatelstvo Fiziko-Mathematicheskoi Literaturi, Moscow, 1961 (English translation: Complemented Modular Lattices and Regular Rings, Oliver and Boyd, 1964.)

[St.] B. Stenström, Rings of Quotients: An Introduction to the Methods of Ring Theory, Springer-Verlag, New York, 1975. MR 52 #10782 RRT 31.01.045

[vN] J. von Neumann, Continuous Geometry, Princeton University, Princeton, 1960.

RESEARCH PAPERS

★[A:73] G. Azumaya, Some properties of TTF-classes, Proceedings of the conference on orders, group rings and related topics, Springer Lectures Notes on Mathematics, No. 353, 1973. MR 49 #2840 RRT 26.03.023

[AA:68] J.S. Alin and E.P. Armendariz, A class of rings having all singular simple modules injective, Math. Scand., 23 (1968), 233-240. MR 40 #4300 RRT 20.04.003.

[AFS:74] E.P. Armendariz, J.W. Fisher, and S.A. Steinberg, Central localizations of regular rings, Proc. Amer. Math. Soc., 46(1974), 315-321. MR 50 #4651 RRT 17.09.024.

[AI:82] J. Ahsan and A.S. Ibrahim, Some homological characterizations of regular rings, Comm. in Algebra, 10(1982), 887-912.

[B:40] R. Baer, Abelian groups that are direct summands of every containing abelian group, Bull. Amer. Math. Soc., 46(1940), 800-806. MR 2 #126.

[B1:83] G.F. Birkenmeier, Idempotents and completely semi-prime ideals, Comm. in Algebra, 11(1983), 567-580.

[B3:73] S.H. Brown, Rings over which every simple module is rationally complete, Canadian J. Math., 25(1973), 693-701. MR52 #492 RRT 30.04.033

[BM:50] B. Brown and N.H. McCoy, Some theorems on groups with applications to ring theory, Trans. Amer. Math. Soc., 69(1950), 302-311.

[C1:74] V.P. Camillo, Semi-hereditary polynomial rings, Proc. Amer. Math. Soc., 45(1974), 173-174. MR 50 #4652 RRT 25.03.035

[C2:60] S.U. Chase, Direct products of modules, Trans. Amer. Math. Soc., 97(1960), 457-473. MR 22 #11017. RRT 20.01.004.

[C3:75] R.R. Colby, Rings which have flat injective modules, J. of Algebra, 35(1975), 235-252. MR 51 # 12938 RRT 16.05.016.

[C4:70] J.H. Cozzens, Homological properties of the ring of differential polynomials, Bull. Amer. Math. Soc., 76(1970), 75-79. MR 41 #3531. RRT 23.18.013.

[CDR:88] I. Choudhuri, M.I. Devi and M.B. Rege, Hopfian and cohopfian modules, NEHU technical report, 1988.

[CE:81] T. Cheatham and E. Enochs, Regular modules, Math. Japon., 26(1981), 9-12.

[CT:73] K. Chiba and H. Tominaga, On strongly regular rings, Proc. Japan Acad., 49(1973), 435-437. MR48 #6161 RRT 17.08.021.

[CS:69] V.C. Cateforis and F.L. Sandomierski, On Commutative rings over which the singular submodule is a direct summand for every module, Pacific J. Math., 31(1969), 289-292.

[D:79] R.F. Damiano, Coflat rings and modules, Pacific J. Math.,81(1979), 349-369. MR 81e : 16017.

*[D1:77] O.I.Domanov, A prime but not primitive regular ring, Uspehi Mat. Nauk, 32(198) (1977), 219-220. MR58 #28058. RRT 24.01.023

[F:66] C. Faith, Rings with ascending condition on annihilators, Nagoya Math. J., 27(1966), 179-191. M R 33 #1328 RRT 19.01.004

[F:72] C. Faith, Modules finite over endomorphism rings, Lectures on rings and modules, Springer Lecture Notes on Mathematics, No. 246, 145-192. MR 49 #7287 RRT 16.04.016

[F1:74] J.W. Fisher, von Neumann regular rings versus V-rings, Ring Theory: proceedings of the Oklahoma Conference (Lecture Notes No. 7, Mareel Dekker, New York, 1974), 101-119.

[FS:74] J.W. Fisher and R.L. Snider, Prime von Neumann regular rings and primitive group algebras, Proc. Amer. Math. Soc., 44(1974), 244-250. MR 49 #7297 RRT 24.01.013.

[G1:73] K.R. Goodearl, Prime ideals in regular self-injective rings, Canadian J. Math., 25(1973), 829-839, MR 48 #8561 RRT17. 03.005.

[G2:84] V. Gupta, A generalization of strongly regular rings,Acta Math. Hungarica, 43(1984),57-61.

[H:81] Y. Hirano, Regular modules and V-modules, Hiroshima Math. J., 11(1981), 125-142.

[IN:54] M.Ikeda and T. Nakayama, On some characteristic properties of quasi-Frobenius and regular rings, Proc. Amer. Math. Soc., 5(1954), 15-19.

[J:66] C.U. Jensen, On homological dimensions of rings with countably generated ideals, Math. Scand., 18(1966), 97-105.MR34 #7611 RRT25.01.013.

[M1:74] R.Y.C.Ming, On simple p-injective modules, Math. Japon., 19(1974), 173-176. MR 53 #3018 RRT 16.05.018.

[M1:76] R.Y.C.Ming, On von Neumann regular rings-II, Math. Scand., 39(1976), 167-170, MR55 #8092 RRT 17.10.025

[M1:81] R.Y.C.Ming, On V-rings and unit -regular rings, Rend. Sem. Mat. Univ. Padova, 64(1981), 127-140.

[M1:80] R.Y.C.Ming, On V-rings and prime rings, J. Algebra, 62(1980), 13-20.

[M2:71] N.H. McCoy, Completely prime and completely semi-prime ideals in rings, Colloquia Mathematica Societalis Janos Bolyai, Vol.6, Rings, Modules and Radicals, Keszthely (Hungary), 1971. MR51 #613 RRT1.05.025

[M3:73] P.J. McCarthy, The ring of polynomialas over a von Neumann regular ring, Proc. Amer. Math. Soc., 39(1973), 253-254. MR 47 #5043 RRT 17.06.003

[M4:85] E. Matlis, Commutative semi-coherent and semi-regular rings, J. Algebra, 95(1985), 34 3-372.

[MV:73] G. Michler and O.E. Villamayor, On rings whose simple modules are injective, J. Algebra, 25(1973), 185-201. MR47 #5052 RRT 23.06.011

[MV:77] F.F. Mbuntum, and K. Varadarajan, Half-exact pre-radicals, Comm. in Algebra, 5(1977), 555-590. MR 55 #10504 RRT 26.06.034.

[O1:71] M. Orzech, Onto endomorphisms are isomorphisms, Amer. Math. Monthly, 78(1971),357-362.

[O2:64] B.L. Osofsky, Rings all of whose finitely generated modules are injective, Pacific J. Math., 14(1964), 645-650. MR 28 #5090 RRT 4.01.007

[O2:68] B.L. Osofsky, Non-injective cyclic modules, Proc.Amer. Math.Soc., 19(1968), 1383-1384. MR 38 #185 RRT 4.01.008

[P1:75] R.D. Peterson, One-sided inverses in rings, Canadian J. Math., 27(1975), 218-224 MR 54 #10322 RRT27.01.047

[P2:80] P. Pillay, On semihereditary noncommutative polynomial rings, Proc. Amer.Math. Soc., 78(1980) 473-474 MR 81g: 16018

[R1:73] V.S. Ramamurthi, Weakly regular rings, Can. Math. Bull., 16(1973), 317-322. MR 48 #11192 RRT 17.09.022

[R1:75] V.S. Ramamurthi, On the injectivity and flatness of certain cyclic modules, Proc. Amer. Math. Soc., 48 (1975), 21-25. MR 50 #7256 RRT 17.06.006

[R2:74] R. Raphael, Some remarks on regular and strongly regular rings, Can. Math. Bull., 17(1974), MR 52 #480 RRT 17.08.029

[R3:75] M.B. Rege, Some problems in algebra, Ph.D. thesis submitted to Bombay University, 1975

[R3:86] M.B. Rege, On von Neumann regular ring s and SF rings, Math.Japon, 31(1986), 927-936 MR 88b:16024

[R4:73] G. Renault, Anneaux reguliers anto-injectifs a droite, Bull. Soc. Math. France, 101(1973), 237-254. MR51 #8171 RRT 17.03.008

[R5:75] E.A. Rutter, Jr., Rings with the principal extension property, Comm. in Algebra, 3(1975), 203-212. MR52 #3222 RRT 19.03.020

[RR:72] K.M. Rangaswamy and V.S. Ramamurthi,Generalized V-rings, Math. Scand., 31(1972), 69-77. MR 48 #339 RRT 16.05.011

[RV1:72] K.M. Rangaswamy and N. Vanaja, Quasi-projectives in abelian and module categories, Pacific J. Math., 43(1972), 221-238

[RV2:77] M.B. Rege and K. Varadarajan, Chain conditions and pure-exactness, Pacific J. Math., 72(1977) 223-225. MR56 #8632 RRT 23.20.015

[RZ:59] A. Rosenberg and D. Zelinsky, Finiteness of the injective hull, Math. Zeitschrift 70(1959), 372-380 MR21 #4176 RRT 16.01.002

[S1:68] H.H. Storrer, Epimorphismen von kommutativen ringen, Comm. Math. Helv., 43(1968), 378-401.

[S2:70] F.L. Sandomierski,Some examples of right self-injective rings which are not left self-injective, Proc. Amer. Math. Soc., 26(1970), 244-245 MR 42 #324 RRT 19.02.023

[S3:73] Gooyong Shin, Prime ideals and sheaf representation of a pseudo-symmetric ring, Trans. Amer. Math. Soc., 184(1973), 43-60. MR 49 #2825 RRT 1.05.023

[S4:80] T. Savage, Generalised inverses in regular rings, Pacific J. Math., 87(1980), 455-466.

[S5:65] E. Snapper, Injective modules under change of rings, Proc. Amer. Math. Soc., 16(1965), 788-793.

[S6:70] J.P. Soublin, Anneaux et modules coherents, J. Algebra, 15(1970), 455-472. MR 41 #5422 RRT 25.02.013

[SV:74] B. Sarath and K. Varadarajan, Injectivity of certain classes of modules, J. Pure and Applied Algebra,5(1974), 293-305. MR 50 #7239 RRT 17.08.025

[T:76] H. Tominaga, On s-unital rings,Math. J. Okayama Univ., 18(1976), 117-133 MR 54 #7532 RRT 17.08.032

[T:77] H. Tominaga, On s-unital rings II, Math. J. Okayama Univ., 19(1977), 171-182. MR 56 #12054 RRT 30.06.053

[U:65] Y. Utumi, On continuous rings and self-injective rings, Trans.Amer. Math.Soc., 118(1965), 158-173. MR 30 #4793 RRT 16.02.002

[V:76] K. Varadarajan,Study of certain radicals,J. Pure Applied Algebra, 9(1976), 107-120 MR 54 #10316 RRT 2.08.011

*[vN:36] J. von Neumann, On regular rings, Proc. Nat. Acad. Sci. U.S.A., 22(1936), 707-713.

[Y:86] M.F. Yousif, SI-modules,Math. J. Okayama Univ., 28(1986), 133-146.

[Z:76] W. Zimmermann, Über die aufsteigende Kettenbedingung fur Annullatoren, Archiv der Math., 27(1976), 261-266 MR 54 #7544 RRT 16.01.049.